Lecture 7: Representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ (part II)

Contents

1. Steinberg representation

We now discuss in more detail the Steinberg representation.

Lemma 1.1. St *is irreducible.*

Proof. The statement is equivalent to showing that the length of $\mathrm{Ind}_B^G(\delta_B^{-1/2})$ is 2. By exactness of the Jacquet functor $\mathrm{Rep}(G) \to \mathrm{Rep}(T)$, the length of the latter representation is that of $\mathrm{Ind}_B^G(\delta_B^{-1/2})_N$, which is indeed 2. $\square$

We now determine the Jacquet module of the Steinberg representation.

Lemma 1.2. *The Jacquet module for* St *is one-dimensional, equal to the character* δ_B *of* T.

Proof. Using exactness again, applied to the definition of St, we obtain

$$0 \to \mathbb{C}_N \to \mathrm{Ind}_B^G(\delta_B^{-1/2})_N \to \mathrm{St}_N \to 0.$$

It is clear that that the Jacquet module $\mathbb{C}_N$ for the constant functions $\mathbb{C}$ is the trivial character 1. On the other hand, the Jacquet module for $\mathrm{Ind}_B^G(\delta_B^{-1/2})$ is $\delta_B^{-1/2}\delta_B^{1/2} \oplus \delta_B^{1/2}\delta_B^{1/2} = 1 \oplus \delta_B$. $\square$

One studies the Steinberg representation by looking at its *Iwahori invariants*. The *Iwahori* subgroup of G is the open compact subgroup I given by inverse image under the reduction mod p map of the Borel subgroup in $\mathrm{SL}_2(\mathbb{F}_p)$. It is the p-adic analogue of the classical $\Gamma_0(p)$ congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 1.3. *We have* $\dim \mathrm{St}^I = 1$.

Proof. We observe that $K = I \cup IwI$. By the Bruhat decomposition of G we obtain $G = BI \cup BwI$. The space of I-invariants of the induced representation $\mathrm{Ind}_B^G(\delta^{-1/2})$ is therefore two-dimensional. As the subspace of constant functions is one-dimensional, and equal to its I-invariants, the I-invariants of the quotient (the Steinberg) is also one-dimensional. $\square$

Proposition 1.1. St *is essentially square-integrable.*

Proof. It is enough to show that one matrix coefficient is essentially square integrable. Take non-zero $v \in \mathrm{St}^I$ and $\tilde{v} \in \widetilde{\mathrm{St}}^I$ and compute

$$\int_{G/Z} |\langle \tilde{v}, \pi(g)v \rangle|^2 dg.$$

Use the affine Bruhat decomposition

$$G = \bigsqcup_{x \in \widetilde{W}} IxI, \qquad \text{where } \widetilde{W} = N_G(T)/T_0.$$

Here $T_0 = T \cap K_0$, and we have $\tilde{W} = T/T_0 \rtimes W$, where $T/T_0 \simeq \mathbb{Z}^2$ and $W = N_G(T)/T$ is the usual Weyl group. Then one calculates

$$\begin{aligned}\int_{G/Z} |\langle \tilde{v}, \pi(g)v \rangle|^2 dg &= \sum_{x \in \widetilde{W}} |\langle \tilde{v}, \pi(x)v \rangle|^2 \mathrm{vol}(IxIZ/Z) \\ &= 2 \sum_{n \geq 0} |\langle \tilde{v}, \pi(t^n)v \rangle|^2 \mathrm{vol}(It^nIZ/Z),\end{aligned}$$

where $t = \mathrm{diag}(\varpi_p, 1)$. One can then show that $\langle \tilde{v}, \pi(t^n)v \rangle = p^{-n}$ and $\mathrm{vol}(It^nIZ/Z) = p^n$. This yields the convergent sum $\sum_{n \geq 0} p^{-n}$. ☐

2. Supercuspidal representations

Definition 1. An infinite dimensional smooth irreducible representation π of G is called **supercuspidal** if $\pi_N = 0$.

We can give a criterion for a smooth irreducible representation to be supercuspidal via its matrix coefficients. Let us introduce some notation to this effect. We denote by $\mathcal{C}(\pi)$ the space of matrix coefficients of π; this is the space spanned by $g \mapsto \langle \tilde{v}, \pi(g)v \rangle$, for $v \in V$, $\tilde{v} \in \tilde{V}$. Note that since π is irreducible, it admits a central character ω_π, so that a matrix coefficient transforms along the center by ω_π.

