Boolean Functions for Cryptography and Coding Theory

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Preface

The present monograph is a merged, reorganized, significantly revised and extensively completed version of two chapters, entitled "Boolean Functions for Cryptography and Error Correcting Codes" [236] and "Vectorial Boolean Functions for Cryptography" [237], which appeared in 2010 as parts of the book "Boolean Models and Methods in Mathematics, Computer Science, and Engineering" [394] (Editors, Yves Crama and Peter Hammer). It is meant for researchers but is accessible to anyone who knows basics in linear algebra and general mathematics. All the other notions needed are introduced and studied (even finite fields are, in Appendix).

Since these chapters were written in 2009, about 1500 papers have been published which deal with this two-fold topic (which is broad as we see), and this version is updated with the main references and their main results (with corrections in the rare cases where they were needed). It also contains original results.

New notions on Boolean and vectorial functions and new ways of using them have also emerged. A chapter devoted to these recent and/or not enough studied directions of research has been included.

In the limit of a book, we tried to be as complete as possible. Of course, we could not go into details as much as do papers, but we made our best to ensure a good trade-off between completeness in scope and in depth. The choice of those papers which are referred and of those results which are developed may seem subjective; it has been difficult, given the large number of papers. We tried, within the length limit of 600 pages and some, to give the proof of a result each time it was short and simple enough, and when it provided a vision (we tried to avoid giving too technical proofs whose only - but of course important - value would have been to convince the reader that the result is true). We would have liked to avoid, when presenting arguments and observations, to refer to results (and concepts) to come later in the text, but the large number of results has made this necessary; otherwise, it would have been impossible to gather in a same place all the facts related to a same notion.

We have limited ourselves to Boolean and vectorial functions in characteristic 2, since these fit better with applications in coding and cryptography, and since dealing with p-ary and generalized functions would have reduced the description of the results on binary functions.

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Research is a collective action and a too long list of names should be cited for acknowledging all the stimulating discussions, collaborations and informations which contributed to this book. A few names are the ten mentioned above and Kanat Abdukhalikov, Benny Applebaum, Thierry Berger, Marco Calderini, Ayça Çeşmelioglu, Xi Chen, Robert Coulter, Diana Davidova, Ulrich Dempwolff, John Dillon, Cunsheng Ding, the late Hans Dobbertin, Yves Edel, Keqin Feng, Caroline Fontaine, Rafael Fourquet, Philippe Gaborit, Faruk Göloglu, Guang Gong, Aline Gouget, Cem Güneri, Tor Helleseth, Xiang-dong Hou, Nikolay Kaleyski, William Kantor, Selçuk Kavut, Jenny Key, Alexander Kholosha, Andrew Klapper, Nicholas Kolokotronis, Gohar Kyureghyan, Philippe Langevin, Gregor Leander, Alla Levina, Chunlei Li, Nian Li, Konstantinos Limniotis, Mikhail Lobanov, Luca Mariot, Subhamoy Maitra, the late James Massey, Gary McGuire, Wilfried Meidl, Willi Meier, Harald Niederreiter, Svetla Nikova, Kaisa Nyberg, Ferruh Özbudak, Daniel Panario, Matthew Parker, Enes Pasalic, George Petrides, Alexander Pott, Mathieu Rivain, Thomas Roche, François Rodier, Neil Sloane, Patrick Solé, François-Xavier Standaert, Pante Stănică, Henning Stichtenoth, Yin Tan, Chunming Tang, Horacio Tapia-Recillas, Faina Solov'eva, Yuriy Tarannikov, Cédric Tavernier, Alev Topuzoğlu, Irene Villa, Arne Winterhof, Satoshi Yoshiara, Xiangyong Zeng, Fengrong Zhang, Victor Zinoviev, as well as the members of the INRIA team, whose CODES project (now called SECRET) has been a nice research environment and has supported the author during his thesis and many years after, and the *Bergen Selmer Center* team which does the same now, with a spirit of kindness and generosity, for his great scientific benefit.

The author also wishes to acknowledge that gathering the bibliography has been considerably eased by web sites like “dblp: computer science bibliography” (https://dblp.uni-trier.de), “ResearchGate” (https://www.researchgate.net) and “Google Scholar” (https://scholar.google.fr/schhp?hl=fr&tab=Xs).

Last but not least, I am so grateful to my wife Madeleine and my family for their support, patience, comprehension and understanding of what a researcher's work is. This is even more true for the last three years, during which the writing of this book, the reviewing of the numerous published papers and the copy editing took so much of my time. I dedicate my book to them, with a special thought for my children and grandchildren, who will have to face the world we leave them.

Notation

$\lvert I\rvert$	size of a set I,
$\lfloor u\rfloor$	integer part (floor) of a real number u,
$\lceil u\rceil$	ceiling of u (the smallest integer larger than or equal to u),
$\phi^{-1}(u)$	pre-image of u by a function ϕ,
1_E	indicator (or characteristic) function of a set E: $1_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ otherwise,} \end{cases}$
δ_a	the Dirac (or Kronecker) symbol at a ($i.e.$ the indicator of $\{a\}$),
$\mathbb{F}_2$	the finite field with two elements $0, 1$ (bits),
$\mathbb{F}_2^n$	the n-dimensional vector space over $\mathbb{F}_2$ (sometimes identified with $\mathbb{F}_{2^n}$),
$\mathcal{L}_{n,m}$	the vector space of linear (n, m)-functions,
0_n	zero vector in $\mathbb{F}_2^n$ or in $\mathbb{F}_q^n$, $n > 1$ (in other groups, we just write 0),
1_n	vector $(1, \ldots, 1)$ in $\mathbb{F}_2^n$,
$+$	addition in characteristic 0 ($e.g.$, in $\mathbb{R}$), and in $\mathbb{F}_2^n$ and $\mathbb{F}_{2^n}$ for $n > 1$,
$\sum_i$	multiple sum of $+$,
$\oplus$	addition in $\mathbb{F}_2$ ($i.e.$ modulo 2); direct sum of two vector spaces,
$\bigoplus_i$	multiple sum of $\oplus$,
$\overline{x}$	$x + 1_n$, where $x \in \mathbb{F}_2^n$,
$a \cdot x$	inner product in $\mathbb{F}_2^n$,
$\ell_a(x)$, $t_a(x)$	$= a \cdot x$, resp. $x + a$, where "$\cdot$" is an inner product in $\mathbb{F}_2^n$,
$\mathbb{F}_2^I$	the vector space over $\mathbb{F}_2$ of all binary vectors whose indices range in I,
$\mathbb{F}_{2^n}$	the finite (Galois) field of order 2^n, identified with $\mathbb{F}_2^n$ as a vector space,
$tr_m^n(x)$	$= x + x^{2^m} + x^{2^{2m}} + \cdots + x^{2^{n-m}}$, trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$ $(m \mid n)$,
$tr_n(x)$	$= tr_1^n(x) = \sum_{i=0}^{n-1} x^{2^i}$ the absolute trace function,
$\mathbb{F}_{2^n}^*$	$\mathbb{F}_{2^n} \setminus \{0\}$, where 0 denotes the zero element of $\mathbb{F}_{2^n}$,
α	primitive element of $\mathbb{F}_{2^n}$,
$\otimes$	convolutional product of two functions over $\mathbb{F}_2^n$ (see page 79),
$f, g, h, \ldots$	Boolean functions,
$\mathcal{BF}_n$	the $\mathbb{F}_2$-vector space of all n-variable Boolean functions $f : \mathbb{F}_2^n \to \mathbb{F}_2$,
$F, G, H, \ldots$	vectorial functions,
$\mathcal{G}_F$	graph of a vectorial function: $\mathcal{G}_F = \{(x, F(x));\ x \in \mathbb{F}_2^n\}$,
$w_H()$	Hamming weight (of a vector, of a function),
$d_H(,)$	Hamming distance (between two vectors, two functions),
$d(C)$	minimum (Hamming) distance of code C,
$supp()$	the support (of a vector, of a function),
$x \preceq y$	"x is covered by y" ($i.e.$ $supp(x) \subseteq supp(y)$),
$x \vee y$	vector such that $supp(x \vee y) = supp(x) \cup supp(y)$,
$x \wedge y$	vector such that $supp(x \wedge y) = supp(x) \cap supp(y)$,
e_i	i-th vector of the canonical basis of $\mathbb{F}_2^n$,

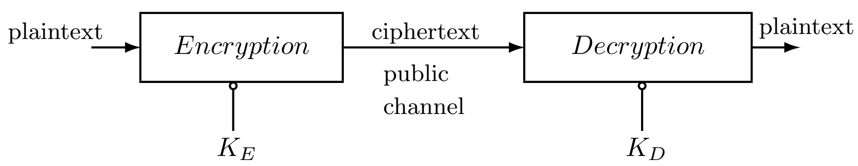
x^I, x^u	$\prod_{i\in I} x_i$, $I \subseteq \{1,\dots,n\}$, $\prod_{i=1}^n x_i^{u_i}$, $u \in \mathbb{F}_2^n$,
$f \mapsto f^\circ$	binary Möbius transform ($f^\circ : u \mapsto a_u$, coef. of x^u in the ANF of f),
$\widehat{\varphi}$	Fourier-Hadamard transform of a real-valued function φ over $\mathbb{F}_2^n$,
f_χ	sign function of a Boolean function f, that is, $x \mapsto (-1)^{f(x)}$,
$W_f()$	Walsh transform of a Boolean function f (*i.e.* $\widehat{f_\chi}$),
$W_F(,)$	Walsh transform of a vectorial function F,
$supp(W_f)$	support of W_f: $\{u \in \mathbb{F}_2^n;\ W_f(u) \neq 0\}$,
N_{W_f}	cardinality of the support of W_f,
$\mathcal{F}(f)$	$\sum_{x\in\mathbb{F}_2^n}(-1)^{f(x)}(= W_f(0_n))$,
$nl()$	nonlinearity of a Boolean or vectorial function,
$nl_r()$	r-th order nonlinearity of a Boolean function,
ln, $\log_2$	natural (Neperian) logarithm, base 2 logarithm,
$d_{alg}(f)$	the algebraic degree of f (*i.e.* the degree of its ANF),
$d_{num}(f)$	the numerical degree of f (*i.e.* the degree of its NNF),
$w_2(j)$	2-weight of integer j (see page 62),
(n,m,t)-function	t-resilient (n,m)-function,
$AI()$	algebraic immunity of a function,
$M_{f,d}$	matrix of the system of equations $\bigoplus_{\substack{I\subseteq\{1,\dots,n\}\\ \lvert I\rvert\le d}} a_I u^I = 0$, $u \in supp(f)$,
$rk(M)$	the rank of a matrix M,
$FAC()$	fast algebraic complexity of a function,
$FAI()$	fast algebraic immunity of a function,
D_af, D_aF	derivatives in the direction a: $x \mapsto f(x) \oplus f(x+a)$, $F(x)+F(x+a)$,
Δ	the symmetric difference between two sets,
$\Delta_f(a)$	autocorrelation function $\Delta_f(a) = \sum_{x\in\mathbb{F}_2^n}(-1)^{D_af(x)}$,
Δ_f	absolute indicator of f: $\Delta_f = \max_{a\in\mathbb{F}_2^n\setminus\{0_n\}} \lvert\Delta_f(a)\rvert$,
$\mathcal{V}(f)$	sum-of-squares indicator of f: $\sum_{e\in\mathbb{F}_2^n}\mathcal{F}^2(D_ef) = \sum_{a,b\in\mathbb{F}_2^n}\mathcal{F}(D_aD_bf)$,
$\mathcal{E}_f$	linear kernel of a Boolean function f,
$RM(r,n)$	Reed-Muller code of order r and length 2^n,
$\rho(r,n)$	covering radius of $RM(r,n)$,
β_f	the symplectic form associated to a quadratic function f,
$\widetilde{f}$	dual of a bent Boolean function (Definition 51, page 221),
$\mathcal{M}$	Maiorana-McFarland's class,
$\mathcal{PS}$	partial spread class,
L^*	adjoint operator of a linear automorphism L,
$Im(F)$	the range (*i.e.* image set) $F(\mathbb{F}_2^n)$ of an (n,m)-function,
$An(f)$	the $\mathbb{F}_2$-vector space of annihilators of a Boolean function f,
$An_d(f)$	restriction of $An(f)$ to those functions of algebraic degree at most d,
$B_{k,l}(f)$	$= \{g \in \mathcal{BF}_n;\ d_{alg}(g) \le k \text{ and } d_{alg}(fg) \le l\}$,
f	defined by $f(x) = \mathsf{f}(w_H(x))$, when f is symmetric,
$\sigma_i(x)$	elementary symmetric Boolean fct., of ANF: $\bigoplus_{I\subseteq\{1,\dots,n\}/\ \lvert I\rvert=i} x^I$,
$S_i(x)$	elementary symmetric pseudo-Boolean fct. NNF: $\sum_{I\subseteq\{1,\dots,n\}/\ \lvert I\rvert=i} x^I$,
δ_F	differential uniformity of an (n,m)-function F,
Nb_F	imbalance of an (n,m)-function, see page 135,
NB_F	derivative imbalance of an (n,m)-function, see page 161,
$\mathbf{x}$	a sharing of x (see page 472),
$\mathbf{F}$	a threshold implementation of function F (see page 472),
$E_{n,k}$	$= \{x \in \mathbb{F}_2^n;\ w_H(x) = k\}$,
$w_H(f)_k$	the Hamming weight of the restriction of function f to $E_{n,k}$,

Contents

1 Introduction to cryptography, codes, Boolean and vectorial functions

1.1 Cryptography

A fundamental objective of *cryptography* is to enable two persons to communicate over an insecure channel (a public channel such as internet) in such a way that any other person is unable to recover their messages (constituting the *plaintext*) from what is sent in its place over the channel (the *ciphertext*). The transformation of the plaintext into the ciphertext is called *encryption*, or enciphering. It is ensured by a *cryptosystem*. Encryption-decryption is the most ancient cryptographic activity (ciphers already existed four centuries B. C.) but its nature has deeply changed with the invention of computers, because the *cryptanalysis* (the activity of the third person, the eavesdropper, aiming at recovering the message, or better, the secret data used by the algorithm, the latter being assumed public) can use their power. Another important change will occur (see *e.g.* [70, 360, 832]), at least for public-key cryptography (see definition below), when quantum computers become operational.
The encryption algorithm takes as input the plaintext and an encryption key K_E, and it outputs the ciphertext. The *decryption* (or deciphering) algorithm takes as input the ciphertext and a private[1] decryption key K_D. It outputs the plaintext.



For being considered robust, a cryptosystem should not be cryptanalysed by an attack needing less than 2^{80} elementary operations (which represent thousands of centuries of computation with a modern computer) and less than billions

[1] According to principles already stated in 1883 by A. Kerckhoffs [688], who cited a still more ancient manuscript by R. du Carlet [207], only the key(s) need absolutely to be kept secret – the confidentiality should not rely on the secrecy of the encryption method – and a cipher cannot be considered secure if it can be decrypted by the designer himself without using the decryption key.

of plaintext-ciphertext pairs. In particular, an exhaustive search of the secret parameters of the cryptosystem (consisting in trying every possible value of them until the data given to the attacker matches with the computed data) should not be feasible in less than 2^{80} elementary operations. In fact, we even most often want that there is no faster cryptanalysis than exhaustive search.
Note that the term of cryptography is often used indifferently for naming the two activities of designing cryptosystems and of cryptanalysing them, while the correct term when dealing with both is *cryptology*.

1.1.1 Symmetric versus public-key cryptosystems

If the encryption key is supposed to be secret, then we speak of *conventional cryptography* or of *private-key cryptography*. We also speak of *symmetric cryptography* since the same key can then be used for K_E and K_D. In practice, the principle of conventional cryptography relies then on the sharing of a private key between the sender of a message (often called Alice) and its receiver (often called Bob). Until the late seventies, only symmetric ciphers existed.
If the encryption key can be public, then we speak of *public-key cryptography* (or *asymmetric cryptography*), which is preferable to conventional cryptography, since it makes possible to securely communicate without having previously shared keys in a secure way: every person who wants to receive secret messages can keep secret a decryption key and publish an encryption key; if n persons want to secretly communicate pairwise using a public key cryptosystem, they need n encryption keys and n decryption keys, when conventional cryptosystems will need $\binom{n}{2} = \frac{n(n-1)}{2}$ keys. Of course, it must be impossible to deduce in reasonable time, even with huge computational power, the private decryption key from the public encryption key. Such requirement is related to the problem of building *one-way functions*, that is, functions such that computing the image of an element is fast (*i.e.* is a problem of polynomial complexity) while the problem of computing the pre-image of an element has exponential complexity.
All known public key cryptosystems, like RSA which uses operations in large rings [846], allow a much lower data throughput; they also need keys of sizes ten times larger than symmetric ciphers for ensuring the same level of security. Some public-key cryptosystems, like those of McEliece and Niederreiter (based on codes) [846] are faster, but have drawbacks, because the ciphertext and the plaintext have quite different lengths, and the keys are still larger than for other public-key cryptosystems[2]. Private-key cryptosystems are then still needed nowadays for ensuring the confidential transfer of large data. In practice, they are widely used for confidentiality in internet, banking, mobile communications, etc. and their study and design are still an active domain of research.

[2] Code-based, lattice-based and other "post-quantum" cryptosystems are however actively studied, mainly because they would be alternatives to RSA and to the cryptosystems based on the discrete logarithm, in case an efficient quantum computer could be built in the future, which would break them.

Thanks to public key cryptosystems, the share-out of the necessary secret keys for the symmetric cipher can be done without using a secure channel (the secret keys for conventional cryptosystems are strings of a few hundreds of bits only and can then be encrypted by public key cryptosystems). The protagonists can then exchange safely, over a public channel such as internet, their common private encryption-decryption key, called a *session key*. Protocols specially devoted to key-exchange can also be used.
The change caused by the intervention of quantum computers will be probably much less important for symmetric than for public-key cryptography. Most current symmetric ciphers seem secure against attacks by quantum computers (Grover's algorithm [576] which, given a black box with N possible inputs and some output, deduces with high probability from the results of $\mathcal{O}(\sqrt{N})$ evaluations the supposedly unique input[3], will probably have as impact to oblige to double the length of the keys).

1.1.2 Block ciphers versus stream ciphers

The encryption in a symmetric cipher can be treated block by block in a so-called *block cipher* (such as the Advanced Encryption Standard AES [403, 404]). The binary plaintext is then divided into blocks of the same size, several blocks being encrypted with the same key (and a public data called initial vector being changed more often). It can also be treated in a *stream cipher* [463], through the addition, most often mod 2, of a *keystream* of the same size as the plaintext, output by a pseudorandom generator (PRG) parameterized by a secret key (the keystream can be produced symbol by symbol, or block by block when the PRG uses a block cipher in a proper mode[4]). A quality of stream ciphers is to avoid error propagation, which gives them an advantage in applications where errors may occur during the transmission.
The ciphertext can be decrypted in the case of block ciphers by inverting the process and in the case of stream ciphers by the same bitwise addition of the keystream, which gives back the plaintext. Stream ciphers are also meant to be faster and to consume less electric power (which makes them adapted to cheap embedded devices). The triple constraint of being lightweight and fast while ensuring security is a difficult challenge for stream ciphers, all the more since they do not have the advantage of involving several rounds like block ciphers (their security is dependent on the PRG, only). And the situation is nowadays still more difficult because modern block ciphers like the AES are very fast. This difficulty has been illustrated by the failure of all six stream ciphers submitted to the 2000-2003 NESSIE project (New European Schemes for Signatures, Integrity and Encryption) [901], whose purpose was to identify secure cryptographic prim-

[3] Or equivalently finds with high probability a specific entry in an unsorted database of N entries.

[4] Note however that stream ciphers are often supposed to be used on lighter devices than block ciphers (typically not needing crypto-processors for instance).

itives. NESSIE has then been followed by the contest eSTREAM [495] organized later, between 2004 and 2008, by the EU ECRYPT network.

As mentioned in [242], the price to pay for the three constraints described above is that security proofs hardly exist for efficient stream ciphers as they do for block ciphers. This is a drawback of stream ciphers, compared to block ciphers[5]. The only practical possibility for verifying the security of efficient stream ciphers (in particular, the unpredictability of the keystream they generate) is to prove that they resist the known attacks. It is then advisable to include some amount of randomness in them, so as to increase the probability of resisting future attacks.[6]

Proving the security of a cipher consists in reducing it to the intractability of a hard problem (a problem which has been extensively addressed by the academic community, and for which only algorithms of exponential or sub-exponential complexity could be found), implying that any potential attack on it could be used for designing an efficient algorithm (whose worst-case complexity would be polynomial in the size of its input) solving the hard problem.

Note that provably secure stream ciphers do exist (some proposals are even unconditionally secure, that is, are secure even if the attacker has unlimited computational power, but limited storage or access); see for instance the proposals by Alexi-Chor-Goldreich-Schnorr (whose security is reducible to the intractability of the RSA problem) or Blum-Blum-Shub [98] (whose security is reducible to the intractability of the quadratic residue problem modulo pq, where p and q are large primes), or the stream cipher QUAD [61] (based on the iteration of a multivariate quadratic system over a finite field, and whose security is reducible to the intractability of the so-called MQ problem). But they are too slow and too heavy for being used in practice. Even in the case of QUAD, which is the fastest, the encryption speed is lower than for the AES. And this is still worse when unconditionally security is ensured. This is why the stream ciphers using Boolean functions (see below) are still much used and studied.

1.2 Error correcting codes

The objective of *error detecting / correcting codes* in *coding theory* is to enable digital communication over a noisy channel, in such a way that the errors of transmission can be detected by the receiver and, in the case of error correcting codes, corrected. General references are [63, 780, 809, 959]. Shannon's paper [1033] is also prominent.

Without correction, when an error is detected, the information needs to be requested again by the receiver and sent again by the sender (such procedure is

[5] However, the security of block ciphers is actually proved under simplifying hypotheses, and it has been said by Lars Knudsen that “what is provably secure is probably not”.

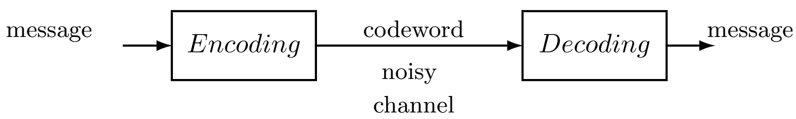
[6] Some stream cipher proposals like the Toyocrypt, LILI-128 and SFINKS ciphers, learned this at their own expense, see [387].

called an “Automatic Repeat reQuest” ARQ). This is what happened with the first computers: working with binary words, they could detect only one error (one bit) in the transmission of $(x_1, \ldots, x_k)$, by adding a parity bit $x_{k+1} = \bigoplus_{i=1}^{k} x_i$ (this transformed the word of length k into a word of length $k+1$ having even *Hamming weight*, *i.e.* even number of nonzero coordinates, which was then sent over the noisy channel; if an error occurred in the transmission, then, assuming that only one could occur, this was detected by the fact that the received word had odd Hamming weight).

With correction, the ARQ is not necessary, but this needs in practice that less errors have occurred than for detection (see below). Hybrid coding techniques exist then which make a trade-off between the two approaches.

The aim of error detection/correction is achieved by using an encoding algorithm which transforms the information (assumed to be a sequence over some alphabet $\mathcal{A}$) before sending it over the channel. In the case of block coding[7], the original sequence (the *message*) is treated as a list of vectors (words) of the same length – say k – called *source vectors* which are encoded into *codewords* of a larger length – say[8] n. If the alphabet over which the words are built is the field $\mathbb{F}_2$ of order 2, we say that the code is binary. If the code is not binary, then the symbols of the alphabet will have to be transformed into binary vectors before being sent over a binary channel.

Thanks to the length extension, called *redundancy*, the codewords sent over the channel are some of all possible vectors of length n. The set C of all codewords is called the *code* (for instance, in the case of the detecting codes using a parity bit as indicated above, the code is made of all binary words of length $n = k + 1$ and of even Hamming weights; it is called the *parity code*). The only information the receiver has, concerning the sent word, is that it belongs to C.



1.2.1 Detecting and correcting capacities of a code

The decoding algorithm of an error-detecting code is able to recognize if a received vector is a codeword. This makes possible to detect errors of transmission if (see [585]), denoting by d the minimum *Hamming distance* between codewords,

[7] We shall not address convolutional coding here.

[8] When dealing with Boolean functions, the symbol n will be often devoted to their number of variables; the length of the codes they will constitute will then not be n but $N = 2^n$, see Section 1.3.

i.e. the minimum number of positions at which codewords differ (called the *minimum distance* of the code), no more than $d-1$ coordinates of the received vector differ from those of the sent codeword (condition for having no risk that a codeword different from the sent one be received and then accepted). In the case of an error-correcting code, the decoding algorithm can additionally correct the errors of transmission, if their number is smaller than or equal to the so-called *correction capacity* of the code. This capacity equals $e = \lfloor \frac{d-1}{2} \rfloor$, where "$\lfloor\ \rfloor$" denotes the integer part (and so, roughly, a code can detect twice more errors than it can correct), since the condition for having no risk that a vector corresponds, as received vector, to more than one sent codeword with at most t errors of transmission in each case is that $2t < d$. Indeed, in order to be always able (theoretically) to recover the correct codeword, we need that, for every word y at distance at most t from a codeword x, there does not exist another codeword x' at distance at most t from y, and this is equivalent to saying that the Hamming distance between any two different codewords is larger than or equal to $2t+1$:
- if there exist a vector y and two codewords x and x' at Hamming distance at most t from y, then we have $d \leq 2t$ by the triangular inequality on distances,
- conversely, if there exist two codewords x and x' at Hamming distance $\delta \leq 2t$ from each other, then there exists a vector y such that $d_H(x,y) \leq t$ and $d_H(x',y) \leq t$ (let I be the set of positions where x and x' coincide; take $y_i = x_i$ when $i \in I$ and among the δ others, take for instance $\lfloor \frac{\delta}{2} \rfloor$ coordinates of y equal to those of x and the $\lceil \frac{\delta}{2} \rceil$ others equal to those of x').

In practice, determining d and then $e = \lfloor \frac{d-1}{2} \rfloor$ and showing that they are large is not sufficient. We still need to have an efficient decoding algorithm to recover the sent codeword. The naive method consisting in visiting all codewords and keeping the nearest one from the received word is inefficient because the number 2^k of codewords is too large, in general. Determining the nearest codeword from a received vector is called *maximum likelihood decoding*.

The correction capacity e is limited by the *Hamming bound* (or *sphere-packing bound*): since all the balls $B(x,e) = \{y \in \mathcal{A}^n; d_H(x,y) \leq e\}$, of radius e and centered in codewords, are pairwise disjoint, and since there are $|C|$ of them, the size of their union equals $|C| \sum_{i=0}^{e} \binom{n}{i} (q-1)^i$, where q is the size of alphabet. This union is a subset of $\mathcal{A}^n$. This implies:

$$|C| \sum_{i=0}^{e} \binom{n}{i} (q-1)^i \leq q^n.$$

The codes which achieve this bound with equality are called *perfect codes*.

Puncturing, shortening and extending codes

The *punctured code* of a code C is the set of vectors obtained by deleting the coordinate at some fixed position i in each codeword of C; we shall call such transformation *puncturing at position i*. This operation can be iterated and we shall still speak of puncturing a code when deleting the codeword coordinates at

several positions.
The *shortened code* of a code C is the set of vectors obtained by keeping only those codewords whose i-th coordinate is null and deleting this i-th coordinate.
The *extended code* of a code C over an additive group is the set of vectors, say, $(c_0, c_1, \ldots, c_n)$, where $(c_1, \ldots, c_n) \in C$ and $c_0 = -(c_1 + \cdots + c_n)$. Note that the extended code of C equals the intersection of the code $\{(c_0, c_1, \ldots, c_n) \in \mathbb{F}_q;\ (c_1, \ldots, c_n) \in C\}$ and of the parity code $(c_0, c_1, \ldots, c_n) \in \mathbb{F}_q;\ \sum_{i=0}^{n} x_i = 0\}$.

1.2.2 Parameters of a code

Sending words of length n over the channel instead of words of length k slows down the transmission of information in the ratio of $\frac{k}{n}$. This ratio, called the *transmission rate*, must be as high as possible, for a given correction capacity, to make possible fast communication. As we see, the three important parameters of a code C are n, k, d (or equivalently $n, |C|, d$ since if q is the alphabet's size, we have $|C| = q^k$), and the first aim[9] of algebraic coding is to find codes minimizing n, maximizing k and maximizing d, for diverse ranges of parameters corresponding to the needs of communication (see tables of best known codes in [570]). It is easily seen that $k \leq n - d + 1$ (this inequality, valid for any code over any alphabet, is called the *Singleton bound*) since erasing the coordinates of all codewords at $d - 1$ fixed positions gives a set of q^k distinct vectors of length $n - d + 1$ where q is the size of the alphabet, and the number of all vectors of length $n-d+1$ equals q^{n-d+1}. Codes achieving the Singleton bound with equality are called *maximum distance separable MDS*. In the case of binary linear codes (see below), it can be shown by using the Pless identities (see *e.g.* [348]) that MDS codes have dimension at most 1 or at least $n - 1$ and, except for such codes, the bound becomes then $k \leq n - d$.
Another important parameter is the *covering radius*, which is the smallest integer ρ such that the spheres of (Hamming) radius ρ centered at the codewords cover the whole space. In other words, it is the minimal integer ρ such that every vector of length n lies at Hamming distance at most ρ from at least one codeword, that is, the maximum number of errors to be corrected when maximum likelihood decoding (see page 20) is used. The book [375] is devoted to its study.
The *sphere covering bound* is the lower bound on the covering radius ρ which expresses that, by definition, the balls $B(x, \rho) = \{y \in \mathcal{A}^n; d_H(x, y) \leq \rho\}$, of radius ρ and centered in codewords, cover the whole space $\mathcal{A}^n$:

$$|C| \sum_{i=0}^{\rho} \binom{n}{i} (q-1)^i \geq q^n.$$

[9] The second aim is to find decoding algorithms for the codes found.

1.2.3 Linear codes

The general class of linear codes gives a simple and wide example of codes and how they can be used in error correction.

Definition 1 *A code is called a* linear code *if its alphabet is a finite field* $\mathbb{F}_q$ *(where* q *is the power of a prime) and if it has the structure of an* $\mathbb{F}_q$*-linear subspace of* $\mathbb{F}_q^n$*, where* n *is its length (see [809]).*

A code which is not necessarily linear is called an *unrestricted code*. The minimum distance of a linear code equals the minimum Hamming weight of all nonzero codewords, since the Hamming distance between two vectors equals the Hamming weight of their difference. We shall write that a linear code[10] over $\mathbb{F}_q$ is an $[n, k, d]_q$*-code* (and if the value of q is clear from the context, an $[n, k, d]$*-code*) if it has length n, dimension k and minimum distance d. The translates of a linear code are called its *coset*s and the elements of minimum Hamming weights in these cosets are called *coset leader*s (there may exist several in some cosets).

Generator matrix

Any linear code can be described by a *generator matrix* G, obtained by choosing a basis of this vector space and writing its elements as the rows of this matrix. The code equals the set of all the vectors of the form $u \times G$, where u ranges over $\mathbb{F}_q^k$ (and $\times$ is the matrix product) and a possible encoding algorithm is therefore the mapping $u \in \mathbb{F}_q^k \mapsto u \times G \in \mathbb{F}_q^n$. When the codeword corresponding to a given source vector u is obtained by inserting so-called *parity check coordinates* in the source vector (whose coordinates are then called *information coordinates*), the code is called *systematic* (it equals then the *graph* $\{(x, F(x)), x \in \mathbb{F}_q^k\}$ of a function, up to coordinate permutation). The corresponding generator matrix is then called a *systematic generator matrix* and has the form $[I_k : M]$ where I_k is the $k \times k$ identity matrix, up to column permutation. It is easily seen that every linear code has such generator matrix: any generator matrix (of rank k) has k linearly independent columns, and if we place these columns at the k first positions, we obtain $G = [A : M]$ where A is a nonsingular $k \times k$ matrix; then $A^{-1} \times G = [I_k : A^{-1} \times M]$ is a systematic generator matrix of the permuted code (since the multiplication by A^{-1} transforms a basis of the permuted code into another basis of the permuted code).

Dual code and parity check matrix

The generator matrix is well suited for generating the codewords, but it is not for checking if a received word of length n is a codeword or not. A characterization of the codewords is obtained thanks to the generator matrix H of the *dual code* $C^\perp = \{x \in \mathbb{F}_q^n; \forall y \in C,\ x \cdot y = \sum_{i=1}^n x_i\, y_i = 0\}$ (such a matrix is called a *parity check matrix* and "$\cdot$" is called the *usual inner product*, or scalar product,

[10] The square brackets around n, k, d specify that the code is linear, contrary to standard parentheses.

in $\mathbb{F}_q^n$): we have $x \in C$ if and only if $x \times H^t$ is the null vector. Consequently, the minimum distance of any linear code equals the minimum number of $\mathbb{F}_q$-linearly dependent columns in one of its parity check matrices (any one). For instance, the binary *Hamming code* of length $n = 2^m - 1$, which has by definition for parity check matrix the $m \times (2^m - 1)$ binary matrix whose columns are all the nonzero vectors of $\mathbb{F}_2^m$ in some order, has minimum distance 3. This code, which by definition is unique up to equivalence, has played an important historical role since it is the first perfect code found. It still plays a role since many computers use it to detect errors in their internal communications. It is the basis on which were built BCH and Reed-Muller codes (see pages 25 and 174). It depends on the choice of the order, but we say that two codes over $\mathbb{F}_q$ are *equivalent codes* if they are equal, up to some permutation of the coordinates of their codewords (and, for nonbinary codes, to the multiplication of each coordinate in each codeword by a nonzero element of $\mathbb{F}_q$ depending only on the position of this coordinate). Note that such codes have the same parameters.
The dual of the binary Hamming code is called the *simplex code*. A generator matrix of this code being the parity check matrix of the Hamming code described above, and the rows of this matrix representing then the *coordinate functions* in $\mathbb{F}_2^m$ (sometimes called dictator functions), on which the order chosen for listing the values is given by the columns of the matrix, the codewords of the simplex code are the lists of values taken on $\mathbb{F}_2^m \setminus \{0_m\}$ by all linear functions.
Note that the dual of a linear code C permuted by some bijection over the indices equals $C^\perp$ permuted by this same bijection, and that, if $G = [I_k : M]$ is a systematic generator matrix of a linear code C, then $[-M^t : I_{n-k}]$ is a parity check matrix of C, where M^t is the transposed matrix of M.
The linear codes which are supplementary with their duals (or equivalently which have trivial intersection with their duals since the dimensions of a code and of its dual are complementary to n) are called complementary dual codes (LCD) and will play an important role in Subsection 12.1.5.

The advantages of linearity

Linearity allows considerably simplifying some main issues about codes. Firstly, the minimum distance being equal to the minimum nonzero Hamming weight, computing it (if it cannot be determined mathematically) needs only to visit $q^k - 1$ codewords instead of $\frac{q^k(q^k-1)}{2}$ pairs of codewords. Secondly, the knowledge of the code is provided by a $k \times n$ generator matrix and needs then the description of k codewords instead of all q^k codewords. Thirdly, a general decoding algorithm is valid for every linear code. This algorithm is not efficient in general but it gives a framework for the efficient decoding algorithms which will have to be found for each class of linear codes. The principle of this algorithm is as follows: let y be the (known) received vector corresponding to the (unknown) sent codeword x. We assume that there has been at most $d - 1$ errors of transmission, where d is the minimum distance, if the code is used for error detection, and at most e errors of transmission, where $e = \lfloor (d-1)/2 \rfloor$ is the correcting capacity of the code, if the code is used for error correction. The error detection is made by checking if the so-called *syndrome* $s = y \times H^t$ is the zero vector. If it is not, then denoting by ϵ the so-called (unknown) *error vector* $\epsilon = y - x$, correcting the errors of transmission is equivalent to determining ϵ. This can be done by visiting all vectors z of Hamming weight at most e in $\mathbb{F}_q^n$ and checking if $z \times H^t = s$ (indeed, by linearity of matrix multiplication, the syndrome of the error vector equals the syndrome of the received vector, which is known). There exists a unique z of Hamming weight at most e in $\mathbb{F}_q^n$ such that $z \times H^t = s$; this unique z equals ϵ.

Concatenating codes

Given an $\mathbb{F}_q$-linear $[n, k, d]$ code C (where n is the length, k is the dimension and d is the minimum distance), where $q = 2^e$, $e \geq 2$, a binary $[n', e, d']$ code C' and an $\mathbb{F}_2$-isomorphism $\phi : \mathbb{F}_q \mapsto C'$, the *concatenated code* C'' equals the $[nn', ke, d'' \geq dd']$ binary code $\{(\phi(c_1), \ldots, \phi(c_n));\ (c_1, \ldots, c_n) \in C\}$. Codes C and C' are respectively called outer code and inner code for this construction.

MDS linear codes

Let C be an $[n, k, d]$ code over a field K, let H be its parity check matrix and G its generator matrix. Then $n - k$ is the rank of H and we have then $d \leq n - k + 1$ since $n - k + 1$ columns of H are always linearly dependent and therefore any set of indices of size $n - k + 1$ contains the support of a nonzero codeword. This proves again the Singleton bound: $d \leq n - k + 1$.
Recall that C is called *MDS* if $d = n - k + 1$. The properties of MDS linear codes are:

1. C is MDS if and only if each set of $n - k$ columns of H has rank $n - k$.
2. If C is MDS, then $C^\perp$ is MDS.
3. C is MDS if and only if each set of k columns of G has rank k (and their positions constitute then an information set, see page 185).

Other properties of linear codes

Puncturing, shortening and extending codes preserve their linearity. Puncturing preserves the MDS property (if $n > k$).
The following lemma will be useful when dealing with Reed-Muller codes in Chapter 4 :

Lemma 1 *Let C be a linear code of length n over $\mathbb{F}_q$ and $\hat{C}$ its extended code. We have $\hat{C}^\perp = \{(y_0, \ldots, y_n) \in \mathbb{F}_q^{n+1};\ (y_1 - y_0, \ldots, y_n - y_0) \in C^\perp\}$.*

Proof. We have $\hat{C}^\perp = \{(y_0, \ldots, y_n) \in \mathbb{F}_q^{n+1};\ \forall (x_1, \ldots, x_n) \in C,\ y_0(-\sum_{i=1}^n x_i) + \sum_{i=1}^n x_i y_i = 0\} = \{(y_0, \ldots, y_n) \in \mathbb{F}_q^{n+1};\ \forall (x_1, \ldots, x_n) \in C,\ \sum_{i=1}^n x_i(y_i - y_0) = 0\} = \{(y_0, \ldots, y_n) \in \mathbb{F}_q^{n+1};\ (y_1 - y_0, \ldots, y_n - y_0) \in C^\perp\}$. $\square$

Uniformly packed codes:
These codes will play a role with respect to APN functions, at page 414.

Definition 2 *[50] Let C be any binary code of length N, with minimum distance $d = 2e + 1$ and covering radius ρ. For any $x \in \mathbb{F}_2^N$, let us denote by $\zeta_j(x)$ the number of codewords of C at distance j from x. The code C is called a* uniformly packed code, *if there exist real numbers h_0, $h_1, \ldots,$ h_ρ such that, for any $x \in \mathbb{F}_2^N$, the following equality holds*

$$\sum_{j=0}^{\rho} h_j \ \zeta_j(x) = 1.$$

As shown in [51], this is equivalent to saying that the covering radius of the code equals its external distance (*i.e.* the number of different nonzero distances between the codewords of its dual).

1.2.4 Cyclic codes

2-error correcting Bose-Chaudhuri-Hocquenghem (BCH) codes
The binary Hamming code of length $n = 2^m - 1$ has dimension $n - m$ and needs m parity check bits for being able to correct 1 error. It happens that 2-error binary correcting codes can be built with $2m$ parity check bits. Let us denote by $W_1, \ldots, W_n$ the nonzero binary vectors of length m written as columns in some order. The parity check matrix of the Hamming code of length $n = 2^m - 1$ is:

$$H = [W_1, \ldots, W_n].$$

To find a 2-error correcting code C of the same length, we consider the codes whose parity check matrices H' are the $2m \times n$ matrices whose m first rows are those of H. These codes being subcodes of the binary Hamming code, they are at least 1-error correcting. For each such matrix H', there exists a function F from $\mathbb{F}_2^m$ to itself such that:

$$H' = \begin{bmatrix} W_1 & W_2 & \ldots & W_n \\ F(W_1) & F(W_2) & \ldots & F(W_n) \end{bmatrix}.$$

Note that, when F is a *permutation* (*i.e.* is bijective), the code of generator matrix H' is a so-called *double simplex code* (and plays a central role in [136]); it is the direct sum of two simplex codes: the standard one and its permutation by F.
Going back to general F, assume that two errors are made in the transmission of a codeword of C, at indices $i \neq j$. The *syndrome* of the received vector equals that of the error vector, that is:

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} W_i \\ F(W_i) \end{bmatrix} + \begin{bmatrix} W_j \\ F(W_j) \end{bmatrix},$$

with $S_1 \neq 0_m$ (where 0_m is the length m all-zero vector) since $i \neq j$. We have then:

$$\begin{cases} W_i + W_j &= S_1 \neq 0_m \\ F(W_i) + F(W_j) &= S_2 \end{cases}.$$

The code is then 2-error correcting if and only if, for every $S_1, S_2 \in \mathbb{F}_2^m$ such that $S_1 \neq 0_m$, this system of equations has either no solution (i,j) (which happens when $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ is not the syndrome of an error vector of Hamming weight 2) or only two solutions (one solution if we impose $i < j$).
Note that since $\{W_1, \ldots, W_n\}$ equals $\mathbb{F}_2^m \setminus \{0_m\}$ and these vectors are all distinct, it is equivalent to consider the system:

$$\begin{cases} x + y &= S_1 \neq 0_m \\ F(x) + F(y) &= S_2 \end{cases},$$

where x and y range over $\mathbb{F}_2^m \setminus \{0_m\}$. This is where finite fields of orders larger than 2 played a historical role in coding theory (see Appendix, page 521, for a description of finite fields): considering such functions F and such systems of equations is easier when we have a structure of field (even though the equations do not involve multiplications). This allows indeed to take $F(x)$ in a polynomial form, and the first polynomials to be tried are of course monomials. The monomials x and x^2, being linear functions, do not satisfy the condition needed for the code to be 2-error correcting, but the next monomial x^3 does satisfy it (this is easily seen since $x^3 + y^3 = (x+y)^3 + x\,y\,(x+y)$ implies that the system is equivalent to $\begin{cases} x + y &= S_1 \neq 0 \\ x\,y &= \frac{S_2 + S_1^3}{S_1} \end{cases}$ and such equation results in an equation of degree 2 which has at most 2 solutions over a finite field).
The condition on F (or more precisely on its extension by taking $F(0) = 0$) is equivalent to saying that it is an almost perfect nonlinear (APN) function. This notion plays a very important role in cryptography, see Chapter 11, page 400.

We need here the notion of *primitive element*, see page 528. Such element α satisfies that $\mathbb{F}_{2^n} = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{2^n-2}\}$ and exists for every n.

Definition 3 *Let α be a primitive element of $\mathbb{F}_{2^m}$. The binary 2-error correcting BCH code of length $n = 2^m - 1$ is the $[n, n-2m, 5]$ code due to Bose, Chaudhuri and Hocquenghem, of parity check matrix:*

$$H' = \begin{bmatrix} \alpha & \alpha^2 & \ldots & \alpha^n \\ \alpha^3 & \alpha^6 & \ldots & \alpha^{3n} \end{bmatrix}.$$

Ordering the elements of $\mathbb{F}_{2^n}^*$ as $\alpha, \alpha^2, \ldots, \alpha^{n-1}, \alpha^n = 1$ (we could have also chosen $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$) implies a property which does not seem so important at first glance but which played a central role in the history of codes and still plays such role nowadays: the code is (globally) invariant under cyclic permutations of the codeword coordinates. This property, when added to the linearity of the code, confers to them a structure of principal ideal, with very nice theoretical and practical consequences.

General cyclic codes

A linear code C of length n is a *cyclic code* if it is (globally) invariant under cyclic shifts of the codeword coordinates (see [809, page 188]). For this, it is enough that it is invariant under one of the primitive cyclic shifts, for instance:

$$(c_0, \ldots, c_{n-1}) \mapsto (c_{n-1}, c_0, \ldots, c_{n-2}).$$

Cyclic codes have been extensively studied in coding theory, because of their strong properties.

Representation of codewords

Each codeword $(c_0, \ldots, c_{n-1})$ is represented by the polynomial $c_0 + c_1X + \cdots + c_{n-1}X^{n-1}$, viewed as an element of the quotient algebra $A = \mathbb{F}_q[X]/(X^n - 1)$ (each element of this algebra is an equivalence class modulo $X^n - 1$, which will be always represented by its unique element of degree at most $n - 1$, equal to the common rest in the division by $X^n - 1$ of the polynomials constituting the class). We shall call $c_0 + c_1X + \cdots + c_{n-1}X^{n-1}$ the *polynomial representation of codeword* $(c_0, \ldots, c_{n-1})$. Then it is easily shown that C is cyclic if and only if it is an ideal of $\mathbb{F}_q[X]/(X^n - 1)$, that is, satisfies $fC \subseteq C$ for every nonzero $f \in A$ (C being assumed linear, it is a subgroup of A).

Generator polynomial

The algebra $\mathbb{F}_q[X]/(X^n - 1)$ is a principal domain. It is easily shown that any (linear) cyclic non-trivial[11] code has a unique monic element $g(X)$ (whose leading coefficient equals 1) having minimal degree, which generates the ideal and is called the *generator polynomial* of the code. In fact, $g(X)$ is a generator of the code in the strong sense that every polynomial of degree at most $n - 1$ is a codeword if and only if it is a multiple of $g(X)$ in $\mathbb{F}_q[X]$ (which implies that it is a multiple of $g(X)$ in $\mathbb{F}_q[X]/(X^n - 1)$). The code equals then the set of all those polynomials which include the zeros of $g(X)$ (in the splitting field of $g(X)$) among their own zeros. It is also easily seen that $g(X)$ is a divisor of $X^n - 1$.

Zeros of the code

In our framework, the length will have the form $n = q^m - 1$ (we call such length a *primitive length*). In such case, since $g(X)$ divides $X^n - 1$, the zeros of $g(X)$ all belong to $\mathbb{F}^*_{q^m}$. The generator polynomial having all its coefficients in $\mathbb{F}_q$, its zeros are of the form $\{\alpha^i, i \in I\}$ (where α is a primitive element of $\mathbb{F}_{q^m}$), where $I \subseteq \mathbb{Z}/n\mathbb{Z}$ is a union of cyclotomic classes of q modulo $n = q^m - 1$ (and *vice versa*). The set I is called the *defining set* of the code. The elements α^i, $i \in I$ are called the *zeros of the cyclic code*, which has dimension $n - |I|$. The elements α^i,

[11] *i.e.* different from $\{0_n\}$; in fact, we shall consider that the trivial cyclic code has also a generator polynomial: $X^n - 1$ itself.

$i \in \mathbb{Z}/n\mathbb{Z} \setminus I$ are called the *nonzeros of the cyclic code*. The generator polynomial of $C^\perp$ is the reciprocal of the quotient of $X^n - 1$ by $g(X)$, and its defining set therefore equals $\{n - i;\, i \in \mathbb{Z}/n\mathbb{Z} \setminus I\}$.
McEliece's theorem [833] states that a binary cyclic code is exactly 2^l-divisible (that is, l is maximum such that all codeword Hamming weights are divisible by 2^l) if and only if l is the smallest number such that $l + 1$ nonzeros of C (with repetitions allowed) have product 1 (and recall that $\alpha^j = 1$ if and only if $2^n - 1$ divides j).

Generating all cyclic codes of some primitive length

Since a polynomial over $\mathbb{F}_q$ is the generator polynomial of a cyclic code of length n if and only if it divides $X^n - 1$, we obtain all cyclic codes from all the divisors of $X^n - 1$ in $\mathbb{F}_q$. Any such divisor is the product of some irreducible factors of $X^n - 1$ in $\mathbb{F}_q$. These irreducible factors are the polynomials of the form $\prod_{j \in \mathcal{C}}(X - \alpha^j)$, where $\mathcal{C}$ is a cyclotomic class of q modulo n. The number of cyclic codes of length n over $\mathbb{F}_q$ is then 2^r where r is the number of these cyclotomic classes (including the trivial cyclic code $\{0_n\}$ and the full one $\mathbb{F}_q^n$). The *Hamming code* has for generator polynomial the irreducible polynomial corresponding to the cyclotomic class containing 1. Its dual, the *simplex code*, has then for generator polynomial the polynomial corresponding to all cyclotomic classes except that of $n - 1$.

Non-primitive length

If the length is not primitive, the zeros of $X^n - 1$ live in its *splitting field* $\mathbb{F}_{q^m}$ (where n divides $q^m - 1$, and m is minimal). If n and q are co-prime, the zeros of $X^n - 1$ are simple since the derivative nX^{n-1} of this polynomial does not vanish on them, and the same theory applies by replacing $\mathbb{F}_{q^m}$ by the group of n-th roots of unity in $\mathbb{F}_{q^m}$ and α by a primitive n-th root of unity.

BCH bound

A very efficient bound on the minimum distance of cyclic codes is the *BCH bound* [809, page 201]: if I contains a "string" $\{l+1, \dots, l+\delta-1\}$ of length $\delta - 1$ of consecutive[12] elements of $\mathbb{Z}/n\mathbb{Z}$, then the cyclic code has minimum distance larger than or equal to δ (which is then called the *designed distance* of the cyclic code). A proof of this bound (in the framework of Boolean functions) is given in the proof of Theorem 23, page 368.

BCH codes

Let n be co-prime with q and $\delta < n$, the BCH codes of length n and designed distance δ are the cyclic codes which have such string of length $\delta - 1$ in their zeros (and have then minimum distance at least δ, according to the BCH bound) and maximal dimension (*i.e.* minimal number of zeros) with such constraint.

[12] Considering of course that 0 is the successor of $n - 1$ in $\mathbb{Z}/n\mathbb{Z}$.

Reed-Solomon codes

When $n = q - 1$, the cyclotomic classes of q modulo n are singletons and the set of zeros of a cyclic code can then be any set of nonzero elements of the field (the generator polynomial can be any divisor of $X^n - 1$); when it is constituted of consecutive powers of a primitive element, this particular case of a BCH code is called a *Reed-Solomon code* (*RS code*). Reed-Solomon codes are important because they achieve the Singleton bound with equality (*i.e.* they are *maximum distance separable MDS*). Indeed, the BCH bound gives: $\delta \leq d \leq n - (n - (\delta - 1)) + 1 = \delta$ and the Singleton bound is then achieved with equality.

Remark. There exists another equivalent definition of Reed-Solomon codes, see the remark at page 62. RS codes are widely used in consumer electronics (CD, DVD, Blu-ray), data transmission technologies (DSL, WiMAX), broadcast systems, computer applications, and deep-space communications. □

Extended Reed-Solomon codes

A cyclic code C of length n being given, recall that the extended code of C is the set of vectors $(c_\infty, c_0, \dots, c_{n-1})$, where $c_\infty = -(c_0 + \cdots + c_{n-1})$. It is a linear code of length $n+1$ and of the same dimension as C. When C is a Reed-Solomon code whose defining set has the form $\{1, 2, \dots, \delta - 1\}$, its extended code is also MDS, because when $(c_0, \dots, c_{n-1})$ is a codeword of C of minimal Hamming weight δ, we have $c_\infty \neq 0$ (again according to the BCH bound: if $c_\infty = 0$, then the polynomial $c_0 + c_1 X + \cdots + c_{n-1} X^{n-1}$ has also $\alpha^0 = 1$ for zero and has then Hamming weight at least $\delta + 1$, thanks to the BCH bound applied with the string $\{0, \dots, \delta - 1\}$). Hence, either $(c_0, \dots, c_{n-1})$ is a codeword of C of minimal Hamming weight δ and then $(c_\infty, c_0, \dots, c_{n-1})$ has Hamming weight $\delta + 1$ or $(c_0, \dots, c_{n-1})$ has Hamming weight at least $\delta + 1$ and $(c_\infty, c_0, \dots, c_{n-1})$ has *a fortiori* Hamming weight at least $\delta + 1$. Hence the minimum distance of the extended code is $\delta + 1 = (n+1) - (n - \delta + 1) + 1$. The extended code is MDS.

Cyclic codes and Boolean functions

Cyclic codes over $\mathbb{F}_2$ and of length $2^m - 1$ can be viewed as sets of m-variable Boolean functions. Indeed, any codeword in such cyclic code with defining set I can be represented in the form $\sum_{i=1}^{l} tr_n(a_i x^{-u_i}), a_i \in \mathbb{F}_{2^m}$, where $u_1, \dots, u_l$ are representatives of the cyclotomic classes lying outside I (see Relation (2.20) in Subsection 2.2.2, page 62).

1.2.5 The MacWilliams identity and the notion of dual distance

Linear codes

A nice relationship, due to F. J. MacWilliams [809, page 127], exists between the Hamming weights in every binary linear code[13] and those in its dual: let C

[13] It exists for every linear code over a finite field and even for more general codes, but we shall need it only for binary codes.

be any binary linear code of length n; consider the polynomial $W_C(X,Y) = \sum_{i=0}^{n} A_i X^{n-i} Y^i$ where A_i is the number of codewords of Hamming weight i. This polynomial is called the *weight enumerator* of C and describes[14] the *weight distribution* $(A_i)_{0\leq i\leq n}$ of C. Then:

$$W_C(X+Y, X-Y) = |C|\, W_{C^\perp}(X,Y). \tag{1.1}$$

We give a sketch of proof[15] of this *MacWilliams' identity*: we observe first that $W_C(X,Y) = \sum_{x\in C}\prod_{i=1}^{n} X^{1-x_i}Y^{x_i}$; substituting X by $X+Y$ and Y by $X-Y$, we deduce that $W_C(X+Y,X-Y) = \sum_{x\in C}\prod_{i=1}^{n}(X+(-1)^{x_i}Y)$. We apply then the classical relation making possible to expand products of sums: for every $\lambda_1,\dots,\lambda_n,\mu_1,\dots,\mu_n$, we have $\prod_{i=1}^{n}(\lambda_i+\mu_i) = \sum_{b\in\mathbb{F}_2^n}\prod_{i=1}^{n}(\lambda_i^{1-b_i}\mu_i^{b_i})$ (indeed, choosing λ_i in the i-th factor when $b_i = 0$ and μ_i when $b_i = 1$, provides when b ranges over $\mathbb{F}_2^n$ all the possible terms in the expansion). This gives here: $W_C(X+Y,X-Y) = \sum_{x\in C}\sum_{b\in\mathbb{F}_2^n}\prod_{i=1}^{n}\left(X^{1-b_i}((-1)^{x_i}Y)^{b_i}\right)$. We obtain then $W_C(X+Y,X-Y) = \sum_{b\in\mathbb{F}_2^n}\left(X^{n-w_H(b)}Y^{w_H(b)}\sum_{x\in C}(-1)^{b\cdot x}\right)$, where "$\cdot$" is the *usual inner product* in $\mathbb{F}_2^n$, and we conclude by observing that, if $b\notin C^\perp$, then the linear form $b\cdot x$ over the vector space C is nonzero, and takes then values 0 and 1 on two complementary hyperplanes, that is, the same number of times (we will find again this in Relation (2.38), page 77). This proves Relation (1.1). Of course, we deduce that $W_C(X,Y) = \frac{1}{|C^\perp|}W_{C^\perp}(X+Y,X-Y)$ and the same method shows, as observed in [37], that for every coset $a+C$, we have $W_{a+C}(X,Y) = \frac{1}{|C^\perp|}\left(2\,W_{C^\perp\cap\{0_n,a\}^\perp}(X+Y,X-Y) - W_{C^\perp}(X+Y,X-Y)\right)$.

Remark. We have $|C| = \sum_{i=0}^{n} A_i = W_C(1,1)$. The fact that the polynomial $\frac{1}{W_C(1,1)}W_C(X+Y,X-Y)$ has non-negative integer coefficients is very specific (among all homogeneous polynomials $P(X,Y)$ whose coefficients are non-negative integers). As far as we know, the characterization of all homogeneous polynomials $P(X,Y)$ over $\mathbb{N}$ such that $\frac{1}{P(1,1)}P(X+Y,X-Y)$ has non-negative integer coefficients has never been investigated in a paper. □

Remark. The average Hamming weight of the codewords of a linear binary code C equals $(W_C)'_Y(1,1)$ (the value at $(1,1)$ of the partial derivative of $W_C(X,Y)$ with respect to Y), divided by $|C|$. MacWilliams' identity writes $W_C(X,Y) = \frac{1}{|C^\perp|}W_{C^\perp}(X+Y,X-Y)$. Differentiating with respect to Y gives $(W_C)'_Y(X,Y) = \frac{1}{|C^\perp|}(W_{C^\perp})'_X(X+Y,X-Y) - \frac{1}{|C^\perp|}(W_{C^\perp})'_Y(X+Y,X-Y)$ and thus $(W_C)'_Y(1,1) = \frac{1}{|C^\perp|}(W_{C^\perp})'_X(2,0) - \frac{1}{|C^\perp|}(W_{C^\perp})'_Y(2,0) = \frac{n2^{n-1}}{|C^\perp|} - \frac{1}{|C^\perp|}(W_{C^\perp})'_Y(2,0)$, and the average Hamming weight of codewords equals $\frac{n}{2} - 2^{-n}(W_{C^\perp})'_Y(2,0)$, which depends on the number of words of Hamming weight 1 in $C^\perp$ (see more in [809, page 131] on the moments of the weight distribution of codes) and is bounded above by $\frac{n}{2}$. In fact, it is easily seen directly that the average Hamming weight of codewords

[14] W_C is a homogeneous version of classical generating series for the weight distribution of C.

[15] The classical proof uses Fourier-Hadamard transform; since this transform will be addressed later in this book, in Section 2.3, we give a proof more coding theory oriented.

equals $\frac{n-r}{2}$, where r is the number of positions where all codewords are null, since if there is a codeword with 1 at position i, the average value of codewords at position i equals $\frac{1}{2}$. $\square$

Remark. Some authors call weight enumerator of C the univariate polynomial $A_C(Z) = \sum_{i=0}^{n} A_i Z^i$. Mac Williams' identity writes then $(1+Z)^n A_C\left(\frac{1-Z}{1+Z}\right) = |C|\, W_{C^\perp}(Z)$, where n is the length of the binary code C. $\square$

The MacWilliams identity gives information on self-dual codes (*i.e.* codes equal to their duals) through the *Gleason theorem* which says that the weight enumerator of a self-dual code is in the ring generated by $X^2 + Y^2$ and $XY - Y^2$ (see [809, page 602]).

Unrestricted codes

The principle of MacWilliams' identity can also be applied to unrestricted codes. When C is not linear, the weight distribution of C has no great relevance. The distance distribution has more interest. We consider the *distance enumerator* of C:

$$D_C(X,Y) = \frac{1}{|C|} \sum_{i=0}^{n} B_i X^{n-i} Y^i,$$

where B_i is the size of the set $\{(x,y) \in C^2;\, d_H(x,y) = i\}$. Note that, if C is linear, then $D_C = W_C$. Similarly as above, we see that:

$$D_C(X,Y) = \frac{1}{|C|} \sum_{(x,y)\in C^2} \prod_{i=1}^{n} X^{1-(x_i \oplus y_i)} Y^{x_i \oplus y_i};$$

we deduce that:

$$D_C(X+Y, X-Y) = \frac{1}{|C|} \sum_{(x,y)\in C^2} \prod_{i=1}^{n} (X + (-1)^{x_i \oplus y_i} Y).$$

Expanding these products by the same method as above, we obtain:

$$D_C(X+Y, X-Y) = \frac{1}{|C|} \sum_{(x,y)\in C^2} \sum_{b \in \mathbb{F}_2^n} \prod_{i=1}^{n} \left(X^{1-b_i} ((-1)^{x_i \oplus y_i} Y)^{b_i}\right),$$

that is:

$$D_C(X+Y, X-Y) = \frac{1}{|C|} \sum_{b\in\mathbb{F}_2^n} X^{n-w_H(b)} Y^{w_H(b)} \left(\sum_{x\in C} (-1)^{b\cdot x}\right)^2. \tag{1.2}$$

Hence, $D_C(X+Y, X-Y)$ has non-negative coefficients (but $D_C(X,Y)$ is not necessarily the weight enumerator of a code; note however that it is one in the case of distance-invariant codes, like Kerdock codes, see Section 6.1.22).

Definition 4 *The smallest nonzero exponent of Y with nonzero coefficient in the polynomial $D_C(X+Y, X-Y)$, that is, the number:*

$$\min\left\{w_H(b);\, b\neq 0_n, \sum_{x\in C}(-1)^{b\cdot x}\neq 0\right\},$$

often denoted by $d^\perp(C)$, is called the dual distance *of C.*

The dual distance of C is strictly larger than an integer t if and only if the restriction to C of any sum of at least one and at most t coordinate functions in $\mathbb{F}_2^n$ is *balanced* (*i.e.*, has uniform distribution), that is, any of the punctured codes of length t of C equals the whole vector space $\mathbb{F}_2^t$ and each vector in $\mathbb{F}_2^t$ is matched the same number of times[16]. Hence, as we shall see again at page 108, the size of a code of dual distance d is divisible by 2^{d-1}; note that for linear codes, this tells more than the Singleton bound applied to the dual.
This notion will play an important role with Boolean functions (see Definition 21, page 105; this is why we include Lemma 2 below)) and with a recent kind of cryptanalysis which plays an important role nowadays: side channel attacks (see Section 12.1, page 460).

Lemma 2 *1. Any coset $a+C$ of a binary unrestricted code has the same dual distance as C. Any union of cosets of a linear code C has at least the same dual distance as C.*

2. The dual distance of a punctured code is larger than or equal to the dual distance of the original code (assuming that the latter has minimum distance at least 2).

3. The dual distance of the Cartesian product of two binary unrestricted codes equals the minimum of their dual distances.

4. Let C_1 and C_2 be binary unrestricted codes of the same length n and

$$C'' = \{(c_1, c_1 + c_2); c_1\in C_1, c_2 \in C_2\},$$

then $d''^\perp = \min(d_1^\perp, 2\, d_2^\perp)$.

The proof of this lemma is also an easy consequence of the properties of the Fourier-Hadamard transform that we shall see in Section 2.3.

Remark. When C is linear, $d^\perp$ equals the minimum distance of the *dual code* $C^\perp$. Hence, since the minimum distance of a linear code over $\mathbb{F}_q$ equals the minimum nonzero number of $\mathbb{F}_q$-linearly dependent columns in its parity check matrix, its dual distance equals the minimum nonzero number of $\mathbb{F}_q$-linearly dependent columns in its generator matrix. □

[16] This is a consequence of the properties of the Fourier-Hadamard transform that we shall see in Section 2.3, applied to the indicator of C, see Corollary 6, page 108 and Theorem 5.

1.3 Boolean functions

We call *Boolean functions* (and sometimes we specify *n-variable* Boolean functions or *Boolean functions in dimension n*) the (single-output) functions from the n-dimensional vector space $\mathbb{F}_2^n$ over $\mathbb{F}_2$, to $\mathbb{F}_2$ itself. Their set is denoted by $\mathcal{BF}_n$. Number n will be named the number of variables, or of input bits. More generally[17], we call n-variable *pseudo-Boolean* functions the functions from $\mathbb{F}_2^n$ to $\mathbb{R}$.

Boolean functions will also be viewed in some cases as taking their input in the field $\mathbb{F}_{2^n}$. Indeed, this field is an n-dimensional vector space over $\mathbb{F}_2$ and it can then be identified with the vector space $\mathbb{F}_2^n$ through the choice of a basis.

Boolean functions play roles in both cryptographic and error correcting coding activities in *information protection*:
- every binary unrestricted code of length 2^n, for some positive integer n, can be interpreted as a set of Boolean functions, since every n-variable Boolean function can be represented by its truth table (an ordering of the set of binary vectors of length n being first chosen) and thus associated with a binary word of length 2^n, and *vice versa*; important codes (Reed-Muller, Kerdock codes, see Sections 4.1 and 6.1.22) can be defined this way as sets of Boolean functions;
- the role of Boolean functions in conventional cryptography is even more important: cryptographic transformations can be designed by appropriate composition of Boolean functions[18].

In both frameworks, n is rarely large, in practice:
- the error correcting codes derived from n-variable Boolean functions have length 2^n; so, taking $n = 10$ already gives codes of length 1024,
- for reason of efficiency, the Boolean functions used in stream ciphers had about 10 variables until algebraic attacks were invented in 2003, and the number of variables is now most often limited to at most 20, except when the functions are particularly fast to compute.

Despite their low numbers of variables, the Boolean functions used in cryptography and satisfying the desired conditions (see Section 3.1 below) can not be determined or studied by an exhaustive computer investigation: the number $|\mathcal{BF}_n| = 2^{2^n}$ of n-variable Boolean functions is too large when $n \geq 6$. We give in Table 1.1 below the values of this number for n ranging between 4 and 8. Assume that visiting an n-variable Boolean function, and determining whether it has the desired properties, needs one nano-second (10^{-9} seconds), then it would need millions of hours to visit all functions in 6 variables, and about one hundred billions times the age of the universe to visit all those in 7 variables. The number

[17] When we will consider Boolean functions as particular pseudo-Boolean functions, by viewing their output values 0 and 1 as elements of $\mathbb{Z}$ rather than $\mathbb{F}_2$ (for instance when defining their numerical normal form in Subsection 2.2.4 or their Fourier-Hadamard transform in Section 2.3), adding their values will be made in $\mathbb{Z}$, with notation $+$; otherwise, it will be made modulo 2, with notation $\oplus$.

[18] Boolean functions play also a role in hash functions, but we shall not develop this aspect, for lack of space, and in the inner protection of some chips.

n	4	5	6	7	8
$\|\mathcal{BF}_n\|$	2^{16}	2^{32}	2^{64}	2^{128}	2^{256}
$\approx$	$6 \cdot 10^4$	$4 \cdot 10^9$	10^{19}	10^{38}	10^{77}

Table 1.1 Number of n-variable Boolean functions

of 8-variable Boolean functions approximately equals the number of atoms in the whole universe! We see that trying to find functions satisfying the desired conditions by simply picking up functions at random is also impossible for these values of n, since visiting a non-negligible part of all Boolean functions in 7 or more variables is not feasible, even when parallelizing. The study of Boolean functions for constructing or studying codes or ciphers is essentially mathematical. But clever computer investigation is very useful to imagine or to test conjectures, and sometimes to generate interesting functions.

Remark. Boolean functions play an important role in computational complexity theory, with the notion of NP-complete decisional problem (where "NP" stands for "nondeterministic polynomial time"), for which satisfiability problems (in particular, the 3-SAT problem) are central. These problems are related to representations of Boolean functions by disjunctive and conjunctive normal forms, which do not ensure uniqueness and are not much used in cryptography and error correcting coding. We refer the reader interested in satisfiability problems and in the related complexity theory of Boolean functions to [31, 81, 1117]. □

A nice site under construction at the moment this book is written can be found at URL http://boolean.h.uib.no/mediawiki.

1.3.1 Boolean functions and stream ciphers

Stream ciphers are based on the so-called *Vernam cipher* (see Figure 1.1) in which the plaintext (a binary string of some length) is bitwise added to a (binary) secret key of the same length, in order to produce the ciphertext. The Vernam cipher is also called the *one time pad* because a new random secret key must be used for every encryption. Indeed, the bitwise addition of two ciphertexts corresponding to the same key equals the addition of the corresponding plaintexts, which gives much information on these plaintexts when they code for instance natural language (it is often enough to recover both plaintexts, even when one of them is reversed; some secret services and spies learned this at their own expense).

The Vernam cipher, which is the only known cipher offering unconditional security (see [1034]) if the key is truly random and if it is changed for every new encryption, was used for the communication between the heads of USA and USSR during the cold war (the keys being carried by diplomats) and by some secret services.

In practice (except in the very sensitive situations indicated above), since in

Figure 1.1 VERNAM CIPHER

the Vernam cipher, the length of the private key must be equal to the length of the plaintext (which is impractical), a so-called *pseudorandom generator* (*PRG*) is used for producing a long *pseudorandom sequence* (the *keystream*, playing the role of the private key in the Vernam cipher) from the short random secret key. Only the latter is actually shared[19]. The unconditional security is then no longer ensured (this is the price to pay for making the cipher lighter). If the keystream only depends on the key (and not on the plaintext), the cipher is called *synchronous*[20]. Stream ciphers, because they operate on data units as small as a bit or a few bits, are suitable for fast telecommunication applications. Having also a very simple construction, they are easily implemented both in hardware and software. They need to resist all known attacks (see in Section 3.1 those which are known so far). The so-called *attacker model* for these attacks (that is, the description of the knowledge the attacker is supposed to have) is as follows: some knowledge on the plaintext may be unavoidable and it is then assumed that the attacker has access to a small part of it. Since the keystream equals the XOR of the plaintext and the ciphertext, the attacker is then assumed to have access to a part of the keystream, and he/she needs to reconstruct the whole sequence.

A first method for generating pseudorandom sequences from secret keys has used *Linear Feedback Shift Registers* (*LFSR*) [550]. In such an LFSR (see Figure 1.2, where $\times$ means multiplication), at every clock-cycle, the bits $s_{n-1}, \ldots, s_{n-L}$ contained in the flip-flops of the LFSR move to the right. The right-most bit is the current output (a keystream of length N will then be produced after N clock-cycles) and the left-most flip-flop is fed with the linear combination $\bigoplus_{i=1}^{L} c_i s_{n-i}$, where the c_i's are bits. Thus, such an LFSR outputs a recurrent

[19] The PRG is supposed to be public since taking a part of the secret for describing it would reduce in practice the length of the key.

[20] There also exist self-synchronizing stream ciphers, in which each keystream bit depends on the private key and on the n preceding ciphertext bits, which makes it possible to re-synchronize after n bits if an error of transmission occurs between Alice and Bob

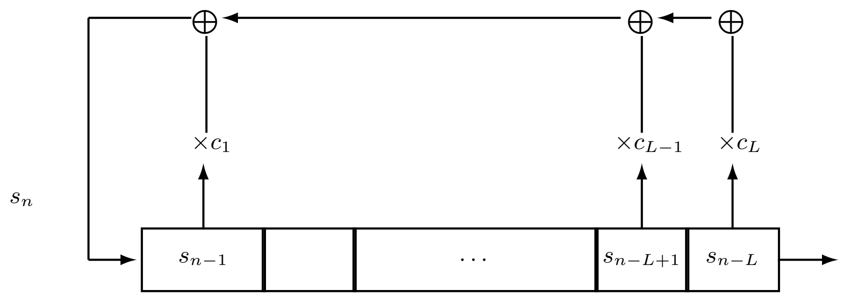


Figure 1.2 LFSR

sequence satisfying the relation

$$s_n = \bigoplus_{i=1}^{L} c_i s_{n-i}.$$

Such sequence is always ultimately periodic[21] (if $c_L = 1$, then it is periodic; we shall assume that $c_L = 1$ in the sequel, because otherwise, the same sequence can be output by an LFSR of a shorter length, except for its first bits, and this can be exploited in attacks) with period at most $2^L - 1$. The generating series $s(X) = \sum_{i\geq 0} s_i X^i$ of the sequence can be expressed in a nice way (see the chapter by Helleseth and Kumar in [959] and Section 10.2 "LFSR sequences and maximal period sequences" by Niederreiter in [890]): $s(X) = \frac{G(X)}{F(X)}$, where $G(X) = \sum_{i=0}^{L-1} X^i \left(\bigoplus_{j=0}^{i} c_{i-j} s_j\right)$ is a polynomial of degree smaller than L and $F(X) = 1 + c_1 X + \cdots + c_L X^L$ is the *feedback polynomial* (an equivalent representation uses the *characteristic polynomial* which is the reciprocal of the feedback polynomial). The minimum length of the LFSR producing a sequence is called the *linear complexity* of the sequence (and sometimes its *linear span*). It equals L if and only if the polynomials F and G above are co-prime and is equal in general to $N - deg\,(\gcd(X^N + 1, S(X)))$, where N is a period and $S(X)$ is the generating polynomial $S(X) = s_0 + s_1 X + \cdots + s_{N-1} X^{N-1}$. An *m-sequence* (or *maximum length sequence*) is a sequence of period $2^{\mathcal{L}} - 1$ where $\mathcal{L}$ is the linear complexity. Assuming that $L = \mathcal{L}$, this corresponds to taking a primitive feedback polynomial (see page 529). The sequence can then be represented in the form $s_i = tr_n(a\alpha^i)$ where α is a primitive element of $\mathbb{F}_{2^n}$ (see page 528) and tr_n is the trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_2$ (see pages 60 and 530). The m-sequences have very strong properties; see the chapter by Helleseth and Kumar in [959].
The initialization $s_0, \ldots, s_{L-1}$ of the LFSR and the values of the *feedback coefficients* c_i must be kept secret (they are then computed from the secret key); if

[21] Conversely, every ultimately periodic sequence can be generated by an LFSR.

the feedback coefficients were public, the observation of L consecutive bits of the keystream would allow recovering all the subsequent sequence.

Berlekamp-Massey attack

The use of LFSRs as pseudorandom generators is cryptographically weak because of an attack found in the late seventies called the *Berlekamp-Massey (BM) algorithm* [826]: let $\mathcal{L}$ be the linear complexity of the sequence, assumed to be unknown from the attacker; if he knows at least $2\mathcal{L}$ consecutive bits of the sequence, the BM algorithm allows him to recover the values of $\mathcal{L}$ and of the feedback coefficients of an LFSR of length $\mathcal{L}$ generating the sequence, as well as the initialization of this LFSR. The BM algorithm has quadratic complexity, that is, works in $\mathcal{O}(\mathcal{L}^2)$ elementary operations. Improvements of the algorithm exist, which have lower complexity: the main idea[22] is to use the extended Euclidean (EE) algorithm (or its variants). The way to use this algorithm is shown in the section "Linearly recurrent sequences" (Section 12.3) of the book "Modern Computer Algebra" by J. von zur Gathen and J. Gerhard [533] (Algorithm 12.9 in this book is essentially EE algorithm). The complexity of EE algorithm being $\mathcal{O}(M(\mathcal{L})\log(\mathcal{L}))$ where $M(\mathcal{L})$ is the cost of the multiplication between two polynomials of degree $\mathcal{L}$, and this latter cost being quasi-linear, the complexity of finding the retroaction polynomial of an LFSR is roughly $\mathcal{O}(\mathcal{L}\log(\mathcal{L}))$. The data complexity is still $2\mathcal{L}$ but these $2\mathcal{L}$ bits of the sequence do not need to be strictly consecutive: having k strings of $2\mathcal{L}/k$ consecutive bits is enough, thanks to a matrix-version of the BM algorithm found by Coppersmith, coupled with an algorithm due to Beckerman and Labahn, or with a simpler (and implemented) one due to Thomé, see more in [1085].

The role of Boolean functions

Many keystream generators still use LFSRs, and to resist the Berlekamp-Massey attack, either combine several LFSRs (and possibly some additional memory) like in the case of E_0, the keystream generator which is part of the Bluetooth standard, or use Boolean functions, see [1006]. The first model which appeared in the literature for such use is the *combiner model* (see Figure 1.3).

Notice that the feedback coefficients of the n LFSRs used in such a generator can be public. The Boolean function is also public, in general, and the (short) secret key is necessary only for the initialization of the n LFSRs (also depending on an initial vector, which being public can be changed more often than the key): if we want to use for instance a 128 bit long secret key, this makes possible using n LFSRs of lengths $L_1, \dots, L_n$ such that $L_1 + \cdots + L_n = 128$.

Such system clearly outputs a periodic sequence whose period is at most the LCM of the periods of the sequences output by the n LFSRs (assuming that $c_L = 1$ in each LFSR; otherwise, the sequence is ultimately periodic and the period is shorter). So, this sequence satisfies a linear recurrence and can therefore

[22] We thank Pierrick Gaudry for his kind explanations.

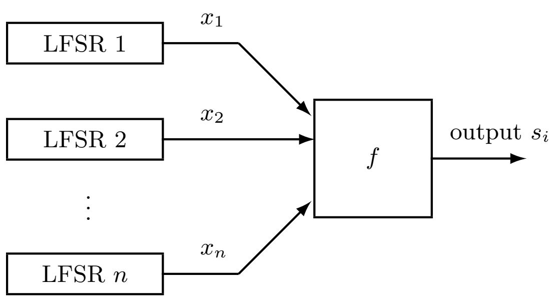


Figure 1.3 COMBINER MODEL

be produced by a single LFSR. However, as we shall see, well-chosen Boolean functions allow the linear complexity of the sequence to be much larger than the sum of the lengths of the n LFSRs. Nevertheless, choosing LFSRs producing sequences of large periods, choosing these periods pairwise co-prime in order to have the largest possible global period, and choosing f such that the linear complexity is large enough too are not sufficient. As we shall see, the combining function should also not leak information about the individual LFSRs and behave as differently as possible from affine functions, in several different ways.
The combiner model is only a model, useful for studying attacks and related criteria. In practice, the systems are more complex (see for instance at URL http://www.ecrypt.eu.org/stream/ how are designed the stream ciphers of the *eSTREAM Project* [495]).

A more recent model is the *filter model*, which uses a single LFSR (of a longer length). A filtered LFSR outputs $f(x_1, \ldots, x_n)$ where f is some n-variable Boolean function, called a filtering function, and where $x_1, \ldots, x_n$ are the bits contained in some flip-flops of the LFSR, see Figure 1.4.

Such system is equivalent to the combiner model using n copies of the LFSR. However, the attacks, even when they apply to both systems, do not work similarly (a first obvious difference is that the lengths of the LFSRs are different in the two models). Consequently, the criteria that the involved Boolean functions must satisfy to allow resistance to these attacks need to be studied for each model (we shall see that they are in practice not so different, except for one criterion which will be necessary for the combiner model but not for the filter model).
Note that in both models, the PRG is made of a *linear part* (constituted by the LFSRs), the linearity allowing speed, and a *nonlinear part* (made of the combiner/filter function) providing confusion (see the meaning of this term in Section 3.1). Generalizations of the two models have been proposed with the same structure "linear part, nonlinear part" [901, 495]. In practice, models will not be used as is; we shall add memory and/or few combinatoric stages and/or initialization registers; a high level security is ensured by the fact that the model,

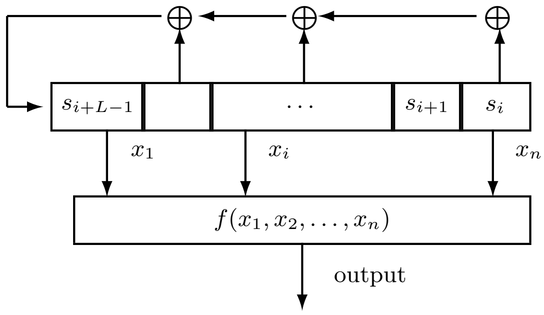


Figure 1.4 FILTER MODEL

as is, is proved resistant to all known attacks, and the additional complexity will make the work of the attacker still more difficult.

Other kinds of pseudorandom generators exist, which are not built on the same principle. A *feedback shift register* (FSR) has the same structure as an LFSR, but the left-most flip-flop is fed with $g(x_{i_1}, \ldots, x_{i_n})$ where $n \leq L$ and $x_{i_1}, \ldots, x_{i_n}$ are bits contained in the flip-flops of the FSR, and where g is some n-variable Boolean function called feedback function (if g is not affine then we speak of *NFSR* where N stands for nonlinear). The linear complexity of the produced sequence can be near 2^L (see [640] for general FSRs and [344] for FSRs with quadratic feedback function, see definition of "quadratic" at page 53). Some finalists of the eSTREAM project [495] like Grain and Trivium use NFSRs. But the theory of NFSRs is not completely understood. The linear complexity is difficult to study in general. Even the period is not easily determined, although some special cases have been investigated [630, 702, 1045, 1046]. Nice results similar to those on the m-sequences exist in the case of FCSRs (Feedback with Carry Shift-Registers), see [703, 559, 30, 560].

1.3.2 Boolean functions and error correcting codes

As explained above, every binary *unrestricted code* whose length equals 2^n for some positive integer n can be interpreted as a set of Boolean functions. A particular class of codes has its very definition given by means of Boolean functions. This class is that of Reed-Muller codes. We shall see in Chapter 2 that an integer lying between 0 and n and called algebraic degree can be associated to every Boolean function over $\mathbb{F}_2^n$. The *Reed-Muller code* of order $k \in \{0, \ldots, n\}$ is made of all Boolean functions over $\mathbb{F}_2^n$ whose algebraic degree is bounded above by k, see Section 4.1. This linear code has length 2^n since each Boolean function is identified to the list of its values over $\mathbb{F}_2^n$, in some order. It is linear and has nice particularities, thanks to which Reed-Muller codes are still used nowadays, even if their parameters are not very good, except for the first-order Reed-Muller code. The second-order Reed-Muller code contains a nonlinear code, called the Kerdock code, which has minimum distance almost the same as that of the first-order Reed-Muller code of the same length and size roughly the square of its size. In fact, the parameters of the Kerdock code are so good that they are provably optimal among all unrestricted codes, see Section 6.1.22.

1.4 Vectorial functions

The functions from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ are called (n,m)-*functions*. Such function F being given, the Boolean functions $f_1,\ldots,f_m$ defined at every $x \in \mathbb{F}_2^n$ by $F(x) = (f_1(x),\ldots,f_m(x))$, are called the *coordinate functions* of F. When the numbers m and n are not specified, (n,m)-functions are called *multi-output Boolean functions* or *vectorial Boolean functions*. Those vectorial functions whose role is to ensure confusion[23] in a cryptographic system are called *substitution boxes* (*S-boxes*).

Note that (n,m)-functions can also be viewed as taking their input in $\mathbb{F}_{2^n}$ as we have seen with Boolean functions, and if m divides n then we shall see that the output can then be expressed as a polynomial function of the input. We shall be in particular interested in *power functions* $F(x) = x^d$, $x \in \mathbb{F}_{2^n}$.

1.4.1 Vectorial functions and stream ciphers

In the pseudorandom generators of stream ciphers, (n,m)-functions can be used to combine the outputs of n linear feedback shift registers (LFSR), or to filter the content of a single one, generating m bits at each clock cycle instead of only one, which increases the speed of the cipher (but risks decreasing its robustness). The attacks, described about Boolean functions are obviously also efficient on these kinds of ciphers. They are in fact often more efficient, see Section 3.3, page 151, since the attacker can combine in any ways the m output bits of the function.

1.4.2 Vectorial functions and block ciphers

Vectorial functions play mainly a role with block ciphers. All known block ciphers are iterative, that is, are the iterations of a transformation depending on a key over each block of plaintext. The iterations are called *rounds* and the key used in an iteration is called a *round key*. The round keys are computed from the secret key (called the *master key*) by a *key scheduling algorithm*. The rounds consist of vectorial Boolean functions combined in different ways and involve the round key.

[23] See Section 3.1 for the meaning of this term.

Remark.
Boolean functions also play an important role in block ciphers, each of which admits as input a binary vector $(x_1, \ldots, x_n)$ (a block of plaintext) and outputs a binary vector $(y_1, \ldots, y_m)$; the coordinates $y_1, \ldots, y_m$ are the outputs of Boolean functions (depending on the key) over $(x_1, \ldots, x_n)$, see Figure 1.5.

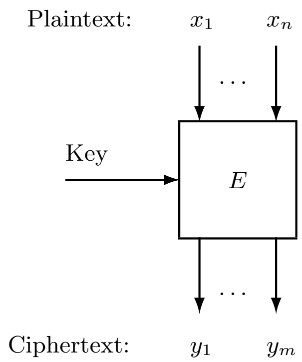


Figure 1.5 Block cipher

But the number n of variables of these Boolean functions being large (often more than a hundred), they are hardly analyzed precisely. □

We give in Figures 1.6 and 1.7 a description of the rounds of the *Data Encryption Standard (DES)* [88] and of the *Advanced Encryption Standard (AES)* [404].

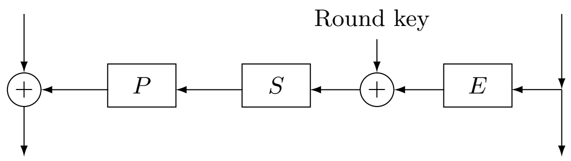


Figure 1.6 A DES round

The input to a DES round is a binary string of length 64, divided into two strings of 32 bits each (in the figure, they enter the round, from above, on the left and on the right); confusion is achieved by the S-box, which is a nonlinear transformation of a binary string of 48 bits[24] into a 32 bit long one. So, 32 Boolean functions on 48 variables are involved. But, in fact, this nonlinear transformation is the concatenation of eight sub-S-boxes, which transform binary strings of 6 bits into 4 bit long ones. Before entering the next round, the two 32 bit

[24] The E-box has expanded the 32 bit long string into a 48 bit long one.

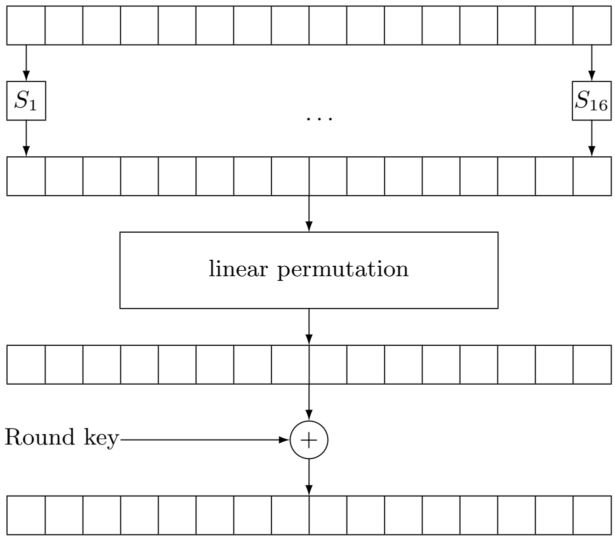


Figure 1.7 An AES round

long halves of data are swapped. Such *Feistel cipher* structure does not need the involved vectorial functions (in particular the S-boxes) to be injective, for the decryption to be possible. Indeed, any function of the form $(x, y) \mapsto (y, x + \phi(y))$ is a permutation. The number of output bits can be smaller than that of input bits like in the DES; it can also be larger, like in the CAST cipher [6], where input dimension is 8 and output dimension is 32. However if the S-boxes are not balanced (that is, if their output is not uniform), this represents a weakness against some attacks and it obliges the designer to complexify the structure (for instance by including expansion boxes), see more in [957].
In the (standard) AES round, the input is a 128 bit long string, divided into 16 strings of 8 bits each; the S-box is the concatenation of 16 sub-S-boxes corresponding to 16×8 Boolean functions in 8 variables. Such *substitution permutation network* (SPN) needs the vectorial functions (in particular the S-boxes) to be bijective, so that decryption is possible. Then $n = m$. Another well-known example of such cipher is PRESENT [100]. A third general structure for block ciphers is ARX structure, see [708].

Remark. Klimov and Shamir [705] have identified a particular kind of vectorial functions usable in stream and block ciphers (and in hash functions), called *T-functions*. These are mappings F from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ such that each i-th bit of $F(x)$ depends only on $x_1, \ldots, x_i$. For example, addition and multiplication in $\mathbb{Z}$, viewed in binary expansion, are T-functions; logical operations (XOR, AND, that is, addition and multiplication in $\mathbb{F}_2$) are T-functions too. Any composition of T-functions is a T-function as well. Their simplicity makes them appealing for lightweight cryptography. But they may be too simple for providing enough confusion; they have suffered attacks. □

1.4.3 Vectorial functions and error correcting codes

We shall see in Chapter 4 that interesting linear subcodes of the Reed-Muller codes and other (possibly nonlinear) codes can be built from vectorial functions.

2 Generalities on Boolean and vectorial functions

The set $\mathbb{F}_2^n$ of all binary vectors[1] of length n will be viewed as an $\mathbb{F}_2$-vector space (with null element 0_n). This vector space will sometimes be also endowed with the structure of the field $\mathbb{F}_{2^n}$ (denoted by $GF(2^n)$ by some authors), with null element 0; indeed, this field being an n-dimensional vector space over $\mathbb{F}_2$, each of its elements can be identified with the binary vector of length n of its coordinates relative to a fixed basis. The set of all Boolean functions $f : \mathbb{F}_2^n \to \mathbb{F}_2$ will be denoted by $\mathcal{BF}_n$. It is a vector space over $\mathbb{F}_2$. The *Hamming weight* $w_H(x)$ of a binary vector $x \in \mathbb{F}_2^n$ being the number of its nonzero coordinates (*i*.e. the size of $supp(x) = \{i \in \{1, \dots, n\};\ x_i \neq 0\}$, the *support of vector* x), the *Hamming weight* $w_H(f)$ of a Boolean function f on $\mathbb{F}_2^n$ is (also) the size of $supp(f) = \{x \in \mathbb{F}_2^n;\ f(x) \neq 0\}$, the *support of function* f. Note that if we denote by Δ the symmetric difference between two sets, we have $supp(f \oplus g) = supp(f)\,\Delta\, supp(g)$. The *Hamming distance* $d_H(f, g)$ between two functions f and g is the size of the set $\{x \in \mathbb{F}_2^n;\ f(x) \neq g(x)\}$. Thus it equals $w_H(f \oplus g)$.

