Lecture 9: The trace formula and the global J-L correspondence

1. A primer on spectral theory

Let us recall some classes of operators on a separable Hilbert space H. First of all, we let $\mathcal{B}(H)$ denote the space of bounded linear operators on H. Recall that a linear operator on a normed vector space is bounded if and only if it is continuous.

Note that the Laplacian of acting on L^2 functions on a Riemannian manifold is not continuous. This is more generally true of differential operators (functions can take on small values although their derivatives can be large). On the other hand, we shall see that a large class of bounded operators arises through the theory of *integral operators*.

1.1. **Compact operators.**

An operator $T \in \mathcal{B}(H)$ is called **compact** if the image of a bounded set under T is relatively compact. Equivalently, T is in the closure, relative to the topology induced by the operator norm, of the finite rank operators. One can further characterize compact operators T in $\mathcal{B}(H)$ as those for which $\lim_{n\to\infty} \|Te_n\| = 0$ for every orthonormal family $\{e_n\}$ in H.

There is a spectral theorem for compact self-adjoint operators, namely, if T is such an operator, there exists a countable orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of H consisting of eigenfunctions for T with eigenvalues λ_n satisfying $\lambda_n \to 0$. (Note that we are using the separability hypothesis on H, since we have a countable orthonormal basis.)

1.2. **Hilbert–Schmidt operators.**

Another important class of operators are the **Hilbert–Schmidt** operators. These are the operators $T \in \mathcal{B}(H)$ for which there is an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H satisfying

$$\sum_{n\in\mathbb{N}} \|Te_n\|^2 < \infty.$$

If T is Hilbert–Schmidt, and $(e_n)_{n\in\mathbb{N}}$ is a basis as above, we define the Hilbert–Schmidt norm $\|T\|_{\mathrm{HS}}$ of T to be the quantity on the left-hand side. One can show that if any Hilbert space basis satisfies the above finiteness statement, and in fact $\|T\|_{\mathrm{HS}}$ is independent of the choice of basis (e_n). For example, if H is finite-dimensional, then the Hilbert–Schmidt norm is the Frobenius norm: the sum of squares of the matrix coefficients of any matrix representing T.

We deduce from the above discussion that *a Hilbert–Schmidt operator is compact.*

Let $\mathcal{B}(H)_{\mathrm{HS}}$ denote the space of Hilbert–Schmidt operators. Then $\mathcal{B}(H)_{\mathrm{HS}}$ is a self-adjoint ideal in $\mathcal{B}(H)$. (A subset of $\mathcal{B}(H)$ is said to be self-adjoint if it is closed under taking adjoints.)

Endowed with the inner product induced by $\|T\|^2_{\mathrm{HS}}$, the space $\mathcal{B}(H)_{\mathrm{HS}}$ becomes a Hilbert space isometric with $\ell^2(\mathbb{N}, H)$, via the map $T \mapsto (Te_n)_{n\in\mathbb{N}}$.

In this way we see that a bounded operator T, with spectrum $\{\lambda_n : n \in \mathbb{N}\}$ is Hilbert–Schmidt if, and only if, $\sum_{n\in\mathbb{N}} \lambda_n^2 < \infty$.

Let (X, μ) be a "nice" locally compact Borel measure space for which $H = L^2(X)$ is separable. A way of producing Hilbert–Schmidt operators on $H = L^2(X)$ is through L^2-integrable kernels on $X \times X$. Namely, let $K \in L^2(X \times X)$ and put

$$(T_K\phi)(y) = \int_X \phi(x)K(x,y)d\mu(x).$$

Then T_K is Hilbert–Schmidt operator with $\|T_K\|_{\mathrm{HS}} = \|K\|_2$. In fact, all Hilbert–Schmidt operators are obtained in this way. We deduce from this that the map $L^2(X \times X) \to \mathcal{B}(L^2(X))_{\mathrm{HS}}$, sending K to T_K, is an isometric isomorphism.

Note furthermore that T_K is self-adjoint if, and only if, $K(x,y) = \overline{K(y,x)}$.

1.3. **Trace class operators.** Finally, a compact operator $T \in \mathcal{B}(H)$ is said to be of **trace class** if

$$\sum_n \langle |T|e_n, e_n\rangle < \infty.$$

for some (hence any) orthonormal basis $\{e_n\}_n$ of H. In that case, the sum $\sum_n \langle Te_n, e_n\rangle$ is absolutely convergent; we call this last sum the **trace** of T, denoted $\operatorname{tr} T$.

