1. Foreward

First we make a few comments related to the last lecture:

1.1. **General linear groups.**

Last time there was a question about why the pair $(\mathrm{GL}_n, \mathrm{GL}_{n+1})$ is excluded in the main theorem of N-V. This was a natural question, since Valentin's motivating examples were taken from (low rank) general linear groups.

I had postulated that perhaps it was due to wanting to take compact automorphic quotients (the unitary and orthogonal groups are assumed to be anisotropic), to avoid complications from continuous spectrum. This is half-true, in the sense that there is another point in the argument (besides the spectral expansion over $\mathbf{H}$) which requires compactness relative to $\mathbf{G}$, namely the application of Ratner's theorem, which Philippe will present. See the Remark in the Assumptions subsection §25.7, as well as Section 27.3 (e.g. the proof of Lemma 2).

At one point I mused aloud that you could take an inner form of GL_n to get compact quotient, and it wasn't clear to me why taking two such inner forms $\mathbf{H}$ and $\mathbf{G}$ wouldn't work. Let me explain why it does not. By Wedderburn's theorem, an inner form of GL_n is necessarily of the form $\mathrm{GL}_m(D)$, where D is a central division F-algebra of rank $d \mid n$ and $n = m^2 d$. Let me take $\mathbf{G} = \mathrm{PGL}_m(D)$ to avoid issues of the center. By a theorem of Borel and Harish-Chandra (and independently Mostow and Tamagawa), the compactness of the automorphic quotient $[\mathbf{G}] = \mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$ is equivalent to $\mathbf{G}$ being anisotropic. In this case, this is true precisely when $d > 1$ and $m = 1$: $\mathbf{G} = \mathrm{PD}^\times$. However, such $\mathbf{G}$ do not admit spherical subgroups $\mathbf{H}$: the F-rational subgroups of $\mathbf{G}$ are of the form $\mathrm{GL}_1(D_1)$, where $D_1 \subset D$ is a division F-subalgebra. If the rank of D_1 is $d_1 < d$ then, at almost all places v, this will give $\mathrm{PGL}_{d_1^2}(F_v) \simeq PD_{1,v}^\times \subset PD_v^\times \simeq \mathrm{PGL}_{d^2}(F_v)$, which is not spherical.

Note, finally, that the general linear GGP pair is a special case of the unitary GGP pair, when the underlying quadratic extension is split. (Unitary groups are *outer forms* of GL_n.) So there are some sections in the N-V paper (invariant theory, for example) in which the pair of unitary groups is allowed to be GL_n inside GL_{n+1}.

1.2. **Local vs global.**

There was some confusion about a step in the proof sketch, which provides for a map

$$\mathcal{S}(\mathfrak{g}^\wedge) \to C^\infty([G]). \tag{1}$$

Here $[G]$ is the S-adic automorphic quotient (the brackets were missing last time). Edgar will present this construction, which is Section 26 in the paper. I would like to clarify this further.

Firstly, at the level of notation, Π will always denote the adelic cuspidal automorphic representation of $\mathbf{G}(\mathbb{A})$. The adelic automorphic quotient $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A})$ is often denoted $[\mathbf{G}]$, but the paper does not work "fully adelically". From the level structure of Π one may work S-adically, where S consists of all archimedean places and a finite number of other places (that one must invert, so that $\mathbf{G}$ has an integral model over $\mathbb{Z}[1/S]$; at all places outside of S, the cuspidal automorphic representation will be unramified). Note that the set S is denoted by R in the paper. The S-adic automorphic quotient is then

$$[G] = \Gamma\backslash(G \times G'),$$

where G is the group of points of $\mathbf{G}$ at the fixed archimedean place v_0 of F (they write $\mathfrak{p}$ for v_0), and G' is the group of F_{S-v_0} points of $\mathbf{G}$. This is rather non-standard notation (as they admit).

From the S-adic viewpoint, the representation Π becomes an irreducible unitary representation $\pi \otimes \pi'$ on $G \times G'$ which admits an isometric intertwining

$$\pi \otimes \pi' \hookrightarrow L^2([G]). \tag{2}$$

We may think of π and π' as "abstract representations" of G and G', respectively, and the embedding (2) as being an additional piece of information, which makes $\pi \otimes \pi'$ "Γ-automorphic". We obtain from (2) an embedding

$$\pi \otimes \overline{\pi} \otimes \pi' \otimes \overline{\pi}' = (\pi \otimes \pi') \otimes (\overline{\pi} \otimes \overline{\pi}') \hookrightarrow L^2([G] \times [G]).$$

In particular, for every fixed $T' \in \pi' \otimes \overline{\pi}'$, we have

$$\pi \otimes \overline{\pi} \hookrightarrow L^2([G] \times [G]). \tag{3}$$

So now we have two different ways to think of the tensor product $\pi \otimes \overline{\pi}$. The first is "abstractly", as $\pi \otimes \overline{\pi} = \mathrm{End}(\pi)$. The second is through the Γ-automorphic map (3). This map (1) uses both the abstract and automorphic pictures:

$$\mathcal{S}(\mathfrak{g}^\wedge) = S^{-\infty} \xrightarrow{\mathrm{Op}} \Psi^\infty \subset \mathrm{End}(\pi) = \pi \otimes \overline{\pi} \to C^\infty([G] \times [G]) \xrightarrow{\mathrm{res}} C^\infty([G]),$$

where Ψ^∞ is the space of "smoothing operators", to be described in my lecture. The fact that we land in $C^\infty([G] \times [G])$ and not $L^2([G] \times [G])$ allows us to take the restriction. This map is denoted $a \mapsto [a]$ in N-V.

1.3. **Some preview of future talks.** The result of this procedure $a \mapsto [a]$ clearly tells us something about the limiting behavior of L^2-norms of restrictions of microlocalized vectors whose microlocal support is described by a, since if $\mathrm{Op}(a) \approx \sum a(\xi) v_\xi \otimes \overline{v_\xi}$, then you get something like

$$[a](x) \approx \sum a(\xi) |v_\xi|^2(x).$$

In practice (it seems), a will be supported about some (regular) $\xi \in \mathfrak{g}^\wedge$ in a "limiting orbit" (or multi-orbit) $\mathcal{O}$ contained in the nilpotent cone, and the above procedure will in fact be upgraded to give you a probability measure on $[G]$:

$$\mathcal{O} \to \{\text{probability measures on } [G]\}, \qquad \xi \mapsto [\delta_\xi].$$

(1) In lecture 5, Philippe will
(2) In lectures 9-10 on "inverse branching", Subhajit will tell us how to pick a so that its micro-local support picks out the necessary information about H.

1.4. **Quantitative and explicit aspects.** I find it hard to make quantitative and explicit the test function construction of N-V. Hopefully this will become clearer by the end of the series. On the other hand, the follow-up paper by Nelson seems to be much more comprehensible in this regard. There, the two jobs that the test function must accomplish (spectral and geometric) are different, since the spectral expansion is on the big group $\mathbf{G}$ not on $\mathbf{H}$. Although we stick with N-V, it might be good to occasionally consult the introduction to the follow-up paper by Nelson, as a point of comparison.

1.5. **My talks.** Now let me give a preview of my two lectures: I will discuss the operator assignment Op which looks at π only from the "abstract" viewpoint, as an irreducible unitary representation of the real Lie group G. In the first of my two lectures, I will emphasize the *operator norms* of this assignment, in the second I will emphasize the *trace norms.* (The latter is related to the Kirillov formula, characters and infinitesimal characters, and co-adjoint orbits and their limits.)

2. Recollections

We motivate the more general picture by recalling the classical one, and its reinterpretation via the Heisenberg group.

2.1. **Classical operator calculus.** Let us first recall the story for $\mathbb{R}^2$. Let f be a function on the real line. Let $f^\wedge$ be its Fourier transform. Let I, J be intervals in $\mathbb{R}$. We have two localization operators:

(1) $f \mapsto \chi_I f$ (space localization);
(2) $f \mapsto (\chi_J f^\wedge)^\vee = \chi_J^\vee * f$ (frequency localization).

