Lecture 6: Representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ (part I)

Contents

1. Smooth representations of $\mathrm{SL}_2(\mathbb{Q}_p)$

Recall that $\mathrm{SL}_2(\mathbb{Q}_p)$ is a locally compact totally disconnected topological group: the identity (and thus every element) admits a neighborhood basis consisting of open compact subgroups. Explicitly, we can take the subgroups $K_p(f)$ of $\mathrm{SL}_2(\mathbb{Z}_p)$ consisting of matrices congruent to the identity mod p^f. These are normal subgroups of $K_0 = \mathrm{SL}_2(\mathbb{Z}_p)$, being the kernel the reduction homomorphism mod p^f. Thus $\mathrm{SL}_2(\mathbb{Q}_p)$ is a locally profinite group.

Here we will be interested in the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{GL}_2(\mathbb{Q}_p)$.

1.1. **Decompositions.** We recall three standard decompositions of $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let Z be the center of G, $K_0 = \mathrm{GL}_2(\mathbb{Z}_p)$, B the standard Borel subgroup of upper triangular matrices, N its unipotent radical.

Lemma 1.1. *We have*

(1) *(Iwasawa)* $G = BK_0$.
(2) *(Bruhat)* $G = B \cup BwN$, *where* $w = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)$.
(3) *(Cartan)* $G = K_0 \Lambda K_0$, *where* $\Lambda = \{\mathrm{diag}(\varpi_p^m, \varpi_p^n) : m \geq n\}$.

1.2. **Smooth representations.** We begin the theory of complex linear representations of $\mathrm{SL}_2(\mathbb{Q}_p)$. In contrast to the case for representations of real groups, such as $\mathrm{SL}_2(\mathbb{R})$, we shall not require that the vector space on which $\mathrm{SL}_2(\mathbb{Q}_p)$ acts be a topological vector space. So for this section, V will just be a complex vector space, usually of infinite dimension. (For this paragraph, the group $\mathrm{SL}_2(\mathbb{Q}_p)$ could be replaced by any locally profinite group.)

Definition 1. A representation (π, V) of $\mathrm{SL}_2(\mathbb{Q}_p)$ is called **smooth** if it verifies one of the following four equivalent conditions:

(1) the stabilizer of every $v \in V$ in $\mathrm{SL}_2(\mathbb{Q}_p)$ is an open subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$;
(2) the stabilizer of every $v \in V$ in $\mathrm{SL}_2(\mathbb{Q}_p)$ contains a compact open subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$;
(3) V can be written as

$$V = \bigcup_K V^K$$

the union going over all compact open subgroups K of G. Here V^K denotes the K-invariants, namely, the subspace of $v \in V$ satisfying $\pi(k)v = v$ for every $k \in K$.

(4) the map $\mathrm{SL}_2(\mathbb{Q}_p) \times V \to V$ is separately continuous, when V is given the *discrete topology.*

The equivalence of these definitions is clear. (That the first implies the second follows from the fact that $\mathrm{SL}_2(\mathbb{Q}_p)$ is totally disconnected.)

Since there are no topological structure on V, the notion of irreducibility is just the most obvious one.

Definition 2. A smooth representation is said to be **irreducible** if there are no proper non-trivial G-invariant subspaces. Two smooth representations (π, V) and (π', V') of $\mathrm{SL}_2(\mathbb{Q}_p)$ are said to be **isomorphic** if there is a linear map $m : V \to V'$ such that $\pi(g) \circ m = m \circ \pi'(g)$ for every $g \in \mathrm{SL}_2(\mathbb{Q}_p)$.

1.3. **Smooth vectors.** In general, given an arbitrary representation (π, V) of $\mathrm{SL}_2(\mathbb{Q}_p)$, the representation (π, V^∞) obtained by restricting the action of G to the subspace

$$V^\infty := \bigcup_K V^K$$

of **smooth vectors** (those admitting open stabilizers) is a smooth representation.

Example 1: A smooth *irreducible* representation (π, V) of a *compact* locally profinite group G, such as $K_0 = \mathrm{SL}_2(\mathbb{Z}_p)$, is finite-dimensional. To see this, take a non-zero vector $v \in V$. By the smoothness assumption, it lies in V^K for some open subgroup K of G. By the irreducibility of V, the right-translates $\{\pi(g)v : g \in G/K\}$ span V. The subgroup $K' = \bigcap_{g \in G/K} gKg^{-1}$ then acts trivially on V. Thus π factors through the quotient group G/K', which, since G is compact, is finite.

Example 2: Let $V = C(\mathrm{SL}_2(\mathbb{Q}_p))$ be the space of continuous complex-valued functions on $\mathrm{SL}_2(\mathbb{Q}_p)$. Then $\mathrm{SL}_2(\mathbb{Q}_p)$ acts on V by the right-regular representation.