A trace class operator is the product of two Hilbert–Schmidt operators; this is in fact a characterization. In particular, since $\mathcal{B}(H)_{\mathrm{HS}}$ is an ideal, this shows that a trace class operator is Hilbert–Schmidt.

If T is a trace class operator on $L^2(X)$, of the form T_K for an L^2-integrable kernel K, then as T is the product of two Hilbert–Schmidt operators, K is the convolution of two L^2-integrable kernels. By Cauchy–Schwarz, K is L^1-integrable along the diagonal copy of X in $X \times X$, and one can compute

$$\operatorname{tr} T_K = \int_X K(x,x)d\mu(x).$$

This is the starting point of the trace formula for G'.

2. The trace formula for G'

Throughout this section, we let B be a division quaternion algebra over F. Let $G' = B^\times$. Denote by Z' the center of G'. Let

$$[G'] = G'(F)Z'(\mathbb{A})\backslash G'(\mathbb{A}).$$

We would like to state and prove the trace formula for G'. This will be rendered rather straightforward, thanks to the following fact.

Theorem 2.1 (Mostow–Tamagawa)**.** *The quotient $[G']$ is compact.*

Proof. We just give the idea: we work with $\mathrm{PB}^\times$ rather than $G' = B^\times$. This allows us to identify $\mathrm{PB}^\times$ with SO_Q, where Q is the norm form on the trace-zero quaternions. By the previous lecture, we know that Q is anisotropic (since B is not split). The inclusion $\mathrm{SO}_Q \subset \mathrm{SL}_3$ induces a map on the corresponding automorphic space $\mathrm{SO}_Q(\mathbb{Q})\backslash\mathrm{SO}_Q(\mathbb{A})$ to $\mathrm{SL}_3(\mathbb{Q})\backslash\mathrm{SL}_3(\mathbb{A})$, the latter being the space of unimodular $\mathbb{Q}$-lattices in $\mathbb{A}^3$. One can show this is a proper map, so that it is enough to prove the compactness of the image. Now we have a compactness criterion, due to Mahler, which states (in this case) that a sequence $L_n = \mathbb{Q}^3.g_n$ in $\mathrm{SL}_3(\mathbb{Q})\backslash\mathrm{SL}_3(\mathbb{A})$ has no convergenct subsequence precisely when for every n there is $x_n \in \mathbb{Q}^3$ with $x_n.g_n \to 0$ in $\mathbb{A}^3$ as $n \to \infty$. If such a sequence L_n exists with $g_n \in \mathrm{SO}_Q(\mathbb{A})$ then $Q(x_n.g_n) = Q(x_n) \to 0$. But $\mathbb{Q}^3$ is discrete in $\mathbb{A}^3$ so for large enough n we have $Q(x_n) = 0$. This is impossible, since Q is anisotropic. $\square$

2.1. **A trace class operator.** We fix throughout a continuous character ω of $Z'(F)\backslash Z'(\mathbb{A})$. For $f \in C_c^\infty(G'(\mathbb{A}), \omega^{-1})$, we let $R(f)$ denote the unitary operator on $L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A}), \omega)$ given by

$$R(f)(\phi)(x) = \int_{Z'(\mathbb{A})\backslash G'(\mathbb{A})} \phi(xy)f(y)dy.$$

This operator "smears" by f the right-regular representation of $G'(\mathbb{A})$ on $L^2(G'(F)\backslash G'(\mathbb{A}), \omega)$. This operator will be of critical importance in the development of the trace formula.

Theorem 2.2. *The unitary operator $R(f)$ is of trace class. For every $f \in C_c^\infty(G'(\mathbb{A}), \omega^{-1})$ we have*

$$\operatorname{tr} R(f) = \int_{[G']} K(x, x)dx,$$

where $K(x, y) = \sum_{\gamma \in G'(F)} f(x^{-1}\gamma y)$.

Proof. Let $d\dot{g}$ be the (quotient) Haar measure on $[G']$. We have

$$\begin{aligned} R(f)(\phi)(x) &= \int_{Z'(\mathbb{A})\backslash G'(\mathbb{A})} \phi(xy)f(y)dy \\ &= \int_{Z'(\mathbb{A})\backslash G'(\mathbb{A})} \phi(y)f(x^{-1}y)dy \\ &= \int_{[G']} \phi(y) \left(\sum_{\gamma \in G'(F)} f(x^{-1}\gamma y) \right) d\dot{y}. \end{aligned}$$

This shows that $R(f)$ has kernel K, with K as in the statement of the theorem.