We put these together to form

$$\chi_I(x)(\chi_J f^\wedge)^\vee(x) = \chi_I(x)\int_{\mathbb{R}} \chi_J(\xi) f^\wedge(\xi) e^{-ix\xi} d\xi = \int_{\mathbb{R}} \alpha_{I,J}(x,\xi) f^\wedge(\xi) e^{-ix\xi} d\xi,$$

where we have put $\alpha_{I,J}(x,\xi) = \chi_I(x)\chi_j(\xi)$. Replacing $\alpha_{I,J}$ with any Schwartz function $\alpha \in \mathcal{S}(\mathbb{R}^2)$ we obtain the standard *pseudo-differential operator*

$$\mathrm{Op}(\alpha) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \mathrm{Op}(\alpha)(f)(x) = \int_{\mathbb{R}} \alpha(x,\xi) f^\wedge(\xi) e^{-ix\xi} d\xi.$$

Remark 1. To understand the term *pseudo-differential operator*, note that if $\alpha(x,\xi) = c(x)p(\xi)$, where $p(\xi) = \sum_{n=0}^m a_n \xi^n$ is a polynomial, then

$$\mathrm{Op}(\alpha)(f)(x) = c(x)\int_{\mathbb{R}} p(\xi) f^\wedge(\xi) e^{-ix\xi} d\xi = c(x)\left(p(\frac{d}{dx}) f\right)(x) = c(x)\sum_{n=0}^m a_n \frac{d^n}{dx^n} f(x).$$

This illustrates the importance of allowing for more general symbols than Schwartz symbols. The standard class of symbols here, as recalled in the Stanford lecture notes, is the space of order $\leq m$ symbols, denoted by S^m. This consists of $\alpha(x,\xi) : \mathbb{R}^2 \to \mathbb{C}$ such that, for all $i, j \geq 0$:

$$\sup_{x,\xi} \frac{|\partial_x^i \partial_\xi^j \alpha(x,\xi)|}{(1+|\xi|)^{m-j}} \leq C_{i,j}. \tag{4}$$

Note the lack of symmetry in the variables: the ξ variable is the differentiating variable. In our polynomial example, we would need the function $c(x)$ to have uniformly bounded derivatives, in which case $\mathrm{Op}(\alpha)$ is an "almost constant coefficient" differential operator.

2.2. **The Heisenberg group.** We saw last term that $\mathrm{Op}(\alpha)$ can be interpreted by means of the Heisenberg group $G = \mathrm{Heis}$ acting unitarily on $L^2(\mathbb{R})$ by means of the oscillatory representation:

$$(A(s,t,z)f)(x) = \begin{pmatrix} 1 & t & z \\ & 1 & s \\ & & 1 \end{pmatrix} f(x) = f(x+t)e^{ixs+iz}.$$

We shall write this representation as $\pi : G \to U(L^2(\mathbb{R}))$, and often write π in place of the space $L^2(\mathbb{R})$. We computed (in Example 4.5.6) the smooth vectors π^∞ to be $\mathcal{S}(\mathbb{R}^2)$.

Remark 2. We remember this formula in the following way: the z variable is the center, so it should act by a character, say e^{iz}. Then, for f to be an approximate eigenfunction of the element $A(s,t,0)$, with eigenvalue $a(t,s)$, means that $\hat{f}$ should be approximately localized at t and f approximately localized at s.

2.3. **The action of the group algebra and linear pullback.** We have an action of the group algebra $L^1(G)$ on π given by

$$L^1(G) \to \mathrm{End}(\pi), \quad A \mapsto \pi(A),$$

where $\pi(A)f = \int_G A(g)\pi(g)f dg$. Thus,

$$\pi(A)f(x) = \int_{\mathbb{R}^3} A(t,s,z)f(x+t)e^{ixs+z}dtdsdz.$$

The operator $\pi(A)$ is in fact a bounded operator, as

$$\|\pi(A)f\| \leq \int_G |A(g)| \|\pi(g)f\| dg = \|f\| \int_G |A(g)| dg = \|A\|_{L^1} \|f\|.$$

Now, the Lie algebra

$$\mathfrak{g} = \mathrm{Lie}(G) = \left\{ \begin{pmatrix} 0 & t & z \\ & 0 & s \\ & & 0 \end{pmatrix} : s,t,z \in \mathbb{R} \right\}$$

maps diffeomorphically to the group G through the exponential map

$$\exp : \begin{pmatrix} 0 & t & z \\ & 0 & s \\ & & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & t & z + \frac{ts}{2} \\ & 1 & s \\ & & 0 \end{pmatrix},$$

preserving Haar measures. Let $\mathfrak{g}^\wedge$ denote the Pontryagin dual of $\mathfrak{g}$. If $A \in C_c^\infty(G)$, let $a \in \mathcal{S}(\mathfrak{g}^\wedge)$ have inverse Fourier transform $a^\vee(X) = A(\exp X)$. Then we may write

$$\pi(A) = \int_{X \in \mathfrak{g}} a^\vee(X)\pi(\exp X)dX.$$

2.4. **The reinterpretation.** For $A \in C_c^\infty(\mathrm{Heis}_{\mathbb{R}})$ we let

$$B(t,s) = \int_{\mathbb{R}} A(t,s,z)e^{iz}dz \in C_c^\infty(\mathbb{R}^2).$$

Write $B^\wedge(x,\xi)$ for the Fourier transform of B. Then we saw last term that

$$(\pi(A)f)(x) = \mathrm{Op}(B^\wedge)(x) = \int_{\mathbb{R}} B^\wedge(x,\xi)f^\wedge(\xi)e^{-ix\xi}d\xi.$$

Now let $\alpha \in \mathcal{S}(\mathbb{R}^2)$. Set $A(t,s,z) = \alpha^\vee(t,s)\delta(z)$, with δ a small bump about 0. Then

$$B(t,s) = \alpha^\vee(t,s)\int_{\mathbb{R}} e^{iz}\delta(z)dz \approx a^\vee(t,s),$$

so that $\alpha(x,\xi) \approx B^\wedge(x,\xi)$. Thus

$$\mathrm{Op}(\alpha) \approx \pi(A) = \int_{X\in\mathfrak{g}} a^\vee(X)\pi(\exp X)dX.$$

Question 1. *We see from the above that* $\mathrm{Op}(\alpha)$ *is a bounded operator on* π *when* α *is Schwartz. What kind of operator is* $\mathrm{Op}(\alpha)$ *for* $\alpha \in S^m$*? How does one define a class of order* $\leq m$ *symbols* S^m *on* $\mathfrak{g}^\wedge$*, which would generalize the classical ones? Is there a characterization in* $\mathrm{End}(\pi)$ *for the class of order* $\leq m$ *pseudo-differential operators?*

3. Basic definitions for N-V

We now fix some basic definitions, to be used throughout N-V.

Definition 3.1. *Here are some definitions to get started:*

(1) G *is a real unimodular Lie group, with Lie algebra* $\mathfrak{g}$*, dimension* d

(2) duals

$$\mathfrak{g}^\wedge = \mathrm{Hom}(\mathfrak{g}, \mathbb{C}^{(1)}), \quad i\mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}^\wedge,\ \xi \mapsto [x \mapsto e^{x\xi}];$$

(3) measures and jacobians:
- dx *(resp.* $d\xi$*,* dg*) Haar measure on* $\mathfrak{g}$ *(resp.* $\mathfrak{g}^\wedge$*,* G*)*
- $\mathcal{G}$ *an open subset of* $\mathfrak{g}$ *containing the origin such that* $\exp : \mathcal{G} \to G$ *is injective*
- $j : \mathcal{G} \to \mathbb{R}_{>0}$ *such that* $dg = j(x)dx$.