Note. Some additions of bits will be considered in $\mathbb{Z}$ (in characteristic 0) and denoted then by $+$, and some will be computed in characteristic 2 and denoted by $\oplus$. These two different notations will be necessary for $\mathbb{F}_2$ because some representations of Boolean functions will live in characteristic 2 and some will live in characteristic 0. But the addition in the finite field $\mathbb{F}_{2^n}$ will be denoted by $+$, as usual in mathematics, as well as the addition in $\mathbb{F}_2^n$ when $n > 1$, since $\mathbb{F}_2^n$ will often be identified with $\mathbb{F}_{2^n}$, and because there will be no ambiguity.

2.1 A hierarchy of equivalence relations over Boolean and vectorial functions

Each notion that we shall study on Boolean or vectorial functions will be preserved by some equivalence relations, that we need to define. It is important to determine precisely, for each notion, those equivalence relations which preserve it. Indeed, if we prove that some function has some property, say $\mathcal{P}$, preserved by a given equivalence relation, this implies automatically that all functions in the equivalence class containing this function share the same property $\mathcal{P}$; to *classify*

[1] Coders say "words".

the set of functions satisfying $\mathcal{P}$, we need to determine all equivalence classes of functions sharing $\mathcal{P}$. This is often a difficult task. Even determining the size of the union of these classes may be quite difficult. If classification is elusive, a possible contribution to the domain is to provide constructions of functions satisfying $\mathcal{P}$. For being able to say that some construction of functions satisfying $\mathcal{P}$ provides new functions, it is needed to prove that at least one function obtained through this construction is inequivalent (for every equivalence relation preserving $\mathcal{P}$) to all known functions satisfying $\mathcal{P}$. This may be a huge work.
There are five main notions of equivalence among vectorial functions and four in the subcase of Boolean functions (because the fifth notion is then equivalent to the fourth one). We give the definitions for vectorial functions; the corresponding definitions for Boolean functions are with $m = 1$ (then all the permutations composed with the functions on their left can be taken equal to identity).

Remark. In the next definition and in the sequel, we present linear functions over $\mathbb{F}_2^n$ in the form $L : (x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, x_n) \times M$, with $(x_1, x_2, \dots, x_n)$ a row vector, as it is usual in information protection, rather than dealing with a column vector as it is usual in mathematics. Applying transposition to the expressions allows to translate a representation into the other. $\square$

Definition 5 *The main notions of equivalence on Boolean and vectorial functions are as follows:*

1. *Two (n, m)-functions F and $\tau \circ F \circ \sigma$, where σ is a permutation of $\{1, \dots, n\}$, extended to a permutation of $\mathbb{F}_2^n$ by:*
$$\sigma : (x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) \in \mathbb{F}_2^n$$
and τ is a permutation of $\{1, \dots, m\}$, similarly extended to a permutation of $\mathbb{F}_2^m$, are called permutation equivalent.
2. *Two (n, m)-functions F and $L' \circ F \circ L$ where*
$$L : (x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n \mapsto (x_1, x_2, \dots, x_n) \times M \in \mathbb{F}_2^n$$
is an $\mathbb{F}_2$-linear automorphism of $\mathbb{F}_2^n$, M being a nonsingular $n \times n$ matrix over $\mathbb{F}_2$, and L' is an $\mathbb{F}_2$-linear automorphism of $\mathbb{F}_2^m$, are called linearly equivalent.
3. *Two (n, m)-functions F and $L' \circ F \circ L$, where*
$$L : (x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n \mapsto (x_1, x_2, \dots, x_n) \times M + (a_1, a_2, \dots, a_n)$$
is an affine automorphism of $\mathbb{F}_2^n$ and L' is an affine automorphism of $\mathbb{F}_2^m$, are called affinely equivalent *or affine equivalent [907].*
4. *Two (n, m)-functions F and $L' \circ F \circ L + L''$, where L is an affine automorphism of $\mathbb{F}_2^n$, L' is an affine automorphism of $\mathbb{F}_2^m$ and $L'' : (x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n \mapsto (x_1, x_2, \dots, x_n) \times M'' + (a_1, a_2, \dots, a_m) \in \mathbb{F}_2^m$ is an affine (n, m)-function, M'' being an $n \times m$ binary matrix, are called (extended affine)* EA equivalent.

5. *Two (n,m)-functions F and G whose graphs $\mathcal{G}_F = \{(x,y) \in \mathbb{F}_2^n \times \mathbb{F}_2^m;\ y = F(x)\}$ and $\mathcal{G}_G = \{(x,y) \in \mathbb{F}_2^n \times \mathbb{F}_2^m;\ y = G(x)\}$ are affinely equivalent (*i.e. *such that $L(\mathcal{G}_F) = \mathcal{G}_G$ for some affine automorphism L on $\mathbb{F}_2^n \times \mathbb{F}_2^m$) are called (Carlet-Charpin-Zinoviev)* CCZ equivalent[2] *(the notion is from [257] and the term from [163]).*

A property or a parameter will be called a permutation invariant *(resp. a* linear invariant*, an* affine invariant*, an* EA invariant*, a* CCZ invariant*) if it is preserved by permutation (resp. linear, affine, extended affine, CCZ) equivalence.*

In [432] is given an asymptotic estimate for the number of EA equivalence classes of Boolean functions.

Note that if F and G are CCZ equivalent and if we write $L = (L_1, L_2)$ where L is the automorphism in Definition 5 (Item 5) with $L_1 : \mathbb{F}_2^n \times \mathbb{F}_2^m \mapsto \mathbb{F}_2^n$ and $L_2 : \mathbb{F}_2^n \times \mathbb{F}_2^m \mapsto \mathbb{F}_2^m$, and if, for every $x \in \mathbb{F}_2^n$, we define $F_1(x) = L_1(x, F(x))$ and $F_2(x) = L_2(x, F(x))$, then function F_1 is bijective because, G being a function, F_1 is surjective. We have $G = F_2 \circ F_1^{-1}$. Note also that, given a function F, finding all functions CCZ equivalent to F consists in finding all affine automorphisms $L = (L_1, L_2)$ such that F_1 is bijective. Moreover, CCZ equivalent functions corresponding to a same F, a same L_1 and different L_2 are EA equivalent, see [163], where is shown that an (n,n)-function G is EA equivalent to a function F or to F^{-1} (if it exists) if and only if there exists an affine permutation $L = (L_1, L_2)$ where L_1 depends only on x or y, and such that $L(\mathcal{G}_F) = \mathcal{G}_G$.
CCZ equivalence can be translated in terms of codes, see the remark, page 411.

Proposition 1 *For n and m ranging over $\mathbb{N}$, each equivalence relation in Definition 5 is a strict particular case of the next one.*

Proof. The only non-obvious facts are that EA equivalence implies CCZ equivalence and that the converse is false. This can be seen as follows:
- if ϕ_1 and ϕ_2 are affine automorphisms of $\mathbb{F}_2^n$, $\mathbb{F}_2^m$, respectively, and if $G = \phi_2 \circ F \circ \phi_1$, then defining $L_1(x,y) = \phi_1^{-1}(x)$ and $L_2(x,y) = \phi_2(y)$, we have that $L = (L_1, L_2)$ is an affine automorphism of $\mathbb{F}_2^n \times \mathbb{F}_2^m$ which maps $\mathcal{G}_F$ onto $\mathcal{G}_G$, since $G(\phi_1^{-1}(x)) = \phi_2(F(x))$, and F and G are then CCZ equivalent[3];
- if $\phi(x)$ is an affine function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ and $G(x) = F(x) + \phi(x)$ then $L(x,y) = (x, y + \phi(x))$ is an affine automorphism which maps $\mathcal{G}_F$ onto $\mathcal{G}_G$ and F and G are CCZ equivalent;
- EA equivalence preserves algebraic degree (see Definition 6, page 52) when it is larger than 1 and it is shown in [162, 163] that CCZ equivalence does not. □
Note that if $m = n$ and $(L_1, L_2)(x,y) = (y,x)$, then $F_2 \circ F_1^{-1}$ is equal to F^{-1}.

[2] This notion has been rediscovered by L. Breveglieri, A. Cherubini and M. Macchetti at Asiacrypt 2004.

[3] Conversely, if F and G are CCZ equivalent and $L_1(x,y)$ and $L_2(x,y)$ depend only on x and y, respectively, say $L_1(x,y) = \phi_1^{-1}(x)$ and $L_2(x,y) = \phi_2(y)$, then ϕ_1 and ϕ_2 are affine automorphisms of $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$, respectively, and $G = \phi_2 \circ F \circ \phi_1$.

2.1.1 Relations between these equivalences

For a lack of space, in this subsection, we shall refer to papers for the proofs. It has been proved in [163] that CCZ equivalence between (n,n)-functions[4] is strictly more general than EA equivalence together with taking inverses of permutations, by exhibiting functions which are CCZ equivalent to the function $F(x) = x^3$ on $\mathbb{F}_{2^n}$, but which cannot be obtained from F by any sequence of applications of EA equivalence and inverse transformation; see also [771]. However, CCZ equivalence coincides with EA equivalence when restricted to some classes of functions (whose definitions will be, in some cases, given after):

1. Boolean (*i.e.* single-output) functions[5], as shown in [149] (on the contrary, CCZ equivalence is shown strictly more general than EA equivalence in the case of (n,m)-*functions* when $n \geq 5$ and m is larger than or equal to the smallest divisor of n different from 1, *e.g.* when n is even and $m \geq 2$),
2. bent functions (see page 297) as proved in [148, 150] and more generally, functions having surjective derivatives (see page 55), as proved in [164],
3. quadratic APN functions (see page 309), as shown in [1139] (extending [119]).

CCZ equivalence also coincides with EA equivalence (see page 309 as well):

- for n even, with plateaued APN functions, one of which is a power function,
- for a quadratic APN function and a power (APN) function (they are then EA equivalent to one of the Gold functions).

And the CCZ equivalence between two power functions coincides with their EA equivalence or with the EA equivalence between one function and the inverse of the other if it is bijective (see Proposition 113, page 310).

Remark. It has been shown in [149] that the CCZ equivalence (*i.e.* the EA equivalence, thanks to 1. above) between the indicators (*i.e.* characteristic functions) of the graphs of two functions coincides with their CCZ equivalence. □

Finding new EA inequivalent functions by using CCZ equivalence is not easy (this could be done in particular cases, see pages 429, 437). If (L_1, L_2) and (L_1, L_2') are linear permutations of $\mathbb{F}_2^n \times \mathbb{F}_2^m$ and $F_1 = L_1(x, F(x))$ is a permutation of $\mathbb{F}_2^n$, then since the functions F' and F'' obtained by CCZ equivalence from F by using (L_1, L_2) and (L_1, L_2') are EA equivalent, finding EA inequivalent functions by using CCZ equivalence needs to find new permutations F_1.

2.2 Representations of Boolean functions and vectorial functions

Among the classical representations of Boolean (resp. vectorial) functions, the most well-known is the *truth-table* (resp. the *look-up table* LUT), equal to the

[4] For (n,m)-functions, see [149, 966].
[5] If one function is Boolean (and viewed as multi-ouput thanks to $\mathbb{F}_2 \subset \mathbb{F}_{2^m}$), this suffices.

list of all pairs of an element of $\mathbb{F}_2^n$ and of the value of the function at this input (an ordering of $\mathbb{F}_2^n$ being chosen).

2.2.1 Algebraic normal form

The truth table is not much used for defining Boolean functions in the frameworks of cryptography and coding theory, because the features of Boolean functions which play a role in these two domains are not easily captured by such representation (except for the Hamming weight). The most used representation in cryptography and coding is the *algebraic normal form* (in brief the *ANF*)[6].

Algebraic normal form of Boolean functions

This is an n-variable polynomial representation over $\mathbb{F}_2$, of the form $f(x) =$

$$\bigoplus_{I\subseteq\{1,\dots,n\}} a_I \left(\prod_{i\in I} x_i\right) = \bigoplus_{I\subseteq\{1,\dots,n\}} a_I\, x^I \in \mathbb{F}_2[x_1,\dots,x_n]/(x_1^2\oplus x_1,\dots,x_n^2\oplus x_n). \tag{2.1}$$

Every coordinate x_i appears in this polynomial with exponents at most 1, because every bit in $\mathbb{F}_2$ equals its own square.

Example: let us consider the function f whose truth-table is

x_1	x_2	x_3	x in hexa	$f(x)$
0	0	0	0	0
0	0	1	1	1
0	1	0	2	0
0	1	1	3	0
1	0	0	4	0
1	0	1	5	1
1	1	0	6	0
1	1	1	7	1

It is the sum (modulo 2 or not, no matter) of the *atomic functions* f_1, f_2, f_3:

x_1	x_2	x_3	x in hexa	$f_1(x)$	$f_2(x)$	$f_3(x)$
0	0	0	0	0	0	0
0	0	1	1	1	0	0
0	1	0	2	0	0	0
0	1	1	3	0	0	0
1	0	0	4	0	0	0
1	0	1	5	0	1	0
1	1	0	6	0	0	0
1	1	1	7	0	0	1

[6] It can have other names in circuit theory, like Zhegalkin polynomial, modulo-2 sum-of-products, Reed-Muller-canonical expansion, positive polarity Reed-Muller form.

The function $f_1(x)$ takes value 1 if and only if $1 \oplus x_1 = 1 \oplus x_2 = x_3 = 1$, that is, $(1 \oplus x_1)(1 \oplus x_2)\, x_3 = 1$. Thus the ANF of f_1 can be obtained by expanding the product $(1 \oplus x_1)(1 \oplus x_2)\, x_3$. After similar observations on f_2 and f_3, we see that the ANF of f equals $(1 \oplus x_1)(1 \oplus x_2)\, x_3 \oplus x_1(1 \oplus x_2)\, x_3 \oplus x_1 x_2 x_3 = x_1 x_2 x_3 \oplus x_2 x_3 \oplus x_3$. $\square$

Another possible representation of this same ANF uses an indexation by means of vectors of $\mathbb{F}_2^n$ instead of subsets of $\{1, \ldots, n\}$; if, for any such vector u, we denote by a_u what is denoted by $a_{supp(u)}$ in Relation (2.1) (where $supp(u)$ denotes the support of u), we have the equivalent representation:

$$f(x) = \bigoplus_{u \in \mathbb{F}_2^n} a_u \left(\prod_{j=1}^{n} {x_j}^{u_j} \right).$$

The monomial $\prod_{j=1}^{n} {x_j}^{u_j}$ is often denoted[7] by x^u. We have $x^u x^v = x^{u \vee v}$ where $supp(u \vee v) = supp(u) \cup supp(v)$.

Existence and uniqueness of the ANF

By applying the method described in the example above, it is a simple matter to show the existence of the ANF of any Boolean function: we have

$$f(x) = \sum_{a \in \mathbb{F}_2^n} f(a)\delta_a(x) = \bigoplus_{a \in \mathbb{F}_2^n} f(a)\delta_a(x) \tag{2.2}$$

where the function δ_a is the Dirac (or Kronecker) symbol at a and equals $\delta_a(x) = \prod_{i=1}^{n}(x_i \oplus a_i \oplus 1)$. Replacing in (2.2) each δ_a by this expression, expanding it and simplifying (mod 2) gives an expression (2.1) for f which shows the existence of an ANF of any Boolean function. This implies that the mapping from polynomials $P \in \mathbb{F}_2[x_1, \ldots, x_n]/(x_1^2 \oplus x_1, \ldots, x_n^2 \oplus x_n)$ to the corresponding functions $x \in \mathbb{F}_2^n \mapsto P(x)$, is onto $\mathcal{BF}_n$. Since the size of $\mathcal{BF}_n$ equals the size of $\mathbb{F}_2[x_1, \ldots, x_n]/(x_1^2 \oplus x_1, \ldots, x_n^2 \oplus x_n)$, this correspondence is one to one[8]. But more can be said:

Relationship between a Boolean function and its ANF

The product $x^I = \prod_{i \in I} x_i$ is nonzero if and only if x_i is nonzero (*i.e.* equals 1) for every $i \in I$, that is, if I is included in the support of x; hence, the Boolean function $f(x) = \bigoplus_{I \subseteq \{1,\ldots,n\}} a_I\, x^I$ takes value

$$f(x) = \bigoplus_{I \subseteq supp(x)} a_I, \tag{2.3}$$

where $supp(x)$ denotes the support of x.
If we use the notation $f(x) = \bigoplus_{u \in \mathbb{F}_2^n} a_u x^u$, we obtain the relation $f(x) =$

[7] The reader should not confuse this notation with a univariate monomial.
[8] Another argument is that this mapping is a linear mapping from a vector space over $\mathbb{F}_2$ of dimension 2^n to a vector space of the same dimension.

$\bigoplus_{u \preceq x} a_u$, where $u \preceq x$ means that $supp(u) \subseteq supp(x)$ (we say that u is *covered* by x). A Boolean function f° can be associated to the ANF of f: for every $x \in \mathbb{F}_2^n$, we set $f^\circ(x) = a_{supp(x)}$, that is, with the notation $f(x) = \bigoplus_{u \in \mathbb{F}_2^n} a_u x^u$: $f^\circ(u) = a_u$. Relation (2.3) shows that f is the image of f° by the so-called *binary Möbius transform*. The converse is also true:

Theorem 1 *Let f be a Boolean function on $\mathbb{F}_2^n$ and let $\bigoplus_{I \subseteq \{1,\ldots,n\}} a_I\, x^I$ be its ANF. We have:*

$$\forall I \subseteq \{1, \ldots, n\},\ a_I = \bigoplus_{x \in \mathbb{F}_2^n;\ supp(x) \subseteq I} f(x). \tag{2.4}$$

Proof. Let us denote $\bigoplus_{x \in \mathbb{F}_2^n;\ supp(x) \subseteq I} f(x)$ by b_I and consider the function $g(x) = \bigoplus_{I \subseteq \{1,\ldots,n\}} b_I\, x^I$. We have

$$\begin{aligned} g(x) &= \bigoplus_{I \subseteq supp(x)} b_I = \bigoplus_{I \subseteq supp(x)} \left(\bigoplus_{y \in \mathbb{F}_2^n;\ supp(y) \subseteq I} f(y) \right) \\ &= \bigoplus_{y \in \mathbb{F}_2^n} f(y) \left(\bigoplus_{I \subseteq \{1,\ldots,n\};\ supp(y) \subseteq I \subseteq supp(x)} 1 \right). \end{aligned}$$

The sum $\bigoplus_{I \subseteq \{1,\ldots,n\};\ supp(y) \subseteq I \subseteq supp(x)} 1$ is null if $y \neq x$. Indeed, if $supp(y) \not\subseteq supp(x)$, then the sum is empty and if $supp(y) \subseteq supp(x)$, then the set $\{I \subseteq \{1, \ldots, n\};\ supp(y) \subseteq I \subseteq supp(x)\}$ contains $2^{w_H(x) - w_H(y)}$ elements. Hence, $g = f$ and, by the uniqueness of the ANF of f, $b_I = a_I$ for every I. □

Algorithm (*Fast binary Möbius transform*)
There exists a simple divide-and-conquer butterfly algorithm to compute the ANF from the truth-table (or *vice-versa*), called the *fast Möbius transform*. For every $u = (u_1, \ldots, u_n) \in \mathbb{F}_2^n$, the coefficient a_u of x^u in the ANF of f equals

$$\bigoplus_{(x_1,\ldots,x_{n-1}) \preceq (u_1,\ldots,u_{n-1})} [f(x_1, \ldots, x_{n-1}, 0)] \quad \text{if } u_n = 0 \text{ and}$$

$$\bigoplus_{(x_1,\ldots,x_{n-1}) \preceq (u_1,\ldots,u_{n-1})} [f(x_1, \ldots, x_{n-1}, 0) \oplus f(x_1, \ldots, x_{n-1}, 1)] \ \text{ if } u_n = 1.$$

Hence if, in the truth-table of f, the binary vectors are ordered in lexicographic order, with the bit of higher weight on the right, the table of the ANF equals the concatenation of the ANFs of the $(n-1)$-variable functions $f(x_1, \ldots, x_{n-1}, 0)$ and $f(x_1, \ldots, x_{n-1}, 0) \oplus f(x_1, \ldots, x_{n-1}, 1)$. This gives the recursive algorithm below. Note that taking the lexicographic order with the bit of higher weight on the left (*i.e.* the standard lexicographic order) would work as well (as would any other order corresponding to a permutation of $\{1, \ldots, n\}$).

1. write the truth-table of f, in which the binary vectors of length n are in lexicographic order with the bit of higher weight on the right;

2. let f_0 and f_1 be the restrictions of f to $\mathbb{F}_2^{n-1} \times \{0\}$ and $\mathbb{F}_2^{n-1} \times \{1\}$, respectively[9]; replace the values of f_1 by those of $f_0 \oplus f_1$;

3. apply recursively step 2, separately to the functions now obtained in the places of f_0 and f_1.

When the algorithm ends (*i.e.* arrives to functions in one variable each), the global table gives the values of the ANF of f. The complexity of this algorithm is of $n\,2^n$ XORs; it is then in $\mathcal{O}(N \log_2 N)$ where $N = 2^n$ is the size of its input f.

```
Data: tt ← truth table, n ← number of variables
Result: anf ← algebraic normal form
for i = 0 to n − 1 do
    for j = 0 to 2^{n−1} − 1 do
        t[j] = tt[2 ∗ j];
        u[j] = tt[2 ∗ j] ⊕ tt[2 ∗ j + 1];
    end
    for k = 0 to 2^{n−1} − 1 do
        anf[k] = t[k];
        anf[2^{n−1} + k] = u[k];
    end
end
```

Algorithm 1: Computing the algebraic normal form

We give in Table 2.1 an example of the computation of the ANF from the truth table using the algorithm of the fast binary Möbius transform, and of the computation of the truth table from the ANF, using this same algorithm.

Remark. The algorithm does not work if the order on $\mathbb{F}_{2^n}$ is not a permuted lexicographic order (for instance, an order by increasing weights of inputs). □

ANF of the graph indicator of a vectorial function

Denoting by $1_{\mathcal{G}_F}(x, y)$ the indicator (*i.e.* the characteristic function) of the *graph* $\mathcal{G}_F = \{(x, F(x));\ x \in \mathbb{F}_2^n\}$ of an (n, m)-function F (sometimes called its *codebook*), Relation (2.4) applied to $1_{\mathcal{G}_F}$ gives that, for every $I \subseteq \{1, \ldots, n\}, J \subseteq \{1, \ldots, m\}$, the coefficient of $x^I y^J$ in its ANF equals:

$$a_{I,J} = |\{x \in \mathbb{F}_2^n;\ supp(x) \subseteq I \text{ and } supp(F(x)) \subseteq J\}| \text{ [mod 2]}.$$

We have also:

Proposition 2 *[253, 254] Let F be any (n, m)-function and let $f_1, \ldots, f_m$ be its*

[9] The truth-table of f_0 (resp. f_1) corresponds to the upper (resp. lower) half of the table of f.

x_1	x_2	x_3	x_4	x in hexa	$f(x)$				$f^\circ(x)$				$f(x)$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	0	0	0	0	0	0	0
0	1	0	0	2	1	1	1	1	1	1	1	1	1
1	1	0	0	3	1	1	1	1	0	0	0	0	1
0	0	1	0	4	0	0	0	0	0	0	0	0	0
1	0	1	0	5	0	0	0	0	0	0	0	0	0
0	1	1	0	6	1	1	0	0	0	0	1	1	1
1	1	1	0	7	0	0	1	1	1	1	1	1	0
0	0	0	1	8	0	0	0	0	0	0	0	0	0
1	0	0	1	9	1	1	1	1	1	1	1	1	1
0	1	0	1	a	1	0	0	0	0	1	1	1	1
1	1	0	1	b	0	1	1	0	0	0	0	1	0
0	0	1	1	c	0	0	0	0	0	0	0	0	0
1	0	1	1	d	1	1	0	0	0	0	1	1	1
0	1	1	1	e	1	0	0	0	0	0	1	1	1
1	1	1	1	f	1	1	0	0	0	1	1	0	1

Table 2.1 ANF of a function from its truth-table and re-calculation of the truth table from ANF (for function $f(x) = x_2 \oplus x_1x_2x_3 \oplus x_1x_4$; $x = (x_1, x_2, x_3, x_4)$)

coordinate functions. We have:

$$1_{\mathcal{G}_F}(x,y) = \prod_{j=1}^{m}(y_j \oplus f_j(x) \oplus 1) = \bigoplus_{J\subseteq\{1,\dots,m\}} y^J \prod_{j\in\{1,\dots,m\}\setminus J}(f_j(x)\oplus 1).$$

Indeed, for every $y, y' \in \mathbb{F}_2^m$, we have $y = y'$ if and only if $\prod_{j=1}^m (y_j \oplus y'_j \oplus 1) = 1$. This, with $y' = F(x)$, proves the first assertion and the rest is straightforward. Note that, if F is a permutation ($m = n$), then $1_{\mathcal{G}_F}(x,y) = 1_{\mathcal{G}_{F^{-1}}}(y,x)$, where F^{-1} is the compositional inverse of F, and thus:

$$\bigoplus_{J\subseteq\{1,\dots,n\}} y^J \prod_{j\in\{1,\dots,n\}\setminus J}(f_j(x)\oplus 1) = \bigoplus_{I\subseteq\{1,\dots,n\}} x^I \prod_{i\in\{1,\dots,n\}\setminus I}(f'_i(y)\oplus 1), \quad (2.5)$$

where the f'_i's are the coordinate functions of F^{-1}.

Algebraic degree of a Boolean function

Definition 6 *The degree of the ANF shall be denoted by $d_{alg}(f)$ and is called the* algebraic degree *of the function*[10]*: $d_{alg}(f) = \max\{|I|;\ a_I \neq 0\}$, where $|I|$ denotes the size of I (with the convention that the zero function has algebraic degree 0).*

This makes sense thanks to the existence and uniqueness of the ANF.
Of course, given two n-variable Boolean functions f, g, we have $d_{alg}(f \oplus g) \leq \max(d_{alg}(f), d_{alg}(g))$ and $d_{alg}(f\,g) \leq d_{alg}(f) + d_{alg}(g)$.

[10] Some authors also call it the nonlinear order of f but this terminology is more or less obsolete.

Note that a Boolean function is affine if and only if it has algebraic degree at most 1. We shall call *quadratic functions* the Boolean functions of algebraic degree at most 2 and *cubic functions* those of algebraic degree at most 3. Note that this means for instance that an affine function is a particular quadratic function (just as, by definition, a constant function is a particular affine function). This may be a little confusing for the reader, but we are obliged to adopt this terminology, since otherwise, we would have sentences like "all derivatives of a Boolean function are affine if and only if the function is quadratic or affine", "all second-order derivatives are affine if and only if the function is cubic or quadratic or affine", etc.

According to Relation (2.4), we have directly:

Proposition 3 *The algebraic degree $d_{alg}(f)$ of any n-variable Boolean function f equals the maximum dimension of the subspaces $\{x \in \mathbb{F}_2^n; supp(x) \subseteq I\}$ on which f takes value 1 an odd number of times. In particular:*
- $d_{alg}(f) = n$ if and only if $w_H(f)$ is odd,
- $d_{alg}(f) = n-1$ if and only if (1) $w_H(f)$ is even and (2) there exists i such that $|\{a \in supp(f); a_i = 0\}|$ is odd, or equivalently thanks to (1), $\sum_{a \in supp(f)} a \neq 0$.

The index i is indeed characterized by $\bigoplus_{a \in supp(f)} a_i \neq 0$. The two latter properties above will be seen under another viewpoint in Corollary 2, page 64.

The algebraic degree is an affine invariant (*i.e.* it is invariant under the action of the general affine group, see Section 2.1): for every affine automorphism $L : (x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n \mapsto (x_1, x_2, \dots, x_n) \times M + (a_1, a_2, \dots, a_n)$, where M is a nonsingular $n \times n$ matrix over $\mathbb{F}_2$, we have $d_{alg}(f \circ L) = d_{alg}(f)$. Indeed, the composition by L clearly cannot increase the algebraic degree, since the coordinates of $L(x)$ have degree 1. Hence we have $d_{alg}(f \circ L) \leq d_{alg}(f)$ (in fact, for every affine homomorphism). And applying this inequality to $f \circ L$ in the place of f and to L^{-1} in the place of L shows the inverse inequality.
Note in particular that, if F is an (n, n)-permutation, then we have $d_{alg}(1_{\mathcal{G}_F}) = d_{alg}(1_{\mathcal{G}_{F^{-1}}})$: these two indicators correspond to each other by swapping x and y.
For functions of algebraic degree strictly larger than 1, the algebraic degree is an *EA invariant* (but not a CCZ invariant, see [162, 163]).
The algebraic degree being an affine invariant, Proposition 3 implies that it also equals the maximum dimension of all the affine subspaces of $\mathbb{F}_2^n$ on which f takes value 1 an odd number of times. Equivalently:

Proposition 4 *A Boolean function has algebraic degree at most d if and only if its restriction to any $(d+1)$-dimensional* flat *(*i.e. *affine subspace) has an even Hamming weight.*

This shows in particular that, given an n-variable Boolean function f and an affine subspace $A = a + E$ of $\mathbb{F}_2^n$ (where E is the vector space equal to the direction of A), the restriction of f to A, viewed as a k-variable function where k is the dimension of A (by identifying the elements of $a + E$ with the vectors of $\mathbb{F}_2^k$

through the choice of a basis of E), has algebraic degree at most $d_{alg}(f)$.

It is shown in [955] that, for every nonzero n-variable Boolean function f, denoting by g the *binary Möbius transform* of f, we have $d_{alg}(f) + d_{alg}(g) \geq n$. This same paper deduces characterizations and constructions of the functions which are equal to their binary Möbius transform, called *coincident functions*.

Remarks. 1. Every atomic function has algebraic degree n, since its ANF equals $(x_1 \oplus \epsilon_1)(x_2 \oplus \epsilon_2) \dots (x_n \oplus \epsilon_n)$, where $\epsilon_i \in \mathbb{F}_2$. Thus, a Boolean function f has algebraic degree n if and only if, in its decomposition as a sum of atomic functions, the number of these atomic functions is odd, that is, if and only if $w_H(f)$ is odd. This property will have an important consequence on the *Reed-Muller codes* and it will be also useful in Chapter 4.
2. If we know that the algebraic degree of an n-variable Boolean function f is bounded above by $d < n$, then the whole function can be recovered from some of its restrictions (*i.e.*, a unique function corresponds to this *partially defined* Boolean function). Precisely, according to the existence and uniqueness of the ANF, the knowledge of the restriction $f_{|E}$ of the Boolean function f (of algebraic degree at most d) to a set E implies the knowledge of the whole function if and only if the system of the equations $f(x) = \bigoplus_{I\subseteq\{1,\dots,n\};\,|I|\leq d} a_I\, x^I$, with indeterminates $a_I \in \mathbb{F}_2$, and where x ranges over E (this makes $|E|$ equations), has a unique solution[11].
This happens with the set E_d of all words of Hamming weights smaller than or equal to d (and then, by affine equivalence, it happens with every set E affinely equivalent to E_d), since Relation (2.4) gives the value of a_I for $|I| \leq d$ and the others are null by hypothesis. And since $|E_d| = |\{I \subseteq \{1,\dots,n\};\ |I| \leq d\}|$, any choice of $f_{|E_d}$ works.
Notice that Relation (2.3) makes possible to express the value of $f(x)$ for every $x \in \mathbb{F}_2^n$ by means of the values taken by f on E. For instance, for $E = E_d$, we have (using the notation a_u instead of a_I, see above, and still using that $d_{alg}(f) \leq d$):

$$f(x) = \bigoplus_{u\preceq x} a_u = \bigoplus_{\substack{u\preceq x\\ u\in E_d}} a_u = \bigoplus_{\substack{y\preceq x\\ y\in E_d}} f(y)\,|\{u\in E_d\,;\, y\preceq u\preceq x|$$
$$= \bigoplus_{\substack{y\preceq x\\ y\in E_d}} f(y)\left[\left[\sum_{i=0}^{d-w_H(y)} \binom{w_H(x)-w_H(y)}{i}\right]\ [\text{mod } 2]\right].$$

These observations generalize to *pseudo-Boolean* (that is, real-valued) functions, if we consider the numerical degree (see below) instead of the algebraic degree,

[11] Note that taking $f(x) = 0, \forall x \in E$, leads to another problem: determine the so-called annihilators f of the indicator 1_E of E (the characteristic function of E, defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ otherwise); this is the core analysis of Boolean functions from the viewpoint of algebraic attacks, see Section 3.1.

cf. [1090]. □

The simplest functions, from the viewpoint of the ANF, are those Boolean functions of algebraic degree at most 1, that is, *affine functions* (the sums of linear and constant functions, sometimes called parity functions, see *e.g.* [914]):

$$f(x) = a_0 \oplus a_1\, x_1 \oplus \cdots \oplus a_n\, x_n; \quad a_i \in \mathbb{F}_2.$$

Denoting by $a \cdot x$ the *usual inner product* $a \cdot x = a_1\, x_1 \oplus \cdots \oplus a_n\, x_n$ in $\mathbb{F}_2^n$ (already encountered in Section 1.2), or any other *inner product* (that is[12], any symmetric bivariate function such that, for every $a \neq 0$, the function $x \to a \cdot x$ is a nonzero linear form[13] on $\mathbb{F}_2^n$), the general form of an n-variable affine function is $a \cdot x \oplus a_0 = \ell_a(x) \oplus a_0$ (with $a \in \mathbb{F}_2^n$; $a_0 \in \mathbb{F}_2$), since the non-degeneracy of the bilinear form implies that the mapping $a \mapsto \ell_a$ is injective and therefore bijective.

Affine functions play an important role in coding (they are involved in the definition of the Reed-Muller code of order 1, see Section 4.1) and in cryptography (the Boolean functions used as "nonlinear functions" in cryptosystems must behave as differently as possible from affine functions, see Section 3.1).

Algebraic degree and derivation

The derivation of Boolean functions must not be confused with the derivation of polynomials:

Definition 7 *Let f be an n-variable Boolean function and let a be any vector in $\mathbb{F}_2^n$. We call* derivative[14] *of f in the direction a (or with the input difference a) the Boolean function $D_a f(x) = f(x) \oplus f(x + a)$.*

For instance, the derivative of a function expressed in the form $g(x_1, \dots, x_{n-1}) \oplus x_n\, h(x_1, \dots, x_{n-1})$ in the direction $(0, \dots, 0, 1)$ equals $h(x_1, \dots, x_{n-1})$.

Proposition 5 *Any derivative of any non-constant Boolean function f has algebraic degree strictly smaller than the algebraic degree of f and there exists at least one derivative of algebraic degree $d_{alg}(f) - 1$.*

Proof. The first assertion can be checked easily for each monomial x^I where $I \neq \emptyset$: we have $x^I \oplus (x + a)^I = \bigoplus_{J \subset I, J \neq I} \left(\prod_{j \in I \setminus J} a_j \right) x^J$. The second assertion is a direct consequence, by affine invariance of the algebraic degree, of the fact observed just above for direction $(0, \dots, 0, 1)$. □

Note that this implies that a function is affine if and only if all its derivatives are constant (this is more generally valid for every function defined over a vector space). And it is quadratic if and only if all its derivatives are affine. For a general

[12] In nonzero characteristic, there is no possible notion of positivity.

[13] *i.e.* "$\cdot$" is a nondegenerate bilinear form.

[14] Some authors write "directional derivative".

function, the sets of those vectors a such that D_af is constant (resp. affine) are vector subspaces of $\mathbb{F}_2^n$, see page 120.

In [275] are studied Boolean functions f whose restrictions to all affine hyperplanes have the same algebraic degree equal to $d_{alg}(f)$ and functions whose derivatives $D_af(x)$, $a \neq 0_n$, have all the same algebraic degree $d_{alg}(f)-1$. Three classes of Boolean functions are presented; the first class satisfies both conditions, the second class satisfies the first condition but not the second and the third class satisfies the second condition but not the first. In this same paper is given, for any fixed positive integer k and for all integers n, p, s such that $p \geq k+1$, $s \geq k+1$ and $n \geq ps$, a class $C_{n,p,s}$ of n-variable Boolean functions whose restrictions to all k-codimensional affine subspaces of $\mathbb{F}_2^n$ have the same algebraic degree as the function.

Higher order derivatives have been introduced by Lai [735].

Definition 8 *Let f be an n-variable Boolean function and let $a_1, \ldots, a_k$ be k vectors in $\mathbb{F}_2^n$. We call* k-th order derivative *of f in the directions $a_1, \ldots, a_k$ the Boolean function $D_{a_1}D_{a_2}\cdots D_{a_k}f(x)$.*

It is easily seen by induction on k that if $a_1, \ldots, a_k$ are linearly independent, then $D_{a_1}D_{a_2}\cdots D_{a_k}f(x) = \bigoplus_{a\in E} f(x+a)$, where E is the $\mathbb{F}_2$-vector space spanned by $a_1, \ldots, a_k$, and otherwise $D_{a_1}D_{a_2}\cdots D_{a_k}f(x) = 0$.

Corollary 1 *Any k-th order derivative of any Boolean function f of algebraic degree at least k has algebraic degree at most $d_{alg}(f)-k$.*

The Algebraic Normal Form of vectorial functions

The notion of algebraic normal form of Boolean functions can easily be extended to (n,m)-functions. Given such function F, each coordinate function of F is uniquely represented by its ANF, which is an element of $\mathbb{F}_2[x_1, \ldots, x_n]/(x_1^2 \oplus x_1, \ldots, x_n^2 \oplus x_n)$. Function F is then represented in a unique way as an element of $\mathbb{F}_2^m[x_1, \ldots, x_n]/(x_1^2 \oplus x_1, \ldots, x_n^2 \oplus x_n)$:

$$F(x) = \sum_{I\subseteq\{1,\ldots,n\}} a_I \left(\prod_{i\in I} x_i\right) = \sum_{I\subseteq\{1,\ldots,n\}} a_I\, x^I, \tag{2.6}$$

where a_I belongs to $\mathbb{F}_2^m$ (maybe should we write $F(x) = \sum_{I\subseteq\{1,\ldots,n\}} \left(\prod_{i\in I} x_i\right) a_I = \sum_{I\subseteq\{1,\ldots,n\}} x^I a_I$, since $\prod_{i\in I} x_i$ is a scalar and a_I is a vector). According to our convention on the notation for additions, we used $\sum$ to denote the sum in $\mathbb{F}_2^m$, but recall that, coordinate by coordinate, this sum is a $\bigoplus$.

This polynomial is called the *algebraic normal form* (ANF) of F. According to Relation (2.3), we have $F(x) = \sum_{I\subseteq supp(x)} a_I$ and according to Relation (2.4), we have $a_I = \sum_{x\in\mathbb{F}_2^n;\; supp(x)\subseteq I} F(x)$ (these sums being calculated in $\mathbb{F}_2^m$).

Remark. An (n,m)-function $F(x)$ being given by its ANF and an (m,r)-function $G(y)$ being given by the ANF of the indicator $1_{\mathcal{G}_G}(y,z)$ of its *graph* $\mathcal{G}_G = \{(y,G(y));\ y\in\mathbb{F}_2^m\}$, the ANF of the indicator $1_{\mathcal{G}_{G\circ F}}(x,z)$ of the graph of the *composite function* $G\circ F$ equals $1_{\mathcal{G}_G}(F(x),z)$, where we denote a function and its ANF the same way.
If we are given the ANF of $1_{\mathcal{G}_F}$ rather than that of $F(x)$, then as observed in [253, 254], $1_{\mathcal{G}_{G\circ F}}(x,z)$ can be obtained by the elimination of y from the two equations $1_{\mathcal{G}_F}(x,y)=1$ and $1_{\mathcal{G}_G}(y,z)=1$. Since for every x, there is exactly one y such that $1_{\mathcal{G}_F}(x,y)=1$, then $1_{\mathcal{G}_{G\circ F}}(x,z)$ equals $\sum_{y\in\mathbb{F}_2^m} 1_{\mathcal{G}_F}(x,y)1_{\mathcal{G}_G}(y,z) = \bigoplus_{y\in\mathbb{F}_2^m} 1_{\mathcal{G}_F}(x,y)1_{\mathcal{G}_G}(y,z)$. This formula can be easily iterated (with more than two functions) and we shall see that it gives an information which is more exploitable than $1_{\mathcal{G}_{G\circ F}}(x,z)=1_{\mathcal{G}_G}(F(x),z)$ because it deals with a multiplication instead of a composition. □

Algebraic degree of a vectorial function

The *algebraic degree* of an (n,m)-function is by definition the global degree of its ANF: $d_{alg}(F)=\max\{|I|;\ I\subseteq\{1,\dots,n\}, a_I\neq 0_m\}$. It therefore equals the maximal algebraic degree of the coordinate functions of F. It also equals the maximal algebraic degree of the *component functions* (in brief, components) of F, that is, the nonzero linear combinations of the coordinate functions, *i.e.* the functions of the form $v\cdot F$, where $v\in\mathbb{F}_2^m\setminus\{0_m\}$ and "$\cdot$" is an inner product in $\mathbb{F}_2^m$. The algebraic degree of vectorial functions is an affine invariant (that is, its value does not change when we compose F, on the right or on the left, by an affine automorphism). For functions of algebraic degree strictly larger than 1, it is an *EA invariant*, but it is not a CCZ invariant. In particular, the algebraic degrees of a permutation and its compositional inverse are in general not equal. It is however observed in [106] that if an (n,n)-permutation F has algebraic degree $n-1$ (the maximum for a permutation), then its inverse has also algebraic degree $n-1$. In fact, this is a direct consequence of Relation (2.5) by considering the terms x^Iy^J where $|I|=|J|=n-1$. Note that, according to Proposition 2 on the *graphs* of (n,m)-functions, writing $1_{\mathcal{G}_F}(x,y)$ in the form $\bigoplus_{J\subseteq\{1,\dots,m\}}\varphi_J(x)y^J$, we have that $d_{alg}(F)=\max_{|J|=m-1} d_{alg}(\varphi_J(x))$ and:

$$d_{alg}(1_{\mathcal{G}_F}) = \max_{J\subseteq\{1,\dots,m\}}\Big(d_{alg}\big(\prod_{j\in\{1,\dots,m\}\setminus J}(f_j\oplus 1)\big)+|J|\Big) \tag{2.7}$$

$$\geq \max(m, m-1+d_{alg}(F)). \tag{2.8}$$

If the algebraic degree of $1_{\mathcal{G}_F}$ is low (*i.e.* close to m), then all the products of a few coordinate functions of F have low algebraic degree.
Proposition 2 and the relation $1_{\mathcal{G}_{G\circ F}}(x,z)=\bigoplus_{y\in\mathbb{F}_2^m}1_{\mathcal{G}_F}(x,y)1_{\mathcal{G}_G}(y,z)$ lead in [254] to the bounds:

$$d_{alg}(G\circ F)\leq d_{alg}(1_{\mathcal{G}_F})+d_{alg}(G)-m \quad \text{and} \tag{2.9}$$

$$d_{alg}(H\circ G\circ F)\leq d_{alg}\left(1_{\mathcal{G}_F}\right)+d_{alg}\left(1_{\mathcal{G}_G}\right)+d_{alg}(H)-m-r, \tag{2.10}$$

for every (n,m)-function F, (m,r)-function G and (r,s)-function H. This is generalized to the composition of any number of functions in [254].
If F is a permutation, then, as observed in [253, 254], $1_{\mathcal{G}_{G\circ F}}(x,z)$ is equal to $\bigoplus_{y\in\mathbb{F}_2^m} 1_{\mathcal{G}_{F^{-1}}}(y,x)1_{\mathcal{G}_G}(y,z)$, that is, according to Proposition 2, page 51, and Proposition 3, page 53:

$$\begin{aligned} 1_{\mathcal{G}_{G\circ F}}(x,z) &= \bigoplus_{\substack{I\subseteq\{1,\dots,n\}\\ K\subseteq\{1,\dots,r\}}} x^I z^K \left(\bigoplus_{y\in\mathbb{F}_2^m}\left[\prod_{i\in I^c}(f_i'(y)\oplus 1)\prod_{k\in K^c}(g_k(y)\oplus 1)\right]\right)\\ &= \bigoplus_{\substack{I\subseteq\{1,\dots,n\},K\subseteq\{1,\dots,r\};\\ d_{alg}(\prod_{i\in I^c}(f_i'\oplus 1)\prod_{k\in K^c}(g_k\oplus 1))=n}} x^I z^K, \end{aligned} \tag{2.11}$$

where $I^c=\{1,\dots,n\}\setminus I$, $K^c=\{1,\dots,r\}\setminus K$ and the f_i''s are the coordinate functions of F^{-1} and the g_k's are those of G. Then, still according to Proposition 2 and as proved in [254], we have directly from (2.11) that:

$$d_{alg}(G\circ F)=\max_{k\in\{1,\dots,r\}}\left(\max\left\{|I|; d_{alg}\left((g_k\oplus 1)\prod_{i\in I^c}(f_i'\oplus 1)\right)=n\right\}\right). \tag{2.12}$$

This is generalized to the composition of any number of functions in [254]. We shall see at page 137 that this leads to an upper bound on $d_{alg}(G\circ F)$. Note that, according to Relation (2.5), page 52, and as observed by [106] (but in a more complex way), for every every integers k,l, the maximal algebraic degree of the product of at most k coordinate functions[15] of F, that we shall denote by $d_{alg}^{(k)}(F)$, satisfies: $d_{alg}^{(k)}(F)<n-l \iff d_{alg}^{(l)}(F^{-1})<n-k$.
The case of functions over $\mathbb{F}_{2^n}$ is also studied in [254].

Another notion of degree is also relevant to cryptography (and is also affine invariant): the minimum algebraic degree of all the component functions[16] of F, often called the *minimum degree*:

$$d_{min}(F)=\min\{d_{alg}(v\cdot F): 0_m\neq v\in\mathbb{F}_2^m\} \quad \leqslant \mathrm{d_{alg}}(F).$$

2.2.2 Univariate and trace representations

A second kind of representation plays an important role in sequence theory, and is also used for defining and studying Boolean functions. For instance, it allows to define the S-box of the AES and leads to the construction of the Kerdock codes (see Section 6.1.22). Recall that, for every n, there exists a (unique up to isomorphism) field $\mathbb{F}_{2^n}$ (also denoted by $GF(2^n)$ in some papers) of order 2^n, see [775, 890]. For making this book self-contained, we recall in Appendix (Chapter 14, page 521), the basics on finite fields, permutation polynomials and equations over finite fields. The vector space $\mathbb{F}_2^n$ can be endowed with the structure

[15] The algebraic degree of the product of k coordinate functions equals n if $k=n$ and is strictly smaller if $k<n$, as can be easily shown and is characteristic of permutations.
[16] Not just the coordinate functions; the notion would then not be affine invariant.

of this field $\mathbb{F}_{2^n}$ (by construction and because $\mathbb{F}_{2^n}$ has the structure of an n-dimensional $\mathbb{F}_2$-vector space; if we choose an $\mathbb{F}_2$-basis $(\alpha_1, \dots, \alpha_n)$ of this vector space, then every element $x \in \mathbb{F}_2^n$ can be identified with $x_1\,\alpha_1 + \cdots + x_n\,\alpha_n \in \mathbb{F}_{2^n}$). We shall still denote by x this element of the field.

Univariate representation of (n, n)-functions

Every mapping from $\mathbb{F}_{2^n}$ into $\mathbb{F}_{2^n}$ (and hence any (n, n)-function[17]) admits a (unique) representation as a polynomial over $\mathbb{F}_{2^n}$ in one variable and of (univariate) degree at most $2^n - 1$:

$$F(x) = \sum_{i=0}^{2^n-1} \delta_i x^i; \quad \delta_i \in \mathbb{F}_{2^n}. \tag{2.13}$$

Indeed, the function mapping every such polynomial to the corresponding polynomial function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^n}$ is $\mathbb{F}_{2^n}$-linear and has trivial kernel since a nonzero polynomial cannot have a number of distinct zeros larger than its degree. Since the dimensions of the $\mathbb{F}_{2^n}$-vector space of such polynomials and of the $\mathbb{F}_{2^n}$-vector space of all (n, n)-functions both equal 2^n, this function is a bijection.

Definition 9 *We call* univariate representation *of an (n, n)-function F the unique polynomial $\sum_{i=0}^{2^n-1} \delta_i X^i$ satisfying (2.13).*

We shall also sometimes write that F is in *univariate form*.

Remark. $\mathbb{F}_{2^n}$ is the set of solutions of equation $x^{2^n} + x = 0$. We can then better view the univariate representation of (n, n)-functions as lying in the quotient ring $\mathbb{F}_{2^n}[X]/(X^{2^n} + X)$, each element of this ring being then represented as the remainder in the division by $X^{2^n} + X$. □

Note that the univariate representation of any (n, n)-function can be obtained by the Lagrange interpolation method or as follows: since every element x in $\mathbb{F}_{2^n}^*$ satisfies $x^{2^n-1} = 1$, the function $x^{2^n-1} + 1$ equals the Dirac (or Kronecker) symbol (*i.e.* the indicator of $\{0\}$), the polynomial $\sum_{a \in \mathbb{F}_{2^n}} F(a)((X + a)^{2^n-1} + 1)$ is the univariate representation of F. Note in particular that the coefficient of x^{2^n-1} in this univariate representation equals the sum of all values $F(a)$. A way of obtaining more directly the univariate representation is by using the so-called Mattson-Solomon polynomial that we shall see at page 61.

Univariate representation of Boolean functions

Any Boolean function on $\mathbb{F}_{2^n}$ is a particular case of a vectorial function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^n}$ (since $\mathbb{F}_2$ is a subfield of $\mathbb{F}_{2^n}$) and has then a (unique) univariate representation. Recall that the mapping $x \mapsto x^2$ is a field automorphism

[17] Note that if m divides n, then any function from $\mathbb{F}_{2^n}$ into $\mathbb{F}_{2^m}$ is a function from $\mathbb{F}_{2^n}$ into $\mathbb{F}_{2^n}$; hence we also cover such (n, m)-functions here. When m does not divide n, we can view the elements of $\mathbb{F}_2^m$ as elements of $\mathbb{F}_2^m \times \{0_{n-m}\} \subset \mathbb{F}_2^n$ and represent them as elements of $\mathbb{F}_{2^n}$, but this is a little more artificial.

called the *Frobenius automorphism*. The polynomial $\sum_{i=0}^{2^n-1}\delta_i X^i$, $\delta_i \in \mathbb{F}_{2^n}$, is the univariate representation of a Boolean function if and only if the functions $\left(\sum_{i=0}^{2^n-1}\delta_i x^i\right)^2$ and $\sum_{i=0}^{2^n-1}\delta_i x^i$ take the same value at every $x \in \mathbb{F}_{2^n}$, that is if and only if $\sum_{i=0}^{2^n-1}\delta_i^2 X^{2i} \equiv \sum_{i=0}^{2^n-1}\delta_i X^i$ [mod $X^{2^n}+X$], that is, $\delta_0, \delta_{2^n-1} \in \mathbb{F}_2$ and, for every $i = 1, \ldots, 2^n-2$, $\delta_{2i} = \delta_i^2$, where the index $2i$ is taken mod 2^n-1.

Absolute trace representation of Boolean functions and vectorial functions

The *absolute trace function* on $\mathbb{F}_{2^n}$, $tr_n(x) = x + x^2 + x^{2^2} + \cdots + x^{2^{n-1}}$, is addressed at page 530 (it is $\mathbb{F}_2$-linear, satisfies $(tr_n(x))^2 = tr_n(x^2) = tr_n(x)$ and is valued in $\mathbb{F}_2$). The function $(x, y) \mapsto tr_n(x\,y)$ is an inner product in $\mathbb{F}_{2^n}$ (recall that this means it is symmetric and, for every $y \neq 0$, the function $x \to tr_n(x\,y)$ is a nonzero linear form over $\mathbb{F}_{2^n}$). Every Boolean function can be written in the form $f(x) = tr_n(F(x))$ where F is a mapping from $\mathbb{F}_{2^n}$ into $\mathbb{F}_{2^n}$ (an example of such mapping F is defined by $F(x) = \lambda\, f(x)$ where $tr_n(\lambda) = 1$ and $f(x)$ is in univariate representation). Thus, every n-variable Boolean function f can be also represented in the form

$$f(x) = tr_n\left(\sum_{i=0}^{2^n-1}\beta_i\, x^i\right), \tag{2.14}$$

where $\beta_i \in \mathbb{F}_{2^n}$. Note that, thanks to the fact that tr_n is $\mathbb{F}_2$-linear and $tr_n(x^2) = tr_n(x)$ for every $x \in \mathbb{F}_{2^n}$, each term $\beta_i\, x^i$ in (2.14) can be replaced by its 2^j-th power, for every j and without changing the value of the expression. We can then transform (2.14) into an expression $tr_n\left(\sum_{i\in I}\gamma_i\, x^i\right)$ where I contains at most one element of each *cyclotomic class* $\{i \times 2^j\,[\bmod\,(2^n-1)]\,;\, j \in \mathbb{N}\}$ of 2 modulo 2^n-1 (but this still does not make the representation unique).

More generally, if m is a divisor of n, then any (n,m)-function F admits a univariate polynomial representation in the form:

$$F(x) = tr_m^n\Big(\sum_{j=0}^{2^n-1}\delta_j x^j\Big), \tag{2.15}$$

where $tr_m^n(x) = x + x^{2^m} + x^{2^{2m}} + x^{2^{3m}} + \cdots + x^{2^{n-m}}$ is the trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$. Indeed, there exists a function G from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^n}$ such that F equals $tr_m^n \circ G$ (for instance, $G(x) = \lambda F(x)$, where $tr_m^n(\lambda) = 1$, since tr_m^n is a $\mathbb{F}_{2^m}$-linear form). But there is no uniqueness of G in this representation as well.

Definition 10 *We shall call the representation (2.14), resp. (2.15), an* absolute trace representation *of Boolean function f (resp. of (n,m)-function F).*

Its use is convenient, with the drawback of non-uniqueness which makes more difficult to determine when two functions are equal.

Subfield trace representation of Boolean functions

We come back to the univariate representation $\sum_{i=0}^{2^n-1} \delta_i X^i$. We have seen that for any Boolean function, we have δ_0, $\delta_{2^n-1} \in \mathbb{F}_2$ and, for every $i = 1, \ldots, 2^n-2$, $\delta_{2i} = \delta_i^2$, where the index $2i$ is taken modulo 2^n-1. Gathering all the elements of a same cyclotomic class of 2 modulo 2^n-1 provides the univariate representation of f in the following form:

$$f(x) = \sum_{j\in\Gamma(n)} tr_{n_j}(\beta_j x^j) + \beta_{2^n-1} x^{2^n-1}, \text{ with } \begin{cases} \forall j \in \Gamma(n), \beta_j \in \mathbb{F}_{2^{n_j}}, \\ \beta_{2^n-1} \in \mathbb{F}_2 \end{cases} \tag{2.16}$$

where $\Gamma(n)$ is a set of representatives of the cyclotomic classes of 2 modulo 2^n-1 (the most usual choice of representative is the smallest element in the cyclotomic class, called the *coset leader* of the class) and n_j is the size of the cyclotomic class containing j. It is easily seen that n_j divides n and that $\beta_j \in \mathbb{F}_{2^{n_j}}$ because $\beta_j^{2^{n_j}} = \beta_j$. We also have that the j-th power of every $x \in \mathbb{F}_{2^n}$ belongs to $\mathbb{F}_{2^{n_j}}$ because $j\, 2^{n_j} \equiv j$ [mod 2^n-1] implies $(x^j)^{2^{n_j}} = x^j$. Hence, tr_{n_j} takes as argument an element of $\mathbb{F}_{2^{n_j}}$, as it should. This representation allows uniqueness.

Definition 11 *We call (2.16) the* subfield trace representation *of function f.*

We shall also sometimes write more simply that f is in *trace form*.

Calculating the univariate and subfield trace representations of a Boolean function from its truth table

Denoting by α a *primitive element* of the field $\mathbb{F}_{2^n}$ (recall that this means that $\mathbb{F}_{2^n} = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{2^n-2}\}$), the *Mattson-Solomon polynomial*[18] of the vector $(f(1), f(\alpha), f(\alpha^2), \ldots, f(\alpha^{2^n-2}))$ is the polynomial [809, page 239]:

$$A(x) = \sum_{j=1}^{2^n-1} A_j x^{2^n-1-j} = \sum_{j=0}^{2^n-2} A_{2^n-1-j} x^j \tag{2.17}$$

with:

$$A_j = \sum_{k=0}^{2^n-2} f(\alpha^k)\alpha^{kj}. \tag{2.18}$$

Note that $A_j = a(\alpha^j)$, where $a(x) = \sum_{k=0}^{2^n-2} f(\alpha^k)x^k$.
We have, for every $0 \le i \le 2^n-2$:

$$A(\alpha^i) = \sum_{j=1}^{2^n-1} A_j \alpha^{-ij} = \sum_{j=1}^{2^n-1}\sum_{k=0}^{2^n-2} f(\alpha^k)\alpha^{(k-i)j} = f(\alpha^i) \tag{2.19}$$

(since, if $1 \le k \ne i \le 2^n-2$, then $\displaystyle\sum_{j=1}^{2^n-1} \alpha^{(k-i)j} = \sum_{j=0}^{2^n-2} \alpha^{(k-i)j} = \frac{\alpha^{(k-i)(2^n-1)}+1}{\alpha^{k-i}+1} =$ 0, and if $k-i = 0$, then $\sum_{j=1}^{2^n-1} \alpha^{(k-i)j} = 1$). Note that, with the usual convention

[18] The Mattson-Solomon transform is a discrete Fourier transform (over $\mathbb{F}_{2^n}$); other discrete Fourier transforms exist (*e.g.* over the complex field, like in [1111]).

$0^0 = 1$, we have $A(0) = A_{2^n-1}$. Hence, if $f(0) = A_{2^n-1} = \sum_{k=0}^{2^n-2} f(\alpha^k)$, that is, if f has even Hamming weight (*i.e.* algebraic degree strictly less than n), the Mattson-Solomon polynomial $A(x)$ equals the univariate representation of $f(x)$. Otherwise, we have $f(x) = A(x)+1+x^{2^n-1}$, since $1+x^{2^n-1}$ equals the Dirac (or Kronecker) function at 0 (*i.e.* takes value 1 at 0 and 0 at every nonzero element of $\mathbb{F}_{2^n}$). This provides the univariate representation:

$$f(x) = f(0) + \sum_{j=1}^{2^n-2} A_j x^{2^n-1-j} + (w_H(f)\ [\text{mod } 2])\, x^{2^n-1}$$

and the subfield trace representation:

$$f(x) = \sum_{j\in\Gamma(n)} tr_{n_j}(A_{2^n-1-j}x^j) + (w_H(f)\ [\text{mod } 2])(1 + x^{2^n-1}).$$

Remark. For any Boolean function f, we have in (2.18) that $A_{2j} = A_j^2$ and this allows to gather the terms corresponding to a same cyclotomic class. This provides the subfield trace representation of f. We can also, thanks to a change of the coefficients, write

$$f(\alpha^j) = \sum_{j=1}^{2^n-2} tr_n(a_j\alpha^{-ij}) \tag{2.20}$$

and obtain the absolute trace representation of f. This shows what was asserted at the end of Subsection 1.2.4. □

Remark on RS codes.
Relations (2.17), (2.18) and (2.19) are valid for every function f from $\mathbb{F}_{2^n}^*$ to $\mathbb{F}_{2^n}$. In this framework, $A(x)$, which according to Relation (2.17) is the polynomial representation (see page 27) of codeword $(A_{2^n-1}, A_{2^n-2}, \dots, A_1)$, belongs to the *Reed-Solomon code* (see page 29) over $\mathbb{F}_{2^n}$ of length $2^n - 1$ and zeros $\alpha^{2^n-\delta}, \alpha^{2^n-\delta+1}, \dots, \alpha^{2^n-2}$ (whose designed distance is δ) if and only if $a(x)$ has degree at most $2^n - 1 - \delta$ (according to Relation (2.19)), and the codeword $(A_{2^n-1}, A_{2^n-2}, \dots, A_1)$ is an evaluation vector of this polynomial over $\mathbb{F}_{2^n}^*$, according to Relation (2.18). The BCH bound in this case corresponds to the fact that a nonzero polynomial of degree at most $2^n - 1 - \delta$ has at most $2^n - 1 - \delta$ zeros in $\mathbb{F}_{2^n}$ and therefore has at least δ nonzeros in $\mathbb{F}_{2^n}^*$. This generalizes to RS codes over $\mathbb{F}_q$. □

Calculating the ANF of a Boolean function or a vectorial function from its univariate representation

We express x in the form $\sum_{i=1}^n x_i\alpha_i$, where $(\alpha_1, \dots, \alpha_n)$ is a basis of the $\mathbb{F}_2$-vector space $\mathbb{F}_{2^n}$. Recall that, for every $j \in \mathbb{Z}/(2^n - 1)\mathbb{Z}$, the *binary expansion* of j has the form $\sum_{s\in E} 2^s$, where $E \subseteq \{0, 1, \dots, n-1\}$. The size of E is often called the *2-weight* of j and written $w_2(j)$. We write more conveniently the binary expansion of j in the form: $\sum_{s=0}^{n-1} j_s 2^s$, $j_s \in \{0,1\}$. We have then:

$$\begin{aligned} F(x) &= \sum_{j=0}^{2^n-1} \delta_j \left(\sum_{i=1}^{n} x_i\alpha_i\right)^j \\ &= \sum_{j=0}^{2^n-1} \delta_j \left(\sum_{i=1}^{n} x_i\alpha_i\right)^{\sum_{s=0}^{n-1} j_s 2^s} \\ &= \sum_{j=0}^{2^n-1} \delta_j \prod_{s=0}^{n-1} \left(\sum_{i=1}^{n} x_i\alpha_i^{2^s}\right)^{j_s}. \end{aligned}$$

Expanding these last products and simplifying gives the ANF of F.

Proposition 6 *Any Boolean function (resp. any (n,n)-function) whose univariate representation equals (2.13) has algebraic degree* $\max_{j=0,\ldots,2^n-1;\,\delta_j\neq 0} w_2(j)$.

Proof. According to the above equalities, the algebraic degree is bounded above by this number, and it cannot be strictly smaller, because the dimension of the $\mathbb{F}_2$-vector space (resp. the $\mathbb{F}_{2^n}$-vector space) of Boolean n-variable functions (resp. of (n,n)-functions) of algebraic degree at most d equals $\sum_{i=0}^{d}\binom{n}{i}$, which is also the dimension of the vector space of those polynomials $\sum_{j=0}^{2^n-1}\delta_j x^j$ such that $\delta_0,\ \delta_{2^n-1} \in \mathbb{F}_2$, $\delta_j \in \mathbb{F}_{2^n}$, $\delta_{2j} = \delta_j^2 \in \mathbb{F}_{2^n}$ for every $j = 1,\ldots,2^n-2$ and $\max_{j=0,\ldots,2^n-1;\,\delta_j\neq 0} w_2(j) \le d$ (resp. of those polynomials $\sum_{j=0}^{2^n-1}\delta_j x^j$ such that $\delta_j \in \mathbb{F}_{2^n}$ for every $j = 0,\ldots,2^n-1$ and $\max_{j=0,\ldots,2^n-1;\,\delta_j\neq 0} w_2(j) \le d$). □

In particular, an (n,n)-function F is $\mathbb{F}_2$-linear (resp. affine) if and only if $F(x)$ is a *linearized polynomial* over $\mathbb{F}_{2^n}$: $F(x) = \sum_{j=0}^{n-1}\beta_j x^{2^j}; x,\beta_j \in \mathbb{F}_{2^n}$ (resp. a linearized polynomial plus a constant).

We have also:

Proposition 7 *[209] Let a be any element of $\mathbb{F}_{2^n}$ and k any integer [mod 2^n-1]. If $f(x) = tr_n(ax^k)$ is not the null function, then it has algebraic degree $w_2(k)$.*

Proof. Let n_k be again the size of the cyclotomic class containing k. Then the univariate representation of $f(x)$ equals

$$\left(a + a^{2^{n_k}} + a^{2^{2n_k}} + \cdots + a^{2^{n-n_k}}\right) x^k + \left(a + a^{2^{n_k}} + a^{2^{2n_k}} + \cdots + a^{2^{n-n_k}}\right)^2 x^{2k}$$
$$+ \cdots + \left(a + a^{2^{n_k}} + a^{2^{2n_k}} + \cdots + a^{2^{n-n_k}}\right)^{2^{n_k-1}} x^{2^{n_k-1}k}.$$

All the exponents of x have 2-weight $w_2(k)$ and their coefficients are nonzero if and only if f is not null. □

Remark. An alternative (more complex but enlightening) way of showing Proposition 7 is also given in [209] as follows: let $r = w_2(k)$; we consider the r-linear function ϕ over the field $\mathbb{F}_{2^n}$ whose value at $(x_1,\dots,x_r)\in(\mathbb{F}_{2^n})^r$ equals the sum of the images by f of all the 2^r possible linear combinations of the x_j's. Then $\phi(x_1,\dots,x_r)$ equals the sum, for all bijective mappings σ from $\{1,\dots,r\}$ onto E (where $k=\sum_{s\in E}2^s$) of $tr_n(a\prod_{j=1}^r x_j^{2^{\sigma(j)}})$. Proving that f has degree r is equivalent to proving that ϕ is not null, and it can be shown that if ϕ is null, then f is null. □

Remark. For calculating the univariate representation from the ANF, we can only propose to calculate the truth table (resp. the LUT) by the fast Möbius transform and then to apply the method of page 61. Note however that the coefficient of $\prod_{i=1}^n x_i$ in the ANF of F is directly linked to the coefficient of x^{2^n-1} in its univariate representation since these two coefficients are equal to each other (up to the correspondence between $\mathbb{F}_2^n$ and $\mathbb{F}_{2^n}$) because they are both equal to the sum of all values $F(x)$. □

To complete this subsection, we give a corollary of Proposition 6 (which for $d=n-2,n-1$ gives back the two last properties in Proposition 3, page 53):

Corollary 2 *A vectorial function $F:\mathbb{F}_{2^n}\mapsto\mathbb{F}_{2^n}$ has algebraic degree at most d if and only if, for every non-negative integer k of 2-weight at most $n-d-1$, we have:*

$$\sum_{x\in\mathbb{F}_{2^n}} x^kF(x)=0.$$

The condition is necessary by applying to function $x^kF(x)$ the fact that, for every (n,n)-function G of algebraic degree at most $n-1$, we have $\sum_{x\in\mathbb{F}_{2^n}}G(x)=0$, and since, for every non-negative integer i, we have $w_2(k+i)\le w_2(k)+w_2(i)$. The condition is also sufficient since, for every (n,n)-function G of algebraic degree n, we have $\sum_{x\in\mathbb{F}_{2^n}}G(x)\neq 0$, and since, for every i of 2-weight strictly larger than d, there exists k of 2-weight at most $n-d-1$ such that $w_2(k+i)=n$; if i is taken with highest possible 2-weight in the univariate representation of F, we can manage that $\sum_{x\in\mathbb{F}_{2^n}}x^{k+j}=0$ for other $j\neq i$ such that x^j has nonzero coefficient in the univariate representation.

See more on the algebraic degree, in particular for composite functions, in ANF or univariate representations, in [253, 254].

2.2.3 Bivariate representation of functions with even number of input bits

The *bivariate representation* of n-variable Boolean functions f and of (n,m)-functions F where n is even and $m=\frac{n}{2}$ is as follows: we identify $\mathbb{F}_2^n$ with $\mathbb{F}_{2^m}\times\mathbb{F}_{2^m}$ and we consider then the input to F as an ordered pair (x,y) of elements of $\mathbb{F}_{2^m}$. There exists a unique bivariate polynomial $\sum_{0\le i,j\le 2^m-1}a_{i,j}x^iy^j$ over $\mathbb{F}_{2^m}$ such that the given function is the bivariate polynomial function over $\mathbb{F}_{2^m}$ associated to it. Then the algebraic degree of the function equals

$\max_{(i,j)\,|\,a_{i,j}\neq 0}(w_2(i) + w_2(j))$, and in the case of a Boolean function, the bivariate representation can be written in the form $f(x, y) = tr_m(P(x, y))$ where $P(x, y)$ is some polynomial in two variables over $\mathbb{F}_{2^m}$. This latter absolute trace representation is not unique. A unique representation uses relative traces, see [245, Section 2.4.2].

Moving from bivariate to univariate representation and *vice versa*

Any bivariate Boolean or vectorial function $F(x, y)$ over $\mathbb{F}_{2^{n/2}}$ and valued in $\mathbb{F}_{2^{n/2}}$ can be represented as a function of $X \in \mathbb{F}_{2^n}$, that we can denote by $F(X)$ by abuse of notation, by posing $x = tr^n_{n/2}(aX) = aX + (aX)^{2^{n/2}}$ and $y = tr^n_{n/2}(bX) = bX + (bX)^{2^{n/2}}$ for some $\mathbb{F}_2^{n/2}$-linearly independent elements $a, b \in \mathbb{F}_{2^n}$ (constituting a basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_{2^{n/2}}$; choosing another basis would result in a linearly equivalent function). The obtained expression can be expressed by means of tr_n by using that, for every $\lambda \in \mathbb{F}_{2^{n/2}}$, we have $tr_{n/2}(\lambda) = tr_n(a\lambda)$ where $tr^n_{n/2}(a) = a + a^{2^{n/2}} = 1$. Conversely, given a Boolean or vectorial function $F(X)$ over $\mathbb{F}_{2^n}$ valued in $\mathbb{F}_{2^{n/2}}$ in univariate representation and a basis (u, v) of $\mathbb{F}_{2^n}$ over $\mathbb{F}_{2^{n/2}}$, we get its bivariate representation by decomposing X over this basis into $X = ux + vy$. The obtained expression can be expressed by means of $tr_{n/2}$ by using that, for every $u \in \mathbb{F}_{2^n}$, we have $tr_n(u) = tr_{n/2}(tr^n_{n/2}(u)) = tr_{n/2}(u + u^{2^{n/2}})$.

2.2.4 Representation over the reals (numerical normal form)

This version over $\mathbb{R}$ (in fact, over $\mathbb{Z}$, for Boolean and integer-valued functions over $\mathbb{F}_2^n$) of the algebraic normal form has proved itself useful for characterizing several cryptographic criteria [220, 292, 293] (see Chapters 6 and 7). When studied in these papers, it was already known in other domains of Boolean functions (see *e.g.* [886, 905]), but rather informally studied.

Definition 12 *[292] We call* numerical normal form *(*NNF*) the representation of pseudo-Boolean functions (*i.e. *real-valued functions over $\mathbb{F}_2^n$) in the quotient ring $\mathbb{R}[x_1, \ldots, x_n]/(x_1^2 - x_1, \ldots, x_n^2 - x_n)$ (or $\mathbb{Z}[x_1, \ldots, x_n]/(x_1^2 - x_1, \ldots, x_n^2 - x_n)$ for integer-valued functions).*

The existence of this representation for every pseudo-Boolean function can be shown with the same arguments as for the ANFs of Boolean functions (writing $1 - x_i$ instead of $1 \oplus x_i$). In the case of a Boolean function, it can also be directly deduced from the existence of the ANF, since, denoting $x^I = \prod_{i \in I} x_i$, we have:

$$\begin{aligned} f(x) = \bigoplus_{I \subseteq \{1,\ldots,n\}} a_I\, x^I &\Longleftrightarrow (-1)^{f(x)} = \prod_{I \subseteq \{1,\ldots,n\}} (-1)^{a_I\, x^I} \\ &\Longleftrightarrow 1 - 2\, f(x) = \prod_{I \subseteq \{1,\ldots,n\}} (1 - 2\, a_I\, x^I) \end{aligned} \quad (2.21)$$

and expanding (2.21) gives the NNF of $f(x)$.
The uniqueness of the NNF of any pseudo-Boolean function is deduced from its existence by the usual argument: the linear mapping from every element of the 2^n-dimensional $\mathbb{R}$-vector space $\mathbb{R}[x_1,\dots,x_n]/(x_1^2-x_1,\dots,x_n^2-x_n)$ to the corresponding pseudo-Boolean function on $\mathbb{F}_2^n$ being surjective, it is therefore one to one (the $\mathbb{R}$-vector space of pseudo-Boolean functions on $\mathbb{F}_2^n$ having also dimension 2^n).

Remark. The NNF does not contain properly speaking more information on a Boolean function than its ANF, since both are unique representations and contain then full information on the function. But the NNF contains more exploitable information in the sense that the coefficients of the ANF contain individually little information on the function, while we shall see that those of the NNF contain more. □

Definition 13 *[292] We call the degree of the NNF of a Boolean or pseudo-Boolean function f its* numerical degree *and denote it by $d_{num}(f)$.*

Since the ANF of a Boolean function is the mod 2 version of its NNF, the numerical degree is always bounded below by the algebraic degree.

It is shown in [905] that, if a Boolean function f has no ineffective variable (*i.e.* if it actually depends on each of its variables), then the numerical degree of f is larger than or equal to $\log_2 n - \mathcal{O}(\log_2\log_2 n)$ (we shall give a proof of this bound - in fact of a slightly more precise and stronger bound - in Proposition 15, page 86).
The numerical degree is *permutation invariant* but is not affine invariant. Nevertheless, the NNF leads to an affine invariant (see a proof of this fact in [293]) which is more discriminant than the algebraic degree:

Definition 14 *[293] Let f be a Boolean function on $\mathbb{F}_2^n$. We call* generalized degree *of f the sequence $(d_i)_{i\geq 1}$ defined as follows:*
for every $i \geq 1$, d_i is the smallest integer $d > d_{i-1}$ (if $i>1$) such that, for every multi-index I of size strictly larger than d, the coefficient λ_I of x^I in the NNF of f is a multiple of 2^i.

Example: the generalized degree of any nonzero affine function is the sequence of all positive integers.

Similarly as for the ANF, a (pseudo-) Boolean function $f(x)=\sum_{I\subseteq\{1,\dots,n\}}\lambda_I\, x^I$ takes value:

$$f(x)=\sum_{I\subseteq supp(x)}\lambda_I. \tag{2.22}$$

But, contrary to what we observed for the ANF, the reverse formula is not identical to the direct formula:

Proposition 8 *[292] Let f be a pseudo-Boolean function on $\mathbb{F}_2^n$ and let its NNF be $\sum_{I\subseteq\{1,\ldots,n\}} \lambda_I\, x^I$. Then:*

$$\forall I \subseteq \{1,\ldots,n\},\ \lambda_I = (-1)^{|I|} \sum_{x\in\mathbb{F}_2^n;\ supp(x)\subseteq I} (-1)^{w_H(x)} f(x). \tag{2.23}$$

In other words, function f and its NNF are related through the *Möbius transform over integers* and its inverse (for which there exist algorithms similar to the fast binary Möbius transform).

Proof. Let us denote the number $(-1)^{|I|} \sum_{x\in\mathbb{F}_2^n;\ supp(x)\subseteq I} (-1)^{w_H(x)} f(x)$ by μ_I and consider the function $g(x) = \sum_{I\subseteq\{1,\ldots,n\}} \mu_I\, x^I$. We have

$$g(x) = \sum_{I\subseteq supp(x)} \mu_I = \sum_{I\subseteq supp(x)} \left((-1)^{|I|} \sum_{y\in\mathbb{F}_2^n;\ supp(y)\subseteq I} (-1)^{w_H(y)} f(y) \right)$$

and thus

$$g(x) = \sum_{y\in\mathbb{F}_2^n} (-1)^{w_H(y)} f(y) \left(\sum_{I\subseteq\{1,\ldots,n\};\ supp(y)\subseteq I\subseteq supp(x)} (-1)^{|I|} \right).$$

The sum $\sum_{I\subseteq\{1,\ldots,n\};\ supp(y)\subseteq I\subseteq supp(x)} (-1)^{|I|}$ is null if $supp(y) \not\subseteq supp(x)$. It is also null if $supp(y)$ is included in $supp(x)$, but different. Indeed, denoting $|I| - w_H(y)$ by i, it equals $\pm \sum_{i=0}^{w_H(x)-w_H(y)} \binom{w_H(x)-w_H(y)}{i} (-1)^i = \pm(1-1)^{w_H(x)-w_H(y)} = 0$. Hence, $g = f$ and, by uniqueness of the NNF, we have $\mu_I = \lambda_I$ for every I. □

Remark. According to Relation (2.4), page 50, the coefficient of x^I in the ANF of a Boolean function f is equal to zero if and only if $supp(f) \cap \{x \in \mathbb{F}_2^n; supp(x) \subseteq I\}$ has even size. According to Relation (2.23), the coefficient of x^I in the NNF of a Boolean function f is equal to zero if and only if $supp(f) \cap \{x \in \mathbb{F}_2^n; supp(x) \subseteq I\} \cap \{x \in \mathbb{F}_2^n; w_H(x) \text{ even}\}$ has same size as $supp(f) \cap \{x \in \mathbb{F}_2^n; supp(x) \subseteq I\} \cap \{x \in \mathbb{F}_2^n; w_H(x) \text{ odd}\}$. □

Remark. Denoting function $\bigoplus_{i=1}^n x_i$ by $\ell(x)$ and taking $I \neq \emptyset$, Relation (2.23) can be interpreted as $\lambda_I = (-1)^{|I|} \sum_{x\in\mathbb{F}_2^n;\ supp(x)\subseteq I} \left(\frac{(-1)^{\ell(x)}}{2} - \frac{(-1)^{f(x)\oplus\ell(x)}}{2} \right)$, and, since I is not empty, ℓ is linear and non-constant over the vector space $E_I = \{x \in \mathbb{F}_2^n;\ supp(x) \subseteq I\}$, and we have $\sum_{x\in\mathbb{F}_2^n;\ supp(x)\subseteq I} \frac{(-1)^{\ell(x)}}{2} = 0$. After replacing $(-1)^{f(x)\oplus\ell(x)}$ by $1 - 2(f \oplus \ell)(x)$, this gives

$$\lambda_I = (-1)^{|I|} \left(w_H((f\oplus\ell)_{|E_I}) - 2^{|I|-1} \right),$$

where $(f \oplus \ell)_{|E_I}$ is the restriction of the Boolean function $f \oplus \ell$ to E_I. Applying this to the function $f \oplus \ell$ instead of f, we can see that the coefficients in the

NNF of $f \oplus \ell$ give the Hamming weights of the restrictions of f to all vector subspaces of $\mathbb{F}_2^n$ of the form $\{x \in \mathbb{F}_2^n;\ supp(x) \subseteq I\}$. $\square$

We have seen that the ANF $f(x) = \bigoplus_{I\subseteq\{1,\dots,n\}} a_I x^I$ of any Boolean function can be deduced from its NNF $f(x) = \sum_{I\subseteq\{1,\dots,n\}} \lambda_I x^I$ by reducing it modulo 2, and that, conversely, the NNF can be deduced from the ANF. The formula is obtained by expanding (2.21) (and has been first obtained in [292] by a slightly more complex way):

$$\lambda_I = \sum_{k=1}^{2^n} (-2)^{k-1} \sum_{\substack{\{I_1,\dots,I_k\}\mid\\ I_1\cup\dots\cup I_k=I}} a_{I_1}\dots a_{I_k}, \tag{2.24}$$

where "$\{I_1,\dots,I_k\};\ I_1\cup\dots\cup I_k = I$" means that the multi-indices $I_1,\dots,I_k$ are all distinct, in indefinite order, and that their union equals I.
For instance, for the Boolean function $f(x) = \bigoplus_{i=1}^n x_i$, we have $\lambda_I = (-2)^{|I|-1}$. This, applied to f_i in the place of x_i, implies that, for every Boolean functions $f_1,\dots,f_k$, we have:

$$\bigoplus_{i=1}^k f_i = \sum_{\emptyset\neq I\subseteq\{1,\dots,k\}} (-2)^{|I|-1} \prod_{i\in I} f_i. \tag{2.25}$$

Applying then Relation (2.25) to each $J \subseteq \{1,\dots,k\}$ instead of $\{1,\dots,k\}$ provides the system of the relations $\bigoplus_{i\in J} f_i = \sum_{\emptyset\neq I\subseteq J}(-2)^{|I|-1}\prod_{i\in I} f_i$ which can be inverted and gives the expression of the product of the f_i's by means of their linear combinations over $\mathbb{R}$:

$$\prod_{i=1}^l f_i = \frac{1}{2^{l-1}} \sum_{\emptyset\neq J\subseteq\{1,\dots,l\}} (-1)^{|J|-1} \left(\bigoplus_{i\in J} f_i\right). \tag{2.26}$$

Indeed, $\sum_{J;\, I\subseteq J\subseteq\{1,\dots,l\}} (-1)^{|J|-1}$ equals $(-1)^{l-1}$ if $I = \{1,\dots,l\}$ and is null otherwise and this shows that the matrices of the two systems of relations are inverses of each other.

A polynomial $P(x) = \sum_{J\subseteq\{1,\dots,n\}} \lambda_J\, x^J$, with real coefficients, is the NNF of some Boolean function if and only if we have $P^2(x) = P(x)$, for every $x \in \mathbb{F}_2^n$ (which is equivalent to $P = P^2$ in $\mathbb{R}[x_1,\dots,x_n]/(x_1^2 - x_1,\dots,x_n^2 - x_n)$), or equivalently, denoting $supp(x)$ by I:

$$\forall I \subseteq \{1,\dots,n\},\ \left(\sum_{J\subseteq I} \lambda_J\right)^2 = \sum_{J\subseteq I} \lambda_J. \tag{2.27}$$

Remark. Imagine that we want to generate a random Boolean function through its NNF (this can be useful, since we will see below that the main cryptographic criteria, on Boolean functions, can be characterized, in simple ways, through their NNFs). Assume that we have already chosen the values λ_J for every $J \subseteq I$

(where $I \subseteq \{1, \ldots, n\}$ is some multi-index) except for I itself. Let us denote the sum $\sum_{J \subseteq I \,|\, J \neq I} \lambda_J$ by μ. Relation (2.27) gives $(\lambda_I + \mu)^2 = \lambda_I + \mu$. This equation of degree 2 has two solutions. One solution corresponds to the choice $P(x) = 0$ (where $I = supp(x)$) and the other one corresponds to the choice $P(x) = 1$. □

Thus, verifying that a polynomial $P(x) = \sum_{I \subseteq \{1,\ldots,n\}} \lambda_I \, x^I$ with real coefficients represents a Boolean function can be done by checking 2^n relations. But it can also be done by verifying a simple condition on P and checking a single equation.

Proposition 9 *[293] Any polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]/(x_1^2 - x_1, \ldots, x_n^2 - x_n)$ is the NNF of an integer-valued function if and only if all of its coefficients are integers. Assuming that this condition is satisfied, then P is the NNF of a Boolean function if and only if: $\sum_{x \in \mathbb{F}_2^n} P^2(x) = \sum_{x \in \mathbb{F}_2^n} P(x)$.*

Proof. The first assertion is a direct consequence of Relations (2.22) and (2.23). If all the coefficients of P are integers, then we have $P^2(x) \geq P(x)$ for every x; this implies that the 2^n equalities (one for each x), expressing that the corresponding function is Boolean, can be reduced to the single one $\sum_{x \in \mathbb{F}_2^n} P^2(x) = \sum_{x \in \mathbb{F}_2^n} P(x)$. □

According to Relation (2.27), the translation of this characterization in terms of the coefficients λ_I of $P(x)$ writes:

$$\sum_{I \subseteq \{1,\ldots,n\}} 2^{n-|I|} \sum_{J,J' \subseteq \{1,\ldots,n\};\; I = J \cup J'} \lambda_J \, \lambda_{J'} = \sum_{I \subseteq \{1,\ldots,n\}} 2^{n-|I|} \lambda_I, \tag{2.28}$$

since the number of those $x \in \mathbb{F}_2^n$ such that $I \subseteq supp(x)$, equals $2^{n-|I|}$.

More results related to the NNF can be found in [292] and [293].

Case of vectorial functions

An extention of the NNF to (n,m)-*functions* is given in [484], but it seems simpler to consider the NNF of the *indicator* $1_{\mathcal{G}_F}$ of the *graph* $\mathcal{G}_F = \{(x, F(x));\; x \in \mathbb{F}_2^n\}$. We obtain a (unique) characterization of the form:

$$\forall x \in \mathbb{F}_2^n, \forall y \in \mathbb{F}_2^m,\; (y = F(x)) \Leftrightarrow \left(\sum_{\substack{I \subseteq \{1,\ldots,n\} \\ J \subseteq \{1,\ldots,m\}}} \lambda_{I,J} x^I y^J = 1 \right).$$

Note that, if we have the NNF of each coordinate function f_j of F, for $j = 1, \dots, m$, then the NNF of $1_{\mathcal{G}_F}$ can be deduced from:

$$\begin{aligned} 1_{\mathcal{G}_F}(x,y) &= \prod_{j=1}^{m} \left(1-(f_j(x)-y_j)^2\right) \\ &= \prod_{j=1}^{m} \left(1-f_j(x)+y_j(2f_j(x)-1)\right) \\ &= \sum_{J\subseteq\{1,\dots,m\}} \left(\prod_{j\in\{1,\dots,m\}\setminus J} (1-f_j(x)) \prod_{j\in J}(2f_j(x)-1)\right) y^J. \end{aligned}$$

Note that in the case of a Boolean function f (*i.e.* in the case of $m = 1$), we have then $1_{\mathcal{G}_F}(x,y) = 1 - f(x) + (2f(x)-1)\,y$, for $x \in \mathbb{F}_2^n$ and $y \in \mathbb{F}_2$.

Remark. As we can see, some representations of Boolean functions (resp. of vectorial function) like the ANF are such that any object having the form of an ANF is the ANF of some function. Some others like the NNF do not have such property. The Fourier-Hadamard and Walsh transforms that we shall see below provide also representations of Boolean and vectorial functions, which are of the latter kind. Some other representations also exist, see *e.g.* [484] where their relationships are studied as well as their behavior with respect to composition, and their eigenanalysis in relation with graphs (see page 89), in the case of representations by square matrices. □

2.3 The Fourier-Hadamard transform and the Walsh transform

2.3.1 Fourier-Hadamard transform of pseudo-Boolean functions

Almost all the characteristics needed for Boolean functions in cryptography and for sets of Boolean functions in coding can be expressed by means of the weights of two kinds of related Boolean functions: $f \oplus \ell$ where ℓ is linear[19], and $D_af(x) = f(x) \oplus f(x+a)$ (the derivatives of f). In this framework, the Fourier-Hadamard transform is an efficient tool: for a given Boolean function f, the Fourier-Hadamard transform of f provides the knowledge of the weights of all the functions $f \oplus \ell$, where ℓ is a linear (or an affine) form, and the weights of the *derivatives* D_af are also directly related to the Fourier-Hadamard transform.

Definition 15 *The* Fourier-Hadamard transform[20] *is the* $\mathbb{R}$*-linear mapping which*

[19] As far as we know, and as reported in [555, 1111], the weights of these functions have been originally considered by S. Golomb [549] to define what he called invariants: given a positive integer $t \le n$, the t-th invariant defined by Golomb is the unordered set of values $\max(w_H(f(x) \oplus u\cdot x), w_H(f(x) \oplus u\cdot x \oplus 1))$, where a ranges over $\mathbb{F}_2^n$.

[20] We write "Fourier-Hadamard" because "Fourier" would be ambiguous (and for the reason that the matrix involved in the transform is the Hadamard matrix [609], see page 214); even "discrete Fourier" would be ambiguous, see *e.g.* [1111].

maps any pseudo-Boolean function φ *on* $\mathbb{F}_2^n$ *to the function* $\widehat{\varphi}$ *defined on* $\mathbb{F}_2^n$ *by*

$$\widehat{\varphi}(u) = \sum_{x\in\mathbb{F}_2^n} \varphi(x)\,(-1)^{u\cdot x}, \tag{2.29}$$

where "$\cdot$" is some chosen inner product in $\mathbb{F}_2^n$. *We call* Fourier-Hadamard spectrum *of* φ *the multi-set of all the values* $\widehat{\varphi}(u)$*, where* $u \in \mathbb{F}_2^n$ *and* Fourier-Hadamard support *of* φ *the set of those* u *such that* $\widehat{\varphi}(u) \neq 0$.

Remark. The most used inner product in $\mathbb{F}_2^n$ is the *usual inner product* $u\cdot x = u_1\,x_1 \oplus\cdots\oplus u_n\,x_n$. If $\mathbb{F}_2^n$ is identified to the finite field $\mathbb{F}_{2^n}$, then $u\cdot x = tr_n(ux)$; $u, x\in\mathbb{F}_{2^n}$, is better used; and if n is even, say $n = 2m$, and $\mathbb{F}_2^n$ is identified to $\mathbb{F}_{2^m}^2$, then it is $(u_1,u_2)\cdot(x_1,x_2) = tr_m(u_1x_1 + u_2x_2)$; $u_1,u_2,x_1,x_2\in\mathbb{F}_{2^m}$. In all cases, the *Walsh functions* $(-1)^{u\cdot x}$ constitute an orthogonal basis of the vector space $\mathbb{R}^{\mathbb{F}_2^n}$ over $\mathbb{R}$, according to properties we shall see at page 77. □

Recall that every linear form over $\mathbb{F}_2^n$ equals $\ell_u : x\mapsto u\cdot x$ for some unique u in $\mathbb{F}_2^n$. If φ is a Boolean function (viewed as an integer-valued function), then $\widehat{\varphi}(0)$ equals $w_H(\varphi)$ and, for $u\neq 0_n$, $\widehat{\varphi}(u) = \sum_{x\in\mathbb{F}_2^n}\varphi(x)\,(1-2\,u\cdot x)$ equals $w_H(\varphi) - 2w_H(\varphi\,\ell_u) = w_H(\varphi\oplus\ell_u) - w_H(\ell_u) = w_H(\varphi\oplus\ell_u) - 2^{n-1}$. This proves what we asserted above. And we shall show a relation between $w_H(D_af)$ and the Fourier-Hadamard transform.