We claim that

$$K \in L^2\left([G'] \times [G']\right).$$

Indeed, as the space the space $[G']$ is compact, it suffices to show K is continuous. This can be seen by observing that, by the discreteness of $G'(F)$ inside $G'(\mathbb{A})$, for any compact subset Ω of $G'(\mathbb{A})$, there are only finitely many $\gamma \in Z'(F)\backslash G'(F)$ lying in $Z'(\mathbb{A})\backslash \Omega^{-1}\mathrm{supp}(f)\Omega$. In other words, when

$x, y \in \Omega$ there are only a finite number of non-zero terms in the sum defining $K(x,y)$, whence the continuity (smoothness, in fact) of K.

From the L^2 integrability of K we deduce that $R(f)$ is Hilbert–Schmidt. To show that $R(f)$ is trace class we shall show that it is the product of two Hilbert–Schmidt operators (cf. the discussion of trace class operators in §1.3). For this one needs to invoke the Dixmier–Malliavin theorem, which assures us that any test function f can be written as a finite sum of convolutions $f_i * g_i$. Thus $R(f)$ is a finite sum of compositions $R(f_i) \circ R(g_i)$. Since each factor is Hilbert–Schmidt, each composition is trace class, and the result follows. □

2.2. **Spectral decomposition.** Let us now calculate the trace of $R(f)$, spectrally, using the following decomposition of $L^2([G'])$.

Proposition 2.1. *Let ω be a continuous unitary character of $Z'(F)\backslash Z'(\mathbb{A})$. The unitary representation of $G'(\mathbb{A})$ on $L^2(G'(F)\backslash G'(\mathbb{A}), \omega)$ decomposes as a countable Hilbert space direct sum of irreducible subrepresentations π' of G', with central character ω, each with finite multiplicity.*

Proof. This is a standard fact from functional analysis, based on the spectral theorem for self-adjoint compact operators, applied to a Dirac sequence of test functions f satisfying $f(g) = \overline{f(g^{-1})}$. (The latter condition implies that $K(g,h) = \overline{K(h,g)}$.) We content ourselves to giving three references:

(1) the classic book by Gelfand–Graev–Piatetski-Shapiro, "Representation theory and automorphic functions", Ch 1, §2.3;
(2) Bump, Chapter 2, Theorem 2.3.3;
(3) Borel, Lemma 16.1 and Theorem 16.2.

Of these, I recommend the first. □

Before continuing, we observe that for an admissible representation $\pi = \otimes'_v \pi_v$ of $G'(\mathbb{A})$ with central character ω, and a factorizable function $f = \otimes_v f_v \in C_c^\infty(Z'(\mathbb{A})\backslash G'(\mathbb{A}), \omega^{-1})$ the operator

$$\pi(f) = \int_{Z'(\mathbb{A})\backslash G'(\mathbb{A})} \pi(g) f(g) dg = \prod_v \int_{Z'_v \backslash G'_v)} \pi_v(g) f_v(g) dg = \otimes_v \pi_v(f_v)$$

is of trace class by purely local considerations.

Corollary 1. *For any $f \in C_c^\infty(Z'(\mathbb{A})\backslash G'(\mathbb{A}), \omega^{-1})$*

$$\operatorname{tr} R(f) = \sum_\pi m(\pi) \operatorname{tr} \pi(f),$$

where the sum absolutely convergent, and is over all irreducible unitary representations of $G'(\mathbb{A})$ with central character ω, and $m(\pi) \in \mathbb{Z}_{\geq 0}$.

We note that the absolute convergence of the sum is a consequence of the definition of a trace class operator (applied to $R(f)$).

2.3. **Geometric expansion.** We recall from the previous lecture the notion of orbital integral: for $\gamma \in G'(\mathbb{A})$ and $f \in C_c^\infty(G'(\mathbb{A}), \omega^{-1})$ we put

$$O_\gamma(f) = \int_{G'_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} f(x^{-1}\gamma x)dx.$$

We now evaluate the trace of $R(f)$ geometrically.

Theorem 2.3. *For $f \in C_c^\infty(G'(\mathbb{A}), \omega^{-1})$ we have*

$$\operatorname{tr} R(f) = \sum_{[\gamma]} \operatorname{vol}(G'_\gamma(F)Z'(\mathbb{A})\backslash G'(\mathbb{A}))O_\gamma(f),$$

with only finitely many terms appearing on the right-hand side.