(4) Fourier transforms

$$\wedge : \mathcal{S}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g}^\wedge) : \phi^\wedge(\xi) = \int_{\mathfrak{g}} \phi(x)e^{x\xi}dx, \quad \vee : \mathcal{S}(\mathfrak{g}^\wedge) \to \mathcal{S}(\mathfrak{g}) : a^\vee(x) = \int_{\mathfrak{g}^\wedge} a(\xi)e^{-x\xi}d\xi;$$

(5) cutoffs

$$\mathcal{X}(\mathcal{G}) = \{\chi \in C_c^\infty(G) : \chi(x) = \chi(-x), \chi(\mathcal{G}) \in [0,1], \chi_U = 1\}$$

(6) π *an unitary representation of* G*, not necessarily irreducible;*

(7) operator assignment: $a \in \mathcal{S}(\mathfrak{g}^\wedge), \chi \in \mathcal{X}(\mathcal{G})$*:*

$$\mathrm{Op}(a) = \mathrm{Op}(a,\chi) = \mathrm{Op}(a,\chi:\pi) = \int_{\mathfrak{g}} \chi(x)a^\vee(x)\pi(\exp(x))dx.$$

Then $\mathrm{Op}(a) \in \mathrm{End}(\pi)$ *is a* bounded operator.

(8) semi-classical parameter $\mathsf{h} \in (0,1]$ *and rescaling:*
- $a_{\mathsf{h}}(\xi) = a(\mathsf{h}\xi)$*;*
- $\mathrm{Op}_{\mathsf{h}}(a) = \mathrm{Op}(a_{\mathsf{h}})$*, which gives*

$$\mathrm{Op}_{\mathsf{h}}(a) = \mathsf{h}^{-d}\int_{\mathfrak{g}} \chi(x)a^\vee(x/\mathsf{h})\pi(\exp x)dx = \int_{\mathfrak{g}} \chi(\mathsf{h}x)a^\vee(x)\pi(\exp \mathsf{h}x)dx.$$

We try to understand the last definition intuitively, in terms of microlocalized vectors. The slogan is

"$\mathrm{Op}_h(a)$ acts by the scalar $a(\xi)$ on vectors microlocalized at ξ."

Indeed, if $v \in \pi$ is such that

$$\pi(\exp \mathsf{h}x)v \approx e^{x\xi} v \text{ for all } |x| \ll 1,$$

then

$$\begin{aligned}
\mathrm{Op}_{\mathsf{h}}(a)v &\approx \int_{|x|=O(1)} a^{\vee}(x)\pi(\exp \mathsf{h}x)v dx \qquad (\chi(\mathsf{h}x) \approx 1_{|x|\ll 1}) \\
&\approx \int_{|x|=O(1)} a^{\vee}(x)e^{x\xi} v dx \qquad (v \text{ is microlocalized at } \xi) \\
&\approx \left(\int_{\mathfrak{g}} a^{\vee}(x)e^{x\xi} dx\right) v \qquad (a^{\vee}(x) \text{ is small on } |x| \gg 1) \\
&= a(\xi)v
\end{aligned}$$

4. The operator calculus of N-V for Schwartz symbols

We record some basic properties of Op_{h}:

(1) the adjoint of $\mathrm{Op}(a,\chi)$ is $\mathrm{Op}(\bar{a},\chi)$. Thus, $\mathrm{Op}(a)$ is self-adjoint precisely when the symbol a is real-valued;

(2) if $g.\mathrm{supp} \subset \mathcal{G}$ then $g.\chi \in \mathcal{X}(\mathcal{G})$ and

$$\begin{aligned}
\pi(g)\mathrm{Op}(a,\chi)\pi(g)^{-1} &= \int_{\mathfrak{g}} \chi(x)a^{\vee}(x)\pi(g\exp(x)g^{-1})dx \\
&= \int_{\mathfrak{g}} \chi(x)a^{\vee}(x)\pi(\exp(g.x))dx \\
&= \int_{\mathfrak{g}} \chi(g.^{-1}x)a^{\vee}(g^{-1}.x)\pi(\exp(x))dx = \mathrm{Op}(g.a, g.\chi);
\end{aligned}$$

(3) Op composition intertwines a "star product" $\star : \mathcal{S}(\mathfrak{g}^{\wedge}) \times \mathcal{S}(\mathfrak{g}^{\wedge}) \to \mathcal{S}(\mathfrak{g}^{\wedge})$:

$$\mathrm{Op}(a_1,\chi) \star \mathrm{Op}(a_2,\chi) = \mathrm{Op}(a_1 \star a_2, \chi').$$

In the next section, we explain what we mean by this last property.

For the moment, we simply recall from last term that the classical star product on $\mathcal{S}(\mathbb{R}^2)$ was calculated as

$$(a \star b)(x,\xi) = \int_{s,t\in\mathbb{R}} a(x,\nu)e^{i(y-x)(\nu-\xi)}b(y,\xi)dyd\nu.$$

Moreover, a Taylor series argument showed that

$$(a \star b)(x,\xi) = \sum_{j\geq 0}(a \star_j b)(x,\xi), \qquad (a \star_j b)(x,\xi) = \frac{i^j}{j!}\partial_\xi^j a(x,\xi)\partial_x^j b(x,\xi). \tag{5}$$

In particular,

$$(a \star_0 b)(x,\xi) = a(x,\xi)b(x,\xi), \qquad (a \star_1 b)(x,\xi) = i\partial_\xi a(x,\xi)\partial_x b(x,\xi)$$

The defect of commutativity is then

$$a \star b - b \star a = i\{a,b\} + (\text{higher order terms}).$$

The asymptotic series (5) allowed us to show that

$$a \approx \chi_{[0,L]}\chi_{[0,M]} \qquad \text{then} \qquad \mathrm{Op}(a)^2 \approx \mathrm{Op}(a),$$

as long as the area of the rectangle satisfies $LM \gg 1$.

5. The star product on Schwartz symbols

The star product on $\mathcal{S}(\mathfrak{g}^\wedge)$ is an infinitesimal version of the convolution product on the group, which itself intertwines composition for operators $\pi(f) = \int_G \pi(g)f(g)dg$ representing the group algebra: for $f_1, f_2 \in C_c^\infty(G)$ we have $\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$.

In what follows, we shall write f for functions in $C_c^\infty(G)$, ϕ for functions in $C_c^\infty(\mathfrak{g})$, and a for Schwartz symbols $\mathcal{S}(\mathfrak{g}^\wedge)$.

We first define a binary operation $*$ on elements of $\mathfrak{g}$ close to the origin. Let $\mathcal{G}$ and $\mathcal{G}'$ be open neigborhoods of the origin in $\mathfrak{g}$ verifying $\exp(\mathcal{G})\exp(\mathcal{G}) \subset \exp(\mathcal{G}')$. We may define

$$* : \mathcal{G} \times \mathcal{G} \to \mathcal{G}', \qquad x * y = \log(\exp(x)\exp(y)). \tag{6}$$

In other words, we require that $*$ make the following diagram commutative:

$$\begin{tikzcd}
\exp(\mathcal{G})^2 \arrow[r, "\text{mult}"] & \exp(\mathcal{G}') \arrow[d, "\log"] \\
\mathcal{G}^2 \arrow[u, "\exp"] \arrow[r, "*"] & \mathcal{G}'
\end{tikzcd}$$

Remark 3. As a binary operation, $*$ is not commutative. The Baker–Campbell–Hausdorff formula describes the difference

$$\{x, y\} := x * y - x - y. \tag{7}$$

Indeed, it describes the solution z to the equation $\exp(x)\exp(y) = \exp(z)$ as an asymptotic series

$$z = x + y + \frac{1}{12}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) - \frac{1}{24}[y, [x, [x, y]]] + \cdots,$$

which converges for x, y sufficiently small. This would reduce to first term $x + y$, if the Lie algebra were commutative.

Remark 4. One should keep in mind that microlocal analysis involves approximate diagonalization into approximate eigenfunctions, for group elements of small norm, say in an $O(\mathsf{h})$-neighborhood of the origin. In this case, the higher quadratic term $\frac{1}{12}[x, y]$ is of order $O(\mathsf{h}^2)$. In such a neighborhood, the product $x * y$ commutes up to an $O(\mathsf{h})$ error.