Let $C^\infty(\mathrm{SL}_2(\mathbb{Q}_p))$ be the subspace of *locally constant* functions: $f \in C^\infty(\mathrm{SL}_2(\mathbb{Q}_p))$ if, for every $g \in \mathrm{SL}_2(\mathbb{Q}_p)$, there exists a compact open subgroup K in $\mathrm{SL}_2(\mathbb{Q}_p)$ (depending on g) such that $f(gk) = f(g)$ for all $k \in K$. In other words, a complex-valued function on $\mathrm{SL}_2(\mathbb{Q}_p)$ is locally constant if it is continuous for the discrete topology on $\mathbb{C}$. Right-translation preserves $C^\infty(\mathrm{SL}_2(\mathbb{Q}_p))$ but this is *not* a smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$. Indeed, the space V^∞ of smooth vectors of $V = C(\mathrm{SL}_2(\mathbb{Q}_p))$ is the subspace of functions f for which there exists an open compact subgroup K (depending only on f) of $\mathrm{SL}_2(\mathbb{Q}_p)$ such that for every $g \in G$ and $k \in K$ we have $f(gk) = f(g)$. Since $\mathrm{SL}_2(\mathbb{Q}_p)$ is not compact, these two notions do not agree.

Now let $V = C_c(\mathrm{SL}_2(\mathbb{Q}_p))$ be the space of continuous *compactly supported* complex-valued functions on $\mathrm{SL}_2(\mathbb{Q}_p)$. Once again $\mathrm{SL}_2(\mathbb{Q}_p)$ acts on V by the right-regular representation. In this case we do have $V^\infty = C_c^\infty(\mathrm{SL}_2(\mathbb{Q}_p))$. Indeed, to see that a locally constant compactly supported function is smooth, for every g in the support of f let K_g be a compact open subgroup such that f is constant on gK_g. Then the collection of gK_g form an open cover of the support of f. Take a finite subcovering $g_iK_{g_i}$ (this is possible since f has compact support). Then $K = \cap_i K_{g_i}$ is a compact open subgroup such that $f(gk) = f(g)$ for all $g \in \mathrm{SL}_2(\mathbb{Q}_p)$ and $k \in K$.

Example 3: Let (π, V) be any smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$. Consider the dual space $V^* = \mathrm{Hom}(V, \mathbb{C})$ of linear functionals into $\mathbb{C}$. Then V^* is a representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ under the action $(g.\ell)(v) = \ell(\pi(g^{-1})v)$. One writes $\widetilde{\pi}$ for the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ on the smooth part $\widetilde{V} = (V^*)^\infty$ of V^*. Then $\widetilde{\pi}$ is called the **contragredient representation**.

Given a representation (π, V) of $\mathrm{SL}_2(\mathbb{Q}_p)$, the subspace V^∞ of smooth vectors could very well be reduced to 0. However, this cannot happen for non-zero *topological* representations V, such as Hilbert space representations.

Theorem 1.1. *If (π, V) is a Hilbert space representation, then V^∞ is dense in V.*

Proof. Let $v \in V$ and take $\epsilon > 0$. We shall show that there is a compact open K and a vector $v' \in V^K$ such that $\|v - v'\| < \epsilon$. By the continuity of the orbit map $(g, v) \mapsto \pi(g)v$, there is an open U in G such that $\|\pi(g)v - v\| < \epsilon$ for all $g \in U$. Since G is locally profinite there is a compact open subgroup K contained in U. Let dk be the probability Haar measure on K. Put $v' = \int_K \pi(k)v\,dk$. Then

$$\|v - v'\| = \left\| \int_K (\pi(k).v - v)dk \right\| \leq \int_K \|\pi(k)v - v\|\,dk \leq \epsilon. \qquad \square$$

A smooth irreducible representation admits at most one homothety class of $\mathrm{SL}_2(\mathbb{Q}_p)$-invariant inner product. Thus a smooth irreducible representation is unitarizable in at most one way. Given a unitarizable smooth irreducible representation V_0, and a choice (unique up to homothety) of invariant inner product, its Hilbert space completion V with respect to this inner product satisfies $V_0 = V^\infty$. For these non-trivial facts, see Section 2.8 of Cartier's survey article in the Corvallis proceedings. One deduces that the study of irreducible unitary representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ reduces to that of smooth irreducible representations. (In fact, smooth irreducible here can be replaced by *admissible*, a concept we shall see in the next section.)

One of the reasons the theory of smooth representations can work so well, despite the absence of topological structures on V, is the following fact.

Proposition 1.1. *An irreducible smooth representation of* $\mathrm{SL}_2(\mathbb{Q}_p)$ *has countable dimension.*

Proof. Let (π, V) be an irreducible smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$. We give a countable generating set of V. Take any non-zero v in V. Then by smoothness, there is an open compact subgroup K of $\mathrm{SL}_2(\mathbb{Q}_p)$ such that $v \in V^K$. By irreduciblity, the span of $\{\pi(g)v : g \in \mathrm{SL}_2(\mathbb{Q}_p)/K\}$ is all of V. But $\mathrm{SL}_2(\mathbb{Q}_p)/K$ is countable: for example, when $K = \mathrm{SL}_2(\mathbb{Z}_p)$, the quotient $\mathrm{SL}_2(\mathbb{Q}_p)/\mathrm{SL}_2(\mathbb{Z}_p)$ is the $(p+1)$-regular tree (more generally, use the Cartan decomposition). □

This can be compared to the case of K-finite vectors in the situation for real groups: the space of $K = \mathrm{SO}(2)$-finite vectors of $L^2(S^1)$ consists of trigonometric polynomials, which is certainly of countable dimension. The difference in the p-adic setting is that smooth representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ are globally defined representations, whereas the space of trigonometric polynomials admits only an infinitesimal action of $\mathrm{sl}_2(\mathbb{R})$.