We shall need the following useful result.

Lemma 2.1. *A smooth irreducible representation whose matrix coefficients are all compactly supported modulo the center is admissible.*

Proof. Suppose there is some K for which V^K has infinite (necessarily countable) dimension. Recall from undergraduate linear algebra that the dual of an infinite dimensional vector space over $\mathbb{C}$ has uncountable dimension, and this in particular applies to $\tilde{V}^K = \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$. We shall find a contradiction by embedding $\tilde{V}^K$ into a space of countable dimension.

Fix a non-zero $v \in V^K$. Consider the map $\tilde{V}^K \to \mathcal{C}(\pi)$, $\tilde{v} \mapsto \langle \tilde{v}, \pi(g)v \rangle$. This map is injective; indeed, since V is irreducible, the translates gv of v span V, and a non-zero $\tilde{v}$ cannot pair trivially with all of V. But the image of the map consists of functions f on G satisfying

$$f(zkgk') = \omega_\pi(z)f(g), \qquad g \in G,\ z \in Z,\ k, k' \in K,$$

and which, by hypothesis, are supported on a finite union of cosets $ZKgK$, a countable set, by the Cartan decomposition. So the image is of countable dimension, as desired. $\square$

Theorem 2.1. *Let π be a smooth representation. Then π is supercuspidal if, and only if, its matrix coefficients are compactly supported modulo the center.*

Proof. We shall prove the converse direction; the forward direction is similar in spirit.

Assume that all matrix coefficients are compactly supported modulo the center. Let $v \in V$. We want to show that $v \in V(N)$. By Lemma 3.4 of the previous lecture, it will be enough to show that there is an open compact subgroup N_0 of N such that

$$\int_{N_0} \pi(n) v dn = 0.$$

Now let $\tilde{v} \in \tilde{V}$. Then there is n large enough so that v and $\tilde{v}$ are fixed by K_n. Since $g \mapsto \langle \tilde{v}, \pi(g) v \rangle$ is compactly supported modulo the center, there is c large enough so that $\langle \tilde{v}, \pi(t^a) v \rangle = 0$ for all $a \geq c$. Since $\tilde{V}$ is admissible, the space $\tilde{V}^{K_n}$ is finite dimensional, so that we may take c uniformly for all $\tilde{V}^{K_n}$. Thus, for all $a \geq c$, we have

$$\langle \tilde{v}, \pi(t^a) v \rangle = 0, \qquad \forall\, \tilde{v} \in \tilde{V}^{K_n}. \tag{2.1}$$

We deduce from this that $\pi(e_{K_n})\pi(t^a)v = 0$. Indeed, to show a vector in a smooth representation is zero, it is enough to show that it pairs to zero on all of $\tilde{V}$ (see Lemma 1.3 of the previous lecture). As $V = V^{K_n} \oplus V(K_n)$ (see proof of Lemma 1.3 from the previous lecture) and $\pi(e_{K_n})\pi(t^a)v \in V^{K_n}$, we deduce from (2.1) that $\pi(e_{K_n})\pi(t^a)v = 0$.

Putting $K_n^{(a)} = t^{-a} K_n t^a$, we have

$$0 = \pi(e_{K_n})\pi(t^a)v = \pi(t^a)\pi(e_{K_n^{(a)}})v.$$

This implies that $\pi(e_{K_n^{(a)}})v = 0$.

Now K_n, for $n \geq 1$, admits an Iwahori decomposition:

$$K_n = N_n T_n \overline{N}_n,$$

where $N_n = \left(\begin{smallmatrix} 1 & \mathfrak{p}^n \\ & 1 \end{smallmatrix}\right)$, $\overline{N}_n = \left(\begin{smallmatrix} 1 & \\ \mathfrak{p}^n & 1 \end{smallmatrix}\right)$, and $T_n = K_n \cap T$. This gives

$$K_n^{(a)} = N_{n-a} T_n \overline{N}_{n+a},$$

and we have an integration formula

$$\int_{K_n'} f(k) dk = \int_{N_{n-a} \times T_n \times \overline{N}_{n+a}} f(nt\bar{n}) dn dt d\bar{n}.$$

Now v is fixed by K_n so up to volume factors v is fixed when integrated over $\overline{N}_{n+a}$ and T_n. So if $K_n^{(a)}$ kills v this is due to the integration over N_{n-a}^+. We take N_0 to be N_{n-a}. $\square$

We now return to Jacquet's theorem, that every smooth irreducible representation is admissible.