Algorithm (*Fast Fourier-Hadamard transform*)

There exists a simple divide-and-conquer butterfly algorithm to compute $\widehat{\varphi}$, called the *fast Fourier-Hadamard transform* (FFT). Let us give it in the case where "$\cdot$" is the usual inner product. For every $a = (a_1,\dots,a_{n-1})\in\mathbb{F}_2^{n-1}$ and every $a_n\in\mathbb{F}_2$, the number $\widehat{\varphi}(a_1,\dots,a_n)$ equals

$$\sum_{x=(x_1,\dots,x_{n-1})\in\mathbb{F}_2^{n-1}} (-1)^{a\cdot x}\left[\varphi(x_1,\dots,x_{n-1},0) + (-1)^{a_n}\varphi(x_1,\dots,x_{n-1},1)\right].$$

Hence, if in the tables of values of the functions, the vectors are ordered for instance in lexicographic order with the bit of highest weight on the right, the table of $\widehat{\varphi}$ equals the concatenation of those of the Fourier-Hadamard transforms of the $(n-1)$-variable functions $\psi_0(x) = \varphi(x_1,\dots,x_{n-1},0)+\varphi(x_1,\dots,x_{n-1},1)$ and $\psi_1(x) = \varphi(x_1,\dots,x_{n-1},0)-\varphi(x_1,\dots,x_{n-1},1)$. We deduce the following algorithm:

1. write the table of the values of φ (its truth-table if φ is Boolean), in which the binary vectors of length n are in lexicographic order with the bit of highest weight on the right;

2. let φ_0 be the restriction of φ to $\mathbb{F}_2^{n-1}\times\{0\}$ and φ_1 the restriction of φ to

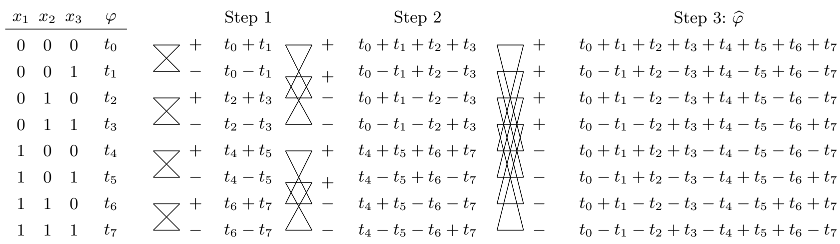


Figure 2.1 Fast Fourier-Hadamard transform

$\mathbb{F}_2^{n-1}\times\{1\}$[21]; replace the values of φ_0 by those of $\varphi_0+\varphi_1$ and those of φ_1 by those of $\varphi_0-\varphi_1$;

3. apply recursively step 2, separately to the functions now obtained in the places of φ_0 and φ_1.

When the algorithm ends (after arriving to functions in one variable each), the global table gives the values of $\widehat{\varphi}$. The complexity of this algorithm is of $n\,2^n$ additions/substractions; it is then in $\mathcal{O}(N\log_2 N)$ where $N=2^n$ is the size of its input f.
As for the fast binary Möbius transform, taking the lexicographic order with the bit of higher weight on the left (*i.e.* the standard lexicographic order) works as well because, for every permutation σ of $\{1,\dots,n\}$, we have $u\cdot x=\sigma(u)\cdot\sigma(x)$ for every u,x, and this implies that $\widehat{\varphi\circ\sigma}(u)=\sum_{x\in\mathbb{F}_2^n}\varphi(\sigma(x))\,(-1)^{u\cdot x}=\sum_{x\in\mathbb{F}_2^n}\varphi(x)\,(-1)^{u\cdot\sigma^{-1}(x)}=\sum_{x\in\mathbb{F}_2^n}\varphi(x)\,(-1)^{\sigma(u)\cdot x}=\widehat{\varphi}\circ\sigma(u)$ and the final values are the same (but not the intermediate ones).

Remark. Here again, the algorithm may not work if the order on $\mathbb{F}_{2^n}$ is not a coordinatewise permuted version of lexicographic order (for instance, if it is an order by increasing Hamming weights of inputs). □

Figure 2.1 illustrates how this algorithm works (with a display of the rows in a different order, better adapted to apprehend the figure).

2.3.2 Fourier-Hadamard and Walsh transforms of Boolean functions

For a given Boolean function f, the Fourier-Hadamard transform can be applied to f itself, viewed as a function valued in $\{0,1\}\subset\mathbb{Z}$ (we denote then by $\widehat{f}$ the corresponding Fourier-Hadamard transform of f). Notice that $\widehat{f}(0_n)$ equals the Hamming weight of f. Thus, the Hamming distance $d_H(f,g)=|\{x\in\mathbb{F}_2^n;\ f(x)\neq$

[21] The table of values of φ_0 (resp. φ_1) corresponds to the upper (resp. lower) half of the table of φ.

$g(x)\}| = w_H(f \oplus g)$ between two functions f and g equals $\widehat{f \oplus g}(0_n)$.
Note that, by linearity of the Fourier-Hadamard transform, Relations (2.25), page 68, and (2.26) imply:

$$\widehat{\bigoplus_{i=1}^{k} f_i} = \sum_{\emptyset \neq I \subseteq \{1,\dots,k\}} (-2)^{|I|-1} \widehat{\prod_{i\in I} f_i}, \tag{2.30}$$

$$\widehat{\prod_{i=1}^{l} f_i} = \frac{1}{2^{l-1}} \sum_{\emptyset \neq J \subseteq \{1,\dots,l\}} (-1)^{|J|-1} \widehat{\bigoplus_{i\in I} f_i}. \tag{2.31}$$

The Fourier-Hadamard transform can also be applied to the pseudo-Boolean function $f_\chi(x) = (-1)^{f(x)}$ (often called the *sign function*[22] of f) instead of f itself.

Definition 16 *We call* Walsh transform[23] *of a Boolean function f the Fourier-Hadamard transform of the sign function f_χ and we denote it*[24] *by W_f:*

$$W_f(u) = \sum_{x\in\mathbb{F}_2^n} (-1)^{f(x)\oplus u\cdot x}.$$

We call Walsh spectrum *of f the multi-set of all the values $W_f(u)$, where $u \in \mathbb{F}_2^n$. We call* extended Walsh spectrum[25] *of f the multi-set of their absolute values, and* Walsh support *of f the set of those u such that $W_f(u) \neq 0$.*

[22] The symbol χ is used here because the sign function is the image of f by the non-trivial character over $\mathbb{F}_2$ (usually denoted by χ).
[23] Some authors specify "Walsh-Hadamard transform" like in signal processing, but most do not, since the risk of ambiguity is weaker than for the Fourier transform; note that some rare authors use "Walsh" or "Hadamard-Walsh" for what we call "Fourier-Hadamard"; we shall use the term of "Walsh" only when dealing with the sign function.
[24] This notation is now widely used; a few years ago, diverse notations were used.
[25] "extended" is in the sense of "extended by the addition of constant Boolean functions to f", since knowing $|W_f(u)|$ is equivalent to knowing the unordered pair $\{W_f(u), W_{f\oplus 1}(u)\}$, because $W_{f\oplus 1}$ and W_f take opposite values; we shall sometimes call extended Walsh transform of f the function $|W_f|$.

```
Data: tt ← truth table, n ← number of variables
Result: wt ← Walsh-Hadamard spectrum
for i = 0 to 2^n − 1 do
    wt[i] = (−1)^tt[i];
end
for i = 1 to n do
    for r = 0 to 2^n − 1 by 2^i do
        t1 = r;
        t2 = r + 2^(i−1);
        for j = 0 to 2^(i−1) − 1 do
            a = wt[t1];
            b = wt[t2];
            wt[t1] = a + b;
            wt[t2] = a − b;
            t1 = t1 + 1;
            t2 = t2 + 1;
        end
    end
end
```

Algorithm 2: Computing the Walsh-Hadamard transform

We give in Table 2.2 an example of the computation of the Walsh transform, when the inner product chosen in $\mathbb{F}_2^n$ is the usual inner product, using the algorithm of the fast Fourier-Hadamard transform[26].

Notice that f_χ being equal to $1-2f$, we have

$$W_f = 2^n\,\delta_0 - 2\widehat{f} \tag{2.32}$$

where δ_0 denotes the *Dirac (or Kronecker) symbol*, *i.e.* the indicator of the singleton $\{0_n\}$, defined by $\delta_0(u) = 1$ if u is the null vector and $\delta_0(u) = 0$ otherwise; see Proposition 10 for a proof of the relation $\widehat{1} = 2^n\,\delta_0$. Relations (2.30) and (2.31) give then:

$$W_{\bigoplus_{i=1}^k f_i}(a) = 2^{n-1}(1+(-1)^k)\delta_0(a) + \sum_{\emptyset\neq I\subseteq\{1,\dots,k\}} (-2)^{|I|-1} W_{\prod_{i\in I} f_i}(a), \tag{2.33}$$

and $W_{\prod_{i=1}^l f_i}(a) =$

$$\left(2^n - 2^{n-l+1}\right)\delta_0(a) + \frac{1}{2^{l-1}} \sum_{\emptyset\neq J\subseteq\{1,\dots,l\}} (-1)^{|J|-1} W_{\bigoplus_{i\in I} f_i}(a), \tag{2.34}$$

since we have $1 - \sum_{\emptyset\neq I\subseteq\{1,\dots,k\}}(-2)^{|I|-1} = 1 - \frac{(1-2)^k-1}{(-2)} = \frac{1+(-1)^k}{2}$ and $1-$

[26] The truth table of the function is first directly calculated; we could also have applied the fast binary Möbius transform to obtain it; this has been done in Table 2.1 for the same function.

x_1	x_2	x_3	x_4	hexa	$x_1x_2x_3$	x_1x_4	$f(x)$	$f_\chi(x)$				$W_f(x)$
0	0	0	0	0	0	0	0	1	2	4	0	0
1	0	0	0	1	0	0	0	1	0	0	0	0
0	1	0	0	2	0	0	1	-1	-2	-4	8	8
1	1	0	0	3	0	0	1	-1	0	0	0	8
0	0	1	0	4	0	0	0	1	2	0	0	0
1	0	1	0	5	0	0	0	1	0	0	0	0
0	1	1	0	6	0	0	1	-1	-2	0	0	0
1	1	1	0	7	1	0	0	1	0	0	0	0
0	0	0	1	8	0	0	0	1	0	0	0	4
1	0	0	1	9	0	1	1	-1	2	4	4	-4
0	1	0	1	a	0	0	1	-1	0	0	0	4
1	1	0	1	b	0	1	0	1	-2	0	4	-4
0	0	1	1	c	0	0	0	1	0	0	0	-4
1	0	1	1	d	0	1	1	-1	2	0	-4	4
0	1	1	1	e	0	0	1	-1	0	0	0	4
1	1	1	1	f	1	1	1	-1	2	-4	4	-4

Table 2.2 truth table and Walsh spectrum of $f(x) = x_1x_2x_3 \oplus x_1x_4 \oplus x_2$

$\frac{1}{2^{l-1}} \sum_{\emptyset \neq I \subseteq \{1,\ldots,l\}} (-1)^{|I|-1} = 1 + \frac{1}{2^{l-1}}((1-1)^l - 1) = 1 - \frac{1}{2^{l-1}}$.
Relation (2.33) has been originally obtained by induction and calculation in [204].
Relation (2.32) gives conversely $\widehat{f} = 2^{n-1}\delta_0 - \frac{W_f}{2}$ and in particular:

$$w_H(f) = 2^{n-1} - \frac{W_f(0_n)}{2}. \tag{2.35}$$

The mapping $f \mapsto W_f(0_n)$ playing an important role, and being applied in the sequel to various functions deduced from f, we shall also use the specific notation

$$\mathcal{F}(f) = W_f(0_n) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)}. \tag{2.36}$$

Relation (2.35) applied to $f \oplus \ell_a$, where $\ell_a(x) = a \cdot x$, gives:

$$d_H(f, \ell_a) = w_H(f \oplus \ell_a) = 2^{n-1} - \frac{W_f(a)}{2}. \tag{2.37}$$

Remark. The Walsh transform represents the correlation between Boolean functions and affine functions and is related to attacks on stream ciphers using LFSR. The *best affine approximations* of $f(x)$ are the functions $a \cdot x \oplus \epsilon$ where $|W_f(a)|$ is maximal and ϵ equals 0 if $W_f(a) > 0$ (since $f(x) \oplus a \cdot x$ has then low Hamming weight), and 1 otherwise.
In [704, 302] is studied the *arithmetic Walsh transform* of Boolean functions, which is based on modular arithmetic and is related to FCSRs (Feedback with Carry Shift-Registers, having the operation of retroaction made with carry). □

The supports of the Walsh transforms of Boolean functions have been studied in [308], among which we find all possible affine subspaces of $\mathbb{F}_2^n$ and the complements of singletons (for $n \geq 10$).

Remark. The relationship between the algebraic degree and the Walsh support needs to be better understood. Of course, any n-variable Boolean function having an odd Hamming weight (i.e. having algebraic degree n) has $\mathbb{F}_2^n$ for Walsh support. Note also that, according to the properties seen in Subsection 2.2.1 and in the present subsection, an n-variable Boolean function f having even Hamming weight has algebraic degree $n-1$ if and only if there is a hyperplane H (that we can take linear) on which f has an odd Hamming weight, and this hyperplane can be taken equal to the set $\{x \in \mathbb{F}_2^n; supp(x) \subseteq supp(u)\}$ where u has Hamming weight $n-1$. Hyperplane H has equation $x \cdot a = 0$ for some $a \neq 0$, and we have then $W_f(a) + W_f(0) = 2\sum_{x\in H}(-1)^{f(x)} \equiv 4 \pmod 8$, assuming that $n \geq 3$, and $W_f(a) + W_f(0)$ is then nonzero. If we take $H = \{x \in \mathbb{F}_2^n; supp(x) \subseteq supp(u)\}$ where u has Hamming weight $n-1$, then a has Hamming weight 1 (i.e. H has equation $x_i = 0$). It is easily shown that the function $a \to W_f(a) - W_f(0) \pmod 8$ is linear from $\mathbb{F}_2^n$ to $\mathbb{Z}/8\mathbb{Z}$ (more precisely to $\{0, 4\}$); the set of those a such that $W_f(a) - W_f(0) \equiv c \pmod 8$ for some c (equal to 0 or 4) is then either empty or is an affine hyperplane or is the whole space. □

In [582] is proposed an algorithm, deduced from the formulae relating NNF and Walsh transform that we shall see in Subsection 2.3.4, page 85, for computing the Walsh transform (for a small set of points) from the ANF when the FFT is not efficient for computing it from the truth table (because the number of variables is too large, which happens when n is significantly larger than 30). For example, it is possible in certain cases to run their algorithm for 50 to 100 variable functions having a few hundreds of terms in their ANF.

In [373] are given concise representations of Walsh transform by binary decision diagrams (BDD) for functions with several hundred variables.

2.3.3 Properties of the Fourier-Hadamard and Walsh transforms of Boolean functions

The Fourier-Hadamard transform, as other Fourier transforms, has very nice and useful properties. The number of these properties and the richness of their mutual relationship are impressive. All of these properties are very useful in practice for studying Boolean functions. We shall often refer to the relations below, by applying them to the Fourier-Hadamard transforms of pseudo-Boolean functions or to the Walsh transforms of Boolean functions (which are a particular case). Almost all properties can be deduced from the next two lemmas and proposition.

Lemma 3 *Let E be any vector space over $\mathbb{F}_2$ and ℓ any nonzero linear form on E. Then $\sum_{x\in E}(-1)^{\ell(x)}$ is null.*

Proof. The linear form ℓ being nonzero, its support is an affine hyperplane of E

and has $2^{dimE-1} = \frac{|E|}{2}$ elements[27]. Thus, $\sum_{x\in E}(-1)^{\ell(x)}$ being the sum of 1's and -1's in equal numbers, it is null[28]. □

Proposition 10 *Let E be any vector subspace of $\mathbb{F}_2^n$. Denote by 1_E its* indicator *(*i.e. *the Boolean function defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ otherwise). Then:*

$$\widehat{1_E} = |E|\, 1_{E^\perp}, \tag{2.38}$$

where $E^\perp = \{x \in \mathbb{F}_2^n;\ \forall y \in E,\ x \cdot y = 0\}$ is the orthogonal space *of E with respect to the* inner product *"$\cdot$".*
In particular, for $E = \mathbb{F}_2^n$, we have $\widehat{1} = 2^n\, \delta_0$.

Proof. For every $u \in \mathbb{F}_2^n$, we have $\widehat{1_E}(u) = \sum_{x\in E}(-1)^{u\cdot x}$. If the linear form $x \in E \mapsto u \cdot x$ is not null on E (*i.e.* if $u \notin E^\perp$) then $\widehat{1_E}(u)$ is null, according to Lemma 3. And if $u \in E^\perp$, then $\widehat{1_E}(u)$ clearly equals $|E|$. □

This proposition leads to the very important Poisson formula below. To be able to state this formula in its general form, we need the:

Lemma 4 *For every pseudo-Boolean function φ on $\mathbb{F}_2^n$ and every elements a, b and u of $\mathbb{F}_2^n$, the value at u of the Fourier-Hadamard transform of the function $(-1)^{a\cdot x}\, \varphi(x + b)$ equals $(-1)^{b\cdot(a+u)}\, \widehat{\varphi}(a + u)$.*

Proof. The value at u of the Fourier-Hadamard transform of the function $x \mapsto (-1)^{a\cdot x}\, \varphi(x + b)$ equals $\sum_{x\in\mathbb{F}_2^n}(-1)^{(a+u)\cdot x}\varphi(x + b) = \sum_{x\in\mathbb{F}_2^n}(-1)^{(a+u)\cdot(x+b)}\varphi(x)$ and thus equals $(-1)^{b\cdot(a+u)}\, \widehat{\varphi}(a + u)$. □

We deduce from Proposition 10 and Lemma 4 the *Poisson summation formula*, which has been used to prove many cryptographic properties in [759], [797], [212] and later in [190, 191], and whose most general statement is:

Corollary 3 *For every pseudo-Boolean function φ on $\mathbb{F}_2^n$, for every vector subspace E of $\mathbb{F}_2^n$, and for every elements a and b of $\mathbb{F}_2^n$, we have:*

$$\sum_{u\in a+E} (-1)^{b\cdot u}\, \widehat{\varphi}(u) = |E|\, (-1)^{a\cdot b} \sum_{x\in b+E^\perp} (-1)^{a\cdot x}\, \varphi(x). \tag{2.39}$$

Proof. For $a = b = 0_n$, the sum $\sum_{u\in E} \widehat{\varphi}(u)$ equals $\sum_{u\in E}\sum_{x\in\mathbb{F}_2^n} \varphi(x)(-1)^{u\cdot x} = \sum_{x\in\mathbb{F}_2^n} \varphi(x)\, \widehat{1_E}(x)$ by definition. Hence, according to Proposition 10:

$$\sum_{u\in E} \widehat{\varphi}(u) = |E| \sum_{x\in E^\perp} \varphi(x). \tag{2.40}$$

We apply this equality to function $(-1)^{a\cdot x}\, \varphi(x + b)$. Using Lemma 4, we deduce

[27] Another way of seeing this is to choose $a \in E$ such that $\ell(a) = 1$ and observe that the mapping $x \mapsto x + a$ is a bijection between $\ker \ell$ and its complement.

[28] Alternatively, choosing again $a \in E$ such that $\ell(a) = 1$, we have $\sum_{x\in E}(-1)^{\ell(x)} = \sum_{x\in E}(-1)^{\ell(x+a)} = (-1)^{\ell(a)} \sum_{x\in E}(-1)^{\ell(x)} = -\sum_{x\in E}(-1)^{\ell(x)}$.

$\sum_{u\in E}(-1)^{b\cdot(a+u)}\,\widehat{\varphi}(a+u)=|E|\sum_{x\in E^\perp}(-1)^{a\cdot x}\,\varphi(x+b)$, that is, (2.39). □

Relation (2.39) applied to $\varphi(x)=f_\chi$, the sign function of f, gives:

$$\sum_{u\in a+E}(-1)^{b\cdot u}\,W_f(u)=|E|\,(-1)^{a\cdot b}\sum_{x\in b+E^\perp}(-1)^{f(x)\oplus a\cdot x}. \tag{2.41}$$

Note that, according to this latter relation, for every Boolean function f, every vector subspace E of $\mathbb{F}_2^n$, and every $a,b\in\mathbb{F}_2^n$, we have $|\sum_{u\in a+E}(-1)^{b\cdot u}W_f(u)|\le 2^n$ (with equality if and only if $f(x)\oplus a\cdot x$ is constant on $b+E^\perp$).
Relation (2.39) with $a=0_n$ and $E=\mathbb{F}_2^n$ gives:

Corollary 4 *For every pseudo-Boolean function φ on $\mathbb{F}_2^n$:*

$$\widehat{\widehat{\varphi}}=2^n\,\varphi. \tag{2.42}$$

Thus, the Fourier-Hadamard transform is a permutation on the set of pseudo-Boolean functions on $\mathbb{F}_2^n$ and is its own inverse, up to the division[29] by the constant 2^n. Relation (2.42) is called the *inverse Fourier-Hadamard transform formula* and writes $\sum_{u\in\mathbb{F}_2^n}\widehat{\varphi}(u)\,(-1)^{u\cdot x}=2^n\varphi(x)$. It means that, viewing φ as a function of $x_\chi=((-1)^{x_1},\dots,(-1)^{x_n})$, the number $2^{-n}\widehat{\varphi}(u)$ is the NNF coefficient indexed by u of the resulting function[30]. Applied to a sign function, Relation (2.42) is called the *inverse Walsh transform formula* and writes:

$$\sum_{u\in\mathbb{F}_2^n}W_f(u)\,(-1)^{u\cdot x}=2^n(-1)^{f(x)}. \tag{2.43}$$

Corollary 4 allows to show easily that a given property observed on the Fourier-Hadamard transform of any pseudo-Boolean function φ having some specificity, is in fact a necessary and sufficient condition for φ having this specificity. For instance, according to Proposition 10, the Fourier-Hadamard transform of any constant function φ takes null value at every nonzero vector. Since the Fourier-Hadamard transform of a function null at every nonzero vector is constant, Corollary 4 implies that the Fourier-Hadamard transform is a bijection between the set of constant functions and the set of those functions null at every nonzero vector. Similarly, φ is constant on $\mathbb{F}_2^n\setminus\{0_n\}$ if and only if $\widehat{\varphi}$ is constant on $\mathbb{F}_2^n\setminus\{0_n\}$.

A classical property of the Fourier transform is to be an isomorphism from the set of functions endowed with the so-called convolutional product (denoted by $\otimes$), into this same set, endowed with the usual product (denoted by $\times$). We recall the definition[31] of the convolutional product between two functions φ and

[29] In order to avoid this division, the Fourier-Hadamard transform is often normalized, that is, divided by $\sqrt{2^n}=2^{\frac{n}{2}}$, so that it becomes its own inverse. We do not use this normalized transform here because the functions we consider are integer-valued, and we want their Fourier-Hadamard transforms to be also integer-valued.

[30] In [693, 779, 914], they call Fourier transform this representation of φ as a polynomial in x_χ.

[31] Since the operations take place in $\mathbb{F}_2^n$, we have a "+" in the formula, where for general groups we would have a "−".

ψ:

$$(\varphi \otimes \psi)(x) = \sum_{y \in \mathbb{F}_2^n} \varphi(y)\psi(x+y).$$

Proposition 11 *Let φ and ψ be any pseudo-Boolean functions on $\mathbb{F}_2^n$. We have:*

$$\widehat{\varphi \otimes \psi} = \widehat{\varphi} \times \widehat{\psi}. \tag{2.44}$$

Consequently:

$$\widehat{\varphi} \otimes \widehat{\psi} = 2^n \, \widehat{\varphi \times \psi}. \tag{2.45}$$

Proof. We have

$$\widehat{\varphi \otimes \psi}(u) = \sum_{x \in \mathbb{F}_2^n} (\varphi \otimes \psi)(x)\,(-1)^{u \cdot x} = \sum_{x \in \mathbb{F}_2^n} \sum_{y \in \mathbb{F}_2^n} \varphi(y)\psi(x+y)\,(-1)^{u \cdot y \oplus u \cdot (x+y)}.$$

Thus, by the change of variable $(x, y) \to (x + y, y)$:

$$\widehat{\varphi \otimes \psi}(u) = \left(\sum_{y \in \mathbb{F}_2^n} \varphi(y)(-1)^{u \cdot y} \right) \left(\sum_{x \in \mathbb{F}_2^n} \psi(x)\,(-1)^{u \cdot x} \right) = \widehat{\varphi}(u)\,\widehat{\psi}(u).$$

This proves the first equality. Applying it to $\widehat{\varphi}$ and $\widehat{\psi}$ in the places of φ and ψ, we obtain $\widehat{\widehat{\varphi} \otimes \widehat{\psi}} = 2^{2n}\; \varphi \times \psi$, according to Corollary 4. Applying again this same corollary, to $\widehat{\varphi} \otimes \widehat{\psi}$, we deduce Relation (2.45). □

Relation (2.45) applied at 0_n gives a relation sometimes called Plancherel's formula:

$$\sum_{x \in \mathbb{F}_2^n} \widehat{\varphi}(x)\widehat{\psi}(x) = 2^n \sum_{x \in \mathbb{F}_2^n} \varphi(x)\psi(x). \tag{2.46}$$

Taking $\psi = \varphi$ in (2.46), we obtain *Parseval's relation*:

Corollary 5 *For every pseudo-Boolean function φ, we have:*

$$\sum_{u \in \mathbb{F}_2^n} \widehat{\varphi}^{\,2}(u) = 2^n \sum_{x \in \mathbb{F}_2^n} \varphi^2(x).$$

If φ takes values ± 1 only, this becomes:

$$\sum_{u \in \mathbb{F}_2^n} \widehat{\varphi}^{\,2}(u) = 2^{2n}. \tag{2.47}$$

This is why, when dealing with Boolean functions, we most often prefer using the Walsh transform of f instead of the Fourier-Hadamard transform of f. Parseval's relation for Walsh transform writes:

$$\sum_{u \in \mathbb{F}_2^n} W_f^2(u) = 2^{2n}. \tag{2.48}$$

According to the inverse Walsh transform and Parseval formulae, we have for every function f that $\left(\sum_{u \in \mathbb{F}_2^n} W_f(u) \right)^2 = \left(2^n (-1)^{f(0_n)} \right)^2 = \sum_{u \in \mathbb{F}_2^n} W_f^2(u)$, that

is, $\sum_{u\neq v} W_f(u)W_f(v) = 0$. Note that this proves (as observed in [312]) that it is impossible, except when the function is affine, *i.e.* when the Walsh transform is null except at one point, that all nonzero values of the Walsh transform have the same sign.
Relation (2.45) applied at $a \neq 0_n$ gives

$$\widehat{\varphi}\otimes\widehat{\psi}(a) = 2^n\,\widehat{\varphi\times\psi}(a) = 2^n \sum_{x\in\mathbb{F}_2^n} \varphi(x)\psi(x)(-1)^{a\cdot x}. \tag{2.49}$$

If φ takes values ± 1 only and $\psi = \varphi$, this becomes:

$$\sum_{u\in\mathbb{F}_2^n} \widehat{\varphi}(u)\widehat{\varphi}(u+a) = 0. \tag{2.50}$$

This provides the relation that some authors call the *Titsworth relation*:

$$\sum_{u\in\mathbb{F}_2^n} W_f(u)W_f(u+a) = 0, \quad \forall a\neq 0_n. \tag{2.51}$$

Note that in some cases (for instance for designing correlation immune functions of low Hamming weights, see Section 7.1.9, page 332) using the Fourier-Hadamard transform of a Boolean function is more convenient.
When $\mathbb{F}_2^n$ is identified to $\mathbb{F}_{2^n}$, with *inner product* $u\cdot x = tr_n(ux)$, *Parseval's relation* is a particular case (corresponding to $a = 1$) of the more general relation:

$$\begin{aligned}\sum_{u\in\mathbb{F}_{2^n}} W_f(u)W_f(au) &= \sum_{u,x,y\in\mathbb{F}_{2^n}} (-1)^{f(y)\oplus f(x)\oplus tr_n(uy+aux)}\\ &= 2^n \sum_{x\in\mathbb{F}_{2^n}} (-1)^{f(x)\oplus f(ax)}.\end{aligned}$$

Relation (2.44) applied with $\psi = \varphi = f_\chi$ implies the *Wiener-Khintchine formula*:

$$\widehat{f_\chi\otimes f_\chi} = W_f^2, \tag{2.52}$$

which involves in fact the *derivatives* of the Boolean function, since for every $a\in\mathbb{F}_2^n$, we have $(f_\chi\otimes f_\chi)(a) = \sum_{x\in\mathbb{F}_2^n}(-1)^{D_af(x)} = \mathcal{F}(D_af)$ (the notation $\mathcal{F}$ was defined at Relation (2.36), page 75).

Definition 17 *The function $a\mapsto\mathcal{F}(D_af)$ is called the* autocorrelation function *of f and denoted by Δ_f.*

Relation (2.52) means that W_f^2 is the Fourier-Hadamard transform of the autocorrelation function of f:

$$\forall u\in\mathbb{F}_2^n,\ \widehat{\Delta_f}(u) = \sum_{a\in\mathbb{F}_2^n} \Delta_f(a)(-1)^{u\cdot a} = W_f^2(u). \tag{2.53}$$

Equivalently, by applying the inverse Fourier transform formula, we have

$$\Delta_f(a) = 2^{-n}\sum_{u\in\mathbb{F}_2^n} W_f^2(u)(-1)^{u\cdot a}. \tag{2.54}$$

This property was first used (as far as we know) in the domain of cryptography in [211] to study the so-called partially-bent functions (see Section 6.2). It leads also to a lower bound on the numerical degree of Boolean functions by means of the Hamming weights of their derivatives (in directions of Hamming weight 1), first given in [905], that we shall give (and prove) as Relation (2.63), page 86. Applied at vector 0_n, Relation (2.53) gives

$$\widehat{\Delta_f}(0_n) = \sum_{a\in\mathbb{F}_2^n} \mathcal{F}(D_a f) = \mathcal{F}^2(f). \tag{2.55}$$

Corollary 3 (the Poisson summation formula), page 77, and Relation (2.53) imply that, for every vector subspace E of $\mathbb{F}_2^n$ and every vectors a and b (cf. [191]):

$$\sum_{u\in a+E} (-1)^{b\cdot u}\, W_f^2(u) = |E|(-1)^{a\cdot b} \sum_{e\in b+E^\perp} (-1)^{a\cdot e}\mathcal{F}(D_e f)\ . \tag{2.56}$$

This leads to an interesting relation, first shown in [191] for Boolean functions (but similar relations exist in other domains like sequences and learning, see *e.g.* [779]), and that, because of its similarity with the Poisson summation formula, we shall call the *second-order Poisson summation formula*[32]:

Proposition 12 *Let E and E' be* supplementary subspaces[33] *of $\mathbb{F}_2^n$ (*i.e. *be two subspaces such that $E\cap E'=\{0_n\}$ and whose direct sum equals $\mathbb{F}_2^n$). For every $a\in E'$, let h_a be the restriction of f to the coset $a+E$ (h_a can be identified with a function on $\mathbb{F}_2^k$ where k is the dimension of E). Then*

$$\sum_{u\in E^\perp} W_f^2(u) = |E^\perp| \sum_{a\in E'} \mathcal{F}^2(h_a)\ . \tag{2.57}$$

Proof. Every element of $\mathbb{F}_2^n$ can be written in a unique way in the form $x+a$ where $x\in E$ and $a\in E'$.
For every $e\in E$, we have that $\mathcal{F}(D_e f) = \sum_{x\in E; a\in E'} (-1)^{f(x+a)\oplus f(x+e+a)} = \sum_{a\in E'} \mathcal{F}(D_e h_a)$. We deduce from Relation (2.56), applied with $E^\perp$ instead of E, and with $a=b=0_n$, that

$$\begin{aligned}\sum_{u\in E^\perp} W_f^2(u) &= |E^\perp| \sum_{e\in E} \mathcal{F}(D_e f) = |E^\perp| \sum_{e\in E}\left(\sum_{a\in E'} \mathcal{F}(D_e h_a)\right)\\ &= |E^\perp| \sum_{a\in E'}\left(\sum_{e\in E} \mathcal{F}(D_e h_a)\right).\end{aligned}$$

Thus, according to Relation (2.55) applied with E in the place of $\mathbb{F}_2^n$ (recall that E can be identified with $\mathbb{F}_2^k$ where k is the dimension of E): $\sum_{u\in E^\perp} W_f^2(u) = |E^\perp| \sum_{a\in E'} \mathcal{F}^2(h_a)$. □

[32] This formula is sometimes more convenient to use than the Poisson summation formula. An example where it helps proving more can be found in Section 10.4.

[33] Some authors say "complementary" but we prefer avoiding the confusion with complementary sets and use "supplementary".

Fourier-Hadamard transform and affine automorphisms

A last relation that must be mentioned shows what the composition with a linear isomorphism implies on the Fourier transform of a pseudo-Boolean function:

Proposition 13 *Let φ be any pseudo-Boolean function on $\mathbb{F}_2^n$. Let M be a non-singular $n\times n$ binary matrix and L the linear automorphism $L:(x_1,x_2,\dots,x_n)\mapsto(x_1,x_2,\dots,x_n)\times M$. Let us denote by M' the transpose of M^{-1} and by L' the linear automorphism $L':(x_1,x_2,\dots,x_n)\mapsto(x_1,x_2,\dots,x_n)\times M'$ (note that L' is the* adjoint operator *of L^{-1}, that is, satisfies $u\cdot L^{-1}(x)=L'(u)\cdot x$ for every x and u, where $\cdot$ is the usual inner product). Then*

$$\widehat{\varphi\circ L}=\widehat{\varphi}\circ L'. \tag{2.58}$$

Proof. By the change of variable $x\mapsto L^{-1}(x)$, we have that for every $u\in\mathbb{F}_2^n$, $\widehat{\varphi\circ L}(u)=\sum_{x\in\mathbb{F}_2^n}\varphi(L(x))(-1)^{u\cdot x}$ equals $\sum_{x\in\mathbb{F}_2^n}\varphi(x)(-1)^{u\cdot L^{-1}(x)}$ and, by the definition of L', equals then $\sum_{x\in\mathbb{F}_2^n}\varphi(x)(-1)^{L'(u)\cdot x}$. □

It is easily deduced from this Relation (2.58) and from Lemma 4, page 77 that the affine equivalence of Boolean functions translates into the affine equivalence of their extended Walsh transforms and in particular of their Walsh supports. Given linear bijections L_1,L_2, a linear function L_3 and vectors a,b,c, the value of $W_{(L_1+a)\circ F\circ(L_2+b)+L_3+c}(u,v)=\pm\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot(L_1(F(L_2(x)+b))+L_3(x))\oplus u\cdot x}$ equals $\pm W_F((L_3\circ {L_2}^{-1})^*(v)+({L_2}^{-1})^*(u),{L_1}^*(v))$, where * is the adjoint operator.

Relationship between *algebraic degree* and Walsh transform

The following bound was shown in [737] (see also [212, Lemma 3]):

Theorem 2 *Let f be an n-variable Boolean function ($n\geq 2$), and let $1\leq k\leq n$. Assume that the Walsh transform values of f are all divisible by 2^k (*i.e.*, according to Relation (2.32), that its Fourier-Hadamard transform takes values divisible by 2^{k-1}, or equivalently, according to Relation (2.37), that all the Hamming distances between f and affine functions are divisible by 2^{k-1}). Then f has algebraic degree at most $n-k+1$.*

Proof. Let us suppose that f has algebraic degree $d>n-k+1$ and, consider a term x^I of degree d in its algebraic normal form. The Poisson summation formula (2.40) applied to $\varphi=f_\chi$ and to the vector space $E=\{u\in\mathbb{F}_2^n;\ \forall i\in I,\ u_i=0\}$ gives $\sum_{u\in E}W_f(u)=2^{n-d}\sum_{x\in E^\perp}f_\chi(x)$. The orthogonal $E^\perp$ of E equals $\{u\in\mathbb{F}_2^n;\ \forall i\notin I,\ u_i=0\}=\{u\in\mathbb{F}_2^n;\ supp(u)\subseteq I\}$. According to Relation 2.4, we have that $\sum_{x\in E^\perp}f(x)$ is not even and therefore $\sum_{x\in E^\perp}f_\chi(x)$ is not divisible by 4. Hence, $\sum_{u\in E}W_f(u)$ is not divisible by 2^{n-d+2} and it is therefore not divisible by 2^k. A contradiction. □

Remark. The result is of course also valid for those vectorial (n,m)-functions whose Walsh transform values are divisible by 2^k. It is shown in [204] that for any (n, m)-function F having such divisibility property and for every (m, r)-function G, we have $d_{alg}(G \circ F) \leq n - k + d_{alg}(G)$. This bound on the algebraic degree of *composite functions* is a direct consequence of Relation (2.34), page 74: all component functions of F having Walsh transform values divisible by 2^k, then for every $l = 1, \ldots, k$, all products of l coordinate functions of F have Walsh transform divisible by 2^{k-l+1}, and thanks to Theorem 2, they have then algebraic degree at most $n - k + l$. If $d_{alg}(G) \leq k$, this completes the proof since $G \circ F$ is a linear combination of such products with $l \leq d_{alg}(G)$, and otherwise, there is nothing to prove since $n - k + d_{alg}(G) > n$. As shown in [254], the bound of [204] is also a direct consequence of Relation (2.9), page 57. □

Remark.
- 1. The converse of Theorem 2 is valid if $k = 1$ (since the Walsh transform values of all Boolean functions are even by definition). It is also valid if $k = 2$, since the n-variable Boolean functions of degrees at most $n-1$ are those Boolean functions of even Hamming weights, and $f(x) \oplus u \cdot x$ has degree at most $n-1$ too for every u, since $n \geq 2$. It is finally also valid for $k = n$, since the affine functions are characterized by the fact that their Walsh transforms take values $\pm 2^n$ and 0 only (more precisely, their Walsh transforms take value $\pm 2^n$ once, and all their other values are null). The converse is false for any other value of k. Indeed, it is false for $k = n-1$ ($n \geq 4$), since there exist *quadratic functions* f whose Walsh transforms take values $\pm 2^{\frac{n}{2}}$ for n even, ≥ 4, and $\pm 2^{(n+1)/2}$ for n odd, ≥ 5 (see Section 5.2, page 193). It is then an easy task to deduce that the converse of Theorem 2 is also false for any value of k such that $3 \leq k \leq n-2$: we choose a quadratic function g in 4 variables, whose Walsh transform value at 0_n equals 2^2, that is, whose weight equals $2^3 - 2 = 6$, and we take $f(x) = g(x_1, x_2, x_3, x_4)\, x_5 \ldots x_l$ ($5 \leq l \leq n$). Such function has algebraic degree $l - 2$ and its weight equals 6; hence its Walsh transform value at 0_n equals $2^n - 12$ and is therefore not divisible by 2^k with $n - 2 \geq k = n - (l-2) + 1 = n - l + 3 \geq 3$.
- 2. It is possible to characterize the functions whose Walsh transform values are all divisible by 2^{n-1} (*i.e.* equal 0, $\pm 2^{n-1}$ and/or $\pm 2^n$): according to Theorem 2, they have algebraic degree at most 2, and the characterization follows from the results of Section 5.2 on quadratic functions (see the last remark of page 196); these functions are the sums of an affine function and of the product of two affine functions (see for instance the observation after Theorem 10, page 195). Determining those Boolean functions (in the *Reed-Muller code* of order $n - k + 1$) whose Walsh transform is divisible by 2^k is an open problem for $3 \leq k \leq n-2$. The *Poisson summation formula* provides some information; it shows by applying the proof of Theorem 2 to $W_f(a + u)$ that, for every supplementary subspaces E and E' of $\mathbb{F}_2^n$ (*i.e.* such that $E \cap E' = \{0_n\}$ and whose direct sum equals $\mathbb{F}_2^n$) where E has dimension $d \geq n - k$, denoting for every $a \in E'$ by h_a the restriction of f to the coset $a + E$, the value of $\mathcal{F}(h_a)$ is divisible by 2^{k+d-n}; and we also have that the arithmetic mean (*i.e.* average) of $\mathcal{F}(h_a)$ when

a ranges over E' is divisible by 2^{k+d-n} (indeed, $\sum_{a\in E'}\mathcal{F}(h_a)=\mathcal{F}(f)=W_f(0_n)$ is divisible by 2^k). The *second-order Poisson summation formula* also provides complementary information on such functions: the quadratic mean (*i.e.* the root mean square) of $\mathcal{F}(h_a)$ when a ranges over E' is also divisible by 2^{k+d-n}. Indeed, $\sum_{a\in E'}\mathcal{F}^2(h_a)=\frac{1}{|E^\perp|}\sum_{u\in E^\perp}W_f^2(u)$ is divisible by 2^{2k+d-n}; hence, the arithmetic mean of $\mathcal{F}^2(h_a)$ is divisible by $2^{2k+2d-2n}$. Summarizing, we have that the integer sequence $2^{-(k+d-n)}\mathcal{F}(h_a)$, of length 2^{n-d}, has integer arithmetic and quadratic means. Note that, according to *McEliece's theorem* (see page 28), given a monomial Boolean function $f(x)=tr_n(x^d)$ where $\gcd(d,2^n-1)=1$, the largest possible exponent of a power of 2 dividing each Walsh transform value of f equals $\min\{w_2(t_0)+w_2(t_1);\ 1\leq t_0,t_1<2^n-1,\ t_0+t_1d\equiv 0\ [\bmod\ 2^n-1]$ (see the definition of w_2 at page 62). See bounds in [674, 676].

- 3. It is possible to characterize the fact that a Boolean function has algebraic degree at most d by means of its Fourier-Hadamard or Walsh transforms: since, as seen in Proposition 4, page 53, a Boolean function has algebraic degree at most d if and only if its restriction to any $(d+1)$-dimensional flat (*i.e.* affine subspace) has even Hamming weight, we can apply Poisson summation formula (2.39). For instance in terms of the Walsh transform, f has algebraic degree at most d if and only if, for every $(n-d-1)$-dimensional vector subspace E of $\mathbb{F}_2^n$ and every $b\in\mathbb{F}_2^n$, the sum $\sum_{u\in E}(-1)^{b\cdot u}W_f(u)$ is divisible by 2^{n-d+1}. But this characterization is not simple. □

Characterizing the Fourier-Hadamard transforms of pseudo-Boolean functions and the Walsh transforms of Boolean functions

According to the *inverse Fourier-Hadamard transform formula* (2.42), the Fourier-Hadamard transforms of integer-valued functions (resp. the Walsh transforms of Boolean functions) are those integer-valued functions over $\mathbb{F}_2^n$ whose Fourier-Hadamard transforms take values divisible by 2^n (resp. take values $\pm 2^n$). Also, according to the *inverse Walsh transform formula* (2.43), page 78, the Walsh transforms of Boolean functions are those integer-valued functions ψ over $\mathbb{F}_2^n$ such that $(\widehat{\psi})^2$ equals the constant function 2^{2n}; they are then those integer-valued functions ψ such that $\widehat{\psi\otimes\psi}=2^{2n}$ (according to Relation (2.44) applied with $\varphi=\psi$), that is $\psi\otimes\psi=2^{2n}\,\delta_0$.

These characterizations need to check 2^n divisibilities by 2^n for the Fourier-Hadamard transforms of integer-valued functions, and 2^n equalities for the Walsh transforms of Boolean functions.

Case of monomial (or power) Boolean functions: So-called *monomial Boolean (univariate) functions* are those functions over $\mathbb{F}_{2^n}$ of the form $f(x)=tr_n(ax^d)$ (recall from page 60 that when $\mathbb{F}_2^n$ is identified with $\mathbb{F}_{2^n}$, an inner product is then $(x,y)\mapsto tr_n(x\,y)$). We shall give at page 92 the known results and a conjecture on the Walsh spectrum of such functions.

2.3.4 Fourier-Hadamard (and Walsh) transform and numerical normal form

Since the main cryptographic criteria on Boolean functions will be characterized as properties of their Fourier-Hadamard/Walsh transforms (see Section 3.1), it is useful to clarify the relationship between these and the NNF representation. Note that there is a similarity between the Fourier-Hadamard transform and the NNF of pseudo-Boolean functions:
- the functions $(-1)^{u\cdot x}$, $u \in \mathbb{F}_2^n$, constitute an orthogonal basis of the space of pseudo-Boolean functions, and the Fourier-Hadamard transform can be seen as a classical decomposition over an orthogonal basis;
- the NNF is defined similarly with respect to the basis of monomials, which is non-orthogonal but allows as well simple calculation of the coefficients in this decomposition.
Let us see now how each representation can be expressed by means of the other. Let $\varphi(x)$ be any pseudo-Boolean function and let $\sum_{I\subseteq\{1,\ldots,n\}} \lambda_I x^I$ be its NNF. For every vector $x \in \mathbb{F}_2^n$, we have: $\varphi(x) = \sum_{I\subseteq supp(x)} \lambda_I$. Setting $1_n = (1,\ldots,1)$, we have $\varphi(x+1_n) = \sum_{I\subseteq\{1,\ldots,n\};\, supp(x)\cap I=\emptyset} \lambda_I$ (since the support of $x+1_n$ equals $\mathbb{F}_2^n \setminus supp(x)$). Hence, $\varphi(x+1_n) = \sum_{I\subseteq\{1,\ldots,n\}} \lambda_I\, 1_{E_I}$, where E_I is the $(n-|I|)$-dimensional vector subspace of $\mathbb{F}_2^n$ equal to $\{x \in \mathbb{F}_2^n;\, supp(x)\cap I = \emptyset\}$, whose orthogonal space equals $\{u \in \mathbb{F}_2^n;\, supp(u) \subseteq I\}$. Applying Lemma 4 with $a = 0_n$ and $b = 1_n$, and Proposition 10, we deduce (as proved in [292]):

$$\widehat{\varphi}(u) = (-1)^{w_H(u)} \sum_{I\subseteq\{1,\ldots,n\};\, supp(u)\subseteq I} 2^{n-|I|}\lambda_I. \tag{2.59}$$

Remark. Applying Relations (2.59) and (2.23), given the NNF $\sum_{I\subseteq\{1,\ldots,n\}} \lambda_I\, x^I$ of φ, we can calculate the NNF $\sum_{I\subseteq\{1,\ldots,n\}} \mu_I\, x^I$ of its Fourier-Hadamard transform $\widehat{\varphi}$. We obtain that μ_I equals $(-1)^{|I|} \sum_{\substack{u\in\mathbb{F}_2^n;\, supp(u)\subseteq I \\ J\subseteq\{1,\ldots,n\};\, supp(u)\subseteq J}} 2^{n-|J|}\lambda_J$, that is, $(-1)^{|I|} \sum_{J\subseteq\{1,\ldots,n\}} 2^{n+|I\cap J|-|J|}\lambda_J$. □

We deduce from Relation (2.59):

$$\lambda_I = 2^{-n}(-2)^{|I|} \sum_{u\in\mathbb{F}_2^n;\, I\subseteq supp(u)} \widehat{\varphi}(u). \tag{2.60}$$

Indeed, according to Relation (2.59), we have $2^{-n}(-2)^{|I|} \sum_{u\in\mathbb{F}_2^n;\, I\subseteq supp(u)} \widehat{\varphi}(u) = 2^{-n}(-2)^{|I|} \sum_{J\subseteq\{1,\ldots,n\}} \left(\sum_{u\in\mathbb{F}_2^n;\, I\subseteq supp(u)\subseteq J} (-1)^{w_H(u)}\right) 2^{n-|J|}\lambda_J$, and the sum inside the parentheses equals 0 if $I \not\subseteq J$ and otherwise is also null if $J \neq I$ since it equals $(-1)^{|I|} \sum_{u\in\mathbb{F}_2^n;\, supp(u)\subseteq J\setminus I} (-1)^{w_H(u)} = (-1)^{|I|}(1-1)^{|J\setminus I|}$.
Relation (2.60) has been proved in [292] in a slightly more complex way. Applied when φ equals a Boolean function f and using that $W_f(u) = 2^n\delta_0(u) - 2\widehat{f}(u)$, we get:

$$W_f(u) = (-1)^{w_H(u)+1} \sum_{I\subseteq\{1,\ldots,n\};\, supp(u)\subseteq I} 2^{n-|I|+1}\lambda_I \text{ if } u \neq 0_n, \tag{2.61}$$

$$W_f(0_n) = 2^n - \sum_{I\subseteq\{1,\dots,n\}} 2^{n-|I|+1}\lambda_I,$$

and

$$\lambda_I = 2^{-n}(-2)^{|I|-1} \sum_{u\in\mathbb{F}_2^n;\, I\subseteq supp(u)} W_f(u) \text{ if } I \neq \emptyset, \tag{2.62}$$

$$\lambda_\emptyset = -2^{-(n+1)}\left(\sum_{u\in\mathbb{F}_2^n} W_f(u) - 2^n\right).$$

Remark. This provides a simpler proof of Theorem 2, page 82: according to Relations (2.61) and (2.62), the hypothesis of the theorem is equivalent to saying that, for every I such that $|I| \geq n-k+1$, the coefficient λ_I of x^I in the NNF of f is divisible by $2^{|I|+k-n-1}$, and this implies that for $|I| \geq n-k+2$, it is even. This gives also more information on those functions whose Walsh transform values are all divisible by 2^k. For instance, if $|I| \geq n-k+3$, then since λ_I is divisible by 4, using Relation (2.24), page 68, we have that $\sum_{\substack{\{I_1,I_2\}\mid\\ I_1\cup I_2=I}} a_{I_1}a_{I_2}$ is even, that is, $\bigoplus_{\substack{\{I_1,I_2\}\mid\\ I_1\cup I_2=I}} a_{I_1}a_{I_2} = 0$. This is exploited in [301] for bounding numbers of functions (see pages 269 and 341). Other similar (but more complex) properties of the coefficients a_I can be obtained by considering the divisibility of λ_I by powers of 2 larger than 4. □

We deduce from Relations (2.59), (2.60), (2.61) and (2.62):

Proposition 14 *Any pseudo-Boolean function φ has* numerical degree *at most d if and only if $\widehat{\varphi}(u) = 0$ for every vector u of Hamming weight strictly larger than d. Any Boolean function f has numerical degree at most d if and only if $W_f(u) = 0$ for every such vector.*

In other words, the numerical degree equals the maximal Hamming weight of those $u \in \mathbb{F}_2^n$ such that $W_f(u) \neq 0$.
This allows proving the fact mentioned at page 66 that, if a Boolean function f has no ineffective variable, then the numerical degree of f is larger than or equal to $\log_2 n - \mathcal{O}(\log_2 \log_2 n)$. In fact, we can prove a little more with the same method as in the sketch of proof given in [905]:

Proposition 15 *Let f be any n-variable Boolean function. Denoting by e_i the i-th vector of the canonical basis of $\mathbb{F}_2^n$, the numerical degree of f satisfies:*

$$d_{num}(f) \geq 2^{-n}\sum_{i=1}^{n} w_H(D_{e_i}f). \tag{2.63}$$

If each variable x_i is effective in $f(x)$, that is, if each derivative *$D_{e_i}f$ is not the zero function, then we have*

$$d_{num}(f) \geq n\,2^{-d_{alg}(f)+1}. \tag{2.64}$$

and a fortiori

$$n \leq d_{num}(f)\, 2^{d_{num}(f)-1}. \tag{2.65}$$

Consequently:

$$d_{num}(f) \geq 1 + \log_2 n - \log_2(1 + \log_2 n). \tag{2.66}$$

Proof. According to Relation (2.54), page 80, we have:

$$\sum_{i=1}^{n} \Delta_f(e_i) = 2^{-n} \sum_{u \in \mathbb{F}_2^n} W_f^2(u) \sum_{i=1}^{n} (-1)^{u \cdot e_i} = 2^{-n} \sum_{u \in \mathbb{F}_2^n} W_f^2(u)[n - 2w_H(u)],$$

and therefore, since $\Delta_f(e_i) = 2^n - 2w_H(D_{e_i} f)$:

$$\sum_{i=1}^{n} w_H(D_{e_i} f) = n2^{n-1} - 2^{-(n+1)} \sum_{u \in \mathbb{F}_2^n} W_f^2(u)[n - 2w_H(u)]. \tag{2.67}$$

Using *Parseval's relation*, we deduce that

$$\sum_{i=1}^{n} w_H(D_{e_i} f) = 2^{-n} \sum_{u \in \mathbb{F}_2^n} W_f^2(u)\, w_H(u) \leq 2^{-n} d_{num}(f) \sum_{u \in \mathbb{F}_2^n} W_f^2(u) = 2^n d_{num}(f).$$

This proves Relation (2.63).
If each derivative $D_{e_i} f$ is nonzero, then $w_H(D_{e_i} f)$ is at least $2^{n-d_{alg}(D_{e_i} f)} \geq 2^{n-d_{alg}(f)+1} \geq 2^{n-d_{num}(f)+1}$, since the minimum nonzero Hamming weight of n-variable Boolean functions of algebraic degree at most r equals 2^{n-r}, as we shall see in Theorem 7, page 176. According to Relation (2.63), we have then Relation (2.64) and that $d_{num}(f)\, 2^{d_{num}(f)} \geq 2n$ and this proves Relation (2.65). Relation (2.66) is directly deduced, since for $x \geq 1$, function $x\, 2^x$ is increasing and we have $x\, 2^x = y \Rightarrow x + \log_2 x = \log_2 y \Rightarrow x \leq \log_2 y$ and therefore $x\, 2^x = y \Rightarrow x = \log_2 y - \log_2 x \geq \log_2 y - \log_2 \log_2 y$. □

The value of $2^{-n} w_H(D_{e_i} f)$ is called in [905, 914] the *influence of variable* x_i and the sum of these values the total influence. See more in [652, 653].
Bound (2.66) is tight up to approximately the term in $\log_2 \log_2 n$. Indeed, the so-called *address function* $f(x, y) = x_{\varphi(y)}$, $x \in \mathbb{F}_2^{2^k}, y \in \mathbb{F}_2^k$, where $\varphi(y) = 1 + \sum_{i=1}^{k} y_i 2^{i-1}$, has for NNF: $\sum_{u \in \mathbb{F}_2^k} x_{\varphi(u)} \delta_u(y) = \sum_{u \in \mathbb{F}_2^k} x_{\varphi(u)} \prod_{i=1}^{k} (1 - y_i - u_i + 2y_i u_i)$. It has then $n = k + 2^k$ variables and numerical degree $1 + k$.

Remark. According to the calculations above, bound (2.63) is an equality if and only if the Walsh support of f is included in the set of vectors of Hamming weight $d_{num}(f)$ (*i.e.* the Walsh transform of f is homogeneous; example: affine functions). Under this condition, Bound (2.65) is an equality if and only if, for every $i = 1, \ldots, n$, we have that $d_{num}(D_{e_i} f) = d_{alg}(f) - 1$ and $D_{e_i} f$ is the indicator of an affine space (see Theorem 8). We do not know if such function exists.
□

Remark. Functions of very low numerical degree $d \approx \log_2 n$ (like the address function) have a Walsh support of size at most $D = \sum_{i=0}^{d} \binom{n}{i} \approx \frac{n^{\log_2 n}}{\lfloor \log_2 n \rfloor !}$, according to Proposition 14. It is interesting to see that the Walsh support's size can be that small, while the function depends on all its variables and can be rather complex. In fact, this size is still smaller when f is the address function above, since for every $a \in \mathbb{F}_2^{2^k}$ and $b \in \mathbb{F}_2^k$, we have then $W_f(a,b) = \sum_{y \in \mathbb{F}_2^k} (-1)^{b \cdot y} \sum_{x \in \mathbb{F}_2^{2^k}} (-1)^{x_{\varphi(y)} \oplus a \cdot x}$ and therefore $W_f(a,b) = 0$ if a has Hamming weight different from 1, and the Walsh support of f has size $2^{2k} < n^2$ (the size is exactly 2^{2k} since, if a is the j-th vector of the canonical basis of $\mathbb{F}_2^{2^k}$, then, for every $b \in \mathbb{F}_2^k$ we have $W_f(a,b) = 2^{2^k}(-1)^{b \cdot y} \neq 0$, where y is the unique element such that $\varphi(y) = j$). The Walsh support is the union of 2^k cosets of the k-dimensional linear subspace $\{0_{2^k}\} \times \mathbb{F}_2^k$ of $\mathbb{F}_2^{2^k} \times \mathbb{F}_2^k$.
The address function is a particular case of a general class of functions called Maiorana-McFarland, that we shall see at page 188, and which can provide more cases of small Walsh supports (see after Proposition 53, page 189). □

Determining, for all n, the exact minimum numerical degree of n-variable Boolean functions depending on all their variables is open.
Of course, if a function does not depend on all its n variables, we still have the bound $d_{num}(f) \geq 1 + \log_2 m - \log_2(1 + \log_2 m)$, where m is the number of effective variables in $f(x)$.

Remark. The NNF presents the interest of being a polynomial representation, but it can also be viewed as the transform which maps any pseudo-Boolean function $f(x) = \sum_{I \subseteq \{1,\ldots,n\}} \lambda_I \, x^I$ to the pseudo-Boolean function g defined by $g(x) = \lambda_{supp(x)}$. Let us denote this mapping by Φ. Three other transforms have also been used for studying Boolean functions:
- the mapping Φ^{-1} (the formulae relating this mapping and the Walsh transform are slightly simpler than for Φ; see [985]);
- a mapping defined by a formula similar to Relation (2.23), but in which $supp(x) \subseteq I$ is replaced by $I \subseteq supp(x)$; see [579];
- the inverse of this latter mapping. □

Remark. An interesting question is: given a Boolean function f, what is the minimum *numerical degree* of all the Boolean functions affine equivalent to f (that is, thanks to the fact that the affine equivalence of functions implies the linear equivalence of their Walsh supports (see page 82), what is the minimum for all the sets S which are linearly equivalent to the Walsh support of f, of the maximum Hamming weight of the elements of S). Note that there exist functions such that this minimum is strictly larger than the algebraic degree (this is the case of bent functions, see Definition 19, page 100, for instance, since for these functions the minimum is n and the algebraic degree equals $n/2$). Note also that if we replace "minimum" by "maximum", the number is n for every non-constant

Boolean function, since given any element of $\mathbb{F}_2^n$, there exists a linear permutation which maps this element to the all-1 vector. $\square$

2.3.5 The size of the support of the Fourier-Hadamard transform and Cayley graphs

In *graph theory*, an undirected graph is an ordered pair (V, E) where V is a set of points called *vertices* or *nodes*, and E is a set of pairs of vertices (that we shall assume distinct) called *edges* (more generally, in the case of *hypergraphs*, edges are subsets of nodes). The degree of a vertex equals the number of edges it is in. Let f be a Boolean function and let G_f be the *Cayley graph* associated to f: the vertices of this graph are the elements of $\mathbb{F}_2^n$ and there is an edge between two vertices u and v if and only if the vector $u+v$ belongs to the support of f. Then (see [68]), the values $\widehat{f}(a)$, $a \in \mathbb{F}_2^n$, of the Fourier-Hadamard transform of f are the eigenvalues of the graph G_f (that is, by definition, the eigenvalues of the adjacency matrix $(M_{u,v})_{u,v\in\mathbb{F}_2^n}$ of G_f, whose term $M_{u,v}$ equals 1 if $u+v$ belongs to the support of f, and equals 0 otherwise). Their product equals then the determinant of the adjacency matrix. Indeed, the matrix is $2^n \times 2^n$ and we have the 2^n linearly independent eigenvectors $((-1)^{a\cdot v})_{v\in\mathbb{F}_2^n}$, each one corresponding to an eigenvalue, since for every $a \in \mathbb{F}_2^n$, we have $\sum_{v\in\mathbb{F}_2^n; u+v\in supp(f)}(-1)^{a\cdot v} = \sum_{x\in supp(f)}(-1)^{a\cdot(u+x)} = \widehat{f}(a)(-1)^{a\cdot u}, \forall u \in \mathbb{F}_2^n$.
As a consequence, the size $N_{\widehat{f}}$ of the support $\{a \in \mathbb{F}_2^n;\ \widehat{f}(a) \neq 0\}$ of the Fourier-Hadamard transform of any n-variable Boolean function f is larger than or equal to the size $N_{\widehat{g}}$ of the support of the Fourier-Hadamard transform of any restriction g of f, obtained by keeping constant some of its input bits. Indeed, the adjacency matrix M_g of the Cayley graph G_g is a submatrix of the adjacency matrix M_f of the Cayley graph G_f; the number $N_{\widehat{g}}$ equals the rank of M_g, and is then smaller than or equal to the rank $N_{\widehat{f}}$ of M_f.
This property can be generalized to any pseudo-Boolean function φ, with a simpler proof using the Poisson summation formula (2.39): let I be any subset of $\{1, \dots, n\}$; let E be the vector subspace of $\mathbb{F}_2^n$ equal to $\{x \in \mathbb{F}_2^n;\ x_i = 0, \forall i \in I\}$; we have $E^\perp = \{x \in \mathbb{F}_2^n;\ x_i = 0, \forall i \in \{1, \dots, n\} \setminus I\}$ and the sum of E and of $E^\perp$ is direct; then, for every $a \in E^\perp$ and every $b \in E$, the equality $\sum_{u\in a+E}(-1)^{b\cdot u}\,\widehat{\varphi}(u) = |E|\,(-1)^{a\cdot b}\,\widehat{\psi}(a)$, where ψ is the restriction of φ to $b+E^\perp$, implies that, if $N_{\widehat{\varphi}} = k$, that is, if $\widehat{\varphi}(u)$ is nonzero for exactly k vectors $u \in \mathbb{F}_2^n$, then clearly $\widehat{\psi}(a)$ is nonzero for at most k vectors $a \in E^\perp$.
Coming back to the case where φ is a Boolean function, say $\varphi = f$ where f has algebraic degree d, choosing for I a multi-index of size d such that x^I is part of the ANF of f, then the restriction $\psi = g$ has odd weight and its Fourier-Hadamard transform takes therefore nonzero values only. We deduce (as proved in [68]) that:

$$N_{\widehat{f}} = |\{a \in \mathbb{F}_2^n;\ \widehat{f}(a) \neq 0\}| \geq 2^{d_{alg}(f)}.$$

Notice that $N_{\widehat{f}}$ equals $2^{d_{alg}(f)}$ if and only if at most one element (that is, exactly one) satisfying $\widehat{f}(u) \neq 0$ exists in each coset of E, that is, in each set obtained by keeping constant the coordinates x_i such that $i \in I$.
The number $N_{\widehat{\varphi}}$ is also bounded above by $\sum_{i=0}^{D} \binom{n}{i}$, where D is the *numerical degree* of φ. This is a direct consequence of Proposition 14.
The graph viewpoint gives insight on those Boolean functions whose Fourier-Hadamard spectra have at most three values, as can be seen in [68]. Bent functions (see Chapter 6) are those Boolean functions whose Cayley graphs are strongly regular of a particular type [68, 69]: those graphs such that, for all distinct vertices u, v, the number of those vertices which are adjacent to both u and v are the same.
A hypergraph (see page 89) can also be related to the ANF of a Boolean function f. A related (weak) upper bound on the nonlinearity of Boolean functions (see definition in Section 3.1) has been pointed out in [1179].

2.3.6 The Walsh transform of vectorial functions

Assuming that an inner product in $\mathbb{F}_2^n$ and an inner product in $\mathbb{F}_2^m$ have been chosen, both denoted by "$\cdot$", we call *Walsh transform* of an (n, m)*-function* F, and we denote by W_F, the function which maps any ordered pair $(u, v) \in \mathbb{F}_2^n \times \mathbb{F}_2^m$ to the value at u of the Walsh transform of the Boolean function $v \cdot F$:

$$W_F(u, v) = \sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot F(x) \oplus u \cdot x};\ u \in \mathbb{F}_2^n, v \in \mathbb{F}_2^m.$$

We call *Walsh spectrum* of F the multi-set of all the values $W_F(u, v)$ where $u \in \mathbb{F}_2^n, v \in \mathbb{F}_2^m$. We call *extended Walsh spectrum* of F the multi-set of their absolute values, and *Walsh support* of F the set of those (u, v) such that $W_F(u, v) \neq 0$.

Remark. If we denote by $\mathcal{G}_F$ the *graph* $\{(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^m;\ y = F(x)\}$ of F, and by $1_{\mathcal{G}_F}$ its *indicator* (taking value 1 on $\mathcal{G}_F$ and 0 outside), then we have $W_F(u, v) = \widehat{1_{\mathcal{G}_F}}(u, v)$. *The Walsh transform of any vectorial function is the Fourier-Hadamard transform of the indicator of its graph.*
The *autocorrelation function* $(a, v) \to \sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot (F(x)+F(x+a))}$ is directly connected to the Fourier transform of the function implementing the *difference table* $D_F(a, b) = |x; F(x) + F(x + a) = b|$ since we have $\sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot (F(x)+F(x+a))} = \sum_{b \in \mathbb{F}_2^n} D_F(a, b)(-1)^{v \cdot b}$, and $D_F(a, b)$ is then recovered from the autocorrelation function by the inverse Fourier transform formula for Boolean functions. □

The *inverse Walsh transform formula* (2.43) for vectorial functions writes:

$$\sum_{u \in \mathbb{F}_2^n} W_F(u, v)\, (-1)^{u \cdot x} = 2^n (-1)^{v \cdot F(x)}. \tag{2.68}$$

There is a simple way of expressing the value of the Walsh transform of the composition of two vectorial functions by means of those of the functions:

Proposition 16 *If we write the values of the function W_F in a $2^m \times 2^n$ matrix $(M_{v,u})_{v\in\mathbb{F}_2^m,u\in\mathbb{F}_2^n}$ where $M_{v,u} = W_F(u,v)$, then, the matrix similarly corresponding to the composite function $F\circ H$, where H is an (r,n)-function, equals $2^{-n}\, M\times N$, where N is defined similarly with respect to H.*

Proof. For every $w\in\mathbb{F}_2^r$ and every $v\in\mathbb{F}_2^m$, we have

$$\begin{aligned}\sum_{u\in\mathbb{F}_2^n} W_F(u,v)W_H(w,u) &= \sum_{u\in\mathbb{F}_2^n;x\in\mathbb{F}_2^r;y\in\mathbb{F}_2^n} (-1)^{v\cdot F(y)\oplus u\cdot(y+H(x))\oplus w\cdot x}\\ &= 2^n \sum_{x\in\mathbb{F}_2^r;y\in\mathbb{F}_2^n\,;\,y=H(x)} (-1)^{v\cdot F(y)\oplus w\cdot x}\\ &= 2^n W_{F\circ H}(w,v),\end{aligned}$$

since $\sum_{u\in\mathbb{F}_2^n}(-1)^{u\cdot(y+H(x))}$ equals 2^n if $y=H(x)$, and is null otherwise. □

Remark. Because of Proposition 16, it could seem more convenient to exchange the positions of u and v in $W_F(u,v)$. But we shall not do so because the common use is to respect the order (input,output). □

Remark. We have $W_F(u,v) = \sum_{b\in\mathbb{F}_2^m}\widehat{\varphi_b}(u)(-1)^{v\cdot b}$, where $\widehat{\varphi_b}$ is the Fourier-Hadamard transform of the indicator function φ_b of the pre-image $F^{-1}(b) = \{x\in\mathbb{F}_2^n;\, F(x)=b\}$. □

In [201] is shown that the possibility of building a function *CCZ equivalent* to a given (n,m)-function F depends on the structure of the set of zeros of its Walsh transform. Given an affine permutation $A = L+(a,b)$ of $\mathbb{F}_2^n\times\mathbb{F}_2^m$ (where $L=(L_1,L_2)$ is a linear permutation and (a,b) a point in $\mathbb{F}_2^n\times\mathbb{F}_2^m$), the image by A of the *graph* $\mathcal{G}_F$ of F is the graph of a function if and only if the image of $\mathbb{F}_2^n\times\{0_m\}$ by the adjoint operator L^* of L is included in the set $W_F^{-1}(0)\cup\{(0_n,0_m)\}$. This is immediate: a necessary and sufficient condition is that $L_1(x,F(x))$ be a permutation and according to Proposition 35, page 134, this is equivalent to $\forall u\neq 0_n, \sum_{x\in\mathbb{F}_2^n}(-1)^{(u,0_m)\cdot L(x,F(x))} = \sum_{x\in\mathbb{F}_2^n}(-1)^{L^*(u,0_m)\cdot(x,F(x))} = 0$. A transformation called twisting allows then to move to another EA equivalence class within the same CCZ equivalence class: the output of F is viewed in the form $(T_y(x),U_x(y))\in\mathbb{F}_2^t\times\mathbb{F}_2^{m-t}$, where $t\le\min(n,m)$, $x\in\mathbb{F}_2^t$, $y\in\mathbb{F}_2^{n-t}$ and where T_y is assumed to be a permutation for every y. Then the t-twisting of F is the function $(T_y^{-1}(x),U_{T_y^{-1}(x)}(y))$, whose graph is obtained from that of F by swapping, in each vector of the graph, the sub-vector of indices $1,\dots,t$ and the sub-vector of indices $n+1,\dots,n+t$. It is shown in [201] that every CCZ equivalent function to F can be obtained from F in three steps: applying EA equivalence, then twisting, then applying EA equivalence again. The number of EA equivalence classes in the CCZ equivalence class of F is bounded above by the number of n-dimensional vector spaces in $W_F^{-1}(0)\cup\{(0_n,0_m)\}$ and below by this same number divided by the order of the *automorphism group of function* F (*i.e.* the group of those affine automorphisms which preserve the graph of F).

The case of power functions

When $\mathbb{F}_2^n$ is identified with $\mathbb{F}_{2^n}$, we have seen at page 40 that *power functions* are those functions of the form $F(x) = x^d$. They usually have a lower implementation cost in hardware. Such F is a permutation of $\mathbb{F}_{2^n}$ if and only if d is co-prime with $2^n - 1$. An inner product being for instance $(x, y) \mapsto tr_n(x\,y)$, for every $(u, v) \in (\mathbb{F}_{2^n}^*)^2$ and every d, we have (by the change of variable $x \mapsto \frac{x}{u}$) that $W_F(u, v) = W_F(1, \frac{v}{u^d})$, and if x^d is a permutation of $\mathbb{F}_{2^n}$, then we have (by the change of variable $x \mapsto \frac{x}{v^{\frac{1}{d}}}$) that $W_F(u, v) = W_F(\frac{u}{v^{\frac{1}{d}}}, 1)$.

It has been conjectured in 1976 by Helleseth[34] in [592] that, for every $n \geq 2$ and every value of d co-prime with 2^n-1, there exists $a \in \mathbb{F}_{2^n}^*$ such that $W_F(a, 1) = 0$. This conjecture is still open. It has been checked for $n \leq 25$ by Langevin and proved for $d = 2^n - 2$ (inverse function) in [733] (see more at page 239). We have:

Proposition 17 *[39] For every power permutation $F(x) = x^d$, there exists $a \in \mathbb{F}_{2^n}^*$ such that $W_F(a, 1) \equiv 0$ [mod 3].*

Proposition 18 *[673] If $\gcd(d, 2^n - 1) = 1$ and if the set $\{W_f(a); a \in \mathbb{F}_{2^n}^*\}$ has three distinct values exactly, then one of these values is 0.*

Proposition 19 *[673] If $\gcd(d, 2^n - 1) = 1$ and n is a power of 2, then the set $\{W_f(a); a \in \mathbb{F}_{2^n}^*\}$ cannot have three distinct values exactly.*

See more in [675]. Some results related to Gauss sums are also given in [38] and we have:

Proposition 20 *[174] For every n equal to a power of 2 and every non-linear power permutation $F(x) = x^d$ over $\mathbb{F}_{2^n}$, there exists $a \in \mathbb{F}_{2^n}^*$ such that $W_F(a, 1)$ is not divisible by $2^{\frac{n}{2}+1}$.*

It is observed in [38] that if n is even and $F(x) = x^d$ is constant over $\mathbb{F}_{2^{\frac{n}{2}}}^*$ and not over $\mathbb{F}_{2^n}^*$, then there exists $a \in \mathbb{F}_{2^n}^*$ such that $W_F(a, 1) = -2^{\frac{n}{2}}$. And using McEliece's theorem, it is proved that:

Proposition 21 *[194, 196] Let l and n be two positive integers. The Walsh values of a power function $F(x) = x^d$ over $\mathbb{F}_{2^n}$ are all divisible by 2^l if and only if, for all $u \in \mathbb{Z}/(2^n - 1)\mathbb{Z}$, $w_2(ud) \leq w_2(u) + n - l$.*

See also [746]. These results are complementary of Theorem 2, page 82 (recall that the algebraic degree of x^d equals $w_2(d)$). The latter will have consequences for the characterization of almost bent functions, see page 415.

[34] The conjecture is stated for every characteristic p such that $d \equiv 1$ [mod $p - 1$].

Relation between Walsh transform and NNF of the graph indicator

We know that $W_F(u,v) = \widehat{1_{\mathcal{G}_F}}(u,v)$. Relation (2.59), page 85, and Relation (2.60) show then that if the NNF of $1_{\mathcal{G}_F}(x,y)$ equals $\sum_{\substack{I\subseteq\{1,\ldots,n\}\\ J\subseteq\{1,\ldots,m\}}} \lambda_{I,J}\, x^I y^J$, then:

$$W_F(u,v) = (-1)^{w_H(u)+w_H(v)} \sum_{\substack{I\subseteq\{1,\ldots,n\}, J\subseteq\{1,\ldots,m\}\\ supp(u)\subseteq I, supp(v)\subseteq J}} 2^{n+m-|I|-|J|}\lambda_{I,J},$$

$$\lambda_{I,J} = 2^{-(n+m)}(-2)^{|I|+|J|} \sum_{\substack{u\in\mathbb{F}_2^n, v\in\mathbb{F}_2^m\\ I\subseteq supp(u), J\subseteq supp(v)}} W_F(u,v).$$

2.3.7 The multidimensional Walsh transform

K. Nyberg defines in [911] a polynomial representation, called the *multidimensional Walsh transform*; let us define:

$$\mathcal{W}(F)(z_1,\ldots,z_m) = \sum_{x\in\mathbb{F}_2^n} \prod_{j=1}^m z_j^{f_j(x)} \in \mathbb{Z}[z_1,\ldots,z_m]/(z_1^2-1,\ldots,z_m^2-1),$$

where $f_1,\ldots,f_m$ are the coordinate functions of F.

The multidimensional Walsh transform maps every linear (n,m)-function L to the polynomial $\mathcal{W}(F+L)(z_1,\ldots,z_m)$. This is a representation with uniqueness of F, since, for every L, the knowledge of $\mathcal{W}(F+L)$ is equivalent to that of the evaluation of $\mathcal{W}(F+L)$ at $(\xi_1,\ldots,\xi_m)$ for every choice of ξ_j, $j=1,\ldots,m$, in the set $\{-1,1\}$ of roots of the polynomial z_j^2-1. For such a choice, let us define the vector $v\in\mathbb{F}_2^m$ by $v_j=1$ if $\xi_j=-1$ and $v_j=0$ otherwise. For every $j=1,\ldots,m$, let us denote by a_j the vector of $\mathbb{F}_2^n$ such that the j-th coordinate of $L(x)$ equals $a_j\cdot x$. We denote then by u the vector $\sum_{j=1}^m v_j a_j \in \mathbb{F}_2^n$. Then this evaluation equals $\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}$. We see that the correspondence between the multidimensional Walsh transform and the Walsh transform is the correspondence between a multi-variate polynomial of $\mathbb{Z}[z_1,\ldots,z_m]/(z_1^2-1,\ldots,z_m^2-1)$ and its evaluation over $\{(z_1,\ldots,z_m)\in\mathbb{Z}^m \,/\, z_1^2-1=\cdots=z_m^2-1=0\}=\{-1,1\}^m$. Consequently, the multidimensional Walsh transform satisfies a relation equivalent to the Parseval's relation (see [911]).

2.4 Fast computation of S-boxes

We shall see in Chapter 11 that substitution boxes are almost always expressed in univariate polynomial form $\sum_{j=0}^{2^n-1} b_j x^j$ (where x, $b_j\in\mathbb{F}_{2^n}$), because the structure of field is needed to generate them, although the multiplication plays no role in the criteria they must satisfy (only the addition playing a role). In such polynomial expression, the additions and scalar multiplications being linear mappings are fast to compute. Those multiplications whose two operands include variables (that we shall exceptionally represent explicitly by $\times$) are more complex to process fastly. Methods exist for multiplication processing (see more in [320]):

- The most efficient in terms of timing is *complete tabulation*, by reading the content of a table in `ROM` containing all the pre-computed results. The size of the table is of $n2^{2n}$ bits, the timing is around 5 cycles.
- The most efficient in terms of memory is direct processing. The timing complexity is of order $\mathcal{O}(n^{\log_3(2)})$ with large constants, thanks to Karatsuba's method repeated recursively until getting low-cost multiplications:

$$m = \left\lceil \frac{n}{2} \right\rceil ; \quad (a_h X^m + a_l) \times (b_h X^m + b_l) = c_h X^{2m} + c_{hl} X^m + c_l \ ,$$

  where a_h, a_l, b_h, b_l, c_h, c_{hl}, c_l are polynomials of degree $\leq m$

$$\begin{aligned} c_h &= a_h \times b_h, \quad c_l = a_l \times b_l, \\ c_{hl} &= (a_h + a_l) \times (b_h + b_l) - c_h - c_l. \end{aligned}$$

- A compromise is the so-called *log-alog* method, which assumes that the functions:

$$log : x \in \mathbb{F}_{2^n} \mapsto i = \log_\alpha(x) \text{ and } alog : i \mapsto x = \alpha^i$$

  have been tabulated in `ROM` for some primitive element α of the field. The processing of $a \times b$ then simply consists in processing:

$$c = alog[(log[a] + log[b]) \bmod 2^n - 1] \ .$$

  Its memory complexity is $n2^{n+1}$ bits and its timing complexity is constant.
- Another compromise is obtained with the *tower field* approach. For $n = 2m$ even, the elements of $\mathbb{F}_{2^n}$ are viewed as elements of $\mathbb{F}_{2^m}[X]/(X^2 + X + \beta)$, where $X^2 + X + \beta$ is a degree-2 polynomial irreducible over $\mathbb{F}_{2^m}$. The field isomorphism mapping an element $a \in \mathbb{F}_{2^n}$ into the pair $(a_h, a_l) \in \mathbb{F}_{2^m}^2$ is denoted by L. The multiplication $a \times b$ is then executed as follows:

$$\begin{array}{rcl} (a_h, a_l) & \leftarrow & L(a); \quad (b_h, b_l) \leftarrow L(b); \quad c_l \leftarrow a_h \times b_h \times \beta + a_l \times b_l \\ c_h & \leftarrow & a_h \times (b_h + b_l) + a_l \times b_h; \quad c \leftarrow L^{-1}(c_h, c_l). \end{array}$$

  This is recursively applied if n is a power of 2.

Methods also exist for evaluating whole polynomials (*e.g.* the cyclotomic method and the Knuth-Eve method) that we shall present in Section 12.1.2 because they play a role with respect to countermeasures against side channel attacks.

3 Boolean functions, vectorial functions and cryptography

The design of conventional cryptographic systems relies on two fundamental principles introduced by Shannon [1034]: *confusion* and *diffusion*. Confusion aims at concealing any algebraic structure in the system. It is closely related to the complexity[1] of the involved (so-called nonlinear) functions. Diffusion consists in spreading out the influence of any minor modification of the input data or of the key over all outputs. These two principles were stated more than half a century ago. Since then, many attacks have been found against the diverse known cryptosystems, and the relevance of these two principles has always been confirmed. In this chapter, we describe the main attacks on symmetric cryptosystems and the related criteria on Boolean and vectorial functions. Two books exist on Boolean and vectorial functions for cryptography [401, 1125], which partly cover the state of the art. Several sections of the Handbook of Finite Fields [890] are also devoted to this same subject (in a reduced format). In the subsequent chapters, we shall develop as completely as possible the study of each criterion.

3.1 Cryptographic criteria (and related parameters) for Boolean functions

The known attacks on stream ciphers lead to criteria [842, 844, 970, 1041] that the implemented cryptographic functions must satisfy to resist attacks [388, 391, 826, 843, 1042]. More precisely, the resistance of the cryptosystems to the known attacks can be quantified through some fundamental characteristics (some, more related to confusion, and some, more related to diffusion) of the Boolean functions used in them; and the design of these cryptographic functions needs to consider various characteristics simultaneously. Some of these characteristics are affine invariants. Of course, all characteristics cannot be optimum at the same time, and trade-offs must be considered (see below).

[1] That is, the cryptographic complexity, which is different from circuit complexity, for instance.

3.1.1 Balancedness

Cryptographic Boolean functions must be *balanced* (their output must be uniformly, *i.e.* equally, distributed over $\{0,1\}$) for avoiding statistical dependence between the plaintext and the ciphertext. Indeed, we wish that it is not possible to distinguish the pair of a random plaintext and of the corresponding ciphertext from a random pair. Notice that f is balanced if and only if $W_f(0_n) = \mathcal{F}(f) = 0$.

3.1.2 Algebraic degree

Cryptographic functions must have high algebraic degrees (see Definition 6, page 52). Indeed, all cryptosystems using Boolean functions for confusion (combining or filtering functions in stream ciphers, functions involved in the S-boxes of block ciphers, ...) can be attacked if the functions have low algebraic degrees. For instance, in the case of combining functions (see Figure 1.3, page 38), if n LFSRs having lengths $L_1, \dots, L_n$ are combined by the function $f(x) = \bigoplus_{I\subseteq\{1,\dots,n\}} a_I \left(\prod_{i\in I} x_i\right)$, then the sequence produced by f has *linear complexity*

$$\mathcal{L} \leq \sum_{I\subseteq\{1,\dots,n\}} a_I \left(\prod_{i\in I} L_i\right)$$

(and $\mathcal{L}$ equals this number under the sufficient condition that the sequences output by the LFSRs are *m-sequences* of pairwise co-prime periods), see [1007, 1183]. In the case of the filter model (see Figure 1.4, page 39), we have a less precise result [1006]: if L is the length of the LFSR and if the feedback polynomial is primitive, then the linear complexity of the sequence satisfies:

$$\mathcal{L} \leq \sum_{i=0}^{d_{alg}(f)} \binom{L}{i}.$$

Moreover, if L is a prime, then $\mathcal{L} \geq \binom{L}{d_{alg}(f)}$, and the fraction of functions f of given algebraic degree which output a sequence of linear complexity equal to $\sum_{i=0}^{d_{alg}(f)} \binom{L}{i}$ is at least $e^{-1/L}$. In both models, the algebraic degree of f has to be high so that $\mathcal{L}$ can have high value (the number of those nonzero coefficients a_I, in the ANF of f, such that I has large size, can also play a role, but clearly a less important one).

When n tends to infinity, random Boolean functions have almost surely algebraic degrees at least $n-1$ (the number $2^{\sum_{i=0}^{n-2}\binom{n}{i}} = 2^{2^n-n-1}$ of Boolean functions of algebraic degree at most $n-2$ is negligible with respect to the number 2^{2^n} of all Boolean functions). But we shall see that the functions of algebraic degree $n-1$ or n do not allow achieving some other characteristics like resiliency.

We have seen in Section 2.2 that the algebraic degree is an affine invariant.

3.1.3 Nonlinearity and higher-order nonlinearity

In order to provide confusion, cryptographic functions must lie at large Hamming distance from all affine functions. Let us explain why.

Correlations with linear functions and attacks

We shall say that there is a nonzero correlation between a Boolean function f and a linear function ℓ if $d_H(f, \ell)$ is different from 2^{n-1} (precisely, the correlation between f and $\ell_a(x) = a \cdot x$, where $a \in \mathbb{F}_2^n$, equals $\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus \ell_a(x)}$, that is $W_f(a)$). Because of Parseval's Relation (2.48), page 79, and of Relation (2.37), page 75, any Boolean function has nonzero correlation with at least one linear function. But all correlations should be small (in magnitude). Indeed, a large positive correlation between a Boolean function f involved in a cryptosystem and a linear function ℓ means that $d_H(f, \ell)$ is small, and f is then efficiently approximated by ℓ; a large negative one means that it is approximated by $\ell \oplus 1$.

The existence of such affine approximations of f allows in various situations (block ciphers, stream ciphers) to build attacks on this system.
In the case of stream ciphers, these attacks are the so-called *fast correlation attacks* [843, 203, 369, 517, 645, 646, 647]: let ℓ be a linear approximation of f (or of $f \oplus 1$, but then we shall study $f \oplus 1$) whose distance to f is smaller than 2^{n-1}, denoting by Prob $[E]$ the probability of an event E, we have:

$$p = \text{Prob}\,[f(x_1, \ldots, x_n) \neq \ell(x_1, \ldots, x_n)] = \frac{d_H(f, \ell)}{2^n} = \frac{1}{2} - \epsilon,$$

where $\epsilon > 0$. The pseudorandom sequence s corresponds then to the transmission with errors of the sequence σ which would be produced by the same model with the same LFSRs, but with ℓ instead of f. Attacking the cipher can be done by correcting the errors as in the transmission of the sequence σ over a noisy channel. Assume that we have N bits $s_u, \ldots, s_{u+N-1}$ of the pseudorandom sequence s, then Prob $[s_i \neq \sigma_i] \approx p$. The set of possible sequences $\sigma_u, \ldots, \sigma_{u+N-1}$ is a vector space, that is, a linear code of length N and dimension at most L, where L is the size of the linear part of the PRG (the length of the LFSR in the case of the filter generator). We then use a decoding algorithm to recover $\sigma_u, \ldots, \sigma_{u+N-1}$ from $s_u, \ldots, s_{u+N-1}$ and since ℓ is linear, the linear complexity of the sequence σ is small and we obtain for instance by the Berlekamp-Massey (BM) algorithm the initialization of the LFSR. We can then compute the whole sequence s.
There are several ways for performing the decoding. The method exposed in [843] and improved by [369] is as follows. We call a *parity check polynomial* any polynomial $a(x) = 1 + \sum_{j=1}^{r} a_j x^j$ ($a_r \neq 0$) which is a multiple of the feedback polynomial of an LFSR generating the sequence σ_i. Denoting by $\sigma(x)$ the generating function $\sum_{i \geq 0} \sigma_i x^i$, the product $a(x)\,\sigma(x)$ is a polynomial of degree less than r. We use for the decoding a set of parity check polynomials satisfying three conditions: their degrees are bounded by some integer m, the number of nonzero coefficients a_j in each of them is at most some number $t \geq 3$ (*i.e.*, each polynomial has Hamming weight at most $t + 1$) and for every $j =$

$1, \ldots, m$, at most one polynomial has nonzero coefficient a_j. Each parity check polynomial $a(x) = 1 + \sum_{j=1}^{r} a_j x^j$ gives a linear relation $\sigma_i = \sum_{j=1}^{r} a_j \sigma_{i-j} = \sum_{j=1,\ldots,r\,;\,a_j \neq 0} \sigma_{i-j}$ for every $i \geq m$ and the relations corresponding to different polynomials involve different indices $i - j$. If we replace the (unknown) σ_i's by the s_i's then some of these relations become false but it is possible by using the method of Gallager [524] to compute a sequence z_i such that $\mathrm{Prob}\,(z_i = \sigma_i) > 1-p$. Then it can be proved that iterating this process converges to the sequence σ (with a speed which depends on m, t and p). The number N of bits needed to be known in the keystream, the off-line time complexity P and the on-line time complexity T are (see [203]):

$$N = 2^{\frac{L}{t-1}} \left(\frac{1}{2\epsilon}\right)^{\frac{2(t-2)}{t-1}} \qquad P = \frac{N^{t-2}}{(t-2)!} \qquad T = 2^{\frac{L}{t-1}} \left(\frac{1}{2\epsilon}\right)^{\frac{2t(t-2)}{t-1}},$$

where L is the length of the LFSR and ϵ is the bias of the nonlinearity with respect to 2^{n-1}, that is, $\epsilon = \frac{2^{n-1}-nl(f)}{2^n}$, where $nl(f)$ is defined below. Note that the number of variables of the function does not play an explicit role.

In the case of block ciphers, we shall see in Section 3.4 that the Boolean functions involved in their S-boxes must also lie at large Hamming distances to *affine functions*, to allow resistance to the linear attacks [829].

The corresponding parameter and criterion for Boolean functions

Definition 18 *The* nonlinearity *of a Boolean function f is the minimum Hamming distance between f and affine functions. We shall denote it by $nl(f)$.*

The larger is the nonlinearity, the larger is p in the fast correlation attack and the less efficient is the attack. Hence, from the designer point of view, the nonlinearity must be large (in a sense that will be clarified below) and we shall see that this condition happens to be necessary against other attacks as well. A high nonlinearity is surely one of the most important cryptographic criteria.