Proof. Inserting the definition of the kernel and interchanging integral and sum, we get

$$\operatorname{tr} R(f) = \int_{[G']} K_f(x,x)dx = \sum_{\gamma \in G'(F)} \int_{[G']} f(x^{-1}\delta x)dx.$$

At this point, we see already that only a finite number of terms contribute to the sum on γ. Indeed, letting $\Omega \subset G'(\mathbb{A})$ be a (compact!) fundamental for the quotient $G'(F)\backslash G'(\mathbb{A})$, if $\gamma \in G'(F)$ is such that there is $x \in \Omega$ with $x^{-1}\gamma x \in \operatorname{supp}(f)$ then γ lies in the intersection of the discrete subgroup $G'(F)$ with a compact set. Continuing, we decompose the sum on γ into conjugacy classes:

$$\begin{aligned}
\operatorname{tr} R(f) &= \sum_{[\gamma]} \int_{[G']} \sum_{\delta \in [\gamma]} f(x^{-1}\delta x)dx \\
&= \sum_{[\gamma]} \int_{[G']} \sum_{\alpha \in G'_\gamma(F)\backslash G'(F)} f(x^{-1}\alpha^{-1}\gamma\alpha x)dx.
\end{aligned}$$

We then unfold the integral to obtain

$$\begin{aligned}
\operatorname{tr} R(f) &= \sum_{[\gamma]} \int_{Z'(\mathbb{A})G'_\gamma(F)\backslash G'(\mathbb{A})} f(x^{-1}\gamma x)dx \\
&= \sum_{[\gamma]} \int_{Z'(\mathbb{A})G'_\gamma(F)\backslash G'_\gamma(\mathbb{A})} \int_{G_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} f(x^{-1}y^{-1}\gamma y x)dxdy \\
&= \sum_{[\gamma]} \operatorname{vol}(Z'(\mathbb{A})G'_\gamma(F)\backslash G'_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A})\backslash G'(\mathbb{A})} f(x^{-1}\gamma x)dx,
\end{aligned}$$

as desired. $\square$

Corollary 2 (Trace formula, compact quotient)**.** *For $f' \in C_c^\infty(G'(\mathbb{A}), \omega^{-1})$ we have*

$$\sum_{\pi'} m(\pi') \operatorname{tr} \pi'(f') = \sum_{[\gamma]} \operatorname{vol}(G'_\gamma(F)Z'(\mathbb{A})\backslash G'(\mathbb{A}))O_\gamma(f'),$$

where the left-hand side is absolutely convergent and the right-hand side involves only finitely many non-zero terms.

Proof. Combine Corollary 1 and Theorem 2.3. □

3. The cuspidal kernel on GL_2

We now pass to the group $G = \mathrm{GL}_2$. In this case the automorphic quotient space $[G] = G(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ is not compact, as one sees already in the case of the standard fundamental domain of the modular group $\mathrm{SL}_2(\mathbb{Z})$.

One of the complications that arise in this non-compact setting is that, although the operator $R(\phi)$ again has kernel

$$K(x,y) = \sum_{\gamma\in G(F)} \phi(x^{-1}\gamma y),$$

this kernel is no longer L^2-integrable along the diagonal copy of $[G]$. Indeed, for $x = y$ going into the cusp, the number of terms coming from $N(F)$, the unipotent radical of the standard Borel subgroup, in the above sum becomes unbounded, regardless of the support of ϕ. Therefore $R(\phi)$ is no longer a compact operator on $L^2(G(F)\backslash G(\mathbb{A}), \omega)$.

However, the situation is rectified as soon as one restricts to the cuspidal subspace, as the next theorem shows.

Theorem 3.1. *For every $\phi \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}), \omega^{-1})$, the restriction of $R(\phi)$ to the cuspidal subspace $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}), \omega)$ is of trace class.*

Proof. We break up the proof into three steps. We only give a sketch. For more details, see Theorem 2.1 of Gelbart–Jacquet, *Automorphic forms from an analytic point of view*, as well as Theorems 9.2 and 9.5 of Borel's book.

Step 1: Establish the bound $\|R(\phi)f\|_\infty \ll \|f\|_2$ for any cuspidal f.

We shall write $R(\phi)$ as an integral operator over $N(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ and estimate the kernel function by Poisson summation. One should think of this quotient space as being the adelization of the vertical strip in the upper half-plane given by $-1/2 \le \operatorname{Re} z \le 1/2$.