We use the product (6) to define a "star composition" of test functions on $\mathfrak{g}$, which are supported close to the identity:

$$\star : C_c^\infty(\mathcal{G}) \times C_c^\infty(\mathcal{G}) \to C_c^\infty(\mathcal{G}'). \tag{8}$$

We make this definition via the Fourier transform, which allows us to transfer the problem to group elements:

$$(\phi_1 \star \phi_2)^\wedge(\xi) := \int_{x,y\in\mathfrak{g}} \phi_1(x)\phi_2(y)e^{(x*y)\xi}dxdy. \tag{9}$$

Remark 5. We note that the assumption $\exp(\mathcal{G})\exp(\mathcal{G}) \subset \exp(\mathcal{G}')$ implies that the map $C_c^\infty(\exp(\mathcal{G})) \times C_c^\infty(\exp(\mathcal{G})) \to C_c^\infty(\exp(\mathcal{G}'))$ given by convolution on functions on the group is well-defined. We denote by $\phi \mapsto f$ the topological isomorphism $C_c^\infty(\mathcal{G}) \to C_c^\infty(\exp \mathcal{G})$ sending ϕ to $f(\exp x) = \phi(x)j(x)^{-1}$. Then the $\star$ product in (8) makes the following diagram commute:

$$\begin{tikzcd}
C_c^\infty(\exp(\mathcal{G}))^2 \arrow[r, "*"] & C_c^\infty(\exp(\mathcal{G}')) \arrow[d, "f\mapsto\phi"', "\simeq"] \\
C_c^\infty(\mathcal{G})^2 \arrow[u, "\simeq", "\phi\mapsto f"'] \arrow[r, "\star"] & C_c^\infty(\mathcal{G}')
\end{tikzcd}$$

Indeed, if $\phi_1, \phi_2 \in C_c(\mathcal{G})$ and $z \in \mathcal{G}'$ we have

$$\begin{aligned}
(f_1 * f_2)(\exp z) &= \int_{\exp \mathcal{G}} f_1(g) f_2(g^{-1}\exp(z))dg \\
&= \int_{\mathcal{G}} f_1(\exp(x)) f_2(\exp(x)^{-1}\exp(z)) j(x)dx \\
&= \int_{\mathcal{G}} \phi_1(x)\phi_2(x' * z) j(x' * z)^{-1} dx,
\end{aligned}$$

where $x' = \log(\exp(x)^{-1})$. Pushing this forward gives

$$(\phi_1 \star \phi_2)(z) = j(z) \int_{\mathcal{G}} \phi_1(x)\phi_2(x' * z) j(x' * z)^{-1} dx.$$

We now take the Fourier transform at $\xi \in \mathfrak{g}^\wedge$ to get

$$(\phi_1 \star \phi_2)^\wedge(\xi) = \int_{z\in\mathfrak{g}} (\phi_1 \star \phi_2)(z) e^{z\xi} dz = \int_{x,z\in\mathfrak{g}} \phi_1(x)\phi_2(x' * z) \left[j(z) j(x' * z)^{-1}\right] e^{z\xi} dx dz.$$

Changing variables

$$y = x' * z = \log(\exp(x')\exp(z)) = \log(\exp(x)^{-1}\exp(z)),$$

so that $z = x * y$, and $j(z)j(x' * z)^{-1}dz = j(x * y)j(y)^{-1}d(x * y) = dy$ recovers (9).

For $a \in \mathcal{S}(\mathfrak{g}^\wedge)$ and $\chi \in \mathcal{X}(\mathcal{G})$ we have $\chi a^\vee \in C_c^\infty(\mathcal{G})$. We may therefore use the star product on the latter space of test functions to define a star product on symbols, as follows.

Definition 5.1. *For $a_1, a_2 \in \mathcal{S}(\mathfrak{g}^\wedge)$ and $\chi \in \mathcal{X}(\mathcal{G})$ we put*

$$a_1 \star a_2 = (\chi a_1^\vee \star \chi a_2^\vee)^\wedge.$$

Note that this definition interacts well with the definition (9), which was given on the "Fourier side". The definition is set up so that we have the following commutative diagram

$$\begin{tikzcd}
C_c^\infty(\exp(\mathcal{G}))^2 \arrow[r, "*"] & C_c^\infty(\exp(\mathcal{G}')) \arrow[d, "f\mapsto\phi"', "\simeq"] \\
C_c^\infty(\mathcal{G})^2 \arrow[u, "\simeq", "\phi\mapsto f"'] \arrow[r, "\star"] & C_c^\infty(\mathcal{G}') \arrow[d, "\phi\mapsto\phi^\wedge"'] \\
\mathcal{S}(\mathfrak{g}^\wedge)^2 \arrow[u, "a\mapsto a^\vee\chi"'] \arrow[r, "\star"] & \mathcal{S}(\mathfrak{g}^\wedge)
\end{tikzcd}$$

Using (9), as well as the Baker–Campbell–Hausdorff remainder notation $\{x, y\}$, we may write the star product on Schwartz symbols as

$$(a_1 \star a_2)(\xi) = \int_{x,y\in\mathfrak{g}} a_1^\vee(x) a_2^\vee(y) e^{x\xi} e^{y\xi} e^{\{x,y\}\xi} \chi(x)\chi(y) dx dy. \tag{10}$$

We now prove the intertwining property stated in the last section.

Lemma 5.2. *Let $\chi \in \mathcal{X}(\mathcal{G})$ and $\chi' \in \mathcal{X}(\mathcal{G}')$ be such that $\chi'(x * y) = 1$ for all $x, y \in \operatorname{supp}(\chi)$. Let $a_1, a_2 \in \mathcal{S}(\mathfrak{g}^\wedge)$. Then*

$$\operatorname{Op}(a_1, \chi)\operatorname{Op}(a_2, \chi) = \operatorname{Op}(a_1 \star a_2, \chi').$$

Proof. Let $\phi_i = \chi a_i^\vee \in C_c(\mathcal{G})$ and write $f_i \in C_c^\infty(\exp \mathcal{G})$ be defined by

$$f_i(\exp x) = \phi_i(x) j(x)^{-1} = \chi(x) a_i^\vee(x) j(x)^{-1}.$$

We have

$$\begin{aligned} \pi(f_i) = \int_G f_i(g)\pi(g) dg &= \int_{\mathcal{G}} f_i(\exp x)\pi(\exp x) j(x) dx \\ &= \int_{\mathcal{G}} \chi(x) a_i^\vee(x)\pi(\exp x) dx = \operatorname{Op}(a_i, \chi). \end{aligned}$$

At the same time, using the commutative diagram defining the star product,

$$\begin{aligned} \pi(f_1 * f_2) = \int_G (f_1 * f_2)(g)\pi(g) dg &= \int_{\mathcal{G}} (f_1 * f_2)(\exp x) j(x)\pi(\exp x) dx \\ &= \int_{\mathcal{G}} (a_1 \star a_2)(x)\pi(\exp x) dx \\ &= \operatorname{Op}(a_1 \star a_2, \chi'), \end{aligned}$$

where in the last equality, we used the hypothesis that $\chi' \equiv 1$ on the support of $\chi a_1^\vee \star \chi a_2^\vee$. The lemma then follows from $\pi(f_1)\pi(f_2) = \pi(f_1 * f_2)$. $\square$

We also have the rescaled star product $a \star_{\mathsf{h}} b$ given by requiring that

$$(a \star_{\mathsf{h}} b)_{\mathsf{h}} = a_{\mathsf{h}} \star b_{\mathsf{h}}. \tag{11}$$

Explicitly, this is

$$(a \star_{\mathsf{h}} b)(\xi) = \int_{x,y\in\mathfrak{g}} a_{\mathsf{h}}^\vee(x) b_{\mathsf{h}}^\vee(y) e^{x\xi/\mathsf{h}} e^{y\xi/\mathsf{h}} e^{\{x,y\}\xi/\mathsf{h}} \chi(x)\chi(y) dx dy.$$

6. Goals ahead

For the remainder of this lecture we want to accomplish the following goals.