By definition, the nonlinearity of any Boolean function is bounded above by its Hamming weight. The set of those Boolean functions which achieve this bound with equality (*i.e.* of all possible *coset leader*s of the first-order Reed-Muller code) is unknown. Some functions belong obviously to it: n-variable Boolean functions of Hamming weight at most 2^{n-2} (since nonzero affine functions have weight at least twice, and according to the triangular inequality); bent functions (see Definition 19 below) of Hamming weight $2^{n-1} - 2^{\frac{n}{2}-1}$; more generally, plateaued functions (see Definition 63, page 285) of amplitude 2^r and Hamming weight $2^{n-1} - 2^{r-1}$. But the set is not completely determined. Note that each coset of the first-order Reed-Muller code contains at least one element of this set. In [1138] are studied the Boolean functions of nonlinearities 2^{n-2} and $2^{n-2} + 1$.

The nonlinearity is an *EA invariant*, since $d_H(f \circ L \oplus \ell', \ell) = d_H(f, (\ell \oplus \ell') \circ L^{-1})$, for every functions f, ℓ and ℓ', and for every affine automorphism L, and since $(\ell \oplus \ell') \circ L^{-1}$ ranges over the whole set of affine functions when ℓ does.

Note also that, given two n-variable Boolean functions f_1 and f_2, we have the inequality $nl(f_1 \oplus f_2) \leq nl(f_1) + nl(f_2)$, because for every affine Boolean functions ℓ_1 and ℓ_2, we have $nl(f_1 \oplus f_2) \leq d_H(f_1 \oplus f_2, \ell_1 \oplus \ell_2) = w_H(f_1 \oplus f_2 \oplus \ell_1 \oplus \ell_2) = w_H(f_1 \oplus \ell_1 \oplus f_2 \oplus \ell_2) \leq w_H(f_1 \oplus \ell_1) + w_H(f_2 \oplus \ell_2) = d_H(f_1, \ell_1) + d_H(f_2, \ell_2)$, and we can choose ℓ_1 and ℓ_2 such that $nl(f_1) = d_H(f_1, \ell_1)$ and $nl(f_2) = d_H(f_2, \ell_2)$. The nonlinearity can be computed through the Walsh transform: let $\ell_a(x) = a_1x_1 \oplus \cdots \oplus a_nx_n = a \cdot x$ be any linear function; according to Relation (2.37), we have $d_H(f, \ell_a) = 2^{n-1} - \frac{1}{2}W_f(a)$ and we deduce $d_H(f, \ell_a \oplus 1) = 2^{n-1} + \frac{1}{2}W_f(a)$; the nonlinearity of f is therefore equal to:

$$nl(f) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} |W_f(a)|. \tag{3.1}$$

Hence a function has high nonlinearity if and only if all of its Walsh values have low magnitudes. The value $\max_{a \in \mathbb{F}_2^n} |W_f(a)|$ is called the "linearity of f" by some authors and its "spectral amplitude" by some others.

Upper and lower bounds, bent functions

Parseval's relation $\sum_{a \in \mathbb{F}_2^n} W_f^2(a) = 2^{2n}$ implies that the arithmetic mean of $W_f^2(a)$ equals 2^n. The maximum of $W_f^2(a)$ being larger than or equal to its arithmetic mean, we deduce that $\max_{a \in \mathbb{F}_2^n} |W_f(a)| \geq 2^{\frac{n}{2}}$. This implies:

Theorem 3 *For every n-variable Boolean function f, we have:*

$$nl(f) \leq 2^{n-1} - 2^{\frac{n}{2}-1}. \tag{3.2}$$

This bound, valid for every Boolean function and tight for every even n as we shall see, is called the *covering radius bound*[2]. It can be improved (*i.e.* lowered) when we restrict ourselves to some subclasses of functions: resilient and correlation immune functions (see Chapter 7); functions $tr(ax^d)$ such that $a \in \mathbb{F}_{2^n}^*$ and $\gcd(d, 2^n - 1) = 1$, since their nonlinearity equals that of the vectorial function x^d and is then bounded above by $2^{n-1} - 2^{\frac{n-1}{2}}$, according to Theorem 6, page 140. A Boolean function will be considered as highly nonlinear if its nonlinearity lies near[3] the upper bound in its class. Note that, for general Boolean functions, there is no direct correlation between the nonlinearity and the algebraic degree: highly nonlinear n-variable functions can have algebraic degree as low as 2 (see Section 5.2) and as large as n (but then the nonlinearity cannot be optimal, see Theorem 13, page 224) and functions with low nonlinearity (*e.g.* functions of Hamming weight at most 2^{n-2}, whose nonlinearity equals the Hamming weight since the minimum distance of the Reed-Muller code of order 1 equals 2^{n-1} and because of the triangular inequality on Hamming distance) can have algebraic degree between 2 and n as well.

Olejár and Stanek [917] have shown that, when n tends to infinity, random Boolean functions on $\mathbb{F}_2^n$ have almost surely nonlinearity larger than $2^{n-1} -$

[2] The covering radius of the *Reed-Muller code* of order 1 equals by definition the maximum nonlinearity of Boolean functions, see Section 4.1.

[3] The meaning of "near" depends on the framework, see [650].

$\sqrt{n}\ 2^{\frac{n-1}{2}}$ (this is easy to prove by counting – or more precisely by bounding from above – the number of functions whose nonlinearities are lower than or equal to a given number, see *e.g.* [224, 229], and using the so-called Shannon effect, see page 125). Rodier [1000] has shown later a more precise and strong result: asymptotically, almost all Boolean functions have nonlinearity between $2^{n-1} - 2^{\frac{n}{2}-1}\sqrt{n}\left(\sqrt{2\,\ln 2} + \frac{4\ln n}{n}\right)$ and $2^{n-1} - 2^{\frac{n}{2}-1}\sqrt{n}\left(\sqrt{2\,\ln 2} - \frac{5\ln n}{n}\right)$ and therefore located in the neighborhood of $2^{n-1} - 2^{\frac{n}{2}-1}\sqrt{2n\ln 2}$, where ln denotes the natural (*i.e.* Neperian) logarithm.
The probability $\mathrm{Prob}\,(\max_w |W_F(w)| \geq y)$ is equal to 1 when $y = 2^{n/2}$; it decreases slowly when y increases, decreases then suddenly to the neighborhood of 0 when y is approaching $2^{n/2}\sqrt{2n\ln 2}$, then it decreases slowly to 0 when y increases to 2^n. Further details can be given around $2^{n/2}\sqrt{2n\ln 2}$. The article [784] provides sharper results by a different method. In particular, for $n > 164$, we have $\mathrm{Prob}\,(\max_w |W_F(w)| \geq 2^{n/2}\sqrt{2n\ln 2}) \leq (1+o(1))/\sqrt{n\pi\ln 2}$.

Equality occurs in (3.2) if and only if $|W_f(a)| = 2^{\frac{n}{2}}$ for every vector a, since the maximum of $W_f^2(a)$ equals the arithmetic mean if and only if $W_f^2(a)$ is constant.

Definition 19 *An n variable Boolean function is called* bent *if its nonlinearity equals $2^{n-1} - 2^{\frac{n}{2}-1}$, or equivalently, $W_f(a) = \pm 2^{\frac{n}{2}}$ for every $a \in \mathbb{F}_2^n$.*

Such functions exist only for even values of n, because $2^{n-1} - 2^{\frac{n}{2}-1}$ must be an integer (in fact, they exist for every n even). Chapter 6 is devoted to them.

For n odd, Inequality (3.2) cannot be tight. The maximum nonlinearity of n-variable Boolean functions, that is, the *covering radius* of $RM(1,n)$, lies then between $2^{n-1} - 2^{\frac{n-1}{2}}$ (which can always be achieved *e.g.* by quadratic functions, see Section 5.2) and $2\lfloor 2^{n-2} - 2^{\frac{n}{2}-2}\rfloor$ [617]. It has been shown in [597, 894] that it equals $2^{n-1} - 2^{\frac{n-1}{2}}$ when $n = 1,\ 3,\ 5,\ 7$, and in [936, 937], by Patterson and Wiedemann[4] (with rotation symmetric functions, see Definition at page 275), that it is strictly larger than $2^{n-1} - 2^{\frac{n-1}{2}}$ if $n \geq 15$ (a review on what was known in 1999 on the best nonlinearities of functions on odd numbers of variables is given in [515], see also [133, 747, 815]). This value $2^{n-1} - 2^{\frac{n-1}{2}}$ is called the *quadratic bound* because, as we already mentioned, such nonlinearity can be achieved by quadratic functions. It is also called the *bent concatenation bound* since it can also be achieved by the concatenation $x_n f(x_1,\ldots,x_{n-1}) \oplus (x_n \oplus 1)g(x_1,\ldots,x_{n-1})$ of two bent functions f,g in $n-1$ variables. It has been later proved by Kavut et al. in [684, 686] (see also [816] where balanced functions are obtained), thanks to rotation symmetric functions as well, that the best nonlinearity of Boolean functions in odd numbers of variables is strictly larger than the quadratic bound for any $n > 7$. See Table 3.1 for the best known nonlinearities for n odd between

[4] It has been later proved (see [1027, 466, 820, 696, 1016]) that balanced functions with nonlinearity strictly larger than $2^{n-1} - 2^{\frac{n-1}{2}}$, and with algebraic degree $n-1$, or satisfying $PC(1)$, see Definition 24, page 118, exist for every odd $n \geq 15$.

5 and 15 compared to the quadratic (or bent concatenation) lower bound and to the upper bound.

Bent functions being not balanced (since we have seen that f is bent if and only

n	5	7	9	11	13	15
$2^{n-1}-2^{\frac{n-1}{2}}$	12	56	240	992	4032	16256
nl	12	56	242	996	4040	16276
$2\lfloor 2^{n-2}-2^{\frac{n}{2}-2}\rfloor$	12	58	244	1000	4050	16292

Table 3.1 Best known nonlinearities nl of Boolean functions in small odd dimension [815]

if $|W_f(a)|$ equals $2^{\frac{n}{2}}$ for every vector a and then $W_f(0_n) \neq 0$), and having too low algebraic degree (as we shall see with Theorem 13, page 224), they are improper for use in cryptosystems. For this reason, even when they exist (for n even), it is also necessary to study those functions which have large nonlinearities, say between $2^{n-1} - 2^{\frac{n-1}{2}}$ and $2^{n-1} - 2^{\frac{n}{2}-1}$, but are not bent, among which some balanced functions exist. The maximum nonlinearity of balanced functions is unknown for any $n \geq 8$. See Table 3.2 for best known nonlinearities for n between 4 and 15 compared to the upper bound $bnd = 2\lfloor 2^{n-2} - 2^{\frac{n}{2}-2}\rfloor$. Note that the 15-variable function can be made 1-resilient (see Definition 21, page 105).

n	4	5	6	7	8	9	10	11	12	13	14	15
nl	4	12	26	56	116	240	492	992	2010	4036	8120	16272
bnd	6	12	28	58	120	244	496	1000	2016	4050	8128	16292

Table 3.2 Best known nonlinearities nl of balanced Boolean functions in small dimension [685, 815, 816, 1016]

As first observed in [1169, 1173], relations exist between the nonlinearity and the *derivatives* of Boolean functions. We give here simpler proofs of these facts. Applying Relation (2.56) to $E = \{0_n, e\}^\perp$, where $e \neq 0_n$, and to $b = 0_n$ and all $a \in \mathbb{F}_2^n$, and using that $\max_{u\in a+E} W_f^2(u) \geq \frac{1}{|E|}\sum_{u\in a+E} W_f^2(u)$, we have:

Proposition 22 *For every $n \geq 1$ and every n-variable Boolean function f, we have:*

$$nl(f) \leq 2^{n-1} - \frac{1}{2}\sqrt{2^n + \max_{e\neq 0_n} |\mathcal{F}(D_e f)|}.$$

This directly proves an important property of bent functions that we shall revisit in Chapter 6: f is bent if and only if all its derivatives $D_e f, e \neq 0_n$, are balanced.

The obvious relation $w_H(f) \geq \frac{1}{2} w_H(D_e f) = \frac{1}{2}\left(2^{n-1} - \frac{1}{2}\mathcal{F}(D_e f)\right)$, valid for every $e \in \mathbb{F}_2^n$, leads when applied to the functions $f(x) \oplus a\cdot x \oplus \epsilon$, where $a \in \mathbb{F}_2^n$ and $\epsilon \in \mathbb{F}_2$, to the inequality $d_H(f, a\cdot x \oplus \epsilon) \geq \frac{1}{2}\left(2^{n-1} - \frac{1}{2}(-1)^{a\cdot e}\mathcal{F}(D_e f)\right) \geq \frac{1}{2}\left(2^{n-1} - \frac{1}{2}|\mathcal{F}(D_e f)|\right)$. Hence, taking the maximum of this last expression when e ranges over $\mathbb{F}_2^n$, we deduce the lower bound:

Proposition 23 *For every positive integer n and every n-variable function f,* *we have:*

$$nl(f) \geq 2^{n-2} - \frac{1}{4} \min_{e \in \mathbb{F}_2^n, e \neq 0_n} |\mathcal{F}(D_e f)|. \tag{3.3}$$

Another lower bound on the nonlinearity is given at the end of the remark located after Theorem 7, page 176, and a further one is given in [248, Subsection 4.2]:

Proposition 24 *Let f be any n-variable Boolean function. Let $S = supp(f) = \{x \in \mathbb{F}_2^n;\ f(x) = 1\}$ be the support of f and let $M_f =$*

$$\frac{\max_{z \in \mathbb{F}_2^n \setminus \{0_n\}} \left|\{(x,y) \in S^2;\ x+y=z\}\right| + \min_{z \in \mathbb{F}_2^n \setminus \{0_n\}} \left|\{(x,y) \in S^2;\ x+y=z\}\right|}{2}$$

and let $E_f =$

$$\frac{\max_{z \in \mathbb{F}_2^n \setminus \{0_n\}} \left|\{(x,y) \in S^2;\ x+y=z\}\right| - \min_{z \in \mathbb{F}_2^n \setminus \{0_n\}} \left|\{(x,y) \in S^2;\ x+y=z\}\right|}{2}.$$

Then:

$$nl(f) \geq 2^{n-1} - \max\left(|2^{n-1} - |S||, \sqrt{|S| - M_f + (2^n - 1)E_f} \right). \tag{3.4}$$

Any bent function achieves (3.4) with equality and it would be interesting to determine all functions f such that (3.4) is an equality.

Nonlinearity and codes

The nonlinearity of a Boolean function f equals the minimum distance of the linear code $RM(1,n) \cup (f \oplus RM(1,n))$. See more in Chapter 4. More generally, the minimum distance of an *unrestricted code* defined as the union of cosets $f \oplus RM(1,n)$ of the Reed-Muller code of order 1, where f ranges over a set $\mathcal{F}$, equals the minimum nonlinearity of the functions $f \oplus g$, where f and g are distinct and range over $\mathcal{F}$, since $d_H(f \oplus h, g \oplus h') = d_H(f \oplus g, h \oplus h')$ and $h \oplus h'$ ranges over $RM(1,n)$ when h, h' do. This observation allows constructing some optimal nonlinear codes such as Kerdock codes (see Section 6.1.22).

Higher-order nonlinearity

Changing one or a few bits in the output (in the truth-table) of a low degree Boolean function gives a function with high degree and does not fundamentally modify the robustness of the system using it (explicit attacks using approximations by low degree functions exist for block ciphers but not for all stream ciphers however, see *e.g.* [707]). A relevant parameter is the *nonlinearity profile*:

Definition 20 *Let n and $r \leq n$ be positive integers. Let f be an n-variable Boolean function. We call r-th* order nonlinearity *(and if r is not specified, the* higher-order nonlinearity*) of f and we denote by $nl_r(f)$, its Hamming distance to the Reed-Muller code of order r. The nonlinearity profile of f is the sequence of its r-th order nonlinearities, for all values of $r < n$.*

Several papers have shown the role played by this EA-invariant parameter against some cryptanalyses (but contrary to the first order nonlinearity, it must have low value for allowing attacks) and studied it from an algorithmic viewpoint [387, 544, 638, 707, 831, 882]. It is related to the minimal distance to functions depending on a subset of variables (which plays a role with respect to the correlation attack, see below in Subsection 3.1.7, and is not EA-invariant) since a function depending on k variables has algebraic degree at most k. Hence the r-th order nonlinearity is a lower bound for the distance to functions depending on at most r variables. The former is much more difficult to study than the latter. The best possible r-th order nonlinearity of Boolean functions equals the covering radius of the r-th order Reed-Muller code, see Subsection 4.1.6, page 180.

Upper and lower bounds and asymptotic behavior
An upper bound on $nl_r(f)$ is given in [309] for $r \geq 2$, that we shall address in Section 4.1 (see page 182). Asymptotically, it gives:

$$nl_r(f) \leq 2^{n-1} - \frac{\sqrt{15}}{2} \cdot (1+\sqrt{2})^{r-2} \cdot 2^{\frac{n}{2}} + \mathcal{O}(n^{r-2}).$$

An asymptotic lower bound, given in [229], is as follows: let $c \in \mathbb{R}$, $c > 0$; for every $r \geq 0$, the density of the set of functions such that:

$$nl_r(f) > 2^{n-1} - c\sqrt{\sum_{i=0}^{r}\binom{n}{i}}\ \ 2^{\frac{n-1}{2}}$$

($i.e.$ the probability for a function to satisfy this inequality) is larger than $1 - 2^{(1-c^2\log_2 e)\sum_{i=0}^{r}\binom{n}{i}}$ and, if $c^2 \log_2 e > 1$, it tends to 1 when n tends to ∞. This is easily proved: the number of functions of algebraic degree at most r equals $2^{\sum_{i=0}^{r}\binom{n}{i}}$. For every such function h, the number of Boolean functions f whose Hamming distance to h is bounded above by some number D equals $\sum_{0\leq i\leq D}\binom{2^n}{i}$. Hence, the number of Boolean functions f such that $d_H(f,h) \leq 2^{n-1} - c\sqrt{\sum_{i=0}^{r}\binom{n}{i}}\ 2^{\frac{n-1}{2}}$ equals $\displaystyle\sum_{0\leq i\leq 2^{n-1}-c\sqrt{\sum_{i=0}^{r}\binom{n}{i}}\ 2^{\frac{n-1}{2}}}\binom{2^n}{i}$. We know from [14] that, for every N, we have $\sum_{0\leq i\leq \lambda N}\binom{N}{i} < 2^N e^{-2N(1/2-\lambda)^2}$. We deduce that the number of Boolean functions f such that $d_H(f,h) \leq 2^{n-1} - c\sqrt{\sum_{i=0}^{r}\binom{n}{i}}\ 2^{\frac{n-1}{2}}$ is bounded above by $2^{2^n - c^2\sum_{i=0}^{r}\binom{n}{i}\log_2 e}$. Thus, the number of those Boolean functions which have r-th order nonlinearity smaller than or equal to $2^{n-1} - c\sqrt{\sum_{i=0}^{r}\binom{n}{i}}\ 2^{\frac{n-1}{2}}$ is smaller than $2^{2^n + (1-c^2\log_2 e)\sum_{i=0}^{r}\binom{n}{i}}$. The rest of the proof is straightforward.
A more precise and more recent result is given by K.-U. Schmidt in [1021], which generalizes the result on $r = 1$ by Rodier [1000] recalled at page 100, and a result from [435] which dealt with $r = 2$: for every $r \geq 1$, the ratio $\frac{2^{n-1} - nl_r(f)}{\sqrt{2^{n-1}\binom{n}{r}\ln 2}}$ tends

to 1 almost surely when n tends to infinity (see more details in [1021]). Unfortunately, this does not help obtaining explicit functions with non-weak r-th order nonlinearity.

Remark. We shall see in Section 4.1 that the minimum Hamming weight of nonzero n-variable Boolean functions of algebraic degree at most r (*i.e.* the minimum distance of the Reed-Muller code $RM(r,n)$) is equal to 2^{n-r} for every $r \leq n$. Hence, applying this property to $r+1$ instead of r, we have $nl_r(f) \geq 2^{n-r-1}$ for every function f of algebraic degree exactly $r+1 \leq n$. Moreover, we shall also see that the minimum weight n-variable Boolean functions of algebraic degree $r+1$ are the characteristic functions of $(n-r-1)$-dimensional flats. Such functions have r-th order nonlinearity 2^{n-r-1} since the null function is the closest function of algebraic degree at most r to such function. □

Computing the r-th order nonlinearity of a given function with algebraic degree strictly larger than r is a hard task for $r > 1$ (for the first order, we have seen that much is known in theory and algorithmically thanks to the Walsh transform, which can be computed by the algorithm of the Fast Fourier-Hadamard Transform; but for $r > 1$, very little is known). Even the second order nonlinearity is known only for a few peculiar functions and for functions in small numbers of variables. Some simple but useful facts are shown in [232]. A nice algorithm due to G. Kabatiansky and C. Tavernier and improved and implemented by Fourquet and Tavernier [518] works well for $r = 2$ and $n \leq 11$ (in some cases, $n \leq 13$), only. It can be applied for higher orders, but it is then efficient only for very small numbers of variables. Proving lower bounds on the r-th order nonlinearity of functions (and therefore proving their good behavior with respect to this criterion) is also a quite difficult task. Until 2008, there had been only one attempt, by Iwata-Kurosawa [638], to construct functions with r-th order nonlinearity bounded from below. But the obtained value, $2^{n-r-3}(r+5)$, of the lower bound was small. Also, lower bounds on the r-th order nonlinearity by means of the algebraic immunity of Boolean functions have been derived (see Chapter 9) but they are small too. In [232] is introduced a method for efficiently bounding from below the nonlinearity profile of a given function when lower bounds exist for the $(r-1)$-th order nonlinearities of the derivatives of f:

Theorem 4 *Let $f \in \mathcal{BF}_n$ and let $0 < r < n$ be an integer. We have:*

$$nl_r(f) \geq \frac{1}{2} \max_{a \in \mathbb{F}_2^n} nl_{r-1}(D_a f) \ , \ and$$

$$nl_r(f) \geq 2^{n-1} - \frac{1}{2}\sqrt{2^{2n} - 2 \sum_{a \in \mathbb{F}_2^n} nl_{r-1}(D_a f)}.$$

The first bound is easily deduced from the inequality $w_H(f) \geq \frac{1}{2} w_H(D_a f)$ applied to $f \oplus h$, $d_{alg}(h) \leq r$, and the second one comes from the equalities $nl_r(f) =$

$$2^{n-1} - \frac{1}{2} \max_{h \in \mathcal{BF}_n\,;\, d_{alg}(h) \leq r} \left| \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus h(x)} \right| \text{ and } \left(\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus h(x)} \right)^2 =$$

$$\sum_{a \in \mathbb{F}_2^n} \sum_{x \in \mathbb{F}_2^n} (-1)^{D_a f(x) \oplus D_a h(x)} = 2^{2n} - 2 \sum_{a \in \mathbb{F}_2^n} d_H(D_a f, D_a h).$$

These bounds ease the determination of efficient lower bounds on the second order nonlinearities of functions in some infinite classes, by reducing the problem to calculations and summations of first-order nonlinearities (often tricky, but feasible). This has been done in a series of papers (see *e.g.* in [714] the references and the table comparing the obtained second order nonlinearities, see also [538]) that we shall not all cite. Such lower bounds were given as examples (about power functions, including the Welch function) in [232], but also bounds for the whole nonlinearity profile of the multiplicative *inverse function* $tr_n(x^{2^n-2})$: the r-th order nonlinearity of this function is approximately bounded below by $2^{n-1} - 2^{(1-2^{-r})\,n}$ and therefore asymptotically equivalent to 2^{n-1}, for every fixed r. Note that the extension of the Weil bound that we shall see in Section 5.6 is efficient for bounding below the r-th order nonlinearity of the inverse function only for $r = 1$. Indeed, already for $r = 2$, the univariate degree of a *quadratic function* in trace form can be bounded above by $2^{\lfloor \frac{n}{2} \rfloor} + 1$ only and this gives a bound in 2^n on the maximum absolute value of the Walsh transform and therefore no information on the nonlinearity. In [240] is similarly studied the (simplest) Dillon bent function $(x, y) \mapsto xy^{2^{n/2}-2}$, $x, y \in F_{2^{n/2}}$ (with an improvement in [1067]) and a univariate function. In [607] is asymptotically studied, for p an odd prime, the Boolean function taking value 0 over the binary expansions of the quadratic residues modulo p.

The relative positions of the two bounds of Theorem 4 with respect to each other have been studied in [872], where it is shown that for $r = 2$, there exist functions for which the first bound is stronger and others where it is *vice versa*.

3.1.4 Correlation immunity and resiliency

We have seen that the Boolean functions used in stream ciphers must be balanced. In both models of *pseudorandom generator*s, there is a stronger condition related to balancedness to satisfy.

In the *combiner model*

Any combining function $f(x)$ must stay balanced when some number of coordinates x_i of x are kept constant.

Definition 21 *Let n be a positive integer and $t \leq n$ a non-negative integer. An n-variable Boolean function f is called an t-th order* correlation immune *function if its output distribution probability is unaltered when at most t (or, equivalently,* *exactly t) of its input bits are kept constant. It is called a t-*resilient *function*[5] *if it is balanced and t-th order correlation immune, that is, if any of its restrictions obtained by fixing at most t (or exactly t) of its input coordinates x_i is balanced.*

Note that, by definition, 0-th order correlation immunity is an empty condition and 0-resiliency means balancedness.
Nota Bene. When we say that a function f is t-th order correlation immune (t-resilient if it is balanced), we do not mean that t is the maximum value of k such that f is k-th order correlation immune. We will call this maximum value the *correlation immunity order* of f (resp. its *resiliency order* if it is balanced).

The notion of correlation immune function has been introduced by Siegenthaler in [1041]. It has been observed later in [181] that the notion existed already in combinatorics and statistics. Indeed, saying that a function f is t-th order correlation immune is equivalent to saying that the array (*i.e.* matrix) whose rows are the vectors of the support of f is a simple binary *orthogonal array*[6] of strength t.

Definition 22 *[988] An array (a matrix) over an alphabet A is an orthogonal array of strength t if, when we select any t columns in it, each vector of A^t appears the same number λ of times as a row in the array restricted to these columns. This orthogonal array is called simple if no two rows are equal. It is often called a $t-(|A|, n, \lambda)$ orthogonal array, where n is the number of columns in the array (in the case of correlation immune functions, the number of variables, with $|A| = 2$).*

Orthogonal arrays play a role in statistics, for the organization of experiments. Each row corresponds to the organization of an experiment and the n columns correspond to parameters. It is necessary to organize the experiments so that any combination of some number k of parameters will appear in the same number of experiments. This is achieved if all possible $|A|^n$ experiments are made, but this is not a solution since the number of rows needs to be minimized (exactly as in the case of countermeasures to side channel attacks, see Subsection 12.1.1, page 467). There exist bounds: the number of rows in a binary orthogonal array of strength k is larger than or equal to $\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n}{i}$ (Rao [988]) and to $2^n \left(1 - \frac{n}{2(k+1)}\right)$ (Friedman, [520]). There exists a monograph on orthogonal arrays [591].

Correlation immunity is a criterion for the resistance to an attack on the combiner model due to Siegenthaler, called *correlation attack* [1042]: if f is not t-th order correlation immune, then there exists a correlation between the output of the function and (at most) t coordinates of its input; if t is small, a divide-and-conquer attack uses this weakness for attacking a system using f as combining function; in the original attack by Siegenthaler, all the possible initializations of

[5] The term of resiliency was introduced in [370], in relationship with another cryptographic problem.
[6] This also relates then correlation immune functions to mutually orthogonal latin squares and threshold secret sharing schemes.

the t LFSRs corresponding to these coordinates are tested (in other words, an exhaustive search of the initializations of these specific LFSRs is done); when we arrive to the correct initialization of these LFSRs, we observe a correlation (before that, the correlation is negligible, as for random pairs of sequences); now that the initializations of the t LFSRs are known, those of the remaining LFSRs can be found with an independent exhaustive search (or by applying again the Siegenthaler attack if possible).

An additional condition
It is shown in [203, 187] that, to make the correlation attack on the combiner model with a t-resilient combining function as inefficient as possible, the coefficient $W_f(u)$ of the function has to be small for every vector u of Hamming weight higher than but close to t. This condition is satisfied under the sufficient condition that the function is highly nonlinear (*i.e.* has high nonlinearity). Hence we see that nonlinearity plays a role with respect to this attack as well.

Characterization of correlation immunity and resiliency by the Walsh transform
Resiliency and correlation immunity have been nicely characterized by means of the Fourier-Hadamard and Walsh transforms of f, first by S. Golomb in [549] (which is not widely known) and later by Xiao and Massey in [1128]. We propose to call this the *Golomb-Xiao-Massey characterization*:

Theorem 5 *[549] Any n-variable Boolean function f is t-th order correlation immune if and only if, for all $u \in \mathbb{F}_2^n$ such that $1 \leq w_H(u) \leq t$, we have $W_f(u) = 0$,* i.e. *$\widehat{f}(u) = 0$. And f is t-resilient if and only if $W_f(u) = 0$ for all $u \in \mathbb{F}_2^n$ such that $w_H(u) \leq t$.*

Proof. Let us prove the first assertion. The second is a direct consequence. By applying the *Poisson summation formula* (2.39), page 77, to $\varphi = f_\chi$, $a = 0_n$ and $E_I = \{x \in \mathbb{F}_2^n;\ x_i = 0,\ \forall i \notin I\}$, b ranging over $\mathbb{F}_2^n$, we obtain since $E_I^\perp = \{x \in \mathbb{F}_2^n;\ x_i = 0,\ \forall i \in I\}$ that f is t-th order correlation immune if and only if, for every I of size t, the value of the sum $\sum_{u\in E_I}(-1)^{b\cdot u}\, W_f(u)$ is independent of b. If, for every nonzero u of weight at most t, we have $W_f(u) = 0$ (that is, $\widehat{f}(u) = 0$ according to Relation (2.32)), then the sum $\sum_{u\in E_I}(-1)^{b\cdot u}\, W_f(u)$ is independent of b. Conversely, if this latter property is satisfied for every I of size t, then since $\sum_{u\in E_I}(-1)^{b\cdot u}\, W_f(u)$ is the Fourier-Hadamard transform of the function equal to $W_f(u)$ if $u \in E_I$ and to 0 otherwise, by the *inverse Fourier-Hadamard transform formula* (2.42), we have $W_f(u) = 0$ for every nonzero u of weight at most t. □

Remark. For f balanced, there is another proof: we apply the second-order Poisson formula (2.57) to $E = \{x \in \mathbb{F}_2^n;\ x_i = 0,\ \forall i \in I\}$ where I is any set of indices of size t; the sum of E and $E^\perp = \{x \in \mathbb{F}_2^n;\ x_i = 0,\ \forall i \notin I\}$ is direct and equals $\mathbb{F}_2^n$; hence we can take $E' = E^\perp$ and we get $\sum_{u\in E^\perp} W_f^2(u) = |E^\perp| \sum_{a\in E^\perp} \mathcal{F}^2(h_a)$, where h_a is the restriction of f to $a+E$, that is, the restriction obtained by fixing the coordinates of x whose indices belong to I to the corresponding coordinates of a. The number $\mathcal{F}(h_a)$ is null if and only if h_a is balanced and clearly, all the numbers $\mathcal{F}(h_a)$, $a \in E^\perp$ are null if and only if all the numbers $W_f(u)$, $u \in E^\perp$ are null. Since this is valid for every multi-index I of size t, this completes the proof.□

Another characterization of correlation immune and resilient functions exists, by the discrete Fourier transform: $j \in \{0, \ldots, 2^n - 1\} \mapsto \sum_{k=0}^{2^n-1} (-1)^{f(k)} \xi^{-kj}$, where j and k are identified respectively with their binary expansions and $\xi = e^{\frac{2\pi\sqrt{-1}}{2^n}}$, see [1111].

Theorem 5 directly implies:

Corollary 6 *Let f be any n-variable Boolean function and $t \leq n$. Then f is t-th order correlation immune if and only if its support, viewed as an unrestricted code, has* dual distance *at least $t+1$.*

Proof. Let C denote the support of f. The dual distance of C equals (by Definition 4, page 32) the number $\min\{w_H(u);\, u \neq 0_n, \sum_{x \in C} (-1)^{u \cdot x} = \widehat{f}(u) \neq 0\}$. □

See more in [422, 423] (see also in [828] a generalization of this result to arrays over finite fields and other related nice results).

Hence, since the Hamming weight of a t-th order correlation immune function is by definition divisible by 2^t, the size of a code of dual distance d is divisible by 2^{d-1}, as we saw at page 32.

Automorphism group

Contrary to the algebraic degree, to the nonlinearity and to balancedness, the correlation immunity and resiliency orders are not *affine invariants* (they are *permutation invariants*), except for the null order (and for the order n, but the set of n-th order correlation immune functions is the set of constant functions and the set of n-resilient functions is empty, because of Parseval's Relation (2.47), page 79). They are both invariant under any translation $x \mapsto x + b$, according to Lemma 4 and Theorem 5. The *automorphism group* of the set of t-resilient functions (that is, the group of all permutations σ of $\mathbb{F}_2^n$ which preserve resiliency) and the orbits under its action have been studied in [622]).

The whole Chapter 7 is devoted to correlation immune and resilient functions.

Remark. An interesting question is: given a Boolean function (resp. a balanced Boolean function) f, what is the best possible correlation immunity (resp. resiliency) order of the Boolean functions affine equivalent to f? Of course, the highest possible power of 2 dividing $w_H(f)$ plays a role, but the reply is not straightforward. □

In the *filter model*

The divide-and-conquer method valid for the combiner model does not apply to the filter model, since there is only one LFSR in this model. The condition of high order resiliency is then not needed. But a stronger condition than balancedness is also necessary in this model, in order to avoid so-called *distinguishing attack*s. These attacks are able to distinguish the pseudorandom sequence, say $(s_i)_{i\in\mathbb{N}}$, from a random sequence. A way of doing so is to observe that the distribution of the sequences $(s_{i+\gamma_1}, \ldots, s_{i+\gamma_n})$ is not uniform, where $\gamma_1, \ldots, \gamma_n$ are (for instance) the positions where the input bits to the filtering function are chosen [20]. Golić [545] has observed that if the feedback polynomial of the LFSR is primitive and if the filtering function has the form $g(x_1, \ldots, x_{n-1}) \oplus x_n$ (up to a permutation of variables), then the property of uniformity is satisfied whatever are the tap positions (where the input bits to the filter function are taken). Canteaut [189] has proved that this condition on the function is also necessary for having uniformity. For choosing a filtering function, we can choose a function g satisfying the cryptographic criteria listed in the present section, and use f defined by means of g in one of the two ways above. But better can be done (see Subsection 9.1.6, page 375). More is said in [567] on the requirements on the filter function.

3.1.5 Algebraic immunity and fast algebraic immunity

A new kind of attacks, called *algebraic attack*s, has been introduced in 2003 (see [391, 497, 388]) and has significantly changed the situation with Boolean functions in stream ciphers. These attacks recover the secret key, or at least the initialization of the system, by solving a system of multivariate algebraic equations.

Shannon's criterion

The idea that the key bits in a cryptosystem can be characterized as the solutions of a system of multivariate equations translating the specifications of the cryptosystem comes from C. Shannon [1034]. Until the invention of algebraic attacks, this bright observation led more to a design criterion (*i.e.* the system should not be solvable in reasonable time with current means) than to an actual attack. Indeed, in practice, for cryptosystems which are robust against the usual attacks (*e.g.* for stream ciphers resisting the Berlekamp-Massey attack), this system is too complex to be solved (its equations being highly nonlinear and the number of unknowns being too large for a nonlinear system of equations). However, in the case of stream ciphers, we can get a very overdefined system (*i.e.* a system with a number of linearly independent equations much larger than the number of unknowns). Let us consider the combiner or the filter model, or any model with a linear part (the n LFSRs in the case of the combiner model, the single LFSR in the case of the filter model) of size N filtered by an n-variable Boolean function f. There exists a linear permutation $L : \mathbb{F}_2^N \mapsto \mathbb{F}_2^N$ updating the current state of the linear part into its next state[7], and a linear function $L' : \mathbb{F}_2^N \mapsto \mathbb{F}_2^n$ mapping the linear part to the n bits selected as input to f. Denoting by $(u_1, \dots, u_N)$ the initialization of the linear part, the current state of the linear part at i-th clock-cycle equals $L^i(u_1, \dots, u_N)$. Denoting by $(s_i)_{i\geq 0}$ the pseudorandom sequence output by the generator, we have, for every $i \geq 0$:

$$s_i = f(L' \circ L^i(u_1, \dots, u_N)). \tag{3.5}$$

These equations have all the same degree $d_{alg}(f)$. The number of those which are exploitable by the attacker equals the number of bits s_i known by him/her, and can then be much larger than the number of unknowns (but of course, the larger the number of equations, the weaker the attack). The system of these equations can then be much overdefined if necessary[8]. This makes less complex the resolution of the system by using Gröbner bases (see [497]), and even allows linearizing the system[9] (*i.e.* obtaining a system of linear equations by replacing every monomial of degree larger than 1 by a new unknown). The linear system obtained after linearization has however too many unknowns: this number is roughly $\sum_{j=0}^{d_{alg}(f)} \binom{N}{j}$.

Courtois' and Meier's improvement for stream ciphers

Courtois and Meier have had a simple but efficient idea. Assume that there exist functions $g \neq 0$ and h of low algebraic degrees (say, of algebraic degree at most d) such that $f\,g = h$ (where $f\,g$ denotes the Hadamard product of f and g, whose support is the intersection of the supports of f and g). For every $i \geq 0$, Relation (3.5) implies:

$$s_i\, g(L' \circ L^i(u_1, \dots, u_N)) = h(L' \circ L^i(u_1, \dots, u_N)). \tag{3.6}$$

This equation in $u_1, \dots, u_N$ has degree at most d, since L and L' are linear, and the system of equations obtained after linearization has then at most $\sum_{j=0}^{d} \binom{N}{j}$ unknowns and may be solved by Gaussian elimination (if d is small enough) in $O\left(\left(\sum_{i=0}^{d} \binom{N}{i}\right)^{\omega}\right)$ operations, where $\omega \approx 3$ is the exponent of the Gaussian reduction[10]. The attack needs about $\sum_{i=0}^{d} \binom{N}{i}$ bits of the keystream.
Low degree relations have been shown to exist for several well known constructions of stream ciphers, which were immune to all previously known attacks. This

7 In the filter model, the matrix of L is simply a companion matrix; in the combiner model, it is a slightly more complex matrix having companion matrices around its diagonal and zeros elsewhere.

8 The probability that N random equations in N variables have rank N equals roughly $\frac{1}{2}$ since the determinant of this system lives in $\mathbb{F}_2$.

9 The known algorithms are, starting from the simplest one: linearization, XL, Buchberger, F4 and F5 (by Faugère); they have different complexities and do not need the same numbers of linearly independent equations.

10 It can be taken equal to $\log_2 7 \approx 2.8$ and the coefficient in the O can be taken equal to 7, according to Strassen [1052]; a still better exponent is due to Coppersmith and Winograd but the multiplicative constant is then inefficiently high for our framework.

was the case for instance with functions whose *ANF*s had only few nonzero coefficients. Such functions had been used as combining/filter functions for reasons of efficiency in the design of some stream ciphers[11] (*e.g.*, LILI-128 and Toyocrypt stream ciphers, see the references in [842]).
Krause and Armknecht [28] extended algebraic attacks to combiners with memory. They studied the algebraic equations satisfied by such combiners and proved an upper bound on their possible degree by means of the input and memory sizes. Courtois [389] generalized their results to multi-output functions.

Algebraic immunity

As observed in [391], if we know the existence of a nonzero low *algebraic degree* multiple h of f, then the support of h being included in that of f, we have $(f \oplus 1)h = 0$, and taking $g = h$, we have the desired relation $fg = h$. But the existence of such multiple h of f is only a sufficient condition for having relation $fg = h$. A necessary and sufficient condition has been found in [842]:

Proposition 25 *Let f be any n-variable Boolean function. The existence of functions $g \neq 0$ and h, both of algebraic degree at most d, such that $fg = h$, is equivalent to the existence of a function $g \neq 0$ of algebraic degree at most d such that $fg = 0$ or $(f \oplus 1)g = 0$.*

Proof. Equality $fg = h$ implies $f^2 g = fh$, that is (since $f^2 = f$), $f(g \oplus h) = 0$, which gives the desired equality of the form $fg = 0$ (with $g \neq 0$) if $g \neq h$ by replacing $g \oplus h$ by g; and if $g = h$ then $fg = h$ is equivalent to $(f \oplus 1)g = 0$. This proves the implication from top to bottom. The converse is straightforward. □
Note that Proposition 25 implies that the existence of a low algebraic degree nonzero multiple of f or of $f \oplus 1$ is a necessary and sufficient condition for the existence of low algebraic degree $g \neq 0$ and h such that $fg = h$ (since being a multiple of f, resp. of $f \oplus 1$, is equivalent to having null product with $f \oplus 1$, resp. with f).

Definition 23 *[842] Let f be any n-variable Boolean function. An n-variable Boolean function g such that $fg = 0$ is called an* annihilator *of f.*
The minimum algebraic degree of nonzero annihilators of f or $f \oplus 1$, i.e. *the minimum algebraic degree of nonzero multiples of $f \oplus 1$ or f, or equivalently, the minimal value d such that there exist $g \neq 0$ and h, both of algebraic degree at most d, such that $fg = h$, is called the* algebraic immunity *of f and is denoted by $AI(f)$.*

This notion has been generalized to functions over general finite fields in [52], with an upper bound on it.

Remark. The set of all annihilators of function f is equal to the ideal of all the

[11] The designers of these stream ciphers had forgotten at their own expense the basic rule of choosing, for cryptosystems, primitives behaving as randomly as possible.

multiples of $f \oplus 1$. □

Remark. Algebraic immunity plays also a role in computational complexity, see [770], where a stronger notion is studied (for symmetric functions). □

The whole Chapter 9 is devoted to algebraic immunity.

Let g be a generic n-variable Boolean function of algebraic degree at most d. Let the ANF of g equal $\bigoplus_{I\subseteq\{1,\dots,n\};|I|\le d} a_I x^I$, where the coefficients a_I can be any elements of $\mathbb{F}_2$. Then g is an annihilator of f if and only if $f(x) = 1$ implies $g(x) = 0$, that is, if and only if the coefficients a_I satisfy the system of homogeneous linear equations $\bigoplus_{I\subseteq\{1,\dots,n\};|I|\le d} a_I u^I$, where u ranges over the support of f. In this system, we have $\sum_{i=0}^{d}\binom{n}{i}$ number of variables (the coefficients of the monomials of degrees at most d) and $w_H(f)$ many equations[12]. We shall denote by $M_{f,d}$ the matrix of this system.

Algebraic immunity is an *affine invariant* but not an EA invariant. More precisely, its automorphism group (that is, the group of all permutations σ of $\mathbb{F}_2^n$ such that $AI(f\circ\sigma) = AI(f)$ for every Boolean function f) equals the general affine group (as for Reed-Muller codes). Indeed, denoting by $An(f)$ the $\mathbb{F}_2$-vector space of annihilators of f, we have $An(f\circ\sigma) = An(f)\circ\sigma$.

A strength of algebraic attack comes from the fact that the algebraic degrees of g and h can always be made lower than or equal to the *Courtois-Meier bound* $\lceil \frac{n}{2}\rceil$:

Proposition 26 *[391] The algebraic immunity of any n-variable Boolean function is bounded above*[13] *by $\lceil \frac{n}{2}\rceil$ and by $d_{alg}(f)$.*

Proof. The number of monomials of algebraic degree at most $\lceil \frac{n}{2}\rceil$ is strictly larger than 2^{n-1}. The disjoint union of the family of these monomials and of the family of the products of f by these monomials has then size strictly larger than 2^n, which is the dimension of the $\mathbb{F}_2$-vector space $\mathcal{BF}_n$. The functions in this disjoint union are then necessarily $\mathbb{F}_2$-linearly dependent. Given a linear combination equal to function 0 and having not all-zero coefficients, let us gather separately the part dealing with the first family and the part dealing with the second. This gives two functions h and g, both of degree at most $\lceil \frac{n}{2}\rceil$, such that $h = f\,g$ and $(g,h)\neq(0,0)$, *i.e.* $g \neq 0$. This proves the first upper bound. The second comes from the fact that f and $f\oplus 1$ are annihilators of each other. □

[12] Those corresponding to u's of small weights may be used to simplify those corresponding to u's of larger weights as shown in [27].

[13] Consequently, it is bounded above by $\lceil k/2\rceil$ if, up to affine equivalence, it depends only on k variables, and by $\lceil k/2+1\rceil$ if it has a linear kernel (see below) of dimension $n-k$, since it is then equivalent, according to Proposition 28, to a function in k variables plus an affine function.

Remark For n odd, according to Proposition 26 and since $2\sum_{i=0}^{\frac{n-1}{2}}\binom{n}{i}=2^n$, we have $AI(f)=\frac{n+1}{2}$ if and only if the family $\{x^I f, |I|\leq\frac{n-1}{2}\}\cup\{x^J(f\oplus 1), |J|\leq\frac{n-1}{2}\}$ is a basis of the $\mathbb{F}_2$-vector space $\mathcal{BF}_n$. Note that this leads to new codes: for every $k\leq\frac{n-1}{2}$, the code $C_{f,k}$ of length 2^n and dimension $2\sum_{i=0}^{k}\binom{n}{i}$ generated by $\{x^I f, |I|\leq k\}\cup\{x^J(f\oplus 1), |J|\leq k\}$. For $k=\frac{n-1}{2}$, it equals the whole space $\mathcal{BF}_n$. For $k=1$, it equals the direct sum of the first-order Reed-Muller code punctured at the positions in the support of f and of the first-order Reed-Muller code punctured at the positions in the co-support, and has minimum distance $\frac{1}{2}nl(f)$, since $w_H(fg)$ equals 2^{n-1} if $g=1$ (because f is balanced, according to Relation (9.5), page 361) and $w_H(fg)=\frac{w_H(f)+w_H(g)-w_H(f\oplus g)}{2}=\frac{2^n-w_H(f\oplus g)}{2}=\frac{w_H(f\oplus g\oplus 1)}{2}$ if g is affine non-constant, and we have the same for $w_H((f\oplus 1)h)$. Note that the known functions f such that $AI(f)=\frac{n+1}{2}$ have diverse nonlinearities. □

Algebraic immunity of random functions

Random functions behave well with respect to algebraic immunity[14]: it has been proved in [437] (see a slightly more complete proof in [307] and its extension to vectorial functions) that, for all $a<1$, when n tends to infinity, $AI(f)$ is almost surely larger than $\frac{n}{2}-\sqrt{\frac{n}{2}\ln\left(\frac{n}{2a\ln 2}\right)}$.

Consequences of the invention of algebraic attack on the design of stream ciphers

A difference by 1 in the algebraic immunity of a function f, used as combiner or filter in a stream cipher, makes a big difference in the efficiency of algebraic attack. The designer needs then to choose f with optimal or near-optimal algebraic immunity. Let then an n-variable function f, with algebraic immunity $\lceil\frac{n}{2}\rceil$, be used for instance as filter on an LFSR of length $N\geq 2k$, where k is the length of the key (otherwise, it is known that the system is not robust against an attack called time-memory-data trade-off attack). Then the complexity of an algebraic attack using one annihilator of degree $\lceil\frac{n}{2}\rceil$ is roughly $7\left(\binom{N}{0}+\cdots+\binom{N}{\lceil\frac{n}{2}\rceil}\right)^{\log_2 7}\approx 7\left(\binom{N}{0}+\cdots+\binom{N}{\lceil\frac{n}{2}\rceil}\right)^{2.8}$ (see [391]). Let us choose $k=128$ (which is usual) and $N=256$, then it is for $n\geq 13$ that the complexity of algebraic attack is at least 2^{80} (which is considered nowadays as just enough); and it is larger than the complexity of an exhaustive search, that is 2^{128}, for $n\geq 15$. If the attacker knows several linearly independent annihilators of degree $\lceil\frac{n}{2}\rceil$, then the number of variables must be enhanced! In practice, the number of variables will have to be near 20 (but this poses then a problem of efficiency of the stream cipher). This has quite changed the situation with Boolean functions at the beginning of this century, since before algebraic attacks, the Boolean functions used had rarely more than 10 variables.

[14] No result is known on the behavior of random functions against fast algebraic attacks.

Fast algebraic attack

A high value of $AI(f)$ is even not a sufficient property for a resistance to algebraic attacks, because other algebraic attacks have been later invented. The *fast algebraic attack* (*FAA*) is an improvement to the standard algebraic attack. It can work even if the algebraic immunity of the function is large[15], provided that there exist n-variable Boolean functions g nonzero of low algebraic degree, and h of reasonable algebraic degree (*i.e.* of algebraic degree possibly larger than $\frac{n}{2}$ but significantly smaller than n) such that $fg = h$, see [388]. This attack is based on the observation that it is possible to obtain a low degree equation from several ones of the form (3.6), by eliminating the large degree terms in the right-hand sides of these equations, and that such elimination may be made off-line by the attacker (that is, before that values of the s_i's are known by him/her) and therefore benefit of a much longer time of computation. The efficiency of the pre-computation and substitution steps has been improved by Hawkes and Rose [590] for the filter model (allowing a complexity of $\mathcal{O}((\sum_{i=0}^{d_{alg}(h)}\binom{N}{i}) \log_2^3(\sum_{i=0}^{d_{alg}(h)}\binom{N}{i}) + (\sum_{i=0}^{d_{alg}(h)}\binom{N}{i}) N \log_2^2 N)$ operations, needing $2(\sum_{i=0}^{d_{alg}(g)}\binom{N}{i})$ bits of stream for the former, and an on-line complexity of $O((\sum_{i=0}^{d_{alg}(g)}\binom{N}{i}))^3 + 2(\sum_{i=0}^{d_{alg}(g)}\binom{N}{i}))(\sum_{i=0}^{d_{alg}(h)}\binom{N}{i})) \log_2((\sum_{i=0}^{d_{alg}(h)}\binom{N}{i}))))$ operations) and by Armknecht [25] for the combiner model, also when they are made more complex by the introduction of memory. Fast algebraic attacks need more data than standard ones (since several values s_i need to be known to obtain one equation), but may also be faster. Armknecht and Ars [26] introduced a variant of the FAA which reduced the data complexity (but not the time complexity).

On the existence of g and h

Given non-negative integers d and e such that $e+d \geq n$, the number of monomials of degrees at most e and the number of monomials of degrees at most d have sum strictly larger than 2^n, and there exist[16] then $g \neq 0$ of algebraic degree at most e and h of algebraic degree at most d such that $fg = h$. An n-variable Boolean function f is then optimal with respect to fast algebraic attacks if there do not exist two functions $g \neq 0$ and h such that $fg = h$, $d_{alg}(g) < \lceil \frac{n}{2} \rceil$ and $d_{alg}(g) + d_{alg}(h) < n$. Since $fg = h$ implies $fh = f^2 g = fg = h$, we see that h is then an annihilator of $f \oplus 1$, and if $h \neq 0$, its algebraic degree is then at least equal to the algebraic immunity of f.

Complexity of the attack and related parameters on Boolean functions

The complexity of FAA is roughly of the order (see [590]):

$$O\left(\min\left\{N^{\max[d_{alg}(g)+d_{alg}(fg), 3d_{alg}(g)]}, g \neq 0\right\}\right).$$

[15] Fast algebraic attack has worked on the eSTREAM [495] proposal SFINKS [390], while the cipher was designed to withstand algebraic attack.

[16] We do not require here that $fg \neq 0$; if such requirement is imposed, the result is no more true, as observed by Gong [553].

It can be seen that FAA with $g = 1$ is less efficient than the Rønjom-Helleseth attack (see below) and that FAA with $d_{alg}(g) \geq AI(f)$ is in fact the algebraic attack. This has led in [324] to studying the so-called *fast algebraic complexity*:

$$FAC(f) = \min \{\max [d_{alg}(g) + d_{alg}(fg), 3d_{alg}(g)] \,; 1 \leq d_{alg}(g) < AI(f)\} ,$$

whose value is invariant by changing f into $f \oplus 1$, and is bounded above by n and below by the so-called *fast algebraic immunity*:

$$FAI(f) = \min (2AI(f), \min \{d_{alg}(g) + d_{alg}(fg);\ 1 \leq d_{alg}(g) < AI(f)\}) ;$$

which had been informally introduced in a preliminary version of the paper [791] and used in [324, 870, 1106]. Note that FAI is also invariant by changing f into $f \oplus 1$, and is easier to study. If this latter parameter is close to n then FAC is too and the function provides then a good resistance to FAA.

Remark. Since, for the resistance against FAA, there must not exist $g \neq 0$ such that $d_{alg}(g)$ is small and $d_{alg}(fg)$ is reasonably large, then if $d_{alg}(f)$ is not large, f does not resist FAA. Because of the Siegenthaler bound (see Proposition 117, page 313) and of the fact that functions in the combiner model must be correlation immune, the combiner model cannot be used nowadays without extra protections. □

Other algebraic attacks

Algebraic attack on the augmented function

Considering now f as a function in N variables, to simplify description, this attack due to [509] works with the vectorial function $F(x)$ whose output equals the vector $(f(x), f(L(x)), \dots, f(L^{m-1}(x)))$, where L is the (linear) update function of the linear part of the generator. This attack can be more efficient than the standard algebraic attack. But the efficiency of the attack not only depends on the function f; it also depends on the update function (and naturally also on the choice of m), since for two different update functions L and L', the vectorial functions $F(x)$ and $F'(x) = (f(x), f(L'(x)), \dots, f({L'}^{m-1}(x))$ are not linearly equivalent; they are even not CCZ equivalent in general. The resistance to this attack is then more a matter with the pair (f, L) rather than with the single function f.

The Rønjom-Helleseth attack

This attack, introduced in [1003] and improved in [1002, 1004, 556, 600, 1001], also adapts the idea of algebraic attacks due to Shannon, but in a different way. An LFSR with a primitive retroaction polynomial (or equivalently a primitive characteristic polynomial) generates a sequence of the form $u_i = tr_N(\lambda\alpha^i)$ where α is a primitive element of $\mathbb{F}_{2^N}$. Essentially, an LFSR generates the field $\mathbb{F}_{2^N}^*$ and a classical filter generator keystream sequence is formed by applying a Boolean function in n variables to n of the N bits of the coefficient vector of the element

u_i. Rønjom and Helleseth then observe that the coefficients in front of a particular monomial in the sequence of multivariate equations expressing the keystream bits form a so-called *coefficient sequence* that inherits highly structural finite field properties from the LFSR. In particular, from this observation they gain fine-grained control over the linear dependencies in the multivariate equation system which enables very efficient reductions. They take advantage of this by proposing an attack whose computational complexity is in about $\sum_{i=0}^{d}\binom{N}{i}$ operations, where d is the algebraic degree of the filter function and N is the size of the LFSR (rather than $O\left(\left(\sum_{i=0}^{AI(f)}\binom{N}{i}\right)^{\omega}\right)$ in the case of standard algebraic attack, where $AI(f)$ is the algebraic immunity of the filter function and $\omega \approx 3$ is the exponent of the Gaussian reduction). It needs about $\sum_{i=0}^{d}\binom{N}{i}$ consecutive bits of the keystream output by the pseudo-random generator (rather than $\sum_{i=0}^{AI(f)}\binom{N}{i}$)). Since d is supposed to be close to the number n of variables of the filter function, the number $\sum_{i=0}^{d}\binom{N}{i}$ is comparable to $\binom{N}{n}$. Since $AI(f)$ is supposed to be close to $\lceil\frac{n}{2}\rceil$, we can see that denoting by C the complexity of the Courtois-Meier attack and by C' the amount of data it needs, the complexity of the Rønjom-Helleseth attack roughly equals $C^{2/3}$ and the amount of data it needs is roughly C'^2. From the viewpoint of complexity, it is more efficient and from the viewpoint of data it is less efficient.

It has been later observed (see [556, 600, 1001]) that the multivariate representation essentially hides away more of the underlying finite field structure stemming from the LFSR, and that it follows straightforwardly from a univariate representation that the equation systems are cyclic Vandermonde-type. In particular, in the univariate representation one has even more complete control over the dependencies of each coefficient and more freedom in comparison to the multivariate case. Here the keystream sequence is simply viewed as $a_i = P(u_i)$ where P is a univariate polynomial over $\mathbb{F}_{2^N}$. Then [556] introduced a parameter on sequences, called *spectral immunity*, an analogue to the algebraic immunity, but related to the approach of the Rønjom-Helleseth attack and to its improvements (in particular, the so-called selective DFT attack, which multiplies the portion of known keystream by a sequence of smaller linear complexity, and which possibly results in a more efficient attack than FAA, or is able to work when the number of known consecutive bits of the keystream is too small for FAA). The spectral immunity $SI(s)$ of a binary sequence s is the lowest linear complexity of a nonzero binary annihilator s (*i.e.* binary sequence a, satisfying $a\,s = 0$). In terms of univariate polynomials, the spectral immunity is equal to the minimal weight of a multiple of P or $P+1$ in $\mathbb{F}_{2^N}$, thus linking security directly to the minimum distance of the associated algebraic codes defined by the univariate filter polynomials. Moving to a univariate representation over finite fields seems to be a more natural representation for this type of generator. For instance, it has been an open question in [188] whether the irregular equation systems resulting from an annihilator attack on the filter generator have full rank. As observed in [1001], from the univariate representation, this directly translates to a question about the singularity of generalized Vandermonde matrices over finite fields, which has already been solved by Shparlinski [1038] (most of such matrices have full rank). It has been shown that univariate cryptanalysis becomes particularly effective in practice in comparison to multivariate attacks when the LFSR is defined over larger fields (*i.e.* word-based stream ciphers), see for instance [1001]. Although filter generators are usually building blocks in more complex designs, the technique has been used to practically break several ciphers, including a large part of the Welch-Gong family of generators and the recent Keccak/Farfalle-based pseudo-random function *Kravatte* [342]. It is an open problem how this change of representation can be used to also improve algebraic attacks on ciphers like SNOW-3G, which use word-based LFSRs as components in more complex designs.

3.1.6 Variants to these criteria in relationship with guess and determine attacks

The *guess and determine* attacks make hypotheses on the values of some bits or some linear combinations of bits in the data processed by the stream cipher. Given the complexity, say C, of the attack when the hypothesis is satisfied, the global complexity of the attack is obtained by dividing C by the probability that the hypothesis is satisfied. There is then a trade-off to be found between this probability and C. In such framework, the input to the Boolean function at one moment in the process belongs, in the simplest case, to an affine subspace of $\mathbb{F}_2^n$ (which may be a different one at each moment). For a given Boolean function f to be used as combiner or filter function, all the criteria introduced in the previous subsections need then to be also studied for the restrictions of f to such affine spaces. It is difficult to say in general which affine spaces exactly are concerned and, as in the case of attacks on the augmented function, such study is hardly viewed as a study of the single Boolean function, except in particular cases. It depends on the whole cryptosystem. We shall see in Section 12.2 another case where functions need to be studied on subsets of $\mathbb{F}_2^n$ (which are no more affine spaces but sets of vectors of fixed Hamming weights).

3.1.7 Avalanche criteria, nonexistence of nonzero linear structure, correlation with subsets of indices

Strict avalanche criterion, propagation criterion and global avalanche criteria

The *strict avalanche criterion* (*SAC*) has been introduced by Webster and Tavares [1116] and this concept was generalized into the *propagation criterion* (*PC*) by Preneel et al. [970] (see also [969]). The SAC, and its generalizations, are based on the properties of the derivatives of Boolean functions. These properties describe the behavior of a function whenever some coordinates of the input are complemented. Thus, they are related to the property of diffusion of the cryptosystems using the function. They concern more the Boolean functions involved in block ciphers.

Definition 24 *Let f be a Boolean function on $\mathbb{F}_2^n$ and E a subset of $\mathbb{F}_2^n$. Function f satisfies the* propagation criterion PC with respect to E *if, for all $a \in E$, the derivative $D_a f(x) = f(x) \oplus f(a+x)$ is balanced. It satisfies $PC(l)$ if it satisfies PC with respect to the set of all nonzero vectors of weight at most l. In other words, f satisfies $PC(l)$ if the autocorrelation coefficient $\mathcal{F}(D_a f)$ is null for every $a \in \mathbb{F}_2^n$ such that $1 \le w_H(a) \le l$. Criterion SAC corresponds to $PC(1)$.*

It is needed, for some cryptographic applications, to have Boolean functions which still satisfy $PC(l)$ when a certain number k of coordinates of the input x are kept constant (whatever are these coordinates and whatever are the constant values chosen for them). We say that such functions satisfy the *propagation criterion $PC(l)$ of order k*. This notion, introduced in [970], is a generalization of the strict avalanche criterion of order k, $SAC(k)$ (which is equivalent to $PC(1)$ of order k), introduced in [516]. Obviously, if a function f satisfies $PC(l)$ of order $k \le n-l$, then it satisfies $PC(l)$ of order k' for any $k' \le k$.
There exists another notion, which is similar to $PC(l)$ of order k, but stronger [970, 968] (see also [219]): a Boolean function satisfies the *extended propagation criterion $EPC(l)$* of order k if every derivative $D_a f$, with $a \neq 0_n$ of weight at most l, is k-resilient.
These parameters are not affine invariants.
A weakened version of the PC criterion has been studied in [721].

Global avalanche criteria: sum-of-squares and absolute indicators
The second moment of the autocorrelation coefficients:

$$\mathcal{V}(f) = \sum_{b \in \mathbb{F}_2^n} \mathcal{F}^2(D_b f) \tag{3.7}$$

has been introduced by Zhang and Zheng [1166] for measuring the *global avalanche criterion* (GAC), and is also called the *sum-of-squares indicator*. The *absolute indicator* $\Delta_f = \max_{b \in \mathbb{F}_2^n,\ b \neq 0_n} \mid \mathcal{F}(D_b f) \mid$ is the other global avalanche criterion. Functions with high absolute indicator are weak against cube attacks [465]. Both indicators are clearly *affine invariants*. In order to achieve good diffusion, cryptographic functions should have low sum-of-squares indicators and absolute indicators. Obviously, we have $\mathcal{V}(f) \ge 2^{2n}$, since $\mathcal{F}^2(D_0 f) = 2^{2n}$. Note that every lower bound of the form $\mathcal{V}(f) \ge V$ straightforwardly implies that the absolute indicator is bounded below by $\sqrt{\frac{V - 2^{2n}}{2^n - 1}}$. The functions achieving $\mathcal{V}(f) = 2^{2n}$ are those functions whose derivatives $D_b f(x)$, $b \neq 0_n$, are all balanced. We shall see in Chapter 6 that these are the bent functions, which are unbalanced. In [1180] and references therein are studied the balanced functions with minimal sum-of-square indicator $2^{2n} + 2^{n+3}$.

If f has a k-dimensional linear kernel $\{e \in \mathbb{F}_2^n;\ D_e f = cst\}$ (see the next subsection), then

$$\mathcal{V}(f) \geq 2^{2n+k} \tag{3.8}$$

(with equality if and only if f is partially-bent, see page 283).
Note that, according to Relation (2.55), page 81, applied to $D_b f$ for every b, we have

$$\mathcal{V}(f) = \sum_{a,b\in\mathbb{F}_2^n} \mathcal{F}(D_a D_b f), \tag{3.9}$$

where $D_a D_b f(x) = f(x) \oplus f(x+a) \oplus f(x+b) \oplus f(x+a+b)$ is the second order *derivative* of f.
Note also that, according to Relation (2.45), page 79 (expressing the convolutional product of Fourier-Hadamard transforms), applied to $\varphi(b) = \psi(b) = \mathcal{F}(D_b f)$, and using that, according to Relation (2.53), the Fourier-Hadamard transform of φ equals W_f^2, we have for any n-variable Boolean function f:

$$\forall a \in \mathbb{F}_2^n,\ \sum_{b\in\mathbb{F}_2^n} W_f^2(b) W_f^2(a+b) = 2^n \sum_{b\in\mathbb{F}_2^n} \mathcal{F}^2(D_b f)(-1)^{b\cdot a}\ ,$$

and thus, for $a = 0_n$:

$$\sum_{b\in\mathbb{F}_2^n} W_f^4(b) = 2^n\, \mathcal{V}(f), \tag{3.10}$$

as observed in [191].
We have: $\sum_{b\in\mathbb{F}_2^n} W_f^4(b) \leq \left(\sum_{b\in\mathbb{F}_2^n} W_f^2(b)\right)\left(\max_{b\in\mathbb{F}_2^n} W_f^2(b)\right) = 2^{2n} \max_{b\in\mathbb{F}_2^n} W_f^2(b)$ (according to Parseval's Relation (2.47), page 79), and we deduce, using Relation (3.10) and inequality $\mathcal{V}(f) \geq 2^{2n}$: $\max_{b\in\mathbb{F}_2^n} W_f^2(b) \geq \frac{\mathcal{V}(f)}{2^n} \geq \sqrt{\mathcal{V}(f)}$; thus, according to Relation (3.1), page 99, relating the nonlinearity to the Walsh transform, we have (as first shown in [1169, 1173]):

Proposition 27 *For every n-variable Boolean function f, we have:*

$$nl(f) \leq 2^{n-1} - 2^{-\frac{n}{2}-1}\sqrt{\mathcal{V}(f)} \leq 2^{n-1} - \frac{1}{2}\sqrt[4]{\mathcal{V}(f)},$$

with equality on the left-hand side if and only if f is plateaued (see Definition 63, page 285), in which case $\mathcal{V}(f) = 2^n\lambda^2$ where λ is the amplitude.

Denoting by N_{W_f} the cardinality of the support $\{a \in \mathbb{F}_2^n;\ W_f(a) \neq 0\}$ of the Walsh transform of f, Relation (3.10) also implies the following relation, first observed in [1173]: $\mathcal{V}(f) \times N_{W_f} \geq 2^{3n}$. Indeed, using for instance the Cauchy-Schwarz inequality, we see that $\left(\sum_{a\in\mathbb{F}_2^n} W_f^2(a)\right)^2 \leq \left(\sum_{a\in\mathbb{F}_2^n} W_f^4(a)\right) \times N_{W_f}$ and we have $\sum_{a\in\mathbb{F}_2^n} W_f^2(a) = 2^{2n}$, according to Parseval's Relation.
According to the observations made above and below Proposition 27, the functions which satisfy $nl(f) = 2^{n-1} - 2^{-\frac{n}{2}-1}\sqrt{\mathcal{V}(f)}$ (resp. $\mathcal{V}(f) \times N_{W_f} = 2^{3n}$) are the functions whose Walsh transforms take one nonzero absolute value (*i.e.* are

plateaued), and the functions satisfying $nl(f) = 2^{n-1} - \frac{1}{2}\sqrt[4]{\mathcal{V}(f)}$ are the bent functions.
Constructions of balanced Boolean functions with low absolute indicators and high nonlinearities have been studied in [813, 1073].

Remark. Zhang and Zheng conjectured that the absolute indicator of any balanced Boolean function of algebraic degree at least 3 is lower-bounded by $2^{\lfloor \frac{n+1}{2} \rfloor}$, but counter-examples were found by many people (Maitra-Sarkar, Burnett et al., Gangopadhyay-Keskar-Maitra, Kavut). $\square$

Remark. A related but different parameter is $\max_{a\in\mathbb{F}_2^n, a\neq 0_n} \Delta_f(a)$ (recall that $\Delta_f(a) = \sum_{x\in\mathbb{F}_2^n}(-1)^{D_af(x)}$ is the autocorrelation function), without absolute value. It has appeared recently in the framework of side channel attacks (see Section 12.1).$\square$

Nonexistence of nonzero linear structure

The set of linear structures of a Boolean functions plays a role in its study, particularly when the function is a *quadratic function* (see Section 5.2).

Definition 25 *The* linear kernel *of an n-variable Boolean function f is the set of those vectors e such that D_ef is a constant function. It is denoted by $\mathcal{E}_f$. Any element of the linear kernel is called a* linear structure[17] *of f.*

More generally a linear structure e of a vectorial function F is such that D_eF equals a constant function. Since, for every n-variable Boolean function f (more generally, any vectorial function) and any $a, b \in \mathbb{F}_2^n$, we have $D_af(x) \oplus D_bf(x) = f(x+a) \oplus f(x+b) = D_{a+b}f(x+a)$, the linear kernel of any Boolean function is an $\mathbb{F}_2$-subspace of $\mathbb{F}_2^n$. Moreover, the restriction f' of f to its linear kernel or to any of its cosets is affine since its derivatives D_af', $a \in \mathcal{E}_f$, are all constant. More generally, for every $r \leq n$, the set of those $e \in \mathbb{F}_2^n$ such that D_ef has algebraic degree at most r is a vector space, and the restriction of f to this vector space has *algebraic degree* at most $r+1$.

Nonlinear cryptographic functions used in block ciphers should have no nonzero linear structure (see [496]). The existence of nonzero linear structures, for the functions implemented in stream ciphers, is a potential risk and is better avoided.

Proposition 28 *Any n-variable Boolean function $f(x_1, \ldots, x_n)$ has a nonzero linear structure if and only if it is* linearly equivalent *to a function of the form*

$$g(x_1, \ldots, x_{n-1}) \oplus \epsilon\, x_n, \tag{3.11}$$

where $\epsilon \in \mathbb{F}_2$. More generally, the linear kernel of f has dimension at least k if and only if f is linearly equivalent to a function of the form

$$g(x_1, \ldots, x_{n-k}) \oplus \epsilon_{n-k+1}\, x_{n-k+1} \oplus \cdots \oplus \epsilon_n\, x_n, \tag{3.12}$$

[17] We also call linear structure a pair $(a, b) \in \mathbb{F}_2^n \times \mathbb{F}_2$ such that D_af equals constant function b.

where $\epsilon_{n-k+1}, \ldots, \epsilon_n \in \mathbb{F}_2$.

Proof. The conditions are clearly sufficient. Conversely, let f have a nonzero linear structure e, then by composing f on the right by a linear automorphism L on $\mathbb{F}_2^n$ such that $L(0,\ldots,0,1) = e$, we have $D_{(0,\ldots,0,1)}(f \circ L)(x) = f \circ L(x) \oplus f \circ L(x + (0,\ldots,0,1)) = f \circ L(x) \oplus f(L(x) + e) = D_e f(L(x))$. And it is easily seen that $D_{(0,\ldots,0,1)}(f \circ L)$ being then constant, $f \circ L$ has the form $g(x_1,\ldots,x_{n-1}) \oplus \epsilon\, x_n$. The case of dimension k is similar. □

Note that, according to Proposition 28, if f admits a nonzero linear structure, then since nonlinearity is an EA invariant, $nl(f)$ equals the nonlinearity of g given by (3.11) and viewed as an n-variable function, which equals $2nl(g)$ where g is now viewed as an $(n-1)$-variable. Hence, according to the covering radius bound (3.2), page 99, applied to this $(n-1)$-variable function, $nl(f)$ is bounded above by the bent concatenation bound $2^{n-1} - 2^{\frac{n-1}{2}}$. This implies that the functions achieving strictly larger nonlinearities (obtained by Patterson and Wiedemann and by Kavut et al., see Section 3.1.3)) cannot have any nonzero linear structure.

Similarly, if k is the dimension of the linear kernel of f, we have that $nl(f) \leq 2^{n-1} - 2^{\frac{n+k-2}{2}}$ as seen in [190], since $nl(f) = 2^k nl(g)$, where g is the $(n-k)$-variable function given in (3.12) and according to the covering radius bound applied on g with $n-k$ in the place of n.

Another characterization of linear structures is by the Walsh transform [736, 486] (see also [193]). In the next proposition, we separate the case where the linear structure e is such that $D_e f$ is the null function and the case where it is function 1.

Proposition 29 *Let f be any n-variable Boolean function. The derivative $D_e f$ equals the null function (resp. function 1) if and only if the support $supp(W_f) = \{u \in \mathbb{F}_2^n;\, W_f(u) \neq 0\}$ of W_f is included in $\{0_n, e\}^\perp$ (resp. in its complement).*

Proof. Relation (2.56), page 81, with $b = 0_n$ and $E = \{0_n, e\}^\perp$, gives the equality

$$\sum_{u \in a+E} W_f^2(u) = 2^{n-1}(2^n + (-1)^{a\cdot e}\mathcal{F}(D_e f)). \tag{3.13}$$

If $D_e f$ is null, then let us fix a such that $a \cdot e = 1$ and if $D_e f = 1$, then let us fix it such that $a \cdot e = 0$. Then $W_f(u)$ is null for every $u \in a + E$, according to Relation (3.13). This proves the implication from top to bottom. The converse is straightforward. □

Notice that, if $D_e f$ is the constant function 1 for some $e \in \mathbb{F}_2^n$, then f is balanced (indeed, the relation $f(x+e) = f(x) \oplus 1$ implies that f takes the values 0 and 1 equally often). Thus, a non-balanced function f has no nonzero linear structure if and only if there is no nonzero vector e such that $D_e f$ is null. According to Proposition 29, we deduce:

Corollary 7 *Any non-balanced function f has no nonzero linear structure if and only if the support of its Walsh transform has rank n.*

A similar characterization exists for balanced functions by replacing the function $f(x)$ by a non-balanced function $f(x) \oplus b \cdot x$. It is deduced in [354] (see more in [1082]) that resilient functions of high orders must have linear structures.

Distance to linear structures
The dimension of the linear kernel is an *affine invariant*. Hence, so is the criterion of nonexistence of nonzero linear structure. But, contrary to the criteria viewed before it, it is an all-or-nothing criterion. Meier and Staffelbach introduced in [844] a related criterion, leading to a characteristic (that is, a criterion which can be satisfied at levels quantified by numbers): a Boolean function on $\mathbb{F}_2^n$ being given, its *distance to linear structures* is its distance to the set of all Boolean functions admitting nonzero linear structures, among which we have all affine functions (hence, this distance is bounded above by the nonlinearity) but also other functions, such as all non bent quadratic functions.

Proposition 30 *[844] The distance to linear structures of any n-variable Boolean function f equals $2^{n-2} - \frac{1}{4} \max_{e \in \mathbb{F}_2^n \setminus \{0_n\}} |\mathcal{F}(D_e f)|,$.*

Proof. Given e in $\mathbb{F}_2^n \setminus \{0_n\}$ and ϵ in $\mathbb{F}_2$, let $\mathcal{L}_{e,\epsilon}$ be the set of those n-variable Boolean functions g such that $D_e g = \epsilon$. Then a function g in $\mathcal{L}_{e,\epsilon}$ lies at minimum Hamming distance from f, among all elements of $\mathcal{L}_{e,\epsilon}$, if and only if, for every $x \in \mathbb{F}_2^n$ such that $D_e f(x) = \epsilon$, we have $g(x) = f(x)$ (and $g(x+e) = f(x+e)$), and for every $x \in \mathbb{F}_2^n$ such that $D_e f(x) = \epsilon \oplus 1$, we have $g(x) = f(x)$ or $g(x+e) = f(x+e)$ (and only one such equality can then happen). The Hamming distance between f and g equals then $\frac{1}{2}|\{x \in \mathbb{F}_2^n;\, D_e f(x) = \epsilon \oplus 1\}| = \frac{1}{2}\left(2^{n-1} - \frac{(-1)^\epsilon}{2}\mathcal{F}(D_e f)\right)$. This completes the proof since the set of functions admitting nonzero linear structures equals $\bigcup_{e \in \mathbb{F}_2^n \setminus \{0_n\}, \epsilon \in \mathbb{F}_2} \mathcal{L}_{e,\epsilon}$. □

Note that Proposition 30 proves again Relation (3.3), page 102, and also proves, according to Theorem 12, page 216, that the distance of f to linear structures equals 2^{n-2} if and only if f is bent.

The maximum correlation with respect to a subset I of indices

This parameter has been introduced in [1155].

Definition 26 *Let f be any n-variable Boolean function and $I \subseteq \{1, \ldots, n\}$. The* maximum correlation with respect to I *equals* $C_f(I) = \max_{g \in \mathcal{BF}_{I,n}} \dfrac{\mathcal{F}(f \oplus g)}{2^n} = \max_{g \in \mathcal{BF}_{I,n}} \dfrac{|\mathcal{F}(f \oplus g)|}{2^n}$*, where $\mathcal{BF}_{I,n}$ is the set of n-variable Boolean functions depending on $\{x_i, i \in I\}$ only.*

According to Relation (2.35), page 75, the Hamming distance from f to $\mathcal{BF}_{I,n}$ is equal to $2^{n-1}(1 - C_f(I))$. As we saw already, denoting the size of I by r, this distance is bounded below by the r-th order nonlinearity of f (*i.e.* the minimum Hamming distance to functions of *algebraic degree* at most r). It can be much larger.

The maximum correlation of any combining function with respect to any subset I of small size should be small (*i.e.* its distance to $\mathcal{BF}_{I,n}$ should be large). It is straightforward to prove, by decomposing the sum $\mathcal{F}(f \oplus g)$ and using that an unrestricted Boolean function over $\mathbb{F}_2^I$ can take any binary value at any input $x \in \mathbb{F}_2^I$, that $C_f(I)$ equals $\sum_{j=1}^{2^{|I|}} \frac{|\mathcal{F}(h_j)|}{2^n}$, where $h_1, \ldots, h_{2^{|I|}}$ are the restrictions of f obtained by keeping constant the x_i's for $i \in I$, and to see that the distance from f to $\mathcal{BF}_{I,n}$ is achieved by the functions g taking value 0 (resp. 1) when the corresponding value of $\mathcal{F}(h_j)$ is positive (resp. negative), and that we have $C_f(I) = 0$ if and only if all h_j's are balanced (thus, f is m-resilient if and only if $C_f(I) = 0$ for every set I of size at most m).

The Cauchy-Schwarz inequality gives $\left(\sum_{j=1}^{2^{|I|}} |\mathcal{F}(h_j)|\right)^2 \leq 2^{|I|} \sum_{j=1}^{2^{|I|}} \mathcal{F}^2(h_j)$, and the second-order Poisson formula (2.57), page 81, directly implies then the following inequality observed in [187]:

$$C_f(I) \leq 2^{-n} \left(\sum_{\substack{u \in \mathbb{F}_2^n\,;\\ supp(u) \subseteq I}} W_f^2(u) \right)^{\frac{1}{2}} \leq 2^{-n+\frac{|I|}{2}} \max_{u \in \mathbb{F}_2^n} |W_f(u)|$$
$$= 2^{-n+\frac{|I|}{2}} \left(2^n - 2\,nl(f)\right) \qquad (3.14)$$

or equivalently:

$$d_H\left(f, \mathcal{BF}_{I,n}\right) \geq 2^{n-1} - \frac{1}{2} \left(\sum_{\substack{u \in \mathbb{F}_2^n\,;\\ supp(u) \subseteq I}} W_f^2(u) \right)^{\frac{1}{2}} \geq 2^{n-1} - 2^{\frac{|I|}{2}-1} \max_{u \in \mathbb{F}_2^n} |W_f(u)|$$
$$= 2^{n-1} - 2^{n+\frac{|I|}{2}-1} + 2^{\frac{|I|}{2}} nl(f).$$

This latter inequality shows that, contrary to the case of approximation by functions of algebraic degree at most r, for avoiding close approximations of f by functions of $\mathcal{BF}_{I,n}$ when I has small size, it is sufficient that the first-order nonlinearity of f be large.

Parameter $\max_{I \subseteq \{1,\ldots,n\}, |I| \leq k} C_f(I)$ is *permutation invariant*. A related (but different) *affine invariant* parameter, also related to the distance to linear structures, is the minimum Hamming distance to those Boolean functions g whose linear kernel $\{e \in \mathbb{F}_2^n;\, D_e g = 0\}$ has dimension at least $n - k$. Indeed, the linear kernels of functions in $\mathcal{BF}_{I,n}$ contain $\mathbb{F}_2^{\{1,\ldots,n\}\setminus I}$. The results on the maximum correlation above generalize to this criterion [187].

Results in the domain of Boolean functions for circuit design and learning express that, if the total influence $2^{-n} \sum_{i=1}^n w_H(D_{e_i} f)$ of an n-variable Boolean function f is low, then the sum $\sum_{u \in \mathbb{F}_2^n; w_H(u) \geq k} W_f^2(u)$ is small for large k (and the function is "essentially determined by few coordinates"), see [519, 914]. This is related to Relation (2.67), page 87.

3.1.8 Complexity parameters

Among the criteria viewed above, the main cryptographic *complexity parameters* (related to Shannon's notion of confusion) are the algebraic degree, the nonlinearity and higher order nonlinearity, the algebraic immunity and the fast algebraic immunity. Other complexity parameters exist. Note that, as pointed out by Meier and Staffelbach in [843], they are supposed to be affine invariants, because the composition by affine automorphisms should not modify the complexity. And indeed, the attacks on cryptosystems using Boolean functions (stream or block ciphers) often work with similar complexities when using two *affinely equivalent* functions (maybe not exactly the same complexity, because diffusion plays also a role and may be different with both functions).

Algebraic thickness

This parameter has been evoked in [844] and later studied in [222, 224, 229].

Definition 27 *Let f be any n-variable Boolean function. The minimum number of terms in the* algebraic normal form*s of all functions affinely equivalent to f, is called the* algebraic thickness *of f. We shall denote it by $AT(f)$.*

As far as we know, this parameter is not directly related to an attack. Note however that if a function has very low algebraic thickness, then it has low algebraic immunity, since, for every set $\mathcal{I}$ of non-empty multi-indices of $\{1,\dots,n\}$, an annihilator of the Boolean function of ANF $\bigoplus_{I\in\mathcal{I}} x^I$ equals $\prod_{I\in\mathcal{I}}(x_{i_I}\oplus 1)$, where, for every $I\in\mathcal{I}$, i_I is an index chosen in I (any one). We deduce that $AI(f)\leq AT(f)$ for every Boolean function.

In the case of affine functions, and more generally of the *indicator*s of flats (in particular, of function $\delta_0(x)=\prod_{i=1}^n(x_i\oplus 1)=\bigoplus_{I\subseteq\{1,\dots,n\}} x^I$ which has all monomials in its ANF), $AT(f)$ equals 1.

In the case of *quadratic functions*, thanks to the existence of the Dickson form of these functions that we shall see in Theorem 10, page 195, $AT(f)$ equals at most $\lceil\frac{n+1}{2}\rceil$, which is also rather small.

Boolean functions of algebraic degree not close to $n-1$ have also moderate algebraic thickness, since $AT(f)\leq\sum_{i=0}^{d_{alg}(f)}\binom{n}{i}$.

But it has been shown that, asymptotically, almost all Boolean functions f (in the sense of probability theory) have large algebraic thickness. This property is related to the fact that the number 2^{2^n} of n-variable Boolean functions is strongly increasing when n grows, which allows proving in some cases the existence of functions possessing some complexity features, without being always able to exhibit any such function. This is possible by bounding above the number of functions which do not possess these features and showing that it is negligible when compared to 2^{2^n}. This phenomenon on Boolean functions, which is also

valid with codes, is the so-called *Shannon effect* (this term has been introduced in [807]): Shannon in [1035] could prove this way the existence of Boolean functions with high circuit complexity.
Concerning algebraic thickness, it has been first proved in [222] that for every number $\lambda < 1/2$, the density in $\mathcal{BF}_n$ of the subset $\{f \in \mathcal{BF}_n \mid AT(f) \geq \lambda\, 2^n\}$ is larger than $1 - e^{-2^{n+1}(1/2-\lambda)^2+(n^2+n)\log_2(e)}$ and tends to 1 when n tends to infinity. A more precise bound has been proved shortly later:

Proposition 31 *[224] Let c be any strictly positive real number. The density in $\mathcal{BF}_n$ of the subset $\{f \in \mathcal{BF}_n \mid AT(f) \geq 2^{n-1} - c\,n\; 2^{\frac{n-1}{2}}\}$ is larger than $1-2^{n^2+n}\, e^{-c^2n^2}$ and, if $c^2\log_2 e > 1$, then this density tends to 1 when n tends to infinity. For every $n \geq 3$, a Boolean function f such that $AT(f) \geq 2^{n-1} - n\, 2^{\frac{n-1}{2}}$ exists.*

Proof. Let k be any positive integer. The number of n-variable Boolean functions whose ANF have at most k monomials equals $1 + \binom{2^n}{1} + \cdots + \binom{2^n}{k}$. The number of affine automorphisms on $\mathbb{F}_2^n$ equals $(2^n-1)(2^n-2)(2^n-4)\cdots(2^n-2^{n-1})\, 2^n < 2^{n^2+n}$. Thus, the number of Boolean functions f such that $AT(f) \leq k$ is smaller than $N(n,k) = \left(1 + \binom{2^n}{1} + \cdots + \binom{2^n}{k}\right) 2^{n^2+n}$. We have seen already at page 103, that, for every N, we have $\sum_{0\leq i\leq \lambda N} \binom{N}{i} < 2^N e^{-2N(1/2-\lambda)^2}$. Hence, applying this with $N = 2^n$ and $\lambda = 1/2 - c\,n\; 2^{-(n+1)/2}$, we deduce that the density of the set $\{f \in \mathcal{BF}_n \mid AT(f) \geq 2^{n-1} - c\,n\; 2^{(n-1)/2}\}$ is larger than $1 - \frac{N(n,2^{n-1}-c\,n\; 2^{(n-1)/2})}{2^{2^n}} > 1 - 2^{n^2+n}\, e^{-c^2n^2} = 1 - 2^{n^2+n-c^2n^2\log_2 e}$, and tends to 1, if $c^2\log_2 e > 1$. The last sentence is easy to check. □

Proposition 31 implies that, for every $\lambda < 1/2$, there exists m such that, for every $n \geq m$, a Boolean function f such that $AT(f) \geq \lambda\, 2^n$ exists. But, unless λ is small, m is greater than 3. We can take $m = 9$ for $\lambda = \frac{1}{4}$ and $m = 12$ for $\lambda = \frac{3}{8}$.

Hence, almost all n-variable Boolean functions have algebraic thickness larger than half the whole number 2^n of monomials (see more in [956]). It may seem surprising that taking the minimum number of terms in the ANFs of all functions affinely equivalent to f does not affect significantly the number of terms in the ANF of a random function. This is due to the small number of affine automorphisms compared to the number of Boolean functions.
The lower bound of Proposition 31 is accompanied by an upper bound:

Proposition 32 *[222] For every $f \in \mathcal{BF}_n$, we have $AT(f) \leq \frac{2}{3}\, 2^n$.*

Proof. The proof is by induction on n. The assertion is clearly valid for $n = 1$. Let n be any integer larger than 1 and assume that the assertion is valid for $n-1$. Let f be any Boolean function in $\mathcal{BF}_n$ and let f_0 and f_1 be the Boolean functions on $\mathbb{F}_2^{n-1}$ such that $f(x_1,\cdots,x_n) = f_0(x_1,\cdots,x_{n-1}) \oplus x_n f_1(x_1,\cdots,x_{n-1})$. We shall denote by $|f|$ the number of terms in the ANF of f. We have $|f| = |f_0| + |f_1|$. By hypothesis, there exists an affine automorphism A of $\mathbb{F}_2^{n-1}$ such that $|f_1 \circ A| \leq$

$\frac{2}{3}\,2^{n-1}$. Thus, we can assume without loss of generality that $|f_1| \leq \frac{2}{3}\,2^{n-1}$. Assume that $|f| = |f_0| + |f_1|$ is larger than $\frac{2}{3}\,2^n$. Let r be the number of terms which are in both ANFs of f_0 and f_1. We have $|f_0| + |f_1| - r \leq 2^{n-1}$, since 2^{n-1} is the total number of monomials in $n-1$ variables. Thus r is larger than or equal to $\frac{2}{3}\,2^n - 2^{n-1} = \frac{1}{3}\,2^{n-1}$. Changing x_n into $x_n \oplus 1$ in the ANF of f keeps f_1 unchanged and replaces f_0 by $f_0 \oplus f_1$. We have $|f_0 \oplus f_1| + |f_1| = (|f_0| + |f_1| - r) - r + |f_1| \leq 2^{n-1} - \frac{1}{3}\,2^{n-1} + \frac{2}{3}\,2^{n-1} = \frac{2}{3}\,2^n$. □

Given Propositions 31 and 32, we can consider that a function f has large thickness if $AT(f)$ equals $\lambda\,2^n$ where λ is near $1/2$. Note that the algebraic degrees of such functions cannot be substantially smaller than $\frac{n}{2}$, since we have seen already that $AT(f) \leq \sum_{i=0}^{d_{alg}(f)} \binom{n}{i}$. There exist functions with low algebraic thicknesses and with highest possible nonlinearity (*e.g.* quadratic bent functions). There also exist functions with high algebraic thicknesses and low nonlinearities, since there exist functions with high algebraic thicknesses and low Hamming weights: take $\lambda < \lambda' < 1/2$; the number of functions of Hamming weights smaller than or equal to $2^n\lambda'$ equals $\sum_{i=0}^{2^n\lambda'} \binom{2^n}{i} \geq \frac{2^{2^n H_2(\lambda')}}{\sqrt{2^{n+3}\lambda'(1-\lambda')}}$ (cf. [809, page 310]), where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the entropy function. We have seen above that the number of functions f such that $AT(f) \leq 2^n\lambda$ is smaller than or equal to:

$$\left(1 + \binom{2^n}{1} + \cdots + \binom{2^n}{k}\right) 2^{\,n^2+n} \leq 2^{2^n\,H_2(\lambda)+n^2+n}\,;$$

thus, the latter is asymptotically smaller than the former and there exist functions of weights smaller than or equal to $2^n\lambda'$ satisfying $AT(f) > 2^n\lambda$.

