Lecture 8: The Jacquet–Langlands correspondence (statement)

1. Abstract theory of quaternion algebras

Let K be a field of characteristic zero (all that we need in what follows is that the characteristic be different from 2). Let B be a quaternion algebra over K. In other words, B is a central simple algebra of dimension 4 over K.

1.1. **Inner forms.**

An example of B is the algebra of matrices $M_2(K)$. All other examples are Galois "twisted" versions of this. Let us explain this. (For a reference, see Serre's *Local fields*, §X.)

For a Galois extension L of K, the Galois group $\mathrm{Gal}(L/K)$ acts on $M_2(L)$. It therefore acts on the automorphism group of $M_2(L)$, by conjugation. By the Skolem-Noether theorem, all automorphisms of a quaternion algebra (or any central simple algebra, for that matter) are inner. In particular, if $B = M_2(K)$, then $\mathrm{Aut}(M_2(L)) = \mathrm{PGL}_2(L)$.

Recall the definition of the pointed set $H^1(L/K, \mathrm{PGL}_2(L))$ in Galois cohomology (see Serre's *Galois Cohomology*, Ch. 1, §5.1). A **1-cocycle** is a map $c : \mathrm{Gal}(L/K) \to \mathrm{PGL}_2(L)$ which satisfies the "twisted multiplicative relation" $c(\sigma\tau) = c(\sigma) \circ \sigma c(\tau)$ (such a c is sometimes called a *crossed homomorphism*). The set $Z(L/K, \mathrm{PGL}_2(L))$ of 1-cocycles is a pointed set, with base point the trivial map. Then $H^1(L/K, \mathrm{PGL}_2(L))$ is, by definition, the quotient of $Z(L/K, \mathrm{PGL}_2(L))$ modulo the following "twisted conjugacy" equivalence relation: $c \sim d$ if there is $\psi \in \mathrm{PGL}_2(L)$ for which $c(\sigma) = \psi^{-1} \circ d(\sigma) \circ \sigma\psi$.

Lemma 1.1. *Quaternion algebras over K which are trivialized by L are parametrized, up to isomorphism, by $H^1(L/K, \mathrm{PGL}_2(L))$.*

Proof. Given a quaternion algebra B over K, trivialized by L, we obtain a 1-cocycle $[c] \in H^1(L/K, \mathrm{PGL}_2(L))$ in the following way. The trivialization of B by L yields an isomorphism $\iota : B \otimes_K L \xrightarrow{\sim} M_2(L)$. Now let $\sigma \in \mathrm{Gal}(L/K)$. Then σ acts on both $B \otimes_K L$ and $M_2(L)$ via its action on L. Put $c(\sigma) = \iota\sigma\iota^{-1}\sigma^{-1}$. Then $c(\sigma)$ is an automorphism of $M_2(L)$, hence an element in $\mathrm{PGL}_2(L)$. One can check that c defines a 1-cocycle. Indeed, since $\sigma c(\tau) = \sigma \circ c(\tau) \circ \sigma^{-1} = \sigma\iota\tau\iota^{-1}\tau^{-1}\sigma^{-1}$, we have

$$c(\sigma)\circ\sigma c(\tau) = \left(\iota\sigma\iota^{-1}\sigma^{-1}\right)\left(\sigma\iota\tau\iota^{-1}\tau^{-1}\sigma^{-1}\right) = \iota\sigma\tau\iota^{-1}\tau^{-1}\sigma^{-1} = \iota\sigma\tau\iota^{-1}(\sigma\tau)^{-1},$$

which is $c(\sigma\tau)$.

Conversely, given the class of a 1-cocycle $[c] \in H^1(L/K, \mathrm{PGL}_2(L))$, we obtain a quaternion algebra B trivialized by L by taking the Galois invariants of $M_2(L)$ under the twisted action by c:

$$B = \{a \in M_2(L) : c(\sigma)\sigma a = a \quad \forall\, \sigma \in \mathrm{Gal}(L/K)\}. \qquad \square$$

1.2. **Relation with quadratic spaces.** Let B be as described in the previous display. If $a \in B$ then

$$\sigma \det(a) = \det(\sigma a) = \det(c(\sigma)^{-1} a) = \det(a),$$

and similarly for the trace. Thus, the trace and determinant on $M_2(L)$, when restricted to B, take values in K and $K^\times$, respectively. We refer to these as the **reduced trace and norm**, denoted $t : B \to K$ and $n : B \to K$.

We define an anti-involution $x \mapsto \bar{x}$ on B by setting $\bar{x} = t(x) - x$. This recovers the expression we gave for the quaternion algebra $B = \left(\frac{a,b}{K}\right)$ in the second week's lecture. Then $n(x) = x\bar{x}$, since that is the case for M_2.