Proof of Jacquet's Theorem. Since we have shown the result to hold for any π with $\pi_N \neq 0$, it remains to show the result for π is supercuspidal. This follows from the above two results. $\square$

Remark 1. One can show, through a computation similar to that of Theorem 2.1, that the space of I-invariants of a supercuspidal representation is zero.

3. Unramified representations and the Satake Isomorphism

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and K a compact open subgroup of G. We denote by $\mathcal{H}(G, K)$ the unital algebra $e_K \mathcal{H}(G) e_K$, where $e_K = \mathrm{vol}(K)^{-1} \mathbf{1}_K$, consisting of compactly supported bi-K-invariant locally constant functions on G.

When $K = K_0$, we call $\mathcal{H}(G, K_0)$ the *spherical (or unramified) Hecke algebra.*

Proposition 3.1. *The spherical Hecke algebra $\mathcal{H}(G, K_0)$ is commutative.*

Proof. The argument we give is due to Gelfand; it is known as the "Gelfand trick". By the Cartan decomposition $G = K_0 \Lambda K_0$, any bi-K_0-invariant function ϕ is determined by its values on Λ. At the same time, one observes that the map $\phi \mapsto \hat{\phi}$, where $\hat{\phi}(g) = \phi({}^t g)$, is an anti-involution on $\mathcal{H}(G, K_0)$. Indeed, the involutivity of $\hat{\phi}(g)$ is clear, and one verifies by checking definitions that

$$\widehat{\phi_1 \star \phi_2} = \hat{\phi}_2 \star \hat{\phi}_1. \tag{3.1}$$

Since transpose fixes Λ pointwise, $\phi \mapsto \hat{\phi}$ is just the identity. From (3.1), this is the same thing as to say that $\mathcal{H}(G, K_0)$ is commutative. $\square$

By contrast, the *Iwahori Hecke algebra* $\mathcal{H}(G, I)$, which appeared in our study of the Steinberg representation, is non-commutative.

Definition 2. An smooth representation π of G is said to be *spherical (or unramified)* if $\pi^{K_0} \neq 0$.

Corollary 1. *Irreducible unramified representations correspond bijectively with the algebra homomorphisms* $\mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathcal{H}(G, K_0), \mathbb{C})$.

Proof. We use the fact that, for any open compact subgroup K, the map $V \mapsto V^K$ is a bijection between irreducible smooth representations (π, V) such that $\pi^K \neq 0$ and simple $\mathcal{H}(G, K)$-modules. Since $\mathcal{H}(G, K_0)$ is commutative, its simple modules are one-dimensional. $\square$

To understand the irreducible unramified representations, we must therefore understand the structure of $\mathcal{H}(G, K_0)$. The Satake isomorphism describes the structure of the spherical Hecke algebra in terms a simpler Hecke algebra, one associated with the split torus of diagonal matrices. The Satake isomorphism lends itself to a wide variety of reinterpretations. These reinterpretations, little by little, pave the way for the ideas behind the functoriality conjectures of Langlands.

3.1. **The Hecke algebra of T.** Let $T_0 = T \cap K_0$. The Hecke algebra $\mathcal{H}(T, T_0)$ is defined to be the convolution algebra of bi-T_0-invariant locally constant functions on T. Clearly, since T is commutative, this is just the convolution algebra of locally constant functions on the quotient T/T_0. The Hecke algebra $\mathcal{H}(T, T_0)$ is therefore rather straightforward to understand.

Indeed, $T/T_0 \simeq \mathbb{Z}^2$, an explicit isomorphism being given once we choice a uniformizer for $\mathbb{Z}_p$. Thus $\mathcal{H}(T, T_0) \simeq \mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$, where X corresponds with $\operatorname{diag}(\varpi_p, 1)$ and Y with $\operatorname{diag}(1, \varpi_p)$. This isomorphism is W-equivariant, where W is the order two symmetry group of T permuting the diagonal entries, and acting on $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$ by permuting the variables. It follows from this that $\mathcal{H}(T, T_0)^W$ is isomorphic to the symmetric polynomials in $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$.