Indeed, for any $f \in L^2(G(F)\backslash G(\mathbb{A}), \omega)$ we have

$$R(\phi)(f)(x) = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \phi(y) f(xy)dy = \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} H(x,y)f(y)dy,$$

where the kernel H is given by

$$H(x,y) = \sum_{\xi\in N(F)} \phi(x^{-1}\xi y).$$

We shall be interested in the behavior of this kernel function for x, y "high in the cusp". For this reason we shall introduce their Iwasawa decompositions $x = u_x a_x k_x$ and $y = u_y a_y k_y$ in the Iwasawa decomposition

$$Z(\mathbb{Z})\backslash G(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K_0.$$

Here $A(\mathbb{A}) = Z(\mathbb{A})\backslash T(\mathbb{A})$. We shall write an arbitrary element $a \in A(\mathbb{A})$ as $a = \mathrm{diag}(t, 1)$, where $t \in \mathbb{A}^\times$.

Note that x lies in $G(F)\backslash G(\mathbb{A})$, but we may take a representative for x lying in a Siegel set

$$\mathfrak{S} = \Omega_N A(\mathbb{A})_c K_0.$$

Here Ω_N denotes a fundamental domain in $N(\mathbb{A})$ for $N(F)$, and $A(\mathbb{A})_c$ consists of diagonal elements $\mathrm{diag}(a, 1)$ with $|a| \geq c$, since for c sufficiently small the map $\mathfrak{S} \to G(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ is surjective.

Now if x, y are such that $H(x, y) \neq 0$, then

$$x^{-1}N(F)y = k_x^{-1}a_x^{-1}u_x^{-1}N(F)u_y a_y k_y \subset k_x^{-1}a_x^{-1}N(\mathbb{A})a_y k_y = k_x^{-1}N(\mathbb{A})a_x^{-1}a_y k_y$$

meets the support of ϕ. Since the support of ϕ is a compact, and since k_x, k_y are themselves constrained to the compact subgroup K_0, we deduce that $a_x^{-1}a_y$ must lie in a compact subset of $A(\mathbb{A})$. From this it follows that y must also lie in a Siegel domain $\mathfrak{S}'$.

We now apply Poisson summation over $N(F)$ to the function $n \mapsto \phi(x^{-1}ny)$. This yields

$$H(x, y) = \sum_{\xi \in N(F)} \int_{N(\mathbb{A})} \phi(x^{-1}ny)\psi_\xi(n)dn.$$

We introduce the following modification of H, in which we subtract its constant term:

$$H'(x, y) = H(x, y) - \int_{N(\mathbb{A})} \phi(x^{-1}ny)dn.$$

Using the Iwasawa decompositions of x and y this becomes

$$\begin{aligned} H'(x, y) &= \sum_{\xi \neq e} \int_{N(\mathbb{A})} \phi(k_x^{-1}a_x^{-1}u_x^{-1}nu_y a_y k_y)\psi_\xi(n)dn \\ &= |t_x| \sum_{\xi \neq e} \psi_\xi(u_x u_y^{-1}) \int_{N(\mathbb{A})} \phi(k_x^{-1}na_x^{-1}a_y k_y)\psi_\xi(a_x n a_x^{-1})dn, \end{aligned}$$

where we used our notational convention on the elements of $A(\mathbb{A})$. Thus

$$|H'(x, y)| \leq |t_x| \sum_{\xi \neq e} \Big| \int_{N(\mathbb{A})} F_{x,y}(n)\psi_\xi(a_x n a_x^{-1})dn \Big|$$

where $F_{x,y}(n) = \phi(k_x^{-1}na_x^{-1}a_y k_y)$.

As before, we note that the elements k_x, k_y, and $a_x^{-1}a_y$ are constrained to fixed compact subsets. Thus, the functions $F_{x,y}$ are themselves in a fixed compact subset of $\mathcal{S}(\mathbb{A})$, where we identify $N(\mathbb{A})$ with $\mathbb{A}$. The same is therefore true of the collection of Fourier transforms of $F_{x,y}$ by the fixed non-trivial additive character ψ_1. The latter then are of rapid decay, *uniformly* in x, y. Using $a_x \left(\begin{smallmatrix} 1 & b \\ & 1 \end{smallmatrix}\right) a_x^{-1} = \left(\begin{smallmatrix} 1 & t_x b \\ & 1 \end{smallmatrix}\right)$, we deduce that there is a Schwartz–Bruhat function $\alpha \in \mathcal{S}(\mathbb{A})$ such that

$$H'(x,y) \ll |t_x| \sum_{\xi \in F^\times} \alpha(t_x \xi).$$

Since t_x is bounded away from zero, the sum on ξ is at most $O_N(|t_x|^{-N})$.