Proposition 1.2 (Schur's lemma)**.** *The algebra of G-endomorphisms of an irreducible smooth representation of G is one-dimensional.*

Proof. First we observe that $\mathrm{End}_G(V)$ is a skew field. Indeed, if $\phi \in \mathrm{End}_G(V)$ is non-zero, then the image of ϕ is a non-zero invariant subspace of V and by irreducibility of V must be equal to V. Thus ϕ is surjective. For the same reason ϕ must be injective and ϕ is therefore invertible.

Now let $\varphi \in \mathrm{End}_G(V)$. Arguing by contradiction, we assume that $\varphi \notin \mathbb{C}.\mathrm{id}_V$. Consider the algebra homomorphism $T : \mathbb{C}[X] \to \mathrm{End}_G(V)$ given by evaluation at φ. We claim that $\ker T = 0$. Indeed, the image is a subalgebra of the skew field $\mathrm{End}_G(V)$, and hence must be an integral domain. If $\ker T$ were non-zero the image $\mathrm{im}(T) \simeq \mathbb{C}[X]/\ker T$ would be a finite degree field extension of $T(\mathbb{C}) \simeq \mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, it would follow that $\mathrm{im}(T) = T(\mathbb{C})$ which contradicts the hypothesis that $\varphi \notin \mathbb{C}.\mathrm{id}_V$. Thus T is an injective homomorphism, and may we extend it to an injective homomorphism from $\mathbb{C}(X)$ to $\mathrm{End}_G(V)$. Now $\mathbb{C}(X)$ is of uncountable dimension over $\mathbb{C}$. But this is not the case for $\mathrm{End}_G(V)$: since V is irreducible, any G-endomorphism is uniquely determined by its value at a non-zero vector $v \in V$ (its G-translates span V), and we have just seen that V is of countable dimension. □

Corollary 1. *Irreducible smooth representations admit central characters.*

Proof. Let (π, V) be an irreducible smooth representation. Let z be in the center of G. Then $\pi(z)$ is an endomorphism of V which commutes with all $\pi(g)$, thus $\pi(z) \in \mathrm{End}_G(V)$. From Schur's lemma, there is a scalar $\omega_\pi(z) \in \mathbb{C}^\times$ such that $\pi(z)v = \omega_\pi(z)v$ and the map $z \mapsto \omega_\pi(z)$ gives a homomorphism $\omega_\pi : Z \to \mathbb{C}^\times$. Then ω_π is a smooth one-dimensional representation, since π is itself smooth: there is a a compact open subgroup K with $V^K \neq 0$, then π is trivial on the compact open subgroup $K \cap Z$ of Z. □

In contrast to the case of real groups, one has the following

Proposition 1.3. *Let V be a smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ and K a compact open subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Then every vector in a smooth representation is K-finite.*

Proof. For any $v \in V$ we wish to show that the space W_v spanned by $\{\pi(k)v : k \in K\}$ is finite dimensional. By smoothness, v is fixed by some open compact subgroup K_v of G. Intersecting K_v with K_0 if necessary, we may assume that K_v is an open subgroup of K_0. Since K_0 is profinite, K_v is of finite index in K_0. So W_v is spanned by $\pi(k_1)v, \ldots, \pi(k_n)v$ for the finite number of coset representatives $k_1, \ldots, k_n$ of K_0/K_v. □

An important feature of the representation theory of $\mathrm{SL}_2(\mathbb{Q}_p)$ is that smooth representations are not always semisimple. We give two examples.

Example 1. The two dimensional representation

$$g \mapsto \begin{pmatrix} 1 & \log|\det(g)| \\ & 1 \end{pmatrix}$$

of $\mathrm{GL}_2(\mathbb{Q}_p)$ is not semisimple.

Example 2. Consider the space of locally constant functions on the projective line $C^\infty(\mathbb{P}^1(\mathbb{Q}_p))$ under the action of $\mathrm{SL}_2(\mathbb{Q}_p)$ by Möbius transformations. Then the subspace of constant functions is stable under this action and forms a subrepresentation. It does not admit a stable complement. The quotient $C^\infty(\mathbb{P}^1(\mathbb{Q}_p))/\{\text{constant functions}\}$ is a representation of $\mathrm{SL}_2(\mathbb{Q}_p)$, called the *Steinberg representation*, denoted St. We shall see later that it is irreducible.