3.2. **Statement of the theorem.** As usual we let N denote the upper triangular unipotent matrices of G. We equip $N \simeq \mathbb{Q}_p$ with the Haar measure dn, normalized so that $N \cap K \simeq \mathbb{Z}_p$ gets volume 1.

Theorem 3.1 (Satake isomorphism)**.** *The map $\phi \mapsto \mathcal{S}(\phi)$ given by*

$$\mathcal{S}(\phi)(t) = \delta_B(t)^{1/2} \int_N \phi(tn)dn$$

is an algebra isomorphism from $\mathcal{H}(G, K_0)$ to $\mathcal{H}(T, T_0)^W$.

Proof. Let $B_0 = B \cap K_0$. The map $\mathcal{S}$ should be thought of as a composition of two maps, first by restriction to the unramified Hecke algebra $\mathcal{H}(B, B_0)$ of B, then from $\mathcal{H}(B, B \cap K_0)$ to $\mathcal{H}(T, T_0)$ by integration over N.

To see that the image of $\mathcal{S}$ lies in $\mathcal{H}(T, T_0)$, it suffices to observe that T_0 is a subset of K_0. Then the bi-K_0-invariance of ϕ implies $\phi(tn) = \phi(t_0tn) = \phi(tt_0n)$ for $t_0 \in T_0$, so that $\mathcal{S}(\phi)(t_0t) = \mathcal{S}(\phi)(t) = \mathcal{S}(\phi)(tt_0)$. We leave it as an exercise (using measure formulae) that each of these two maps are algebra homomorphisms: one must check that it respects convolution products. A direct computation shows that $\mathcal{S}$ sends the unit in $\mathcal{H}(G, K_0)$ to the unit in $\mathcal{H}(T, T_0)$. Indeed,

$$\begin{aligned}\mathcal{S}(\mathbf{1}_{K_0})(\operatorname{diag}(a,b)) &= \delta_B^{1/2}(\operatorname{diag}(a,b)) \int_N \mathbf{1}_{K_0}(\operatorname{diag}(a,b)n)dn \\ &= |a/b|^{1/2} \int_{\mathbb{Q}_p} \mathbf{1}_{K_0}\left(\begin{pmatrix} a & ax \\ 0 & b \end{pmatrix}\right) dx.\end{aligned}$$

This vanishes unless $a, b \in \mathbb{Z}_p$ and the determinant ab lies in $\mathbb{Z}_p^\times$; in other words, $a, b \in \mathbb{Z}_p^\times$, in which case $|a/b| = 1$ and x should lie in $\mathbb{Z}_p$. Since the Haar measure on N gives $\mathbb{Z}_p$ volume 1, we obtain $\mathcal{S}(\mathbf{1}_{K_0}) = \mathbf{1}_{T_0}$.

The fact that $\mathcal{S}(\phi)(wt) = \mathcal{S}(\phi)(t)$ is a consequence of the normalization by $\delta_B(t)^{1/2}$ in front of the integral. Rather than give a proof, let us see that this holds on an example, namely, when applied to $\mathbf{1}_{K_0 \mathrm{diag}(\varpi_p,1)K_0}$. Now, in order for

$$\begin{pmatrix} a & ax \\ 0 & b \end{pmatrix} \in K_0 \begin{pmatrix} \varpi_p & \\ & 1 \end{pmatrix} K_0,$$

we must have $a, b, ax \in \mathbb{Z}_p$ and the determinant ab should have p-valuation 1. We deduce that either a has p-valuation 1 and b is a unit, or *vice versa*. From this we deduce that $\mathcal{S}(\mathbf{1}_{K_0 \mathrm{diag}(\varpi_p,1)K_0})$ is supported on the cosets $\mathrm{diag}(\varpi_p, 1)T_0$ and $\mathrm{diag}(1, \varpi_p)T_0$. These cosets are W-conjugate to each other, so we want to see that they give the same values. An important part of the general proof is to expand into left cosets, and in the present case we have