Goal (1a). We shall define, for $m \in \mathbb{R}$, a more general symbol class S^m, of complex-valued functions on $\mathfrak{g}^\wedge$ of "order m", which will generalize polynomial symbols, and for which $\mathcal{S}(\mathfrak{g}^\wedge) = S^{-\infty}$. The class

$$S^\infty = \bigcup_{m\in\mathbb{R}} S^m$$

will then be the "finite order symbols". They are h-independent, and oscillate "dyadically". We then wish to extend the star product, initially defined on $S^{-\infty} \times S^{-\infty}$, to a continuous bilinear map

$$\star : S^\infty \times S^\infty \to S^\infty,$$

which will then induce continuous bilinear maps

$$\star : S^{m_1} \times S^{m_2} \to S^{m_1+m_2}$$

for all m_1, m_2. Here, continuity is take with respect to the "evident" topologies, which we will define.

Goal (1b). Fixing $m \in \mathbb{R}$ and now an additional parameter $\delta \in [0,1)$, we shall define a class of h-dependent symbols S^m_δ, which specialize to S^m when $\delta = 0$. Functions in this symbol class should oscillate "subdyadically", on an interval of the form $\xi(1 + O(\mathsf{h}^\delta))$. In practice, one cannot prove anything for $\delta \in [1/2, 1)$; the value $\delta = 1/2$ corresponds to the "Planck scale". For example, for $\delta \in [0, 1/2)$, we wish to prove that the star product extends to a continuous bilinear map

$$\star : S^{m_1}_\delta \times S^{m_2}_\delta \to S^{m_1+m_2}_\delta.$$

Goal (2). We shall define, for $m \in \mathbb{Z}$, a class Ψ^m (Ψ for *pseudo*) of unbounded, densely defined operators, on π of order $\leq m$. Then the union

$$\Psi^\infty = \bigcup_{m\in\mathbb{Z}} \Psi^m$$

will be the class of "finite order operators", and the intersection

$$\Psi^{-\infty} = \bigcap_{m\in\mathbb{Z}} \Psi^m$$

the space of "smoothing operators".

We must also define the Hilbert space completion of the (dense) domain of each Ψ^m, which will be a Sobolev space $\pi^{\mathrm{Sob},m}$ on π. Taking the intersection over all m gives the space of smooth vectors

$$\pi^\infty = \pi^{\mathrm{Sob},\infty} = \bigcap_{m\in\mathbb{Z}} \pi^{\mathrm{Sob},m}$$

and taking the union will give

$$\pi^{-\infty} = \pi^{\mathrm{Sob},-\infty} = \bigcup_{m\in\mathbb{Z}} \pi^{\mathrm{Sob},m},$$

the space of "distributional vectors".

Goal (3). Extend the operator calculus $\mathrm{Op}_h : S^{-\infty} \to \Psi^{-\infty}$ to the class of finite order symbols S^{∞}, and showing that $\mathrm{Op}(S^m)$ takes values in Ψ^m.

Remark 6. Let us make some comments on these goals

(1) Points (1a) and (1b), which have nothing to do with the representation π, are addressed in Section 4 in N-V, whose main result is Theorem 1. The proof of the latter is given in Section 7.
(2) The operator classes $\Psi^m = \Psi^m(\pi)$ in (2) will be defined by the L^2-boundedness of certain iterated (or nested) commutators, following a characterization of the operator map on order $\leq m$ symbols for the Heisenberg group due to Beals (1977). These classes are defined in Section 3 of N-V. There are no major statements to be proved in this section.
(3) This is the contents of Section 5 in N-V. The fact that $\mathrm{Op}(S^m) \subset \Psi^m$ is Theorem 2, proved in Section 8, using a Beals like characterization of membership in Ψ^m, given in Proposition 1 in §8.5

7. Order m symbols on $\mathfrak{g}^\wedge$

We shall use standard multi-index notation throughout (sometimes implicitly), which involves making an identification of $\mathfrak{g}$ with $\mathbb{R}^d$ and $\mathfrak{g}^\wedge$ with $i\mathbb{R}^d$, in such a way that $x\xi = \sum_i x_i \xi_i$. We shall also write $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

7.1. h-independent symbols.

Definition 7.1. *Let $m \in \mathbb{R}$. Define the* class of order m symbols S^m *as the space of smooth complex-valued functions a on $\mathfrak{g}^\wedge$ verifying the following property: for every multi-index α there is $C_\alpha(a) \geq 0$, depending on a, such that for all $\xi \in \mathfrak{g}^\wedge$:*

$$|\partial^\alpha a(\xi)| \leq C_\alpha(a) \langle \xi \rangle^{m-|\alpha|}.$$

Let

$$S^{\infty} = \bigcup_m S^m \qquad \text{and} \qquad S^{-\infty} = \bigcap_m S^m,$$

and note that $S^{-\infty} = \mathcal{S}(\mathfrak{g}^\wedge)$. The space S^∞ is called the class of finite order symbols.

Example 1. When $m \geq 0$ is an integer, S^m contains all polynomials of degree at most m.

Example 2. Let $f \in C_c^\infty(\mathbb{R}_{>0})$ and define $a(\xi) := f(|\xi| - 1)$. Thus a is supported on $1 < |\xi| \leq 1 + C$, and the dilated $a_{\mathsf{h}}(\xi) = a(\mathsf{h}\xi)$ is supported on $\mathsf{h}^{-1} < |\xi| \leq (1 + C)\mathsf{h}^{-1}$.

Remark 7. When applied to G the Heisenberg group, the above definition does not seem to generalize the classical symbols on $\mathbb{R}^2$ described in (4). In that case the ξ-variable was privileged over the x variable. The above definition would seem to require differentiating both variables.

For $m \in \mathbb{R} \cup \{-\infty\}$, the spaces S^m are given their natural Fréchet topology, relative to the family of seminorms

$$\nu_{m,\alpha}(a) = \sup_{\xi \in \mathfrak{g}^\wedge} \langle \xi \rangle^{|\alpha| - m} |\partial^\alpha a(\xi)|.$$

The space S^∞ is given the inductive limit topology.

7.2. h-dependent symbols.

Definition 7.2. *Let $m \in \mathbb{R}$ and $\delta \in [0, 1)$. The space S^m_δ consists of smooth complex-valued h-dependent functions $a(\xi, \mathsf{h})$ on $\xi \in \mathfrak{g}^\wedge$ verifying the following property: for every multi-index α there is $C_\alpha(a) \geq 0$, depending on a independently of h, such that for all $\mathsf{h} \in (0, 1]$:*

$$\nu_{m,\alpha}(a) \leq C_\alpha(a)\mathsf{h}^{-\delta|\alpha|}.$$

Example 3. Let $f \in C_c^\infty(\mathbb{R}_{>0})$ and define

$$a(\xi) := f\left(\frac{|\xi| - 1}{\mathsf{h}^\delta}\right).$$

Thus a is supported on $1 < |\xi| \leq 1 + O(\mathsf{h}^\delta)$, and the dilated $a_\mathsf{h}(\xi) = a(\mathsf{h}\xi)$ is supported on $\mathsf{h}^{-1} < |\xi| \leq \mathsf{h}^{-1} + O(\mathsf{h}^{-1+\delta})$.

Remark 8. In Nelson's follow-up article, proving subconvex bounds on the same L-functions treated in N-V, but by spectrally expanding over the big group $\mathbf{G}$, a refined symbol class is introduced (see his section 9.4).

Nelson's refinement allows for concentration along thin rectangles, which will approximate long pieces of co-adjoint orbits. (The refined symbols concentrate in directions transverse to the foliation of $\mathfrak{g}^\wedge$ by co-adjoint orbits.) So, while the symbol class S^m above allows for bump functions on sets of the form

$$\{\tau + \xi : |\xi| \ll T^{1/2+\epsilon}\} \qquad (|\tau| \approx T),$$

the refined symbol class allows sets of the form

$$\{\tau + \xi : |\xi'| \ll T^{1/2+\epsilon}, |\xi''| \ll T^\epsilon\} \qquad (|\tau| \approx T),$$

where $\xi = (\xi', \xi'')$ is a suitable choice of coordinates on $\mathfrak{g}^\wedge$ relative to the shape of the co-adjoint orbit $\mathcal{O}_\pi$. Here $T = \mathsf{h}^{-1}$, so that $T^{1/2+\epsilon}$ is $\mathsf{h}^{-1+\delta}$ with $\delta = 1/2 - \epsilon$.