One can profitably study quaternion algebras by the underlying quadratic space of B, equipped with its norm form n. In fact, it is enough to consider the trace-zero subspace

$$B^0 = \{x \in B : t(x) = 0\}$$

and the quadratic form $n_0 = n|_{B^0}$ on it given by restriction of n. Note that if $B = \left(\frac{a,b}{K}\right)$, then

$$B^0 = \{x = \beta i + \gamma j + \delta k : \beta, \gamma, \delta \in K\}$$

and

$$n_0(x) = x\bar{x} = -(\beta i + \gamma j + \delta k)^2 = -a\beta^2 - b\delta^2 + ab\gamma^2,$$

which is of discriminant $(-a)(-b)(ab) = a^2b^2 \in K^2$. Viewing the discriminant of n_0 as an invariant in $K/(K^\times)^2$, we can then say that n_0 is of discriminant $1 \in K/(K^\times)^2$.

Lemma 1.2. *The map $B \mapsto (B^0, n_0)$ induces a bijection from the isomorphism classes of quaternion algebras to equivalence classes of quadratic forms on K^3 of discriminant $1 \in K/(K^\times)^2$.*

Proof. It is clear that the stated map sends isomorphic quaternion algebras to equivalent quadratic forms, since such an isomorphism preserves the norm. The surjectivity follows from the fact that any non-degenerate quadratic form of square discriminant has the form $-ax_1^2 - bx_2^2 + abx_3^2$, where $a, b \in K^\times$.

For the injectivity, suppose that B and $\left(\frac{a,b}{K}\right)$ induce equivalent quadratic forms on K^3. To show that B and $\left(\frac{a,b}{K}\right)$ are isomorphic, we must exhibit two elements $x, y \in B$ such that $x^2 = a$, $y^2 = b$, and $xy = -yx$. In fact, we shall find these elements in B^0.

Note that for any $x \in B^0$ we have $n_0(x) = x\bar{x} = -x^2$. Thus we need

$$n_0(x) = -a \qquad \text{and} \qquad n_0(y) = -b. \tag{1.1}$$

Clearly $\left(\frac{a,b}{K}\right)^0$ equipped with $-ax_1^2 - bx_2^2 + abx_3^2$ contains such elements, namely, i and j. Since n_0 is isometric to this quadratic space, by hypothesis, we obtain $x, y \in B^0$ verifying (1.1). Moreover, $x \perp y$ in B^0 since $i \perp j$

in $\left(\frac{a,b}{K}\right)^0$. But orthogonality of x and y is equivalent with the remaining property $xy = -yx$, since $x\bar{y} + \bar{y}x = -(xy + yx)$ is the symmetric bilinear form associated with n_0. □

We note that, in the correspondence described by the above lemma, the split quaternion algebra $M_2(K) \simeq \left(\frac{1,1}{K}\right)$ is mapped to the isotropic quadratic space $(K^3, -x^2 - y^2 + z^2)$, the unique rank 3 quadratic space having Witt index 1 and discriminant 1. By the Witt decomposition theorem, all other (inequivalent) non-degenerate discriminant 1 quadratic forms on K^3 have Witt index 0, and so are anisotropic.

2. Conjugacy classes

Let $G = B^\times$. When $B = M_2$ we obtain $G = \mathrm{GL}_2$. When $B \not\simeq M_2$ we denote $B^\times$ by G'. Similarly to the discussion above, we may parametrize the unit groups G' by the first cohomology group $H^1(K, \mathrm{PGL}_2)$, where we now recognize PGL_2 as the group of *inner automorphisms* of GL_2. The groups G' are therefore called the **inner forms** of G.

Remark 1. In contrast to M_2, not every automorphism of GL_2 is inner. For example, if L/K is a quadratic extension with non-trivial Galois automorphism σ, then $g \mapsto {}^t\sigma(g)$ is an outer automorphism. It gives rises to a unitary group, through the process outlined in the proof of Lemma 1.1. We call such unitary groups **outer forms** of GL_2.

Let us now describe the conjugacy classes of $G = \mathrm{GL}_2$ and its inner forms G'. We shall use the notation $\{g\}$ to denote the conjugacy class of the element g. Let $\{G\}$ denote the set of conjugacy classes of G, and similarly for G'.

2.0.1. *Conjugacy classes of* GL_2*.* Let us first describe the conjugacy classes $\{g\}$ in $G = \mathrm{GL}_2$. They can be organized according to the characteristic polynomial χ_g, as follows:

(1) $\chi_g(x) = (x-\lambda)^2$, where $\lambda \in K$. In this case, we have two possibilities
 (a) $g = \lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ is **central**;
 (b) g is conjugated to the **non-semisimple** $\lambda \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$.
(2) $\chi_g(x) = (x - \lambda_1)(x - \lambda)$ with $\lambda_1 \neq \lambda_2$, in which case g is conjugated to the **hyperbolic** $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$
(3) χ_g is irreducible over K, and we call g **regular elliptic**.

All but the case (1b) are semisimple. All but the case (1) are *regular*. Let G_{rs} denote the regular, semi-simple locus of G; it is the set of g whose characteristic polynomial has two distinct roots over an algebraic closure. Let $G_{\mathrm{reg,ell}} \subset G_{\mathrm{rs}}$ denote the set of regular elliptic elements of G. We shall also refer to the central elements $g = \lambda \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ as being *irregular elliptic*; in this way the g in (1a) and (3) form the set of elliptic elements G_{ell}.