Now, taking f cuspidal, the integral

$$\int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} f(y) \int_{N(\mathbb{A})} \phi(x^{-1}ny)dndy = \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} \phi(x^{-1}y) \int_{N(\mathbb{A})} f(ny)dndy$$

vanishes for every x. We may therefore write

$$R(\phi)|_{\text{cusp}}(f)(x) = \int_{N(F)Z(\mathbb{A})\backslash G(\mathbb{A})} H'(x,y)f(y)dy.$$

Inserting the estimate on $H'(x,y)$, we obtain

$$R(\phi)|_{\text{cusp}}(f)(x) \ll_N |t_x|^{-N} \int_{\mathfrak{S}'} |f(y)|dy.$$

Since the projection $\mathfrak{S}' \to G(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ has finite fibers,

$$\int_{\mathfrak{S}'} |f(y)|dy \ll \int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} |f(y)|dy.$$

Finally, by the Cauchy–Schwarz inequality, we obtain

$$\int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} |f(y)|dy \leq m(G(F)Z(\mathbb{A})\backslash G(\mathbb{A}))^{1/2} \left(\int_{\mathfrak{S}'} |f(y)|^2 dy \right)^{1/2},$$

which is $O(\|f\|_2)$, as desired.

Step 2: Show that $R(\phi)|_{\text{cusp}}$ is of Hilbert–Schmidt class.

From Step 1, we deduce for every $x \in G(\mathbb{A})$ the map $f \mapsto R(\phi)(f)(x)$ is a continous linear form on $L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}), \omega)$. Thus there is some (inexplicit!) kernel $k_x \in L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}), \omega^{-1})$ (depending also on ϕ, of course) such that

$$R(\phi)(f)(x) = \int_{G(F)Z(\mathbb{A})\backslash G(\mathbb{A})} f(y)k_x(y)dy, \quad f \in L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}), \omega).$$

The L^2-norm of k_x can be estimated by Step 1. Indeed,

$$\|k_x\|_2^2 = R(\phi)(k_x) \ll \|k_x\|_2,$$

so that $\|k_x\|_2 \ll 1$. Now write $k(x,y) = k_x(y)$, so that $R(\phi)|_{\text{cusp}}$ has kernel k. From the above, and the finite volume of the quotient $G(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ we estimate

$$\int |k(x,y)|^2 dxdy = \int \int |k_x(y)|^2 dxdy \ll 1,$$

as desired.

Step 3: Upgrade to trace class.

Now write ϕ as a finite linear combination of convolutions, using the Dixmier–Malliavin theorem. In this way $R(\phi)|_{\text{cusp}}$ becomes a finite linear combination of products of Hilbert–Schmidt operators. $\square$

As with Proposition 2.1, we conclude from the above (only the compactness of the operator is needed) the following discrete decomposition of the cuspidal subpace.

Corollary 3. *The right-regular representation of $G(\mathbb{A})$ on the cuspidal subspace $L^2_{\text{cusp}}(G(F)\backslash G(\mathbb{A}), \omega)$ decomposes as a discrete Hilbert space direct sum of irreducible subrepresentations with finite multiplicities.*

In fact, the multiplicities appearing in the above decomposition are known, after the work of Jacquet–Langlands, to be either 0 or 1. This famous result is referred to as *Multiplicity One*. Although we shall not prove it here (it relies on Whittaker models, which we have not really discussed), we shall need it in the proof of the Jacquet–Langlands correspondence. In fact, as one of the first steps in the proof of the Jacquet–Langlands correspondence, one deduces Multiplicity One for G' from that of G.

4. The "simple" trace formula for GL_2

We now turn to the trace formula for $G = \mathrm{GL}_2$. This is significantly more complicated than for G', due to the fact that the quotient $G(F)Z(\mathbb{A})\backslash G(\mathbb{A})$ is no longer compact. This manifests itself in several different ways:

(1) on the spectral side, the space $L^2(G(F)\backslash G(\mathbb{A}), \omega)$ no longer decomposes discretely. Instead one encounters continuous spectrum coming from Eisenstein series;

(2) on the geometric side, centralizer volumes are not always finite. Indeed if $\gamma = \mathrm{diag}(a, b)$, the centralizer is $T(\mathbb{A})$ and the quotient $Z(\mathbb{A})T(F)\backslash T(\mathbb{A})$ is isomorphic to $F^\times\backslash\mathbb{A}^\times$, which has infinite volume;

(3) finally, since $R(\phi)$ is no longer of trace class, nor even compact, one cannot speak of the "trace of $R(\phi)$".