There also exist functions with algebraic degree at least $n-1$, nonlinearity at least $2^{n-1} - 2^{\frac{n-1}{2}}\sqrt{n}$ and algebraic thickness at least $\lambda\,2^n$, with $\lambda < 1/2$ as close to $1/2$ as we wish, since the probabilities that f has algebraic degree at most $n-2$, resp. nonlinearity at most $2^{n-1} - 2^{\frac{n}{2}-1}\sqrt{n}\left(\sqrt{2\,\ln 2} + \frac{4\,\ln n}{n}\right)$, resp. algebraic thickness at most $\lambda\,2^n$ tend all three to 0 (see Section 3.1).

Non-normality

Hans Dobbertin has introduced in [466] the following notion: for any n even, an n-variable Boolean function is a *normal function* (resp. a weakly normal function) if it is constant (resp. affine) on at least one $\frac{n}{2}$-dimensional flat. He used this notion for constructing balanced functions with high nonlinearities (see more at page 325). The notion has been generalized and extended (see *e.g.* [222, 224]):

Definition 28 *Let n and $k \leq n$ be positive integers. An n-variable Boolean function f is called a* k-normal function *(resp. a* k-weakly-normal function*) if there exists a k-dimensional flat on which f is constant (resp. affine). For n even, $\frac{n}{2}$-normal functions are simply called normal.*

The notion of normality has been later related to an attack on stream ciphers

[881]. The related parameter is studied in [276] as well as two other parameters which complete the information it gives. The notion of k-nonnormal function is a particular case of that of *affine disperser* of dimension k and is also related to that of affine extractor, a stronger notion needed for the extraction of randomness from few independent sources; see more precise definitions and constructions in [110, 1036]. It is also related to a similar notion coming from computational number theory: that of k-wise independent random variables, see [12].

The complexity criterion we are interested in is k-nonnormality with small k. Even if almost all Boolean functions satisfy it as we shall see, it is not satisfied by simple ones:

- Every quadratic Boolean function f on $\mathbb{F}_2^n$ is $\frac{n}{2}$-normal if n is even and $\frac{n+1}{2}$-weakly-normal if n is odd, according to the properties of quadratic functions that we shall see in Section 5.2.
- Every symmetric Boolean function (*i.e.* every function whose output is invariant under permutation of its input bits, and depends then only on the Hamming weight of the input, see Section 10.1) is $\lfloor \frac{n}{2} \rfloor$-normal and $\lceil \frac{n}{2} \rceil$-weakly-normal since its restriction to the $\lceil \frac{n}{2} \rceil$-dimensional flat:

$$\left\{(x_1, \ldots x_n) \in \mathbb{F}_2^n \mid x_{i+\lfloor \frac{n}{2} \rfloor} = x_i \oplus 1, \forall i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}$$

is constant if n is even and affine if n is odd. Indeed, if n is even, all the elements of this flat have same Hamming weight $\frac{n}{2}$ and $f(x)$ takes therefore constant value; if n is odd, we have $f(x) = f(x_1, \cdots, x_{n-1}, 0) \oplus x_n[f(x_1, \cdots, x_{n-1}, 0) \oplus f(x_1, \cdots, x_{n-1}, 1)]$ where the functions $f(x_1, \cdots, x_{n-1}, 0)$ and $f(x_1, \cdots, x_{n-1}, 1)$ are constant on this flat.
- Every Boolean function on $\mathbb{F}_2^n$ with $n \leq 7$ is $\lfloor \frac{n}{2} \rfloor$-normal, as can be checked by computer investigation.

There is a mutual upper bound on k and on the nonlinearity of the function:

Proposition 33 *Let f be a k-weakly-normal Boolean function on $\mathbb{F}_2^n$. Then*

$$nl(f) \leq 2^{n-1} - 2^{k-1},$$

or equivalently $k \leq \log_2[2^{n-1} - nl(f)] + 1$.

Proof. Applying the *Poisson summation formula* (2.39), page 77, to the sign function f_χ, we see that if $f \oplus a \cdot x$ is constant on the flat $b \oplus E^\perp$, then the mean of $(-1)^{b \cdot u} W_f(u)$ when u ranges over $a \oplus E$ equals $\pm |E^\perp|$. And the maximum absolute value of a sequence of numbers is larger than or equal to the absolute value of its arithmetic mean. □

Hence, k-normality with large k implies low nonlinearity. Notice that, since every Boolean function has nonlinearity bounded above by $2^{n-1} - 2^{\frac{n}{2}-1}$, Proposition 33 gives no information if $k \leq \frac{n}{2}$. But the high nonlinearity $2^{n-1} - 2^{\frac{n}{2}-1}$ of bent functions implies that they cannot be $(\frac{n}{2} + 1)$-weakly-normal.

Remark. A more general result due to Zhang, Zheng and Imai, proved in a complex way in [1179], can be proved similarly: let A be any k-dimensional flat ($k \leq n$). Let f be a Boolean function on $\mathbb{F}_2^n$ and f' its restriction to A. Denote by $nl(f')$ the nonlinearity of f' (*i.e.* the minimum Hamming distance between f' and any affine function on A). Then we have[18]:

$$nl(f) - nl(f') \leq 2^{n-1} - 2^{k-1}.$$

Indeed, according to the Poisson summation formula applied to f_χ with $A = b \oplus E^\perp$, we have: $\max_{u\in\mathbb{F}_2^k} |W_{f'}(u)| \leq \max_{v\in\mathbb{F}_2^n} |W_f(v)|$ which completes the proof.
In fact, a little more can be said, as seen in [191]. Recall that, given two subspaces E of dimension k and E' of $\mathbb{F}_2^n$ such that $E \cap E' = \{0_n\}$ and whose direct sum equals $\mathbb{F}_2^n$, and denoting for every $a \in E'$ by h_a the restriction of f to the coset $a+E$, the second-order Poisson formula (2.57) in Proposition 12 (page 81) implies

$$\max_{u\in\mathbb{F}_2^n} W_f^2(u) \geq \sum_{a\in E'} \mathcal{F}^2(h_a)$$

(indeed, the maximum of $W_f^2(u)$ is larger than or equal to its mean). Hence we have: $\max_{u\in\mathbb{F}_2^n} W_f^2(u) \geq \mathcal{F}^2(h_a)$ for every a. Applying this property to $f \oplus \ell$, where ℓ is any linear function, and using Relation (3.1), page 99, between the nonlinearity and the maximum absolute value of the Walsh transform, we deduce:

$$\forall a \in E',\ nl(f) \leq 2^{n-1} - 2^{k-1} + nl(h_a). \tag{3.15}$$

The approaches by the first and the second Poisson formulae lead to two different necessary conditions for the case of equality in (3.15), see [224], where the case of equality is studied. The proof above shows that, if equality occurs in the inequality $nl(f) \leq 2^{n-1} - 2^{k-1}$ for a given function f which coincides with an affine function ℓ on a k-dimensional flat, then $f \oplus \ell$ is balanced on every other coset of this flat. □

As a consequence of Proposition 33, the maximum possible nonlinearity of *quadratic functions* (*i.e.* the covering radius of the Reed-Muller code $RM(1,n)$ in the Reed-Muller code $RM(2,n)$) is bounded above by $2^{n-1} - 2^{\frac{n}{2}-1}$ if n is even, which tells nothing, and by $2^{n-1} - 2^{\frac{n-1}{2}}$ if n is odd (these values are in fact the exact ones).

For every $\alpha > 1$, when n tends to infinity, random Boolean functions are almost surely $[\alpha \log_2 n]$-non-normal:

Proposition 34 *[222] Let c be larger than 1. Let $(k_n)_{n\in\mathbb{N}^*}$ be a sequence of positive integers such that $c\log_2 n \leq k_n \leq n$. The density in $\mathcal{BF}_n$ of the set of all Boolean functions on $\mathbb{F}_2^n$ which are not k_n-weakly-normal is larger than $1 - 2^{n(k_n+1)-2^{k_n}}$. This density tends to 1 when n tends to infinity. Therefore,*

[18] Note that in Proposition 33, we have $nl(f') = 0$.

there exists a positive integer N such that, for every $n \geq N$, k_n-nonnormal functions exist. For $k_n = \lfloor \frac{n}{2} \rfloor$ we can take $N = 12$.

Proof. Let λ_n be the number of k_n-dimensional flats in $\mathbb{F}_2^n$. Fix such a flat A. Let μ_n be the number of Boolean functions whose restrictions to A are affine (clearly, this number does not depend on the choice of A). The number of k_n-weakly-normal functions on $\mathbb{F}_2^n$ is smaller than or equal to $\lambda_n \, \mu_n$.
The number of k_n-dimensional vector subspaces of $\mathbb{F}_2^n$ equals (cf. *e.g.* [809]):

$$\begin{bmatrix} n \\ k_n \end{bmatrix} = \frac{(2^n - 1)(2^n - 2)(2^n - 2^2) \cdots (2^n - 2^{k_n - 1})}{(2^{k_n} - 1)(2^{k_n} - 2)(2^{k_n} - 2^2) \cdots (2^{k_n} - 2^{k_n - 1})}$$

and the number of k_n-dimensional flats in $\mathbb{F}_2^n$ is: $\lambda_n = 2^{n-k_n} \begin{bmatrix} n \\ k_n \end{bmatrix}$.
We choose now as particular k_n-dimensional flat the set $\mathbb{F}_2^{k_n} \times \{0_{k_n}\}$. The restriction to $\mathbb{F}_2^{k_n} \times \{0_{k_n}\}$ of a Boolean function on $\mathbb{F}_2^n$ is affine if and only if the algebraic normal form of the function contains no monomial of degree at least 2 involving the coordinates $x_1, \cdots, x_{k_n}$ only. The number of such functions is $\mu_n = 2^{\nu_n}$, where $\nu_n = 2^n - 2^{k_n} + 1 + k_n$. The number of k_n-weakly-normal functions on $\mathbb{F}_2^n$ is then smaller than or equal to $2^{n-k_n} \begin{bmatrix} n \\ k_n \end{bmatrix} 2^{\nu_n}$. The number of Boolean functions on $\mathbb{F}_2^n$ being equal to 2^{2^n}, the density of the subset $\mathcal{A}_n$ in $\mathcal{BF}_n$ of all Boolean functions on $\mathbb{F}_2^n$ which are not k_n-weakly-normal is larger than or equal to: $1 - 2^{n-k_n} \begin{bmatrix} n \\ k_n \end{bmatrix} 2^{\nu_n - 2^n}$.
We have $\begin{bmatrix} n \\ k_n \end{bmatrix} < 2^{nk_n - k_n^2 + k_n}$, since every factor in the numerator of $\begin{bmatrix} n \\ k_n \end{bmatrix}$ is smaller than 2^n and every factor in its denominator is larger than or equal to $2^{k_n - 1}$. Thus, the density of $\mathcal{A}_n$ is larger than or equal to

$$1 - 2^{n(k_n+1) + k_n + 1 - k_n^2 - 2^{k_n}} > 1 - 2^{n(k_n+1) - 2^{k_n}}.$$

The exponent $n(k_n + 1) - 2^{k_n}$ is smaller than or equal to $2^{k_n/c}(k_n + 1) - 2^{k_n}$ and thus tends to $-\infty$ when n tends to $+\infty$. The last sentence of the proposition can be checked by computation (the sequences $1 - 2^{n-k_n} \begin{bmatrix} n \\ k_n \end{bmatrix} 2^{\nu_n - 2^n}$, n even and n odd are increasing and positive respectively for $n \geq 12$ and $n \geq 13$). □

Remark 1. The result of Proposition 34 is easy to prove but pretty astonishing: the size of a k_n-dimensional flat is close to n.
2. Proposition 34 also remains essentially valid (except for the number "12") if, in the definition of k-weakly-normal functions, we replace "there exists a k-dimensional flat on which the function is affine" by "there exists a k-dimensional flat such that the restriction of the function to this flat has degree $\leq l$", where l is some fixed positive integer: the value of ν_n has then to be changed into

$2^n - 2^{k_n} + 1 + \binom{k_n}{1} + \ldots + \binom{k_n}{l}$. □

The deterministic function with asymptotically lowest known normality, due to Shaltiel [1036], has normality $2^{\log^{0.9} n}$. Other constructions are given in [47].

The behavior of normality for fixed algebraic degree functions is also interesting to determine. X.-D. Hou has shown in [616] that, for any odd $n \leq 13$, the maximum nonlinearity of all cubic functions is the same as for quadratic functions: $2^{n-1} - 2^{\frac{n-1}{2}}$. So we can wonder whether cubic Boolean functions behave for generic n as quadratic functions with respect to maximum nonlinearity or to normality. For nonlinearity, this is an open problem. But for normality, k_n-nonnormal Boolean functions of algebraic degree 3 exist, where k_n is negligible with respect to n (this confirms the feeling that cubic functions behave merely as general functions, considering their Hamming weights, see Section 5.3, page 204). Indeed, it has been shown in [222] that for every $\lambda > \frac{1}{2}$ and any sequence $(k_n)_{n\in\mathbb{N}^*}$ of positive integers such that $n^\lambda \leq k_n \leq n$, the density of the set of all Boolean functions of algebraic degree at most 3 on $\mathbb{F}_2^n$ which are not k_n-weakly-normal in the set of all Boolean functions of algebraic degree at most 3 is larger than or equal to $1 - 2^{n(k_n+1)-k_n^2-\binom{k_n}{2}-\binom{k_n}{3}}$. This density tends to 1 when n tends to infinity.
As proved later in [377] (and recalled in [111]), for any constant d, a random algebraic degree d Boolean function has normality $\Omega(n^{1/(d-1)})$.

Remark
1. All the results above are essentially valid if we restrict ourselves to balanced functions. Indeed, the number of balanced functions on $\mathbb{F}_2^n$ equals $\binom{2^n}{2^{n-1}} = \frac{(2^n)!}{((2^{n-1})!)^2} \sim \frac{\sqrt{2\pi 2^n}(2^n)^{2^n} e^{-2^n}}{\left(\sqrt{2\pi 2^{n-1}}(2^{n-1})^{2^{n-1}} e^{-2^{n-1}}\right)^2} = \sqrt{\frac{2}{\pi}}\; 2^{2^n - \frac{n}{2}}$, according to Stirling's formula, and all our arguments can be used, replacing the number of functions, 2^{2^n}, by $\binom{2^n}{2^{n-1}}$.
2. We can also deal with the distance to linear structures. Since the existence of a linear structure for a function f is equivalent to the existence of a Boolean function g on $\mathbb{F}_2^{n-1}$ and of a linear function l on $\mathbb{F}_2$ such that $f(x_1, \ldots, x_n)$ is affinely equivalent to the function $g(x_1, \ldots, x_{n-1}) \oplus l(x_n)$, the number of functions admitting linear structures is smaller than or equal to $2^{2^{n-1}}$, times the number of affine automorphisms, times 2. Thus, it is smaller than $2^{2^{n-1}+n^2+n+1}$. Moreover, let ρ be a positive number smaller than $1/2$. The number of Boolean functions on $\mathbb{F}_2^n$ which lie at distance smaller than or equal to $\rho\, 2^n$ from this set is smaller than or equal to $2^{2^{n-1}+n^2+n+1} \sum_{i=0}^{\rho\, 2^n} \binom{2^n}{i} \leq 2^{2^{n-1}+n^2+n+1+2^n H_2(\rho)}$. Thus, this number is negligible with respect to 2^{2^n} if $H_2(\rho) < 1/2$ and, asymptotically, almost all functions lie then at distance greater than $\rho\, 2^n$ from the set of all Boolean functions admitting linear structures. □

We have seen that a low algebraic degree of Boolean functions does not imply their normality. Conversely, k-normality does not imply either low algebraic degree: take a function of high algebraic degree on $\mathbb{F}_2^{n-1}$ (considered as a subspace of $\mathbb{F}_2^n$) and complete it by 0 to obtain a function on $\mathbb{F}_2^n$.

There exist functions f with low algebraic thicknesses (*e.g.* functions of algebraic degree 3) which are k-nonnormal with small k; and there exist functions with high algebraic thicknesses which are k-normal with large k: take a function g on $\mathbb{F}_2^{n-1}$ with high $AT(g)$ and complete it by 0 to obtain a function f on $\mathbb{F}_2^n$; it is a simple matter to check that $AT(f) \geq AT(g)$. In [111, 377] (and references therein) is studied the relationship between algebraic thickness and non-normality. The most interesting is that *almost all functions have high algebraic degrees, nonlinearities and algebraic thicknesses and are non-k-normal with small k's.*

Spectral complexity

The size of the support of the Walsh transform of an n-variable function f, that is, 2^n minus the number of its zeros, is called the *spectral complexity* of f. We shall denote it by $SC(f)$. This criterion has been studied in [968, 1008]. Since, according to the *inverse Walsh transform formula* (2.43), page 78, the Walsh transform values $W_f(u)$ provide the decomposition of the sign function of f over the basis of the so-called *Walsh functions* $(-1)^{u\cdot x}$, and since these functions are realized by simple circuits, the spectral complexity is related to the circuit complexity of Boolean functions.

Note that, for every n-variable Boolean function f, an easy lower bound can be derived from the Cauchy-Schwarz inequality:

$$SC(f) \geq \frac{\left(\sum_{u\in\mathbb{F}_2^n} W_f^2(u)\right)^2}{\sum_{u\in\mathbb{F}_2^n} W_f^4(u)} = \frac{(2^{2n})^2}{2^n \sum_{\substack{(x,y,z,t)\in(\mathbb{F}_2^n)^4\\ x+y+z+t=0_n}} (-1)^{f(x)\oplus f(y)\oplus f(z)\oplus f(t)}}$$
$$= \frac{2^{3n}}{\sum_{(x,y,z)\in(\mathbb{F}_2^n)^3}(-1)^{f(x)\oplus f(y)\oplus f(z)\oplus f(x+y+z)}}.$$

Note that, for any nonzero function f and according to Relation (2.32), page 74, we have that $SC(f)$ equals $N_{\widehat{f}}$ if f is not balanced and $N_{\widehat{f}}-1$ if f is balanced. According to what we have seen at page 89, we have then:

$$SC(f) \geq 2^{d_{alg}(f)} - 1 \text{ if } f \text{ is balanced}, SC(f) \geq 2^{d_{alg}(f)} \text{ otherwise.}$$

The average spectral complexity of n-variable Boolean functions, equal to $2^n - 2^{-2^n}\sum_{f\in\mathcal{BF}_n} |\{u \in \mathbb{F}_2^n; W_f(u) = 0\}|$, is also easily determined: for every $f \in \mathcal{BF}_n$ and $u \in \mathbb{F}_2^n$, we have $W_f(u) = 0$ if and only if function $f(x) \oplus u\cdot x$ is balanced. We have then $|\{f \in \mathcal{BF}_n; W_f(u) = 0\}| = \binom{2^n}{2^{n-1}}$ for every u. Hence, the average number of zeros of the Walsh transform equals $\frac{2^n\binom{2^n}{2^{n-1}}}{2^{2^n}} \sim \sqrt{\frac{2}{\pi}}\, 2^{\frac{n}{2}}$ and the average spectral complexity equals $2^n - \frac{2^n\binom{2^n}{2^{n-1}}}{2^{2^n}}$.

Ryazanov in [1008] shows the more precise result that the random variable equal

to $\left(\frac{\pi}{2^{n+3}}\right)^{\frac{1}{2}}$ times the number of zeros of the Walsh transform tends in distribution to the constant function $\frac{1}{2}$ over $\{0,1\}$. The proof is too long for being included here. He also studies the number of zeros of the Walsh transform of functions of even Hamming weights and shows then that the same random variable converges to 1 (in particular, functions of even Hamming weights have in average twice more zeros than general Boolean functions; this can be simply proved with the same method as above).
The evaluation can also be done for random (n,m)-functions. When F ranges over the set of (n,m)-functions and v ranges over $\mathbb{F}_2^m \setminus \{0_m\}$, the component function $v \cdot F$ ranges $2^m - 1$ times over the set of n-variable Boolean functions. Since for $v = 0_m$ we have $W_F(u,v) = 0$ for every $u \neq 0_n$ and $W_F(0_n, 0_m) = 2^n$, we deduce that the average number of zeros of the Walsh transform of (n,m)-functions equals $2^n - 1 + (2^m - 1)\frac{2^n\binom{2^n}{2^{n-1}}}{2^{2^n}}$.
And when restricting ourselves to (n,n)-permutations, we know that when $v \neq 0_n$, the component function $v \cdot F$ ranges uniformly over the set of balanced functions when F ranges over the set of permutations. Distinguishing the cases "$u = v = 0_n$", "$u = 0_n, v \neq 0_n$", "$u \neq 0_n, v = 0_n$" and "$u \neq 0_n, v \neq 0_n$", we obtain an average of $2(2^n - 1) + \frac{(2^n-1)^2\binom{2^{n-1}}{2^{n-2}}^2}{\binom{2^n}{2^{n-1}}}$, since $|\{f \in \mathcal{BF}_n,\ f \text{ balanced}; W_f(u) = 0\}|$ equals $\binom{2^{n-1}}{2^{n-2}}^2$ for every $u \neq 0_n$, because $u \cdot x$ and $f(x) \oplus u \cdot x$ need to be both balanced, that is, we need $w_H(f(x)(u \cdot x)) = w_H(f(x)(u \cdot x \oplus 1)) = 2^{n-2}$, where $f(x)(u \cdot x)$ is the product of $f(x)$ and $u \cdot x$.

As in the case of Boolean functions above, by the Cauchy-Schwarz inequality, the spectral complexity of (n,m)-functions $SC(F) = |\{(u,v) \in \mathbb{F}_2^n \times \mathbb{F}_2^m;\ W_F(u,v) \neq 0\}|$ of F satisfies:

$$SC(F) \geq 1 + \frac{\left(\sum_{u \in \mathbb{F}_2^n, v \in \mathbb{F}_2^m, v \neq 0_m} W_F^2(u,v)\right)^2}{\sum_{u \in \mathbb{F}_2^n, v \in \mathbb{F}_2^m, v \neq 0_m} W_f^4(u,v)} =$$

$$1 + \frac{(2^m-1)^2\, 2^{4n}}{2^{n+m}\left|\left\{(x,y,z,t) \in (\mathbb{F}_2^n)^4;\ \begin{array}{l} x+y+z+t = 0_n \\ F(x)+F(y)+F(z)+F(t) = 0_m \end{array}\right\}\right| - 2^{4n}} =$$

$$1 + \frac{(2^m-1)^2\, 2^{3n}}{2^m|\{(x,y,z) \in (\mathbb{F}_2^n)^3;\ F(x)+F(y)+F(z)+F(x+y+z) = 0_m\}| - 2^{3n}}.$$

In the case of an APN (n,n)-function (see Definition 41, page 159), this gives:

$$SC(F) \geq 1 + \frac{(2^n-1)^2\, 2^{2n}}{2 \cdot 2^{2n} - 2^{n+1}} = 1 + (2^n - 1)\, 2^{n-1} \approx 2^{2n-1}.$$

Nonhomomorphicity
For every even integer k such that $4 \leq k \leq 2^n$, the k-th order *nonhomomorphicity* [1171] of a Boolean function equals the number of k-tuples $(u_1, \ldots, u_k)$ of vectors of $\mathbb{F}_2^n$ such that $u_1 + \cdots + u_k = 0_n$ and $f(u_1) \oplus \cdots \oplus f(u_k) = 0$. It is a simple matter to show that it equals $2^{(k-1)n-1} + 2^{-n-1} \sum_{u \in \mathbb{F}_2^n} W_f^k(u)$. This parameter should be small (but no related attack exists on stream ciphers). It is maximum and equals $2^{(k-1)n}$ if and only if the function is affine. It is minimum and equals $2^{(k-1)n-1} + 2^{\frac{nk}{2}-1}$ if and only if the function is bent.

Conclusion of this section

As we can see, there are numerous cryptographic criteria for Boolean functions to be used in stream ciphers. The ones which must be necessarily satisfied are balancedness, a high algebraic degree, a high nonlinearity, a high algebraic immunity and a good resistance to fast algebraic attacks. It is difficult but not impossible to find functions satisfying good trade-offs between all these criteria (see Chapter 9). Achieving additionally resiliency of a sufficient order, which is necessary for the combiner model, is impossible because of the Siegenthaler bound[19]. Hence, the filter model is more appropriate.
We saw that, asymptotically, almost all Boolean functions (in the sense of probability theory) have high algebraic degree, high nonlinearity and high algebraic immunity. They have also high algebraic thickness and low normality. The related following randomness criteria for n-variable Boolean functions seem then appropriate:

- algebraic degree close to $n-1$ (since the number of functions of algebraic degree at most $n-2$ is negligible compared to 2^{2^n}),
- nonlinearity lying within the interval
$$\left[2^{n-1} - 2^{\frac{n}{2}-1}\sqrt{n}\left(\sqrt{2\,\ln 2} + \frac{4\,\ln n}{n}\right); 2^{n-1} - 2^{\frac{n}{2}-1}\sqrt{n}\left(\sqrt{2\,\ln 2} - \frac{5\,\ln n}{n}\right)\right]$$
(according to Rodier's results, see Subsection 3.1.3),
- algebraic immunity at distance at most $\ln n$ from $\frac{n}{2}$ (according to Didier's results, see Subsection 3.1.5),
- algebraic thickness equal to $\lambda 2^n$ with λ near $\frac{1}{2}$.

Of course, these criteria make really sense asymptotically only.

3.2 Cryptographic criteria for vectorial functions in stream and block ciphers

Vectorial functions can be used (in the place of Boolean functions) as combiners or filters in stream ciphers (they allow then the PRG to generate several bits at

[19] But to render f 1-resilient by composing it with a linear automorphism - which preserves the other features - we just need that there exist n linearly independent vectors at which the Walsh transform vanishes.

each clock cycle, which increases the speed of the cipher), or as *S-boxes* in block ciphers. These two situations are very different, but some criteria of resistance to attacks are the same. We study them in this section. We shall study in the two next sections those criteria and parameters which are specific to each use.

3.2.1 Balancedness of vectorial functions

Recall that an (n,m)-function is called *balanced* if its output distribution is uniformly distributed (with $m \leq n$), that is, if it takes every value of $\mathbb{F}_2^m$ the same number 2^{n-m} of times. By definition, F is then balanced if every Boolean function $\varphi_b = 1_{\{b\}} \circ F$ has *Hamming weight* 2^{n-m}. A vectorial function used as combiner or as filter needs to be balanced because any combination of its output bits can be made and for avoiding such combination to give statistical information allowing to distinguish when a pair of texts is a pair (plaintext,ciphertext), this needs the vectorial function to be balanced.
S-boxes in block ciphers are also better balanced. In every SPN (see Subsection 1.4.2), the S-boxes need to be permutations (with $m = n$) and are then balanced. In *Feistel ciphers*, we have seen that the S-boxes do not need to be balanced, but that it has been shown for instance in [957] that an attack exists then, which obliges the designer to complexify the encryption algorithm, for instance with expansion boxes. Hence, balanced S-boxes are preferred.

Characterization through the *component functions*

The balanced S-boxes (and among them, the permutations) can be nicely characterized by the balancedness of their component functions:

Proposition 35 *[775] An (n,m)-function F is balanced if and only if its component functions $v \cdot F$, $v \neq 0_m$, are all balanced, that is, if and only if, for every nonzero $v \in \mathbb{F}_2^m$, we have $W_F(0_n, v) = 0$.*

Proof. The relation:

$$\sum_{v \in \mathbb{F}_2^m} (-1)^{v \cdot (F(x)+b)} = \left\{ \begin{array}{l} 2^m \text{ if } F(x) = b \\ 0 \text{ otherwise} \end{array} \right. = 2^m \, \varphi_b(x), \tag{3.16}$$

is valid for every (n,m)-function F, every $x \in \mathbb{F}_2^n$ and every $b \in \mathbb{F}_2^m$, since the function $v \mapsto v \cdot (F(x) + b)$ being linear, it is either balanced or null. Thus, we have:

$$\sum_{x \in \mathbb{F}_2^n; v \in \mathbb{F}_2^m} (-1)^{v \cdot (F(x)+b)} = 2^m \, |F^{-1}(b)| = 2^m \, w_H(\varphi_b), \tag{3.17}$$

where w_H denotes the Hamming weight. Hence, the Fourier-Hadamard transform of the function $v \mapsto \sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot F(x)}$ equals the function $b \mapsto 2^m \, |F^{-1}(b)|$. We know that a pseudo-Boolean function has constant Fourier-Hadamard transform if and only if it is null at every nonzero vector. We deduce that F is balanced if and only if the function $v \mapsto \sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)}$ is null on $\mathbb{F}_2^m \setminus \{0_m\}$. $\square$

Equivalently, F is balanced if and only if $\sum_{a\in\mathbb{F}_2^n}\mathcal{F}(D_a(v\cdot F)) = 0$ for every $v \neq 0_m$ (according to Wiener-Khintchine's formula (2.53), page 80). Note that, for $m = n$, F is a permutation if and only if $\sum_{v\in\mathbb{F}_2^n}\mathcal{F}(D_a(v\cdot F)) = 0$ for every $a \neq 0_n$ (since $\sum_{v\in\mathbb{F}_2^m}\mathcal{F}(v\cdot G) = 2^m|G^{-1}(0_m)|$ for every (n,m)-function G).

If F is balanced, then the f_i's ($1\leqslant i\leqslant m$) being balanced, we have $d_{\text{alg}}(F)\leqslant n-1$. Much more can be said, in particular for permutations: F is a permutation if and only if the product of strictly less than n coordinate functions of F has even Hamming weight, that is, algebraic degree strictly less than n, and the product of all n coordinate functions has algebraic degree n. The condition is clearly necessary and it is easily seen that it is sufficient (since "$|F^{-1}(a)|$ is odd for every $a \in \mathbb{F}_2^n$" implies F bijective). Note that the relation between this characterization and Proposition 35 is given by Relations (2.25) and (2.26).

There is a nice property of the Walsh transform of permutations:

$$\forall v \neq w,\ \sum_{u\in\mathbb{F}_2^n} W_F(u,v)W_F(u,w) = 0. \tag{3.18}$$

Indeed, we have $\sum_{u\in\mathbb{F}_2^n} W_F(u,v)W_F(u,w) = \sum_{u,x,y\in\mathbb{F}_2^n}(-1)^{u\cdot(x+y)\oplus v\cdot F(x)\oplus w\cdot F(y)} = 2^n\sum_{x\in\mathbb{F}_2^n}(-1)^{(v+w)\cdot F(x)}$. Note that for $v = w$, the sum in (3.18) equals 2^{2n} (this is Parseval's relation on the Boolean function $v\cdot F$). Of course, Relation (3.18) can be also applied to F^{-1} and since

$$W_{F^{-1}}(u,v) = W_F(v,u),$$

we obtain:

$$\forall v \neq w,\ \sum_{u\in\mathbb{F}_2^n} W_F(v,u)W_F(w,u) = 0.$$

Imbalance of an (n,m)-function

A natural way of quantifying the fact that some (n,m)-*function* F is unbalanced is by the variance of the random variable $b \to |F^{-1}(b)|$, where $|F^{-1}(b)|$ denotes the size of the pre-image of b by F. In [267], the variance is multiplied by 2^m to give the following integer-valued parameter[20], that we shall call the *imbalance* of F:

$$Nb_F = \sum_{b\in\mathbb{F}_2^m}\left(|F^{-1}(b)| - 2^{n-m}\right)^2 = \sum_{b\in\mathbb{F}_2^m}\left|F^{-1}(b)\right|^2 - 2^{2n-m}. \tag{3.19}$$

It has the following properties:

[20] The framework of [267] is functions from Abelian groups to Abelian groups; we stick here to Boolean functions.

- $Nb_F \geq 0$, for every vectorial function F, and $Nb_F = 0$ if and only if F is balanced;
- Nb_F is invariant under composition of F by permutations (on the right and on the left); in particular, it is *affine invariant*;
- $Nb_F = |\{(x,y) \in (\mathbb{F}_2^n)^2\,;\, F(x) = F(y)\}| - 2^{2n-m} \leq 2^{2n} - 2^{2n-m}$ and $Nb_F = 2^{2n} - 2^{2n-m}$ if and only if F is constant;
- $Nb_F = \sum_{a\in\mathbb{F}_2^n} \left|(D_aF)^{-1}(0_m)\right| - 2^{2n-m}$.

Parameter Nb_F can be expressed by means of the Walsh transform. We have

$$\sum_{v\in\mathbb{F}_2^m} W_F^2(0_n, v) = \sum_{x,y\in\mathbb{F}_2^n} \left(\sum_{v\in\mathbb{F}_2^m} (-1)^{v\cdot(F(x)+F(y))} \right)$$

$$= 2^m|\{(x,y) \in \mathbb{F}_2^n \,|\, F(x) = F(y)\}| = 2^m(Nb_F + 2^{2n-m}).$$

Hence:

$$Nb_F = 2^{-m} \sum_{v\in\mathbb{F}_2^m, v\neq 0_m} W_F^2(0_n, v). \tag{3.20}$$

3.2.2 Algebraic degree of vectorial functions

The algebraic degree of vectorial functions has been defined at page 56. The output of the function used in a stream cipher being also the output of the PRG, the output bits can be combined and used in a Berlekamp-Massey attack. The algebraic degree is then an important parameter.
In block ciphers, the algebraic degree is a security parameter against structural attacks, such as integral [709], higher-order differential, cube [465] or, recently, attacks based on the division property[21] [1086] (see also the two first sections of [106] and the references therein). In particular, the *higher-order differential attack* [735, 706] (see also [204]) exploits the fact that the algebraic degree of the S-box F is low, or more generally that there exists a low dimensional vector subspace V of $\mathbb{F}_2^n$ such that the function $D_VF(x) = \sum_{v\in V} F(x+v)$ (*i.e.* $D_{a_1}\cdots D_{a_k}F(x)$ where $\{a_1,\ldots,a_k\}$ is a basis of V) is constant. A probabilistic version of this attack [638] allows the derivative not to be constant and the S-box must then have high *higher-order nonlinearity* (notion defined for Boolean functions in Definition 20, page 102; for vectorial functions, see page 381 in Subsection 9.2.4). *Stricto sensu*, the higher-order differential attack has been proved efficient for *quadratic functions* only. But since cryptographers like to have some security margin, even *cubic functions* may be viewed as weak (unless, as usual in cryptography, some precautions are taken with the global cipher). Quadratic S-boxes, if used, need care. It is observed in [204, 108, 106] (see page 82 and below) that the algebraic degree of the function resulting from the first rounds

[21] A very elementary notion, from a viewpoint of Boolean functions, whose properties given in diverse papers are in fact well-known properties of Reed-Muller codes.

of the cipher may increase less than expected.
The algebraic degree of the computational inverse of a permutation plays also a role in the algebraic degree of the iterated rounds implementing it. This is shown in [106] by proving that $d_{alg}(G \circ F) < n - \left\lfloor \frac{n-1-d_{alg}(G)}{d_{alg}(F^{-1})} \right\rfloor$ for every (n,n)-permutation F and every (n,r)-function G. We do not recall the proof given in [106] for this bound[22], since as seen in [254] we have directly from Relation (2.12), page 58, that $d_{alg}(G \circ F) \le n - \left\lceil \frac{n-d_{alg}(G)}{d_{alg}(F^{-1})} \right\rceil$, implied by $n = d_{alg}\left((g_k \oplus 1)\prod_{i\in I^c}(f_i' \oplus 1)\right) \le d_{alg}(G) + (n-|I|)\, d_{alg}(F^{-1})$. And $d_{alg}(G \circ F)$ is bounded above by $\max\{t;\, d_{G,F^{-1}}(n-t) = n\}$, where $d_{G,F^{-1}}(n-t)$ equals the maximal numerical degree of the linear combinations in $\mathcal{BF}_n$ of at most one coordinate function of G and at most $n-t$ coordinate functions of F^{-1} (or more precisely of the parts of the NNFs of these functions which are not divisible by 2^{n-t}). Indeed, in the framework of Relation (2.12) again, we have $n = d_{alg}\left((g_k \oplus 1)\prod_{i\in I^c}(f_i' \oplus 1)\right) \le d_{num}\left((g_k \oplus 1)\prod_{i\in I^c}(f_i' \oplus 1)\right) \le d_{G,F^{-1}}(n-|I|)$, the latter inequality being due to Relation (2.26), page 68. This is generalized to the composition of any number of functions in [254].

Remark It is an open problem to know whether those high algebraic degree functions which are *CCZ equivalent* to low algebraic degree functions could be attacked by a modification of the higher-order differential attack. Thus, it is not clear whether the designer should also avoid functions CCZ equivalent to quadratic functions. □

3.2.3 Nonlinearity of vectorial functions

In stream ciphers, since the output bits can be combined by the attacker, the nonlinearity of all *component functions* must be large, and the minimum of these nonlinearities, called the nonlinearity of the vectorial function, is then a parameter related to the resistance to the fast correlation attack [843]. But nonlinear combinations of the output bits can also be used by the attacker and this will lead in Subsection 3.3.2 to the introduction of a parameter more adapted to this framework.
In block ciphers, the *linear attack*, introduced by Matsui [829], is based on an idea from [1084]. It may have been unknown by the NSA at the time it was introduced; this could explain why it works better[23] than the differential attack on the DES. It seems that it was known or partly known from the USSR. It is, with the differential attack that we shall describe at page 157, one of the two most powerful general purpose cryptographic attacks known to date. Its most common version is an attack on the *reduced cipher*, that is, the cipher obtained from the

[22] Note that a nice simpler proof is given in Udovenko's PhD thesis "Design and Cryptanalysis of Symmetric-Key Algorithms in Black and White-box Models".
[23] The differential attack needs 2^{47} pairs (plaintext, ciphertext) while the linear attack needs "only" 2^{43} pairs.

original one by removing its last round[24] (or more generally an attack on a round whose inputs and outputs can be computed from the plaintext and ciphertext and a number of key bits hopefully "small"). We describe the principle of the attack in the case it is applied to the reduced cipher. In Figure 3.1 below, $Y(r-1)$ denotes the output of the reduced cipher corresponding to a plaintext $Y(0)$, and $Y(r)$ denotes the ciphertext. Assume that it is possible to distinguish the outputs

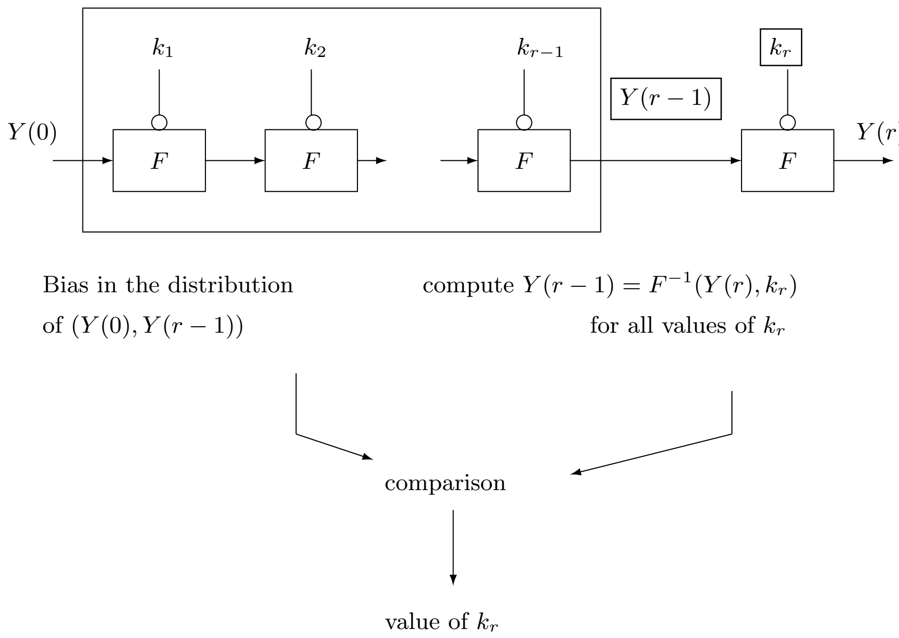


Figure 3.1 LAST ROUND ATTACKS

of the reduced cipher from random outputs, by observing some statistical bias in their value distribution. The existence of such *distinguisher* allows recovering the key used in the last round, either by an exhaustive search, which is efficient if this key is shorter than the master key, or by using specificities of the cipher allowing replacing the exhaustive search by, for instance, solving algebraic equations.
We describe now the attack in the case of exhaustive search, which is simpler to describe. The attacker, who knows a number of pairs (plaintext, ciphertext) of the (complete) cipher, visits all possible last round keys. For each try, he/she applies

[24] The output of the reduced cipher is unknown if the last round key is unknown, but it is convenient to name this reduced cipher for describing the attack.

to all the ciphertexts in these pairs the inverse of what is the last round when the key corresponds to the try (this is possible since all except the key is supposed known to him/her; if not, say, if some parameter is unknown, he will have to try all possibilities). He obtains in the case of the correct key guess the output of the reduced cipher and has then a number of pairs (plaintext,ciphertext) of the reduced cipher, on which he can observe the statistical bias. In all the other cases (incorrect guesses), the obtained pairs (plaintext, ciphertext) correspond to a cipher equal to the original cipher with an additional round whose round key is random, and the pairs are then assumed random, with no observable bias. Such assumption is verified in practice. The number of pairs (m, c) which are known to him needs then to be large enough to distinguish the bias (the smaller the bias, the larger the number of known pairs needed).

For distinguishing pairs (plaintext,ciphertext) of the reduced cipher, the linear attack uses triples (α, β, γ) of binary strings such that, a block m of plaintext and a key k being randomly chosen, the bit $\alpha \cdot m \oplus \beta \cdot c \oplus \gamma \cdot k$, where "$\cdot$" denotes the *usual inner product* (between two strings of the same length) and c denotes the (reduced) ciphertext related to m, has a probability different from $1/2$ of being null. The more distant from $1/2$ the probability, the more efficient the attack. Note that when searching for triples (α, β, γ), both m and k are supposed ranging uniformly over their definition spaces (indeed, the plaintext can be any binary string of a given length and the round key can be as well any string of a given length), while during the attack, m still ranges uniformly but k is fixed.

The related criterion on any S-box F used in the cipher for allowing resistance to the attack is that the component functions $v \cdot F$, $v \neq 0_m$, be at Hamming distance to affine Boolean functions $u \cdot x \oplus \epsilon$ as close to 2^{n-1} as possible. In other words, the nonlinearities of all these component functions must be as large as possible. The generalization to vectorial functions of the notion of nonlinearity introduced by Nyberg [907] and studied by Chabaud and Vaudenay [341], is then:

Definition 29 *The* nonlinearity *of an (n, m)-function is the minimum nonlinearity of its component functions:*

$$nl(F) = 2^{n-1} - \frac{1}{2} \max_{\substack{v \in \mathbb{F}_2^m \setminus \{0_m\} \\ u \in \mathbb{F}_2^n}} |W_F(u, v)| \,; \; W_F(u, v) = \sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot F(x) \oplus u \cdot x}. \quad (3.21)$$

Note that "$\max_{v \in \mathbb{F}_2^m \setminus \{0_m\};\, u \in \mathbb{F}_2^n}$" can be replaced by "$\max_{(u,v) \in \mathbb{F}_2^n \times \mathbb{F}_2^m; (u,v) \neq (0_n, 0_m)}$", since we have $\sum_{x \in \mathbb{F}_2^n} (-1)^{u \cdot x} = 0$ for every nonzero u.

Nonlinearity is an EA invariant (see Definition 5, page 45), that is, does not change when we compose the function by affine automorphisms nor when we add an affine function to it (this implies for instance that if A is a surjective affine function from $\mathbb{F}_2^r$ into $\mathbb{F}_2^n$, then $nl(F \circ A) = 2^{r-n} nl(F)$, since by affine invariance, we can assume without loss of generality that A is a projection and the equality is then easily shown). Nonlinearity is more strongly a *CCZ invariant*. Indeed, in Relation (3.21), $W_F(u, v)$ equals the *Fourier-Hadamard transform* of the graph $\{(x, F(x)), x \in \mathbb{F}_2^n\}$ of F and $\max_{v \in \mathbb{F}_2^{m\,*}, u \in \mathbb{F}_2^n} |W_F(u, v)|$ is then invariant under affine transformation of this graph.

S. Dib has shown in [436] that for $0 < \beta < 1/4$ and $m \leq n$, when n tends to infinity, the nonlinearity of almost all (n,m)-functions (in terms of probability) is bounded above by $2^{n-1} - 2^{\frac{n-1}{2}}\sqrt{(n+m)\log 2}\ (1-\beta)$ and that for $\beta > 0$, when $n+m$ tends to infinity, the nonlinearity of almost all (n,m)-functions is bounded below by $2^{n-1} - 2^{\frac{n-1}{2}}\sqrt{(n+m)\log 2}\ (1+\beta)$.

The *covering radius bound* $2^{n-1} - 2^{\frac{n}{2}-1}$ (see page 99) on the nonlinearity of any n-variable Boolean function is obviously valid for (n,m)-functions. Naturally, this has led researchers to extend the notion of bentness to vectorial functions:

Definition 30 *Given two integers n and m (with n necessarily even), an (n,m)-function F is called bent if its nonlinearity $nl(F)$ achieves the optimum $2^{n-1} - 2^{n/2-1}$.*

We shall see with Proposition 104, page 296, that bent (n,m)-functions do not exist if $m > \frac{n}{2}$. This has led to asking whether better upper bounds than the covering radius bound could be proved in this case. Such bound has been found by Chabaud and Vaudenay in [341]. In fact, a bound on sequences due to Sidelnikov [1040] is equivalent for power functions to the bound obtained by Chabaud and Vaudenay and its proof is valid for all functions. This is why the bound is now called the *Sidelnikov-Chabaud-Vaudenay bound* (*SCV bound*):

Theorem 6 *Let n and m be any positive integers such that $m \geq n-1$. Let F be any (n,m)-function. Then:*

$$nl(F) \leq 2^{n-1} - \frac{1}{2}\sqrt{3 \times 2^n - 2 - 2\frac{(2^n-1)(2^{n-1}-1)}{2^m-1}}.$$

Proof. Recall that $nl(F) = 2^{n-1} - \frac{1}{2}\max_{v\in\mathbb{F}_2^m\setminus\{0_m\};\ u\in\mathbb{F}_2^n} |W_F(u,v)|$. We have:

$$\max_{\substack{v\in\mathbb{F}_2^m\setminus\{0_m\}\\ u\in\mathbb{F}_2^n}} W_F^2(u,v) \geq \frac{\sum_{\substack{v\in\mathbb{F}_2^m\setminus\{0_m\}\\ u\in\mathbb{F}_2^n}} W_F^4(u,v)}{\sum_{\substack{v\in\mathbb{F}_2^m\setminus\{0_m\}\\ u\in\mathbb{F}_2^n}} W_F^2(u,v)}. \tag{3.22}$$

Parseval's relation states that, for every $v \in \mathbb{F}_2^m$:

$$\sum_{u\in\mathbb{F}_2^n} W_F^2(u,v) = 2^{2n}. \tag{3.23}$$

Using that any character sum $\sum_{x\in E}(-1)^{\ell(x)}$ associated to a linear function ℓ

over any $\mathbb{F}_2$-vector space E is nonzero if and only if ℓ is null on E, we have:

$$\begin{aligned}
&\sum_{v\in\mathbb{F}_2^m,\ u\in\mathbb{F}_2^n} W_F^4(u,v)\\
&= \sum_{x,y,z,t\in\mathbb{F}_2^n}\left[\sum_{v\in\mathbb{F}_2^m}(-1)^{v\cdot(F(x)+F(y)+F(z)+F(t))}\right]\left[\sum_{u\in\mathbb{F}_2^n}(-1)^{u\cdot(x+y+z+t)}\right]\\
&= 2^{n+m}\left|\left\{(x,y,z,t)\in\mathbb{F}_2^{4n};\ \left\{\begin{array}{ll} x+y+z+t & =0_n\\ F(x)+F(y)+F(z)+F(t) & =0_m\end{array}\right.\right\}\right|\\
&= 2^{n+m}|\{(x,y,z)\in\mathbb{F}_2^{3n};\ F(x)+F(y)+F(z)+F(x+y+z)=0_m\}| \quad (3.24)\\
&\geq 2^{n+m}|\{(x,y,z)\in\mathbb{F}_2^{3n};\ x=y \text{ or } x=z \text{ or } y=z\}|. \quad (3.25)
\end{aligned}$$

Clearly, $|\{(x,y,z);\ x=y \text{ or } x=z \text{ or } y=z\}|$ equals:

$$3\cdot|\{(x,x,y);\ x,y\in\mathbb{F}_2^n\}|-2\cdot|\{(x,x,x);\ x\in\mathbb{F}_2^n\}|=3\cdot 2^{2n}-2\cdot 2^n.$$

Hence, according to Relation (3.22):

$$\max_{v\in\mathbb{F}_2^m\setminus\{0_m\};\ u\in\mathbb{F}_2^n} W_F^2(u,v)\geq$$

$$\frac{2^{n+m}(3\cdot 2^{2n}-2\cdot 2^n)-2^{4n}}{(2^m-1)\,2^{2n}}=3\times 2^n-2-2\frac{(2^n-1)(2^{n-1}-1)}{2^m-1}$$

and this gives the desired bound, according to Relation (3.21), page 139. □

The condition $m\geq n-1$ is assumed in Theorem 6 to make non-negative the expression located under the square root. Note that for $m=n-1$, this bound coincides with the covering radius bound. For $m\geq n$, it strictly improves upon it. For $m>n$, the square root in it cannot be an integer (see [341]). Hence, the SCV bound can be tight only if $n=m$ with n odd, in which case it states:

$$nl(F)\leq 2^{n-1}-2^{\frac{n-1}{2}}. \quad (3.26)$$

We shall see that, under this condition, it is actually tight.

Definition 31 *[341] The (n,n)-functions F which achieve (3.26) with equality are called* almost bent *(AB).*

Remark. The term of *almost* bent is a little misleading. It gives the feeling that these functions are not optimal. But they are, according to Theorem 6. Proposition 104, page 296, will give the values of n and m such that bent (n,m)-functions exist. □

According to Inequality (3.22), page 140, the AB functions are those (n,n)-functions such that, for every $u,v\in\mathbb{F}_2^n$, $v\neq 0_n$, the sum $\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}=W_F(u,v)$ equals 0 or $\pm 2^{\frac{n+1}{2}}$ (indeed, the maximum of a sequence of non-negative and not all null integers equals the ratio of the sum of their squares over the sum of their values if and only if these integers take one nonzero value exactly). We shall see at page 289 that this is equivalent to saying that all component functions are *near-bent*. Note that this condition does not depend on the choice of the inner product.

We shall see that AB functions exist for every odd $n \geq 3$. Function $F(x) = x^3$, $x \in \mathbb{F}_{2^n}$, is the simplest one. Chapter 11 covers their topic.

Bounds on nonlinearity by means of imbalance

We follow [239] in this subsection. A bound is given on the nonlinearity of (n,m)-functions, by means of their imbalance (see definition at page 135):

Proposition 36 *Let F be any (n,m)-function. The nonlinearity of F satisfies:*

$$nl(F) \leq 2^{n-1} - \frac{1}{2}\sqrt{\frac{2^m}{2^m-1} Nb_F}.$$

Proof. We have, using Relation (3.20), page 136:

$$\max_{\substack{v\in\mathbb{F}_2^m \,/\, v\neq 0_m\\ u\in\mathbb{F}_2^n}} W_F^2(u,v) \geq \max_{v\in\mathbb{F}_2^m \,/\, v\neq 0_m} W_F^2(0_n,v) \geq \frac{1}{2^m-1}\sum_{v\in\mathbb{F}_2^m \,/\, v\neq 0_m} W_F^2(0_n,v)$$
$$= \frac{2^m}{2^m-1} Nb_F.$$

Relation (3.21), page 139, completes the proof. □

This bound shows that, to have a chance of having a high nonlinearity, a function must not differ too much from a balanced function.

The bound of Proposition 36 is tight (it is achieved with equality for instance by bent functions, since both inequalities above are equalities in that case). Moreover, it can be applied to $F+L$ (which has the same nonlinearity as F) for every linear (n,m)-function L. Note that we have in general $Nb_{F+L} \neq Nb_F$. Proposition 36 implies, denoting by $\mathcal{L}_{n,m}$ the set of linear (n,m)-functions:

$$nl(F) \leq 2^{n-1} - \frac{1}{2}\sqrt{\frac{2^m}{2^m-1} \max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}}, \tag{3.27}$$

which is obviously tight too.

Remark. We have $v \cdot L(x) = L^*(v) \cdot x$ where L^* is the adjoint operator of L. Hence $\max_{\substack{v\in\mathbb{F}_2^m \,/\, v\neq 0_m\\ u\in\mathbb{F}_2^n}} W_F^2(u,v) = \max_{\substack{v\in\mathbb{F}_2^m \,/\, v\neq 0_m\\ L\in\mathcal{L}_{n,m}}} W_{F+L}^2(0_n,v)$. □

Relation (3.27) raises the question of determining the mean of Nb_{F+L}:

Proposition 37 *[239] Let F be any (n,m)-function. The mean of the random variable $L \in \mathcal{L}_{n,m} \to Nb_{F+L}$ equals $2^n - 2^{n-m}$. We have $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L} \geq 2^n - 2^{n-m}$, with equality if and only if F is bent.*

Proof. For every $L \in \mathcal{L}_{n,m}$, we have:

$$Nb_{F+L} = \sum_{a\in\mathbb{F}_2^n} |(D_a(F+L))^{-1}(0_m)| - 2^{2n-m}$$
$$= \sum_{a\in\mathbb{F}_2^n} |(D_aF)^{-1}(L(a))| - 2^{2n-m}. \tag{3.28}$$

The size of $\mathcal{L}_{n,m}$ equals 2^{mn}. Given any nonzero element a of $\mathbb{F}_2^n$ and any element b of $\mathbb{F}_2^m$, the number of linear functions L such that $L(a) = b$ equals $2^{m(n-1)}$. We have then, distinguishing the case $a = 0_n$ from the others:

$$\sum_{L\in\mathcal{L}_{n,m}} \sum_{a\in\mathbb{F}_2^n} |(D_aF)^{-1}(L(a))| = 2^{mn}\, 2^n + 2^{m(n-1)} \sum_{\substack{a\in\mathbb{F}_2^n\\ a\neq 0_n}} \sum_{b\in\mathbb{F}_2^m} |(D_aF)^{-1}(b)|$$
$$= 2^{(m+1)n} + 2^{m(n-1)}(2^n-1)2^n.$$

The mean $\frac{1}{|\mathcal{L}_{n,m}|}\sum_{L\in\mathcal{L}_{n,m}}\sum_{a\in\mathbb{F}_2^n} |(D_aF)^{-1}(L(a))|$ equals $2^n + 2^{n-m}(2^n-1) = 2^{2n-m} + 2^n - 2^{n-m}$. This proves the first assertion. The second is then straightforward and the case of equality is when the function $(a,b) \in (\mathbb{F}_2^n\setminus\{0_n\})\times\mathbb{F}_2^m \mapsto |(D_aF)^{-1}(b)|$ is constant and we shall see in Section 6.4 that this is characteristic of bent functions. □

Remark. The definition of nonlinearity given in Definition 29, page 139, is related to Matsui's linear attack [829], but the term of nonlinearity can also evoke the behavior of the functions $F+L$ where L is any linear (n,m)-function, which could lead to other "nonlinearity" notions. We see with Proposition 37 that bent functions, which are related to the classical notion of nonlinearity, are also related to the imbalance of functions $F+L$. □

Proposition 37 and Relation (3.27) give the covering radius bound, and show that the constancy of function $L \in \mathcal{L}_{n,m} \to Nb_{F+L}$ is characteristic of bent functions.
The fact that the average value of Nb_{F+L} is the same for all (n,m)-functions is not surprising: Relation (3.20) applied to the function $F+L$ gives

$$Nb_{F+L} = 2^{-m} \sum_{v\in\mathbb{F}_2^m, v\neq 0_m} \left(\sum_{x\in\mathbb{F}_2^n} (-1)^{v\cdot F(x)\oplus L^*(v)\cdot x}\right)^2,$$

where L^* is the adjoint operator of L. Summing up this equality when L ranges over $\mathcal{L}_{n,m}$ allows, for every $v \neq 0_m$, the vector $L^*(v)$ to cover uniformly $\mathbb{F}_2^n$, and Parseval's relation leads then to the mean.

Remark.
The number $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}$ is, after $nl(F)$, a second parameter quantifying the non-affineness of F (in a different way from $nl(F)$ but in a coherent one, according to Relation (3.27)). We shall see that it is also closely related to a

third parameter NB_F that we shall introduce at page 161. Some easily proved properties of $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}$ are:
- if F is affine, that is, if $F+L_0$ is constant for some linear function L_0, then we know that $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L} = Nb_{F+L_0} = 2^{2n} - 2^{2n-m}$ is maximal;
- if, on the opposite side, F is bent then, for every L, we have $Nb_{F+L} = 2^n - 2^{n-m}$ and $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L} = 2^n - 2^{n-m}$ is minimal (according to Proposition 37); we can say that, for every L, the function $F+L$ is "almost balanced", which is the best which can be achieved for every linear function L;
- $F \mapsto \max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}$ is EA-invariant since Nb is affine-invariant.
For $m=n=5$, $\max\limits_{L\in\mathcal{L}_{n,m}} Nb_{F+L} = 52 < 2\,(2^n-1) = 62$ for every AB function. □

Other bounds

Bounds have been obtained in relation with codes [267]:

$$nl(F) < 2^{n-1} - \frac{m}{2} \times \frac{2^{n-1}}{2^{n-1}-1};\ m < 2^n - 2, \tag{3.29}$$

and using the sphere packing bound:

$$\sum_{i=0}^{\left\lfloor \frac{nl(F)-1}{2} \right\rfloor} \binom{2^n}{i} \leq 2^{2^n-n-m-1}, \tag{3.30}$$

and the Griesmer bound:

$$\sum_{i=0}^{m+n} \left\lceil \frac{nl(F)}{2^i} \right\rceil \leq 2^n. \tag{3.31}$$

A construction using concatenated codes (see page 24) is given in [53], which allows approaching these bounds. Precisely, a $(2e-1,(k-2)e)$-function F is obtained for every $e\geq 2, k\geq 3$, such that $nl(F) = 2^{e-2}(2^e-k+1)$.
A lower bound on the nonlinearity of vectorial functions is given in [234] and upper bounds in [1133] by means of parameter Nb_F of page 135, under particular conditions, in some cases. A table of the best known nonlinearities is given in [53].

Another notion of nonlinearity of vectorial functions, sometimes denoted by nl_v, has been introduced in [266] and studied further in [788]: their minimum Hamming distance to affine vectorial functions.

Higher-order nonlinearity

This notion (see Definition 20, page 102) can be extended to vectorial functions by taking the minimum r-th order nonlinearity of *component functions*: $nl_r(F) = \min_{v\neq 0_m} nl_r(v\cdot F)$. We can more generally consider F composed by functions of higher degrees:

Definition 32 *For every (n,m)-function F, for every positive integers $s\leq m$ and $t\leq n+m$, and every non-negative integer $r\leq n$, we define:*

$$nl_{s,r}(F) = \min\{nl_r(f\circ F);\ f\in\mathcal{BF}_m,\ d_{alg}(f)\leq s,\ f\neq cst\},$$

$$\textit{and} \quad NL_t(F) = \min\{w_H(h(x,F(x)));\, h \in \mathcal{BF}_{n+m},\, d_{alg}(h) \le t,\, h \ne cst\}.$$

Definition 32 excludes $f = cst$ and $h = cst$ for obvious reasons.
Clearly, for every function F and every integers $t \le t'$, $s \le s'$ and $r \le r'$, we have $NL_t(F) \ge NL_{t'}(F)$ and $nl_{s,r}(F) \ge nl_{s',r'}(F)$. Note also that we have $NL_1(F) = nl_{1,1}(F) = nl(F)$.
As recalled in [233, Section 3], which is devoted to these notions, T. Shimoyama and T. Kaneko have exhibited in [1037] several quadratic functions h and pairs (f,g) of quadratic functions showing that the nonlinearities NL_2 and $nl_{2,2}$ of some sub-S-boxes of the DES are null (and therefore that the global S-box of each round of the DES has the same property). They deduced a "higher-order non-linear" attack (an attack using the principle of the linear attack by Matsui but with non-linear approximations) which needs 26% less data than Matsui's attack. This improvement is not very significant, practically, but the notions of NL_t and $nl_{s,r}$ may be related to potentially more powerful attacks. Note that we have $NL_{\max(s,r)}(F) \le nl_{s,r}(F)$ by taking $h(x,y) = g(x) \oplus f(y)$ (since $f \ne cst$ implies then $h \ne cst$) and the inequality can be strict if $s > 1$ or $r > 1$ since it may happen that a function $h(x,y)$ of low algebraic degree and such that $w_H(h(x,F(x)))$ is small exists while no such function exists with separated variables x and y. This is the case, for instance, of the S-box of the AES for $s = 1$ and $r = 2$ (see below).

Proposition 38 *[233] For every positive integers n, m, $r \le n$ and $s \le m$ and every (n,m)-function F, we have: $NL_s(F) \le 2^{n-s}$ and $nl_{s,r}(F) \le 2^{n-s}$. These inequalities are strict if F is not balanced (that is, if its output is not uniformly distributed over $\mathbb{F}_2^m$).*

Indeed, there necessarily exists an $(m-s)$-dimensional affine subspace A of F_2^m (whose indicator 1_A has algebraic degree s) such that $|F^{-1}(A)| \le 2^{n-s}$, and we can take $f(y) = h(x,y) = 1_A(y)$. See in [233] the rest of the proof.
The bound $nl_{s,r}(F) \le 2^{n-s}$ is asymptotically almost tight (in a sense which will be specified in Proposition 40, page 146) for permutations when $r \le s \le .227\,n$.

Existence of permutations with higher-order nonlinearities bounded from below

The case of permutations is more interesting and useful than that of general functions when dealing with higher-order nonlinearity, but it is more delicate.

Proposition 39 *Let n and s be positive integers and let r be a non-negative integer. Let D be the greatest integer such that*

$$\sum_{t=0}^{D} \binom{2^n}{t} \le \frac{\binom{2^n}{2^{n-s}}}{2^{\sum_{i=0}^{s}\binom{n}{i} + \sum_{i=0}^{r}\binom{n}{i}}}.$$

There exist (n,n)-permutations F whose higher-order nonlinearity $nl_{s,r}(F)$ is strictly larger than D.

Proof. We recall the proof from [233]. Given a number D, a permutation F of $\mathbb{F}_2^n$ and two n-variable Boolean functions f and g, let us consider the support $E = supp((f \circ F) \oplus g)$, that is, $E = (F^{-1}(supp(f)))\,\Delta\, supp(g)$, where Δ is the symmetric difference operator. Then F^{-1} maps $supp(f)$ onto $supp(g)\,\Delta\, E$ (since the equality $1_E = f \circ F \oplus g$ implies $f \circ F = g \oplus 1_E$) and $\mathbb{F}_2^n \setminus supp(f)$ onto $(\mathbb{F}_2^n \setminus supp(g))\,\Delta\, E$. If we have $d_H(f \circ F, g) \leq D$ then E has size at most D. For every integers $i \in [0, 2^n]$ and r, let us denote by $A_{r,i}$ the number of codewords of Hamming weight i in the Reed-Muller code of order r. If i is the size of $supp(f)$ (with $0 < i < 2^n$, since $f \neq cst$), then for every set E such that $|supp(g)\,\Delta\, E| = |supp(f)| = i$ and $|(\mathbb{F}_2^n \setminus supp(g))\,\Delta\, E| = |\mathbb{F}_2^n \setminus supp(f)| = 2^n - i$, the number of permutations whose restriction to $supp(f)$ is a one-to-one function onto $supp(g)\,\Delta\, E$ and whose restriction to $\mathbb{F}_2^n \setminus supp(f)$ is a one-to-one function onto $(\mathbb{F}_2^n \setminus supp(g))\,\Delta\, E$ equals $i!\,(2^n - i)!$. We deduce that the number of permutations F such that $nl_{s,r}(F) \leq D$ is bounded above by

$$\sum_{t=0}^{D} \binom{2^n}{t} \sum_{i=1}^{2^n-1} \sum_{j=0}^{2^n} A_{s,i} A_{r,j}\, i!\,(2^n - i)!$$

Since the non-constant codewords of the Reed-Muller code of order s have Hamming weights between 2^{n-s} and $2^n - 2^{n-s}$, we deduce that the probability $P_{s,r,D}$ that a permutation F chosen at random (with uniform probability) satisfies $nl_{s,r}(F) \leq D$ is bounded above by

$$\sum_{t=0}^{D} \binom{2^n}{t} \sum_{j=0}^{2^n} A_{r,j} \sum_{2^{n-s} \leq i \leq 2^n - 2^{n-s}} A_{s,i} \frac{i!\,(2^n - i)!}{2^n!} =$$

$$\sum_{t=0}^{D} \binom{2^n}{t} \sum_{j=0}^{2^n} A_{r,j} \sum_{2^{n-s} \leq i \leq 2^n - 2^{n-s}} \frac{A_{s,i}}{\binom{2^n}{i}}$$

$$< \frac{\left(\sum_{t=0}^{D} \binom{2^n}{t}\right) 2^{\sum_{i=0}^{s} \binom{n}{i} + \sum_{i=0}^{r} \binom{n}{i}}}{\binom{2^n}{2^{n-s}}}. \tag{3.32}$$

We deduce that, under the hypothesis of Proposition 39, we have $P_{s,r,D} < 1$ and there exist permutations F from $\mathbb{F}_2^n$ to itself whose higher-order nonlinearity $nl_{s,r}(F)$ is strictly larger than D. This completes the proof. □

This lemma is translated into a table for small values of n in [233]. Let us see now what happens when n tends to ∞. Let $H_2(x) = -x \log_2(x) - (1-x)\log_2(1-x)$ be the binary entropy function.

Proposition 40 *Let $\frac{s_n}{n}$ tend to a limit ρ such that $1 - H_2(\rho) > \rho$ (which is approximately equivalent to $\rho \leq .227$) when n tends to ∞. If $r_n \leq \mu\, n$ for every n, where $1 - H_2(\mu) > \rho$* (e.g. *if r_n/s_n tends to a limit strictly smaller than 1), then for every $\rho' > \rho$, almost all permutations F of $\mathbb{F}_2^n$ satisfy $nl_{s_n,r_n}(F) \geq 2^{(1-\rho')n}$.*

Proof. We recall the proof from [233]. We know (see *e.g.* [809, page 310]) that, for every integer n and every $\lambda \in [0, 1/2]$, we have $\sum_{i \le \lambda n} \binom{n}{i} \le 2^{nH_2(\lambda)}$. According to the Stirling formula, we have also, when i and j tend to ∞: $i! \sim i^i e^{-i}\sqrt{2\pi i}$ and $\binom{i+j}{i} \sim \frac{(\frac{i+j}{i})^i(\frac{i+j}{j})^j}{\sqrt{2\pi}}\sqrt{\frac{i+j}{ij}}$. For $i + j = 2^n$ and $i = 2^{n-s_n}$, this gives

$$\begin{aligned}\binom{2^n}{2^{n-s_n}} &\sim \frac{(2^{s_n})^{2^{n-s_n}}}{\sqrt{2\pi}(1-2^{-s_n})^{2^n-2^{n-s_n}}}\sqrt{\frac{2^{s_n}}{2^n-2^{n-s_n}}}\\ &= \frac{2^{s_n 2^{n-s_n}}}{\sqrt{2\pi}\, 2^{(2^n-2^{n-s_n})\ln(1-2^{-s_n})\log_2 e}}\sqrt{\frac{2^{s_n}}{2^n-2^{n-s_n}}}.\end{aligned}$$

We deduce then from Inequality (3.32), page 146:

$$\begin{aligned}\log_2 P_{s_n,r_n,D_n} = O\Bigg(2^n \bigg[H_2\left(\frac{D_n}{2^n}\right) &+ 2^{-n(1-H_2(s_n/n))} + 2^{-n(1-H_2(r_n/n))}\\ &-2^{-s_n+\log_2(s_n)} - 2^{-s_n}(1-2^{-s_n})\log_2 e\bigg]\Bigg)\end{aligned}$$

(we omit $-\frac{s_n}{2^{n+1}} + \frac{n}{2^{n+1}}\log_2(1-2^{-s_n})$ inside the brackets above; it is negligible). If $\lim \frac{s_n}{n} = \rho$ where $1-H_2(\rho) > \rho$, then there exists $\rho' > \rho$ such that $1-H_2(\rho') > \rho'$ and such that asymptotically we have $s_n \le \rho' n$; hence $2^{-n(1-H_2(s_n/n))}$ is negligible with respect to 2^{-s_n}. And if $r_n \le \mu n$ where $1 - H_2(\mu) > \rho$, then we have $2^{-n(1-H_2(r_n/n))} = o(2^{-s_n})$ and for $D_n = 2^{(1-\rho')n}$ where ρ' is any number strictly larger than ρ, we have $H_2\left(\frac{D_n}{2^n}\right) = H_2\left(2^{-\rho' n}\right) = \rho' n 2^{-\rho' n} - (1 - 2^{-\rho' n})\log_2(1-2^{-\rho' n}) = o(2^{-\rho n}) = o(2^{-s_n})$. We obtain that, asymptotically, $nl_{s_n,r_n}(F) > 2^{(1-\rho')n}$ for every $\rho' > \rho$. □

The inverse S-box

For $F_{inv}(x) = x^{2^n-2}$ and $f_{inv}(x) = tr_n(F_{inv}(x))$, we have $nl_r(F_{inv}) = nl_r(f_{inv})$ as for any power permutation. Recall that, for $r = 1$, this parameter equals $2^{n-1} - 2^{\frac{n}{2}}$ when n is even and is close to this number when n is odd, and that for $r > 1$, it is approximately bounded below by $2^{n-1} - 2^{(1-2^{-r})n}$ (see more in [232]). We have $NL_2(F_{inv}) = 0$, since we have $w_H(h(x, F_{inv}(x))) = 0$ for the bilinear function $h(x, y) = tr_n(axy)$ where a is any nonzero element of null trace and xy denotes the product of x and y in $\mathbb{F}_{2^n}$. Indeed we have $x F_{inv}(x) = 1$ for every nonzero x. As observed in [392], we have also $w_H(h(x, F_{inv}(x))) = 0$ for the bilinear functions $h(x, y) = tr_n(a(x + x^2y))$ and $h(x, y) = tr_n(a(y + y^2x))$ where a is now any nonzero element, and for the quadratic functions $h(x, y) = tr_n(a(x^3 + x^4y))$ and $h(x, y) = tr_n(a(y^3 + y^4x))$. These properties are the core properties used in the tentative algebraic attack on the AES by Courtois and Pieprzyk.

It is proved in [233] that, for every ordered pair (s, r) of strictly positive integers, we have:

- $nl_{s,r}(F_{inv}) = 0$ if $r + s \ge n$;
- $nl_{s,r}(F_{inv}) > 0$ if $r + s < n$;

and that, in particular, for every ordered pair (s, r) of positive integers such that $r + s = n - 1$, we have $nl_{s,r}(F_{inv}) = 2$. The other values are unknown when $r + s < n$, except for small values of n.

3.2.4 Algebraic immunities of vectorial functions

Algebraic attacks can be performed on stream ciphers and on block ciphers; this is why we address the algebraic immunities of vectorial functions in the present section. But there are several definitions and the relevant ones are not the same in both frameworks. Algebraic attacks can be applied to those stream ciphers which, for increasing the speed, use as combiners or filters vectorial (n, m)-functions F instead of single-output Boolean functions. Figures 3.2 and 3.3 below display how vectorial functions can be used in the pseudorandom generators of stream ciphers to speed up the ciphers.

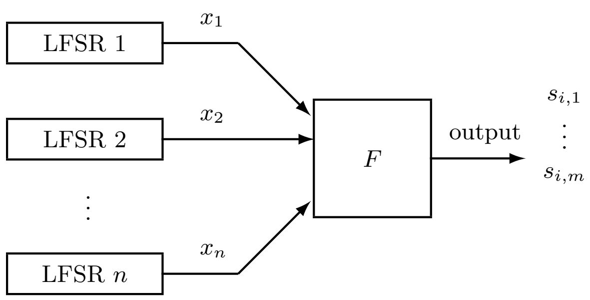


Figure 3.2 *combiner model*

The output bits of F can be combined in any way, that is by applying any m-variable Boolean function h, and the algebraic attack can be performed on the combiner or filter model using the resulting Boolean function $h \circ F$. The minimum algebraic immunity of all these functions clearly equals the minimum algebraic immunity of the indicators of the pre-images $F^{-1}(z)$ for $z \in \mathbb{F}_2^m$. This will lead to Definition 34.

Algebraic attacks also exist on block ciphers (see [392]), exploiting the existence of multivariate equations involving the input x to the S-box and its output y. In the case of the AES, whose S-box is the power function $x \in \mathbb{F}_{2^8} \to x^{2^8-2} \in \mathbb{F}_{2^8}$, an example of such equation is $x^2 y = x$, where $x, y \in \mathbb{F}_{2^8}$. The main parameter playing a role in the complexity of algebraic attacks, to be studied for a given S-box F in a cipher, is the lowest algebraic degree d of Boolean relations between inputs and ouputs to F. If these are viewed in $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$, the simplest relations to be considered are of the form $\sum_{I\subseteq\{1,\dots,n\}, J\subseteq\{1,\dots,m\}} a_{I,J}\, x^I (F(x))^J = 0$; $a_{I,J} \in \mathbb{F}_2$. Another parameter is the number of linearly independent relations of degree d. Since, for an (n, m)-function, the number of unknowns $a_{I,J}$ in the equations above equals $\sum_{i=0}^{d} \binom{n+m}{i}$ and the number of equations is 2^n, the number of

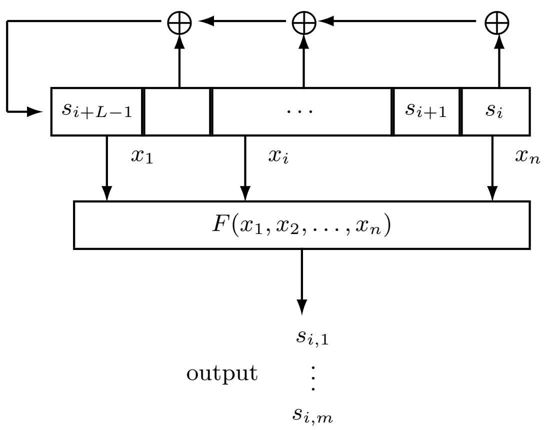


Figure 3.3 *filter model*

linearly independent relations of degree d is at least $\sum_{i=0}^{d} \binom{n+m}{i} - 2^n$.
But the actual efficiency of algebraic attacks on block ciphers is difficult to study. The global number of variables in the large system of equations expressing the whole cipher, that is, the number of data bits and key bits in all the rounds of the cipher, is much larger than for stream ciphers and the resulting systems of equations are not as overdefined as for stream ciphers; nobody is able to predict correctly the complexity of solving such polynomial systems. The AES allowing bilinear relations between the input and the output bits of the S-boxes, this may represent a threat, if an idea is found which would reduce the number of unknowns without increasing too much the degrees of the equations. In [392] was written that "it is not completely unreasonable to believe, that the structure of Rijndael and Serpent could allow attacks with complexity growing slowly with the number of rounds" and the authors added "In this paper, it seems that we have found such an attack" but it is widely believed today that such attack is not efficient on these two cryptosystems.

Several notions of *algebraic immunity* of vectorial functions have been studied in [29, 32]. We first need to recall the definition of annihilator and give the definition of the algebraic immunity of a set:

Definition 33 *We call* annihilator *of a subset E of $\mathbb{F}_2^n$ any n-variable Boolean function vanishing on E. We call algebraic immunity of E, and we denote by $AI(E)$, the minimum algebraic degree of all the nonzero annihilators of E.*

Note that the algebraic immunity of a Boolean function f equals by definition

$\min(AI(f^{-1}(0)), AI(f^{-1}(1)))$.
The first generalization of algebraic immunity to *S-boxes* is its direct extension:

Definition 34 *The* basic algebraic immunity *of an* (n,m)*-function* F *is defined as:*

$$AI(F) = \min\{AI(F^{-1}(z));\ z \in \mathbb{F}_2^m\}.$$

Note that $AI(F)$ also equals the minimum algebraic immunity of all the *indicators* φ_z of the pre-images $F^{-1}(z)$ since, the algebraic immunity being a non-decreasing function over sets, we have $AI(\mathbb{F}_2^n \setminus F^{-1}(z)) \geq AI(F^{-1}(z'))$ for every distinct $z, z' \in \mathbb{F}_2^m$.
This version of algebraic immunity is relevant to stream ciphers. A second notion of algebraic immunity of S-boxes, more relevant to S-boxes in block ciphers, has been called the *graph algebraic immunity* and is defined as follows:

Definition 35 *The graph algebraic immunity of an* (n,m)-function F *is the algebraic immunity of the graph* $\{(x, F(x));\ x \in \mathbb{F}_2^n\}$ *of the S-box, and is denoted by* $AI_{gr}(F)$.

Two other notions studied in [32] are essentially different expressions for the same $AI(F)$ and $AI_{gr}(F)$.
A third notion seems also natural:

Definition 36 *The* component algebraic immunity *of an* (n,m)*-function* F *is defined as:*

$$AI_{comp}(F) = \min\{AI(v \cdot F);\ v \in \mathbb{F}_2^m \setminus \{0_m\}\}.$$

Properties and relative bounds
It has been observed in [29] that, for any (n,m)-function F, we have $AI(F) \leq AI_{gr}(F) \leq AI(F) + m$. The left-hand side inequality is straightforward (by restricting an annihilator of the graph to a value of y such that the annihilator does not vanish) and is shown tight in [235], and the right-hand side inequality comes from the fact that, since there exists z and a nonzero annihilator $g(x)$ of $F^{-1}(z)$ of algebraic degree $AI(F)$, the function $g(x)\prod_{i=1}^m (y_j \oplus z_j \oplus 1)$ is an annihilator of algebraic degree $AI(F) + m$ of the graph of F.
It has been also observed in [29] that, denoting by d the smallest integer such that $\sum_{i=0}^d \binom{n}{i} > 2^{n-m}$, we have $AI(F) \leq d$ (indeed, there is at least one z such that $|F^{-1}(z)| \leq 2^{n-m}$; the annihilators of $F^{-1}(z)$ are the solutions of $|F^{-1}(z)|$ linear equations in $\sum_{i=0}^d \binom{n}{i}$ unknowns - which are the coefficients in the *ANF* of an unknown annihilator of algebraic degree at most d - and the number of equations being strictly smaller than the number of unknowns, the system must have non-trivial solutions). It has been proved in [500] that this bound is tight. Note that it shows that for having a chance that $AI(F)$ be large, we need m small enough: we know (see [809, page 310]) that $\sum_{i=0}^d \binom{n}{i} \geq \frac{2^{nH_2(d/n)}}{\sqrt{8d(1-d/n)}}$, where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$; for $AI(F)$ being possibly larger than a

number k, we must have $\sum_{i=0}^{k}\binom{n}{i} \leq 2^{n-m}$, and therefore $\frac{2^{nH_2(k/n)}}{\sqrt{8k(1-k/n)}} \leq 2^{n-m}$, that is, $m \leq n\,(1 - H_2(k/n)) + \frac{1}{2}\,(3 + \log_2(k(1 - k/n)))$. It also implies that $AI(F) \leq n - m$, see more in [235].
Finally, it has also been proved in [29] that, denoting by D the smallest integer such that $\sum_{i=0}^{D}\binom{n+m}{i} > 2^n$, we have $AI_{gr}(F) \leq D$ (the proof is similar, considering annihilators in $n + m$ variables of the graph) but it is not known whether this bound is tight (it is shown in [29] that it is tight for $n \leq 14$). This implies that $AI_{gr}(F) \leq n$, see more in [235].

Since the algebraic immunity of any Boolean function is bounded above by its algebraic degree, the component algebraic immunity of any vectorial function is bounded above by its minimum degree and therefore by its algebraic degree:

$$AI_{comp}(F) \leq d_{alg}(F).$$

We have also:

$$AI_{comp}(F) \geq AI(F),$$

since $AI_{comp}(F)$ equaling the algebraic immunity of the Boolean function $v \cdot F$ for some $v \neq 0_m$, it equals $AI(F^{-1}(H))$ for some affine hyperplane H of $\mathbb{F}_2^m$, and AI is a non-decreasing function over sets. We have:

$$AI_{comp}(F) \geq AI_{gr}(F) - 1$$

since:
- if g is a nonzero annihilator of $v \cdot F$, $v \neq 0_m$, then the product $h(x, y) = g(x)\,(v \cdot y)$ is a nonzero annihilator of the graph of F;
- if g is a nonzero annihilator of $v \cdot F \oplus 1$ then $h(x, y) = g(x)\,(v \cdot y) \oplus g(x)$ is a nonzero annihilator of the graph of F.
More bounds on these three parameters are given in [235].

Remark As in the case of Boolean functions, see Subsection 3.1.6, page 117, the variants of these parameters (and of the ones to come in the next sections) in relationship with guess and determine attacks should be studied as well. □

3.3 Cryptographic criteria and parameters for vectorial functions in stream ciphers

3.3.1 Correlation immunity and resiliency of vectorial functions

The notion of resilient Boolean function, when extended to vectorial functions, is relevant in cryptology to quantum cryptographic key distribution (see [58]) and to stream ciphers with multi-output combiners or filters.
Recall that an (n, m)-function is called balanced if the distribution of $F(x)$ when x ranges over $\mathbb{F}_2^n$ is uniform over $\mathbb{F}_2^m$.

Definition 37 *Let n and m be two positive integers. Let t be an integer such that $0 \leq t \leq n$. An (n,m)-function $F(x)$ is called t-th order* correlation immune *if its output distribution does not change when at most t coordinates x_i of x are kept constant. It is called t-*resilient *if it is balanced and t-th order correlation immune, that is if it stays balanced when at most t coordinates x_i of x are kept constant.*

This notion has a relationship with another notion which plays also a role in cryptography: an (n,m)-function F is called a *multipermutation* (see [1095]) if any two ordered pairs $(x, F(x))$ and $(x', F(x'))$, such that x, $x' \in \mathbb{F}_2^n$ are distinct, differ in at least $m+1$ distinct positions (that is, collide in at most $n-1$ positions); such (n,m)-function ensures then a perfect diffusion; an (n,m)-function is a multipermutation if and only if the indicator of its graph $\{(x, F(x));\, x \in \mathbb{F}_2^n\}$ is an n-th order correlation immune Boolean function (see [179]).

Since *S-boxes* must be balanced, we shall focus on resilient functions, but most of the results below can also be stated for correlation immune functions.
We call an (n,m) function which is t-resilient an (n,m,t)-function. Clearly, if such a function exists, then $m \leq n-t$, since balanced (n,m)-functions can exist only if $m \leq n$. This bound is weak (it is tight if and only if $m=1$ or $t=1$). It is shown in [370] (see also [79]) that, if an (n,m,t)-function exists, then $m \leq n - \log_2\left[\sum_{i=0}^{t/2}\binom{n}{i}\right]$ if t is even and $m \leq n - \log_2\left[\binom{n-1}{(t-1)/2} + \sum_{i=0}^{(t-1)/2}\binom{n}{i}\right]$ if t is odd. This can be deduced from the bound on orthogonal arrays due to Rao [988], see page 106. But, as shown in [79] (see also [760]), potentially better bounds can be deduced from the linear programming bound due to Delsarte [421]: if an (n,m,t)-function exists, then $t \leq \left\lfloor \frac{2^{m-1}n}{2^m-1}\right\rfloor - 1$ and $t \leq 2\left\lfloor \frac{2^{m-2}(n+1)}{2^m-1}\right\rfloor - 1$.
Note that composing a t-resilient (n,m)-function by a permutation on $\mathbb{F}_2^m$ does not change its resiliency order (this obvious result was first observed in [1168]). Also, the t-resiliency of S-boxes can be expressed by means of the t-resiliency and t-th order correlation immunity of Boolean functions:

Proposition 41 *Let n and m be two positive integers and $0 \leq t \leq n$. Let F be an (n,m) function. Then F is t-resilient if and only if one of the following conditions is satisfied:*
1. for every nonzero vector $v \in \mathbb{F}_2^m$, the Boolean function $v \cdot F(x)$ is t-resilient, that is, $W_F(u,v) = 0$, for every $u \in \mathbb{F}_2^n$ such that $w_H(u) \leq t$,
2. for every balanced m-variable Boolean function g, the n-variable Boolean function $g \circ F$ is t-resilient, that is, $\sum_{x\in\mathbb{F}_2^n}(-1)^{g(F(x))\oplus u\cdot x} = 0$, for every $u \in \mathbb{F}_2^n$ such that $w_H(u) \leq t$,
3. for every vector $b \in \mathbb{F}_2^m$, the Boolean function $\varphi_b = \delta_{\{b\}} \circ F$ is t-th order correlation immune and has Hamming weight 2^{n-m}.

Proof. We prove that the t-resiliency of F implies Condition 2, which implies Condition 1, which implies Condition 3, which implies that F is t-resilient.
- If F is t-resilient, then, for every balanced m-variable Boolean function g, the function $g \circ F$ is t-resilient, by definition; hence Condition 2 is satisfied.

- Condition 2 clearly implies Condition 1, since the function $g(x) = v \cdot x$ is balanced for every nonzero vector v.
- If Condition 1 is satisfied, then Relation (3.16), page 134, implies that, for every nonzero vector $u \in \mathbb{F}_2^n$ such that $w_H(u) \leq t$ and for every $b \in \mathbb{F}_2^m$, we have $\widehat{\varphi_b}(u) = 2^{-m} \sum_{x \in \mathbb{F}_2^n, v \in \mathbb{F}_2^m} (-1)^{v \cdot (F(x)+b) \oplus u \cdot x} = 0$, and φ_b is t-th order correlation immune for every b. Also, according to Proposition 35, page 134, Condition 1 implies that F is balanced, *i.e.* φ_b has Hamming weight 2^{n-m}, for every b. These two conditions obviously imply, by definition, that F is t-resilient. □
Consequently, the t-resiliency of vectorial functions is invariant under the same transformations as for Boolean functions.

3.3.2 Unrestricted nonlinearity of vectorial functions

The classical notions of nonlinearity of vectorial functions (Definition 29, page 139) and higher-order nonlinearity (Definition 32, page 144), have been introduced in the framework of block ciphers: due to the iterative structure of these ciphers, the knowledge of a function f such that $nl(f \circ F)$ or $nl_r(f \circ F)$ is low does not necessarily lead to an attack, unless the algebraic degree of f is low, and r is low too in the latter case. This is why, in Definition 32, the algebraic degree of f is also specified.
On the contrary, the structure of pseudorandom generators in stream ciphers is not iterative, and all of the m output bits of the (n, m)-function used as combiner or filter can be combined by a linear or nonlinear (but non-constant) m-variable Boolean function f to perform (fast) correlation attacks. Consequently, a second generalization to (n, m)-functions of the notion of nonlinearity has been introduced (in [318], directly related to the Zhang-Chan attack [1156]).

Definition 38 *Let F be an (n, m)-function. The* unrestricted nonlinearity *of F, denoted by $unl(F)$, is the minimum Hamming distance between all non-constant* affine function*s and all Boolean functions $g \circ F$, where g is a non-constant Boolean function in m variables.*

If $unl(F)$ is small, then one of the linear or nonlinear (non-constant) combinations of the output bits of F has high correlation to a non constant affine function of the input, and a (fast) correlation attack is feasible.

Remark.
1. In Definition 38, the considered affine functions are non-constant, because the minimum distance between all Boolean functions $g \circ F$ (g non-constant) and all constant functions equals $\min_{b \in \mathbb{F}_2^m} |F^{-1}(b)|$ (each number $|F^{-1}(b)|$ is indeed equal to the distance between the null function and $g \circ F$, where g equals the indicator of the singleton $\{b\}$); it is therefore an indicator of the balancedness of F. It is bounded above by 2^{n-m} (and it equals 2^{n-m} if and only if F is balanced).
2. We can replace "non constant affine functions" by "nonzero linear functions"

in the statement of Definition 38 (replacing g by $g \oplus 1$, if necessary).
3. Thanks to the fact that the affine functions considered in Definition 38 are non-constant, we can relax the condition that g is non-constant: the distance between a constant function and a non-constant affine function equals 2^{n-1}, and $unl(F)$ is clearly always smaller than 2^{n-1}. □

The unrestricted nonlinearity of any (n,m)-function F is obviously unchanged when F is right-composed with an affine invertible mapping. Moreover, if A is a surjective linear (or affine) function from $\mathbb{F}_2^p$ (where p is some positive integer) into $\mathbb{F}_2^n$, then it is easily shown that $unl(F \circ A) = 2^{p-n} unl(F)$. Also, for every (m,p)-function ϕ, we have $unl(\phi \circ F) \geq unl(F)$ (indeed, the set $\{g \circ \phi,\, g \in \mathcal{BF}_p\}$, where $\mathcal{BF}_p$ is the set of p-variable Boolean functions, is included in $\mathcal{BF}_m$), and if ϕ is a permutation on $\mathbb{F}_2^m$, then we have $unl(\phi \circ F) = unl(F)$ (by applying the inequality above to $\phi^{-1} \circ F$).