We note that we have the *characteristic polynomial map* $\{G\} \to K \times K^\times$ in which we associate with each conjugacy class $\{g\}$ the pair $(\operatorname{tr} g, \det g)$. This map is surjective: take the companion matrix. It is injective on the semisimple classes. In this way, we may identify the space $X := K \times K^\times$ with the semisimple conjugacy classes of G.

2.0.2. *Conjugacy classes of G'.* We now describe the conjugacy classes in G'. In analogy with the general linear group G, we shall associate with each $g \in G'$ a characteristic polynomial $x^2 - \mathrm{t}(g)x + n(g)$.

In contrast to the case of G, where the identity element shares the same characteristic polynomial as an upper-triangular unipotent matrix, the characteristic polynomial of any non-central element in G' is irreducible over K. Indeed, if $g \in G'$ has split characteristic polynomial, then the discriminant of the latter is a square, so that $t(g)^2 = 4n(g) + \lambda^2$ for some $\lambda \in K$. Thus $(g + \bar{g})^2 = 4g\bar{g} + \lambda^2$ which simplifies to $(g - \bar{g})^2 = \lambda^2$. We write this as $(b - \lambda)(b + \lambda) = 0$, where $b = g - \bar{g}$. Now B is assumed to be a division algebra, so we have either $b = \lambda$ or $b = -\lambda$. But $b \in B_0$, so in either case $g - \bar{g} = b = 0$. This shows that g is central, as claimed.

We in fact have the following result for G':

Lemma 2.1. *Two elements in G' are conjugate if and only if their characteristic polynomials coincide.*

Proof. The forward direction is clear, since the norm and trace are invariant under conjugation.

Conversely, assume that $x, y \in G'$ have the same characteristic polynomial. If x is in the center $K^\times$ then its characteristic polynomial splits over K and so y must also lie in $K^\times$, since otherwise the characteristic polynomial of y would be irreducible (by the discussion preceding the lemma). But then we must have $x = y$, as desired.

Now assume that neither x nor y is central. As they have the same characteristic polynomial, they are conjugate over a finite Galois extension L (splitting the quaternion algebra). In other words, there is $g \in G' \otimes_K L$ such that $gxg^{-1} = y$. Let T denote the centralizer of x. We want to find an element $t \in T$, such that the product gt lies in G' (in which case x and y will be conjugate inside G'). In other words, we would like $t \in T$ such that for any $\sigma \in \mathrm{Gal}(L/K)$ we have $gt = \sigma(gt)$, or what is the same, $g^{-1}\sigma(g) = t\sigma(t)^{-1}$.

Let $\sigma \in \mathrm{Gal}(L/K)$. Applying σ to the identity $gxg^{-1} = y$, and using the fact that x and y are fixed under $\mathrm{Gal}(L/K)$, we obtain $\sigma(g)x\sigma(g)^{-1} = y$. Together these two equations show that $g^{-1}\sigma(g) \in T$. This defines a map $c : \mathrm{Gal}(L/K) \to T$, $\sigma \mapsto g^{-1}\sigma(g)$, which verifies the twisted multiplicativity of a 1-cocycle:

$$\begin{aligned} c(\sigma) \circ \sigma c(\tau) = g^{-1}\sigma(g) \circ \sigma(g^{-1}\tau(g)) &= g^{-1}\sigma(g) \circ \sigma(g^{-1})(\sigma\tau)(g) \\ &= g^{-1}(\sigma\tau)(g) = c(\sigma\tau). \end{aligned}$$

Now T is a torus; indeed $T = K[x]^\times$. This torus is non-split over K, as x is non-central so that its characteristic polynomial is irreducible over K. We may therefore write $T = (\mathrm{Res}_{E/K}\mathbf{G}_m)(K)$, for a quadratic field extension E of K. But by Shapiro's lemma and then Hilbert 90 we have

$$H^1(L/K, T) = H^1(K, \mathrm{Res}_{E/K}\mathbf{G}_m) = H^1(E, \mathbf{G}_m) = 0.$$

Thus $H^1(L/K, T) = 0$ and $g^{-1}\sigma(g)$ is therefore a co-boundary, which was to be shown. □

Remark 2. We could also has argued as follows in the above proof: when neither x nor y is central, we may apply the Skolem–Noether theorem to the embeddings of $K[x]$ and $K[y]$ into B. This theorem states that these embeddings are conjugate over K, so the same is true of x and y. Note, however, that this particular instance of Skolem–Noether can also be proved via the triviality of the H^1 of a torus (see Gille–Szamuely, *Galois Cohomology and Central Simple Algebras*, Theorem 2.7.2), putting this proof on the same standing as the one we have presented.

We now enumerate the conjugacy classes in G'.

(1) the central elements in $K^\times$, whose characteristic polynomial is $(x - \lambda)^2$, for $\lambda \in K^\times$.
(2) a non-central element has an irreducible (over K) characteristic polynomial.