$$K_0 \begin{pmatrix} \varpi_p & \\ & 1 \end{pmatrix} K_0 = \begin{pmatrix} 1 & \\ & \varpi_p \end{pmatrix} K_0 \bigcup \left(\bigcup_{b \in \mathbb{Z}_p/\varpi_p\mathbb{Z}_p} \begin{pmatrix} \varpi_p & b \\ & 1 \end{pmatrix} K_0 \right)$$

as in the definition of the classical T_p Hecke operator. We now calculate $\mathcal{S}(\mathbf{1}_{K_0 \mathrm{diag}(\varpi_p,1)K_0})$ on $\mathrm{diag}(\varpi_p, 1)$ to be

$$\begin{aligned}
\delta_B^{1/2}(\mathrm{diag}(\varpi_p, 1)) & \int_N \mathbf{1}_{K_0 \mathrm{diag}(\varpi_p,1)K_0}(\mathrm{diag}(\varpi_p, 1)n)dn \\
&= |\varpi_p|^{1/2} \int_{\mathbb{Q}_p} \mathbf{1}_{K_0 \mathrm{diag}(\varpi_p,1)K_0} \left(\begin{pmatrix} \varpi_p & \varpi_p x \\ 0 & 1 \end{pmatrix} \right) dx \\
&= p^{-1/2} \sum_{b \in \mathbb{Z}_p/\varpi_p\mathbb{Z}_p} \int_{\mathbb{Q}_p} \mathbf{1}_{\begin{pmatrix} \varpi_p & b \\ & 1 \end{pmatrix} K_0} \left(\begin{pmatrix} \varpi_p & \varpi_p x \\ 0 & 1 \end{pmatrix} \right) dx \\
&= p^{-1/2} \sum_{b \in \mathbb{Z}_p/\varpi_p\mathbb{Z}_p} \int_{\varpi_p^{-1} b + \mathbb{Z}_p} dx.
\end{aligned}$$

Using the invariance of Haar measure on N and its normalization, we obtain

$$p^{-1/2} \sum_{b \in \mathbb{Z}_p/\varpi_p\mathbb{Z}_p} 1 = p^{-1/2} p = p^{1/2}.$$

On the other hand, on $\mathrm{diag}(1, \varpi_p)$ it is

$$\begin{aligned}
\delta_B^{1/2}(\mathrm{diag}(1, \varpi_p)) & \int_N \mathbf{1}_{K_0 \mathrm{diag}(1,\varpi_p)K_0}(\mathrm{diag}(1, \varpi_p)n)dn \\
&= |\varpi_p|^{-1/2} \int_{\mathbb{Q}_p} \mathbf{1}_{K_0 \mathrm{diag}(\varpi_p,1)K_0} \left(\begin{pmatrix} 1 & x \\ 0 & \varpi_p \end{pmatrix} \right) dx \\
&= p^{1/2} \int_{\mathbb{Q}_p} \mathbf{1}_{\begin{pmatrix} 1 & \\ & \varpi_p \end{pmatrix} K_0} \left(\begin{pmatrix} 1 & x \\ 0 & \varpi_p \end{pmatrix} \right) dx,
\end{aligned}$$

which is $p^{1/2} \int_{\mathbb{Z}_p} dx = p^{1/2}$, as desired.

Accepting that $\mathcal{S}$ lies in the W-invariants, the proof of the bijectivity goes roughly as follows. From the Cartan decomposition of G, the functions

$$f_{m,n} = \mathbf{1}_{K_0 \mathrm{diag}(\varpi_p^m, \varpi_p^n) K_0} \qquad (m \geq n)$$

form a basis for $\mathcal{H}(G, K_0)$, with $f_{0,0} = \mathbf{1}_{K_0}$ the unit element in the algebra. In the same way, the functions

$$g_{m,n} = \mathbf{1}_{\mathrm{diag}(\varpi_p^m, \varpi_p^n) T_0}, \qquad (m, n) \in \mathbb{Z}^2,$$

form a basis for $\mathcal{H}(T, T_0)$, with $g_{0,0} = \mathbf{1}_{T_0}$ the unit element in the algebra. Thus the symmetrized functions $\{e_{m,n}\}_{m \geq n}$, where

$$e_{m,n} = g_{m,n} + g_{n,m} \qquad (m > n), \qquad e_{n,n} = g_{n,n} \quad (n \in \mathbb{Z}),$$

form a basis for $\mathcal{H}(T, T_0)^W$. We expand

$$\mathcal{S}(f_{m,n}) = \sum_{a \geq b} c_{m,n}(a, b) e_{a,b},$$

for certain coefficients $c_{m,n}(a, b)$. One then proves that

- $c_{m,n}(m, n) \neq 0$, and
- $c_{m,n}(a, b) = 0$ whenever $(a, b) \succ (m, n)$.