7.3. Star product expansion.

We now wish to discuss the asymptotic expansion of the star product. We expand the function $e^{\{x,y\}\xi}$ appearing in the definition (10). Its multi-variable Taylor expansion can be written

$$e^{\{x,y\}\xi} = \sum_{\alpha,\beta,\gamma} c_{\alpha\beta\gamma} x^\alpha y^\beta \xi^\gamma.$$

Here x, y are thought of as being small.

Lemma 7.3. *The coefficient $c_{\alpha\beta\gamma} = 0$ unless $|\gamma| \leq \min\{|\alpha|, |\beta|\}$.*

Proof. This apparently follows from the estimate $\{x, y\} = O(|x|\,|y|)$, coming from the definition (7). I have not checked the details (which are surely elementary)! □

We expand according to the homogeneity degree $j = |\alpha| + |\beta| - |\gamma|$ and formally insert this into the definition (10), to obtain

$$a \star b \sim \sum_{j \geq 0} a \star^j b,$$

where (blithely ignoring the presence of the cutoff functions χ)

$$(a \star^j b)(\xi) = \sum_{|\alpha|+|\beta|-|\gamma|=j} c_{\alpha\beta\gamma} \xi^\gamma \partial^\alpha a(\xi) \partial^\beta b(\xi).$$

Each term $a \star^j b$ is a finite bidifferential operator and satisfies the homogeneous relation $a_{\mathsf{h}} \star^j b_{\mathsf{h}} = \mathsf{h}^j (a \star^j b)_{\mathsf{h}}$. This suggests the rescaled expansion

$$a \star_{\mathsf{h}} b \sim \sum_{j \geq 0} \mathsf{h}^j (a \star^j b).$$

It is clear that

$$\star^j : S^{m_1} \times S^{m_2} \to S^{m_1+m_2-j}$$

and

$$\star^j : S^{m_1}_\delta \times S^{m_2}_\delta \to \mathsf{h}^{-2\delta j} S^{m_1+m_2-j}_\delta.$$

The question is whether these formulae "pass to the limit" to extend $a \star b$ to such symbol classes. From the first of these it follows that for large J the difference

$$a \star b - \sum_{0 \leq j \leq J} a \star^j b \in S^{m_1+m_2-J}$$

decays very rapidly (using the definition of S^m). The same is true of the h dependent classes, as long as $\delta \in [0, 1/2)$, in which case, $\mathsf{h}^{-2\delta j}$ decays with h.

These observations render plausible the following theorem.

Theorem 7.4 (NV, Theorem 1)**.** *The star product $\star$, initially defined on $S^{-\infty} \times S^{-\infty}$, extends to a continuous bilinear map on the space of finite order symbols $S^\infty \times S^\infty$, and induces by restriction continuous bilinear maps for*

$$\star : S^{m_1} \times S^{m_2} \to S^{m_1+m_2}.$$

Similar statements hold for S^m_δ, as long as $\delta \in [0, 1/2)$.

8. Operator classes

Fix a basis $\mathcal{B}$ of $\mathfrak{g}$ and set

$$\Delta = 1 - \sum_{x \in \mathcal{B}} x^2 \in \mathfrak{U}.$$

We conflate Δ and $\pi(\Delta) \in \mathrm{End}(\pi^\infty)$. We shall need Δ along with its inverse. For this reason we quote the following facts:

(1) Δ is densely defined on π, and has bounded inverse Δ^{-1};
(2) $\pi^\infty = \bigcap_{n \geq 0} \mathrm{Domain}(\Delta^n)$, and so Δ^{-1} acts on π^∞ (expand $(1-x)^{-1} = \sum_{n \geq 0} x^n$).

For $s \in \mathbb{Z}$ let π^s be the Hilbert space completion of π^∞ endowed with the inner product

$$\langle v_1, v_2 \rangle_{\pi^s} := \langle \Delta^s v_1, v_2 \rangle.$$

Then, since it is harder to be in π^{s+1} than in π^s, we have inclusions

$$\pi^\infty = \bigcap_{s \in \mathbb{Z}} \pi^s \subset \cdots \subset \pi^{s+1} \subset \pi^s \subset \cdots \pi^{-\infty} = \bigcup_{s \in \mathbb{Z}} \pi^s.$$

We call $\pi^{-\infty}$ the "distributional vectors" of π.

Definition 8.1. *An operator on π is a linear map $T : \pi^\infty \to \pi^{-1}$.*

Question 2. *How to define an operator of "order at most m"?*

Following Beals, we use iterated commutators. For each $x \in \mathfrak{g}$ we put

$$\theta_x(T) := [\pi(x), T].$$

Fact 1. *$\theta_x(T)$ is an operator on π.*

We may therefore view θ_x as an element in End({operators on π}). The map $x \mapsto \theta_x$ extends to an algebra homomorphism

$$\mathfrak{U} \to \mathrm{End}(\{\text{operators on } \pi\}), \qquad u \mapsto \theta_u.$$

For example, if $x_1, \dots, x_n \in \mathfrak{g}$, we have

$$\theta_{x_1 \cdots x_n}(T) = [\pi(x_1), [\pi(x_2), \dots, [\pi(x_n), T]]].$$

Definition 8.2. *Let $m \in \mathbb{Z}$. An operator T is* of order $\leq m$ *if for each $s \in \mathbb{Z}$ and $u \in \mathfrak{U}$, the operator $\theta_u(T)$ induces a* bounded map

$$\theta_u : \pi^s \to \pi^{s-m}.$$

Let Ψ^m denote the space of operators of order $\leq m$ on π. We put

$$\Psi^{-\infty} = \bigcap_{m \in \mathbb{Z}} \Psi^m \qquad \Psi^{\infty} = \bigcup_{m \in \mathbb{Z}} \Psi^m,$$

the spaces of "smoothing operators" and "finite order operators", respectively.

There is also a definition of the space Ψ^m_δ, given in Section 5.3. We refrain from spelling this out.

Theorem 8.3 (Theorems 2 & 3, N-V)**.** *For $m \in \mathbb{Z}$, $\delta \in [0, 1/2)$, and $\mathsf{h} \in (0, 1]$, we have*

$$\mathrm{Op}(S^m) \subset \Psi^m \qquad \textit{and} \qquad \mathrm{Op}_{\mathsf{h}}(S^m) \subset \mathsf{h}^{\min(0,m)} \Psi^m_\delta.$$

In fact, the first inclusion follows from the second, by taking $\mathsf{h} = 1$ and $\delta = 0$.

8.1. **Sketch of Theorem 2.** Here is a rough sketch, following Sections 8.5, 8.6 and 8.7 (and assuming the contents of Section 7).

We only treat the case $m = 0$, the others being reduced to this by a composition law (for $m < 0$) and an induction argument ($m > 0$). See the end of Section 8.7.

Then we are to prove $\mathrm{Op}(S^0) \subset \Psi^0$. Firstly, the characterization of the class Ψ^0 in Section 8.5 states that $T \in \Psi^0$ if, and only if, T is a bounded operator. We shall in fact prove that for $a \in S^0_\delta$, where $\delta \in [0, 1/2)$, the operator $\mathrm{Op}_{\mathsf{h}}(a)$ is a bounded operator, with operator norm bounded uniformly in h. Taking $\delta = 0$ gives the result.