Let $\{G'\}$ denote the set of conjugacy classes of G'. Then, similarly to the case for G, we have the *characteristic polynomial map* $\{G'\} \to K \times K^\times$ in which with each conjugacy class $\{g\}$ is sent to the pair $(t(g), n(g))$. This map is injective by Lemma 2.1. It is not surjective: indeed, by (2) above, we see that the image is

$$X' = K \times K^\times - \{(t, n) \in K \times K^\times : t^2 - 4n \in (K^\times)^2\}.$$

In this way, we identify the space X' with the conjugacy classes of G'.

2.0.3. *Transfer of conjugacy.* We deduce the following important fact:

Corollary 1. *There is an injective map $\{G'\} \to \{G\}$, preserving characteristic polynomials, whose image is G_{ell}.*

3. Local and global quaternion algebras

The previous sections were valid for any field of characteristic different from 2. We now specialize to the fields of interest, namely local and global fields of characteristic zero.

Lemma 3.1. *Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$. Then, if $K = \mathbb{C}$ we have $|H^1(\mathbb{C}, \mathrm{PGL}_2)| = 1$ and if $K \neq \mathbb{C}$ we have $H^1(K, \mathrm{PGL}_2) = \mathbb{Z}/2\mathbb{Z}$. In particular, for $K \neq \mathbb{C}$ there is a unique division quaternion algebra over K.*

Proof. The cases of $\mathbb{C}$ and $\mathbb{R}$ follow from Lemma 1.2 and the well-known classification of quadratic forms over those fields. For example, over $\mathbb{R}$, there are only two rank 3 signatures giving positive discriminant: $(1,2)$ and $(3,0)$. The former corresponds with $B = M_2(\mathbb{R})$, the latter Hamilton's quaternions.

When K is a finite extension of $\mathbb{Q}_p$, in view of Lemma 1.2 and the discussion following it, the result is equivalent to statement that there is a unique anisotropic discriminant 1 quadratic space on K^3. For $K = \mathbb{Q}_p$, this statement can be deduced from the classification of non-degenerate quadratic forms by the discriminant $d \in \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ and the invariant $\epsilon \in \{-1, 1\}$, as described, for example, in Serre, *Course in Arithmetic*, Ch. IV, §2.3. Indeed, after setting $d = 1$, only $\epsilon = -1$ is anisotropic. □

We now pass to the global situation, where F is a number field. Let B be a quaternion algebra over F. We say that B is *ramified* at a place v if the localization $B_v = B \otimes_F F_v$ – a quaternion algebra over F_v – is not split. Let $\operatorname{Ram}(B)$ denote the set of places of F at which B is ramified.

Theorem 3.1. *Let F be a number field. Let B be a quaternion algebra over F. Then $\operatorname{Ram}(B)$ is finite and of even cardinality. Conversely, if S is a finite set of places of F of even cardinality, there exists a unique (up to isomorphism) quaternion algebra B over F such that $\operatorname{Ram}(B) = S$.*

Proof. This follows again by Lemma 1.2, the Hasse principle (which reduces the question to the preceding classification theorem, and is called the Albert–Brauer–Hasse–Noether theorem in this setting), and Hilbert's reciprocity law, which states that the product over all places of the local Hilbert symbol (giving rise to the local ϵ-invariant) is 1. □

4. Characters

Recall that for a finite dimensional representation ρ of a group G, the character of ρ is defined to be $\chi_\rho(g) = \operatorname{tr} \rho(g)$. This is a conjugation invariant function on G. If G is a compact Lie group (in which case all continuous representations of G are finite-dimensional), then the Peter–Weyl theorem states that the irreducible characters of G form an orthonormal basis for the space of L^2-integrable class functions. We have a similar theory for smooth irreducible representations of compact p-adic Lie groups, such as $Z\backslash G'$, where G' is the unit group of division quaternion algebra.

We now develop the theory of characters for admissible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Note that these are infinite-dimensional the moment they are not one-dimensional. *In the remainder of this subsection §1.5, we shall denote by $G = \mathrm{GL}_2(\mathbb{Q}_p)$. We let Z denote the center of G.*

Let π be an admissible representation of G, admitting a central character ω. Recall from our discussion on smooth representations, and associated Hecke modules, that for $\phi \in C_c^\infty(G, \omega^{-1})$, we have an operator $\pi(f)$ on the space of π defined via the smearing operation

$$\pi(\phi)(v) = \int_{Z\backslash G} (\pi(g)v)\phi(g)dg, \qquad v \in V_\pi.$$

Since ϕ is locally constant of compact support, there is a compact open subgroup K of G such that ϕ is right-K-invariant. Furthermore, since π is smooth, for any $v \in V_\pi$ there is some open compact subgroup K of G such that $v \in V_\pi^K$. Replacing K by $K \cap K_0$, we may assume that $K \subset K_0$. We deduce that for any $v \in V_\pi$ we have $\pi(\phi)(v) \in V_\pi^K \subset V_\pi^{K_0}$. Since π is assumed to be admissible, the dimension of $V_\pi^{K_0}$ is finite, and we deduce that $\pi(\phi)$ is a finite rank operator.