A further generalization of the Zhang-Chan attack, called the *generalized correlation attack* has been introduced in [299]: considering implicit equations which are linear in the input variable x and of any degree in the output variable $z = F(x)$, the following probability is considered, for any non-constant function g and every functions $w_i : \mathbb{F}_2^m \to \mathbb{F}_2$:

$$\mathrm{Prob}\,[g(z) + w_1(z)\,x_1 + w_2(z)\,x_2 + \cdots + w_n(z)\,x_n = 0], \qquad (3.33)$$

where $z = F(x)$ and where x uniformly ranges over $\mathbb{F}_2^n$.
The knowledge of such approximation g with a probability significantly higher than $1/2$ leads to an attack, because $z = F(x)$ corresponding to the output keystream is known, and therefore $g(z)$ and $w_i(z)$ are known for all $i = 1, \ldots, n$. This led to a new notion of generalized nonlinearity:

Definition 39 *Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$. The* generalized Hadamard transform *$\hat{F} : (\mathbb{F}_2^{2^m})^{n+1} \to \mathbb{R}$ is defined as:*

$$\hat{F}(g(\cdot), w_1(\cdot), \ldots, w_n(\cdot)) = \sum_{x \in \mathbb{F}_2^n} (-1)^{g(F(x)) + w_1(F(x))\,x_1 + \cdots + w_n(F(x))\,x_n},$$

where the input is in $\mathcal{BF}_m^{n+1}$.
Let $\mathcal{W}$ be the set of all n-tuple functions $w(\cdot) = (w_1(\cdot), \ldots, w_n(\cdot)) \in \mathcal{BF}_m^n$, where $w(z) \neq 0_n$ for all $z \in \mathbb{F}_2^m$.
The generalized nonlinearity *is defined as:*

$$gnl(F) = \min\Big\{ \min_{0 \neq u \in \mathbb{F}_2^m} \Big(w_H(u \cdot F), 2^n - w_H(u \cdot F) \Big), nl_{gen}F \Big\},$$

where

$$nl_{gen}F = 2^{n-1} - \frac{1}{2} \max_{g \in \mathcal{BF}_m, w \in \mathcal{W}} \hat{F}(g(\cdot), w_1(\cdot), \ldots, w_n(\cdot)). \qquad (3.34)$$

The generalized nonlinearity can be much smaller than the other nonlinearity measures and provides linear approximations with better bias for (fast) correlation attacks.

Relations to the Walsh transforms and lower bounds

The unrestricted nonlinearity of F can be related to the values of the *Fourier-Hadamard transforms* of the functions $\varphi_b = 1_{\{b\}} \circ F$ (see page 134), and a lower bound (observed in [1156]) depending on $nl(F)$ can be directly deduced:

Proposition 42 *For every (n, m)-function, we have*

$$unl(F) = 2^{n-1} - \frac{1}{2} \max_{u \in \mathbb{F}_2^n \setminus \{0_n\}} \sum_{b \in \mathbb{F}_2^m} |\widehat{\varphi_b}(u)| \geq 2^{n-1} - 2^{m/2} \left(2^{n-1} - nl(F)\right). \tag{3.35}$$

This bound does not give an idea of the best possible unrestricted nonlinearities: even if $nl(F)$ is close to the nonlinearity of bent functions $2^{n-1} - 2^{\frac{n}{2}-1}$, it implies that $unl(F)$ is approximately larger than $2^{n-1} - 2^{\frac{n+m}{2}-1}$, whereas there exist balanced $(n, \frac{n}{2})$-functions F such that $unl(F) = 2^{n-1} - 2^{\frac{n}{2}}$ (see below).

Proposition 43 *[299] Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ and let $w(\cdot)$ denote the n-tuple of m-bit Boolean functions $(w_1(\cdot), \ldots, w_n(\cdot))$. Then*

$$\begin{aligned} nl_{gen} F &= 2^{n-1} - 1/2 \sum_{z \in \mathbb{F}_2^m} \max_{w(z) \in \mathbb{F}_2^n \setminus \{0_n\}} |\widehat{\varphi_b}(w(z))| \\ &= 2^{n-1} - \frac{1}{2^{m+1}} \sum_{z \in \mathbb{F}_2^m} \max_{\substack{0 \neq w(z) \in \\ \mathbb{F}_2^n}} \left| \sum_{v \in \mathbb{F}_2^m} (-1)^{v \cdot z} W_F(w(z), v) \right|, \end{aligned}$$

where W_F denotes the Walsh transform. Hence

$$gnl(F) \geq 2^{n-1} - (2^m - 1) \left(2^{n-1} - nl(F)\right).$$

Upper bounds

If F is balanced, the minimum distance between the component functions $v \cdot F$ and the affine functions cannot be achieved by constant affine functions, because $v \cdot F$, which is balanced, has distance 2^{n-1} to constant functions. Hence:

Proposition 44 (*covering radius bound*) *For every balanced S-box F:*

$$unl(F) \leq nl(F) \leq 2^{n-1} - 2^{\frac{n}{2}-1}. \tag{3.36}$$

Another upper bound:

$$unl(F) \leq 2^{n-1} - \frac{1}{2} \left(\frac{2^{2m} - 2^m}{2^n - 1} + \sqrt{\frac{2^{2n} - 2^{2n-m}}{2^n - 1} + \left(\frac{2^{2m} - 2^m}{2^n - 1} - 1 \right)^2 - 1} \right)$$

has been obtained in [318]. It improves upon (*i.e.* is lower than) the covering radius bound only for $m \geq \frac{n}{2} + 1$, and the question of knowing whether it is possible to improve upon the covering radius bound for $m \leq \frac{n}{2}$ is open. In any case, this improvement will not be dramatic, at least for $m = \frac{n}{2}$, since it is shown (by using Relation (3.35)) in this same paper that the balanced function $F(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ x & \text{if } y = 0 \end{cases}$ satisfies $unl(F) = 2^{n-1} - 2^{\frac{n}{2}}$ (see other examples of S-boxes in [698], whose unrestricted nonlinearities seem low, however). It is pretty astonishing that an S-box with such high unrestricted nonlinearity exists; but it can be shown that this balanced function does not contribute to a good resistance to algebraic attacks and has null generalized nonlinearity (see below).

Proposition 45 *Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$. Then the following inequality holds.*

$$nl_{gen}F \leq 2^{n-1} - \frac{1}{4} \sum_{z \in \mathbb{F}_2^m} \sqrt{\frac{2^{n+2}|F^{-1}(z)| - 4|F^{-1}(z)|^2}{2^n - 1}}.$$

Furthermore if $F(x)$ is balanced, then we have:

$$gnl(F) \leq 2^{n-1} - 2^{n-1}\sqrt{\frac{2^m - 1}{2^n - 1}}.$$

It is proved in [300] that the balanced function $F(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0 \\ x & \text{if } y = 0 \end{cases}$ has null generalized nonlinearity. Hence, a vectorial function may have very high unrestricted nonlinearity and have zero generalized nonlinearity. Some functions with good generalized nonlinearity are given in [300]:

1. $F(x) = tr_m^n(x^k)$ where $k = 2^i + 1$, $\gcd(i,n) = 1$, is a Gold exponent;
2. $F(x) = tr_m^n(x^k)$ where $k = 2^{2i} - 2^i + 1$ is a Kasami exponent, $3i \equiv 1$ [mod] n,

where m divides n and n is odd, and where tr_m^n is the trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$, have generalized nonlinearity satisfying $gnl(F) \geq 2^{n-1} - 2^{(n-1)/2+m-1}$.

Power functions and sums of *power functions* represent for the designer of the cryptosystem using them the interest of being more easily computable than general functions (which makes possible using them with more variables while keeping a good efficiency). Power functions have the peculiarity that, denoting the set $\{x^d;\ x \in \mathbb{F}_{2^n}^*\}$ by U, two functions $tr_n(ax^d)$ and $tr_n(bx^d)$ such that $a/b \in U$ are *linearly equivalent*. It is not clear whether this is more an advantage for the designer or for the attacker of a system using such function.

3.4 Cryptographic criteria and parameters for vectorial functions in block ciphers

We have seen in Subsection 3.2.3 a first example of the role played by *S-box*es in the robustness of the block ciphers in which they are involved, and of how the main attacks on block ciphers result in design criteria for the S-boxes they implement. We shall see now a second example, whose importance is comparable.

3.4.1 Differential uniformity

The *differential attack*, introduced by Biham and Shamir [82] (but which was already known since 1974 by IBM and still earlier by the NSA and kept secret), is anterior to the linear attack. It assumes the existence of ordered pairs (α, β), $\alpha \neq 0$, of binary strings of the same length as the blocks (which are binary strings too), such that, a block m of plaintext being randomly chosen and c and c' being the ciphertexts related to m and $m+\alpha$, the bitwise difference $c+c'$ (recall that $+$ denotes the bitwise addition/difference in $\mathbb{F}_2^n$) has a larger probability to equal β than if c and c' were randomly chosen binary strings. Such an ordered pair (α, β) corresponding to a *bias in the output distribution* is called a *differential* and can be exploited in differential attacks; the larger the probability of the differential, the more efficient the attack. As for the linear attack, there are several ways to mount such *differential cryptanalysis*. The most common (and most efficient) is to use differentials for the *reduced cipher* (see Figure 3.1, page 138). The existence of a differential allows to distinguish, in a *last round attack*, the reduced cipher output from a random permutation. The existence of such *distinguisher* allows recovering the key used in the last round, by an exhaustive search, which is efficient if this key is shorter than the master key, or by using specificities of the cipher allowing replacing the exhaustive search by, for instance, solving algebraic equations.
Here also, we describe the attack in the case of exhaustive search, which is simpler to describe. Similarly to what we have seen at page 138, the attacker, who knows a number of pairs (plaintext, ciphertext), corresponding to the original cipher and of the form (m, c) and $(m + \alpha, c')$ where (α, β) is a differential for the reduced cipher, visits all possible last round keys. For each try of such a candidate as last round key, he inverts the last round and obtains in the case of the correct key guess the output of the reduced cipher; he observes then the statistical bias of the differential. In all the other cases (incorrect guesses), the obtained binary string is considered as random, with no observable bias. The number of pairs (m, c) and $(m + \alpha, c')$ which are known to him/her needs then to be large enough to distinguish the bias. This number depends on how the probability of the differential is larger than for a random pair. In the case of DES, the number was 2^{47} (which is huge and made the attack impractical).
The existence of differential attacks leads to a criterion on (n, m)-functions F, when used as S-boxes in the round functions of the cipher, which corresponds to minimizing the possibilities for the attacker to find differentials whose probability is large. Since the differentials cannot be determined by direct computer investigation and must then be approximately evaluated by "chaining" differentials inside each round, the criterion is that the output of the derivatives $D_aF(x) = F(x) + F(x + a)$; $x, a \in \mathbb{F}_2^n$, $a \neq 0_n$, be as uniformly distributed as possible. This leads to the following parameter.

Definition 40 *[906, 912, 907] Let n, m, δ be positive integers. An (n, m)-function F is called* differentially δ-uniform *if, for every nonzero $a \in \mathbb{F}_2^n$ and* *every $b \in \mathbb{F}_2^m$, the equation $F(x) + F(x + a) = b$ has at most δ solutions. The minimum of those values δ having such property, that is, the maximum number of solutions of such equations, is denoted by δ_F and called the* differential uniformity *of F.*

The differential uniformity δ_F is necessarily even since the solutions of equation $D_aF(x) = b$ go by pairs: if x is a solution of $F(x) + F(x + a) = b$ then $x + a$ is also a solution. The lower is δ_F, the better is the contribution of the S-box to the resistance to the differential attack, as shown in [912, 908]. The differential uniformity δ_F of any (n, m)-function F is bounded below by 2^{n-m} (as observed by Nyberg) since D_aF being an (n, m)-function, at least one element of $\mathbb{F}_2^m$ has at least 2^{n-m} pre-images by D_aF. The differential uniformity equals 2^{n-m} if and only if every derivative D_aF, $a \neq 0_n$, is balanced. We say then that F is *perfect nonlinear* and we shall see in Chapter 6 that this is equivalent to saying that F is bent. According to a result from Nyberg that we shall see in Proposition 104, page 296, (n, m)-functions have differential uniformity strictly larger than 2^{n-m} when n is odd or $m > n/2$.

Differential uniformity is in fact a notion on the graph $\mathcal{G}_F = \{(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^m;\ y = F(x)\}$ of the function: it is the maximum number of solutions $(X, Y) \in \mathcal{G}_F^2$ of the equation $X + Y = (a, b)$ when $(a, b) \in (\mathbb{F}_2^n \setminus \{0_n\}) \times \mathbb{F}_2^m$. For this reason, differential uniformity is a *CCZ invariant* (see Definition 5, page 45). The differential uniformity of an S-box being determined, its differential spectrum also affects the security of the corresponding cipher. The *differential spectrum* is the multiset of the values:

$$\delta_F(a, b) = |\{x \in \mathbb{F}_2^n;\ D_aF(x) = F(x)+F(x+a) = b\}| = (1_{\mathcal{G}_F} \otimes 1_{\mathcal{G}_F})(a, b), \quad (3.37)$$

(where $1_{\mathcal{G}_F}$ is the graph indicator of F, see page 51) and the *difference distribution table* (*DDT*) is the table which displays them (note that, given a permutation F, all these data are the same for F and F^{-1}, up to exchanging a and b, since $1_{\mathcal{G}_F}(x, y) = 1_{\mathcal{G}_{F^{-1}}}(y, x)$).
For every $u \in \mathbb{F}_2^n$ and $v \in \mathbb{F}_2^m$, we have $\sum_{a\in\mathbb{F}_2^n, b\in\mathbb{F}_2^m} \delta_F(a, b)(-1)^{u\cdot a\oplus v\cdot b} = \sum_{a,x\in\mathbb{F}_2^n} (-1)^{u\cdot a\oplus v\cdot D_aF(x)} = \sum_{a\in\mathbb{F}_2^n} \Delta_{v\cdot F}(a)(-1)^{u\cdot a}$, by the change of variable $b = D_aF(x)$ and since $v \cdot D_aF = D_a(v \cdot F)$, and the Wiener-Khintchine formula (2.53), page 80 (or Property (2.44), page 79 applied to expression (3.37)), shows that the Fourier transform of function δ_F equals W_F^2.
The necessary and sufficient condition, recalled from [201], that we reported at page 91, ensuring that the image by an affine permutation $A = L + (a, b)$ of the *graph* $\mathcal{G}_F$ of F is the graph of a function, is equivalent to the fact that the image of $\{0_n\} \times \mathbb{F}_2^m$ by L^{-1} is included in the set $\delta_F^{-1}(0) \cup \{(0_n, 0_m)\}$: we have $W_F(L^*(u, 0_m)) = 0$ for all $u \neq 0_n$ if and only if we have $\sum_{u\in\mathbb{F}_2^n} W_F^2(L^*(u, 0_m)) = 2^{2n}$, and according to the Poisson summation formula (2.39), page 77, this gives $\sum_{(\alpha,\beta)\in(L^*(\mathbb{F}_2^n\times\{0_m\}))^\perp} \delta_F(\alpha, \beta) = 2^n$, which proves the result since $(L^*(\mathbb{F}_2^n \times \{0_m\}))^\perp = L^{-1}(\{0_n\} \times \mathbb{F}_2^m)$, because $(\alpha, \beta) \cdot L^*(u, 0_m) = L(\alpha, \beta) \cdot (u, 0_m)$ equals

0 for every u if and only if $L(\alpha,\beta)\in\{0_n\}\times\mathbb{F}_2^m$.

It is observed in [910, 1060] that, because of the truncated differential attack [706], the differential uniformity of the (so-called chopped) functions obtained by withdrawing a few coordinate functions should also be considered and can be low for some vectorial functions having good differential uniformity.

Note that if a function has good nonlinearity then it does not have necessarily a good differential uniformity too: take an (n,m)-function F and consider the $(n+1,m)$-function F' such that $F'(x,x_{n+1})=F(x)$ for every $x\in\mathbb{F}_2^n, x_{n+1}\in\mathbb{F}_2$; the nonlinearity of F' is twice that of F and can then be rather good while the differential uniformity of F' equals 2^n and is then bad. The converse is not true either: take any (n,m)-function F and consider the $(n,m+1)$-function F' obtained by adding a null coordinate function; the nonlinearity of F' is null while the differential uniformity equals that of F and can then be good.

The asymptotic behavior of δ_F for general (n,n)-functions F has been studied in [1098], after being studied in [643] for power functions over $\mathbb{F}_{2^n}$:

Proposition 46 *[1098] For any $d>4$ with $d\equiv 0,3$ [mod 4], the limit when n to infinity of the ratio:*

$$\frac{\left|\left\{F\in\mathbb{F}_{2^n}[x];\ deg(F)=d \text{ and } \delta_F=\begin{array}{l} d-1 \text{ for } d \text{ odd}\\ d-2 \text{ for } d \text{ even}\end{array}\right\}\right|}{|\{F\in\mathbb{F}_{2^n}[x];\ deg(F)=d\}|},$$

where $deg(F)$ denotes the polynomial degree, equals 1.

For more general (n,m)-functions, see [405, 589, 913]; the average differential uniformity of (n,m)-functions is much larger than 2^{n-m}.

Almost perfect nonlinear functions
The smaller the differential uniformity, the better the contribution to the resistance against differential cryptanalysis. When $m\geq n$, the smallest possible value of δ_F (which is always even) is 2, and differentially 2-uniform functions can exist only when $m\geq n$ (indeed, we need $m\geq n-1$ and $m=n-1$ is impossible except if $n\leq 2$ since differentially 2-uniform $(n,n-1)$-functions are perfect nonlinear and we would then need to have $n-1\leq n/2$ as we shall see in Proposition 104, page 296). We use the term of APN function only when $m=n$. Note that the notion of APN function and the differential property of the multiplicative *inverse function* had been investigated starting from 1968 by V. Bashev and B. Egorov in USSR.

Definition 41 *[912, 71, 908] An (n,n)-function F is called* almost perfect nonlinear *(*APN*) if it is differentially 2-uniform, that is, if for every $a\in\mathbb{F}_2^n\setminus\{0_n\}$ and every $b\in\mathbb{F}_2^n$, the equation $F(x)+F(x+a)=b$ has 0 or 2 solutions (*i.e.

$|\{D_aF(x), x \in \mathbb{F}_2^n\}| = 2^{n-1}$*). Equivalently, for distinct elements* x, y, z, t *of* $\mathbb{F}_2^n$*, the equality* $x+y+z+t = 0_n$ *implies* $F(x)+F(y)+F(z)+F(t) \neq 0_n$*, that is, the restriction of* F *to any 2-dimensional flat (*i.e. affine plane*) of* $\mathbb{F}_2^n$ *is non-affine.*

We have already encountered APN functions when proving the SCV bound and the equivalence between these three properties is easily seen: Inequality (3.25), page 141, is an equality if and only if $F(x) + F(y) + F(z) + F(x + y + z) = 0_n$ can be achieved only when $x = y$ or $x = z$ or $y = z$ and this is equivalent to any of the following properties:
- the restriction of F to any 2-dimensional flat (*i.e.* affine plane) of $\mathbb{F}_2^n$ is non-affine, that is, does not sum up to 0_n (indeed, the set $\{x, y, z, x + y + z\}$ is a flat and it is 2-dimensional if and only if $x \neq y$ and $x \neq z$ and $y \neq z$; and $F(x)+F(y)+F(z)+F(x+y+z) = 0_n$ is equivalent to saying that the restriction of F to this flat is affine, since we know that a function F is affine on a flat A if and only if, for every x, y, z in A we have $F(x + y + z) = F(x) + F(y) + F(z)$);
- for every distinct nonzero (that is, $\mathbb{F}_2$-linearly independent) vectors a and a', the second order derivative $D_aD_{a'}F(x) = F(x)+F(x+a)+F(x+a')+F(x+a+a')$ takes only nonzero values;
- the equality $F(x) + F(x + a) = F(y) + F(y + a)$ (obtained from $F(x) + F(y) + F(z)+F(x+y+z) = 0_n$ by denoting $x+z$ by a) can be achieved only for $a = 0_n$ or $x = y$ or $x = y + a$;
- for every $a \in \mathbb{F}_2^n \setminus \{0_n\}$ and every $b \in \mathbb{F}_2^n$, the equation $D_aF(x) = F(x) + F(x + a) = b$ has at most 2 solutions (that is, 0 or 2 solutions, since if it has one solution x, then it has $x + a$ for second solution).

Remark. As in the case of AB functions, the term of *almost* perfect nonlinear gives the feeling that these functions are almost optimal while they are optimal. □

Chapter 11 covers the whole topic of APN functions.

Related nonlinearity parameters

- We have seen at page 143 the nonlinearity parameter alternative to the classical nonlinearity, equal to the maximum imbalance of the sums of F and linear functions: $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}$.

If $m = n$ and F is APN, then according to the properties seen at page 135, $Nb_{F+L} = \sum_{a\in\mathbb{F}_2^n} \left|(D_aF)^{-1}(L(a))\right| - 2^n = \sum_{a\in\mathbb{F}_2^n\setminus\{0_n\}} \left|(D_aF)^{-1}(L(a))\right|$ is bounded above by $2\,(2^n - 1)$, for every L, which implies that $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}$ lies in the interval $]2^n - 1; 2\,(2^n - 1)]$ (since we know from Proposition 37, page 142, that it is larger than or equal to $2^n - 2^{n-m}$ and we know that it cannot equal $2^n - 2^{n-m}$ since F would be bent). Moreover, when n is even, the maximum $2\,(2^n - 1)$ is achieved by all APN power functions; indeed, Dobbertin proved (and we shall see in Proposition 165, page 417) that for any APN power function F, there are $\frac{2^n-1}{3}$ elements of $\mathbb{F}_{2^n}^*$ having three pre-images each by F, and

all the other elements of $\mathbb{F}_{2^n}^*$ have no pre-image (see e.g. [237]), which implies, using (3.19), that $Nb_F = 1 + 9 \cdot \frac{2^n-1}{3} - 2^n = 2\,(2^n - 1)$.
APN functions in 5 variables have been classified under EA equivalence and CCZ equivalence in [134]. When F is the inverse function, $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}$ equals 56. There is no other APN and non-AB function for $n = 5$.
For $m = n = 6$ and $m = n = 8$, the functions CCZ equivalent to x^3, found in [162, 163] match the maximum $2\,(2^n-1)$ as do the APN power functions. We do not know if some APN functions can have smaller value of $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}$ for n even. And it is not clear to us whether $\max_{L\in\mathcal{L}_{n,m}} Nb_{F+L}$ can take diverse values when F is AB for n odd and whether it is CCZ-invariant.

- The bentness/perfect nonlinearity of a function being characterized by the balancedness of its derivatives, the following nonlinearity indicator has been introduced in [267]:

$$NB_F = \sum_{a\in\mathbb{F}_2^n\setminus\{0_n\}} Nb_{D_aF} = \sum_{a\in\mathbb{F}_2^n\setminus\{0_n\}} \sum_{b\in\mathbb{F}_2^m} |(D_aF)^{-1}(b)|^2 - (2^n-1)2^{2n-m}. \tag{3.38}$$

This indicator is directly related to Nyberg's and Chabaud-Vaudenay's results and proofs; it allows clarifying some properties found by them (see *e.g.* Relation (3.40) below) and saying a bit more. We shall call it the *derivative imbalance* of F. It has the following properties, as mentioned in [267, 239]:

- $NB_F \geq 0$, for every function F, and $NB_F = 0$ if and only if F is bent/perfect nonlinear;
- NB is *CCZ invariant* since NB_F equals $\sum_{a\in\mathbb{F}_2^n\setminus\{0_n\}} |\{(x,y)\in(\mathbb{F}_2^n)^2 \,/\, D_aF(x) = D_aF(y)\}| - (2^n-1)2^{2n-m}$ and equals therefore:
$$\left|\left\{(x,x',y,y')\in(\mathbb{F}_2^n)^4 \,/\, \begin{array}{c} x+x'=y+y'\neq 0_n \\ F(x)+F(x')=F(y)+F(y') \end{array}\right\}\right| - (2^n-1)2^{2n-m}$$
$$= \left|\left\{(X,X',Y,Y')\in\mathcal{G}_F^4 \,/\, X+X'=Y+Y'\neq 0_{n+m}\right\}\right| - (2^n-1)2^{2n-m},$$
where $\mathcal{G}_F = \{(x,F(x))\in\mathbb{F}_2^n\times\mathbb{F}_2^m\}$ is the graph of F;
- $NB_F \geq (2^n-1)(2^{n+1}-2^{2n-m})$ (this inequality comes from the Cauchy-Schwarz inequality $\sum_{b\in\mathbb{F}_2^m} \left|F^{-1}(b)\right|^2 \geq \frac{\left(\sum_{b\in\mathbb{F}_2^m}|F^{-1}(b)|\right)^2}{|Im(F)|} = \frac{2^{2n}}{|Im(F)|}$ applied to D_aF and from $|Im(D_aF)| \leq 2^{n-1}$; see an improvement in [522, Proposition 3]); note that this proves again that (n,n)-functions cannot be perfect nonlinear; there is equality if and only if, for every $a\neq 0_n$, $|Im(D_aF)|$ equals 2^{n-1} and $|(D_aF)^{-1}(b)|$ is constant for $b\in Im(D_aF)$;
for $n = m$, there is then equality if and only if F is APN;
- $NB_F = \sum_{\substack{a,a'\in\mathbb{F}_2^n \\ \text{linearly indept}}} \left|(D_aD_{a'}F)^{-1}(0_m)\right| - (2^n-1)(2^{2n-m}-2^{n+1})$;
- $NB_F \leq (2^n-1)(2^{2n}-2^{2n-m})$, for every function $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ (see a

refinement in [522, Proposition 4]) and $NB_F = (2^n-1)(2^{2n}-2^{2n-m})$ if and only if F is affine.

Remark. A parameter[25] has been introduced afterwards in [925, 922] and studied further[26] in [923, 921, 924], without comparing it to NB_F. We give here its definition for (n,m)-functions but, as NB_F, it can be defined for any function from an Abelian group to an Abelian group: the *ambiguity* $\mathcal{A}(F)$ equals:

$$\sum_{i\geq 0}\binom{i}{2}|\{(a,b)\in(\mathbb{F}_2^n\setminus\{0_n\})\times\mathbb{F}_2^m;\ |(D_aF)^{-1}(b)|=i\}|.$$

$\mathcal{A}(F)$ is the same as NB_F, up to a constant and to the multiplication by $\frac{1}{2}$:

$$\begin{aligned}\mathcal{A}(F) &= \frac{1}{2}\sum_{(a,b)\in(\mathbb{F}_2^n\setminus\{0_n\})\times\mathbb{F}_2^m}|(D_aF)^{-1}(b)|^2-\frac{1}{2}\sum_{(a,b)\in(\mathbb{F}_2^n\setminus\{0_n\})\times\mathbb{F}_2^m}|(D_aF)^{-1}(b)|\\ &= \frac{1}{2}(NB_F+(2^n-1)2^{2n-m})-\frac{(2^n-1)2^n}{2}\\ &= \frac{1}{2}NB_F+(2^n-1)(2^{2n-m-1}-2^{n-1}).\end{aligned}$$

In [522] is made the necessary work of unification of the results on NB_F and on ambiguity. The known bounds on NB_F and those on ambiguity are compared and all the results are translated from one definition to the other. More results are also given. □

Parameter NB_F can be expressed by means of the Walsh transform. Thanks to Relation (3.20), page 136, we have:

$$NB_F = 2^{-m}\sum_{a\in\mathbb{F}_2^n,a\neq 0_n}\ \sum_{v\in\mathbb{F}_2^m,v\neq 0_m}W_{D_aF}^2(0_n,v). \tag{3.39}$$

Chabaud-Vaudenay's calculations recalled in the proof of Theorem 6, more precisely at page 141, show that: $\sum_{\substack{v\in\mathbb{F}_2^m,v\neq 0_m\\ u\in\mathbb{F}_2^n}}W_F^4(u,v)=$

$$\begin{aligned}&2^{n+m}|\{(x,y,a)\in\mathbb{F}_2^{3n}/\,F(x)+F(x+a)=F(y)+F(y+a)\}|-2^{4n}=2^{3n+m}+\\ &2^{n+m}\sum_{a\in\mathbb{F}_2^n,a\neq 0_n}|\{(x,y)\in\mathbb{F}_2^{3n}/\,D_aF(x)=D_aF(y)\}|-2^{4n}=\\ &2^{3n+m}-2^{4n}+2^{n+m}\sum_{a\in\mathbb{F}_2^n,a\neq 0_n}\sum_{b\in\mathbb{F}_2^m}|D_aF^{-1}(b)|^2=\\ &2^{3n}(2^m-1)+2^{n+m}NB_F.\end{aligned} \tag{3.40}$$

[25] A second parameter called deficiency is also introduced and studied in the same papers: $\mathcal{D}(F)=|\{(a,b)\in(\mathbb{F}_2^n\setminus\{0_n\})\times\mathbb{F}_2^m;\ |(D_aF)^{-1}(b)|=0\}|$; it plays a less important role.

[26] In particular for functions from $\mathbb{Z}/n\mathbb{Z}$ (resp. from the additive / multiplicative group of a finite field) to itself, and for some specific functions over finite fields, including all permutation polynomials over finite fields up to degree 6 and reversed Dickson polynomials (that we shall see more in details at page 422).

The Sidelnikov-Chabaud-Vaudenay bound can then be specified as follows:

$$nl(F) \leq 2^{n-1} - \frac{1}{2}\sqrt{2^n + \frac{2^{m-n}}{2^m - 1} NB_F}, \tag{3.41}$$

(this obviously implies the covering radius bound since $NB_F \geq 0$, and the SCV bound, because of the inequality $NB_F \geq (2^n-1)(2^{n+1}-2^{2n-m})$ recalled above). We can immediately see that the bound in (3.41) is tight for $m \leq n/2$, n even (since the covering radius bound is then tight) and for $m = n$, n odd (since the Sidelnikov-Chabaud-Vaudenay bound is then tight). In fact, it is tight for all values of n and m: the proof in [267] shows that it is an equality for a given F if and only if F is *plateaued with single amplitude* (see Definition 67, page 302). It would be interesting to determine for which triples (n, m, NB_F), or equivalently for which triples $(n, m, nl(F))$, the bound is tight (which would be determined if we know all possible amplitudes for plateaued (n, m)-functions with single amplitude).
Two other bounds on the nonlinearity involving the imbalance are given in [1133].

We have seen with Proposition 37, page 142, that the mean of the random variable $L \to Nb_{F+L}$ is the same for every function. We shall see now that its variance equals NB_F, up to a multiplicative factor.

Proposition 47 *[267, 239] Let F be any (n, m)-function. The variance of the random variable $L \in \mathcal{L}_{n,m} \to Nb_{F+L}$ equals $2^{-m} NB_F$.*

Proof. Let us denote by V_F the variance of the random variable $L \in \mathcal{L}_{n,m} \to Nb_{F+L}$, equal to that of the random variable $L \to \sum_{a \in \mathbb{F}_2^n \setminus \{0_n\}} |(D_aF)^{-1}(L(a))|$ according to Relation (3.28), page 143, whose mean equals $2^{2n-m} - 2^{n-m}$, according to Proposition 37. Hence V_F equals:

$$\frac{1}{|\mathcal{L}_{n,m}|} \sum_{\substack{L \in \mathcal{L}_{n,m} \\ a, a' \in \mathbb{F}_2^n \setminus \{0_n\}}} |(D_aF)^{-1}(L(a))|\, |(D_{a'}F)^{-1}(L(a'))| - \left(2^{2n-m} - 2^{n-m}\right)^2.$$

Let us distinguish the case where $a = a' \neq 0_n$ and the case where a and a' are linearly independent. We have seen that, when a is a fixed nonzero vector, the number of linear functions L such that $L(a) = b$ equals $2^{m(n-1)} = 2^{-m}|\mathcal{L}_{n,m}|$, for every vector b; similarly, when a, a' are fixed linearly independent vectors, the number of linear functions L such that $L(a) = b$ and $L(a') = b'$ equals $2^{m(n-2)} = 2^{-2m}|\mathcal{L}_{n,m}|$, for every vectors b, b'. We obtain:

$$V_F = 2^{-m} \sum_{a\in\mathbb{F}_2^n\setminus\{0_n\}} \sum_{b\in\mathbb{F}_2^m} |(D_aF)^{-1}(b)|^2 + 2^{-2m}\mu_F - \left(2^{2n-m} - 2^{n-m}\right)^2, \text{ where}$$

$$\begin{aligned}\mu_F &= \sum_{\substack{a,a'\in\mathbb{F}_2^n\setminus\{0_n\}\\ a\neq a'}} \sum_{b,b'\in\mathbb{F}_2^m} |(D_aF)^{-1}(b)|\,|(D_{a'}F)^{-1}(b')| \\ &= \sum_{\substack{a,a'\in\mathbb{F}_2^n\setminus\{0_n\}\\ a\neq a'}} \left(\sum_{b\in\mathbb{F}_2^m} |(D_aF)^{-1}(b)|\right)\left(\sum_{b\in\mathbb{F}_2^m} |(D_{a'}F)^{-1}(b)|\right) \\ &= (2^n-1)(2^n-2)\,2^{2n}.\end{aligned}$$

Then by the definition of NB_F, $V_F = 2^{-m}(NB_F + (2^n-1)2^{2n-m}) + 2^{4n-2m} - 3\cdot 2^{3n-2m} + 2\cdot 2^{2n-2m} - (2^{4n-2m} - 2\cdot 2^{3n-2m} + 2^{2n-2m}) = 2^{-m}NB_F$. □

Remark. In [239], it is shown that the mean of Nb_{F+L} when L ranges over the subset of balanced linear (n,m)-functions is the highest when F is balanced, but that its value is then not much larger than the mean in Proposition 37. □.

A recent stronger criterion for permutations

Boomerang attacks [1101] (and their variants called sandwich attacks) are a possible alternative to differential attacks when differentials (see Subsection 3.4.1, page 157) having sufficiently large probability are not known. The parameter which quantifies the contribution of an (n,n)-permutation F to the resistance to these attacks (the smaller the parameter, the better the resistance), see [371], is the so-called *Boomerang uniformity* (see [107]):

$$\begin{aligned}&\max_{(a,b)\in(\mathbb{F}_2^n\setminus\{0_n\})^2} |\{x\in\mathbb{F}_2^n; F(F^{-1}(x)+a)+F(F^{-1}(x+b)+a)=b\}| = \\ &\max_{(a,b)\in(\mathbb{F}_2^n\setminus\{0_n\})^2} |\{y\in\mathbb{F}_2^n; F^{-1}(F(y)+b)+F^{-1}(F(y+a)+b)=a\}| = \\ &\max_{(a,b)\in(\mathbb{F}_2^n\setminus\{0_n\})^2} |\{(x,y)\in{\mathbb{F}_2^n}^2; F(x+a)+F(y+a)=b \text{ and } F(x)+F(y)=b\}|\end{aligned} \tag{3.42}$$

(the first equality being shown by using that $F(F^{-1}(x)+a)+F(F^{-1}(x+b)+a)=b$ is equivalent to $F^{-1}(x+b)+a = F^{-1}(F(F^{-1}(x)+a)+b)$ and setting $y = F^{-1}(x)+a$). It is easily shown that the Boomerang uniformity is affine invariant and as we can see, it is also invariant when changing F into F^{-1}, but it is not EA-invariant (see [107]) and therefore not CCZ-invariant. We have that, denoting $y = F^{-1}(x)+a$ and $z = F^{-1}(x+b)+a$, the Boomerang uniformity equals $\max_{(a,b)\in(\mathbb{F}_2^n\setminus\{0_n\})^2} |\{(x,y)\in(\mathbb{F}_2^n)^2; F(y)+F(z)=F(y+a)+F(z+a)=b\}|$. The necessary condition $F(y)+F(z)=F(y+a)+F(z+a)$ being equivalent to $D_aF(y)=D_aF(z)$, if F is APN then, since this latter equality implies $y=z$ or $y=z+a$ and since $y=z$ is impossible because $b\neq 0_n$, the Boomerang uniformity equals 2. But APN permutations are known only for n odd and $n=6$. For general permutations, we can see by considering the particular case $z=y+a$ that the Boomerang uniformity is larger than or equal to the differential uniformity δ_F (see Definition 40, page 157). In [107] is shown that the boomerang

uniformity of the multiplicative inverse (n,n)-function for n even equals 6 if 4 divides n and 4 otherwise. Its value is characterized when $n=4$ for all differentially 4-uniform permutations (showing that it is at least 6). It is shown that if F is differentially 4-uniform and quadratic then its Boomerang uniformity is at most 12.
Quadratic permutations whose Boomerang Connectivity Table (BCT) is optimal (in the sense that the maximal value in the BCT equals the lowest known differential uniformity) have been derived in [875]. Moreover, boomerang uniformities of some specific permutations (mainly the ones with low differential uniformity) as well as a characterization by means of the Walsh transform of those functions F from $\mathbb{F}_{2^n}$ to itself with boomerang uniformity δ_F have been considered in [762].

3.4.2 Other features also related to attacks

Univariate degree

The *interpolation attack* [639] is efficient when the degree of the univariate polynomial representation of the S-box over $\mathbb{F}_{2^n}$ is low or when the distance of the S-box to the set of low univariate degree functions is small. A vectorial function should then not have low degree *univariate representation* nor be approximated by such function.

Attacks without related criteria on Boolean functions

The *slide attack* [89], when it can be mounted, has a complexity independent of the number of rounds in the block cipher, contrary to the attacks previously described. It analyzes the weaknesses of the key schedule (the most common case of weakness being when round keys repeat in a cyclic way) to break the cipher. The slide attack is efficient when the cipher can be decomposed into multiple rounds of an identical F function vulnerable to a known-plaintext attack.

3.5 Search for functions achieving the desired features

3.5.1 The difficulty of designing good S-boxes

Substitution boxes in block ciphers need to satisfy many criteria.

- The S-boxes for SPN networks must be bijective. The S-boxes for Feistel cryptosystems are better surjective and in fact balanced, see [995, 957].
- The S-boxes are better APN or differentially 4-uniform, or at least differentially 6-uniform.
- They have better high nonlinearity, say near $2^{n-1}-2^{n/2}$.
- They have better not too low algebraic degree; degree 2 is often too small because of the higher-order differential attack [706, 735].

- For reason of efficiency (see page 434), in software, n is better even, $n/2$ too ... that is, n is better a power of 2. In hardware, n can be any number. But general purpose cryptosystems must be implementable in both hardware and software. Then $n = 4, 8$ are preferred ($n = 4$ for lightweight ciphers).
- The S-box should be easy to protect against physical (side channel and fault injection) attacks, see Section 12.1.1, page 467. Hence, the number of nonlinear multiplications in $\mathbb{F}_{2^n}$ to compute the output (when the S-box is expressed over this field) should be small.

Examples of S-boxes used in practice:

- $(4, 4)$-S-boxes: Serpent, PRESENT, CLEFIA, NOEKEON, LED, RECTANGLE,
- $(6, 4)$-S-boxes: DES
- $(8, 8)$-S-boxes (inverse function): AES, CLEFIA, CAMELLIA
- $(9, 9)$ and $(7, 7)$-S-boxes, combined (AB functions Gold x^5 and Kasami $x^{13} \sim x^{81}$): MISTY, KASUMI
- $(8, 32)$-S-boxes: CAST

Other examples:

- Key-dependent S-boxes: CAST, Twofish
- Pseudorandomly generated $(4, 4)$-S-boxes: KHAZAD
- Round function based on x^3 in $\mathbb{F}_{2^{37}}$ or $\mathbb{F}_{2^{33}}$ according to the versions: KN
- Mixing operations from different groups: IDEA, CAST, RC6.

3.5.2 Constructions versus computer investigations of Boolean and vectorial functions

We shall give in Chapters 5-11 constructions of Boolean and vectorial functions satisfying the criteria we have seen in the present chapter. We shall study how these constructions can allow obtaining functions providing good trade-offs between several criteria. Such constructions provide in general infinite classes of functions (in any numbers of variables ranging over some infinite sets). These functions are rather well structured, compared with random functions satisfying the same criteria. This is a quality (it simplifies the study of criteria) but also a drawback (the structure may be usable by attackers).
It is then also useful to search by computer investigation for functions, in numbers of variables small enough for search to be feasible, meeting one or several criteria, and if possible to classify as in [124] these functions under proper notions of equivalence (which needs mathematical tools as well). Of course, such searches are also useful to guess infinite classes and constructions. They often show that the functions built by algebraic constructions have peculiarities.

General classification

The classification of Boolean functions dates back to the fifties [549] and sixties [588]. It has been realized in [66] under affine equivalence for all Boolean functions up to 5 variables (with 48 equivalence classes), for all 6-variable Boolean functions in [812] (see also Fuller's thesis "Analysis of affine equivalent Boolean functions for cryptography", Queensland University of Technology, 2003), for those 7-variable Boolean functions of algebraic degree at most 3 modulo those of degree 1 in [613, 102], for those 8-variable Boolean functions of algebraic degree 4 modulo those of degree 3 in [739], and under CCZ, EA, affine and permutation equivalences for $(4,4)$-functions in [756, 183, 1009, 1158]. Note that Burnside's Lemma [171] states that, if G is a group of permutations acting on a set X, then the number of orbits induced on X is given by $\frac{1}{|G|}\sum_{\sigma\in G}|\{x\in X\,;\,\sigma(x)=x\}|$.

More targeted computer investigations

Many papers report computer searches of specific functions (made after mathematical work). A few first examples are [1005] for 6-variable bent Boolean functions, [741] for 8-variable bent Boolean functions, [134] for APN $(5,5)$-functions (with a classification) and [135] for APN (n,n)-functions with $n=6,7,8$.

The survey of the recent literature shows that many results using heuristics (providing specific instances of Boolean functions, not general ones that algebraic constructions can give, but allowing to create many different solutions satisfying certain properties) are now obtained with evolutionary algorithms and to a lesser extent with other methods used for diverse kinds of searches. For instance, [694] implements hill climbing algorithms, which are a different type of heuristics than evolutionary algorithms (even if evolutionary algorithms can work as hill climbing algorithms). Other examples are [1051], which uses SAT solvers and [74], which uses similarly a satisfiability modulo theory tool.

Usually, there is no guarantee that the solutions are not equivalent to each others, and a hard part of the work (when it is done) is to check inequivalence.

A list of recent papers making computer investigations can be found in [952].

As far as we know, the first application of genetic algorithms (GA) to the evolution of cryptographically suitable Boolean functions has been done in [884], where the aim was to reach high nonlinearity. The authors worked up to 16 variables and concluded that GA combined with hill climbing is much faster than random search.

In [946, 945] are used several types of evolutionary algorithms to find correlation immune Boolean functions with minimal Hamming weight, and [951] is the first attempt (as far as we are aware) to mathematically show why finding balanced Boolean functions with high nonlinearity is hard for evolutionary algorithms.

In [947] are evolved *secondary constructions* of bent Boolean functions (*i.e.* of bent functions from bent functions), the goal being to reach many dimensions; there is no further analysis whether such obtained constructions are valid for an infinite number of dimensions or whether they are new, up to equivalence.

These results show that techniques with heuristics can compete with algebraic constructions of Boolean functions when the numbers of inputs are not too big (for larger n, it becomes a computationally intensive process to examine a large number of functions generated by heuristics).

For vectorial Boolean functions, the situation is less positive. In [883] are used genetic algorithms to evolve S-boxes with high nonlinearity and low autocorrelation value. The selection of the appropriate genetic algorithm parameters is discussed. In [372] are used simulated annealing and hill climbing algorithms to evolve bijective S-boxes of sizes up to 8×8 with high nonlinearity values. In [170] is used a heuristic method to generate MARS-like S-boxes, generating a number of S-boxes of appropriate size that satisfy all the requirements placed on the MARS S-box and even managing to find S-boxes with improved nonlinearity values. Bent (n, m)-functions are obtained in [948] with evolutionary computation. Picek et al. use several types of evolutionary algorithms to find differentially-6 uniform $(n, n-2)$ functions but are not able to report success for any previously unknown size [949]. In [682] are searched functions with particular symmetries. The results for vectorial Boolean functions obtained with heuristics cannot really compete with algebraic constructions even when considering the nonlinearity property. While algebraic constructions reach nonlinearity of 112 for 8×8 S-box size, the best result for heuristics is currently 104. Optimal values of nonlinearity and differential uniformity have been obtained with heuristics only recently for sizes larger than 4×4 (see [950] where proper cellular automata rules are found and used to construct S-boxes). The biggest advantage of using heuristics in the design of S-boxes lies in the fact that such techniques can account for properties, like resistance against side-channel attacks, that algebraic constructions cannot (see *e.g.* [297]). Finally, if we consider not only cryptographic properties of S-boxes but also their implementation cost (like area and power), then the heuristics could have an advantage over algebraic constructions. As an example of such a direction, Picek et al. use evolutionary algorithms to construct S-boxes that are either area or power efficient [953].

3.6 Boolean and vectorial functions for diffusion, secret sharing, authentication

Designing diffusion layers for block ciphers is related to codes and to Boolean vectorial functions. It is addressed in Subsection 4.2.3, page 185. The motivation for secret sharing is cryptographic and that is why we cover it in this chapter, but it could have also been covered in the next one.

3.6.1 Secret sharing, access structures and minimal codes

In [1030], Shamir has introduced a simple and elegant way to (probabilistically) split a secret $a \in \mathbb{F}_q$ into a number n of shares so that no set of shares with cardinality (strictly) less than m gives any information on a, where m is some positive integer smaller than or equal to n, and at least m shares allow reconstructing (deterministically) the secret. Such scheme is called an (n, m) threshold *secret sharing scheme*. Blakeley in [90] presented independently an idea for realizing the same; we shall not describe his slightly less efficient scheme. Shamir's scheme associates the secret a with a polynomial $P_a(X)$ over $\mathbb{F}_q$ defined as $P_a(X) = a + \sum_{i=1}^{m-1} u_i X^i$, where the u_i's denote random coefficients. Then, $n \geq m$ distinct non-zero elements $\alpha_0, \ldots, \alpha_{n-1}$ are publicly chosen in $\mathbb{F}_q^*$ and the polynomial $P_a(X)$ is evaluated in the α_i to construct a so-called *n-sharing* $(a_0, a_1, \cdots, a_{n-1})$ of a such that $a_i = P_a(\alpha_i)$ for every $i \in [0, \ldots, n-1]$. To re-construct a from at least m shares $(\alpha_i, a_i); i \in I$, Lagrange's polynomial interpolation is first applied to re-construct $P_a(X) = \sum_{i \in I} a_i \prod_{k \in I, k \neq i} \frac{X - \alpha_k}{\alpha_i - \alpha_k}$. Then, the polynomial is evaluated in 0. This allows a *dealer* to distribute the shares to n *player*s so that at least m of them are able to reconstruct the secret, while less have no information on it. We have

$$a = \sum_{i \in I} a_i \cdot \beta_i \ , \tag{3.43}$$

where the constants β_i are defined as follows: $\beta_i = \displaystyle\prod_{k \in I, k \neq i} \frac{\alpha_k}{\alpha_k - \alpha_i}$, and can be precomputed once for all and be public.
Shamir's scheme is related to a problem in distributed storage systems, the *exact repair problem*, described in [583]: a file (cut into blocks) to be stored is interpreted as a degree d polynomial F over a field $\mathbb{F}$, each block being a coefficient of the polynomial; to distribute the file over n nodes, n elements $\alpha_1, \ldots, \alpha_n$ of $\mathbb{F}$ are chosen, and $F(\alpha_i)$ is sent to node i; if a node fails, we may recover it by polynomial interpolation from the information on any m other nodes; it is possible to organize the distribution by breaking symbols into sub-symbols belonging to subfields, in order that for repairing a failed node is needed only a part of the information from other nodes (more than m nodes, but with a smaller amount of information needed globally). A lower bound is given in [583] on the amount of information needed (the repair bandwidth), and empirical constructions are proposed which allow to approach it.
Secret sharing schemes play also a central role in *multiparty computation* protocols, first introduced in [1136], in which n participants (also called players) are supposed to compute the image of a given function by making computations on the shares of the input provided by a secret sharing scheme, each player having one share. Such protocol is supposed to enable the coalition of players to securely evaluate the function, while some of the players are corrupted by an adversary. The protocol is called t-private if any t players cannot get from the protocol execution more information than their own shares; this is possible for any function when the number of players is at least $2t+1$. This happens to be closely related to the problematics of masking functions (S-boxes) and of probing security that we shall see in Chapter 12, and is also connected with threshold implementation (see the same chapter).

Shamir's scheme is a *linear secret sharing scheme* in the sense that the set $\{(a, a_0, a_1, \cdots, a_{n-1}) \in \mathbb{F}_q^{n+1}\}$ of those vectors of all possible $a \in \mathbb{F}_q$ concatenated with all possible sharings of a is a vector subspace of $\mathbb{F}_q^{n+1}$ (a linear code) and a is a linear function of the vector of its shares. As observed by Massey[27] in [827], given any linear $[n+1, k, d]_q$-code with (in the framework of the present book) $q = 2^n$ (and assuming in practice that $d \geq 2$ and that the corresponding *dual code* has a minimum distance $d^\perp \geq 2$, even if this is not specified by Massey), one can define a (linear) n-sharing over $\mathbb{F}_q$. Indeed, let G denote a generator matrix of the code; we assume that its first column is non-zero (this can be ensured by permuting the codeword coordinates if necessary, thanks to the fact that $d^\perp \geq 2$), then the sharing $(a_0, a_1, \cdots, a_{n-1})$ of a is built from a k-tuple $(r_1, \cdots, r_k)$ such that a equals the (usual) inner product between $(r_1, \cdots, r_k)$ and the first column of G, and chosen with a uniform probability under this constraint, and the sharing $(a_0, a_1, \cdots, a_{n-1})$ is defined by $(a, a_0, a_1, \cdots, a_{n-1}) = (r_1, \cdots, r_k) \times G$. For simplicity, up to a permutation of codeword coordinates again, and to a change of generator matrix, we can assume that the first column of G equals the first vector $(1, 0, \ldots, 0)$ of the canonical basis of $\mathbb{F}_q^k$ (we can even assume that G is in *systematic form* $G = [I_k \mid M]$, where I_k is the k-dimensional identity matrix over $\mathbb{F}_q$), and we have then $r_1 = a$ (and the other r_i's are random). The reconstruction of a from its sharing $(a_0, \cdots, a_{n-1})$ is obtained by choosing a row of a parity check matrix whose first coordinate is nonzero (which exists because we have $d \geq 2$: otherwise the vector $(1, 0, \ldots, 0)$ would belong to the code; note that it is not the only nonzero one since $d^\perp \geq 2$) and writing that the (usual) inner product between this row and $(a, a_0, a_1, \cdots, a_{n-1})$ equals 0. The next proposition is from [991].

Proposition 48 *Given a linear $[n+1, k, d]_q$-code used for secret sharing as described above, with $d, d^\perp \geq 2$, the knowledge of any $d^\perp - 2$ shares gives no information on the secret and $n - d + 2$ shares allow reconstructing the secret.*

Indeed, those vectors of length $d^\perp - 1$ whose first term equals a generic secret and the other ones are the shares of this secret at $d^\perp - 2$ positions cover uniformly the whole vector space $\mathbb{F}_q^{d^\perp - 1}$, by definition of the dual distance (see the observations and footnote after Definition 4, page 32), and the code of length $n - d + 1$ built the same way with $n - d + 2$ shares instead of $d^\perp - 2$ has straightforwardly minimum distance at least 2 and has dual distance at least 2 as well (since the dual distance of this punctured code is larger than or equal to the dual distance of the original code, see Lemma 2, page 32).
Some of such known secret sharing schemes use Boolean or vectorial functions, as initiated in [269] and developed in other papers, see [456, 867, 874] and the references therein. And as already observed in 1996 in [461, page 148 and Figure 7.1], correlation immune and resilient Boolean functions (see Definition 21, page 105) being related to the dual distance of codes by Corollary 6, page 108, they

[27] We shall borrow much from his paper in the present subsection.

can, in accordance with Proposition 48, be employed for secret sharing.
But the determination of the so-called *qualified coalitions* of players, which are able to (uniquely) reconstruct the secret, is more difficult to do in general than for Shamir's construction (which is equivalent to using as code a possibly punctured Reed-Solomon code, and is simple because such code being *MDS*, any set of k positions is an information set, see pages 24 and 185, and any k positions of a codeword determine then the full codeword uniquely). As also observed in [461] (see Theorem 2.5), MDS codes lead then to so-called *threshold secret sharing schemes* (in which qualified sets are exactly those of sufficient sizes), and conversely. For general codes, the set of all qualified coalitions satisfies the monotone property. An important notion is then the *access structure* of a secret sharing scheme, that is, the class of minimal qualified coalitions (for which, if any share is removed, the remaining shares give no information about the secret).
Let us recall how the access structure of a code can be determined[28]. Recall that we say that a vector u over a finite field $\mathbb{F}_q$ covers a vector v and we write $v \preceq u$ if $supp(v) \subseteq supp(u)$. A nonzero codeword u of a code $\mathcal{C}$ is called a *minimal codeword* of $\mathcal{C}$ if it covers no codeword of $\mathcal{C}$ different from au, with $a \in \mathbb{F}_q$ (*i.e.* no $\mathbb{F}_q$-linearly independent codeword) [103, 104]. Minimum weight codewords are minimal, but the converse is in general not true, except for MDS codes, in which the minimal codewords are the codewords of weight $n-k+1$.
As observed by Massey, no two $\mathbb{F}_q$-linearly independent minimal codewords of a linear code can have the same support, since otherwise any linear combination would be a codeword that both of the former codewords would cover. This means that each support of a minimal codeword corresponds uniquely to this minimal codeword, up to linear dependency. In fact, as shown in [33], a set I of indices is the support of a minimal codeword if and only if a parity check matrix H restricted to the columns indexed by I has rank $|I|-1$ (the fact that it has rank less than $|I|$ is equivalent to the existence of a codeword whose support is included in I, and the fact that it has rank $|I|-1$, say that the columns indexed in $I' \subset I$ with $|I'| = |I| - 1$ are linearly independent, is then equivalent to the fact that this codeword has support I and is minimal, since otherwise we could find by linear combination a codeword whose support would be I', a contradiction) and this is a condition on I which does not require to know the minimal codeword of support I. This also proves that any minimal codeword has Hamming weight at most $n-k+1$.
Every codeword is a linear combination of minimal codewords, since if a codeword u is not minimal, it covers a minimal codeword v and there exists a linear combination $u + cv$, $c \in \mathbb{F}_q$, which has Hamming weight strictly smaller than $w_H(u)$; the process can continue (with $u + cv$) a finite number of steps and, when it ends, it provides a linear decomposition of u over minimal codewords it covers. Hence, for every nonzero position in a codeword, this codeword covers a minimal codeword which is nonzero at this position.

[28] In practice, this is often a very hard task.

As shown by Massey [827] (and recalled in [33]), the access structure of the secret sharing scheme corresponding to a linear code is specified by those minimal codewords in the dual code, whose first component is nonzero (the set of shares corresponding then to the locations where this minimal codeword is non-zero, except the first). Indeed, as we saw, the secret is a linear combination of the shares and the vector of the coefficients of the resulting null linear combination belongs to the dual. Note that this property also proves that codewords of Hamming weight at most $2d-1$ in a binary $[n,k,d]$ code are minimal. More generally, it is shown in [33] for every $[n,k,d]_q$ code that the codewords of Hamming weight at most $\frac{qd}{q-1}-1$ are all minimal, since given such codeword u, and supposing the existence of a nonzero codeword $v \preceq u$ linearly independent of u, we have $\sum_{c\in\mathbb{F}_q^*} w_H(u-cv) = (q-1)w_H(u) - w_H(v) \leq (q-1)w_H(u) - d$ and the average of $w_H(u-cv)$ when $c \in \mathbb{F}_q^*$ is then at most $w_H(u) - \frac{d}{q-1} \leq \frac{qd}{q-1} - 1 - \frac{d}{q-1} = d-1$; one of these codewords has then Hamming weight at most this value, a contradiction since none can be the zero codeword by hypothesis.
The minimal codewords have been determined in [33] for the (not necessarily binary) Hamming codes and for the binary Reed-Muller codes of order at most 2 (all nonzero codewords in $RM(1,n)$ are minimal except the all-1 codeword, and all codewords in $RM(2,n)$ are minimal except those of Hamming weight $2^{n-1}+2^{n-1-h}$ for $h=0,1,2$ and for some of those of Hamming weight 2^{n-1}; the proof, too technical for being included here, is based on the facts that $2d = 2^{n-1}$ and that any non-minimal codeword in a binary code equals the sum of two codewords with disjoint supports).
The codes whose nonzero codewords are all minimal are particularly interesting (this makes the code easily decodable and simplifies the access structures of the secret sharing scheme; it plays also a role in multiparty computation, see *e.g.* [339]). A code having such property is called a *minimal code*. We have:

Proposition 49 *[33] Let $\mathcal{C}$ be a linear code over $\mathbb{F}_q$. If $\frac{w_{min}}{w_{max}} > \frac{q-1}{q}$, where w_{min} and w_{max} denote respectively the minimum and maximum nonzero weights in $\mathcal{C}$, then $\mathcal{C}$ is minimal.*

We do not give the proof. The hypothesis of Proposition 49 seems very strong as soon as q is large, but many examples of codes satisfying it exist and no example of a non-binary minimal code not satisfying it is known in characteristic 2. Infinite families of minimal binary linear codes (related to Boolean functions) not meeting this condition have been recently found in [345, 456].
A recent necessary and sufficient condition for linear codes to be minimal is:

Proposition 50 *[456, 602] A linear code $\mathcal{C}$ over $\mathbb{F}_q$ is minimal if and only if, for each pair of $\mathbb{F}_q$-linearly independent codewords u and v in $\mathcal{C}$, we have:*

$$\sum_{c\in\mathbb{F}_q^*} w_H(u+cv) \neq (q-1)w_H(u) - w_H(v). \tag{3.44}$$

Hence, the minimality of $\mathcal{C}$ is completely determined by the weights of its code-

words, and it is more easily handled if the numbers of these weights is small. Minimal codes derived from finite geometry (hyperovals, see page 245) are given in [862] in relation with bent vectorial functions.
A code is called a *two-weight code* if its nonzero elements have two possible weights only. Examples related to Boolean functions will be seen with Proposition 68, page 219. It is shown in [602] that if $\mathcal{C}$ is a two-weight linear code with length N and weights w_1 and w_2, such that $0 < w_1 < w_2 < N$ and $jw_1 \neq (j-1)w_2$ for every integer j such that $2 \leq j \leq q$, then $\mathcal{C}$ is minimal.
Binary three weight minimal codes are also investigated in [456].

3.6.2 Authentication schemes

The framework is as follows: Alice wishes to transmit to Bob a message m (a vector over the field $\mathbb{F}_{2^n}$) in the form (m,t) where t is a tag corresponding to m and depending on a secret key k shared between Alice and Bob, in order that Bob can verify the validity of the signature and nobody else than Alice and Bob can forge a valid message.
A *systematic authentication scheme* is a tuple $(M, T, K, \{E_k : k \in K\})$, where $E_k : M \mapsto T$ is the encoding rule related to k. To transmit an information $m \in M$ to Bob, Alice calculates $t = E_k(m)$ and sends the tuple (m,t) to Bob over the public channel. Bob verifies that the relation $t = E_k(m)$ is satisfied. There exist two kinds of attacks: the attacker can try to forge (m,t) from scratch, hoping that it is accepted by Bob, this is called the "impersonation attack", or he can observe a valid tuple (m,t) and try to modify it, this is called the "substitution attack". The maximal success probabilities of these attacks are denoted by P_I and P_S, respectively.

$$P_I = \max_{m\in M, t\in T} \frac{|\{k \in K; E_k(m) = t\}|}{|K|};$$

$$P_S = \max_{m\neq m'\in M, t,t'\in T} \frac{|\{k \in K; E_k(m) = t \text{ and } E_k(m') = t'\}|}{||\{k \in K; E_k(m) = t\}|}.$$

An example

Let $\mathbb{F}_{2^h}$ be a subfield of $\mathbb{F}_{2^n}$ and F a vectorial function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^n}$. We define the following scheme from [268]: $M = \mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$, $T = \mathbb{F}_{2^h}$, $K = \mathbb{F}_{2^n} \times \mathbb{F}_{2^h}$, $E = \{E_k : k \in K\}$, where for every $k = (k_1, k_2) \in K$ and $m = (a, b) \in M$, we have $E_k(m) = tr^n_h(aF(k_1) + bk_1) + k_2$, where tr^n_h is the trace function from $\mathbb{F}_{2^n}$ into $\mathbb{F}_{2^h}$: $tr^n_h(a) = \sum_{j=0}^{\frac{n}{h}-1} a^{2^{hj}}$.
Function $k \mapsto E_k$ is a bijection from K to E and:

$$P_I = \frac{1}{2^h} \quad , \quad P_S \leq \frac{1}{2^h} + \left(1 - \frac{1}{2^h}\right)\left(1 - \frac{nl(F)}{2^{n-1}}\right)$$

where $nl(F)$ denotes the nonlinearity of F and $|M| = 2^{2n}$, $|T| = 2^h$, $|K| = |E| = 2^{n+h}$. Other examples can be found for instance in [268, 303, 460].

4 Boolean functions, vectorial functions and error correcting codes

Nota Bene: Symbol n being traditionally used to denote the number of variables of Boolean functions, what was denoted by m in Section 1.2 is changed in this chapter into n. The codes have then length 2^n, and not n as it is usual in coding.

4.1 Reed-Muller codes

Reed-Muller codes have been introduced by David Muller in [891] and their decoding algorithm has been given by Irving Reed in [989]. They have played an important role in the history of error correcting codes. For instance, they were used in the sixties and early seventies for the transmission of the first photographs of Mars by the Mariner series of spacecrafts. A Reed-Muller code of length 32, dimension 6 and minimum distance 16 was used (precisely, the first-order Reed-Muller code $RM(1,5)$). Each codeword corresponded to a level of darkness, this made 64 different levels and up to $\lfloor \frac{16-1}{2} \rfloor = 7$ errors could be corrected in the transmission of each codeword. Reed-Muller codes were also used in the 3rd generation (3G) of mobile phones (developed in the late 1990s for release in the early 2000s), in the so-called "Transport Format Combination Indicator" TFCI (part of the initial "handshake" between the mobile device and the base station, designed to inform the receiver of what type of communication will come next), for which it is extremely important to get information correct. The same code as for Mariner spacecrafts was first used and it was later replaced by a punctured subcode of the second-order Reed-Muller code $RM(2,5)$, which had dimension 10 and minimum distance 12.

Reed-Muller codes still play an important role thanks to their specific properties (see *e.g.* [1, 457]) and their roles with respect to new problematics (like Locally Correctable Codes [778]), despite the fact that their parameters are not good[1], except for the lowest and largest orders. They also constitute a useful framework for the study of Boolean functions.

Definition 42 *For every non-negative integer r and every positive integer $n \geq r$, the Reed-Muller code $RM(r,n)$ of order r and length 2^n is the binary linear code of all words of length 2^n corresponding to the evaluations over $\mathbb{F}_2^n$ (on which*

[1] In the late seventies, for transmitting color photographs of Mars, the Voyager spacecrafts used the extended binary Golay code and later *Reed-Solomon codes*.

some order has been chosen) of all n-variable Boolean functions of algebraic degree *at most* r.

In other words, codewords are the last columns in the *truth-tables* of these functions. *By abuse of language, we shall say that* $RM(r,n)$ *is the* $\mathbb{F}_2$*-vector space of all* n*-variable Boolean functions of algebraic degree at most* r.
For $r = 0$, $RM(0,n)$ equals the pair of constant functions.
For $r = 1$, $RM(1,n)$ equals the vector space of all *affine functions*. Note that we have seen in Section 1.2 that the codewords of the simplex code are the lists of values taken on $\mathbb{F}_2^n \setminus \{0_n\}$ by all linear functions. Hence $RM(1,n)$ is the $\mathbb{F}_2$-vector space generated by the extended simplex code and the constant function 1.
For $r = n$, $RM(n,n)$ equals the whole space of n-variable Boolean functions, since every n-variable Boolean functions has an ANF and then algebraic degree at most n.
The dimension of $RM(r,n)$ equals $1+n+\binom{n}{2}+\cdots+\binom{n}{r}$ since this is the number of monomials of degrees at most r, which constitute a basis of $RM(r,n)$.

Remark Let $\mathcal{G} = \mathbb{F}_2[\mathbb{F}_2^n]$ be the so-called *group algebra* of $\mathbb{F}_2^n$ over $\mathbb{F}_2$, consisting of the formal sums $\sum_{g\in\mathbb{F}_2^n} a_g\, g$ where $a_g \in \mathbb{F}_2$. The algebra $\mathcal{G}$ has only one maximal ideal, called its *radical*:

$$\mathcal{R} = \Big\{ \sum_{g\in\mathbb{F}_2^n} x_g X^g \; ; \; \sum_{g\in\mathbb{F}_2^n} x_g = 0 \Big\},$$

whose elements correspond to the words of even Hamming weight. The ideals $\mathcal{R}^j$, $j \geq 1$, generated by the products of j elements of $\mathcal{R}$, provide the decreasing sequence

$$\mathcal{G} \supset \mathcal{R} \supset \cdots \supset \mathcal{R}^n = \{0_{2^n}, 1_{2^n}\} \ ,$$

with $\mathcal{R}^i\mathcal{R}^j = \mathcal{R}^{i+j}$. Berman [67] observed that, for any r, $RM(r,n) = \mathcal{R}^{n-r}$. $\square$

$RM(r,n)$ being a linear code, it can be described by a generator matrix G. For instance, a generator matrix of the Reed-Muller code $RM(1,4)$ can be:

$$G = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&1&0&0&0&1&1&1&0&0&0&1&1&1&0&1 \\ 0&0&1&0&0&1&0&0&1&1&0&1&1&0&1&1 \\ 0&0&0&1&0&0&1&0&1&0&1&1&0&1&1&1 \\ 0&0&0&0&1&0&0&1&0&1&1&0&1&1&1&1 \end{bmatrix}.$$

The first row corresponds to the monomial of degree 0 (the constant function 1) and the other rows correspond to the monomials of degree 1 (the coordinate functions $x_1, \dots, x_4$), when ordering the words of length 4 by increasing Hamming weights (we could choose other orderings, we have seen that this would lead to so-called equivalent codes, see page 23).

4.1.1 Minimum distance and minimum weight codewords

Theorem 7 *The minimum distance of $RM(r,n)$ equals 2^{n-r}.*

This was historically proved by double induction over r and n (see [809, page 375]), but there exists a simpler proof.

Proof. Code $RM(r,n)$ being linear, its minimum distance d equals the minimum nonzero Hamming weight of codewords. Let us first prove that $d \geq 2^{n-r}$. Since 2^{n-r} is a decreasing function of r, it is sufficient to show the bound for functions of algebraic degree r. Let $\prod_{i\in I} x_i$ be a monomial of degree r in the ANF of a Boolean function f of algebraic degree r; consider the 2^{n-r} restrictions of f obtained by keeping fixed the $n-r$ coordinates of x whose indices lie outside I. Each of these restrictions, viewed as a function on $\mathbb{F}_2^r$, has an ANF of degree r because, when fixing these $n-r$ coordinates, the monomial $\prod_{i\in I} x_i$ is unchanged and all the monomials different from $\prod_{i\in I} x_i$ in the ANF of f give either 0 or monomials of degrees strictly less than r. Thus, any such restriction has an odd (and hence a nonzero) Hamming weight (see Section 2.2). The Hamming weight of f being equal to the sum of the Hamming weights of its restrictions, f has Hamming weight at least 2^{n-r}.

To complete the proof, we just need to exhibit a codeword of Hamming weight 2^{n-r}. The simplest example is the Boolean function $f(x) = \prod_{i=1}^{r} x_i$, that is, the indicator of the affine space$\{(1,\ldots,1)\} \times \mathbb{F}_2^{n-r}$. □

Remark.

1. The proof of Theorem 7 shows in fact that, if a monomial $\prod_{i\in I} x_i$ has coefficient 1 in the ANF of f, and if every other monomial $\prod_{i\in J} x_i$ such that $I \subset J$ has coefficient 0 (*i.e.* if I is maximal), then the function has Hamming weight at least $2^{n-|I|}$. Applying this observation to the Möbius transform f° of f - whose definition has been given after Relation (2.3), page 49, - shows that, if there exists a vector $x \in \mathbb{F}_2^n$ such that $f(x) = 1$ and $f(y) = 0$ for every vector $y \neq x$ whose support contains $supp(x)$ (*i.e.* if x is maximal in the support of f), then the ANF of f has at least $2^{n-w_H(x)}$ terms; this has been first observed in [1179]. Indeed, the Möbius transform of f° is f.

2. The d-dimensional subspace $E = \{x \in \mathbb{F}_2^n\,;\, x_i = 0, \forall i \notin I\}$, in the proof of Theorem 7, is a *maximal odd weighting* subspace: the restriction of f to E has odd Hamming weight (*i.e.* has algebraic degree equal to the dimension d when viewed as a d-variable function), and the restriction of f to any of its proper superspaces has even Hamming weight (*i.e.* the restriction of f to any coset of E has odd Hamming weight). Similarly as above, it can be proved, see [1179], that any Boolean function admitting a d-dimensional maximal odd weighting subspace E has Hamming weight at least 2^{n-d}, and if $d \geq 2$, applying this observation to $f \oplus \ell$ where ℓ is affine, we have that f has nonlinearity at least 2^{n-d}. Indeed, the restriction of f to a d-dimensional affine space has algebraic degree d if and only if the restriction of $f \oplus \ell$ does. See more in [191], where the proofs are given in terms of group rings/algebras (see page 175; see [283, 323] for other

examples where these are used). □

Notice that all non-constant affine functions have Hamming weight 2^{n-1}, their supports being affine hyperplanes. Thus, non-constant affine functions are the codewords of minimum Hamming weight in $RM(1,n)$. More generally, the codewords of minimum Hamming weight in $RM(r,n)$ have been characterized (see *e.g.* [809]). We give below another proof, more Boolean function oriented, of this characterization.

Theorem 8 *The Boolean functions of algebraic degree r and of* Hamming weight 2^{n-r} *are the* indicator*s of $(n-r)$-dimensional flats (*i.e. *the functions whose supports are $(n-r)$-dimensional affine subspaces of $\mathbb{F}_2^n$).*

Proof. The indicators of $(n-r)$-dimensional flats have clearly Hamming weight 2^{n-r} and they have algebraic degree r, since they are affinely equivalent to the function $\prod_{i=1}^r x_i$, because two affine subspaces of $\mathbb{F}_2^n$ of the same dimension are affinely equivalent (and recall from page 53 that the algebraic degree is an affine invariant). Conversely, let f be a function of algebraic degree r and of Hamming weight 2^{n-r}. Let $\prod_{i\in I} x_i$ be a monomial of degree r in the ANF of f and let $I^c = \{1,\ldots,n\}\setminus I$. For every vector $\alpha \in \mathbb{F}_2^{I^c}$, let us denote by f_α the restriction of f to the flat $\{x \in \mathbb{F}_2^n;\ \forall j \in I^c,\ x_j = \alpha_j\}$, viewed as a function over $\mathbb{F}_2^I$. According to the proof of Theorem 7, and since f has Hamming weight 2^{n-r}, each function f_α is the indicator δ_{a_α} of a singleton $\{a_\alpha\}$ of $\mathbb{F}_2^I$. Moreover, the mapping $\alpha \in \mathbb{F}_2^{I^c} \to a_\alpha \in \mathbb{F}_2^I$ is affine, *i.e.*, for every $\alpha,\beta,\gamma \in \mathbb{F}_2^{I^c}$, we have $a_{\alpha+\beta+\gamma} = a_\alpha + a_\beta + a_\gamma$. Indeed, the r-variable function $f_\alpha \oplus f_\beta \oplus f_\gamma \oplus f_{\alpha+\beta+\gamma}$ being the restriction to $\mathbb{F}_2^I$ of $D_{\alpha+\beta}D_{\alpha+\gamma}f(x+\alpha)$ where $D_{\alpha+\beta}D_{\alpha+\gamma}f$ is a second-order derivative of f, it has algebraic degree at most $r-2$, according to Corollary 1, page 56. And it is easily seen by using that $\delta_a(x) = \prod_{i=1}^n (x_i \oplus a_i \oplus 1)$ or by using Relation (2.3), page 49, that, for every $i \in \{1,\ldots,n\}$, the coefficient of the degree $r-1$ monomial $\prod_{j\neq i} x_j$ in $(f_\alpha \oplus f_\beta \oplus f_\gamma \oplus f_{\alpha+\beta+\gamma})(x)$ (which is null) equals the i-th coordinate of $a_\alpha + a_\beta + a_\gamma + a_{\alpha+\beta+\gamma}$. This completes the proof since, denoting by x_I (resp. x_{I^c}) the restriction of x to I (resp. to I^c), the support of f equals the set $\{x \in \mathbb{F}_2^n\,;\, x_I = a_{x_{I^c}}\}$ and that the equality $x_I = a_{x_{I^c}}$ is equivalent to r linearly independent linear equations. □

See more in [34], from a design viewpoint. The minimum weight codewords of $RM(r,n)$ generate the code over $\mathbb{F}_2$, see [420].

4.1.2 Dual

The dual of a *Reed-Muller code* is a Reed-Muller code:

Theorem 9 *For every positive n and every non-negative $r < n$, the dual*

$$RM(r,n)^\perp = \{f \in \mathcal{BF}_n;\ \forall g \in RM(r,n),\ f\cdot g = \bigoplus_{x\in\mathbb{F}_2^n} f(x)\,g(x) = 0\}$$

equals $RM(n-r-1,n)$.

Proof. We have seen in Section 2.2 that the n-variable Boolean functions of even Hamming weights are the elements of $RM(n-1,n)$ (which equals then the parity code of length 2^n). Thus, $RM(r,n)^\perp$ is the set of those functions f such that, for every function g of algebraic degree at most r, the product function fg (whose value at any $x \in \mathbb{F}_2^n$ equals $f(x)g(x)$) has algebraic degree at most $n-1$. This is clearly equivalent to the fact that f has algebraic degree at most $n-r-1$. □

Note that, since $RM(1,n)$ is the $\mathbb{F}_2$-vector space generated by the extended *simplex code* and the constant function 1, its dual $RM(n-2,n)$ is the intersection of the dual of the extended simplex code and the parity code. It also equals the extended *Hamming code*, according to Lemma 1, page 24, applied to $RM(1,n)$.

Characterization in the field $\mathbb{F}_{2^n}$

If the vector-space $\mathbb{F}_2^n$ is identified with the field $\mathbb{F}_{2^n}$, then the family of those functions $tr_n(ax^j)$ such that $a \in \mathbb{F}_{2^n} \setminus \{0\}$ and $w_2(j) \leq n-r-1$ generates $RM(n-r-1,n)$ (according to what we have seen on the trace representation of Boolean functions). We have then that a Boolean function f belongs to $RM(r,n)$ if and only if, for every nonzero j such that $w_2(j) \leq n-r-1$, we have $\sum_{x\in\mathbb{F}_{2^n}} f(x)\, tr_n(ax^j) = tr_n(a\sum_{x\in\mathbb{F}_{2^n}} f(x)x^j) = 0$ for every $a \in \mathbb{F}_{2^n}$, that is:

Corollary 8 *For every positive n and every non-negative $r < n$, a Boolean function f over $\mathbb{F}_{2^n}$ belongs to $RM(r,n)$ if and only if, for every nonzero j such that $w_2(j) \leq n-r-1$, we have $\sum_{x\in\mathbb{F}_{2^n}} f(x)\, x^j = 0$.*

4.1.3 Automorphism group

The Reed-Muller codes are invariant under the action of the *general affine group* (*i.e.* the group of affine permutations over $\mathbb{F}_2^n$). More precisely, it is a simple matter to show that:

Proposition 51 *For any $1 \leq r \leq n-1$, the automorphism group of $RM(r,n)$ (that is, the group of all permutations σ of $\mathbb{F}_2^n$ such that $f \circ \sigma \in RM(r,n)$ for every $f \in RM(r,n)$) equals the general affine group.*

The sets $RM(r,n)$ or $RM(r,n)/RM(r',n)$ have been classified under this action for some values of r, of $r' < r$ and of n, see [611, 613, 615, 130, 812, 1053, 1057].

4.1.4 Cyclicity of the punctured code $R^*(r,n)$

Let us identify $\mathbb{F}_2^n$ with the finite field $\mathbb{F}_{2^n}$. The *punctured code* $R^*(r,n)$ obtained from $RM(r,n)$ by erasing in each codeword f the coordinate at zero input, and ordering the resulting vector as $(f(1), f(\alpha), f(\alpha^2), \ldots, f(\alpha^{2^n-2}))$, where α is a *primitive element* of $\mathbb{F}_{2^n}$, is a *cyclic code*. Indeed, the cyclic shift $(f(1), f(\alpha), f(\alpha^2), \ldots, f(\alpha^{2^n-2})) \mapsto (f(\alpha^{2^n-2}), f(1), f(\alpha), \ldots, f(\alpha^{2^n-3}))$ is equivalent to changing function $f(x)$ into $f(\frac{x}{\alpha})$, and such transformation on Boolean functions does not change the algebraic degree since it is linear bijective. For any $r < n$, the Reed-Muller code $RM(r,n)$ is then an extended cyclic code [809, page 383].

Proposition 52 *For every $r < n$, the zeros of the punctured Reed-Muller code $R^*(r,n)$ of order r and length $2^n - 1$ are the elements α^i such that $1 \leq i \leq 2^n - 2$ and such that the 2-weight of i is at most $n - r - 1$.*

Proof. We have seen that any Boolean function f of algebraic degree at most r has a univariate polynomial representation of the form $\sum_{\substack{0 \leq j \leq 2^n-2 \\ w_2(j) \leq r}} f_j\, x^j$. The codeword $(f(1), f(\alpha), f(\alpha^2), \ldots, f(\alpha^{2^n-2}))$ of the cyclic code $R^*(r,n)$ is represented by the polynomial $\sum_{0 \leq l \leq 2^n-2} f(\alpha^l) X^l$ (see Section 1.2), whose value at α^i equals

$$\sum_{0 \leq l \leq 2^n-2} f(\alpha^l)\alpha^{li} = \sum_{\substack{0 \leq j \leq 2^n-2 \\ w_2(j) \leq r}} f_j \left(\sum_{0 \leq l \leq 2^n-2} \alpha^{l(i+j)} \right).$$

The sum $\sum_{0 \leq l \leq 2^n-2} \alpha^{l(i+j)}$ equals 0 when $w_2(i) \leq n-r-1$ and $w_2(j) \leq r$ since $i + j \geq i \geq 1$ cannot equal $2^n - 1$ since $w_2(i+j) \leq w_2(i) + w_2(j)$, and then, α^{i+j} cannot equal 1, and then $\sum_{0 \leq l \leq 2^n-2} \alpha^{l(i+j)} = \frac{1+\alpha^{(2^n-1)(i+j)}}{1+\alpha^{i+j}} = 0$. Hence, the α^i's such that $1 \leq i \leq 2^n - 2$ and $w_2(i) \leq n - r - 1$ are zeros of the code. Since their number equals the co-dimension of the code, they are the only zeros of the code. □

4.1.5 The problem of determining the weight distributions of Reed-Muller codes

What are in $RM(r,n)$ the possible Hamming distances between codewords, or equivalently the possible Hamming weights (or better, the weight distribution)? The answer, useful for improving the efficiency of the decoding algorithms, for evaluating their complexities, and for many other issues, is known for every n if $r \leq 2$: see Section 5.2. For $r \geq n - 3$, it can also be deduced from the MacWilliams identity (1.1), which theoretically allows to deduce the weight distribution of $RM(n-r-1,n)$ from the weight distribution of $RM(r,n)$. Practically, it is necessary to be able to explicitly expand the factors $(X+Y)^{2^n-i}(X-Y)^i$ and to simplify the obtained expression for $W_C(X+Y, X-Y)$; this is possible up to some value of n (around 35) by running a computer.
The cases $3 \leq r \leq n-4$ remain unsolved (except for small values of n, see [66], and for $n = 2r$, because the code is then self-dual, see [809, 959]). Asymptotically tight bounds exist [679].
McEliece's theorem [833] or *Ax's theorem* [41] (see also the *Stickelberger theorem*, *e.g.* in [740, 746]) shows that the Hamming weights (and thus the distances) in $RM(r,n)$ are all divisible by $2^{\lceil \frac{n}{r} \rceil - 1} = 2^{\lfloor \frac{n-1}{r} \rfloor}$, where $\lceil u \rceil$ denotes the ceiling (the smallest integer larger than or equal to u) and $\lfloor u \rfloor$ denotes the integer part. For instance, it is shown in [677] (see also [623]) that if $d_{alg}(g) \leq d_{alg}(f)$, then

$d_H(f,g) \equiv w_H(f) \left[\bmod 2^{\left\lceil \frac{n-d_{alg}(g)}{d_{alg}(f)} \right\rceil} \right]$ and this proves the McEliece's divisibility property by taking $g = f$.

McEliece's divisibility bound is tight and can also be shown by using the properties of the *NNF*; it is deduced in [292] from the fact that, if s is the number of monomials of degree $d_{alg}(f) > 0$ in the ANF of f, then the coefficient λ_u of x^u in its NNF is a multiple of $2^{\left\lfloor \frac{w_H(u)-1}{d_{alg}(f)} \right\rfloor}$ if $w_H(u) > 0$ and of $2^{\left\lfloor \frac{w_H(u)-s-1}{d_{alg}(f)-1} \right\rfloor}$ if $w_H(u) > s$ and $d_{alg}(f) > 1$. Moreover, it is also shown in [292] that if $s < \frac{n}{d_{alg}(f)}$, then the Hamming weight of f is a multiple of $2^{\left\lceil \frac{n-s}{d_{alg}(f)-1} \right\rceil - 1}$ (larger than what gives McEliece's theorem).

Further properties of Hamming weights are given in [185] within the coset $f \oplus RM(1,n)$.

Kasami and Tokura [670] have shown that, for $r \geq 2$, the only Hamming weights in $RM(r,n)$ occurring in the range $[2^{n-r}; 2^{n-r+1}[$ are of the form $2^{n-r+1} - 2^i$ for some i; and they have completely characterized the codewords: the corresponding functions are affinely equivalent either to $x_1 \cdots x_{r-2}(x_{r-1}x_r \oplus x_{r+1}x_{r+2} \oplus \cdots \oplus x_{r+2l-3}x_{r+2l-2})$, $2 \leq 2l \leq n-r+2$, or to $x_1 \cdots x_{r-l}(x_{r-l+1} \cdots x_r \oplus x_{r+1} \cdots x_{r+l})$, $3 \leq l \leq \min(r, n-r)$. The functions whose Hamming weights are strictly less than 2.5 times the minimum distance 2^{n-r} have later been studied in [671].

It is shown in [210] (and reported in Section 5.3 below, page 204) that for every Boolean function f on $\mathbb{F}_2^n$, there exists an integer m and a Boolean function g of algebraic degree at most 3 on $\mathbb{F}_2^{n+2m}$ whose Walsh transform satisfies: $W_g(0_{n+2m}) = 2^m W_f(0_n)$. Hence, the Hamming weight of f is related in a simple way to the Hamming weight of a cubic function (in a number of variables which can be exponentially larger). This shows that the distances in $RM(3,n)$ can be very diverse, contrary to those in $RM(2,n)$. See also [65].

4.1.6 Covering radius

The *covering radius* of $RM(r,n)$, that we shall denote by $\rho(r,n)$, equals by definition (see Section 1.2) the maximum, when f ranges over $\mathcal{BF}_n$, of the minimum Hamming distance between f and all n-variable Boolean functions of algebraic degree at most r (*i.e.* of the distance between f and $RM(r,n)$; this distance is called the r-th order nonlinearity of f, and more simply its nonlinearity when $r = 1$, see Section 3.1).

- We have $\rho(1,n) = 2^{n-1} - 2^{\frac{n}{2}-1}$ when $n \geq 2$ is even (see Chapter 6). When n is odd, as we already saw in Section 3.1, $\rho(1,n)$ is unknown, except for $n \leq 7$, in which case it equals $2^{n-1} - 2^{\frac{n-1}{2}}$ [894]. For $n \geq 9$ odd, $\rho(1,n)$ lies strictly between $2^{n-1} - 2^{\frac{n-1}{2}}$ and $2\lfloor 2^{n-2} - 2^{\frac{n}{2}-2} \rfloor$ [936, 937, 684, 686, 617].
- We have $lim_{n\to\infty} \left(2^{n/2} - \frac{\rho(1,n)}{2^{n/2-1}}\right) = 1$ (this fact, conjectured by Patterson and Wiedemann in 1983, has been proved by Schmidt [1022] in 2019, who also proved the same limit when restricting to balanced functions).

- $\rho(2,n)$ is known for $n \leq 7$ (see [1104]). In [715] is calculated the second order nonlinearity of all Boolean functions in the infinite class of those cubic functions[2] whose degree 3 part, up to affine equivalence, has the form $\bigoplus_{i=1}^{s} x_i q_i(x)$, $s \leq n$, where s is minimal and the q_i's are quadratic on separate sets of variables, and where each q_i does not depend on $x_1, \ldots, x_i$. This is done by translating in a systematic way what is known on the best affine approximations of quadratic functions, and deducing formulae allowing a direct computation of the second order nonlinearity of the cubic functions above, without needing the Walsh transform. This provides a lower bound on $\rho(2,n)$ (more precisely, on the covering radius of $RM(2,n)$ in $RM(3,n)$). This lower bound is compared with the upper bound from [309] that we shall recall as Relation (4.1) below; for $n \leq 20$ the lower and upper bounds are not that far from each other, and the lower bound performs also well asymptotically. These results are extended to more general Maiorana-McFarland functions in [714], with a focus on functions $f(x,y) = x \cdot \phi(y)$ where ϕ is perfect nonlinear, showing that some of these functions have best quadratic approximation achieved by affine functions and that the lower bound of [715] on $\rho(2,n)$ can be improved. See also [1109].
- $\rho(n,n)$, $\rho(n-1,n)$ and $\rho(n-2,n)$ equal respectively 0, 1 and 2.
- $\rho(n-3,n)$, $n \geq 3$, has been determined in [837]: it equals $n+1$ if n is odd and $n+2$ if n is even.
- More results can be found in [610, 612, 614, 616].

We summarize what is known for small numbers of variables in Table 4.1.

$r \backslash n$	1	2	3	4	5	6	7	8	9
1	0	1	2	6	12	28	56	120	242-244[686]
2		0	1	2	6	18[1020]	40[1104]	84-100	171-220
3			0	1	2	8	20-23[610]	43-67	111-167
4				0	1	2	8	22-31	58-98
5					0	1	2	10	23-41
6						0	1	2	10
7							0	1	2
8								0	1
9									0

Table 4.1 Lower and upper bounds on the covering radii of Reed-Muller codes for small n

General lower and upper bounds and more results are given in [375, 378, 379]. A first lower bound is simply the translation of the sphere covering bound: $2^{1+n+\binom{n}{2}+\cdots+\binom{n}{r}} \sum_{i=0}^{\rho(r,n)} \binom{n}{i} \geq 2^{2^n}$, and two other lower bounds are due to [378]: $\rho(r,n) \geq \begin{cases} 2^{n-r-3}(r+4), r \text{ even} \\ 2^{n-r-3}(r+5), r \text{ odd} \end{cases}$ for $r \leq n-3$ and $\rho(r,n) \geq 2^{n-r}$ for $2 \leq r \leq$

[2] These functions are closely related to Maiorana-McFarland's (MM) functions, see page 188; in the case of the so-called separable functions, they are MM (up to quadratic functions).