On the other hand, we have the following

Lemma 1.2. *Let (V, π) be a smooth representations of $\mathrm{SL}_2(\mathbb{Q}_p)$. Let K be an open compact subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Then V is K-semisimple.*

Proof. Let $v \in V$ be arbitrary. Let W be the space spanned by $\{\pi(k)v : k \in K\}$. By smoothness, v is fixed by some open compact subgroup K', which may be taken to be normal. Thus W is a finite dimensional (complex) representation of the finite group K/K'. It is therefore semi-simple. □

Denote by $\mathrm{Irrep}(K)$ the set of equivalence classes of irreducible smooth representations of K. We have seen that irreducible smooth representations of a compact subgroup K of $\mathrm{SL}_2(\mathbb{Q}_p)$ are finite dimensional. Thus a smooth irreducible representation of K is just a continuous irreducible representation of K on a finite dimensional vector space, where this vector space is given the standard complex topology. If (π, V) is a smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\rho \in \mathrm{Irrep}(K)$, write $V(\rho)$ for the ρ-isotypic component of V. By definition $V(\rho)$ is the sum of all subspaces of V which, under the restriction of π to K, are irreducible representations of K in the equivalence class of ρ.

Proposition 1.4. *Let V be a smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ and K a compact open subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Then V is the algebraic direct sum*

$$V = \bigoplus_{\rho \in \mathrm{Irrep}(K)} V(\rho)$$

of its K-isotypic components.

Proof. Using the previous lemma we may decompose V as a direct sum $\bigoplus_i U_i$ of irreducible K-representations. We may organize the U_i according to their isomorphism classes to obtain

$$V|_K = \bigoplus_{\rho \in \mathrm{Irrep}(K)} U(\rho),$$

where $U(\rho)$ is the direct sum of all U_i isomorphic to ρ. But clearly $V(\rho) = U(\rho)$. $\square$

Lemma 1.3. *The canonical map $V \to \widetilde{\widetilde{V}}$ which sends $v \in V$ to the functional $\ell \mapsto \ell(v)$ on $\widetilde{V}$ is injective.*

Proof. The thing to worry about is the passage to smooth vectors, since the result is certainly true if we simply considered the map V into V^{**}.

Let us show that, for any open compact subgroup K of G, we have $(\tilde{V})^K \simeq (V^K)^*$ as vector spaces. Note that V^K is the isotypic component for the trivial representation of K. From the above proposition V^K admits a K-stable complement $V(K) = \oplus_{\rho \neq 1} V(\rho)$. Now a smooth functional $\tilde{v} \in \tilde{V}$ is fixed by K precisely when it annihilates $V(K)$. We may therefore define an injective map $(\tilde{V})^K \to (V^K)^*$ by restriction. In the other direction we can extend any functional on V^K trivially across $V(K)$ to obtain a functional ℓ on V, which is clearly smooth. This proves the claim.

Now returning to the statement of the lemma, we have to show that for every non zero $v \in V$ there is $\ell \in \widetilde{V}$ such that $\ell(v) \neq 0$. Let K be an open compact subgroup such that $v \in V^K$. Let $\ell_0 \in (V^K)^*$ be non zero on v. We view $\ell \in \tilde{V}$ by the above isomorphism. $\square$

Remark 1. We shall later see an additional condition (see Proposition 2.1) that one can place on (π, V) which ensures that V is isomorphic to the contragredient of its contragredient.

1.4. **Hecke algebras and their modules.** Let $\mathcal{H}(\mathrm{SL}_2(\mathbb{Q}_p))$ denote the Hecke algebra of compactly supported locally constant functions on $\mathrm{SL}_2(\mathbb{Q}_p)$ (with multiplication given by convolution); it is a non-unital ring.

Given a smooth representation (π, V) of $\mathrm{SL}_2(\mathbb{Q}_p)$ one defines an action of $\mathcal{H}(\mathrm{SL}_2(\mathbb{Q}_p))$ on V in the usual way, by smearing out the action of π:

$$\pi(\phi)v = \int_{\mathrm{SL}_2(\mathbb{Q}_p)} \phi(g)\pi(g)v dg.$$

From the smoothness of π and ϕ it follows that the above integral is a finite sum. One can verify that $\pi(\phi_1 * \phi_2) = \pi(\phi_1) \circ \pi(\phi_2)$. In other words, we have associated with a smooth representation V a representation $\mathcal{H}(\mathrm{SL}_2(\mathbb{Q}_p)) \to \mathrm{End}(V)$ of the Hecke algebra. Note that if $v \in V^K$ and ϕ is right-K-invariant, then we have

$$\pi(\phi)v = \sum_{g \in \mathrm{SL}_2(\mathbb{Q}_p)/K} \mathrm{vol}(K)\phi(g)\pi(g)v,$$

where only finitely many terms are non-zero, since ϕ has compact support.

Note that the Hecke module $M = \mathrm{End}(V)$ inherits the following smoothness property from V. Given a vector $v \in V$, by the smoothness of V there is a compact open subgroup K of $\mathrm{SL}_2(\mathbb{Q}_p)$ such that v is fixed under π by K. This implies that $\pi(e_K)v = v$, where $e_K = \mathrm{vol}(K)^{-1}\mathbf{1}_K$ be the normalized characteristic function of K. We abstract this property for M by calling an $\mathcal{H}(\mathrm{SL}_2(\mathbb{Q}_p))$-module M *smooth (or non-degenerate)* if for every finite subset X of V there exists a compact open subgroup K of $\mathrm{SL}_2(\mathbb{Q}_p)$ such that $\pi(e_K)v = v$ for every $v \in X$. Given a smooth Hecke module M we obtain a smooth representation π of $\mathrm{SL}_2(\mathbb{Q}_p)$ by sending declaring that g acts on $v \in V^K$ by $\pi(g)v = \mathrm{vol}(K)^{-1}\mathbf{1}_{gK}v$.