In particular, when $G = \mathrm{GL}_2(K)$ and π is unramified, and ϕ is the characteristic function of $K = K_0$, then the operator $\pi(\phi)$ in this case is the projection onto the line of K_0 invariants of π. Its trace is equal to 1 in this case.

Theorem 4.1 (Harish-Chandra)**.** *Let π be a smooth irreducible representation of non-archimedean G with central character ω. Then there exists a unique smooth (complex-valued) function Θ_π on G_{rs} such that for all $\phi \in C_c^\infty(G, \omega^{-1})$ we have*

$$\operatorname{tr} \pi(\phi) = \int_{Z\backslash G} \phi(g)\widetilde{\Theta}_\pi(g)dg,$$

where $\widetilde{\Theta}_\pi$ is any extension of Θ_π to G.

We only sketch the proof for supercuspidal π. We follow Taibi's notes: see his Theorem 3.12. Before embarking on the proof, we make a few important remarks.

4.1. **Orbital integrals.** Let π be an irreducible supercuspidal representation of G. To prove Theorem 4.1 for π, we shall attempt to construct Θ_π as an *orbital integral* of a normalized matrix coefficient.

By definition, given $\gamma \in G$ and $f \in C_c^\infty(G, \omega)$, where ω is a smooth character of Z, the orbital integral of f at γ is

$$O_\gamma(f) = \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)d\dot{g}, \tag{4.1}$$

where G_γ is the centralizer of γ in G, and $d\dot{g}$ is the right-G-invariant measure on $G_\gamma\backslash G$ given by the quotient of a choice of Haar measures on G_γ and G. This integral converges for any semisimple γ.

This strategy will barely fail for convergence reasons, but we shall be able to salvage (after a limiting process) a weighted version. Namely, we shall show that for $g \in G_{rs}$ we have $\Theta_\pi(g) = WO_g(f_\pi)$, where f_π is a "normalized" matrix coefficient for π, and

$$WO_\gamma(f) = \int_{N\times K_0} f(k^{-1}n^{-1}\gamma nk)w(n)dndk \tag{4.2}$$

is the *weighted orbital integral* of a function $f \in C_c^\infty(G,\omega)$ at γ, where $w\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) = \min\{0, 2\mathrm{val}(x)\}$.

For a nice exposition, see the Introduction to Kottwitz's article, *Harmonic Analysis on Reductive p-adic Groups and Lie Algebras*, in the Toronto Summer School Clay Proceedings.

4.2. **Formal degree.** To properly express the normalization of the matrix coefficients we consider, we shall need to introduce the idea of *formal degree*. This is an integer associated with any essentially square-integrable representation (π, V_π) which generalizes the notion of dimension. By definition, it is the unique positive real number such that, for any $u, v \in V_\pi$ and $\tilde{u}, \tilde{v} \in \tilde{V_\pi}$ we have

$$\int_{Z\backslash G} \langle \pi(g)u, \tilde{u}\rangle \langle \pi(g^{-1})v, \tilde{v}\rangle dg = d_\pi^{-1}\langle u, \tilde{v}\rangle\langle v, \tilde{u}\rangle.$$

This definition should of course be justified, so let us briefly sketch why d_π exists. Firstly, to see why the left-hand side is proportional to the right, by a *complex* scalar, we may argue using Schur's lemma (since π is irreducible), in the form $\dim \mathrm{End}_G(V) = \dim(V \otimes \tilde{V})^G = 1$. Indeed, fixing u and $\tilde{v}$, the above integral is G-invariant, and so defines an element in the line $(V \otimes \tilde{V})^G$, and thus a multiple (depending on u and $\tilde{v}$) of the canonical pairing $\langle v, \tilde{u}\rangle$. Applying the same argument, while fixing v and $\tilde{u}$, shows that the integral is proportional to the pairing $\langle \tilde{v}, u\rangle$. Together, these show the left-hand side is $c_\pi\langle u, \tilde{v}\rangle\langle v, \tilde{u}\rangle$, for some $c_\pi \in \mathbb{C}$. To show that this scalar is actually a *positive real number*, one uses the fact that an irreducible essentially square-integrable representation is unitarizable, and then choosing a Hermitian inner product, and choosing vectors correctly, one can show the positivity of all quantities except c_π. (See Lemma 2.32 and Proposition 2.34 of Taibi's notes.)

5. Proof of Theorem 4.1

Let (π, V_π) be irreducible supercuspidal. As π is, in particular, essentially square-integrable, we may choose $v \in V_\pi$ and $\tilde{v} \in \tilde{V}$ such that $\langle v, \tilde{v}\rangle = d_\pi$.