$n-3$ and $n \geq 6$. The best known upper bound, from [309], is as follows:
- a bound is first obtained for $r = 2$:

$$\rho(2,n) \leq \left\lfloor 2^{n-1} - \frac{\sqrt{15}}{2} \cdot 2^{\frac{n}{2}} \cdot \left(1 - \frac{122929}{21 \cdot 2^n} - \frac{155582504573}{4410 \cdot 2^{2n}}\right) \right\rfloor \tag{4.1}$$

- this bound is generalized to every r by using the inequality $\rho(r,n) \leq \rho(r-1,n-1) + \rho(r,n-1)$ which is easily proved,
- and this implies that, asymptotically, $\rho(r,n)$ is bounded above by:

$$2^{n-1} - \frac{\sqrt{15}}{2} \cdot (1+\sqrt{2})^{r-2} \cdot 2^{\frac{n}{2}} + \mathcal{O}(n^{r-2}).$$

The principle of the proof of (4.1) is to use that, for any two n-variable Boolean functions f and g, we have $\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus g(x)} = 2^n - 2\, d_H(f,g)$, which shows:

$$\rho(2,n) = 2^{n-1} - \frac{1}{2} \min_{f \in \mathcal{BF}_n} \max_{g \in RM(2,n)} \left| \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus g(x)} \right|$$

and to use that:

$$\max_{g \in RM(2,n)} \left| \sum_{x \in \mathbb{F}_2 n} (-1)^{f(x) \oplus g(x)} \right| \geq \sqrt{\frac{\sum_{g \in RM(2,n)} \left(\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus g(x)} \right)^{2k+2}}{\sum_{g \in RM(2,n)} \left(\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus g(x)} \right)^{2k}}}.$$

We have:

$$\sum_{g \in RM(2,n)} \left(\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus g(x)} \right)^{2k} =$$

$$\sum_{x_1, \dots, x_{2k} \in \mathbb{F}_2^n} (-1)^{\bigoplus_{i=1}^{2k} f(x_i)} \left(\sum_{g \in RM(2,n)} (-1)^{\bigoplus_{i=1}^{2k} g(x_i)} \right),$$

and the mapping $g \in RM(2,n) \mapsto \bigoplus_{i=1}^{2k} g(x_i)$ being an $\mathbb{F}_2$-linear form over $RM(2,n)$, the sum $\sum_{g \in RM(2,n)} (-1)^{\bigoplus_{i=1}^{2k} g(x_i)}$ equals the size of $RM(2,n)$ when $\bigoplus_{i=1}^{2k} g(x_i)$ is the null function, and otherwise, this sum equals 0. We refer to [309] for the rest of the proof, which is more technical.
We have seen at page 103 that the suitably normalised r-th order nonlinearity of a random Boolean function converges strongly for all $r \geq 1$ as shown in [1021], but no limit on $\rho(r,n)$ similar to the one recalled above for $\rho(1,n)$ is known yet.

Remark. The so-called "Gowers norm" (whose definition involves *k-th order derivatives* of Boolean functions) is related to the *covering radius* of Reed-Muller codes. We devote Section 12.4 to it. □

A notion on cosets of the first-order Reed-Muller code called *orphan* or *urcoset* is related to the notion of *plateauedness* of Boolean functions, see page 289.

4.2 Other codes related to Boolean functions

4.2.1 Linear codes

There exist mainly two principles of constructions of linear codes (which are binary[3]) from Boolean functions and vectorial functions (surveys can be found in [454] and [455]):

- *Codes from Boolean functions*: Let f be an n-variable Boolean function. Recall that we denote its support by $supp(f)$. We choose an order on it and assume that it has rank n. We define the linear code $C_{supp(f)}$ whose codewords are the lists of values of the restrictions to $supp(f)$ of the linear functions $v \cdot x$, where $v \in \mathbb{F}_2^n$ and "$\cdot$" is an *inner product* in $\mathbb{F}_2^n$. In other words, $C_{supp(f)}$ equals the code of all linear functions punctured at all the positions which are not in $supp(f)$. Any linear code whose generator matrix G has its columns all different [4] can be obtained by this construction, introduced in the early 1970s and called nowadays the defining-set construction. Indeed, the codewords of such code are obtained as $(v \times G)_{v\in\mathbb{F}_2^k}$. The support of f being assumed to have rank n, the parameters of this code are $[w_H(f), n, d]$, where d needs to be determined for each function f. A generator matrix is made of the elements of $supp(f)$ put in columns. When f is a bent function in $n \geq 4$ variables (n even), the code has two weights (this property is characteristic) and d is their minimum (see [1120] and other papers by Wolfmann written in French, whose results have been rediscovered in [453] among many other results); we recall why in Chapter 6 at page 219 and give more characterizations. More generally we can consider the code obtained from any Reed-Muller code by puncturing it at all positions outside $supp(f)$. Note that $\mathbb{F}_2^n$ can be identified with $\mathbb{F}_{2^n}$, and the inner product can then be $v \cdot x = tr_n(vx)$.
  Cyclic codes are also related to algebraic immunity, see page 357.
- *Codes from vectorial functions*:
  - Given inner products in $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$ (that we shall both denote by "$\cdot$" for simplicity), can be associated to each vectorial function $F : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$ having no affine component (*i.e.* having strictly positive nonlinearity), the subcodes C'_F and C''_F of $RM(r,n)$ where $r \geq 2$ is the algebraic degree of F, whose codewords are the Boolean functions $v \cdot F(x) \oplus u \cdot x$, respectively $v \cdot F(x) \oplus u \cdot x \oplus \epsilon$, where u ranges over $\mathbb{F}_2^n$, v over $\mathbb{F}_2^m$ and ϵ over $\mathbb{F}_2$. More precisely, the codewords are the lists of values of these functions, some order being chosen on $\mathbb{F}_2^n$. The Hamming weight of codeword $v \cdot F(x) \oplus u \cdot x$ (resp. $v \cdot F(x) \oplus u \cdot x \oplus \epsilon$) equals $2^{n-1} - \frac{1}{2}W_F(u,v)$ (resp. $2^{n-1} - \frac{(-1)^\epsilon}{2}W_F(u,v)$). Code C''_F equals the union of the cosets $v \cdot F + RM(1,n)$, where v ranges over $\mathbb{F}_2^m$. The parameters of C''_F are $[2^n, n+m+1, d]$ where d is the nonlinearity

[3] There also exist constructions of non-binary codes from so-called *p-ary functions*, that is, Boolean-like functions in characteristic p.

[4] Such codes are sometimes called projective codes.

of F, see more in [257, 269, 1147, 1099, 53]. A generator matrix of C'_F is $\begin{bmatrix} \dots & x & \dots \\ \dots & F(x) & \dots \end{bmatrix}$, and a generator matrix of C''_F is $\begin{bmatrix} \dots & 1 & \dots \\ \dots & x & \dots \\ \dots & F(x) & \dots \end{bmatrix}$, where x and $F(x)$ are column vectors and x ranges over $\mathbb{F}_2^n$. Conversely, let C be a linear $[2^n, k, d]$ binary code such that $k > n + 1$ and including the Reed-Muller code $RM(1, n)$ as a subcode. Let $(b_1, \dots, b_k)$ be a basis of C completing a basis $(b_1, \dots, b_{n+1})$ of $RM(1, n)$. The n-variable Boolean functions corresponding to the vectors $b_{n+2}, \dots, b_k$ are the coordinate functions of an $(n, k - n - 1)$-function whose nonlinearity is d.
The CCZ equivalence between (n, m)-functions can be expressed in terms of these codes (see the remark at page 411).
Often, we have $m = n$ and $\mathbb{F}_2^n$ is identified with $\mathbb{F}_{2^n}$, the inner product being then $u \cdot x = tr_n(ux)$.
- When $m = n$, the dual of the code ${C'_F}^*$, equal to C'_F punctured at the zero position, plays an important role with respect to APN functions F (defined in Definition 41, page 159) such that $F(0_n) = 0_m$, see Proposition 160, page 410; the dual of C''_F plays a similar role with respect to general APN functions (see the remark at page 411). When F is a *power function*, ${C'_F}^*$ is cyclic; we find among such codes related to APN functions in particular the dual of the historical 2-error-correcting BCH code of length $2^n - 1$.
- Codes (which are constant-weight) are deduced in [862] from o-polynomials in relation with vectorial bent functions (see Definition 30, page 140).
- The other notion of nonlinearity of vectorial functions nl_v introduced at page 144 has been studied in [788] in relation with codes.

A hybrid construction is proposed in [1072] and other constructions of cyclic codes from vectorial (possibly APN) functions are given in [452, 464].
We have seen (in the remark at page 113) connections between algebraic immunity and linear codes; connections exist with cyclic codes, see page 357.

4.2.2 Unrestricted codes

Boolean functions play an important role with nonlinear codes, as we shall see in Section 6.1.22 about Kerdock codes.
Vectorial functions also play a role. Given any (n, m)-function F, we can consider the code $\mathcal{G}_F = \{(x, F(x)); x \in \mathbb{F}_2^n\}$ (the graph of F viewed as a code). When F is linear, $\mathcal{G}_F$ is a linear code, but it happens that nonlinear functions F provide better parameters, as in the case of Kerdock and Preparata codes.
Codes of the form $\mathcal{G}_F$ are *systematic*: the set of n first indices has the property that every possible n-tuple occurs in exactly one codeword within the coordinates of indices $1, \dots, n$. We call $\{1, \dots, n\}$ an *information set* of C. Conversely, if a subset I of $\{1, \dots, N\}$, where N is the length C, is an information set, then,

up to permutation of the coordinates, it has the form $\mathcal{G}_F$. It is easily shown that all linear codes have such property: the generator matrix having rank k, it has k linearly independent columns and placing them in the k first positions, we can multiply the resulting permuted generator matrix on the left by the inverse of the invertible square matrix made of its first k columns; this provides a systematic permuted generator matrix.
Such codes play a role in relation with countermeasures to side channel attacks, see Section 12.1.1, page 467. They need then to be complementary information set codes (CIS) (see the same page) in the sense that they admit two complementary information sets. This is a necessary and sufficient condition so that F can be a permutation.

4.2.3 Codes and diffusion layers in block ciphers

The diffusion (see definition at page 95) ensured by a mapping F can be studied by analyzing the pairs $(x-y, F(x)-F(y))$. In practice, q will be a power of 2 and "$-$" will be the same as "$+$".
These pairs play also a role with respect to the differential attack.

Definition 43 *Let q be a power of a prime. The differential branch number of a function $F : \mathbb{F}_q^n \mapsto \mathbb{F}_q^m$ is defined as: $\beta(F) = \min_{x,y\in\mathbb{F}_q^n, x\neq y}\{d_H(x,y) + d_H(F(x),F(y))\}$, the minimum distance of code $\mathcal{G}_F$. The differential branch number of a linear function $F : \mathbb{F}_q^n \mapsto \mathbb{F}_q^m$ (or of its matrix) is then defined as: $\beta(F) = \min_{x\in\mathbb{F}_q^n, x\neq 0_n}\{w_H(x) + w_H(F(x))\}$.*

$\beta(F)$ quantifies the level of diffusion induced by F when it is used as a diffusion layer in a block cipher. When $q = 2^n$ and F diffuses the outputs of (n,n)-S-boxes, $\beta(F)$ indicates the minimum number of *active* S-boxes.
Also, the larger $\beta(F)$, the more difficult the research of *characteristics* needed for mounting differential attacks (see page 157). An r-round characteristic constitutes an $(r+1)$-tuple of difference patterns: $(\Delta X_0, \Delta X_1, \ldots, \Delta X_r)$. The probability of this characteristic is the probability that an initial difference pattern ΔX_0 propagates to difference patterns $\Delta X_1, \ldots, \Delta X_r$ after $1, 2, \ldots, r$ rounds.

If F is linear[5], then $\mathcal{G}_F$ is linear and the diffusion is studied by analyzing the pairs $(a, F(a))$.
It is easily shown that the differential branch number of a linear permutation equals that of its inverse and that, for every $F : \mathbb{F}_q^n \mapsto \mathbb{F}_q^m$, we have $\beta(F) \leq m+1$, with equality if and only if the code $\mathcal{G}_F = \{(x, F(x));\, x \in \mathbb{F}_q^n\}$ is *MDS*.
If $\mathcal{G}_F$ is an MDS $[N,k,d]$-code such that $N > k$ then any *punctured code* obtained, for instance, by erasing the last coordinate of each codeword is an MDS $[N-1, k, d-1]$ code. For every prime power q, every $N < q$ and every $k \leq N$, we know

[5] Contrary to a substitution layer, a diffusion layer does not need to be nonlinear; for reasons of speed, it is then better to choose it linear.

that there exist MDS codes over $\mathbb{F}_q$ of parameters $[N, k, N-k+1]$ (Reed-Solomon codes, for instance). This allows to build optimal diffusion layers.

Definition 44 *The linear branch number of a linear function $F : \mathbb{F}_q^n \mapsto \mathbb{F}_q^m$ is defined as:*

$$\beta'(F) = \min_{\substack{a,b \in \mathbb{F}_q^n,\ (a,b)\neq(0_n,0_n) \\ a\cdot x \oplus b\cdot F(x)\, unbalanced}} \{w_H(a) + w_H(b)\}.$$

The linear branch number of a function $F : \mathbb{F}_q^n \mapsto \mathbb{F}_q^m$ is the *dual distance* of the code $\{(x, F(x)),\ x \in \mathbb{F}_q^n\}$. Then if F is linear and F' is the linear mapping whose matrix is the transpose of that of F, we have $\beta'(F) = \beta(F')$.
In [795] is proposed a nonlinear diffusion layer based on Kerdock codes.

4.2.4 Codes and association schemes

Association schemes, originated in statistics, have been used in coding theory and combinatorics in the seventies by Delsarte, McEliece and others, to obtain strong upper bounds on the size of codes and other combinatorial objects, and to characterize those objects (such as perfect codes) which meet these bounds. They have also been studied in relation to Boolean functions. They are related to graphs (that we encountered at page 89). For more details, the reader is referred to [411, 424].

Definition 45 *Let V be a finite set of vertices and $\{G_0, G_1, \ldots, G_d\}$ be binary relations on V with $G_0 = \{(x, x) : x \in V\}$. Then the decomposition $(V; G_0, G_1, \ldots, G_d)$ (represented in short as $(V, \{G_i\}_{0\le i\le d})$) is called an association scheme of class d on V provided that the following properties hold:*

- *$V \times V = G_0 \cup G_1 \cup \cdots \cup G_d$ and $G_i \cap G_j = \emptyset$ for $i \neq j$;*
- *${}^tG_i = G_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where ${}^tG_i = \{(x, y) : (y, x) \in G_i\}$; (If $i' = i$, then we call G_i symmetric.)*
- *for $i, j, k \in \{0, 1, \ldots, d\}$ and $x, y \in V$ with $(x, y) \in G_k$, the number $p_{ij}^k := \#\{z \in V : (x, z) \in G_i, (z, y) \in G_j\}$ is a constant.*

An association scheme is said to be symmetric if each G_i is symmetric.

One of the well-known construction methods of association schemes is to use Schur rings. Constuctions of association schemes from bent functions (in odd characteristic) have been considered in the literature (see e.g. [964]). In [506] have been studied Boolean functions arising in some popular association schemes.

4.2.5 Codes and secret sharing

We have seen in Subsection 3.6.1, page 168, how codes play a role with respect to secret sharing and that Boolean functions can play a role in this domain.

5 Functions with weights, Walsh spectra and nonlinearities easier to study

In this chapter, we visit diverse types of Boolean and vectorial functions, whose study is simpler than for general functions. We will encounter them again in almost all subsequent chapters.

5.1 Affine functions and their combinations

Affine functions are weak cryptographically (see Sections 3.1 and 3.4), and many criteria seen in Chapter 3 quantify the difference between cryptographic functions and affine functions. However, good functions can be obtained by combining affine functions in different ways. Before presenting them, we briefly address affine functions themselves.

Affine Boolean functions

The Hamming weights and the Walsh spectra of affine Boolean functions (*i.e.* of the codewords of $RM(1,n)$) are peculiar.

The Hamming weight of any non-constant affine function is 2^{n-1} since this is the size of any affine hyperplane. The Hamming weights of the two constant functions are of course 0 and 2^n.

Recall from page 55 that, given any *inner product* "$\cdot$", any affine Boolean function can be written in the form $\ell(x) = a \cdot x \oplus \epsilon$, where $a \in \mathbb{F}_2^n$ and $\epsilon \in \mathbb{F}_2$. The Walsh transform of such function takes null value at every vector $u \neq a$ and takes value $2^n\,(-1)^\epsilon$ at a. The Walsh support is then a singleton.

Conversely, every Boolean function whose Walsh support is a singleton is an affine function, according to the inverse Walsh transform formula (2.43), page 78, and to Parseval's Relation (2.47), page 79.

Of course, the nonlinearity of any affine Boolean function is null and this is characteristic of affine functions.

Affine vectorial functions

The component functions of affine (n,m)-functions are affine Boolean functions (this property is characteristic of affine vectorial functions). If $F(x) = L(x) + a$ where L is a linear (n,m)-function and $a \in \mathbb{F}_2^m$, then, for every $(u,v) \in \mathbb{F}_2^n \times \mathbb{F}_2^m$, $W_F(u,v) = \sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot L(x)\oplus u\cdot x\oplus v\cdot a} = \sum_{x\in\mathbb{F}_2^n}(-1)^{(L^*(v)+u)\cdot x\oplus v\cdot a}$ equals

$2^n(-1)^{v\cdot a}$ if $u = L^*(v)$ and is null otherwise, where L^* is the adjoint operator of L, that is, where $v \cdot L(x) = L^*(v) \cdot x$ for every v and x (in the case where "$\cdot$" is the *usual inner product*, the matrix of L^* is simply the transpose of that of L). Of course, the nonlinearity of any affine vectorial function is null, but this is not characteristic of affine vectorial functions (it is characteristic of the fact that at least one component function of F is affine).

5.1.1 Maiorana-McFarland functions

Since the Walsh transform of affine functions behaves so simply, it is natural to try building more robust functions by using them as building blocks in constructions. A first way is based on the additive structure of $\mathbb{F}_2^n$ as an $\mathbb{F}_2$-vector space. This leads to considering those functions whose restrictions to each coset $a + E$ of some $\mathbb{F}_2$-vector subspace E of $\mathbb{F}_2^n$ are affine. Up to affine equivalence, we can take $E = \mathbb{F}_2^r \times \{0_{n-r}\}$. Then the corresponding functions are called *Maiorana-McFarland* (MM) functions, since originally, the idea of such functions comes from Maiorana and McFarland [834], as reported in [441]. The general class, obtained by considering all *affinely equivalent* functions to Maiorana-McFarland functions, is called the *completed Maiorana-McFarland class*.

Maiorana-McFarland Boolean functions

They have been first investigated for building bent functions, see Section 6.1.15, page 233, and later been considered in [181] for constructing correlation immune and resilient functions, see Subsection 7.1.8, page 319. Recall that every affine Boolean function has the form $a \cdot x \oplus \epsilon$. The idea of Maiorana-McFarland's construction corresponds to making a and ϵ vary. For convenience, instead of denoting the input to the global function by $x = (x_1, \ldots, x_n)$, we denote it then by (x, y), where $x = (x_1, \ldots, x_r)$ and $y = (x_{r+1}, \ldots, x_n)$.

Definition 46 *Let n and r be any positive integers such that $r \leq n$. We call Maiorana-McFarland's function any n-variable Boolean function of the form:*

$$f(x,y) = x \cdot \phi(y) \oplus g(y); \quad x \in \mathbb{F}_2^r,\ y \in \mathbb{F}_2^{n-r}, \tag{5.1}$$

where ϕ is a function from $\mathbb{F}_2^{n-r}$ to $\mathbb{F}_2^r$ and g is an $(n-r)$-variable Boolean function. We denote by MM_r the corresponding class.

The size of this class roughly equals $2^{(r+1)2^{n-r}}$.
An example already seen of Maiorana-McFarland Boolean function is the address function (see page 87), with $r \approx n - \log_2(n)$. Note that, for every $r < n$, we have $\mathrm{MM}_{r+1} \subseteq \mathrm{MM}_r$ (this can be seen directly with Relation (5.1) or by the fact that the restriction of an affine function to an affine subspace is affine) and that $\mathrm{MM}_1 = \mathcal{BF}_n$ (since every function in one variable is necessarily affine).
The algebraic degree of f in (5.1) is at most $n - r + 1$ (and at most $n - r$ if $\sum_{y\in\mathbb{F}_2^{n-r}} \phi(y) = 0_r$) since the algebraic degree of ϕ is at most $n-r$ (and at most $n - r - 1$ if $\sum_{y\in\mathbb{F}_2^{n-r}} \phi(y) = 0_r$). We shall see in Section 5.2 that all *quadratic*

functions belong to the completed $MM_{\lceil \frac{n}{2} \rceil}$ class.

Remark Maiorana-McFarland functions can be viewed as the concatenations of *affine functions*. Indeed, let us order all the binary words of length n in lexicographic order, with the bit of higher weight on the right handside. Then, the *truth-table* of f is the concatenation of the truth-tables of its restrictions obtained by fixing the values of the $n-r$ last bits of the input and letting the r first input bits freely range over $\mathbb{F}_2$. And f is an MM_r function if and only if all these restrictions are affine. □

The calculation of Hamming weight, Walsh spectrum and nonlinearity are easier for functions in MM_r, $r \geq 2$, than for general Boolean functions, and in some cases can be completely determined. Note that since the input to f is written in the form (x, y) where $x \in \mathbb{F}_2^r$, $y \in \mathbb{F}_2^{n-r}$, the input to W_f is better written (u, v), where $u \in \mathbb{F}_2^r$, $v \in \mathbb{F}_2^{n-r}$.

Proposition 53 *Let f be the function given by Relation (5.1). Then, assuming that the* inner product *in $\mathbb{F}_2^r \times \mathbb{F}_2^{n-r}$ writes $(u, v) \cdot (x, y) = u \cdot x \oplus v \cdot y$ (where we use the same notation "$\cdot$" for denoting inner products in $\mathbb{F}_2^r$ and $\mathbb{F}_2^{n-r}$), we have:*

$$W_f(u, v) = 2^r \sum_{y \in \phi^{-1}(u)} (-1)^{g(y) \oplus v \cdot y}; \quad u \in \mathbb{F}_2^r,\, v \in \mathbb{F}_2^{n-r},$$

where $\phi^{-1}(u)$ denotes the pre-image of u by ϕ. Hence:

$$w_H(f) = 2^{n-1} - 2^{r-1} \sum_{y \in \phi^{-1}(0_r)} (-1)^{g(y)}$$

and

$$nl(f) = 2^{n-1} - 2^{r-1} \max_{u \in \mathbb{F}_2^r,\, v \in \mathbb{F}_2^{n-r}} \left| \sum_{y \in \phi^{-1}(u)} (-1)^{g(y) \oplus v \cdot y} \right|.$$

Proof. We have

$$\begin{aligned} W_f(u, v) &= \sum_{x \in \mathbb{F}_2^r, y \in \mathbb{F}_2^{n-r}} (-1)^{f(x,y) \oplus u \cdot x \oplus v \cdot y} \\ &= \sum_{y \in \mathbb{F}_2^{n-r}} \left((-1)^{g(y) \oplus v \cdot y} \sum_{x \in \mathbb{F}_2^r} (-1)^{(\phi(y) + u) \cdot x} \right) \\ &= 2^r \sum_{y \in \phi^{-1}(u)} (-1)^{g(y) \oplus v \cdot y}, \end{aligned}$$

since $\sum_{x \in \mathbb{F}_2^r} (-1)^{(\phi(y)+u) \cdot x}$ is null when $\phi(y) \neq u$. □

Proposition 53 shows that the Walsh support of f is included in $Im(\phi) \times \mathbb{F}_2^{n-r}$. Note that this Walsh support can be made very small (minimizing the size of

$Im(\phi)$ and the value of $n-r$), even while ensuring some properties like the nonexistence of linear structure.

We shall see that MM_r class provides easy constructions of bent (or highly nonlinear) functions, correlation immune functions and resilient functions. It will then be important to be able to say if a given Boolean function is in the completed MM_r class or not. The following proposition is an easy extension of an observation from [441]:

Proposition 54 *An n-variable Boolean function f belongs to the completed MM_r class if and only if there exists an r-dimensional vector space E such that D_aD_bf is the null function for every $a,b\in E$.*

Proof. The condition is clearly necessary. It is also sufficient since it means that each restriction of f to a coset of E is affine. $\square$

Note that for such function, E is in general not the linear kernel of f (see Definition 25, page 120); it can be a superset of the linear kernel.

Maiorana-McFarland vectorial functions

It is easily seen that an r-variable vectorial function is linear if and only if all its component functions are linear. Let n,r and m be positive integers such that $r\le n$. Let F be any function of the form:

$$F:(x,y)\in\mathbb{F}_2^r\times\mathbb{F}_2^{n-r}\mapsto\psi(x,y)+G(y)\in\mathbb{F}_2^m, \tag{5.2}$$

where G is any function from $\mathbb{F}_2^{n-r}$ to $\mathbb{F}_2^m$ and $\psi:\mathbb{F}_2^r\times\mathbb{F}_2^{n-r}\mapsto\mathbb{F}_2^m$ is such that, for every $y\in\mathbb{F}_2^{n-r}$, the function $x\mapsto\psi(x,y)$ is linear. Then, for every $y\in\mathbb{F}_2^{n-r}$ and $w\in\mathbb{F}_2^m$, there exists $\phi(y,w)\in\mathbb{F}_2^r$ such that $w\cdot\psi(x,y)=\phi(y,w)\cdot x$, and this property is characteristic of the functions of the form (5.2). For every $(u,v,w)\in\mathbb{F}_2^r\times\mathbb{F}_2^{n-r}\times\mathbb{F}_2^m$, we have:

$$\begin{aligned}W_F((u,v),w)&=\sum_{(x,y)\in\mathbb{F}_2^r\times\mathbb{F}_2^{n-r}}(-1)^{(\phi(y,w)+u)\cdot x\oplus w\cdot G(y)\oplus v\cdot y}\\&=2^r\sum_{y\in\mathbb{F}_2^{n-r};\ \phi(y,w)=u}(-1)^{w\cdot G(y)\oplus v\cdot y}.\end{aligned}$$

Remark If r divides n, then we can endow $\mathbb{F}_2^n$ with the structure of the field $\mathbb{F}_{2^n}$ and $\mathbb{F}_2^r$ with the structure of subfield $\mathbb{F}_{2^r}$ of $\mathbb{F}_{2^n}$. In particular, if $r=\frac{n}{2}$ (which will be well suited for designing bent functions), we can identify $\mathbb{F}_2^n$ with $\mathbb{F}_{2^{\frac{n}{2}}}\times\mathbb{F}_{2^{\frac{n}{2}}}$ and we consider the functions of the form

$$F(x,y)=L(x\,\phi(y))+G(y), \tag{5.3}$$

where the product $x\,\phi(y)$ is calculated in $\mathbb{F}_{2^{\frac{n}{2}}}$ and L is any linear or affine function from $\mathbb{F}_{2^{\frac{n}{2}}}$ to $\mathbb{F}_2^m$, ϕ is any function from $\mathbb{F}_{2^{\frac{n}{2}}}$ to itself and G is any $(\frac{n}{2},m)$-function. $\square$

5.1.2 Niho and $\mathcal{PS}_{ap}$-like functions

When $\mathbb{F}_2^n$ is identified with $\mathbb{F}_{2^n}$, we can also use the multiplicative structure of $\mathbb{F}_{2^n}^*$ to build Boolean functions from affine functions. Similarly to the case of Maiorana-McFarland functions, in which we considered additive subgroups of $\mathbb{F}_2^n$ and their cosets, we can consider multiplicative subgroups of $\mathbb{F}_{2^n}^*$ and their cosets. A natural choice[1] as a subgroup is the multiplicative group of a subfield $\mathbb{F}_{2^m}$ of $\mathbb{F}_{2^n}$ (where m is a divisor of n). We can view $\mathbb{F}_{2^n}^*$ as the union of the cosets $\mu\,\mathbb{F}_{2^m}^*$ of $\mathbb{F}_{2^m}^*$, where μ ranges over a subset U of $\mathbb{F}_{2^n}^*$ containing one representative of each coset of $\mathbb{F}_{2^m}^*$ and one only (U has then $\frac{2^n-1}{2^m-1}$ elements). Under some condition, it is possible to take U equal to the multiplicative subgroup of $\mathbb{F}_{2^n}^*$ of order $\frac{2^n-1}{2^m-1}$. This is possible when 2^m-1 and $\frac{2^n-1}{2^m-1}$ are co-prime (which is always the case if n is even and $m=\frac{n}{2}$, in which case the representation of the elements of $\mathbb{F}_{2^n}^*$ in the form μx, $x\in\mathbb{F}_{2^m}^*$, is often called *polar representation*[2]), since there exist then relative integers i,j such that $i(2^m-1)+j\frac{2^n-1}{2^m-1}=1$, and given a *primitive element* α of $\mathbb{F}_{2^n}$, we have then $\alpha=(\alpha^{2^m-1})^i\left(\alpha^{\frac{2^n-1}{2^m-1}}\right)^j$ and $\alpha^{2^m-1}\in U$, $\alpha^{\frac{2^n-1}{2^m-1}}\in\mathbb{F}_{2^m}^*$. It is observed in [804] that, if[3] $n=2m$, any (n,n)-function F (and therefore any n-variable Boolean function) can then be uniquely represented by a polynomial in the form $F(\mu x)=\sum_{s=0}^{2^m-2}\sum_{t=0}^{2^m}a_{s,t}\mu^t x^s$, where $a_{s,t}\in\mathbb{F}_{2^n}$, $\mu\in U, x\in\mathbb{F}_{2^m}^*$ (with additionally the indication of the value of $F(0)$), and that if $F(0)=\sum_{\mu\in U,x\in\mathbb{F}_{2^m}^*}F(\mu x)$, its algebraic degree equals the maximal 2-weight of $s(2^m+1)u+t(2^m-1)v$ [mod 2^n-1] such that $a_{s,t}\neq 0$, where $(2^m+1)u+(2^m-1)v=1$.

We consider then those n-variable Boolean functions whose restrictions to the cosets $\mu\,\mathbb{F}_{2^m}^*$, where m divides n, coincide with affine functions:

$$f(\mu\,x)=tr_m(x\,\phi(\mu))+g(\mu);\quad \mu\in U,\ x\in\mathbb{F}_{2^m}^*,\tag{5.4}$$

where ϕ is a function from U to $\mathbb{F}_{2^m}$ and g is a Boolean function over U. And a value must still be chosen for $f(0)$. Note that if each restriction to $\mu\mathbb{F}_{2^m}$ has algebraic degree less than m (in particular if $d_{alg}(f)<m$), then the univariate representation $f(z)=\sum_{i=0}^{2^n-2}a_iz^i$ of f satisfies "$(i\neq 0$ and $a_i\neq 0)\Rightarrow$ $(i$ [mod $2^m-1]\in I)$", where $I=\{2^j;j=0,\ldots,m-1\}$: this sufficient condition is indeed necessary since, assuming without loss of generality that $a_0=0$, for every $\omega\in\mathbb{F}_{2^n}^*$, the function $x\in\mathbb{F}_{2^m}\mapsto f(\omega x)=\sum_{i=0}^{2^n-2}a_i\omega^i x^{i[\mathrm{mod}\ 2^m-1]}$ being linear, we have that $k\notin I\Rightarrow \displaystyle\sum_{\substack{0\le i\le 2^n-2\\ i\equiv k\ [\mathrm{mod}\ 2^m-1]}}a_i\omega^i=0$, and by uniqueness of the univariate representation of the functions of $\omega\in\mathbb{F}_{2^n}$, this completes the proof [311].

Recall that functions tr_n, tr_m and tr_m^n (see page 60) satisfy that, for every

[1] But not the only one; investigations could be made on other subgroups, like those of order $\frac{2^n-1}{2^m-1}$, where m divides n (for which affinity would no more be the property on which the functions would be built).

[2] A slightly different representation is the trace 0/trace 1 representation, see [547].

[3] More general cases are studied there.

$a \in \mathbb{F}_{2^n}$, we have $tr_n(a) = tr_m(tr_m^n(a))$, and that, for every $u \in \mathbb{F}_{2^n}$ and $x \in \mathbb{F}_{2^m}$, we have $tr_m^n(ux) = x\, tr_m^n(u)$. Therefore, for $\mu \in U$, $x \in \mathbb{F}_{2^m}$, we have $tr_n(u\mu x) = tr_m(xtr_m^n(u\mu))$. We have then, for every $u \in \mathbb{F}_{2^n}$:

$$\begin{aligned} W_f(u) &= (-1)^{f(0)} + \sum_{\mu \in U, x \in \mathbb{F}_{2^m}^*} (-1)^{tr_m(x\,[\phi(\mu)+tr_m^n(u\mu)]) \oplus g(\mu)} \\ &= (-1)^{f(0)} - \sum_{\mu \in U}(-1)^{g(\mu)} + 2^m \sum_{\mu \in U; \phi(\mu)+tr_m^n(u\mu)=0} (-1)^{g(\mu)}, \quad (5.5) \end{aligned}$$

$$w_H(f) = 2^{n-1} - \frac{1}{2}\left((-1)^{f(0)} - \sum_{\mu \in U}(-1)^{g(\mu)} + 2^m \sum_{\mu \in \phi^{-1}(0)} (-1)^{g(\mu)}\right)$$

and

$$nl(f) = 2^{n-1} - \frac{1}{2}\max_{u \in \mathbb{F}_{2^n}} \left|(-1)^{f(0)} - \sum_{\mu \in U}(-1)^{g(\mu)} + 2^m \sum_{\mu \in U; \phi(\mu)+tr_m^n(u\mu)=0} (-1)^{g(\mu)}\right|.$$

A subcase is when function g is null in (5.4) (*i.e.* when the restrictions to the cosets $\mu\,\mathbb{F}_{2^m}^*$ coincide with linear functions), which leads (when $n = 2m$) to the so-called Niho Boolean functions (the name comes from a theorem by Niho [902] dealing with power functions; see a survey on their applications in [769]):

$$f(\mu\, x) = tr_m(x\,\phi(\mu)); \quad \mu \in U,\, x \in \mathbb{F}_{2^m}^*, \tag{5.6}$$

among which are bent functions, see Subsection 6.1.15. Another subcase is (also when $n = 2m$) when function ϕ is null in (5.4) (*i.e.* when the restrictions to the cosets $\mu\,\mathbb{F}_{2^m}^*$ coincide with constant functions), which leads to the so-called $\mathcal{PS}_{ap}$-like class of Boolean functions:

$$f(\mu\, x) = g(\mu); \quad \mu \in U,\, x \in \mathbb{F}_{2^m}^*, \tag{5.7}$$

among which are also bent functions, see Subsection 6.1.15 as well. In [769] are studied Niho power functions with few Walsh values.

Niho and $\mathcal{PS}_{ap}$-like classes in bivariate form

The last sum in (5.5) is not always easily simplified further since it deals with U, which has no additive structure in general. This can be circumvented when $n = 2m$ (this case can be generalized) by representing the elements of $\mathbb{F}_{2^n}$ by ordered pairs of elements of $\mathbb{F}_{2^m}$ (which is possible since $\mathbb{F}_{2^n}$ is a plane over $\mathbb{F}_{2^m}$). It is then easily seen that the subset U introduced at the beginning of the present subsection can be taken equal to $\{(0,1)\} \cup \{(1,\lambda), \lambda \in \mathbb{F}_{2^m}\}$ and one of the cosets of $\mathbb{F}_{2^m}^*$ becomes then $\{(0,y), y \in \mathbb{F}_{2^m}^*\}$ and the others become the sets $\{x, \lambda x), x \in \mathbb{F}_{2^m}^*\}$ where $\lambda \in \mathbb{F}_{2^m}$. We have then:

$$f(x,y) = \begin{cases} tr_m\left(x\,\phi(\frac{y}{x})\right) + g(\frac{y}{x}); & x \in \mathbb{F}_{2^m}^*, y \in \mathbb{F}_{2^m} \\ tr_m(a\,y) + \epsilon; & x = 0, y \in \mathbb{F}_{2^m}^* \\ f(0,0); & x = y = 0, \end{cases} \tag{5.8}$$

where $a \in \mathbb{F}_{2^m}$, $\epsilon \in \mathbb{F}_2$, ϕ is a function from $\mathbb{F}_{2^m}$ to $\mathbb{F}_{2^m}$, g is a Boolean function over $\mathbb{F}_{2^m}$, and where the products $x\,\phi(\frac{y}{x})$ and $a\,y$ are calculated in $\mathbb{F}_{2^m}$.
We have then, for every $u, v \in \mathbb{F}_{2^m}$, that $W_f(u,v)$ equals:

$$(-1)^{f(0)} + \sum_{y\in\mathbb{F}_{2^m}^*} (-1)^{tr_m(y\,(a+v))+\epsilon} + \sum_{x\in\mathbb{F}_{2^m}^*, y\in\mathbb{F}_{2^m}} (-1)^{tr_m(x\,[\phi(\frac{y}{x})+u+v\frac{y}{x}])+g(\frac{y}{x})}$$

$$= (-1)^{f(0)} - (-1)^{\epsilon} + \sum_{y\in\mathbb{F}_{2^m}} (-1)^{tr_m(y\,(a+v))+\epsilon} + \sum_{\substack{x\in\mathbb{F}_{2^m}^*\\ z\in\mathbb{F}_{2^m}}} (-1)^{tr_m(x\,[\phi(z)+u+vz])+g(z)}$$

$$= (-1)^{f(0)} + (2^m\delta_a(v)-1)(-1)^{\epsilon} + 2^m \sum_{\substack{z\in\mathbb{F}_{2^m};\\ \phi(z)+u+vz=0}} (-1)^{g(z)} - \sum_{z\in\mathbb{F}_{2^m}} (-1)^{g(z)}. \quad (5.9)$$

We have then: $\quad w_H(f) =$

$$2^{n-1} - \frac{1}{2}\left((-1)^{f(0)} + (2^m\delta_0(a)-1)(-1)^{\epsilon} + 2^m \sum_{z\in\phi^{-1}(0)} (-1)^{g(z)} - \sum_{z\in\mathbb{F}_{2^m}} (-1)^{g(z)}\right)$$

and

$$nl(f) = 2^{n-1} - \frac{1}{2}A,$$

where A equals

$$\max_{u,v\in\mathbb{F}_{2^m}} \left|(-1)^{f(0)} + (2^m\delta_a(v)-1)(-1)^{\epsilon} + 2^m \sum_{\substack{z\in\mathbb{F}_{2^m};\\ \phi(z)+u+vz=0}} (-1)^{g(z)} - \sum_{z\in\mathbb{F}_{2^m}} (-1)^{g(z)}\right|.$$

5.2 Quadratic functions and their combinations

The next functions to be naturally considered after affine ones are quadratic ones. We shall see that they offer a compromise between robustness and simplicity. They will play roles in almost all domains addressed in the subsequent chapters.

5.2.1 Quadratic Boolean functions

The behavior of quadratic Boolean functions (*i.e.* of the codewords of $RM(2,n)$) is rather simple (less, though, than that of affine functions). There are many results on their Walsh transform, that we shall try to present completely, but without being able to give all proofs, since this would take too much space.

Absolute value of the Walsh transform

Recall that Relation (2.55), page 81, states that, for every Boolean function f, we have $\mathcal{F}^2(f) = \sum_{b\in\mathbb{F}_2^n} \mathcal{F}(D_b f)$, where $\mathcal{F}(f) = \sum_{x\in\mathbb{F}_2^n} (-1)^{f(x)}$. If f is quadratic, then D_bf is affine for every $b \in \mathbb{F}_2^n$, and is therefore either balanced or constant. Since $\mathcal{F}(g) = 0$ for every balanced function g, we deduce:

$$\mathcal{F}^2(f) = 2^n \sum_{b\in\mathcal{E}_f} (-1)^{D_bf(0_n)}, \tag{5.10}$$

where $\mathcal{E}_f$ is the linear kernel (*i.e.* the set of all $b \in \mathbb{F}_2^n$ such that D_bf is constant, see Section 3.1). Since f is quadratic, $\mathcal{E}_f$ is also the kernel $\{x \in \mathbb{F}_2^n;\ \forall y \in \mathbb{F}_2^n,\ \beta_f(x,y) = 0\}$ of the symplectic[4] form associated to f:

$$\beta_f(x,y) = f(0_n) \oplus f(x) \oplus f(y) \oplus f(x+y).$$

In other words, $\mathcal{E}_f$ is the *radical* of the quadratic form.
The restriction of the function $b \mapsto D_bf(0_n) = f(b) \oplus f(0_n)$ to $\mathcal{E}_f$ being linear, since we have already seen after Definition 25, page 120, that the restriction of f to $\mathcal{E}_f$ is affine, we deduce from (5.10) that $\mathcal{F}^2(f)$ equals $2^n\,|\mathcal{E}_f|$ if $f(b) \oplus f(0_n)$ is null on $\mathcal{E}_f$ (*i.e.* if f is constant on $\mathcal{E}_f$), and is null otherwise. Note that in the former case, f is constant on any coset $a + \mathcal{E}_f$ of $\mathcal{E}_f$, since f and D_af are constant on $\mathcal{E}_f$. According to Relation (2.35), page 75 (and since the linear kernel of $f(x)\oplus a\cdot x$ equals that of f), this proves the following proposition, which shows in particular that the absolute value of the Walsh transform of every quadratic Boolean function takes only two values, one of which is 0 (such functions will be called plateaued in Section 6.2).

Proposition 55 *[209] Let n be any positive integer. Any n-variable quadratic function f is unbalanced if and only if its restriction to its linear kernel $\mathcal{E}_f$ (*i.e.* the kernel of its associated symplectic form) is constant, or equivalently, if every constant derivative of f is null. Then, f is constant on any coset of $\mathcal{E}_f$ and the Hamming weight of f equals $2^{n-1} \pm 2^{\frac{n+k}{2}-1}$ where k is the dimension of $\mathcal{E}_f$.
For every $a \in \mathbb{F}_2^n$ and every n-variable quadratic function f, $W_f(a)$ is nonzero if and only if the restriction of $f(x) \oplus a \cdot x$ to $\mathcal{E}_f$ is constant. Then, $W_f(a)$ equals $\pm 2^{\frac{n+k}{2}}$.*

Note that Proposition 55 implies that f is balanced if and only if there exists $b \in \mathbb{F}_2^n$ such that the derivative $D_bf(x) = f(x)\oplus f(x+b)$ equals the constant function 1. For non-quadratic Boolean functions, this condition for f to be balanced is sufficient but not necessary.
Note that, according to Parseval's relation, there exists a such that $W_f(a) \neq 0$. Proposition 55 implies then that $\frac{n+k}{2}$ is an integer (because the Hamming weight is an integer), and then that the co-dimension of $\mathcal{E}_f$ must be even. This co-dimension is the *rank of* β_f, also called by abuse of language the rank of f. Note that, given two quadratic functions f and g, we have $|rank(f\oplus g) - rank(f)| \le rank(g)$ because the rank of matrices is sub-additive: $rank(A+B) \le rank(A) + rank(B)$.
We also deduce:

[4] Bilinear, symmetric, and null for $x = y$; the associated matrix is called a *symplectic matrix*.

Corollary 9 *Let n be any positive integer and f any n-variable quadratic function. The nonlinearity of f equals $2^{n-1} - 2^{\frac{n+k}{2}-1} = 2^{n-1} - 2^{n-\frac{r_k(f)}{2}-1}$, where k is the dimension of the linear kernel of f and $r_k(f)$ is the rank of β_f.*

The Hamming weight of an n-variable quadratic Boolean function belongs then to the set $\{2^{n-1}\} \cup \{2^{n-1} \pm 2^i; i = \lceil \frac{n}{2} \rceil - 1, \ldots, n-1\}$ and can be any element of this set, since it is easily seen that the dimension of the linear kernel in the case of function $x_1x_2 \oplus x_3x_4 \oplus \cdots \oplus x_{2r-1}x_{2r}$ equals $n - 2r$. The nonlinearity of an n-variable quadratic Boolean function can be any element of the set $\{2^{n-1} - 2^i; i = \lceil \frac{n}{2} \rceil - 1, \ldots, n-1\}$, and if f has Hamming weight $2^{n-1} \pm 2^i$, then for every affine function l, the Hamming weight of the function $f \oplus l$ belongs to the set $\{2^{n-1} - 2^i, 2^{n-1}, 2^{n-1} + 2^i\}$.

The method seen above is particularly simple[5], but it does not allow determining whether the Hamming weight is $2^{n-1} - 2^i$ or $2^{n-1} + 2^i$ when the function is not balanced, nor determining the sign of the Walsh transform. It may be much more difficult to calculate this sign than the absolute value. Such calculation is sometimes necessary. This is the case for instance when trying to determine the absolute value of the Walsh transform of a cubic function by applying Relation (2.55), page 81, or when we calculate the size of the pre-image of an element $u \in \mathbb{F}_2^m$ by a quadratic function $F : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$, thanks to the formula $|\{x \in \mathbb{F}_2^n; F(x) = u\}| = 2^{-m} \sum_{x\in\mathbb{F}_2^n,\, v\in\mathbb{F}_2^m} (-1)^{v\cdot(F(x)+u)}$.

Dickson form of a quadratic function

A first important step, anterior to the method above, has been made by Dickson for calculating explicitly the Hamming weight of *quadratic functions*, by showing as described in [809, page 438] that any non-affine quadratic Boolean function f over $\mathbb{F}_2^n$ is affinely equivalent to:

$$x_1x_2 \oplus \cdots \oplus x_{2r-1}x_{2r} \oplus \ell, \tag{5.11}$$

where $2r$ is the rank of the quadratic function and ℓ is an affine function (which can be taken equal, up to affine equivalence, to 0, 1 or x_{2r+1}). This is easily shown: by hypothesis, f has a monomial of degree 2 in its ANF, and we can assume without loss of generality that this monomial is x_1x_2. The function has then the form $x_1x_2 \oplus x_1f_1(x_3, \ldots, x_n) \oplus x_2f_2(x_3, \ldots, x_n) \oplus f_3(x_3, \ldots, x_n)$ where f_1, f_2 are affine functions and f_3 is quadratic. Then, $f(x) = (x_1 \oplus f_2(x_3, \ldots, x_n))(x_2 \oplus f_1(x_3, \ldots, x_n)) \oplus f_1(x_3, \ldots, x_n)f_2(x_3, \ldots, x_n) \oplus f_3(x_3, \ldots, x_n)$ is affinely equivalent to the function $x_1x_2 \oplus f_1(x_3, \ldots, x_n)f_2(x_3, \ldots, x_n) \oplus f_3(x_3, \ldots, x_n)$. Applying this method recursively shows:

Theorem 10 *Every quadratic non-affine function is affinely equivalent to*

$$x_1x_2 \oplus \cdots \oplus x_{2r-1}x_{2r} \oplus x_{2r+1} \tag{5.12}$$

[5] Theoretically; in practice, calculating the dimension of the linear kernel is not always an easy task.

(where $r \leq \frac{n-1}{2}$) if it is balanced, to

$$x_1x_2 \oplus \cdots \oplus x_{2r-1}x_{2r} \tag{5.13}$$

(where $r \leq \frac{n}{2}$) if it has Hamming weight smaller than 2^{n-1} and to

$$x_1x_2 \oplus \cdots \oplus x_{2r-1}x_{2r} \oplus 1 \tag{5.14}$$

(where $r \leq \frac{n}{2}$) if it has Hamming weight larger than 2^{n-1}.

The unique expressions (5.12), (5.13) and (5.14) are called the *Dickson form* of the quadratic function. They allow describing precisely the weight distribution of $RM(2,n)$ [809, page 441].

Walsh transform when the function is given by its ANF

We have seen how a quadratic Boolean function can be put in the form $g(L(x)+b)$ where L is a linear automorphism and g is in Dickson form. Thanks to Relation (2.58), page 82 and to Lemma 4, page 77, it is then enough to be able to calculate $\sum_{x\in\mathbb{F}_2^n}(-1)^{x_1x_2\oplus\cdots\oplus x_{2r-1}x_{2r}\oplus u\cdot x}$, for every $u \in \mathbb{F}_2^n$ and every $r \in \{1,\ldots,\lfloor\frac{n}{2}\rfloor\}$. This sum equals:

$$\sum_{x\in\mathbb{F}_2^n}(-1)^{\bigoplus_{i=1}^{r}[(x_{2i-1}\oplus u_{2i})(x_{2i}\oplus u_{2i-1})\oplus u_{2i-1}u_{2i}]\oplus u_{2r+1}x_{2r+1}\oplus\cdots\oplus u_nx_n}$$

and equals then $2^{n-r}(-1)^{\bigoplus_{i=1}^{r}u_{2i-1}u_{2i}}$ if $u_{2r+1}=\cdots=u_n=0$ and 0 otherwise. Since the dimension k of the kernel $\mathcal{E}_f$ of the symplectic form $\beta_f(x,y)$ equals $n-2r$, this shows again that the Walsh transform $W_f(u)$ lies in $\{0,\pm 2^{(n+k)/2}\} = \{0,\pm 2^{n-r}\}$ for every u. But we have also the sign of $W_f(u)$.

Remark. Any quadratic function belongs to the *completed Maiorana-McFarland class*; this can be easily seen from its Dickson form. Note however that given a quadratic function in Maiorana-McFarland form $f(x,y) = x\cdot(L(y)+b)\oplus g(y)$, where $x\in\mathbb{F}_2^k, y\in\mathbb{F}_2^{n-k}$ and L is linear, the linear kernel of f is not $E=\mathbb{F}_2^k\times\{0_k\}$, in general, despite the fact that $D_aD_{a'}f$ is null for $a,a'\in E$. Indeed, writing $a=(a_1,a_2)$, we have $D_af(x,y) = x\cdot L(a_2)\oplus a_1\cdot L(y+a_2)\oplus a_1\cdot b\oplus D_{a_2}g(y)$, and we do not have necessarily that D_af is contant for $a\in E$. □

Remark. According to Theorem 2, page 82, the functions whose Walsh transform values are all divisible by 2^{n-1} are quadratic. According to Theorem 10, they are the sums of an affine function and of the product of two affine functions. This proves one of the points that we asserted at page 83. □

More general approach on the Walsh transform

Calculating the Dickson form of a quadratic Boolean function in generic number n of variables is most often impossible when the function is given by its trace representation. As originally shown by Dillon and Dobbertin in [448, Appendix A] for the case of functions $tr_n(x^{2^i+1})$ and generalized by Hou to all quadratic functions, there is a possibility of relating all the values of the Walsh transform to one of them (which needs of course to be nonzero; we know by the Parseval relation that such nonzero value necessarily exists). If the sign of one of these nonzero Walsh values is known, then all will be deduced.

X. Hou in [625] calculates the product of two nonzero Walsh values, instead of calculating the square of one value as we did in Proposition 55. This has the interest of providing the sign of every value $W_f(u)$, knowing one of them. Hou works with quadratic functions in trace form. This does not reduce theoretically the generality of his results since any function admits a trace form. However, in practice, if a Boolean function is given by its ANF, it is non-negligible work to first determine its trace representation; and if, instead of working with a particular function in a particular number of variables, we work with all functions with an ANF of some form in arbitrary number of variables, it is most often impossible. We shall then revisit Hou's result in a way which will not depend on a particular representation of the functions. Subsequently, we shall see what this result gives in trace representation.

Hou needs to assume that $W_f(0_n) \neq 0$ (*i.e.* that f is unbalanced). This does not reduce the generality since, if f is balanced, we can apply the result to one of the unbalanced functions $f(x) \oplus b \cdot x$. We assume then that $W_f(0_n) \neq 0$. This means according to Proposition 55 that any constant derivative of f is null on $\mathbb{F}_2^n$, *i.e.*, for every $x \in \mathcal{E}_f$, we have $f(x) = f(0_n)$.

We have $W_f(0_n)W_f(a) = \sum_{x,y \in \mathbb{F}_2^n} (-1)^{f(x)\oplus f(y)\oplus a\cdot y} = \sum_{x,y \in \mathbb{F}_2^n} (-1)^{f(x+y)\oplus f(y)\oplus a\cdot y}$.

For every $x \in \mathbb{F}_2^n$, function $y \mapsto f(x+y) \oplus f(y) \oplus a \cdot y$ is affine. We are then in the same situation as in Proposition 55, but with the advantage that we shall know the product of the signs of $W_f(0_n)$ and $W_f(a)$ when $W_f(a)$ will be nonzero. The sum $\sum_{y\in\mathbb{F}_2^n} (-1)^{f(x+y)\oplus f(y)\oplus a\cdot y}$ is nonzero if and only if function $y \mapsto f(x+y) \oplus f(y) \oplus a \cdot y$ is constant over $\mathbb{F}_2^n$. The set of those $x \in \mathbb{F}_2^n$ having such property either is empty or is a coset of the linear kernel $\mathcal{E}_f$, since $f(x+y) \oplus f(y) \oplus a \cdot y \oplus f(x'+y) \oplus f(y) \oplus a \cdot y = D_{x+x'}f(x+y)$. Moreover, the constant values of $f(x+y) \oplus f(y) \oplus a \cdot y$ are the same for all those x which belong to this coset, since $W_f(0_n)$ being nonzero, $D_x f$ is the zero function for every $x \in \mathcal{E}_f$ (according to Proposition 55). The next proposition is a version made as general as possible of the main result from [625].

Proposition 56 *Let n be any positive integer and f any unbalanced quadratic n-variable function. Let W_f be the Walsh transform associated to some inner product "$\cdot$". Then, for every $a \in \mathbb{F}_2^n$, the value of $W_f(a)$ is nonzero if and only if there exists x in $\mathbb{F}_2^n$ such that the function $y \mapsto f(x+y) \oplus f(y) \oplus a \cdot y$ is constant on $\mathbb{F}_2^n$. The set of such x is then a coset of $\mathcal{E}_f$ and we have $W_f(0_n)W_f(a) = 2^{n+dim\,\mathcal{E}_f}(-1)^{f(x)\oplus f(0_n)}$.*

The determination, for given a, of the set of those x such that $a \cdot y$ coincides with function $D_x f(y)$ or with function $D_x f(y) \oplus 1$ leads in Hou's method to the resolution of an equation, which is over $\mathbb{F}_{2^n}$ in his paper since f is taken in trace representation, and that we shall see below. This determination is necessary for calculating $W_f(a)$ explicitly and is the difficult part of this method in practice.

We now introduce a slightly different viewpoint (which has never been addressed as is, as far as we know). We start with the vector space $\{a \in \mathbb{F}_2^n; \exists x \in \mathbb{F}_2^n; \forall y \in \mathbb{F}_2^n, a \cdot y = \beta_f(x,y)\}$, that we denote by $\mathcal{E}'_f$, for reasons which will appear below. After identification between $\mathbb{F}_2^n$ and the vector space of its linear forms[6] through the correspondence $a \longleftrightarrow (y \to a \cdot y)$, we can view $\mathcal{E}'_f$ as the image of $\mathbb{F}_2^n$ by the linear function $x \to (y \to \beta_f(x,y))$. This linear function having kernel $\mathcal{E}_f$, the dimension of $\mathcal{E}'_f$ equals $n - dim\, \mathcal{E}_f$. Moreover, if $a \in \mathcal{E}'_f$ then $a \cdot y$ is null over $\mathcal{E}_f$, and since the dimension of the vector space $\mathcal{E}_f^\perp = \{a \in \mathbb{F}_2^n; \forall y \in \mathcal{E}_f, a \cdot y = 0\}$ is equal to $n - dim\, \mathcal{E}_f$ as well, we have:

$$\mathcal{E}'_f = \mathcal{E}_f^\perp.$$

According to Proposition 55, for every $b \in \mathbb{F}_2^n$, $W_f(b)$ is nonzero if and only if the function $x \mapsto f(x) \oplus f(0_n) \oplus b \cdot x$ is null on $\mathcal{E}_f$, and the Walsh support of f equals then $b + \mathcal{E}_f^\perp = b + \mathcal{E}'_f$. For every $a \in \mathcal{E}'_f$, choosing $x \in \mathbb{F}_2^n$ such that $a \cdot y = \beta_f(x,y)$, we have:

$$\begin{aligned} W_f(a+b) &= \sum_{y \in \mathbb{F}_2^n} (-1)^{f(y) \oplus (a+b) \cdot y} = \sum_{y \in \mathbb{F}_2^n} (-1)^{f(y) \oplus \beta_f(x,y) \oplus b \cdot y} \\ &= \sum_{y \in \mathbb{F}_2^n} (-1)^{f(x+y) \oplus f(x) \oplus f(0_n) \oplus b \cdot y} = \sum_{y \in \mathbb{F}_2^n} (-1)^{f(y) \oplus f(x) \oplus f(0_n) \oplus b \cdot (x+y)} \\ &= (-1)^{f(x) \oplus f(0_n) \oplus b \cdot x} \sum_{y \in \mathbb{F}_2^n} (-1)^{f(y) \oplus b \cdot y} = (-1)^{f(x) \oplus f(0_n) \oplus b \cdot x}\, W_f(b). \end{aligned}$$

Proposition 57 *Let f be any quadratic n-variable Boolean function and let W_f be its Walsh transform associated to some inner product "$\cdot$". Let $\beta_f(x,y) = f(x+y) \oplus f(x) \oplus f(y) \oplus f(0_n)$ be the symplectic form associated to f. Let b be any element of $\mathbb{F}_2^n$ such that $W_f(b) \neq 0$. Then, for every $a \in \mathbb{F}_2^n$, we have $W_f(a+b) \neq 0$ if and only if $a \in \mathcal{E}_f^\perp = \{u \in \mathbb{F}_2^n;\ u \cdot y = 0, \forall y \in \mathcal{E}_f\}$, which is equivalent to saying that there exists $x \in \mathbb{F}_2^n$ such that the functions $y \mapsto a \cdot y$ and $y \mapsto \beta_f(x,y)$ coincide over $\mathbb{F}_2^n$, and we have then:*

$$W_f(a+b) = (-1)^{f(x) \oplus f(0_n) \oplus b \cdot x}\, W_f(b).$$

Quadratic functions in trace form

We know (see Subsection 2.2.2, page 58) that any quadratic function $f(x)$ over $\mathbb{F}_{2^n}$ can be written in a unique way under the form:

$$f(x) = tr_n\left(\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} a_k x^{2^k+1}\right) \oplus q(x); \quad a_k \in \mathbb{F}_{2^n}, \tag{5.15}$$

[6] This vector space is called in mathematics the dual space (here of $\mathbb{F}_2^n$), but we shall avoid using this denomination for obvious reasons.

where:

$$\begin{cases} \text{if } n \text{ is even, } q(x) = tr_{n/2}\left(a_{n/2}x^{2^{n/2}+1}\right) + \ell(x);\ a_{n/2} \in \mathbb{F}_{2^{n/2}},\ \ell \text{ affine,} \\ \text{if } n \text{ is odd, } q(x) = \ell(x);\ \ell \text{ affine.} \end{cases} \tag{5.16}$$

We have then, using that, for every $u \in \mathbb{F}_2^n$ and $j \in \mathbb{N}$, we can replace $tr_n(u)$ by $tr_n(u^{2^j})$ and u^{2^n} by u:

$$\beta_f(x,y) = tr_n\left(y\Big[\sum_{k=1}^{\lfloor (n-1)/2\rfloor}\left(a_k x^{2^k} + a_k^{2^{n-k}} x^{2^{n-k}}\right)\Big]\right) + \beta_q(x,y),$$

where $\beta_q(x,y) = tr_{n/2}(a_{n/2}(x^{2^{n/2}}y + xy^{2^{n/2}})) = tr_{n/2}(a_{n/2}tr^n_{n/2}(x^{2^{n/2}}y)) = tr_n(a_{n/2}yx^{2^{n/2}})$ for n even and $\beta_q(x,y) = 0$ for n odd. We have then:

$$\mathcal{E}_f = \begin{cases} \left\{x \in \mathbb{F}_{2^n};\ \sum_{k=1}^{\frac{n}{2}-1}\left(a_k x^{2^k} + a_k^{2^{n-k}} x^{2^{n-k}}\right) + a_{n/2}x^{2^{n/2}} = 0\right\}, \text{ for } n \text{ even,} \\ \left\{x \in \mathbb{F}_{2^n};\ \sum_{k=1}^{\frac{n-1}{2}}\left(a_k x^{2^k} + a_k^{2^{n-k}} x^{2^{n-k}}\right) = 0\right\}, \text{ for } n \text{ odd.} \end{cases}$$

Hou has observed a useful property for evaluating the size of $\mathcal{E}_f$, and therefore the nonlinearity of any quadratic Boolean function in trace form:

Proposition 58 *[625] Let f be any quadratic n-variable function in the form (5.15), with $q = 0$. Denoting by K the maximal value of k such that $a_k \neq 0$, $|\mathcal{E}_f|$ equals the degree of the following polynomial:*

$$gcd\left(\left(\sum_{k=1}^{\lfloor (n-1)/2\rfloor}\left(a_k x^{2^k} + a_k^{2^{n-k}} x^{2^{n-k}}\right)\right)^{2^K}, x^{2^n} + x\right).$$

Indeed, $x^{2^n} + x$ splits completely over $\mathbb{F}_{2^n}$ and $|\mathcal{E}_f|$ equals the number of solutions in $\mathbb{F}_{2^n}$ of the equation $\sum_{k=1}^{\lfloor (n-1)/2\rfloor}\left(a_k x^{2^k} + a_k^{2^{n-k}} x^{2^{n-k}}\right) = 0$, which has no repeated root, because its derivative (as a polynomial) has no common zero with the equation.

Let us see now how the method we introduced works in univariate representation. According to Proposition 57, and taking for inner product $a \cdot x = tr_n(ax)$, we have the following:

Proposition 59 *Let $f(x)$ be any quadratic function. Let*

$$tr_n\left(\sum_{k=1}^{\lfloor (n-1)/2\rfloor} a_k x^{2^k+1}\right) + q(x)$$

be its trace form, where $q(x)$ is defined in Relation (5.16). Let:

$$P_f(x) = \sum_{k=1}^{\lfloor (n-1)/2\rfloor}\left(a_k x^{2^k} + a_k^{2^{n-k}} x^{2^{n-k}}\right) \text{ if } n \text{ is odd,} \tag{5.17}$$

and

$$P_f(x) = \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \left(a_k x^{2^k} + a_k^{2^{n-k}} x^{2^{n-k}} \right) + a_{n/2} x^{2^{n/2}} \text{ if } n \text{ is even.} \tag{5.18}$$

Let b be any element of $\mathbb{F}_{2^n}$ such that $W_f(b)$ is nonzero. For every $a \in \mathbb{F}_{2^n}$, $W_f(a+b)$ is nonzero if and only if there exists $x \in \mathbb{F}_{2^n}$ such that $a = P_f(x)$ and we have then:

$$W_f(a+b) = (-1)^{f(x)\oplus f(0)\oplus b\cdot x}\, W_f(b).$$

Remark The observation that $\mathcal{E}_f^\perp$ is at the same time equal to $\{a \in \mathbb{F}_{2^n}; \exists x \in \mathbb{F}_{2^n}; \forall y \in \mathbb{F}_{2^n}, a\cdot y = \beta_f(x,y)\}$ and to $\{a \in \mathbb{F}_{2^n}; \forall y \in \mathcal{E}_f, a \cdot y = 0\}$ gives, when applied to function f in (5.15), that $P_f(a) = 0$ if and only if there exists $x \in \mathbb{F}_{2^n}$ such that $a = P_f(x)$, where P_f is defined by (5.17) (resp. (5.18)). This gives a parameterized form of the set of the solutions of this equation. □

Particular classes of quadratic function*s*

Particular quadratic Boolean functions have been successfully investigated in the 70's. For some of them, the explicit Walsh transform could be given in a rather simple statement. This begun with Kerdock [689] when he constructed the so-called Kerdock codes (but the question of the sign was not posed, because his code is a union of cosets of the first-order Reed-Muller code, and two complementary functions f and $f \oplus 1$ have opposite Walsh transforms). Then Carlitz showed in [332] the following equalities on so-called *cubic sums* $\sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n(wx^3+ux)}$, $w \neq 0$, (this name being a reference to the polynomial degree of the functions, not to their algebraic degree which is 2):

- Let n be an odd integer and $u \in \mathbb{F}_{2^n}$. For $tr_n(u) = 1$, we denote by $\gamma \in \mathbb{F}_{2^n}$ any element in $\mathbb{F}_{2^n}$ such that $u = \gamma^4 + \gamma + 1$. We have:

$$\sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n(x^3+ux)} = \begin{cases} (-1)^{tr_n(\gamma^3+\gamma)} \left(\frac{2}{n}\right) 2^{\frac{n+1}{2}} & \text{when } tr_n(u) = 1 \\ 0 & \text{when } tr_n(u) = 0, \end{cases}$$

  where $(\frac{2}{n})$ denotes the *Jacobi symbol* which equals $(-1)^{\frac{n^2-1}{8}}$ when n is odd. If we know the sign of the Walsh transform at 1, this can be deduced from Proposition 55 and Proposition 59, after observing that the linear kernel of function $tr_n(x^3)$ equals $\{x \in \mathbb{F}_{2^n}; x^2 + x^{2^{n-1}} = 0\} = \{0\} \cup \{x \in \mathbb{F}_{2^n}; x^3 = 1\}$ which equals $\mathbb{F}_2$ since n is odd. The additional information we have thanks to Carlitz is the sign of the Walsh transform at 1.

  The value of $\sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n(wx^3+ux)}$ equals $\sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n\left(x^3 + \frac{u}{w^{1/3}}x\right)}$ (by the change of variable $x \mapsto \frac{x}{w^{\frac{1}{3}}}$; note that since n is odd, function x^3 is a permutation of $\mathbb{F}_{2^n}$; we denote the inverse function by $x^{\frac{1}{3}}$; the value of $\frac{1}{3}$ can be found in [907, 731]).
- Let n be an even integer. Then we have two cases according to whether w is a cube or not:

- If $w \neq 0$ is a cube, say $w = v^3$, then for $tr_2^n(uv^{-1}) = 0$, we denote by γ_0 any element in $\mathbb{F}_{2^n}$ such that $\gamma_0^4 + \gamma_0 = u^2v^{-2}$. We have:

$$\sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n(wx^3+ux)} = \begin{cases} (-1)^{\frac{n}{2}+1+tr_n(\gamma_0^3)}2^{\frac{n}{2}+1} & \text{when } tr_2^n(uv^{-1}) = 0 \\ 0 & \text{when } tr_2^n(uv^{-1}) \neq 0. \end{cases}$$

If we know the sign of the Walsh transform at 0, this is deduced from Propositions 55 and 59, after observing that the linear kernel of function $tr_n(wx^3)$ equals $\{x \in \mathbb{F}_{2^n}; wx^2 + (wx)^{2^{n-1}} = 0\} = \{0\} \cup \{x \in \mathbb{F}_{2^n}; wx^3 = 1\} = v\mathbb{F}_4$.
- If w is not a cube, then let γ_1 be the unique element in $\mathbb{F}_{2^n}$ such that $w^2\gamma_1^4 + w\gamma_1 = u^2$. Such γ_1 exists and is unique because the linear function $\gamma \mapsto w^2\gamma^4 + w\gamma$ has a trivial kernel (the linear kernel of function $tr_n(wx^3)$ equals $\{0\}$ since w is not a cube) and is then bijective. Then, we have:

$$\sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n(wx^3+ux)} = (-1)^{\frac{n}{2}+tr_n(w\gamma_1^3)}2^{\frac{n}{2}}.$$

Coulter [384, 385] and Dillon-Dobbertin [448] generalized Carlitz' results to exponents of the form 2^k+1 instead of 3. Their results can be deduced from Proposition 59 as well. To illustrate how, let us assume that n is odd and $\gcd(k,n) = 1$. Then x^{2^k+1} is a permutation. We can then reduce ourselves to the sums $\sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n(x^{2^k+1}+ux)}$. The linear kernel of quadratic function $tr_n(x^{2^k+1})$ has equation $x^{2^k} + x^{2^{-k}} = 0$, that is, $x^{2^{2k}} + x = 0$, which has for solutions in $\mathbb{F}_{2^n}$ the elements of $\mathbb{F}_{2^{\gcd(2k,n)}} = \mathbb{F}_2$, and $tr_n(x^{2^k+1} + ux)$ is then balanced if and only if $tr_n(u) = 0$. We assume then $tr_n(u) = 1$, and there exists a such that $u = 1 + a^{2^k} + a^{2^{-k}}$. Then since $tr_n((x+a)^{2^k+1}) = tr_n(x^{2^k+1} + (a^{2^k} + a^{2^{-k}})\,x + a^{2^k+1})$, we have $\sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n(x^{2^k+1}+ux)} = (-1)^{tr_n(a^{2^k+1})} \sum_{x\in\mathbb{F}_{2^n}} (-1)^{tr_n(x^{2^k+1}+x)}$.

As we wrote, much work has been done on the Walsh transform of quadratic functions in univariate form. We shall give the next ones without giving clues on their proofs.
For n odd, the quadratic functions of nonlinearity $2^{n-1} - 2^{\frac{n-1}{2}}$ (called *semi-bent functions* or *near-bent* functions; their extended Walsh spectra only contain values 0 and $2^{\frac{n+1}{2}}$, see Section 6.2) of the form $tr_n(\sum_{i=1}^{(n-1)/2} c_ix^{2^i+1})$ have been studied by Khoo, Gong and Stinson [697, 699]. The study of such function is simplified when all coefficients c_i belong to $\mathbb{F}_2$ since the linearized polynomial $\sum_{i=1}^{(n-1)/2}(c_ix^{2^i} + (c_ix)^{2^{n-i}}) = \sum_{i=1}^{(n-1)/2} c_i(x^{2^i} + x^{2^{n-i}})$ is then a 2-polynomial over $\mathbb{F}_2$ (see page 532) and its study can be done through its 2-associate polynomial $c(x) = \sum_{i=1}^{(n-1)/2} c_i(x^i + x^{n-i})$, more precisely, its gcd with $x^n + 1$ (*e.g.* near-bentness is equivalent to $\gcd(c(x), x^n + 1) = x + 1$), and the factorization of $x^n + 1$ (see [697, 699] and see more in [19, 510, 672, 840, 841]). If n and $2n + 1$ are primes, the function is near-bent for all non-all zero c_i's. This study has been generalized to n even by Charpin, Pasalic and Tavernier in [355] ("$\gcd(c(x), x^n + 1) = x + 1$" is then replaced by $\gcd(c(x), x^n + 1) = x^2 + 1$) and non-quadratic bent functions have been deduced by concatenation of such

near-bent functions. Further functions of this kind have been given and studied in [629, 672, 699, 840, 841].

In [734] is studied the sign of the values of the Walsh transform of AB Gold and Kasami functions (see pages 230 and 256). The former are quadratic (the latter are not but they are related to quadratic functions). In [548], the result of [734] is generalized: for every AB power function x^d over $\mathbb{F}_{2^n}$ whose restriction to any subfield of $\mathbb{F}_{2^n}$ is also AB, the value $\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(x^d+x)}$ equals $2^{\frac{n+1}{2}}$ if $n\equiv\pm1$ [mod 8] and $-2^{\frac{n+1}{2}}$ if $n\equiv\pm3$ [mod 8]. In [383] are studied the Walsh transform values of the functions $tr_n(x^{2^a+1}+x^{2^b+1})$, $\gcd(b-a,n)=\gcd(b+a,n)=1$.

X. Hou in [625] has been able to address whole subclasses of quadratic functions (and even more since he could view such functions over every field extension of $\mathbb{F}_{2^n}$). With the method of calculating $W_f(0)W_f(a)$, he determined the Walsh transform of any quadratic function whose trace form involves exponents of the form 2^k+1, where k has fixed 2-valuation.

X. Zhang, X. Cao and R. Feng in [1165] use that, given a linear function $L(x)=\sum_{k=0}^{n-1}a_kx^{2^k}\in\mathbb{F}_{2^n}[x]$, and denoting $\widetilde{L}(x)=\sum_{k=0}^{n-1}a_k^{2^{n-k}}x^{2^{n-k}}$, we have $tr_n(xL(y)))=tr_n(y\widetilde{L}(x))$ for every $x,y\in\mathbb{F}_{2^n}$. For every linear permutation L and linear function L', we have $tr_n\big(x\,(\widetilde{L}\circ L'\circ L(x))\big)=tr_n\big(L(x)\,(L'\circ L(x))\big)$, and then:

$$\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n\left(x(\widetilde{L}\circ L'\circ L(x))\right)}=\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n\left(L(x)\,(L'\circ L(x))\right)}=\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n\left(xL'(x)\right)}.$$

The functions $f(x)=tr_n\left(x\,(\widetilde{L}\circ L'\circ L(x))\right)$ and $g(x)=tr_n\left(x\,L'(x)\right)$ satisfy then $\mathcal{F}(f)=\mathcal{F}(g)$. This provides in fact an equivalence relation between quadratic functions, which preserves the mapping $f\mapsto\mathcal{F}(f)$.

5.2.2 Concatenations of quadratic functions

Concatenated quadratic functions (instead of affine functions) generalize the Maiorana-McFarland construction. These functions are a little harder to study than Maiorana-McFarland's functions, but they are more numerous and they avoid the property of null second-order derivatives seen in Proposition 54, page 190, which may be a cryptographic weakness. There are at least two classes:

- A first class [221] is built on the Dickson form of quadratic functions:

$$f_{\psi,\phi,g}(x,y)=\bigoplus_{i=1}^{t}x_{2i-1}x_{2i}\,\psi_i(y)\oplus x\cdot\phi(y)\oplus g(y),\qquad(5.19)$$

  with $x\in\mathbb{F}_2^r$, $y\in\mathbb{F}_2^s$, where $n=r+s$, $t=\lfloor\frac{r}{2}\rfloor$, and where $\psi:\mathbb{F}_2^s\to\mathbb{F}_2^t$, $\phi:\mathbb{F}_2^s\to\mathbb{F}_2^r$ and $g:\mathbb{F}_2^s\to\mathbb{F}_2$ can be chosen arbitrarily.
  The size of this class roughly equals $2^{(t+r+1)2^s}$. The Walsh transform is easily deduced from the observation that, for every quadratic Boolean function of the form $f(x)=\bigoplus_{i=1}^{t}u_ix_{2i-1}x_{2i}\oplus\sum_{j=1}^{2t}v_jx_j\oplus c$, where $u_i,v_j,c\in\mathbb{F}_2,x\in\mathbb{F}_2^{2t}$, and for every element a of $\mathbb{F}_2^{2t}$, if there exists $i=1,\cdots,t$

such that $u_i = 0$ and $v_{2i-1} \neq a_{2i-1}$ or $v_{2i} \neq a_{2i}$, then we have $W_f(a) = 0$, and otherwise, $W_f(a)$ is equal to $2^{2t-w_H(u)}(-1)^{\sum_{i=1}^{t}(v_{2i-1}\oplus a_{2i-1})(v_{2i}\oplus a_{2i})\oplus c}$, where $u = (u_1, \ldots, u_t)$. This implies that, for every function f of the form (5.19), for every $a \in \mathbb{F}_2^r$ and every $b \in \mathbb{F}_2^s$, we have:

$$W_{f_{\psi,\phi,g}}(a,b) = \sum_{y\in E_a} 2^{r-w_H(\psi(y))}(-1)^{\sum_{i=1}^{t}(\phi_{2i-1}(y)\oplus a_{2i-1})(\phi_{2i}(y)\oplus a_{2i})\oplus g(y)\oplus y\cdot b},$$

where E_a is the superset of $\phi^{-1}(a)$ equal if r is even to

$$\{y \in \mathbb{F}_2^s / \ \forall i \leq t,\ \psi_i(y) = 0 \Rightarrow (\phi_{2i-1}(y) = a_{2i-1} \text{ and } \phi_{2i}(y) = a_{2i})\},$$

and if r is odd to

$$\left\{ y \in \mathbb{F}_2^s / \left\{ \begin{array}{l} \forall i \leq t,\ \psi_i(y) = 0 \Rightarrow (\phi_{2i-1}(y) = a_{2i-1} \text{ and } \phi_{2i}(y) = a_{2i}) \\ \phi_r(y) = a_r \end{array} \right. \right\}.$$

- A second class [317] has for elements the concatenations of quadratic functions of rank at most 2, of the form:

$$f_{\phi_1,\phi_2,\phi_3,g}(x,y) = (x\cdot\phi_1(y))\,(x\cdot\phi_2(y)) \oplus x\cdot\phi_3(y) \oplus g(y), \tag{5.20}$$

  with $x \in \mathbb{F}_2^r$, $y \in \mathbb{F}_2^s$, where ϕ_1, ϕ_2 and ϕ_3 are three functions from $\mathbb{F}_2^s$ into $\mathbb{F}_2^r$ and g is any Boolean function on $\mathbb{F}_2^s$. The size of this class roughly equals $2^{(3r+1)2^s}$ (the exact number, which is unknown, is smaller since a function can be represented in this form in several ways) and is larger than for the first class.
  The Walsh transform is deduced from the fact that, for every positive integer r and every Boolean function f on $\mathbb{F}_2^r$ of the form $(u\cdot x)(v\cdot x)\oplus w\cdot x$; $u, v, w \in \mathbb{F}_2^r$:
  - if u and v are $\mathbb{F}_2$-linearly independent (*i.e.* $u \neq 0_r, v \neq 0_r$ and $u \neq v$), then f is balanced if and only if w is outside the vectorspace $< u, v >= \{0_r, u, v, u+v\}$ spanned by u and v, and otherwise, if $w \in \{0_r, u, v\}$, then $\sum_{x\in\mathbb{F}_2^r}(-1)^{f(x)}$ equals 2^{r-1}, and if $w = u+v$, then it equals -2^{r-1};
  - if u and v are $\mathbb{F}_2$-linearly dependent, then if we have $w = 0_r$ and $u = 0_r$ or $v = 0_r$, or if we have $u = v = w$, then $\sum_{x\in\mathbb{F}_2^r}(-1)^{f(x)}$ equals 2^r; otherwise, $\sum_{x\in\mathbb{F}_2^r}(-1)^{f(x)}$ is null.
  We deduce that for any function $f_{\phi_1,\phi_2,\phi_3,g}$ of the form (5.20) with $\phi_2(y) \neq 0_r$ for every $y \in \mathbb{F}_2^s$, denoting by E the set of all $y \in \mathbb{F}_2^s$ such that the vectors $\phi_1(y)$ and $\phi_2(y)$ are $\mathbb{F}_2$-linearly independent, for every $a \in \mathbb{F}_2^r$ and every $b \in \mathbb{F}_2^s$, $W_{f_{\phi_1,\phi_2,\phi_3,g}}(a,b)$ equals

$$2^{r-1} \sum_{\substack{y\in E;\\ \phi_3(y)+a\in\{0_r,\phi_1(y),\phi_2(y)\}}} (-1)^{g(y)\oplus b\cdot y} -$$

$$2^{r-1} \sum_{\substack{y\in E;\\ \phi_3(y)+a=\phi_1(y)+\phi_2(y)}} (-1)^{g(y)\oplus b\cdot y} + 2^r \sum_{\substack{y\in \mathbb{F}_2^s\setminus E;\\ \phi_3(y)+a=\phi_1(y)}} (-1)^{g(y)\oplus b\cdot y}.$$

5.3 Cubic functions

The *Hamming weights* and the Walsh spectra of non-quadratic cubic Boolean functions (*i.e.* of the codewords in $RM(3,n) \setminus RM(2,n)$) behave in a much less peculiar way than quadratic functions[7]. This has been shown in [210] as follows. Let f_1, f_2 and f_3 be any Boolean functions on $\mathbb{F}_2^n$. Define the function on $\mathbb{F}_2^{n+2}$: $f(x, y_1, y_2) = y_1y_2 \oplus y_1f_1(x) \oplus y_2f_2(x) \oplus f_3(x)$. Then we have

$$\begin{aligned}\mathcal{F}(f) &= \sum_{x\in\mathbb{F}_2^n;\, y_1,y_2\in\mathbb{F}_2} (-1)^{(y_1\oplus f_2(x))(y_2\oplus f_1(x))\oplus f_1(x)f_2(x)\oplus f_3(x)} \\ &= \sum_{x\in\mathbb{F}_2^n;\, y_1,y_2\in\mathbb{F}_2} (-1)^{y_1y_2\oplus f_1(x)f_2(x)\oplus f_3(x)} = 2\sum_{x\in\mathbb{F}_2^n} (-1)^{f_1(x)f_2(x)\oplus f_3(x)}.\end{aligned}$$

So, starting with a function $g = f_1f_2 \oplus f_3$, we can relate $\mathcal{F}(g)$ to $\mathcal{F}(f)$, in two more variables, in which the term f_1f_2 has been replaced by $y_1y_2 \oplus y_1f_1(x) \oplus y_2f_2(x)$. Applying this repeatedly ("breaking" this way all the monomials of degrees at least 4), this shows that, for every Boolean function g on $\mathbb{F}_2^n$, there exists an integer m and a Boolean function f of algebraic degree at most 3 on $\mathbb{F}_2^{n+2m}$ whose Walsh transform takes value $W_f(0_{n+2m}) = 2^m\, W_g(0_n)$ at zero. This proves that the functions of algebraic degree 3 can have Hamming weights much more diverse than functions of degrees at most 2, since function g from which we started can have for Hamming weight any integer between 0 and 2^n, and then $W_g(0_n)$ can take any even value between -2^n and 2^n.

Note however that the weights of some cubic functions (and even some quartic ones) are easily determined. The weight of the product fg of two quadratic functions and of its sum with any affine function can be deduced from $fg = \frac{f+g-(f\oplus g)}{2}$. And the Fourier-Hadamard transform being $\mathbb{R}$-linear, we have $\widehat{fg} = \frac{\widehat{f}+\widehat{g}-\widehat{f\oplus g}}{2}$. This works for instance for $\sigma_3 = \sigma_1\sigma_2$, where σ_i is the i-th elementary symmetric Boolean function. See also [745].

5.4 Indicators of flats

As we have already seen, a Boolean function f is the *indicator* of a flat A of co-dimension r if and only if it has the form $f(x) = \prod_{i=1}^{r}(a_i \cdot x \oplus \epsilon_i)$ where $a_1, \ldots, a_r \in \mathbb{F}_2^n$ are $\mathbb{F}_2$-linearly independent and $\epsilon_1, \ldots, \epsilon_r \in \mathbb{F}_2$. Then f has Hamming weight 2^{n-r}. Moreover, for any $a \in \mathbb{F}_2^n$, if a is $\mathbb{F}_2$-linearly independent of $a_1, \ldots, a_r$, then the function $f(x) \oplus a \cdot x$ is balanced (and hence $W_f(a) = 0$), since it is linearly equivalent to a function of the form $g(x_1, \ldots, x_r) \oplus x_{r+1}$. If a is $\mathbb{F}_2$-linearly dependent of $a_1, \ldots, a_r$, say $a = \sum_{i=1}^{r} \eta_i\, a_i$ with $\eta_i \in \mathbb{F}_2$, then $a \cdot x$ takes constant value $\bigoplus_{i=1}^{r} \eta_i\,(a_i \cdot x) = \bigoplus_{i=1}^{r} \eta_i\,(\epsilon_i \oplus 1)$ on the flat; hence, $\widehat{f}(a) = \sum_{x\in A}(-1)^{a\cdot x}$ equals $2^{n-r}(-1)^{\bigoplus_{i=1}^{r}\eta_i\,(\epsilon_i\oplus 1)}$. Thus, if $a = \sum_{i=1}^{r} \eta_i\, a_i \neq 0_n$,

[7] Except that, according to McEliece's theorem, the Hamming weights are divisible by $2^{\lceil \frac{n}{3}\rceil - 1}$ and the Walsh transform values are divisible by $2^{\lceil \frac{n}{3}\rceil}$).

then we have $W_f(a) = -2^{n-r+1}(-1)^{\bigoplus_{i=1}^r \eta_i\,(\epsilon_i \oplus 1)}$; and we have $W_f(0_n) = 2^n - 2|A| = 2^n - 2^{n-r+1}$.
Note that the nonlinearity of f equals 2^{n-r} and is bad as soon as $r \geq 2$. But indicators of flats can be used to design Boolean functions with good nonlinearities, by concatenating sums of indicators of flats and of affine functions, see below.

Remark As recalled in Section 4.1, the functions of $RM(r,n)$ whose weights occur in the range $[2^{n-r}; 2^{n-r+1}[$ have been characterized by Kasami and Tokura [670]; any such function is the product of the indicator of a flat and of a quadratic function or is the sum (modulo 2) of two indicators of flats. The Walsh spectra of such functions can also be precisely computed. □

5.4.1 Concatenations of sums of indicators of flats and affine functions

Concatenating sums of indicators of flats and of affine functions gives another super-class, studied in [226], of Maiorana-McFarland's class. The functions of this generalized class are of the form:

$$f(x,y) = \prod_{i=1}^{t(y)} (x \cdot \phi_i(y) \oplus g_i(y) \oplus 1) \oplus x \cdot \phi(y) \oplus g(y); \quad (x,y) \in \mathbb{F}_2^r \times \mathbb{F}_2^s, \tag{5.21}$$

where t is a function from $\mathbb{F}_2^s$ into $\{0,1,\dots,r\}$, and where $\phi_1,\dots,\phi_r,\phi$ are functions from $\mathbb{F}_2^s$ into $\mathbb{F}_2^r$ such that, for every $y \in \mathbb{F}_2^s$, the vectors $\phi_1(y),\dots,\phi_{t(y)}(y)$ are linearly independent; $g_1,\dots,g_r$ and g are Boolean functions on $\mathbb{F}_2^s$.
Let f be defined by (5.21). For every $a \in F_2^r$ and every $b \in F_2^s$, we have

$$W_f(a,b) = 2^r \sum_{y\in\phi^{-1}(a)} (-1)^{g(y)\oplus b\cdot y} - \sum_{y\in F_a} 2^{r-t(y)+1}(-1)^{g(y)\oplus b\cdot y \oplus \bigoplus_{i=1}^{t(y)} \eta_i(a,y)\, g_i(y)},$$

where F_a is the set of all the vectors y of the space $\mathbb{F}_2^s$ such that a belongs to the flat $\phi(y)+ < \phi_1(y),\dots,\phi_{t(y)}(y) >$ (by convention equal to $\{\phi(y)\}$ if $t(y)=0$), and where $\eta_i(a,y)$ is defined (with uniqueness) for every $i \leq t(y)$ by the relation $a + \phi(y) = \sum_{i=1}^{t(y)} \eta_i(a,y)\,\phi_i(y)$.
The cryptographic parameters of such functions are studied in [226, Section 5].

5.5 Functions admitting (partial) covering sequences

5.5.1 The case of Boolean functions

The notion of covering sequence of a Boolean function has been introduced in [326].

Definition 47 *Let f be an n-variable Boolean function. An integer-valued*[8] *sequence $(\lambda_a)_{a\in\mathbb{F}_2^n}$ is called a* covering sequence *of f if the integer-valued function*

[8] or real-valued, or complex-valued; but taking real or complex sequences instead of integer-valued ones has no practical sense.

$\sum_{a\in\mathbb{F}_2^n} \lambda_a D_a f(x)$ *takes a constant value. This constant value is called the* level of the covering sequence*. If the level is nonzero, we say that the covering sequence is a* non-trivial covering sequence*.*

For instance, any balanced quadratic function admits a non-trivial atomic covering sequence (see page 194). Note that the sum $\sum_{a\in\mathbb{F}_2^n} \lambda_a D_a f(x)$ involves both kinds of additions: the addition $\sum$ in $\mathbb{Z}$ and the addition $\oplus$ in $\mathbb{F}_2$ (which is concealed inside $D_a f$). It has been shown in [326] that any function admitting a non-trivial covering sequence is balanced (see Proposition 61 below for a proof) and that any balanced function admits the constant sequence 1 as covering sequence (the level of this sequence is 2^{n-1}).
A characterization of covering sequences by means of the Walsh transform was also given in [326]: denote again by $supp(W_f)$ the support $\{u \in \mathbb{F}_2^n \mid W_f(u) \neq 0\}$ of W_f; then:

Proposition 60 *Let f be any n-variable Boolean function and $\lambda = (\lambda_a)_{a\in\mathbb{F}_2^n}$ an integer-valued sequence. Then f admits λ as covering sequence if and only if the Fourier-Hadamard transform $\widehat{\lambda}$ of the function $a \mapsto \lambda_a$ takes a constant value on $supp(W_f)$. This constant value is $\left(\sum_{a\in\mathbb{F}_2^n} \lambda_a - 2\rho\right)$, where ρ is the level of the covering sequence.*

Proof. Replacing $D_a f(x)$ by $\frac{1}{2} - \frac{1}{2}(-1)^{D_a f(x)} = \frac{1}{2} - \frac{1}{2}(-1)^{f(x)}(-1)^{f(x+a)}$ in the equality $\sum_{a\in\mathbb{F}_2^n} \lambda_a D_a f(x) = \rho$, we see that f admits the covering sequence λ with level ρ if and only if, for every $x \in \mathbb{F}_2^n$, we have $\sum_{a\in\mathbb{F}_2^n} \lambda_a (-1)^{f(x+a)} = \left(\sum_{a\in\mathbb{F}_2^n} \lambda_a - 2\rho\right)(-1)^{f(x)}$. These two integer-valued functions are equal if and only if their Fourier-Hadamard transforms are equal to each other, that is, if for every $b \in \mathbb{F}_2^n$, the sum $\sum_{a,x\in\mathbb{F}_2^n} \lambda_a (-1)^{f(x+a)\oplus x\cdot b}$, which by changing x into $x+a$ equals $\left(\sum_{a\in\mathbb{F}_2^n} \lambda_a (-1)^{a\cdot b}\right) W_f(b) = \widehat{\lambda}(b)\, W_f(b)$, equals $\left(\sum_{a\in\mathbb{F}_2^n} \lambda_a - 2\rho\right) W_f(b)$. The characterization follows. □

Any Boolean function f on $\mathbb{F}_2^n$ is balanced (*i.e.* satisfies $0_n \notin supp(W_f)$) if and only if it admits at least one non-trivial covering sequence: the condition is clearly sufficient according to Proposition 60 (since $\widehat{\lambda}(0_n) = \sum_{a\in\mathbb{F}_2^n} \lambda_a$ and $\rho \neq 0$), and it is also necessary since the constant sequence 1 is a covering sequence for all balanced functions. See more in [308].

We shall see in Chapter 7 that covering sequences play a role with respect to correlation immunity and resiliency. But knowing a covering sequence for f gives no information on the nonlinearity of f, since it gives only information on the support of the Walsh transform, not on the nonzero values it takes. In [231] is weakened the definition of covering sequence, so that it can help computing the (nonzero) values of the Walsh transform.