Proposition 1.5. *The category of smooth representations of* $\mathrm{SL}_2(\mathbb{Q}_p)$ *is equivalent to the category of smooth* $\mathcal{H}(\mathrm{SL}_2(\mathbb{Q}_p))$*-modules.*

Proof. See Section 4.2 of the book by Bushnell-Henniart. □

The above result is the analog for $\mathrm{SL}_2(\mathbb{Q}_p)$ of the Casselman-Wallach theorem for Harish-Chandra modules of $\mathrm{SL}_2(\mathbb{R})$. The former is much easier since there is no need to globalize: smooth representations are already globally defined as representations of the group $\mathrm{SL}_2(\mathbb{Q}_p)$.

2. Admissible representations of $\mathrm{SL}_2(\mathbb{Q}_p)$

In the theory of automorphic forms for SL_2, the open compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$ give rise to level structures. For example, the groups

$$K_{1,p}(p^f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : c \equiv 0,\ d \equiv 1 \bmod p^f \right\}$$

and

$$K_{0,p}(p^f) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_p) : c \equiv 0 \bmod p^f \right\}$$

are open and compact in $\mathrm{SL}_2(\mathbb{Z}_p)$, and correspond to the standard congruence subgroups $\Gamma_1(p^f)$ and $\Gamma_0(p^f)$ in the classical theory of modular forms. An important property that one wants to ensure is that, having fixed a level structure such as $\Gamma_1(N)$ and a character χ of $\mathcal{Z}(\mathscr{U}(\mathfrak{sl}_2(\mathbb{C})))$ and K-type $\chi \in \widehat{\mathrm{SO}(2)}$, the space of automorphic forms on $\Gamma_1(N)\backslash\mathrm{SL}_2(\mathbb{R})$ having infinitesimal character χ and K-type χ is finite dimensional.

With this motivation in mind, it is natural to make the following definition.

Definition 3. A representation (π, V) of $\mathrm{SL}_2(\mathbb{Q}_p)$ is said to be **admissible** if for every compact open subgroup K of $\mathrm{SL}_2(\mathbb{Q}_p)$ the space of K-invariants V^K of V is finite dimensional.

Lemma 2.1. *Let V be a smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$. Then V is admissible if, and only if, for every open compact subgroup K and every equivalence class $\rho \in \mathrm{Irrep}(K)$ the ρ-isotypic component $V(\rho)$ of V is of finite dimension.*

Proof. Suppose V admissible. Then since $V(\rho)$ is contained in $V^{\ker \rho}$, and the latter is finite dimensional, we deduce that $V(\rho)$ is finite dimensional. Conversely, suppose that there is some open compact K for which V^K is infinite dimensional. We may take K to be a normal subgroup of K_0. Then K_0/K is a finite group and V^K is a K_0-invariant subspace. From the direct sum decomposition

$$V^K = \bigoplus_{\rho \in \mathrm{Irrep}(K/K_0)} V(\rho),$$

we deduce that some $V(\rho)$ is infinite dimensional. $\square$

Proposition 2.1. *Let (π, V) be a smooth admissible representation of $\mathrm{SL}_2(\mathbb{Q}_p)$. Then*

1. *(π, V) is canonically isomorphic to $(\widetilde{\widetilde{\pi}}, \widetilde{\widetilde{V}})$; in particular $(\widetilde{\pi}, \widetilde{V})$ is admissible;*
2. *(π, V) is irreducible if and only if $(\widetilde{\pi}, \widetilde{V})$ is irreducible.*

Proof. See Bushnell-Henniart, Propositions 2.9 and 2.10. $\square$

An example of a smooth representation which is non-admissible is the right-regular representation on $V = C_c^\infty(\mathrm{SL}_2(\mathbb{Q}_p))$. To see this, take $K = \mathrm{SL}_2(\mathbb{Z}_p)$ and the trivial representation $\rho = 1$ of $\mathrm{SL}_2(\mathbb{Z}_p)$. Then the $\rho = 1$-isotypic component of V is

$$V^K = C_c^\infty(\mathrm{SL}_2(\mathbb{Q}_p)/\mathrm{SL}_2(\mathbb{Z}_p)),$$

which is nothing other than the space of complex valued functions of compact support on the $(p+1)$-regular tree. This is clearly an infinite dimensional space.

The non-admissibility of the above smooth representation was, in a sense, due to the fact that it is highly reducible. In the other direction, we have the following theorem, due to Jacquet.

Theorem 2.1. *Every* **irreducible** *smooth representation is admissible.*

We shall prove this theorem shortly, after we have discussed some basic classes of representations. In fact, Bernstein has shown a much stronger statement than the above theorem, called the *uniform admissibility theorem.* When applied to our case, it states that given an open compact subgroup K of $\mathrm{SL}_2(\mathbb{Q}_p)$, there is a constant N such that for any smooth irreducible representation V of $\mathrm{SL}_2(\mathbb{Q}_p)$ one has $\dim V^K \leq N$.