Step 1 (Taibi's notes, Lemma 3.14). We first establish that the function

$$\dot{g} \mapsto \int_G \phi(h)\langle \pi(g^{-1}hg)v, \tilde{v}\rangle dh$$

is compactly supported on G/Z, and

$$\mathrm{tr}\,\pi(\phi) = \int_{G/Z}\int_G \phi(h)\langle \pi(g^{-1}hg)v, \tilde{v}\rangle dh dg.$$

To see this, we first examine the inner integral. We have

$$\int_G \phi(h)\langle \pi(g^{-1}hg)v, \tilde{v}\rangle dh = \langle \pi(g^{-1})\pi(\phi)\pi(g)v, \tilde{v}\rangle = \langle \pi(\phi)\pi(g)v, \tilde{\pi}(g)\tilde{v}\rangle.$$

We want to expand the right-hand side over a basis of V_π and $\tilde{V}_\pi$. Take a basis $\{e_n\}$ of K_0-types of V_π. Let $\{\tilde{e}_n\}$ be the dual basis in $V_{\tilde{\pi}}$. Recall that we have shown that π smooth irreducible implies admissible. Moreover, we have highlighted the fact that for admissible representations the double contragredient of π is isomorphic to π itself through the evaluation map. This shows that the dual basis is well-defined and satisfies $\tilde{e}_i(e_j) = \delta_{i,j}$.

We may expand any $w \in V_\pi$ as $w = \sum_i \langle w, \tilde{e}_i \rangle e_i$. Thus

$$\pi(\phi)w = \sum_i \langle w, \tilde{e}_i \rangle \pi(\phi) e_i.$$

We now expand each $\pi(\phi)e_i = \sum_j a_{i,j} e_j$, where $a_{ij} = \langle \pi(\phi)e_i, e_j \rangle$ are "distributional matrix coefficients". Recall that $\pi(\phi)$ is a finite rank operator (again, since smooth irreducible implies admissible) so all but finitely many of the a_{ij} are zero. This yields

$$\pi(\phi)w = \sum_i \langle w, \tilde{e}_i \rangle \sum_j a_{i,j} e_j.$$

We apply this with $w = \pi(g)v$ to get

$$\begin{aligned}\langle \pi(\phi)\pi(g)v, \tilde{\pi}(g)\tilde{v} \rangle &= \sum_{i,j} a_{i,j} \langle \pi(g)v, \tilde{e}_i \rangle \langle e_j, \tilde{\pi}(g)\tilde{v} \rangle \\ &= \sum_{i,j} a_{i,j} \langle \pi(g)v, \tilde{e}_i \rangle \langle \pi(g^{-1})e_j, \tilde{v} \rangle.\end{aligned}$$

As π is supercuspidal, each term on the right-hand side is compactly supported on G/Z. Since all but finitely many $a_{i,j}$ are zero, the same is true of the entire right-hand side, proving the first assertion.

Now we integrate each term in the last expression over G/Z, and invoke the definition of the formal degree. This gives

$$\sum_{i,j} a_{i,j} d_\pi^{-1} \langle v, \tilde{v} \rangle \langle v_j, \tilde{v}_i \rangle = \sum_i a_{i,i} = \operatorname{tr} \pi(\phi),$$

as desired.

Step 2. Next, one shows that the order of integration can be interchanged, as long as one restricts the integration over the *regular elliptic* elements $G_{\text{reg,ell}}$. Note that the centralizer G_h of an element $h \in G_{\text{reg,ell}}$ is compact modulo the center, so that its volume in G/Z is finite. Put $f_\pi(g) = \langle \pi(g)v, \tilde{v} \rangle$ with $\langle v, \tilde{v} \rangle = d_\pi$. Then f_π is compactly supported modulo the center since π is supercuspidal, and one has

$$\Theta_\pi(h) = \int_{G/Z} f_\pi(g^{-1}hg) d\dot{g} = \operatorname{vol}(Z \backslash G_h) O_h(f_\pi) \qquad (h \in G_{\text{rs}}^{\text{ell}}),$$

which is well-defined. Note that this argument fails to work for hyperbolic $h = \operatorname{diag}(a, b)$, where $a \neq b$, as in this case the centralizer is the diagonal torus T, which has infinite volume. (Recall that such hyperbolic elements are precisely the non-elliptic regular semisimple elements in $G = \mathrm{GL}_2$.)

Extending the integration over all of G_{rs} is the analytically hard part of the argument, where we encounter problems of convergence, and the weighted orbital integral makes its appearance. We will not work out the details. One of the key ingredients is the Selberg principle, which shows that the orbital integrals of cuspidal functions (such as matrix coefficients of supercuspidal representations) on the complement $G_{\text{rs}} \setminus G_{\text{reg,ell}}$ vanish.

6. Statement of the local Jacquet-Langlands correspondence

Let K be either $\mathbb{R}$ or a finite extension of $\mathbb{Q}_p$ and put $G = \mathrm{GL}_2(K)$. Let $\mathscr{E}_2(G)$ denote the set of equivalence classes of essentially square-integrable smooth irreducible representations of G. Let $G' = B^\times$ with B the unique (up to isomorphism) quaternion division algebra over K, and write $\Pi(G')$ for the set of equivalence classes of smooth irreducible representation of G. (With the obvious notation, we have $\Pi(G') = \mathscr{E}_2(G')$, since G' is compact modulo the center.)