This gives a upper triangular system with non-zero entries along the diagonal. □

3.3. **Parametrization of spherical representations.** We may use the Satake isomorphism to explicate the parametrization of irreducible spherical representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ given by Corollary 1.

Note that, on one hand, the algebra homomorphisms into $\mathbb{C}$ of the symmetric functions in $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$ into $\mathbb{C}$ are given by a pair of complex numbers $\alpha, \beta \in \mathbb{C}^\times$, up to permutation. On the other, the algebra homomorphisms of $\mathcal{H}(T, T_0)^W$ into $\mathbb{C}$ can be identified with the *unramified characters* $\mu = \mu_1 \otimes \mu_2$ of T – those which are trivial on $T_0 = T \cap K_0$ – up to permutation of μ_1 and μ_2. Under the parametrization of irreducible spherical representations induced by the Satake isomorphism, the representation corresponding to $\{\alpha, \beta\}$ is the principal series representation $\mathrm{Ind}_B^G(\mu)$, where

$$\mu = \mu_1 \otimes \mu_2, \quad \mu_1(x) = |x|^\alpha, \quad \mu_2(x) = |x|^\beta.$$

The fact that we take Weyl group invariants corresponds to the fact that (the semisimplifications of) $\mathrm{Ind}(\mu)$ and $\mathrm{Ind}(\mu^w)$ are isomorphic.

We can take another point of view. We identify the symmetric functions in $\mathbb{C}[X^{\pm 1}, Y^{\pm 1}]$ with $\mathbb{C}[T, S^{\pm 1}]$, via $T = X + Y$ and $S = XY$. In the above proof, we saw that the Satake isomorphism takes $f_{1,0}$ to $p^{1/2}T$ and $f_{1,1}$ to S, since T is identified with $e_{1,0}$ in $\mathcal{H}(T, T_0)$ and S with $e_{1,1}$. On the other hand, the representation ring of $\mathrm{GL}_2(\mathbb{C})$ is given by $\mathbb{C}[\mathrm{tr}, \det^{\pm 1}]$. Under $\mathbb{C}[T, S^{\pm 1}] = \mathbb{C}[\mathrm{tr}, \det^{\pm 1}]$, with $T = \mathrm{tr}$ and $S = \det$, we see that the Satake isomorphism identifies $\mathcal{H}(G, K_0)$ with the representation ring of $\mathrm{GL}_2(\mathbb{C})$.

Note that under $\mathcal{H}(G, K_0)$ the function $f_{m,0}$ is not sent to the character of the $p^{m/2}\mathrm{sym}^m$, as one might naïvely expect: for example

$$\mathcal{S}(f_{2,0} + f_{1,1}) = p\,\mathrm{tr}(\mathrm{sym}^2).$$

Indeed, on the Hecke algebra side

$$f_{1,0} * f_{1,0} = (f_{2,0} + f_{1,1}) + pf_{1,1},$$

corresponding to the classical Hecke relation $T_p^2 = T_{p^2} + 1$. On the representation ring side we have

$$\mathrm{tr.tr} = \mathrm{tr}(\mathrm{sym}^2) + \det$$

(think of the decomposition of the tensor product representation, then take traces). Thus

$$\mathcal{S}(f_{2,0} + f_{1,1}) = \mathcal{S}(f_{1,0} * f_{1,0} - pf_{1,1}) = p\,\mathrm{tr.tr} - p\det = p\,\mathrm{tr}(\mathrm{sym}^2).$$

4. Local Langlands Correspondence

Let F be a finite extension of $\mathbb{Q}_p$ with residue cardinality q. Recall that we have defined the Weil group W_F in a previous lecture. It is a dense subgroup of $G_F = \mathrm{Gal}(\bar{F}/F)$ which fits into an exact sequence

$$1 \longrightarrow I \longrightarrow W_F \longrightarrow \mathbb{Z} \longrightarrow 1. \tag{4.1}$$

We put a topology on W_F by first giving I its subspace topology in G_F and then declaring that I should be open in W_F. This makes W_F a locally profinite group, which, unlike G_F, is not compact. From this definition, W_F has a neighborhood basis of open subgroups of I.