There are three key inputs:

(1) the *Cotlar–Stein lemma*, which states that if $T_j : V_1 \to V_2$ are bounded linear operators on Hilbert spaces such that

$$\sup_j \sum_k \|T_j^* T_k\|^{1/2} \leq C,$$

then $\sum T_j$ converges to a bounded linear operator $T : V_1 \to V_2$ with operator norm at most C;

(2) the trivial bound $\|\mathrm{Op}_{\mathsf{h}}(a)\| \leq \mathcal{N}(a)$, where $\mathcal{N}(a) = \|a^\wedge\|_{L^1(\mathfrak{g})}$ for $a \in \mathcal{S}(\mathfrak{g}^\wedge)$;

(3) the bound $\mathcal{N}(a) \ll 1$ for *localized* a. Here, a is localized at $w \in \mathfrak{g}^\wedge$ (see Definition 7.6.) if a is supported on

$$U_\omega = \left\{ \xi \in \mathfrak{g}^\wedge : |\xi - \omega| \leq \frac{1}{2} h^\delta \langle \omega \rangle \right\}.$$

With these ingredients, we sketch the proof. We write $a = \sum_{w\in\Omega} a_\omega$, with $a \in S^0_\delta$ localized at ω. Thus

$$\mathrm{Op}_{\mathsf{h}}(a) = \sum_{\omega} \mathrm{Op}_{\mathsf{h}}(a_\omega).$$

Since $\mathrm{Op}_{\mathsf{h}}(a_\omega)^* = \mathrm{Op}_{\mathsf{h}}(\overline{a_\omega})$, we have

$$\mathrm{Op}_{\mathsf{h}}(a_{\omega_1})^*\mathrm{Op}_{\mathsf{h}}(a_{\omega_2}) = \mathrm{Op}_{\mathsf{h}}(\overline{a_\omega})\mathrm{Op}_{\mathsf{h}}(a_{\omega_2}) = \mathrm{Op}_{\mathsf{h}}(\overline{a_\omega} \star_{\mathsf{h}} a_{\omega_2}, \chi').$$

According to the first two inputs, if we can prove

$$\sup_{\omega_1\in\Omega} \sum_{\omega_2\in\Omega} \mathcal{N}(\overline{a_{\omega_1}} \star_h a_{\omega_2})^{1/2} \ll 1,$$

then we are done. We expand

$$\overline{a_{\omega_1}} \star_h a_{\omega_2} = \sum_{0\le j<J} \mathsf{h}^j \overline{a_{\omega_1}} \star^j a_{\omega_2} + (\text{small}),$$

for some large enough J. For all but finitely many ω_2 the main term is zero. The contributing terms are localized at ω_1. The third input yields the result. □

9. Representations of real reductive groups

We want to review a lot of ingredients to define the Kirillov formula for tempered representations on real reductive G.

9.1. **Tempered.** Many properties of unitary representations can be read off on their matrix coefficients:

$$c_{v,w} : g \mapsto \langle \pi(g).v, w\rangle,$$

where v and w are assumed fixed, sometimes assumed to have nice properties. For example, the basic building blocks of the representation theory of p-adic Lie groups are the supercuspidal representations, which are irreducible representations whose matrix coefficeints (it turns out) are compactly supported (modulo the center). Matrix coefficients are local analogues to automorphic forms, which realize vectors in an automorphic representation as automorphic functions. The global analogue of supercuspidal represenations are cuspidal automorphic representations over function fields, whose automorphic forms (it turns out) are compactly supported.

For real groups there are no supercuspidal representations, but discrete series representations can be characterized as those whose matrix coefficients are L^2-integrable over G (modulo the center). Discrete series representations are somewhat rare: some groups (such at $\mathrm{GL}_n(\mathbb{R})$ for $n \ge 3$ or $\mathrm{GL}_n(\mathbb{C})$ for $n \ge 2$) do not even admit them.

A unitary representation π of G will be called tempered if its matrix coefficients (for K-finite vectors v, w, where K is a maximal compact subgroup of G) just fail to be L^2 integrable: $c_{v,w} \in L^{2+\epsilon}(G/Z)$ for every $\epsilon > 0$.

Tempered representations are abundant: they always exist, and all irreducible representations of G can be expressed in terms of tempered representations of small groups, in the following sense: there is a unique G-conjugacy class of pairs (P, σ), where P is a parabolic subgroup (not necessarily proper!) of G and σ is a representation of the Levi quotient $M = P/U_P$, such that

- σ is tempered on $M^{\mathrm{der}} = [M, M]$;

- the central character of σ is strictly dominant;
- π is the unique irreducible quotient of the induced rep $\iota_P^G\sigma$.

This is the so-called *Langlands classification*. Thus π is tempered when $P = G$.

Example 4. On $\mathrm{SL}_2(\mathbb{R})$ we take P the Borel, M the diagonal matrices, so that $M^{\mathrm{der}} = 1$. Then the above procedure gives the principal series representations obtained from unitarily inducing the character $\mathrm{diag}(y, 1/y) \mapsto y^s$ where $\Re s > 0$. (The last condition corresponds to strictly dominant.) Then two cases present themselves. These are irreducible (and non-tempered) when s is not integral. For $s = k-1 \geq 1$ an odd integer these are reducible with unique irreducible quotient the $(k-2)$-th symmetric power of the standard representation. (Note that π_k contains as a subrepresentation the direct sum $D_k^+ \oplus D_k^-$.) In particular, the trivial representation is a quotient of $I(1)$.

9.2. **Characters on the group.** Now let us review a few different notions of characters associated with an irreducible representations of a real reductive Lie group. Sometimes we will take G semisimple with finite center.

The first is the *distributional character*, which we have seen before. Let $A \in C_c^\infty(G)$ then the operator $\pi(A)$ is of trace class, and we write $\operatorname{tr}\pi(A)$ for its trace.

The second is much subtler, which is the *Harish-Chandra character*. It is an L^1_{loc}-function χ_π on the group G (supported on the regular semisimple elements) which represents the distributional character, in the sense that

$$\operatorname{tr}\pi(A) = \int_G \chi_\pi(g)A(g)dg.$$

We think of the above two characters as being on the group. There is another notion of characte, the *infinitesimal character*, that is defined on the Lie algebra, or rather the center of the universal enveloping algebra. This will require some time and preparation.

9.3. **GIT quotients.** Recall that $G = \mathbf{G}(\mathbb{R})$ acts on $\mathfrak{g}_{\mathbb{C}}^*$ via the coadjoint action. Its orbit space has a non-separated topology: the closures of certain orbits intersect. For example, the nilpotent cone for $\mathrm{SL}_2(\mathbb{R})$ consists of three orbits, with $\{0\}$ being in the closure of both the upper and lower half-cones. We consider a more well-behaved spaced, which will identify orbits of this nature.

We introduce the GIT (geometric invariant theoretic) quotient

$$[\mathfrak{g}_{\mathbb{C}}^*] = \mathfrak{g}_{\mathbb{C}}^* /\!\!/ \mathbf{G}.$$

This is an affine algebraic variety defined as the prime spectrum of the G-invariant polynomials on $\mathfrak{g}_{\mathbb{C}}^*$.

Example 5. Consider the action of $\mathbb{G}_m$ on the affine plane $\mathbb{A}^2$ given by $t.(x, y) = (tx, t^{-1}y)$. The orbits are the hyperbolas $xy = a$ for $a \neq 0$ as well as the following degenerate orbits:

- the punctured axes $\{x = 0\} \setminus \{(0, 0)\}$, $\{y = 0\} \setminus \{(0, 0)\}$;
- the origin $\{(0, 0)\}$

The GIT quotient $\mathbb{A}^2 /\!\!/ \mathbb{G}_m$ is the prime spectrum of $\mathbb{C}[x, y]^{\mathbb{G}_m} = \mathbb{C}[xy]$. The latter is the affine line $\mathrm{spec}(\mathbb{C}[xy]) = \mathbb{A}^1$. The map $\mathbb{A}^2 \to \mathbb{A}^1$ sends the union of the two coordinate axes $xy = 0$ to the origin in $\mathbb{A}^1$.

Example 6. Going back to the example relevant to us, we now take $G = \mathrm{GL}_n$. Then $[\mathfrak{g}^*_\mathbb{C}]$ can be identified, via the characteristic polynomial, with the monic polynomials of degree n. This is a consequence of Jordan normal form. Note that (for GL_2, say) the characteristic polynomials of the elements $\left(\begin{smallmatrix}\lambda & x\neq 0\\ & \lambda\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}\lambda & \\ x\neq 0 & \lambda\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)$ are all the same, so they are identified in the quotient. Notice that $[\mathfrak{g}^*_\mathbb{C}]$ is naturally identified with $\mathbb{C}^n$.