Theorem 6.1 (Local Jacquet-Langlands)**.** *Then there is a bijective map*

$$\mathrm{JL} : \Pi(G') \to \mathscr{E}_2(G)$$

such that if $\{g\} \in X$ *and* $\{g'\} \in X'$ *are regular elliptic elements which coincide under Corollary 1 then*

$$\theta_{\mathrm{JL}(\pi')}(g) + \theta_{\pi'}(g') = 0.$$

6.1. **Examples.** For a more detailed discussion of the following results, see Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Appendix A.

(1) Let $K = \mathbb{R}$. Then $G' = \mathbb{H}^\times$, where $\mathbb{H}$ are Hamilton's quaternions, and $G = \mathrm{GL}_2(\mathbb{R})$. Since $\mathbb{H}^\times = \mathbb{R}_+\mathrm{SU}(2)$, we have

$$\Pi(G') = \{\pi'_{s,k} = |\det|^s \otimes \mathrm{sym}^k(\mathbb{C}^2) : s \in \mathbb{C}, k \geq 0\}.$$

Moreover, we have

$$\mathscr{E}_2(G) = \{\pi_{s,k} = |\det|^s D_k : s \in \mathbb{C}, k \geq 2\}.$$

Then the Jacquet-Langlands correspondence over $\mathbb{R}$ sends $\pi'_{s,k}$ to $\pi_{s,k+2}$. In particular, the trivial representation of G' is sent to the weight 2 discrete series representation D_2.

To see the character relation we have to check that on regular elliptic elements g in G, we have $\Theta_{\mathrm{Sym}^{k-2}}(g) + \Theta_{D_k}(g) = 0$. We have an exact sequence

$$0 \longrightarrow D_k \longrightarrow \mathrm{Ind}_B^G(\mu_k) \longrightarrow \mathrm{Sym}^{k-2}(\mathbb{C}^2) \longrightarrow 0,$$

for the non-unitary character $\mu_k = \mathrm{sgn}^\epsilon|\cdot|^{(k-1)/2} \otimes |\cdot|^{-(k-1)/2}$ on T, where $k \equiv \epsilon \bmod 2$. Now characters are additive on exact sequences, and the character of an induced representation vanishes on regular elliptic elements, whence the result.

(2) For K non-archimedean, the trivial representation of G' is sent to the Steinberg representation of G. This is similar to the above real case, as it results from the exact sequence

$$0 \to 1 \to \mathrm{Ind}_B^G(\delta_B^{-1/2}) \to \mathrm{St} \to 0.$$

7. Statement of the global Jacquet-Langlands correspondence

Let B be a quaternion division algebra over a number field F and let $G' = B^\times$. Let S be the set of places of F where B is non-split. Note that $G'_v \simeq \mathrm{GL}_2(F_v)$ for every $v \notin S$.

Let $\mathcal{A}'_0$ be the set of isomorphism classes of infinite dimensional irreducible automorphic representations of G'. Let $\mathcal{A}_0$ be the set of isomorphism classes of irreducible cuspidal automorphic representations of G.

We shall prove the following theorem.

Theorem 7.1. *There exists a injective map*

$$\mathrm{JL} : \mathcal{A}'_0 \longrightarrow \mathcal{A}_0$$

with the following properties:

(1) *the image of* JL *is the set of* $\pi = \otimes'_v \pi_v \in \mathcal{A}_0$ *for which* $\pi_v \in \mathscr{E}^2(G_v)$ *for every* $v \in S$*;*

(2) *for every* $v \notin S$*,* $\mathrm{JL}(\pi')_v \simeq \pi'_v$ *under the isomorphism* $\mathrm{GL}_2(F_v) \simeq G'_v$*;*

(3) *for every* $v \in S$*, we have* $\mathrm{JL}(\pi')_v \simeq \mathrm{JL}(\pi'_v)$*.*

7.1. **Example.** Let us describe what this means in classical terms, via an example.

Let $B = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ be the Hamilton quaterions over $\mathbb{Q}$, with ramification set $\mathrm{Ram}(B) = \{2, \infty\}$. Note that the quadratic space associated with the norm form on the trace-zero subspace B^0 is $(\mathbb{Q}^3, x^2 + y^2 + z^2)$. Consider $\mathbf{PB}^\times$, the projective group of units in B. Then $\mathbf{PB}^\times = \mathrm{SO}_3$, the special orthogonal group of the quadratic form $x^2 + y^2 + z^2$, as an algebraic group over $\mathbb{Q}$. In particular $\mathbf{PB}^\times(\mathbb{R}) = \mathbb{R}^\times \backslash \mathbb{H}^\times \simeq \mathrm{SO}(3)$.