We shall circumvent these difficulties by using what is called a "simple trace formula" which resembles the trace formula for compact quotient, but only when the test function is subject to certain local conditions. See Gelbart–Jacquet, Section 7.F. We recall the local invariant orbital integral $O_\gamma(f_v)$ as defined in the last lecture, and put $O_\gamma(f) = \prod_v O_\gamma(f_v)$.

Theorem 4.1 (Simple Trace Formula for GL_2)**.** *Let $f \in C_c^\infty(Z(\mathbb{A})\backslash G(\mathbb{A}), \omega^{-1})$ be factorizable $f = \prod_v f_v$, and such that the local hyperbolic orbital integral*

$O_\gamma(f_w) = 0$ vanishes at two distinct places $w_1 \neq w_2$. Then

$$\begin{aligned}\operatorname{tr} R_{\mathrm{cusp}}(f) + \operatorname{tr} R_{\mathrm{sp}}(f) = & m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))f(1) \\ & + \sum_{[\gamma]\ \textit{elliptic}} \operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbb{A}))O_\gamma(f),\end{aligned}$$

where (using Multiplicity One)

$$\operatorname{tr} R_{\mathrm{cusp}}(f) = \sum_{\pi \subset L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}),\omega)} \operatorname{tr} \pi(f),$$

and the trace of the ***special contribution*** *is given by*

$$\operatorname{tr} R_{\mathrm{sp}}(f) = m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})) \sum_{\mu^2=\omega} \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} f(g)\mu(\det g)dg.$$

We shall not give a proof of this theorem, since it is a consequence of the full trace formula, which is a subject unto itself (even in this first non-trivial case of GL_2). We only give an idea for why the vanishing condition in the theorem allows one to simplify the full formula. In the full formula, there are two types of terms on the geometric side which do not appear for compact quotients, namely those associated with *hyperbolic* and *unipotent* conjugacy classes.

A hyperbolic class $\gamma = \operatorname{diag}(a, b)$ makes the following global contribution

$$\sum_{v \in S_f} \Big(\prod_{w \neq v} O_\gamma(f_w) \Big) WO_\gamma(f_v),$$

where S_f is the set of places of F outside of which f_v is the characteristic function of $Z_v K_v$, and WO_γ is the local weighted orbital integral, as defined in the previous lecture. If there are two places $w_1 \neq w_2$ such that $O_\gamma(f_w) = 0$, then for each $v \in S_f$ at least one of w_1, w_2 is assured to lie in set $w \neq v$, so that the interior product vanishes.

The contributions from the unipotent classes vanishes by a limiting argument from the hyperbolic ones (recall that the nilpotent cone in $\mathfrak{sl}_2$ is the limit of the one-sheeted hyperbola, itself the conjugacy class of $\operatorname{diag}(x, -x)$, as $x \to 0$).

5. Sketch of proof of the global J-L correspondence

Recall the statement of the global Jacquet–Langlands correspondence from last lecture.

Theorem 5.1. *There exists a injective map*

$$\mathrm{JL} : \mathcal{A}'_0 \longrightarrow \mathcal{A}_0$$

with the following properties:

1. *the image of* JL *is the set of $\pi = \otimes'_v \pi_v \in \mathcal{A}_0$ for which $\pi_v \in \mathscr{E}^2(G_v)$ for every $v \in S$;*

(2) *for every* $v \notin S$, $\mathrm{JL}(\pi')_v \simeq \pi'_v$ *under the isomorphism* $\mathrm{GL}_2(F_v) \simeq G'_v$*;*
(3) *for every* $v \in S$*, if* $\{g\} \in X$ *and* $\{g'\} \in X'$ *are regular elliptic elements which coincide under the natural inclusion* $X' \hookrightarrow X$*, then*

$$\theta_{\mathrm{JL}(\pi')_v}(g) + \theta_{\pi'_v}(g') = 0.$$

We now briefly sketch the proof of Theorem 5.1.

The basis of the proof is a comparison of the two trace formulae in Corollary 2 and Theorem 5.1. Notice the similarity between the right-hand sides of these two trace formulae. The proof proceeds in three steps, which we now outline.

5.1. **Matching.** Establish a "matching of test functions". This is a problem in local harmonic analysis, since our test functions factor place by place.