3. Induced representations

This is a vast subject, which is really the heart of the representation theory of p-adic reductive groups such as $G = \mathrm{GL}_2(\mathbb{Q}_p)$. By necessity, we shall be brief.

3.1. **Induced representations.** We first discuss (parabolically) induced representations. Here the idea is to construct a representation of G from a representation of a "sufficiently large" subgroup. Namely, our starting point will be a representation of the Borel subgroup B of G, since it is cocompact: $G/B = \mathbb{P}^1$.

In the following definition we shall need the following *modular character*

$$\delta_B : B \to \mathbb{C}^\times, \qquad \begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \mapsto |x/y|$$

of the Borel subgroup. It is the homomorphism of B into $\mathbb{R}_+^\times$ arising in the following way: if db denotes a left-Haar measure for B, and $a \in B$, then uniqueness of Haar measure implies

$$\int_B f(ba)db = \delta_B(a) \int_B f(b)db,$$

for any $f \in C_c^\infty(B)$.

Lemma 3.1. *With notations as above, $\delta_B(b)db$ is a right-invariant Haar measure.*

Proof. Let $f \in C_c^\infty(B)$ and $a \in B$. Then

$$\begin{aligned}\int_B f(ba)\delta(b)db = \int_B f(a^{-1}ba)\delta(a^{-1}b)db &= \delta_B^{-1}(a) \int_B f(a^{-1}ba)\delta(b)db \\ &= \delta_B^{-1}(a) \int_B f(a^{-1}ba)\delta(a^{-1}ba)db \\ &= \delta_B^{-1}(a)\delta_B(a) \int_B f(a^{-1}b)\delta(a^{-1}b)db \\ &= \int_B f(a^{-1}b)\delta(a^{-1}b)db \\ &= \int_B f(b)\delta(b)db,\end{aligned}$$

as claimed. □

Definition 4. Let μ_1, μ_2 be smooth characters of $\mathbb{Q}_p^\times$. Let μ be character of the diagonal torus T of G given by $\mu(\mathrm{diag}(x, y)) = \mu_1(x)\mu_2(y)$. We extend μ trivially across U in $B = TU$ to obtain a character of B.

We let $\mathrm{Ind}_B^G(\mu)$ denote the space of smooth complex-valued functions f on G such that

(1) $f(bg) = \delta_B(b)^{1/2}\mu(b)f(g)$ for all $b \in B$ and $g \in G$.

(2) $f(gk) = f(g)$ for all $g \in G$ and all k in some open compact subgroup K of G, depending on f.

We endow $\mathrm{Ind}_B^G(\mu)$ with the structure of a representation by allowing G to act on it by right-translation. The second property above guarantees that this representation is smooth.

Remark 2. The presence of the modular character in the above definition is there to assure that when μ is a unitary character (takes values on the unit circle) then $\mathrm{Ind}_B^G(\mu)$ is unitarizable. In particular, when μ is unitary, $\mathrm{Ind}_B^G(\mu)$ is semisimple. One refers to this normalization as *unitary induction.*

One can show that, when μ is unitary, the inner product given by $\|f\|^2 = \int_{K_0} |f(k)|^2 dk$ is a G-invariant Hermitian form on $\mathrm{Ind}_B^G(\mu)$. One can think of the formula defining this inner product as a sort of substitute, arising from the Iwasawa decomposition $G = BK_0$, for the natural definition one would have taken were B unimodular: integation of $|f|^2$ over the $B\backslash G$ with respect to Haar measure.

The representation $\mathrm{Ind}_B^G(\mu)$ is not necessarily irreducible. Nevertheless, it is clear that they admit central characters: the central character of $\mathrm{Ind}_B^G(\mu)$ is the restriction of μ to the center of G.

Remark 3. The Steinberg representation St introduced in Example 2 can be realized as the quotient of $\mathrm{Ind}_B^G(\delta_B^{-1/2})$ (the space of smooth functions on $B\backslash G = \mathbb{P}^1$) modulo the constant functions (the trivial representation of G). In other words, we have the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Ind}_B^G(\delta_B^{-1/2}) \longrightarrow \mathrm{St} \longrightarrow 0.$$

More generally, for a unitary character χ of $\mathbb{Q}_p^\times$, one can realize the *twisted Steinberg representation* $\mathrm{St} \otimes (\chi \circ \det)$ (often abbreviated to $\mathrm{St} \otimes \chi$) via the exact sequence

$$0 \longrightarrow \chi \longrightarrow \mathrm{Ind}_B^G(\delta_B^{-1/2}) \otimes \chi \longrightarrow \mathrm{St} \otimes \chi \longrightarrow 0.$$

Note that in both these cases the inducing character $\delta_B^{-1/2}$ is not unitary.

Some sources define the Steinberg representation dually as the subrepresentation of $\mathrm{Ind}_B^G(\delta_B^{1/2})$ consisting of functions $f \in \mathrm{Ind}_B^G(\delta_B^{1/2})$ for which $\int_{K_0} f(k)dk = 0$.

In some other sources (such as Bushnell–Henniart) non-unitary induction is used, so there is no factor of $\delta_B^{1/2}$ in the definition of the induced representation. Under this convention, the Steinberg representation is the unique irreducible quotient of the induction of the trivial representation.