Here we are interested in the 2-dimensional (non-necessarily irreducible) complex representations of W_F. One can produce examples of such representations by taking the Galois action on the Tate module of an elliptic curve defined over F. However, if this elliptic curve has potential multiplicative reduction[1] the resulting representation will not be trivial on inertia.[2] It follows from the above remarks on the topology of W_F that such a representation would not be continuous. We seek to enlarge W_F so as to account for such representations.

Definition 3. The Weil–Deligne group W_F' is the semi-direct product $W_F' = W_F \ltimes \mathbb{C}$, where the action of W_F on $\mathbb{C}$ is given by

$$gzg^{-1} = \omega(g)z,$$

[1]Recall that multiplicative reduction means that the curve has a nodal singularity, so that the non-singular points of the curve form a multiplicative group. Any elliptic over F acquires either good or multiplicative reduction over a finite extension, and potentially multiplicative reduction refers to the latter. An example of an elliptic curve over $\mathbb{Q}_p$, with $p \geq 5$, with potentially multiplicative reduction is $y^2 = x^3 + x^2 + p$. For more on this, see Silverman, book 1, Chapter VII.5.

[2]This is nicely explained in Rohrlich, *Elliptic curves and the Weil–Deligne group*, Section 15.

where $\omega : W_F \to \mathbb{C}^\times$ is the unramified character of W_F such that $\omega(\text{Frob}) = q$, for any Frobenius element Frob (sent to 1 under (4.1), in other words, to $x \mapsto x^q$ on the absolute Galois group of the residue field).

We now must define a representation of the Weil–Deligne group.

Definition 4. A representation of W_F' on a finite-dimensional complex vector space V is a continuous homomorphism

$$\sigma' : W_F' \to \mathrm{GL}(V)$$

such that the restriction to $\mathbb{C}$ is complex analytic.

We give an alternative description of such representations.

Lemma 4.1. *To give a representation σ' of W_F as above is the same as to give a pair (σ, N), where*

(1) *$\sigma : W_F \to \mathrm{GL}(V)$ is a continuous representation of W_F;*
(2) *$N \in \mathrm{End}(V)$ is nilpotent, satisfying $\sigma(g)N\sigma(g)^{-1} = \omega(g)N$ for all $g \in W_F$.*

Proof. To go from (σ, N) to σ' we put $\sigma'(g, z) = \sigma(g)\exp(zN)$. In the other direction, given σ' we put $\sigma = \sigma'|_{W_F}$ and $N = (\log \sigma'(1, z))/z$, for any $z \in \mathbb{C}^\times$. For a proof that N is nilpotent and independent of z, consult Rohrlich, Section 3. □

Remark 2. One can express 2-dimensional Weil–Deligne representations in terms of compatible systems of 2-dimensional ℓ-adic representations of G_F. See Section 4 of Rohrlich, or Section 35 of Bushnell–Henniart, for more details.

We now come to the local Langlands correspondence. Let $\mathcal{G}_{2,F}$ denote the equivalence classes of complex 2-dimensional semisimple representations of W_F'. (Semisimplicity of $\sigma' = (\sigma, N)$ refers to the semisimplicity of σ.) Let $\mathcal{A}_{2,F}$ denote the equivalence classes of infinite dimensional smooth irreducible representations of $\mathrm{GL}_2(F)$.

Theorem 4.1 (Kutzko, Henniart)**.** *Let ψ be a non-trivial additive character of F. There is a unique bijective correspondence*

$$\mathcal{L} : \mathcal{A}_{2,F} \to \mathcal{G}_{2,F}$$

preserving local L- and ϵ-factors for all character twists.

Remark 3. We have chosen to express the theorem in this way because of its conciseness. We have not, however, defined the local L- and ϵ-factors for semisimple Weil–Deligne representations. (Note that these local constants depend on the choice of additive character ψ.) This was done by Langlands, and subsequently Deligne. The uniqueness statement was proved by Henniart.