Now, since $x^2 + y^2 + z^2$ has genus one, strong approximation gives

$$\mathrm{SO}_3(\mathbb{A}_f) = \mathrm{SO}_3(\mathbb{Q})\mathrm{SO}_3(\widehat{\mathbb{Z}}).$$

This in turn allows us to identify

$$\mathrm{SO}_3(\mathbb{Z})\backslash\mathrm{SO}_3(\mathbb{R}) = \mathrm{SO}_3(\mathbb{Q})\backslash\mathrm{SO}_3(\mathbb{A})/\mathrm{SO}_3(\widehat{\mathbb{Z}})$$

Since $S^2 = \mathrm{SO}_3(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ we deduce

$$\mathrm{SO}_3(\mathbb{Z})\backslash S^2 = \mathrm{SO}_3(\mathbb{Q})\backslash\mathrm{SO}_3(\mathbb{A})/\mathrm{SO}_2(\mathbb{R})\mathrm{SO}_3(\widehat{\mathbb{Z}}).$$

We have thus identified $\mathrm{SO}_3(\mathbb{Z})\backslash S^2$ with an adelic quotient.

We can do better, in fact: we can identify the sphere itself with an adelic quotient, by adding in level structure to the latter. Indeed, note that $\mathrm{SO}_3(\mathbb{Z})$

is a finite group, isomorphic to $\mathrm{SO}_3(\mathbb{Z}/3\mathbb{Z})$. This allows us to add level structure to $\mathrm{SO}_3(\widehat{\mathbb{Z}})$ to identify

$$S^2 = \mathrm{SO}_3(\mathbb{Q})\backslash \mathrm{SO}_3(\mathbb{A})/K = \mathbf{PB}^\times(\mathbb{Q})\backslash \mathbf{PB}^\times(\mathbb{A})/K, \tag{7.1}$$

where $K = K_\infty K_f[3]$, and $K_\infty = \mathrm{SO}(2)$ and $K_f[3]$ is the principal congruence subgroup $\ker(\mathrm{SO}_3(\hat{\mathbb{Z}}) \to \mathrm{SO}_3(\hat{\mathbb{Z}}/3\hat{\mathbb{Z}}))$. This idea of replacing the quotient by $\mathrm{SO}_3(\mathbb{Z})$ by an additional congruence level structure was borrowed from Section 5.5 of the paper *Linnik's ergodic method and the distribution of integral points on spheres,* by Ellenberg–Michel–Venkatesh.

Now $L^2(S^2) = \oplus_{k\geq 0}\mathcal{H}_k$, where $\mathcal{H}_k$ the space of degree k spherical harmonics. For each $k \geq 0$, the space $\mathcal{H}_k$ forms an irreducible representation of $\mathbf{PB}^\times(\mathbb{R}) = \mathrm{SO}(3)$ of dimension $2k+1$. Now, each $\mathcal{H}_k$ can furthermore be decomposed under the $\mathrm{SO}(2)$-action by rotations as $\mathcal{H}_k = \oplus_{|\ell|\leq k}\mathbb{C}Z_{k,\ell}$. But rather than working with these classical functions $Z_{k,\ell}$, we would like to diagonalize $\mathcal{H}_k$ by the Hecke operators on S^2. These Hecke operators preserve $\mathcal{H}_k$ since they commute with the Laplacian on the sphere, and can either be written down classically, as in the work of Lubotzky–Phillips–Sarnak (see Lubotzky's book, *Discrete groups, expanding graphs and invariant measures*), or via the adelic description of S^2 as in (7.1) above.

In this way, we obtain $\mathcal{H}_k = \oplus_{|\ell|\leq k}\mathbb{C}h_{k,\ell}$, where each $h_{k,\ell}$ generates an irreducible automorphic (adelic) representation $\sigma_{k,\ell}$ of $\mathbf{PB}^\times$ such that $\sigma_{k,\ell}^K \neq 0$. This gives

$$L^2(S^2) = \bigoplus_{k\geq 0}\bigoplus_{|\ell|\leq k}\sigma_{k,\ell}$$

The global Jacquet-Langlands correspondence then sends $\sigma_{k,\ell}$ to an irreducible cuspidal automorphic representation $\pi_{k,\ell}$ of $\mathbf{PGL}_2(\mathbb{A})$, which is

(1) unramified outside of $\{2,3,\infty\}$, since B is unramified outside of $\{2,\infty\}$, and the local component of K is $\mathbf{PGL}_2(\mathbb{Z}_p)$ for $p \notin \{2,3\}$;
(2) is square-integrable at $p = 2$, since B is ramified at 2;
(3) has $K(3)$-invariants at $p = 3$;
(4) discrete series at infinity, since B is ramified at ∞. Moreover, $\pi_{k,\ell}$ is of weight $2k+2$. Indeed, as we saw in §6.1, the $2k+1$-dimensional representation sym^{2k} of $\mathrm{SU}(2)$ goes to D_{2k+2}, and sym^{2k} is induced from $\mathcal{H}_k$ under the degree 2 covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

In other words, each $h_{k,\ell}$ is sent to a classical holomorphic modular form with trivial central nebentypus, of weight $2k+2$ with respect to $\Gamma_1(2^a 3^2)$, for some a (come from the level structure of K at 2) that I haven't worked out.