We can verify Theorem 2.1 for any irreducible representation of G which appears as a subquotient of $\mathrm{Ind}_B^G(\mu)$ by means of the following

Lemma 3.2. *The representation* $\mathrm{Ind}_B^G(\mu)$ *is admissible.*

Proof. Let K be a compact open subgroup of G. We can assume that $K \subset K_0$. The space $\mathrm{Ind}_B^G(\mu)^K$ consists of smooth functions $f : G \to \mathbb{C}$ satisfying

$$f(bgk) = \delta_B(b)^{1/2}\mu(b)f(g), \qquad b \in B, g \in G, k \in K.$$

But the double quotient $B\backslash G/K$ is finite (since $B\backslash G$ is compact), and the functions supported on a given double coset BgK is one-dimensional. $\square$

3.2. **Jacquet module.** In the previous paragraph, we started with a smooth representation of T, extended it trivially across N, and defined, through parabolic induction, a smooth representation of G. We thus have a map

$$\mathrm{Ind}_B^G : \mathrm{Rep}(T) \longrightarrow \mathrm{Rep}(G).$$

The Jacquet module allows us to go in the opposite direction.

Definition 5. Let (π, V) be a smooth representation of G. The **Jacquet module** of π, denoted V_N, is the space of co-invariants of N: the largest quotient of V on which N acts trivially.

Concretely we may realize V_N as the quotient

$$V_N = V/V(N), \quad \text{where } V(N) = \mathrm{span}\{\pi(n)v - v : v \in V, n \in N\}.$$

Now $V(N)$ is a B-submodule of V: indeed, if $b \in B$ and $\pi(n)v - v \in V(N)$, then, using the fact that B normalizes N and setting $v' = \pi(b)v \in V$, we obtain

$$\pi(b)(\pi(n)v - v) = \pi(bn)v - \pi(b)v = \pi(n'b)v - \pi(b)v = \pi(n')v' - v' \in V(N).$$

We obtain a representation of B on V_N, which can be shown to be smooth. We denote it by π_N. Since N acts trivially, we may in fact view π_N as a smooth representation of $T = B/N$. We have obtained a map

$$J_N : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(T), \quad J_N(V) = V_N,$$

which bears the following relation to Ind_B^G.

Lemma 3.3. *For $V \in \mathrm{Rep}(G)$ and $W \in \mathrm{Rep}(T)$ we have*

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G W) \simeq \mathrm{Hom}_T(V_N, W).$$

Proof. Frobenius reciprocity for smooth representations states that

$$\mathrm{Hom}_G(V, \mathrm{Ind}_B^G W) \simeq \mathrm{Hom}_B(V, W),$$

where W is viewed as a representation of B by letting N act trivially. So we have only to show that $\mathrm{Hom}_B(V, W) \simeq \mathrm{Hom}_T(V_N, W)$.

We have an obvious inclusion $\mathrm{Hom}_T(V_N, W) \to \mathrm{Hom}_B(V, W)$ given by composition with the quotient $V \to V_N$.

Conversely, first observe that any $\varphi \in \mathrm{Hom}_B(V, W)$ vanishes on $V(N)$: indeed, since N acts trivially on W, we have $\varphi(n.v - v) = n.\varphi(v) - \varphi(v) = 0$. Since V_N is the largest quotient of V on which N acts trivially, there is a linear map $\tilde{\varphi} : V_N \to W$ such that $\varphi : V \to V_N \xrightarrow{\tilde{\varphi}} W$. Moreover, $\tilde{\varphi}$ intertwines the T action, since

$$\tilde{\varphi}(t.(v + V(N))) = \tilde{\varphi}(t.v + V(N)) = \varphi(t.v) = t.\varphi(v).$$

Thus $\tilde{\varphi} \in \mathrm{Hom}_T(V_N, W)$, and the map $\varphi \to \tilde{\varphi}$ is inverse to the first. $\square$

We now arrive at an important consequence.

Corollary 2. *Let (π, V) be a smooth irreducible representation of G. The Jacquet module V_N is non-zero if, and only if, there is a smooth character μ of T such that V embeds as a subrepresentation of $\mathrm{Ind}_B^G(\mu)$.*

Proof. One direction is clear: if V embeds as a subrepresentation of $\mathrm{Ind}_B^G(\mu)$ then $\mathrm{Hom}_G(V, \mathrm{Ind}_B^G(\mu)) \neq 0$ and hence $\mathrm{Hom}_T(V_N, \mu) \neq 0$ by Lemma 3.3. This implies $V_N \neq 0$.

For the other implication, the idea is to find a smooth irreducible representation W of T such that $\mathrm{Hom}_T(V_N, W) \neq 0$. By Lemma 3.3, this will show that there is some non-zero element in $\mathrm{Hom}_F(V, \mathrm{Ind}_B^G(W))$, which must be injective since V is irreducible. Moreover, since W is irreducible, Schur's theorem shows that W is one-dimensional.