To better understand the correspondence, it is helpful to identify where the various representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ we have encountered are sent. We denote by $r_F : W_F^{\mathrm{ab}} \xrightarrow{\sim} F^\times$ is the reciprocity map of local class field theory (the inverse of the map $\theta_F : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$ from the second lecture).

(1) If $\mathrm{Ind}_B^G(\mu)$ is irreducible, with $\mu = \mu_1 \otimes \mu_2$, then
$$\mathcal{L}(\mathrm{Ind}_B^G(\mu)) = \mu_1 \circ r_F \oplus \mu_2 \circ r_F;$$
(2) $\mathcal{L}(\mathrm{St} \otimes \chi) = \chi \otimes \mathrm{Sp}(2)$, where $\mathrm{Sp}(2)$ is the 2-dimensional *special representation* of W_F', acting on $\mathbb{C}.e_1 \oplus \mathbb{C}.e_2$ by
  (a) $\mathrm{Sp}(2)(g)e_i = \omega(g)^{i-1}e_i$;
  (b) $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$;
  Note that $\mathrm{Sp}(2)$ is indeed semisimple on W_F (of the form $1 \oplus \omega$).
(3) π is supercuspidal if, and only if, $\mathcal{L}(\pi)$ is irreducible.

Moreover $\det \mathcal{L}(\pi) = \omega_\pi \circ r_F$, where ω_π is the central character of π.

5. Local base change

We now provide an application of the non-abelian reciprocity law for the group GL_2 stated in Theorem 4.1. It is an instance of Langlands *functoriality*, which relates the representation theory of one group to that of another, when these two groups "share some common features".

Let E be a finite extension of F. Then the inclusion of the Weil–Deligne groups $W_E' \subset W_F'$ induces an injective map
$$\mathcal{G}_{2,F} \to \mathcal{G}_{2,E}, \qquad \sigma \mapsto \sigma|_{W_E'},$$
given by restriction. Thanks to Theorem 4.1, this in turn yields an injective map
$$BC_{E/F} : \mathcal{A}_{2,F} \to \mathcal{A}_{2,E}, \qquad \pi \mapsto \mathcal{L}^{-1}\left(\mathcal{L}(\pi)|_{W_E'}\right).$$
We call the latter map the *Base Change* lift from $\mathrm{GL}_2(F)$ to $\mathrm{GL}_2(E)$.

It is slightly more convenient to view both of these groups over the same field F, so we write $G = \mathrm{GL}_2(F)$ and $H = (\mathrm{Res}_{E/F}\mathrm{GL}_2)(F)$. The common feature shared by these two groups G and H is as follows.

Recall from §3.3 that the spherical Hecke algebra $\mathcal{H}(\mathrm{GL}_2(F), \mathrm{GL}_2(\mathcal{O}_F))$ of G is isomorphic to the representation ring $\mathbb{C}[\mathrm{tr}, \det^{\pm 1}]$ of $\mathrm{GL}_2(\mathbb{C})$. We call $\mathrm{GL}_2(\mathbb{C})$ the *Langlands dual group* of G, denote by $\widehat{G}$.

The same is true for the spherical Hecke algebra $\mathcal{H}(\mathrm{GL}_2(E), \mathrm{GL}_2(\mathcal{O}_E))$ of $\mathrm{GL}_2(E)$, but since we are considering $\mathrm{GL}_2(E)$ as a group over F, by restriction of scalars, we should take into account the Galois group (of the Galois closure E'/F). For that reason we define the *L-group* of H as
$$\widehat{H} = \underbrace{\mathrm{GL}_2(\mathbb{C}) \times \cdots \times \mathrm{GL}_2(\mathbb{C})}_{[E':F] \text{ times}}, \quad \text{and} \quad {}^L H = \widehat{H} \rtimes \mathrm{Gal}_{E'/F},$$
where $\mathrm{Gal}_{E'/F}$ acts by permuting the factors.

If we similarly write ${}^L G$ for the direct product $\widehat{G} \times \mathrm{Gal}_{E'/F}$, then we obtain a homomorphism

$$ {}^L G \to {}^L H $$

given by the diagonal map $\widehat{G} \xrightarrow{\Delta} \widehat{H}$ and the identity morphism on $\mathrm{Gal}_{E'/F}$.

An important question is then to understand the *image* of the base change map (or, for that matter, other instances of local or global functoriality). We will return to this topic in the last lecture.