We may assume that v is a place for which $G'(F_v)$ is non-split, for otherwise we $G'(F_v)$ and $G(F_v)$ are isomorphic, and we may transfer test functions via this isomorphism. We note that almost all places v fall into the latter case, and we require that almost everywhere the test functions f_v and f'_v be the characterstic function of $K_{0,v}$, multiplied by $\omega_v^{-1}(z)$.

Note that we have already set up a matching of conjugacy classes of $G'(F_v)$ with the elliptic conjugacy classes in $G(F_v)$. For the *regular* elliptic conjugacy classes (in either G or G'), these correspond to quadratic field extensions E_v/F_v, the splitting field of their irreducible characteristic polynomials. The matching of γ with γ' means that E_v embeds into both $M_2(F_v)$ and B'_v. The unit group $E_v^\times$ in both cases can be viewed as the centralizer in $G(F_v)$ and $G'(F_v)$, respectively.

Let us now state the precise matching result, which is due to Langlands. For every $f' \in C_c^\infty(G'(F_v), \omega^{-1})$ there exists $f \in C_c^\infty(G(F_v))$ such that

(1) for every hyperbolic γ we have $O_\gamma(f) = 0$;
(2) for every pair of corresponding regular elliptic conjugacy classes $\{\gamma\}$ and $\{\gamma'\}$, we have $O_\gamma(f) = O_{\gamma'}(f')$.

5.2. **Lining up geometric sides.** As a result of this matching of test functions, one deduces an equality of the corresponding spectral sides

$$\sum_{\pi \subset L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}),\omega)} \mathrm{tr}\,\pi(f) = \sum_{\pi' \in L^2_0(G'(F)\backslash G'(\mathbb{A}),\omega)} m(\pi')\,\mathrm{tr}\,\pi'(f'),$$

where $L^2_0(G'(F)\backslash G'(\mathbb{A}),\omega)$ denotes the orthocomplement of the space spanned by functions of the form $\chi \circ n_B$, where n_B is the reduced norm on B and $\chi^2 = \omega$.

Let us see how to obtain the above spectral equality. We must match up the two geometric sides of the two trace formulae we have quoted (the full trace formula for G' and the simple trace formula for G).

Recall that S is the set of places of F where the global quaternion algebra B is not split. Recall furthermore that S is of even cardinality, and since B is assumed to be a division algebra, it is not empty. Let v_1, v_2 be two distinct places in S. Then $G'(F_v)$ is non-split and by the first matching property we deduce that f_{v_1} and f_{v_2} have vanishing hyperbolic integrals. We may therefore quote the simple trace formula Theorem 4.1 for G.

Furthermore, when γ is globally regular elliptic for G, we have the factorization $O_\gamma(f) = \prod_v O_\gamma(f_v)$. For every place v of F let E_v be the quadratic étale extension of F_v defined by the characteristic polynomial of γ over F_v. If $v \in S$ and E_v is split, then γ is locally hyperbolic at v while $G'(F_v)$ is non split, and again by the first matching condition we have that $O_\gamma(f_v) = 0$. We may therefore assume that the elliptic terms γ appearing on the geometric side of the simple trace formula for G are such that for any $v \in S$ the regular semisimple element γ is elliptic. Its conjugacy class therefore corresponds to one for G'.

Finally, the identity contributions for both groups align as well, as long as we take for our global measures m and m' on $G(\mathbb{A})$ and $G'(\mathbb{A})$ the Tamagawa measure, for which

$$m(Z(\mathbb{A})G(F)\backslash G(\mathbb{A})) = m'(Z'(\mathbb{A})G'(F)\backslash G'(\mathbb{A})).$$

We have therefore equated the geometric sides of the two trace formulae, when we insert into these a pair of matching test functions. We obtain an equality of the spectral sides

$$\sum_{\pi \subset L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A}),\omega)} \operatorname{tr}\pi(f) + \operatorname{tr} R_{\mathrm{sp}}(f) = \sum_{\pi \subset L^2(G'(F)\backslash G'(\mathbb{A}),\omega)} m(\pi') \operatorname{tr}\pi'(f'),$$

where the characters are included. We match up the contributions of the characters by our conventions on f_v, f'_v away from S, yielding the claim.

5.3. **Linear independence of characters.** One uses "linear independence of characters" in a sophisticated way to go from this equality of spectral distributions to the stated global correspondence (including the deduction of multiplicity one for G' from that of G). We will not provide details on this last step, which is quite elaborate, but send the reader to Gelbart–Jacquet, Section 8.