To produce W, we first note that V_N is finitely generated as representation of T. To see this, it suffices to show V is finitely generated as a B-representation. To see this, let $v \in V$ be any non-zero vector. Let K be an open compact subgroup of K_0 stabilizing v. Let $R = \{k_1, \ldots, k_r\}$ be a set of representatives of the finite quotient K/K_0. Then since $G = BK_0$ we have $G = BRK$. Since V is irreducible we have $V = G.v = BR.v$ so that $v_1 = \pi(k_1)v, \ldots, v_r = \pi(k_r)v$ generate V over B.

Then Zorn's lemma allows us to find an irreducible quotient. To see this, consider the set S of proper subrepresentations of V_N. We claim that if one has an increasing sequence $U_1 \subset \cdots U_k \subset \cdots$ of elements in S then $U = \cup_i U_i$ is in S. Indeed, since V_N is finitely generated we can find an index i_0 such that $U = U_{i_0}$. An application of Zorn's lemma is therefore justified, and we can take a maximal element $U_{\max}$ in S. The quotient $V_N/U_{\max}$ will then be irreducible. $\square$

The following characterization of $V(N)$ will be useful for us.

Lemma 3.4. *Let (π, V) be a smooth representation of N. Then $v \in V(N)$ if, and only if, there is an open compact subgroup N_0 of N such that*

$$\int_{N_0} \pi(n) v dn = 0.$$

Proof. We show the reverse direction. If N_1 is an open compact subgroup of N_0 which fixes v then the above integral can be written as a sum over cosets, namely

$$\mathrm{vol}(N_1) \sum_{n \in N_0/N_1} \pi(n) v.$$

Let $1, n_1, \dots, n_m$ be a set of representatives for N_0/N_1. The above shows that $v + \sum_{i=1}^m \pi(n_i)v = 0$. Thus

$$(m+1)v = \sum_{i=1}^{m} (v - \pi(n_i)v),$$

proving $v \in V(N)$, which we recall is $\mathrm{span}\{\pi(n)u - u : n \in N, u \in V\}$. $\square$

3.3. **Jacquet module of the principal series.** It is an important exercise to calculate the Jacquet module for principal series representations $\mathrm{Ind}_B^G(\mu)$.

Lemma 3.5. *Let $\mu = \mu_1 \otimes \mu_2$ be a character of T and put $\mu^w = \mu_2 \otimes \mu_1$. Then*

(1) *if $\mu^w \neq \mu$, then the Jacquet module of $\mathrm{Ind}_B^G(\mu)$ is the two-dimensional semisimple representation $\mu\delta_B^{1/2} \oplus \mu^w\delta_B^{1/2}$ of T;*
(2) *if $\mu^w = \mu$, then the Jacquet module of $\mathrm{Ind}_B^G(\mu)$ is the two-dimensional indecomposable representation*

$$\begin{pmatrix} \mu\delta_B^{1/2} & * \\ & \mu\delta_B^{1/2} \end{pmatrix}$$

of T.

In both cases, the ***semisimplification*** *of $\mathrm{Ind}_B^G(\mu)_N$ is isomorphic to the direct sum $\mu\delta_B^{1/2} \oplus \mu^w\delta_B^{1/2}$.*

Proof. Define a surjective map $\mathrm{Ind}_B^G(\mu) \to \mu\delta_B^{1/2}$ by $f \mapsto f(e)$. Let V be its kernel, so we obtain an exact sequence

$$0 \to V \to \mathrm{Ind}_B^G(\mu) \to \mu\delta_B^{1/2} \to 0$$

of B-representations. This is trivial on N and so factors through a map from $\mathrm{Ind}_B^G(\mu)_N$. Thus we get an exact sequence

$$0 \to V_N \to \mathrm{Ind}_B^G(\mu)_N \to \mu\delta_B^{1/2} \to 0$$

of T representations. We claim that $V_N = \mu^w\delta_B^{1/2}$.

First we examine the space V, viewed as a subspace of $\mathrm{Ind}_B^G(\mu)$. Note that the map $\mathrm{Ind}_B^G(\mu) \to \mu\delta_B^{1/2}$, $f \mapsto f(e)$ is just restriction to B. From the Bruhat decomposition (2) it follows that $f \in V$ precisely when f is supported on the big cell BwN. By Lemma 3.4, the space $V(N)$ consists precisely of those f on BwN killed under the map

$$V \to \mathbb{C}, \qquad f \mapsto \int_N f(wn)dn.$$

The above map is $\mu^w \delta_B^{1/2}$ equivariant, as can be seen through the following computation:

$$\begin{aligned}
\int_N f(wnt)dn &= \int_N f(wtt^{-1}nt)dn \\
&= \delta_B(t)^{-1} \int_N f(wtn)dn \\
&= \delta_B(t)^{-1} \int_N f(wtwwn)dn \\
&= (\delta_B^{1/2}\mu^w)(t) \int_N f(wn)dn.
\end{aligned}$$

This establishes $V_N = \mu^w \delta_B^{1/2}$, and produces the exact sequence

$$0 \to \mu^w \delta_B^{1/2} \to \mathrm{Ind}_B^G(\mu)_N \to \mu\delta_B^{1/2} \to 0.$$

For a proof of the splitting of the exact sequence according to whether $\mu^w = \mu$, see §2.4 of Olivier Taïbi's course notes, *The Jacquet–Langlands correspondence for* $\mathrm{GL}_2(\mathbb{Q}_p)$. $\square$