Example 7. When $G = \mathrm{SL}_2(\mathbb{R})$ we have $[\mathfrak{g}^*_\mathbb{C}]$ identified with monic traceless polynomials of degree 2, so of the form $x^2 - \lambda$, where $\lambda \in \mathbb{C}$. Here we have a natural identification with $\mathbb{C}$.

9.4. **Auxiliary Cartan.** The map $\mathfrak{g}^*_\mathbb{C} \to [\mathfrak{g}^*_\mathbb{C}], \xi \mapsto [\xi]$ is surjective. In fact, it is already surjective when restricted to $\mathfrak{t}^*_\mathbb{C}$, where $\mathfrak{t}_\mathbb{C}$ is a Cartan (=maximal abelian diagonalizable) subalgebra of $\mathfrak{g}^*_\mathbb{C}$. For example, when $G = \mathrm{GL}_n(\mathbb{R})$, this is a consequence of Jordan form (see the previous remark) and fundamental theorem of algebra (factor your polynomial over $\mathbb{C}$ and create a diagonal matrix by putting the roots on the diagonal). One sees from this example that in fact

$$[\mathfrak{g}^*_\mathbb{C}] \simeq \mathfrak{t}^*_\mathbb{C}/W_\mathbb{C} \tag{12}$$

(one can put the roots on the diagonal in any order).

There is a notion of regular elements in $[\mathfrak{g}^*_\mathbb{C}]$, which can be described through the above identification as those $[\xi] \in \mathfrak{t}^*_\mathbb{C}/W_\mathbb{C}$ having $|W_\mathbb{C}|$ distinct preimages in $\mathfrak{t}^*_\mathbb{C}$. Equivalently (think of the GL_n example) $\lambda \in [\mathfrak{g}^*_\mathbb{C}]$ is regular if and only if all of its preimages in $\mathfrak{g}^*_\mathbb{C}$ are regular semisimple (for GL_n this means no repeated roots). Let $[\mathfrak{g}^*_\mathbb{C}]_{\mathrm{reg}}$ denote the regular subset. For $G = \mathrm{SL}_2(\mathbb{R})$ the condition is that $\lambda \neq 0$.

9.5. **Harish-Chandra homomorphism.** Let $\mathcal{Z}$ be the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_\mathbb{C})$ of $\mathfrak{g}_\mathbb{C}$. The Harish-Chandra homomorphism describes $\mathcal{Z}$ concretely, given a choice of Cartan subalgebra. It provides in fact an algebra isomorphism

$$\gamma_{HC} : \mathcal{Z} \xrightarrow{\sim} \mathcal{U}(\mathfrak{t}_\mathbb{C})^W = \mathrm{Sym}(\mathfrak{t}_\mathbb{C})^W = \mathrm{Pol}(\mathfrak{t}^*_\mathbb{C})^W.$$

In view of (12), the map γ_{HC} identifies $\mathcal{Z}$ with the algebra of regular functions on $[\mathfrak{g}^*_\mathbb{C}]$. This induces an identification

$$[\mathfrak{g}^*_\mathbb{C}] \simeq \mathfrak{t}^*_\mathbb{C}/W \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathcal{Z}, \mathbb{C})$$

9.6. **Infinitesimal character.** We finally arrive at the notion of infinitesimal character.

If π is an irreducible representation of G then a version of Schur's lemma says that $\mathcal{Z}$ acts on π by scalars. Thus π defines an element $\lambda_\pi \in \mathrm{Hom}_{\mathbb{C}-\mathrm{alg}}(\mathcal{Z}, \mathbb{C})$. From the Harish-Chandra isomorphism, this defines an element in $[\mathfrak{g}^*_\mathbb{C}]$, which we again denote by λ_π.

Example 8. If $G = \mathrm{SL}_2(\mathbb{R})$ then $\mathcal{Z}$ is a polynomial algebra in one variable, with generator the Casimir operator Ω. Let $\pi = \pi_s$ (using the previous notation, but now allowing s to be arbitrary). Then Ω acts on π_s via the scalar $\lambda_s = (1 - s^2)/4$. Using the description of $[\mathfrak{g}^*_\mathbb{C}]$ from a previous remark, the corresponding element in $[\mathfrak{g}^*_\mathbb{C}]$ is the monic traceless polynomial $x^2 - \lambda_s$.

Now if you take $D_k^\pm$ the holomorphic or anti-holomorphic discrete series of even weight $k \geq 2$ then this is a subrepresentation of π_{k-1} and so it has the same infinitesimal character as the unique irreducible quotient, which is sym^{k-2}, and this infinitesimal character maps to $x^2 - \lambda_{k-1}$ in $[\mathfrak{g}^*_\mathbb{C}]$. Note that $\lambda_{k-1} = \frac{k}{2}(1 - \frac{k}{2})$. This is zero when $k = 2$, as it should be, as it corresponds to the infinitesimal character of the trivial representation.

Note that the infinitesimal character does not completely determine the representation, as we just saw.

9.7. **The real form of $[\mathfrak{g}_{\mathbb{C}}^*]$.** The complex affine variety $[\mathfrak{g}_{\mathbb{C}}^*]$ has a real form which we shall denote by $[i\mathfrak{g}^*]$. It is the prime spectrum of the G-invariant polynomials taking real values on $i\mathfrak{g}^*$. It can be described explicitly as

$$[i\mathfrak{g}^*] = \{\lambda \in [\mathfrak{g}_{\mathbb{C}}^*] : \lambda = -\bar{\lambda}\}.$$

It is a real affine space of dimension n. In the $\mathrm{GL}_n(\mathbb{R})$ example, this pics out the monic polynomials of degree n having real coefficients. In the $\mathrm{SL}_2(\mathbb{R})$ example, it yields the monic traceless quadratic polynomials with real coefficients $\{x^2 - \lambda : \lambda \in \mathbb{R}\}$.

Fact: if π is unitary then $\lambda_\pi \in [i\mathfrak{g}^*]$.

This is not an equivalence: π_s is unitary only if $s \in i\mathbb{R} \cup [-1, 1]$. These are precisely the values of $s \in \mathbb{C}$ for which $\lambda_s \geq 0$. Thus, $[i\mathfrak{g}^*]$ is strictly larger.

9.8. **Criterion for temperedness.** We have the following

Fact: if the infinitesimal character of an an irreducible unitary π is sufficiently away from irregular locus, then π is tempered.

Example 9. Let's see this with the $\mathrm{SL}_2(\mathbb{R})$ example. By unitarity we have either have $\lambda_\pi = (1 - s^2)/4 \geq 0$ or $\lambda_\pi = \frac{k}{2}(1 - \frac{k}{2})$, the latter corresponding to the discrete series representations which are found as subrepresentations of the non-tempered reducible principal series representation π_{k-1}, where $k \geq 2$ is even. The irregular locus is then $\lambda_\pi = 0$, which to the three "borderline" representations D_2^+, D_2^-, triv. The tempered reps are those π_s with $s \in i\mathbb{R}$ or D_k^+, D_k^-; in terms of their infinitesimal characters, this corresponds to $\lambda_\pi \geq 1/4$ or $\lambda_\pi = \frac{k}{2}(1 - \frac{k}{2})$. In this example, if you take a compact subset of the positive reals (which might initially contain the non-tempered segment $(0, 1/4)$) will, when rescaled by a sufficiently large positive parameter, eventually be strictly contained in the tempered part $[1/4, \infty)$.

Remark 9. The parameter s (or the pair of parameters $s, -s$) in the above example is the more natural parameter from the representation theoretic point of view, rather than the constant term $-\lambda_\pi$. Similarly, the roots (or "eigenvalues") of the monic degree polynomials (rather than their coefficients) are more natural from a representation theoretic point of view. In the GIT sections that Bart will cover, and the Satake parameter section that Asbjørn will cover, this relationship will be explored.