Definition 48 *Let f be a Boolean function on $\mathbb{F}_2^n$. A partial covering sequence for f is a sequence $(\lambda_a)_{a\in\mathbb{F}_2^n}$ such that $\sum_{a\in\mathbb{F}_2^n} \lambda_a D_a f(x)$ takes two values ρ and*

ρ' (distinct or not) called the levels *of the sequence. The partial covering sequence is called* non-trivial *if one of the constants is nonzero.*

The interest of non-trivial partial covering sequences is that they give information on the Hamming weight and the Walsh transform.

Proposition 61 *Let $(\lambda_a)_{a\in\mathbb{F}_2^n}$ be a partial covering sequence of a Boolean function f, of levels ρ and ρ'.*
Let $A = \{x \in \mathbb{F}_2^n\,;\ \sum_{a\in\mathbb{F}_2^n} \lambda_a D_a f(x) = \rho'\}$ (assuming that $\rho' \neq \rho$; otherwise, when λ is in fact a covering sequence of level ρ, we set $A = \emptyset$).
Then, for every vector $b \in \mathbb{F}_2^n$, we have:

$$\left(\widehat{\lambda}(b) - \widehat{\lambda}(0_n) + 2\,\rho\right)\ W_f(b) = 2\,(\rho - \rho')\ \sum_{x\in A}(-1)^{f(x)\oplus b\cdot x}.$$

Hence, if $\rho \neq 0$, we have:

$$2^n - 2w_H(f) = W_f(0_n) = \left(1 - \frac{\rho'}{\rho}\right)\sum_{x\in A}(-1)^{f(x)}.$$

Proof. By definition, we have, for every $x \in \mathbb{F}_2^n$:

$$\sum_{a\in\mathbb{F}_2^n} \lambda_a D_a f(x) = \rho'\,1_A(x) +\, \rho\,1_{A^c}(x)$$

and therefore:

$$\sum_{a\in\mathbb{F}_2^n} \lambda_a(-1)^{D_af(x)} = \sum_{a\in\mathbb{F}_2^n} \lambda_a(1 - 2\,D_af(x)) = \sum_{a\in\mathbb{F}_2^n} \lambda_a - 2\,\rho'\,1_A(x) - 2\,\rho\,1_{A^c}(x).$$

We deduce:

$$\sum_{a\in\mathbb{F}_2^n} \lambda_a(-1)^{f(x+a)} = (-1)^{f(x)}\left(\sum_{a\in\mathbb{F}_2^n} \lambda_a - 2\,\rho'\,1_A(x) - 2\,\rho\,1_{A^c}(x)\right). \qquad (5.22)$$

We have already seen that the Fourier-Hadamard transform of the function $(-1)^{f(x+a)}$ maps every vector $b \in \mathbb{F}_2^n$ to the value $(-1)^{a\cdot b}\,W_f(b)$. Hence, taking the Fourier-Hadamard transform of both terms of equality (5.22), we get:

$$\left(\sum_{a\in\mathbb{F}_2^n} \lambda_a(-1)^{a\cdot b}\right) W_f(b) =$$

$$\left(\sum_{a\in\mathbb{F}_2^n} \lambda_a\right) W_f(b) - 2\,\rho'\,\sum_{x\in A}(-1)^{f(x)\oplus b\cdot x} - 2\,\rho\,\sum_{x\in A^c}(-1)^{f(x)\oplus b\cdot x},$$

that is

$$\widehat{\lambda}(b)\ W_f(b) = \widehat{\lambda}(0_n)\ W_f(b) - 2\,\rho\,W_f(b) + 2\,(\rho - \rho')\ \sum_{x\in A}(-1)^{f(x)\oplus b\cdot x}.$$

Hence:

$$\left(\widehat{\lambda}(b) - \widehat{\lambda}(0_n) + 2\,\rho\right)\; W_f(b) = 2\,(\rho - \rho') \sum_{x\in A}(-1)^{f(x)\oplus b\cdot x}. \qquad \square$$

A simple example of non-trivial partial covering sequence is as follows: let $\mathcal{E}$ be any set of derivatives of f. Assume that $\mathcal{E}$ contains a nonzero function and is stable under addition (*i*.e. is a non-trivial $\mathbb{F}_2$-vector space). Then $\sum_{g\in\mathcal{E}} g$ takes on values 0 and $\frac{|\mathcal{E}|}{2}$. Thus, if $\mathcal{E} = \{D_a f;\; a \in E\}$ (where we choose E minimum, so that any two different vectors of the set E give different functions of $\mathcal{E}$), then 1_E is a non-trivial partial covering sequence.

Corollary 10 *Let $\mathcal{E}$ be any set of derivatives of an n-variable Boolean function f. Assume that $\mathcal{E}$ contains a nonzero function and is stable under addition (*i.e. *is a non-trivial $\mathbb{F}_2$-vector space). Then*

$$2^n - 2w_H(f) = W_f(0_n) = \sum_{x\in A}(-1)^{f(x)}.$$

See more in [231], with the notion of *linear set of derivatives* (which are sets of derivatives stable under addition and provide partial covering sequences), combined with Proposition 61, and applied to the computation of the Hamming weights and Walsh spectra of quadratic and Maiorana-McFarland functions and of other examples of functions.

5.5.2 The case of vectorial functions

The generalization of the notion of covering sequence to vectorial functions has been studied in [319]. A covering sequence for a Boolean function can be seen as a function φ from $\mathbb{F}_2^n$ into $\mathbb{R}$ such that $\sum_{a\in\mathbb{F}_2^n;\; D_aF(x)=1}\varphi(a) = \rho$, for every $x \in \mathbb{F}_2^n$. This generalizes to vectorial functions:

Definition 49 *We call covering sequence of an (n,m)-function F, a pair of functions (φ,ψ) from, respectively, $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$ into $\mathbb{R}$, such that:*

$$\forall x \in \mathbb{F}_2^n, \forall b \in \mathbb{F}_2^m, \sum_{a\in\mathbb{F}_2^n;\; D_aF(x)=b}\varphi(a) = \psi(b)\,. \tag{5.23}$$

Note that this equality between functions $b \mapsto \sum_{a\in\mathbb{F}_2^n;\; D_aF(x)=b}\varphi(a)$ and $b \mapsto \psi(b)$ is equivalent to the equality between their Fourier transforms, that is:

$$\forall v \in \mathbb{F}_2^m,\; \forall x \in \mathbb{F}_2^n,\; \sum_{a\in\mathbb{F}_2^n}\varphi(a)(-1)^{v\cdot D_aF(x)} = \sum_{b\in\mathbb{F}_2^m}\psi(b)(-1)^{v\cdot b},$$

which is equivalent to:

$$\sum_{a\in\mathbb{F}_2^n}\varphi(a)(-1)^{v\cdot F(x+a)} = (-1)^{v\cdot F(x)}\widehat{\psi}(v),$$

that is:

$$\varphi \otimes \chi_F(\cdot, v) = \widehat{\psi}(v)\, \chi_F(\cdot, v), \tag{5.24}$$

where $\chi_F(\cdot, v)$ denotes function $x \mapsto \chi_F(x, v) = (-1)^{v \cdot F(x)}$ and $\otimes$ is the convolutional product.

Proposition 62 *An (n, m)-function F is balanced if and only if it admits at least one covering sequence (φ, ψ) satisfying $\widehat{\psi}(v) \neq \widehat{\varphi}(0_n)$ for every nonzero vector v of $\mathbb{F}_2^m$. Any balanced (n, m)-function F admits the pair of constant functions $(1, 2^{n-m})$ for covering sequence.*

Proof. Assume that (φ, ψ) is a covering sequence of F, then Equation (5.24) is satisfied and by applying the Fourier transform at 0_n to both sides of this functional equality, we obtain:

$$\forall v \in \mathbb{F}_2^m,\ \widehat{\varphi}(0_n) W_F(0_n, v) = \widehat{\psi}(v) W_F\left(0_n, v\right),$$

that is, $(\widehat{\varphi}(0_n) - \widehat{\psi}(v)) W_F(0_n, v) = 0$ for every $v \in \mathbb{F}_2^m$. This gives $\widehat{\varphi}(0_n) = \widehat{\psi}(0_m)$. If $\widehat{\varphi}(0_n) - \widehat{\psi}(v)$ is nonzero for every nonzero $v \in \mathbb{F}_2^m$, then the function $v \mapsto W_F\left(0_n, v\right)$ is null on $\mathbb{F}_2^m \setminus \{0_m\}$, which implies that F is balanced, according to Proposition 35.
Conversely, if F is balanced, then, for every pair $(b, x) \in \mathbb{F}_2^m \times \mathbb{F}_2^n$, the cardinality of the set $\{a \in \mathbb{F}_2^n; D_a F(x) = b\}$ is constant equaling 2^{n-m} since the equation $D_a F(x) = b$ is equivalent to $F(x + a) = b + F(x)$. Let $\varphi : \mathbb{F}_2^n \mapsto \mathbb{R}$ and $\psi : \mathbb{F}_2^m \mapsto \mathbb{R}$ be respectively the constant function $x \mapsto 1$ and the constant function $y \mapsto 2^{n-m}$, then the pair (φ, ψ) is a covering sequence of F satisfying the relation $\widehat{\psi}(v) = 0 \neq \widehat{\psi}(0_m) = \widehat{\varphi}(0_n) = 2^{n-m}$ for every element v of $\mathbb{F}_2^m \setminus \{0_m\}$. □

Remark Finding a second covering sequence is often a difficult problem. It is shown in [319] that the Maiorana-McFarland functions which satisfy the hypothesis of Proposition 128, page 345, admit several covering sequences. □

Definition 50 *A covering sequence (φ, ψ) of an (n, m)-function F is said to be* non-trivial *if $\widehat{\psi}(v)$ never equals $\widehat{\varphi}(0_n)$ (that is $\widehat{\psi}(0_m)$) when v ranges over $\mathbb{F}_2^m \setminus \{0_m\}$.*

Thus, according to Proposition 62, an (n, m)-function F is balanced if and only if it admits a non-trivial covering sequence. This definition and this observation generalize what was known for Boolean functions.

Remark If ψ is a function from $\mathbb{F}_2^m$ into $\mathbb{R}_+$, then we have $\widehat{\psi}(v) \neq \widehat{\psi}(0_m)$ for every element v of $\mathbb{F}_2^m \setminus \{0_m\}$ if and only if the support of ψ has rank m (*i.e.* spans the whole vector space $\mathbb{F}_2^m$). Indeed, we have

$$\forall v \in \mathbb{F}_2^m \setminus \{0_m\},\ \widehat{\psi}(v) \neq \widehat{\psi}(0_m) \Longleftrightarrow \forall v \in \mathbb{F}_2^m \setminus \{0_m\}, \sum_{b \in \mathbb{F}_2^m, b \in (v^\perp)^c} \psi(b) \neq 0$$

and, since $\psi(b) \geq 0$, $\forall b \in \mathbb{F}_2^m$, this relation is equivalent to saying that the support of ψ is not included in a linear hyperplane of $\mathbb{F}_2^m$. $\square$

Let us now generalize to vectorial functions the characterization of covering sequences of Boolean functions by means of their Fourier transforms and of the Walsh support of F.

Proposition 63 *Let F be an (n,m)-function and let (φ, ψ) be any pair of real-valued functions respectively defined on $\mathbb{F}_2^n$ and on $\mathbb{F}_2^m$. Then, F admits (φ, ψ) for covering sequence if and only if, for every pair (u,v) belonging to Supp W_F, we have $\widehat{\varphi}(u) = \widehat{\psi}(v)$.*

Proof. Thanks to the bijectivity of the Fourier transform, for every nonzero vector $v \in \mathbb{F}_2^m$, the functions $\widehat{\psi}(v)\,\chi_F(\cdot, v)$ and $\varphi \otimes \chi_F(\cdot, v)$ of Relation (5.24) are equal if and only if their Fourier transforms on $\mathbb{F}_2^n$ are equal, that is:

$$\forall v \in \mathbb{F}_2^m,\ \forall u \in \mathbb{F}_2^n,\ \widehat{\varphi}(u) W_F(u,v) = \widehat{\psi}(v)\, W_F(u,v),$$

that is, if and only if

$$((u,v) \in Supp\ W_F) \Longrightarrow \left(\widehat{\varphi}(u) = \widehat{\psi}(v)\right). \qquad \square$$

Corollary 11 *Let F be an* (n,m)-function *admitting (φ, ψ) for covering sequence. If the sets $\widehat{\varphi}\left(\{u \in \mathbb{F}_2^n /\, w_H(u) \leq t\}\right)$ and $\widehat{\psi}(\mathbb{F}_2^m \setminus \{0_m\})$ are disjoint, then F is t-resilient.*

It is deduced in [319] from Proposition 63 that if an (n,m)-function F admits a covering sequence (φ, ψ) such that the functions φ and ψ are, respectively, different from the zero function on $\mathbb{F}_2^n$ and different from the zero function on $\mathbb{F}_2^m \setminus \{0_m\}$, then, for every vector $u \in \mathbb{F}_2^n$, there exists $v \in \mathbb{F}_2^m \setminus \{0_m\}$ such that $W_F(u,v) = 0$, and for every vector $v \in \mathbb{F}_2^m$, there exists a vector $u \in \mathbb{F}_2^n$ such that $W_F(u,v) = 0$.

We show now that the notion of covering sequence behaves well with respect to composition.

Proposition 64 *[319] Let $F : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$ and $G : \mathbb{F}_2^m \mapsto \mathbb{F}_2^k$ be two functions admitting respectively (φ, ψ) and (ψ, θ) for covering sequences. Then, (φ, θ) is a covering sequence of $G \circ F$.*

Proof. For every pair $(x,a) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$, we have, denoting $D_aF(x)$ by b:

$$D_a[G\circ F](x) = G(F(x)) + G(F(x+a)) = G(F(x)) + G(F(x)+b) = (D_bG)(F(x)).$$

Thus, for every pair $(x,c) \in \mathbb{F}_2^n \times \mathbb{F}_2^k$, we have:

$$\sum_{a\in\mathbb{F}_2^n, D_a[G\circ F](x)=c} \varphi\,(a) = \sum_{b\in\mathbb{F}_2^m, (D_bG)(F(x))=c} \left(\sum_{a\in\mathbb{F}_2^n, D_aF(x)=b} \varphi(a) \right).$$

For every pair $(x,b)\in\mathbb{F}_2^n\times\mathbb{F}_2^m$, we have $\sum_{a\in\mathbb{F}_2^n,D_aF(x)=b}\varphi(a)=\psi(b)$ and thus:

$$\sum_{a\in\mathbb{F}_2^n,D_a[G\circ F](x)=c}\varphi(a)=\sum_{b\in\mathbb{F}_2^m,(D_bG)(F(x))=c}\psi(b)\,.$$

Let y denote $F(x)$, then:

$$\sum_{a\in\mathbb{F}_2^n,D_a[G\circ F](x)=c}\varphi(a)=\sum_{b\in\mathbb{F}_2^m,D_bG(y)=c}\psi(b)\quad. \tag{5.25}$$

Since (ψ,θ) is a covering sequence of G, the sum $\sum_{b\in\mathbb{F}_2^m,D_bG(y)=c}\psi(b)$ takes constant value $\theta(c)$ for every pair $(y,c)\in\mathbb{F}_2^m\times\mathbb{F}_2^k$ and we deduce

$$\forall x\in\mathbb{F}_2^n,\ \forall c\in\mathbb{F}_2^k,\quad\sum_{a\in\mathbb{F}_2^n,D_aG\circ F(x)=c}\varphi(a)=\theta(c)\,.\qquad\square$$

In [319] is given a similar (more technical) result on the concatenation of functions, with consequences on Maiorana-McFarland functions. An attack on ciphers using functions admitting covering sequences is also presented.

5.6 Functions with low univariate degree and related functions

The following Weil's Theorem is very well-known in finite field theory (*cf.* [775, Theorem 5.38]):

Theorem 11 *Let q be a prime power and $F(x)\in\mathbb{F}_q[x]$ a univariate polynomial of degree $d\geq 1$ with $\gcd(d,q)=1$. Let χ be a non-trivial character of $\mathbb{F}_q$. Then*

$$\left|\sum_{x\in\mathbb{F}_q}\chi(F(x))\right|\leq (d-1)\,q^{1/2}.$$

For $q=2^n$, this *Weil's bound* means that, for every nonzero $a\in\mathbb{F}_{2^n}$, we have: $\left|\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(aF(x))}\right|\leq (d-1)\,2^{\frac{n}{2}}$. And since adding a linear function $tr_n(bx)$ to the function $tr_n(aF(x))$ corresponds to adding $(b/a)\,x$ to $F(x)$ and does not change its univariate degree, we deduce that, if $d>1$ is odd and $a\neq 0$, then:

$$nl(tr_n(aF))\geq 2^{n-1}-(d-1)\,2^{\frac{n}{2}-1}.$$

An extension of the Weil bound to the character sums of functions of the form $F(x)+G(1/x)$ (where $1/x=x^{2^n-2}$ takes value 0 at 0), among which are the so-called *Kloosterman sums* $\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(1/x+ax)}$, has been first obtained by Carlitz and Uchiyama [333] and extended by Shanbhag, Kumar and Helleseth [1032]: if F and G have odd univariate degrees, then

$$\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(F(x^{-1})+G(x))}\leq (d_{alg}(F)+d_{alg}(G))\,2^{\frac{n}{2}}.$$

More can also be found in [678] for the case a function with sparse univariate representation is added to F.

6 Bent functions and plateaued functions

Bent functions are fascinating extremal mathematical objects. Bent Boolean functions play a role in coding theory, with Kerdock codes (see Subsection 6.1.22, page 280), and in other domains of communications (for instance, they are used to build the so-called bent function sequences for telecommunications [919] and are related to Golay Complementary Sequences [416]). Bent vectorial functions allow constructing good codes [453, 866, 865] and pose interesting problems related to coding theory [278, 854].

The role of bent Boolean functions in cryptography is less obvious nowadays since, because of fast *algebraic attack*s and Theorem 22, page 363 (which shows that Boolean functions obtained from bent functions by modifying a few values can not allow resisting them), we do not know an efficient construction using bent functions which would provide Boolean functions having all the necessary features for being used in stream ciphers. Concerning block ciphers, since bent vectorial functions are not balanced and do not exist when $m > \frac{n}{2}$, they are rarely used as substitution boxes in block ciphers[1]. Bijectivity is mandatory in the kind of ciphers called Substitution-Permutation Networks, and unbalancedness can represent a weakness in the other kind of ciphers called Feistel (see *e.g.* [957]). But vectorial bent functions can however be used in block ciphers at the cost of additional diffusion/compression/expansion layers, or as building blocks for constructions of substitution boxes. Moreover, constructions of bent Boolean functions are often transposable into constructions of Boolean functions for stream ciphers and bent vectorial functions are used to construct algebraic manipulation detection codes (see Section 12.1.6), which play an important role in cryptography. Hence, even from a cryptographic viewpoint, it seems important to devote a chapter to them. This is all the more true that bent (Boolean or vectorial) functions possess properties which are cryptographically very interesting: they have optimal nonlinearity, by definition, and their derivatives are balanced (in other words, changing the input to a bent function by the addition of a nonzero vector induces a uniform change among the 2^n outputs; this has of course relationship with the differential attack on block ciphers). And it often

[1] But the S-boxes in the block ciphers CAST-128 and CAST-256 are modified from bent functions, as well as the round functions in the cryptographic hash algorithms MD4, MD5 and HAVAL, and the nonlinear-feedback shift registers (NLFSR) in the stream cipher Grain.

happens that the cryptographic interest of notions on Boolean functions be renewed with the apparition of new ways of using them (see Section 12.1).
The notion of bent function has been generalized to functions over $\mathbb{Z}_4$ and to the wider domain of generalized bent functions. The page limit of this book does not allow us to address them.

Plateaued functions are a generalization of bent functions which free themselves from some cryptographic weaknesses inherent to bent functions (in particular their unbalancedness, the fact that their numbers of variables are necessarily even, and for vectorial functions the nonexistence of bent (n, m)-functions when $m > n/2$) but not all of them (for instance, they also have *limited algebraic degree*, which represents a weakness with respect to *fast algebraic attacks*).

The history of bent functions begins in the sixties[2]. The first paper in English on bent Boolean functions has been written by O. Rothaus in 1966 and published ten years later [1005]. It seems that, already in 1962, bent functions had been studied in the Soviet Union under the name of minimal functions, as mentioned by Tokareva in [1089]. V.A. Eliseev and O.P. Stepchenkov had proved that their algebraic degree is bounded above by half the number of variables (except in the case of two variables); they had also proposed an analogue of the Maiorana-McFarland construction. Their technical reports have never been declassified. The extension of the notion to vectorial (n, m)-functions is due to Kaisa Nyberg [906]. A book by S. Mesnager [865] that we recommend and a slightly more recent survey [313] exist on bent functions.

The introduction of plateaued Boolean functions is due to Zheng and Zhang [1173] as a generalization of partially-bent functions [211]. Recently has been shown in [247] that plateaued vectorial functions share with quadratic vectorial functions most of their nice properties, which considerably simplify in particular the study of their APNness, see Chapter 11 (but the property of plateauedness is not easy to prove in general).

6.1 Bent Boolean functions

We first recall for the convenience of the reader what we have seen on bent functions in Section 3.1, and we add some observations:

- A Boolean function f on $\mathbb{F}_2^n$ (n even) is called bent if its Hamming distance to the code $RM(1, n)$ of n-variable *affine functions* (the nonlinearity of f) equals $2^{n-1} - 2^{\frac{n}{2}-1}$ (*i.e.* is optimal).
- f is bent if and only if its Walsh transform W_f (with respect to some inner

[2] But the supports of bent Boolean functions being difference sets [651] in elementary Abelian 2-groups, bent functions had been studied before the adjective “bent” was invented; nevertheless, mathematicians were not much interested in such groups at that time.

product) takes values $\pm 2^{\frac{n}{2}}$ only[3]. This characterization is independent of the choice of the inner product on $\mathbb{F}_2^n$, since any other inner product has the form $\langle x, s\rangle = x \cdot L(s)$, where L is an auto-adjoint linear automorphism, *i.e.*, when "$\cdot$" is the *usual inner product*, an automorphism whose associated matrix is symmetric. The condition in this characterization can be slightly weakened, without losing the property of being necessary and sufficient:

Lemma 5 *Let $n \geq 2$ be even. Any n-variable Boolean function f is bent if and only if, for every $a \in \mathbb{F}_2^n$, we have $W_f(a) \equiv 2^{\frac{n}{2}} \left[\ mod\ 2^{\frac{n}{2}+1}\right]$, or equivalently $\widehat{f}(a) \equiv 2^{\frac{n}{2}-1} \left[\ mod\ 2^{\frac{n}{2}}\right]$.*

Proof. This necessary condition is also sufficient, since, if it is satisfied, then writing $W_f(a) = 2^{\frac{n}{2}}\lambda_a$, where λ_a is odd for every a, Parseval's Relation (2.47) implies $\sum_{a\in\mathbb{F}_2^n} \lambda_a^2 = 2^n$, and hence $\lambda_a^2 = 1$ for every a. □

A slightly different viewpoint on bent functions is that of bent sequences: for each vector X in $\{-1,1\}^{2^n}$, define: $\hat{X} = \frac{1}{\sqrt{2^n}} H_n X$, where H_n is the Walsh-Hadamard matrix, recursively defined by:

$$H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}, H_0 = [1].$$

The vectors X such that $\hat{X}$ belongs to $\{-1,1\}^{2^n}$ are called bent sequences. They are the images by character $\chi = (-1)^{\cdot}$ of the bent functions on $\mathbb{F}_2^n$. In [993] are considered some generalized bent notions (among which the *nega-bent* notion) from the domain of quantum error correcting codes, corresponding to flat spectra with respect to some unitary transforms (whose matrices U are such that $UU^{\dagger}$ equals the identity matrix, where "$\dagger$" means transpose-conjugate, and generalize Walsh-Hadamard matrices).

- An n-variable Boolean function f is bent if and only if its Hamming distance to any affine function equals $2^{n-1} \pm 2^{\frac{n}{2}-1}$; then half of the elements of the Reed-Muller code of order 1 lie at distance $2^{n-1} + 2^{\frac{n}{2}-1}$ from f and half lie at distance $2^{n-1} - 2^{\frac{n}{2}-1}$ (since if ℓ lies at distance $2^{n-1} + 2^{\frac{n}{2}-1}$ from f, then $\ell \oplus 1$ lies at distance $2^{n-1} - 2^{\frac{n}{2}-1}$ and *vice versa*).
  Conversely, a Boolean function is affine if and only if it lies at maximal Hamming distance from the set of bent functions (this is shown in [1088] but was probably known earlier by Dillon, Dobbertin and others, although maybe not explicitly written). In other words, the set of affine functions and the set of bent functions are *metric complements* of each other and constitute a so-called *pair of metrically regular sets* in the Boolean hypercube. Indeed, let us first observe that this maximal distance is at least $2^{n-1} - 2^{\frac{n}{2}-1}$ (the distance from affine functions to the set of bent functions), and that it cannot be more, because bentness being stable under the addition of affine functions, all the elements of the coset of $RM(1,n)$ containing

[3] In [1093] is shown that Boolean functions with two Walsh values are affine functions and bent functions, possibly modified at 0_n.

a function at maximal distance to the set of bent functions would be at the same distance to this set, and Parseval's relation prevents any such coset to be at a distance larger than $2^{n-1} - 2^{\frac{n}{2}-1}$ from any bent function. Only affine functions lie at such distance, because, as observed by Tokareva, for any non affine Boolean function f in even dimension, there exists a bent function g such that $f \oplus g$ is not bent, that is, whose distance to affine functions is strictly smaller than $2^{n-1} - 2^{\frac{n}{2}-1}$, and it is easily shown that f lies then at distance strictly smaller than $2^{n-1} - 2^{\frac{n}{2}-1}$ from bent functions, thanks once again to the fact that bentness is stable under the addition of affine functions.

- Bent Boolean functions are not balanced. As soon as n is large enough (say $n \geq 20$), the difference $2^{\frac{n}{2}-1}$ between their Hamming weights and the weight 2^{n-1} of balanced functions is very small with respect to this weight. However, according to [42, Theorem 6], 2^n bits of the pseudorandom sequence output by f in a combiner or a filter model are enough to distinguish it from a random sequence. Nevertheless, we shall see that highly nonlinear balanced functions can be built from bent functions.

Remark. Given a bent Boolean function f, the functions $f \oplus \ell$ where ℓ is affine are not balanced, but their weights are globally as close to 2^{n-1} as possible: according to Parseval's relation, there do not exist functions f such that the functions $f \oplus \ell$ have all weights closer to 2^{n-1}. □

6.1.1 Extended affine invariance of bentness and automorphism group of a function

The nonlinearity being an *EA invariant*, so is the notion of bent function. A class of bent functions shall be called a *complete class* if it is preserved by EA equivalence.

The *automorphism group* of the set of bent functions (let us denote it by G) is the general affine group. It indeed contains the general affine group, and the reverse inclusion is a direct consequence of the property that, given a Boolean function g, if for every bent function f, function $f \oplus g$ is also bent, then g is affine (which shows that the image of any affine function by an element of G is affine and then that G is included in the automorphism group of all affine functions; Proposition 51, page 178, completes then the proof).

Other notions of equivalence between bent functions come from design theory, see Subsection 6.1.9.

Given a (Boolean or vectorial) function f, recall that the group (already seen at page 91) of those affine automorphisms A which preserve f (alternatively[4], those which preserve its graph) is called the *automorphism group of function* f and is denoted by $Aut(f)$. The determination of $Aut(f)$ for f bent is often a difficult problem, see [56, 426, 659]. In [1162] is only studied the so-called

[4] In the cases of Boolean functions and of bent functions, it makes less difference [149, 150].

symmetric group (the sub-group of those input coordinate permutations which preserve the function).

6.1.2 Characterization of bentness by the derivatives

Characterization by first-order derivatives

Thanks to Relation (2.53), page 80, and to the fact that the Fourier-Hadamard transform of a pseudo-Boolean function is constant if and only if the function equals δ_0 times some constant, we have:

Theorem 12 *Any n-variable Boolean function (n even[5]) is bent if and only if, for any nonzero vector a, the Boolean function $D_af(x) = f(x) \oplus f(x+a)$ is balanced, that is, if and only if f satisfies $PC(n)$.*

In [190, 191] (see also [353]) is observed that, for every linear hyperplane H of $\mathbb{F}_2^n$, the condition of Theorem 12 can be weakened into "for any nonzero a in H, function D_af is balanced". Indeed, for $H = \{0_n, \alpha\}^\perp$, we have $W_f^2(\alpha) = \sum_{a\in\mathbb{F}_2^n}(-1)^{a\cdot\alpha}\mathcal{F}(D_af)$, $W_f^2(0_n) + W_f^2(\alpha) = 2\sum_{a\in H}\mathcal{F}(D_af)$ and this necessary condition is also sufficient since n being even, the sum of these two squares equals 2^{n+1} if and only if each square equals 2^n (see *e.g.* [191]). The functions whose derivatives D_af, $a \in H$, $a \neq 0_n$ are all balanced for n odd are also characterized in [190, 191] as well as, for every n, the functions whose derivatives D_af, $a \in E$, $a \neq 0_n$ are all balanced, where E is a vector subspace of $\mathbb{F}_2^n$ of dimension $n-2$.

Because of Theorem 12, bent (Boolean) functions are also called *perfect nonlinear* functions[6]. Equivalently, as noted by Rothaus and Welch, f is bent if and only if the $2^n \times 2^n$ matrix $H = [(-1)^{f(x+y)}]_{x,y\in\mathbb{F}_2^n}$ is a Hadamard matrix (*i.e.* satisfies $H \times H^t = 2^n\, I$, where I is the identity matrix). This implies that the Cayley graph G_f (see Subsection 2.3.5, page 89) is strongly regular (see [68] for more precision and for a characterization).

Characterization by second-order derivatives and second-order covering sequences

Proposition 65 *[317] An n-variable Boolean function f is bent if and only if:*

$$\forall x \in \mathbb{F}_2^n,\ \sum_{a,b\in\mathbb{F}_2^n}(-1)^{D_aD_bf(x)} = 2^n. \tag{6.1}$$

Proof. If we multiply both terms of Relation (6.1) by $f_\chi(x) = (-1)^{f(x)}$, we obtain the (equivalent) relation: $\forall x \in \mathbb{F}_2^n, f_\chi \otimes f_\chi \otimes f_\chi(x) = 2^n\, f_\chi(x)$; indeed, we have $f_\chi \otimes f_\chi \otimes f_\chi(x) = \sum_{b\in\mathbb{F}_2^n}\left(\sum_{a\in\mathbb{F}_2^n}(-1)^{f(a)\oplus f(a+b)}\right)(-1)^{f(b+x)} =$

[5] In fact, according to the observations above, "n even" is implied by "f satisfies $PC(n)$"; functions satisfying $PC(n)$ do not exist for odd n.

[6] The characterization of Theorem 12 leads to a generalization of the notion of bent function to non-binary functions. In fact, several generalizations exist [16, 718, 802] (see [266] for a survey); the equivalence between being bent and being perfect nonlinear is no more valid if we consider functions defined over residue class rings (see *e.g.* [271]).

$\sum_{a,b\in\mathbb{F}_2^n}(-1)^{f(a+x)\oplus f(a+b+x)\oplus f(b+x)}$. According to the bijectivity of the Fourier-Hadamard transform and to Relation (2.44), page 79, this is equivalent to :

$$\forall u \in \mathbb{F}_2^n,\ W_f^3\,(u) = 2^n\, W_f\,(u)\,.$$

Thus, we have $\sum_{a,b\in\mathbb{F}_2^n}(-1)^{D_aD_bf(x)} = 2^n$ if and only if, for every $u \in \mathbb{F}_2^n$, $W_f(u)$ equals $\pm\sqrt{2^n}$ or 0. According to Parseval's relation, the value 0 cannot be achieved by W_f and this is therefore equivalent to the bentness of f. □

Relation (6.1) is equivalent to the relation $\sum_{a,b\in\mathbb{F}_2^n}(1-2D_aD_bf(x)) = 2^n$, that is $\sum_{a,b\in\mathbb{F}_2^n} D_aD_bf(x) = 2^{2n-1}-2^{n-1}$, and hence to the fact that f admits the *second order covering sequence* with all-1 coefficients and with level $2^{2n-1}-2^{n-1}$.

6.1.3 Characterization of bentness by power moments of the Walsh transform

For every even integer $w \geq 4$, bent functions are characterized by the property that the sum $\sum_{a\in\mathbb{F}_2^n} W_f{}^w(a)$ is minimum:

Proposition 66 *Let n be any positive integer and f be any n-variable Boolean function. Then, for every even integer $w \geq 4$, we have*

$$\sum_{u\in\mathbb{F}_2^n} W_f^w(u) \geq 2^{\left(\frac{w}{2}+1\right)n},$$

with equality if and only if f is bent.

This is straightforward for $w = 4$ by using for instance the Cauchy-Schwarz inequality and its case of equality, and for $w \geq 6$, it is a direct consequence of the well-known inequality on the L^w norm: if $w' \geq w$ then $(\sum_{i\in I}|\lambda_i|^{w'})^{\frac{1}{w'}} \geq |I|^{\frac{1}{w'}-\frac{1}{w}}(\sum_{i\in I}|\lambda_i|^{w})^{\frac{1}{w}}$.

Such sums (for even or odd w) play a role with respect to *fast correlation attacks* [203, 189] (when these sums have small magnitude for low values of w, this contributes to a good resistance to fast correlation attacks).

Note that for $w = 4$, we have, according to (3.10) and (3.9), page 119:

$$\sum_{u\in\mathbb{F}_2^n} W_f^4(u) = 2^n\mathcal{V}(f) = 2^n \sum_{x,a,b\in\mathbb{F}_2^n}(-1)^{D_aD_bf(x)}.$$

Hence:

Corollary 12 *Let n be any positive integer and f any n-variable Boolean function. Then*

$$\sum_{x,a,b\in\mathbb{F}_2^n}(-1)^{D_aD_bf(x)} \geq 2^{2n},$$

with equality if and only if f is bent.

Remark. There is no such characterization for w odd, except in particular cases, like Niho functions, see Proposition 82, at page 248. □

Corollary 13 *Let n be any positive integer, w any even integer larger than or equal to 4 and E an $\mathbb{F}_2$-vector space of n-variable Boolean functions. There exists an (n,m)-function F such that $E \setminus \{0\}$ is the set of component functions of F. All functions except the null one in E are bent if and only if F is bent and this happens if and only if*

$$|\{(x_1,\dots,x_w) \in (\mathbb{F}_2^n)^w;\ \sum_{i=1}^{w} F(x_i) = 0_m \text{ and } \sum_{i=1}^{w} x_i = 0_n\}| =$$

$$2^{(w-1)n-m} + (2^m-1)\cdot 2^{\frac{wn}{2}-m}.$$

Proof. The two first assertions are by definition, and, according to Proposition 66, the component functions $v\cdot F$, $v \in \mathbb{F}_2^m \setminus \{0_m\}$, are all bent if and only if we have $\sum_{u\in\mathbb{F}_2^n, v\in\mathbb{F}_2^m} W_F^w(u,v) = 2^{wn} + (2^m-1)\cdot 2^{\left(\frac{w}{2}+1\right)n}$ (distinguishing the case $v = 0_m$ from the cases $v \neq 0_m$), that is, if and only if we have:

$$\sum_{u,x_1,\dots,x_w\in\mathbb{F}_2^n, v\in\mathbb{F}_2^m} (-1)^{v\cdot\sum_{i=1}^{w} F(x_i) \oplus u\cdot \sum_{i=1}^{w} x_i} =$$

$$2^{n+m}|\{(x_1,\dots,x_w) \in (\mathbb{F}_2^n)^w;\ \sum_{i=1}^{w} F(x_i) = 0_m \text{ and } \sum_{i=1}^{w} x_i = 0_n\}| =$$

$$2^{wn} + (2^m-1)\cdot 2^{\left(\frac{w}{2}+1\right)n}.$$ □

6.1.4 Characterization of bentness by the *NNF*

The *ANF* does not allow directly characterizing bent functions, but the *NNF* does, and this provides then a possible characterization through the ANF by using Relation (2.24), page 68 (however, this characterization is complex and we then do not state it explicitly).
The direct relationship between the Walsh transform values and the coefficients of the NNF gives:

Proposition 67 *[292] Let $f(x) = \sum_{I\subseteq\{1,\dots,n\}} \lambda_I\, x^I$ be the NNF of a Boolean function f on $\mathbb{F}_2^n$. Then f is bent if and only if:*
1. for every I such that $\frac{n}{2} < |I| < n$, the coefficient λ_I is divisible by $2^{|I|-\frac{n}{2}}$;
2. $\lambda_{\{1,\dots,n\}} \equiv 2^{\frac{n}{2}-1}$ [mod $2^{\frac{n}{2}}$].

Proof. According to Lemma 5, page 214, f is bent if and only if, for every $a \in \mathbb{F}_2^n$, $\widehat{f}(a) \equiv 2^{\frac{n}{2}-1}\left[\bmod 2^{\frac{n}{2}}\right]$. We deduce that, according to Relation (2.59), page 85, applied with $\varphi = f$, Conditions *1.* and *2.* are sufficient for f to be bent.
Conversely, Condition *1.* is necessary, according to Relation (2.60). Condition *2.* is also necessary since $\widehat{f}(1_n) = (-1)^n \lambda_{\{1,\dots,n\}}$, from Relation (2.59) or (2.60). □

The related characterization of bent functions by the ANF mentioned above implies conditions on the coefficients of the ANFs of bent functions, which have been observed and used in [301] (see more at page 269) and also partially observed by Hou and Langevin in [627].

Point 1 in Proposition 67 can be expressed by a single equation, see [293]. It is proved in this same reference that bentness can also be characterized by the generalized degree introduced at page 66.

6.1.5 Characterization of bentness by codes

A way of looking at bent functions deals with linear codes (as we mentioned in Section 4.1, at page 183): let f be any n-variable Boolean function (n even); we write its support $supp(f) = \{x \in \mathbb{F}_2^n;\ f(x) = 1\}$ as $\{u_1, \dots, u_{w_H(f)}\}$; we consider the matrix G whose columns are all the vectors of $supp(f)$, without repetition, and call C the linear code generated by the rows of this matrix. Then C is the set of all the vectors $U_v = (v \cdot u_1, \dots, v \cdot u_{w_H(f)})$, where v ranges over $\mathbb{F}_2^n$, and:

Proposition 68 *[1120] Let $n \geq 4$ be an even integer. Any n-variable Boolean function f is bent if and only if the linear code C whose generator matrix is the matrix whose columns are all the vectors of $supp(f)$ has dimension n, and has exactly two nonzero* Hamming weight*s: 2^{n-2} and $w_H(f) - 2^{n-2}$.*

Indeed, for every nonzero v in $\mathbb{F}_2^n$, the Hamming weight of codeword U_v equals $\sum_{x\in\mathbb{F}_2^n} f(x) \times v\cdot x = \sum_{x\in\mathbb{F}_2^n} f(x)\,\frac{1-(-1)^{v\cdot x}}{2} = \frac{\widehat{f}(0_n)-\widehat{f}(v)}{2}$. Hence, according to Relation (2.32), page 74, relating Fourier-Hadamard and Walsh transforms, $w_H(U_v)$ equals $2^{n-2} + \frac{W_f(v)-W_f(0_n)}{4}$. Thus, if f is bent, this weight is never null and C has then dimension n; moreover, either $W_f(v) = W_f(0_n)$ and $w_H(U_v) = 2^{n-2}$, or $W_f(v) = -W_f(0_n)$ and $w_H(U_v) = 2^{n-2} - \frac{W_f(0_n)}{2} = 2^{n-2} - \frac{2^n - 2\,w_H(f)}{2} = w_H(f) - 2^{n-2}$. Conversely, if C has dimension n and has exactly the two nonzero Hamming weights 2^{n-2} and $w_H(f) - 2^{n-2}$, then according to the relation $w_H(U_v) = 2^{n-2} + \frac{W_f(v)-W_f(0_n)}{4}$, for every v we have either $W_f(v) = W_f(0_n)$ or $W_f(v) = W_f(0_n) + 4w_H(f) - 2^{n+1} = -W_f(0_n)$ and, according to Parseval's Relation (2.48), page 79, $W_f(v)$ equals then $\pm 2^{\frac{n}{2}}$ for every v, *i.e.* f is bent.

C being linear, the minimum distance of C equals the minimum of these two nonzero weights: 2^{n-2} if $w_H(f) = 2^{n-1} + 2^{\frac{n}{2}-1}$ and $2^{n-2} - 2^{\frac{n}{2}-1}$ if $w_H(f) = 2^{n-1} - 2^{\frac{n}{2}-1}$.

There exist two other characterizations by Wolfmann [1120] dealing with C:
1. C has dimension n and C has exactly two weights, whose sum equals $w_H(f)$;
2. The length $w_H(f)$ of C is even, C has exactly two weights, and one of these weights is 2^{n-2}.

Of course, any bent Boolean function f can also be viewed as a (vectorial) $(n, 1)$-function and be related to the code C'_f seen at page 183, which has then

Hamming weight	Multiplicity
0	1
2^{n-1}	$2^n - 1$
$2^{n-1} - 2^{\frac{n}{2}-1}$	$2^{n-1} + (-1)^{f(0_n)} 2^{\frac{n}{2}-1}$
$2^{n-1} + 2^{\frac{n}{2}-1}$	$2^{n-1} - (-1)^{f(0_n)} 2^{\frac{n}{2}-1}$

Table 6.1 Weight distribution of C_f for f bent

weight distribution given by Table 6.1 (deduced from the Parseval and inverse Walsh transform formulae).

In [633, 634] is introduced the so-called near weight enumerator of a bent function f, equal to $W_{C_f}(X,Y) + 2^{\frac{n}{2}-1}X^n$, where W_{C_f} is the weight enumerator (see page 30) of the code $C_f = supp(f)$. A related Mac Williams-like identity is shown between dual bent functions (see Definition 51, page 221), leading to a notion of formally self-dual bent function and a Gleason-type theorem (see Gleason's theorem at page 31). As an application is proved in [634] the non-existence of bent functions in $2n$ variables with lowest degree of nonconstant terms in their ANF equal to $n-k$, for any nonnegative integer k and $n \geq N$, where N is the smallest integer satisfying $\binom{N+k+1}{k+1} < 2^{N-1} - 1$.

6.1.6 Characterization of bentness by difference sets, relative difference sets and structures of finite geometries

A subset D of a finite additive group G is called a $(|G|, |D|, \lambda)$-difference set in G if every nonzero element in G can be written in exactly λ ways as the difference between two elements of D (which implies $\lambda(|G|-1) = |D|(|D|-1)$). Equivalently, the incidence matrix $[D]$ defined by $[D]_{u,v} = 1$ if $u + v \in D$ and $[D]_{u,v} = 0$ otherwise satisfies $[D]^2 = (|D| - \lambda)I + \lambda J$ where I is the identity matrix and J the all-1 matrix [440]. Then $G \setminus D$ is also a difference set. Moreover, for any $g \in G$, $g + D$ is a difference set, called translate of D (we shall see in the next subsection that the set of all translates forms a symmetric block design).
It is observed in [441, 1005] that a Boolean function $f : \mathbb{F}_2^n \mapsto \mathbb{F}_2$ is bent if and only if its support $supp(f)$ is a non-trivial *difference set* in the elementary Abelian 2-group $\mathbb{F}_2^n$. It is known from Mann [824] that the parameters of such difference set must then be $(|G|, |D|, \lambda) = (2^n, 2^{n-1} \pm 2^{\frac{n}{2}-1}, 2^{n-2} \pm 2^{\frac{n}{2}-1})$. Such a difference set is called a *Hadamard difference set*.
Note that the EA equivalence of two bent functions does not necessarily imply the equivalence of the related difference sets (see *e.g.* [695, page 265]).

A subset R of a finite additive group G is called a $(\frac{|G|}{|N|}, |N|, |R|, \lambda)$ *relative difference set* in G relative to a subgroup N of G if every element in $G \setminus N$ can be written in exactly λ ways as the difference between two elements of R and no nonzero element of N can be written this way. An n-variable Boolean function is bent if and only if its graph is a relative difference set relative to $\{0_n\} \times \mathbb{F}_2$. This property extends to vectorial functions. See more in [965] on the connections between Boolean or vectorial functions and such structures.

In [428] are also characterized some bent functions by means of the notion of dimensional doubly dual hyperoval, in finite geometry.

6.1.7 The dual of a bent Boolean function

As linear codes, bent functions go by pairs:

Definition 51 *For every n even and every bent n-variable Boolean function f, the* dual function *$\widetilde{f}$ of f, is defined by:*

$$\forall u \in \mathbb{F}_2^n,\ W_f(u) = 2^{\frac{n}{2}}(-1)^{\widetilde{f}(u)}.$$

Proposition 69 *[441, 1005] The dual of any bent function is also bent and its own dual is f itself.*

Indeed, the inverse Walsh transform property (2.43), page 78, gives, for every $a \in \mathbb{F}_2^n$: $\sum_{u\in\mathbb{F}_2^n}(-1)^{\widetilde{f}(u)\oplus a\cdot u} = 2^{\frac{n}{2}}(-1)^{f(a)}$.

Let f and g be two bent functions, then Relation (2.46), page 79, applied with $\varphi(x) = f_\chi(x) = (-1)^{f(x)}$ and $\psi = g_\chi$ shows that

$$\mathcal{F}(\widetilde{f} \oplus \widetilde{g}) = \mathcal{F}(f \oplus g). \tag{6.2}$$

Thus, $f \oplus g$ and $\widetilde{f} \oplus \widetilde{g}$ have the same Hamming weight and:

Proposition 70 *[209, 212] The mapping $f \mapsto \widetilde{f}$ is an isometry of the class of bent n-variable Boolean functions.*

Remark. This isometry clearly cannot be extended into an isometry of the whole space $\mathcal{BF}_n$. Indeed, there would exist then a permutation π of $\mathbb{F}_2^n$ and an n-variable Boolean function g such that $\widetilde{f} = f \circ \pi \oplus g$ for every bent function f, and the examples of duals of bent functions we know (with Maiorana-McFarland functions for instance) show that such π, g do not exist. □

The mapping $f \mapsto \widetilde{f}$ also preserves EA equivalence, as originally observed in [441] in different terms. Indeed, for every linear automorphism L, we have according to Relation (2.58), page 82, that $\widetilde{f \circ L} = \widetilde{f} \circ L'$, where L' is the adjoint operator of L^{-1}, and, for every $a, b \in \mathbb{F}_2^n$, we have according to Lemma 4, page 77, that $\widetilde{f \circ t_b \oplus \ell_a} = \widetilde{f} \circ t_a \oplus \ell_b \oplus a \cdot b$, where t_a is the translation by a.
Denoting $b \cdot x$ by $\ell_b(x)$, Relation (6.2), applied with $g(x) = f(x+b) \oplus a \cdot x$, gives

$$\mathcal{F}(D_a\widetilde{f} \oplus \ell_b) = (-1)^{a\cdot b}\mathcal{F}(D_b f \oplus \ell_a), \tag{6.3}$$

and applied with $g(x) = f(x) \oplus \ell_a(x)$ and with $f(x+b)$ in the place of $f(x)$, it gives the following property, first observed in [219] (and rediscovered in [193]):

$$\mathcal{F}(D_a\widetilde{f} \oplus \ell_b) = \mathcal{F}(D_b f \oplus \ell_a) \tag{6.4}$$

This implies in particular the following relation that we shall need in the sequel:

$$\sum_{a,b\in\mathbb{F}_2^n} \mathcal{F}(D_a\widetilde{f}\oplus\ell_b) = \sum_{a,b\in\mathbb{F}_2^n} \mathcal{F}(D_bf\oplus\ell_a). \tag{6.5}$$

Moreover, from Relations (6.3) and (6.4), we deduce;

Proposition 71 *[236] Let f be any n-variable bent function. For every $a,b\in\mathbb{F}_2^n$, let us denote $\ell_b(x)=b\cdot x$ and $\ell_a(x)=a\cdot x$. Then $D_a\widetilde{f}$ and D_bf satisfy Relation (6.4). Moreover, if $a\cdot b=1$, then $\mathcal{F}(D_a\widetilde{f}\oplus\ell_b)=\mathcal{F}(D_bf\oplus\ell_a)=0$.*

In fact, Relation (6.4) is in a way characteristic of bent functions:

Proposition 72 *[236] If a pair of n-variable Boolean functions f and f' satisfies the relation $\mathcal{F}(D_af'\oplus\ell_b)=\mathcal{F}(D_bf\oplus\ell_a)$ for every $a,b\in\mathbb{F}_2^n$, then these functions are bent and are the dual of each other, up to the addition of a constant function.*

Proof. Taking $a=0_n$ in the equality $\mathcal{F}(D_af'\oplus\ell_b)=\mathcal{F}(D_bf\oplus\ell_a)$ shows that D_bf is balanced for every $b\neq 0_n$ and taking $b=0_n$ shows that D_af' is balanced for every $a\neq 0_n$. This proves the first assertion. Let us sum up the relations $\mathcal{F}(D_af'\oplus\ell_b)=\mathcal{F}(D_bf\oplus\ell_a)$ for b ranging over $\mathbb{F}_2^n$. We obtain the equalities $\sum_{x,b\in\mathbb{F}_2^n}(-1)^{f'(x)\oplus f'(x+a)\oplus b\cdot x}=\sum_{x,b\in\mathbb{F}_2^n}(-1)^{f(x)\oplus f(x+b)\oplus a\cdot x}=\sum_{x,y\in\mathbb{F}_2^n}(-1)^{f(x)\oplus f(y)\oplus a\cdot x}=W_f(0_n)\times W_f(a)$, and this gives $2^n(-1)^{f'(0_n)\oplus f'(a)}=2^n(-1)^{\widetilde{f}(0_n)\oplus\widetilde{f}(a)}$, that is, $f'(0_n)\oplus f'(a)=\widetilde{f}(0_n)\oplus\widetilde{f}(a)$, for every a. □

Notice that, for every a and b, we have $D_bf=\ell_a\oplus\epsilon$ if and only if $D_a\widetilde{f}=\ell_b\oplus\epsilon$.

Rothaus already observed that "many" bent functions are equal to their duals, *i.e.* are *self-dual bent functions*. The characterization of self-dual bent functions is an open problem, partially addressed in [265, 502, 626] (the latter reference classifies self-dual bent quadratic functions under the action of the orthogonal group, *i.e.* the group of $n\times n$ matrices M such that $MM^t=I$). See also [806]. It is observed in [265] that a Boolean n-variable function is self-dual bent or anti-self-dual bent (*i.e.* bent such that $\widetilde{f}=f\oplus 1$) if and only if its so-called *Rayleigh quotient* $\sum_{x,y\in\mathbb{F}_2^n}(-1)^{f(x)\oplus f(y)\oplus x\cdot y}=\sum_{x\in\mathbb{F}_2^n}(-1)^{f(x)}W_f(x)$ has maximal modulus (that is, has modulus $2^{\frac{3n}{2}}$), which is easier to handle in the case of quadratic functions: [626] uses that the associate *symplectic matrix* (see the footnote at page 194) is then involutive.

Remark. Since Boolean functions can be expressed in different forms, the question of moving from one form to another is important. For general functions, we have addressed this question at page 65. Regarding the duals, we have the easily proved following lemma (see *e.g.* [311]), in which an autodual basis is a pair (u,v) such that $tr^n_{n/2}(u)=tr^n_{n/2}(v)=1$ and $tr^n_{n/2}(uv)=0$. □

Lemma 6 *Let n be even and $m=\frac{n}{2}$. Let (u,v) be an autodual basis of $\mathbb{F}_{2^n}$ over*

$\mathbb{F}_{2^m}$. Let f be bent over $\mathbb{F}_{2^n}$ and $g(x,y) = f(ux+vy)$, $x, y \in \mathbb{F}_{2^m}$. Then:

$$W_f(au+bv) = W_g(a,b),$$

where W_f is calculated with respect to the inner product $X \cdot Y = tr_n(XY)$ and W_g is calculated with respect to the inner product $(x,y)\cdot(x',y') = tr_m(xx'+yy')$. Hence, if f is bent, then $\widetilde{f}(au+bv) = \widetilde{g}(a,b)$.

Numerical normal form of the dual

The *numerical normal form* of $\widetilde{f}$ can be deduced from that of f. Indeed, using equality $\widetilde{f} = \frac{1-(-1)^{\widetilde{f}}}{2}$, we have $\widetilde{f} = \frac{1}{2} - 2^{-\frac{n}{2}-1} W_f = \frac{1}{2} - 2^{\frac{n}{2}-1}\delta_0 + 2^{-\frac{n}{2}}\widehat{f}$. Applying now to $\varphi = f$ Relation (2.59), page 85, expressing the value of the Fourier-Hadamard transform by means of the coefficients of the NNF, we deduce that if $\sum_{I\subseteq\{1,\ldots,n\}} \lambda_I x^I$ is the NNF of f then:

$$\widetilde{f}(x) = \frac{1}{2} - 2^{\frac{n}{2}-1}\delta_0(x) + (-1)^{w_H(x)} \sum_{I\subseteq\{1,\ldots,n\};\ supp(x)\subseteq I} 2^{\frac{n}{2}-|I|}\lambda_I.$$

Changing I into $\{1,\ldots,n\}\setminus I$ in this relation, and observing that $supp(x)$ is included in $\{1,\ldots,n\}\setminus I$ if and only if $x_i = 0, \forall i \in I$, we obtain the NNF of $\widetilde{f}$ by expanding the following relation: $\widetilde{f}(x) =$

$$\frac{1}{2} - 2^{\frac{n}{2}-1}\prod_{i=1}^{n}(1-x_i) + (-1)^{w_H(x)} \sum_{I\subseteq\{1,\ldots,n\}} 2^{|I|-\frac{n}{2}}\lambda_{\{1,\ldots,n\}\setminus I}\prod_{i\in I}(1-x_i). \quad (6.6)$$

We deduce again that, for every $I \neq \{1,\ldots,n\}$ such that $|I| > \frac{n}{2}$, the coefficient of x^I in the NNF of $\widetilde{f}$ (resp. of f) is divisible by $2^{|I|-\frac{n}{2}}$.
Reducing Relation (6.6) modulo 2 proves Rothaus' bound (see Theorem 13 below) and the following fact:

Proposition 73 *[1005] Let $n \geq 4$ be even and f be any n-variable bent Boolean function. For every $I \subset \{1,\ldots,n\}$ such that $|I| = \frac{n}{2}$, the coefficient of x^I in the* ANF *of $\tilde{f}$ equals the coefficient of $x^{\{1,\ldots,n\}\setminus I}$ in the ANF of f.*

Using Relation (2.24), page 68, expressing the NNF by means of the ANF, Equality (6.6) can be related to the main result of [619] (but this result by Hou is stated in a complex way).

The *Poisson summation formula* (2.39), page 77, applied to $\varphi = f_\chi = (-1)^f$ gives (see [212]) that for every vector subspace E of $\mathbb{F}_2^n$, and for every elements a and b of $\mathbb{F}_2^n$, we have:

$$\sum_{x\in a+E}(-1)^{\widetilde{f}(x)\oplus b\cdot x} = 2^{-\frac{n}{2}}|E|\,(-1)^{a\cdot b}\sum_{x\in b+E^\perp}(-1)^{f(x)\oplus a\cdot x}. \quad (6.7)$$

In particular, $f(x)\oplus a\cdot x$ is constant on $b+E^\perp$ if and only if $\sum_{x\in a+E}(-1)^{\widetilde{f}(x)\oplus b\cdot x} = \pm 2^{\frac{n}{2}}$, and if E has dimension $\frac{n}{2}$, this is equivalent to the fact that $\widetilde{f}(x)\oplus b\cdot x$ is constant (with the same value on $a+E$ as $f(x)\oplus a\cdot x$ on $b+E^\perp$ if $a\cdot b=0$). Note that if $f(0_n)=0$ and $b=0_n$, this means that the constant value of $\widetilde{f}(x)$ on $a+E$ is zero. This is particularly interesting when f is self-dual.

6.1.8 Bound on algebraic degree and related properties

The algebraic degree of any Boolean function f being equal to the maximum size of the multi-index I such that x^I has an odd coefficient in the NNF of f, Proposition 67, page 218, gives:

Theorem 13 *[441, 1005] Let $n\geq 4$ be an even integer. The algebraic degree of any bent function on $\mathbb{F}_2^n$ is at most $\frac{n}{2}$.*

In the case that $n=2$, the bent functions have degree 2, since they have odd Hamming weight (in fact, they are the functions of odd weights).
The minimal possible Hamming distance between two bent n-variable functions is $2^{\frac{n}{2}}$, since this is the minimum distance of $RM(\frac{n}{2},n)$ (see Theorem 7, page 176), and since such distance is achieved by bent functions.
The bound of Theorem 13 is called *Rothaus' bound*. It shows, as observed by Dillon and Rothaus, that n-variable bent functions of algebraic degree $n/2$ can not be the direct sums (see page 258) of (necessarily bent) functions in less variables. Theorem 13 can also be proved with the same method as for proving Theorem 2, page 82, which also allows obtaining a bound, shown in [620], relating the gaps between $\frac{n}{2}$ and the algebraic degrees of f and $\widetilde{f}$:

Proposition 74 *The algebraic degrees of any n-variable bent function and of its dual satisfy:*

$$\frac{n}{2}-d_{alg}(f)\geq\frac{\frac{n}{2}-d_{alg}(\widetilde{f})}{d_{alg}(\widetilde{f})-1}. \tag{6.8}$$

Proof of Proposition 74 and alternative proof of Theorem 13: Let us denote by d (resp. by $\widetilde{d}$) the algebraic degree of f (resp. of $\widetilde{f}$) and consider a term x^I of degree d in the ANF of f. The Poisson summation formula (2.40), page 77, applied to $\varphi=f_\chi$ and to the vector space $E=\{u\in\mathbb{F}_2^n;\ \forall i\in I,\, u_i=0\}$ gives $\sum_{u\in E}(-1)^{\widetilde{f}(u)}=2^{\frac{n}{2}-d}\sum_{x\in E^\perp}f_\chi(x)$. The orthogonal $E^\perp$ of E equals $\{u\in\mathbb{F}_2^n;\ \forall i\notin I,\, u_i=0\}$. According to Relation (2.4), page 50, the restriction of f to $E^\perp$ has odd Hamming weight w, thus $\sum_{x\in E^\perp}f_\chi(x)=2^d-2w$ is not divisible by 4. Hence, $\sum_{u\in E}(-1)^{\widetilde{f}(u)}$ is not divisible by $2^{\frac{n}{2}-d+2}$.
We deduce first Theorem 13: suppose that $d>\frac{n}{2}$, then $\sum_{u\in E}(-1)^{\widetilde{f}(u)}$ is not even, a contradiction since E has an even size (indeed, we have $I\neq\{1,\ldots,n\}$, because f has algebraic degree smaller than n, since it has even Hamming weight).
We prove now Proposition 74: according to McEliece's theorem (or Ax's theorem), see page 179, $\sum_{u\in E}(-1)^{\widetilde{f}(u)}$ is divisible by $2^{\lceil\frac{n-d}{\widetilde{d}}\rceil}$. We deduce the inequality $\frac{n}{2}-d+2>\left\lceil\frac{n-d}{\widetilde{d}}\right\rceil$, that is, $\frac{n}{2}-d+1\geq\frac{n-d}{\widetilde{d}}$, which is equivalent to (6.8).□

Using Relation (2.22), page 66, instead of Relation (2.4) gives a more precise result than Theorem 13, shown in [292], which will be given in Section 6.1.18.

Proposition 74 can also be deduced from Proposition 67 and from some divisibility properties, shown in [292], of the coefficients of the NNFs of Boolean functions of algebraic degree d.

More on the algebraic degree of bent functions can be said for *homogeneous functions* (see page 274).

Remark. The *numerical degree* of a bent function equals n since the Walsh transform does not vanish. □

6.1.9 Bent Boolean functions and designs

A *balanced incomplete block design* (BIBD), or *2-design*, is a collection of subsets (called blocks) of the same size in some finite set, such that each pair of distinct elements is included in the same number λ of blocks (then, any element is contained in the same number of blocks as well). A BIBD is symmetric if the number of block equals the number of elements[7]. Symmetric designs are the 2-designs having the smallest number of blocks, given their number of elements. As recalled in [313], at least two designs are associated with any bent function f (cf. [441, 450, 656]):

1. the *difference set design* $D(f)$, in which the blocks are the translates $c + D$, $c \in \mathbb{F}_2^n$, of the support $D = supp(f) = f^{-1}(1)$ (or of the *co-support* $f^{-1}(0_n)$). Suppose for instance that f has Hamming weight $2^{n-1} + 2^{n/2-1}$ and that $D = f^{-1}(0_n)$; given a pair $\{x, y\}$ of distinct elements, the number of c such that $\{x, y\} \subset c + D$ equals $|\{c \in \mathbb{F}_2^n; f(x + c) = f(y + c) = 0\}|$, that is, $w_H(f \oplus 1) - \frac{w_H(D_{x+y}f \oplus 1)}{2} = 2^{n-2} - 2^{n/2-1}$ (since we have $|(x+D) \cap (y+D)| = |D| - \frac{(x+D)\Delta(y+D)}{2}$).
2. the *code design* $C(f)$, in which the blocks are the supports D'_c of the functions $f(x) \oplus c \cdot x \oplus \epsilon$, where ϵ is chosen such that $w_H(f(x) \oplus c \cdot x \oplus \epsilon) = 2^{n-1} - 2^{n/2-1}$, that is, $\epsilon = \widetilde{f}(c)$; hence $D'_c = \{x; f(x) \oplus c \cdot x \oplus \widetilde{f}(c) = 1\}$; this design has the same parameters as the difference set design (designs with such parameters are called *Menon designs*): denoting $l_x(c) = c \cdot x$, the number of those c such that a pair $\{x, y\}$ of distinct elements is included in D'_c equals $w_H((\widetilde{f} \oplus l_x \oplus f(x))(\widetilde{f} \oplus l_y \oplus f(y)) = \frac{w_H(\widetilde{f} \oplus l_x \oplus f(x)) + w_H(\widetilde{f} \oplus l_y \oplus f(y)) - w_H(l_{x+y} \oplus f(x) \oplus f(y)}{2} = \frac{2^{n-1} - 2^{n/2-1} + 2^{n-1} - 2^{n/2-1} - 2^{n-1}}{2} = 2^{n-2} - 2^{n/2-1}$.
   $D(f)$ admits all translations as automorphisms, but $C(f)$ has no obvious automorphism.

Related notions of equivalence can then be studied: two bent functions f and g could be called "difference set design equivalent" if $D(f)$ and $D(g)$ are isomorphic designs, and "code design equivalent" if $C(f)$ and $C(g)$ are isomorphic designs.

[7] when $\lambda = 1$, we have a projective plane; the blocks are the lines.

Note that the designs $D(f)$ and $C(f)$ are equal if and only if f is quadratic. Indeed, the quadratic bent functions have the property that for every linear function $l(x)$, the function $f(x) \oplus l(x)$ equals $f(x+a) \oplus \epsilon$, for some $a \in \mathbb{F}_2^n$ and some $\epsilon \in \mathbb{F}_2$. The set $\{D_a f, a \in \mathbb{F}_2^n\} + \mathbb{F}_2$ equals then the Reed-Muller code of order 1; this allows proving that $D(f) = C(f)$. Conversely, $D(f) = C(f)$ for a bent Boolean function implies that all derivatives have algebraic degree at most 1, which is equivalent to "f is quadratic".

6.1.10 Bent Boolean functions and affine subspaces

The Poisson summation formula (2.39), page 77, applied on f or on f_χ with $a = 0_n$, shows that the intersection between the support D of an n-variable bent function and a k-dimensional affine subspace $b+E$ of $\mathbb{F}_2^n$, where $k \geq n/2$, equals $2^{k-1} - 2^{k-n/2-1} \sum_{u \in E^\perp} (-1)^{\widetilde{f}(u) \oplus b \cdot u}$ and lies then between $2^{k-1} - 2^{n/2-1}$ and $2^{k-1} + 2^{n/2-1}$, as observed by Dillon. This implies that D can contain $b+E$ or be disjoint from $b+E$ only if $k = n/2$, and that if D contains $b+E$ (resp. is disjoint from $b+E$), then D has balanced intersection with any proper coset, and $D \setminus (b+E)$ (resp. $D \cup (b+E)$) is also a difference set. Studying the intersection of the supports of bent functions and affine spaces results in studying the sums of bent functions and indicators of flats:

Theorem 14 *[212] Let $b+E$ be any flat in $\mathbb{F}_2^n$ (E being a linear subspace of $\mathbb{F}_2^n$). Let f be any bent function on $\mathbb{F}_2^n$ (n even). The function $f^* = f \oplus 1_{b+E}$ is bent if and only if one of the following equivalent conditions is satisfied:*

1. *For any a in $\mathbb{F}_2^n \setminus E$, the function $D_a f$ is balanced on $b+E$;*
2. *The restriction of the function $\widetilde{f}(x) \oplus b \cdot x$ to any coset of $E^\perp$ is either constant or balanced.*

If f and f^* are bent, then E has dimension larger than or equal to $\frac{n}{2}$ and the algebraic degree *of the restriction of f to $b+E$ is at most* $\dim(E) - \frac{n}{2} + 1$.
If f is bent, E has dimension $\frac{n}{2}$, and the restriction of f to $b+E$ has algebraic degree at most $\dim(E) - \frac{n}{2} + 1 = 1$, i.e. *is affine, then conversely f^* is bent too.*

Proof. The equivalence between Condition 1 and the bentness of f^* is directly deduced from the fact that $\mathcal{F}(D_a f^*)$ equals $\mathcal{F}(D_a f)$ if $a \in E$, and equals $\mathcal{F}(D_a f) - 4\sum_{x \in b+E} (-1)^{D_a f(x)}$ otherwise (since when $a \notin E$, the cosets $b+E$ and $b+a+E$ are disjoint and $D_a f$ takes the same values on both of them). Condition 2 is also necessary and sufficient, since we have $W_f(a) - W_{f^*}(a) = 2\sum_{x \in b+E} (-1)^{f(x) \oplus a \cdot x}$, and using Relation (6.7), page 223, applied with $E^\perp$ in the place of E, we have then, for every $a \in \mathbb{F}_2^n$:

$$\sum_{u \in a+E^\perp} (-1)^{\widetilde{f}(u) \oplus b \cdot u} = 2^{\dim(E^\perp) - \frac{n}{2} - 1} (-1)^{a \cdot b} \left(W_f(a) - W_{f^*}(a) \right).$$

Then $W_f(a) - W_f^*(a)$ takes value 0 or $\pm 2^{\frac{n}{2}+1}$ for every a (which is necessary, and is sufficient according to Lemma 5, page 214) if and only if Condition 2 is

satisfied.
Let us now assume that f and f^* are bent. Then $1_{b+E} = f^* \oplus f$ has algebraic degree at most $\frac{n}{2}$, according to Rothaus' bound, and thus $\dim(E) \geq \frac{n}{2}$.
The values of the Walsh transform of the restriction of f to $b+E$ being equal to those of $\frac{1}{2}(W_f - W_{f^*})$, they are divisible by $2^{\frac{n}{2}}$ and thus the restriction of f to $b+E$ has algebraic degree at most $\dim(E) - \frac{n}{2} + 1$, according to Theorem 2.
If f is bent, E has dimension $\frac{n}{2}$, and the restriction of f to $b+E$ is affine, then the relation $W_f(a) - W_f^*(a) = 2\sum_{x\in b+E}(-1)^{f(x)\oplus a\cdot x}$ shows that f^* is bent too, according to Lemma 5. □

Remark. Relation (6.7) applied to $E^\perp$ in the place of E, where E is some $\frac{n}{2}$-dimensional subspace, shows that, if f is a bent function on $\mathbb{F}_2^n$, then $f(x) \oplus a\cdot x$ is constant on $b+E$ if and only if $\widetilde{f}(x) \oplus b\cdot x$ is constant on $a+E^\perp$. The same relation shows that $f(x)\oplus a\cdot x$ is then balanced on every other coset of E and $\widetilde{f}(x)\oplus b\cdot x$ is balanced on every other coset of $E^\perp$. Notice that Relation (6.7) shows also that $f(x)\oplus a\cdot x$ cannot be constant on a flat of dimension strictly larger than $\frac{n}{2}$ (*i.e.* that f cannot be k-weakly-normal with $k > \frac{n}{2}$). □

Remark. Let f be bent on $\mathbb{F}_2^n$. Let a and a' be two linearly independent elements of $\mathbb{F}_2^n$. Let us denote by E the orthogonal of the subspace spanned by a and a'. According to Condition 2 in Theorem 14, the function $f \oplus 1_E$ is bent if and only if $D_aD_{a'}\widetilde{f}$ is null (indeed, a 2-variable function is constant or balanced if and only if it has even Hamming weight, and $\widetilde{f}$ has even weight on any coset of the vector subspace spanned by a and a' if and only if, for every vector x, we have $f(x)\oplus f(x+a)\oplus f(x+a')\oplus f(x+a+a') = 0$). This result, stated in [193] and used in [198, Corollary 15] to design a new class of bent functions, is then a direct consequence of Theorem 14. □

6.1.11 Affine spaces of bent Boolean functions

It is observed in [210] that k-dimensional affine spaces of bent Boolean n-variable functions with k even correspond to bent functions in $n+k$ variables of a particular form. We shall denote these affine spaces in the form $f + < f_1,\ldots,f_k >$, where $< f_1,\ldots,f_k >$ denotes the vector space over $\mathbb{F}_2$ spanned by $\mathbb{F}_2$-linearly independent functions $f_1,\ldots,f_k$.

Proposition 75 *[210] For every positive even integers n,k, a k-dimensional affine space of Boolean n-variable functions $f + < f_1,\ldots,f_k >$ contains only bent functions if and only if the Boolean function*

$$h : (x,y) \in \mathbb{F}_2^n \times \mathbb{F}_2^k \mapsto \bigoplus_{i=1}^{\frac{k}{2}} (y_{2i-1}\oplus f_{2i-1}(x))(y_{2i}\oplus f_{2i}(x)) \oplus f(x)$$

is bent.

The proof is a generalization of the calculations made in Section 5.3, page 204.
Proof. For every $(a,b) \in \mathbb{F}_2^n \times \mathbb{F}_2^k$, we have $W_h(a,b) =$

$$\sum_{(x,y)\in\mathbb{F}_2^n\times\mathbb{F}_2^k} (-1)^{\bigoplus_{i=1}^{\frac{k}{2}}(y_{2i-1}\oplus f_{2i-1}(x)\oplus b_{2i})(y_{2i}\oplus f_{2i}(x)\oplus b_{2i-1})\oplus f(x)\oplus a\cdot x}$$

$$\cdot(-1)^{\bigoplus_{i=1}^{\frac{k}{2}}[b_{2i}b_{2i-1}\oplus b_{2i}f_{2i}(x)\oplus b_{2i-1}f_{2i-1}(x)]} =$$

$$\sum_{(x,y)\in\mathbb{F}_2^n\times\mathbb{F}_2^k} (-1)^{\bigoplus_{i=1}^{\frac{k}{2}} y_{2i-1}y_{2i}\oplus f(x)\oplus a\cdot x\oplus\bigoplus_{i=1}^{\frac{k}{2}}[b_{2i}b_{2i-1}\oplus b_{2i}f_{2i}(x)\oplus b_{2i-1}f_{2i-1}(x)]} =$$

$$2^{\frac{k}{2}}\sum_{x\in\mathbb{F}_2^n}(-1)^{\bigoplus_{i=1}^{\frac{k}{2}}[b_{2i}b_{2i-1}\oplus b_{2i}f_{2i}(x)\oplus b_{2i-1}f_{2i-1}(x)]\oplus f(x)\oplus a\cdot x} =$$

$$\pm 2^{\frac{k}{2}} W_{\bigoplus_{j=1}^{k} b_j f_j(x)\oplus f(x)}(a)$$

(by making the changes of variables $y_{2i-1} \mapsto y_{2i-1} \oplus f_{2i-1}(x) \oplus b_{2i}$ and $y_{2i} \mapsto y_{2i} \oplus f_{2i}(x) \oplus b_{2i-1}$ and using that $\sum_{y\in\mathbb{F}_2^k}(-1)^{\bigoplus_{i=1}^{\frac{k}{2}} y_{2i-1}y_{2i}} = 2^{\frac{k}{2}}$). Hence h is bent if and only if each function $\bigoplus_{j=1}^{k} b_j f_j(x) \oplus f(x)$ is bent. □

Remark. The situation with k-dimensional affine spaces of bent functions is quite different from what we have with k-dimensional vector spaces of Boolean functions whose nonzero elements are all bent: these latter vector spaces are in correspondence with bent (n,k)-functions: their nonzero elements are the component functions of these bent vectorial functions (see Section 6.4, page 295) and can then exist only if $k \le \frac{n}{2}$ (see Proposition 104, page 296). □

An example of application of Proposition 75 is given in [210], providing a large number of $(m-2)$-variable bent functions of algebraic degree 4 from any m-variable cubic bent function: let h be any such function, we have that each derivative $D_u h(x)$ is quadratic and balanced for every $u \neq 0_m$, since h is bent. According to Proposition 55, page 194 (see also the few lines following the proposition), for each $u \neq 0_m$, there exists v such that $D_v D_u h$ equals the constant function 1, that is, $D_u h(x+v) = D_u h(x) \oplus 1$, that is, $h(x+u+v) = h(x) \oplus h(x+u) \oplus h(x+v)] \oplus 1$, and hence:

$$\forall x \in \mathbb{F}_2^m, \forall y_1, y_2 \in \mathbb{F}_2, h(y_1 u + y_2 v + x) = h(x) \oplus y_1 D_u h(x) \oplus y_2 D_v h(x) \oplus y_1 y_2$$

(this can be checked for each value of (y_1, y_2)). We can then see that Proposition 75 can be applied with $n = m-2$, $k = 2$, by taking for f the restriction of $h(x) + D_u h(x) D_v h(x)$ to an $(m-2)$-dimensional vector space E not containing u, v nor $u+v$ (identifying then this vector space with $\mathbb{F}_2^n$), for f_1 the restriction of $D_u h$ to E and for f_2 the restriction of $D_v h$ to E. We deduce that the 2-dimensional affine space $(h \oplus D_u h D_v h)_{|E} + < D_u h_{|E}, D_v h_{|E} >$ contains only bent functions. These bent functions have algebraic degree 4 in general.

6.1.12 A graph related to bent functions

In [716] is studied the graph G whose vertices are the bent functions and whose edges connect vertices at Hamming distance $2^{\frac{n}{2}}$ of each other. It is shown that the degree of any vertex is not more than $2^{\frac{n}{2}} \prod_{i=1}^{n/2}(2^i+1)$ and that this bound is achieved with equality by quadratic bent functions, and only by them.
The minimal codewords of Reed-Muller codes being indicators of affine spaces (see Theorem 8, page 177), if two bent functions lie at distance $2^{n/2}$ from each other, then according to Rothaus' bound and to Theorem 14, page 226, they are weakly normal and they differ by the indicator of the $\frac{n}{2}$-dimensional space on which they are affine. Hence, if a bent function is not weakly normal, there is no bent function at Hamming distance $2^{\frac{n}{2}}$ from it. According to the existence of bent functions for $n \geq 14$ which are not weakly normal, G is disconnected if $n \geq 14$ (it is connected if $n \leq 6$; the question whether it is disconnected for $8 \leq n \leq 12$ seems open). Does it remain disconnected when we take off all vertices corresponding to functions being not weakly normal ? See more in [716].

6.1.13 Bent Boolean functions of low algebraic degrees

Quadratic bent functions

All the quadratic bent functions are known. According to the properties recalled in Section 5.2, any quadratic function

$$f(x) = \bigoplus_{1\leq i<j\leq n} a_{i,j}\, x_i\, x_j \oplus h(x) \ (h \text{ affine}, a_{i,j} \in \mathbb{F}_2)$$

is bent if and only if one of the following equivalent properties is satisfied:

1. its Hamming weight is equal to $2^{n-1} \pm 2^{\frac{n}{2}-1}$;
2. its associated symplectic form: $\beta_f : (x,y) \mapsto f(0_n) \oplus f(x) \oplus f(y) \oplus f(x+y)$ is non-degenerate (*i*.e. has kernel $\{0_n\}$);
3. the matrix of this symplectic form, that is, the skew-symmetric matrix $M = (m_{i,j})_{i,j\in\{1,\ldots,n\}}$ over $\mathbb{F}_2$, defined by: $m_{i,j} = a_{i,j}$ if $i < j$, $m_{i,j} = 0$ if $i = j$, and $m_{i,j} = a_{j,i}$ if $i > j$, is non-singular (*i*.e. has determinant 1);
4. $f(x)$ is equivalent, up to an affine nonsingular transformation, to the function: $x_1x_2 \oplus x_3x_4 \oplus \cdots \oplus x_{n-1}x_n \oplus \epsilon$ $(\epsilon \in \mathbb{F}_2)$.

Hence, there is a unique EA equivalence class of bent functions, as Rothaus and Dillon already observed in different terms.

Remark According to these characterizations, there exist (quadratic) bent functions for every even positive n (we can take the simplest one $x_1x_2\oplus\cdots\oplus x_{n-1}x_n$). Thus, the *covering radius* of the Reed-Muller code of order 1 equals $2^{n-1} - 2^{\frac{n}{2}-1}$ when n is even. □

Note that when f is bent in Proposition 57, page 198, that is, when $\mathcal{E}_f$ is the trivial vector space, $\mathcal{E}_f^\perp$ equals the whole space $\mathbb{F}_2^n$ and the linear functions

$y \mapsto \beta_f(x,y)$ cover then all linear forms on $\mathbb{F}_2^n$ (once each) when x ranges over $\mathbb{F}_2^n$. Examples of quadratic bent functions over $\mathbb{F}_2^n$ are:

- the so-called Maiorana-McFarland (see below) quadratic bent functions $f(x,y) = x \cdot \pi(y) \oplus h(y)$ where $x, y \in \mathbb{F}_2^{n/2}$ and π is an affine permutation of $\mathbb{F}_2^{n/2}$,
- the elementary quadratic *symmetric Boolean function* $\sigma_2(x) = \binom{w_H(x)}{2}$ [mod 2] $= \bigoplus_{1\leq i<j\leq n} x_i x_j$ (which is, up to the addition of an affine symmetric function, the only symmetric bent function, see Section 10.1); this function is bent because the kernel of it associated symplectic form $\varphi(x,y) = \bigoplus_{1\leq i\neq j\leq n} x_i y_j$, equal to $\{(x_1,\dots,x_n) \in \mathbb{F}_2^n;\ \forall i = 1,\dots,n,\ \bigoplus_{j\neq i} x_j = 0\}$ is reduced to $\{0_n\}$, since n is even.

Quadratic bent functions in trace representation

We have seen at page 199 how the Hamming weight and Walsh transform values of quadratic Boolean functions in trace form can be calculated.

A generic quadratic function $tr_n\left(\sum_{k=1}^{\frac{n}{2}-1} a_k x^{2^k+1}\right) + tr_{n/2}\left(a_{n/2} x^{2^{n/2}+1}\right) + \ell(x)$, where $a_1,\dots,a_{\frac{n}{2}-1} \in \mathbb{F}_{2^n}$, $a_{n/2} \in \mathbb{F}_{2^{n/2}}$ and ℓ is affine, is bent if and only if the equation $\sum_{k=1}^{\frac{n}{2}-1}\left(a_k x^{2^k} + a_k^{2^{n-k}} x^{2^{n-k}}\right) + a_{n/2} x^{2^{n/2}} = 0$ has 0 for only solution, that is, the linearized polynomial on the left-hand side is a permutation polynomial.

In the case of *Gold Boolean functions* $f(x) = tr_n(ax^{2^i+1});\quad a \in \mathbb{F}_{2^n}^*$, Carlitz' result shows that, for $i = 1$, f is bent if and only if a is not a cube (see page 201). For general i, raising the equation $ax^{2^i} + (ax)^{2^{n-i}} = 0$ to the 2^i-th power gives $a^{2^i} x^{2^{2i}} + ax = 0$. Hence, $x \neq 0$ is a solution if and only if $(ax^{2^i+1})^{2^i-1} = 1$, that is, $ax^{2^i+1} \in \mathbb{F}_{2^n} \cap \mathbb{F}_{2^i} = \mathbb{F}_{2^{\gcd(i,n)}}$ and since 2^i+1 and 2^i-1 are co-prime and $x \mapsto x^{2^i+1}$ is then a permutation in $\mathbb{F}_{2^{\gcd(i,n)}}$, the existence of such x is equivalent to that of x such that $x^{2^i+1} = \frac{1}{a}$; function $tr_n(ax^{2^i+1})$ is then bent if and only if $a \notin \{x^{2^i+1}, x \in \mathbb{F}_{2^n}\}$. Such a exists if and only if function x^{2^i+1} is not a permutation on $\mathbb{F}_{2^n}$, that is, $\gcd(2^i+1, 2^n-1) \neq 1$, and since $2^{2i}-1 = (2^i-1)(2^i+1)$ and 2^i-1 and 2^i+1 are co-prime, we have $\gcd(2^i+1, 2^n-1) = \frac{\gcd(2^{2i}-1, 2^n-1)}{\gcd(2^i-1, 2^n-1)} = \frac{2^{\gcd(2i,n)}-1}{2^{\gcd(i,n)}-1}$; the condition is then that $\frac{n}{\gcd(i,n)}$ is even. Being quadratic, these functions belong to the completed Maiorana-McFarland class.

Another classical example of quadratic bent function is:

$$f(x) = tr_n\Bigg(\sum_{i=1}^{\frac{n}{2}-1} x^{2^i+1}\Bigg) + tr_{\frac{n}{2}}\left(x^{2^{\frac{n}{2}}+1}\right);$$

the equation $\sum_{i=1}^{\frac{n}{2}-1}(x^{2^i} + x^{2^{n-i}}) + x^{2^{n/2}} = 0$, that is, $x + tr_n(x) = 0$ has indeed clearly 0 as the only solution, since $tr_n(x) \in \mathbb{F}_2$ and $tr_n(1) = 0$.

Quadratic bent functions are studied in [355, 629, 632, 699, 1144] (with the viewpoint of linearized permutation polynomials in the latter reference) and deduced in [768] from generalized bent functions.

An example of bivariate bent function over $\mathbb{F}_{2^n}$ for every n is from [236]. Function:

$$f(x,y) = tr_n(x^{2^i+1} + y^{2^i+1} + xy); \quad x,y \in \mathbb{F}_{2^n}, \quad \gcd(n,3) = \gcd(n,i) = 1,$$

is bent. Its associated symplectic form equals $\beta_f : ((x,y),(x',y')) \to f(0,0) \oplus f(x,y) \oplus f(x',y') \oplus f(x+x',y+y') = tr_n(x^{2^i}x' + xx'^{2^i} + y^{2^i}y' + yy'^{2^i} + xy' + x'y)$. The kernel of β_f equals $\left\{(x,y) \in \mathbb{F}_{2^n}^2 ; \left\{ \begin{array}{c} x^{2^i} + x^{2^{n-i}} + y = 0 \\ y^{2^i} + y^{2^{n-i}} + x = 0 \end{array} \right\}\right.$, equal to $\{(0,0)\}$ since denoting $z = x+y$ we have $z^{2^i} + z^{2^{n-i}} + z = 0$ which implies $z^{2^{2i}} = z^{2^i} + z$ and therefore $z^{2^{3i}} = z$, that is $z \in \mathbb{F}_{2^{3i}}$, and therefore $z \in \mathbb{F}_2$ and $z = 1$ being not a solution of the equation $z^{2^i} + z^{2^{n-i}} + z = 0$, we have $x = y$ and $x^{2^i} + x^{2^{n-i}} + x = 0$, that is, $x = 0$.

Remark Another representation of Boolean functions, in which, instead of identifying $x = (x_1, \ldots, x_n)$ with a field element (by the use of a basis of the vector space $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$), we identify $(x_1, \ldots, x_{n-1})$ with a field element in $\mathbb{F}_{2^{n-1}}$, and keep x_n in $\mathbb{F}_2$, leads to the Kerdock code; see Section 6.1.22 where the bent functions leading to this code are given. The so-called *cyclic bent functions*, such that, for any $a \neq b \in \mathbb{F}_{2^{n-1}}$ and any $\epsilon \in \mathbb{F}_2$, $f(ax_1, x_2) + f(bx_1, x_2 + \epsilon)$ is bent (as well as f itself), are proposed in [458], with applications in codes, codebooks, designs, mutually unbiased bases and sequences. $\square$

The unique EA equivalence class of quadratic bent functions has simplest representative $tr_n(x^{2^m+1})$ in univariate representation and $tr_m(xy)$ in *bivariate representation*.

Cubic bent functions

Any Boolean function f being bent if and only if every derivative of f in a nonzero direction is balanced, and every quadratic Boolean function being balanced if and only if one of its derivatives is the constant function 1, we have:

Proposition 76 *Let n be any positive integer and f any cubic n-variable Boolean function. Then f is bent if and only if, for every nonzero $a \in \mathbb{F}_2^n$, there exists $b \in \mathbb{F}_2^n$ such that the second-order derivative D_aD_bf equals constant function 1.*

Up to an affine transformation, we may assume in Proposition 76 that $a = (1,0,\ldots,0)$ and $b = (0,1,0,\ldots,0)$ and any cubic bent function is then affinely equivalent to a function of the form $x_1x_2 \oplus x_1f_1(x_3,\ldots,x_n) \oplus x_2f_2(x_3,\ldots,x_n) \oplus f_3(x_3,\ldots,x_n)$, but it seems difficult to go further in the determination of cubic bent functions.

The characterization given by Corollary 12, page 217, simplifies itself in the case of a cubic function: denoting the set $\{(a,b) \in \mathbb{F}_2^n;\ D_aD_bf = cst\}$ by $\mathcal{E}_f^{(2)}$, we have $\displaystyle\sum_{x,a,b \in \mathbb{F}_2^n} (-1)^{D_aD_bf(x)} = 2^n \sum_{(a,b) \in \mathcal{E}_f^{(2)}} (-1)^{D_aD_bf(0_n)}$, since the second-order

derivatives of f, which are affine, are constant or balanced. Then f is bent if and only if $\sum_{a,b\in\mathcal{E}_f^{(2)}}(-1)^{D_aD_bf(0_n)} = 2^n$. Note that, for every $a \in \mathbb{F}_2^n$, the section $\{b \in \mathbb{F}_2^n;\, (a,b) \in \mathcal{E}_f^{(2)}\}$ of $\mathcal{E}_f^{(2)}$ at a equals $\mathcal{E}_{D_af}$ and is then a linear subspace of $\mathbb{F}_2^n$. Hence, $\mathcal{E}_f^{(2)}$ is a bilinear space. Moreover, according to the property observed for quadratic functions at page 194, and since D_af is quadratic, function $b \mapsto D_aD_bf(0_n)$ is linear over $\mathcal{E}_{D_af}$ for every a, that is, function $(a,b) \mapsto D_aD_bf(0_n)$ is bilinear over $\mathcal{E}_f^{(2)}$.
We deduce that $\sum\limits_{a,b\in\mathcal{E}_f^{(2)}}(-1)^{D_aD_bf(0_n)} = \sum\limits_{\substack{a\in\mathbb{F}_2^n\\ \forall b\in\mathcal{E}_{D_af},\, D_aD_bf(0_n)=0}} |\mathcal{E}_{D_af}|$ and that f is bent if and only if $\sum\limits_{\substack{a\in\mathbb{F}_2^n\\ \forall b\in\mathcal{E}_{D_af},\, D_aD_bf(0_n)=0}} |\mathcal{E}_{D_af}| = 2^n$, and since for $a = 0_n$ we have $\mathcal{E}_{D_af} = \mathbb{F}_2^n$ and $\forall b \in \mathcal{E}_{D_af}, D_aD_bf(0_n) = 0$, this proves again Proposition 76.

But it is still an open problem to characterize the bent functions of algebraic degree 3 (that is, classify them under the action of the general affine group). This has been done for $n \leq 6$ in [1005] (see also [968] where the number of bent functions is computed for these values of n). For $n = 8$, it has been done in [618] (and completed in [9]); all of these functions have at least one affine derivative D_af, $a \neq 0_n$ (it has been proved in [193] that this happens for $n \leq 8$ only).

6.1.14 Bent Boolean functions in few variables

Bent functions in 2 variables are the Boolean functions of odd Hamming weight, *i.e.* of algebraic degree 2.
Bent functions in 4 variables are quadratic and therefore known.

Bent Boolean functions in 6 variables

The determination of all bent 6-variable functions has been done in [1005], where a search for all cubic bent functions in 6 variables was made, *i.e.* of all 6-variable bent functions of maximal algebraic degree. Rothaus determined, up to affine equivalence, four possible degree 3 parts of cubic Boolean functions in 6 variables. Determining all 6-variable bent functions was then possible by visiting all $2^{\binom{6}{2}} = 2^{15}$ quadratic parts for each of these four cases. Bart Preneel in his thesis [968] made this work again and found a fifth class of degree 3 parts, but this fifth class did not give any bent function. It was also proved by R.E. Kibler (as mentioned by Dillon in [440]) and rediscovered thirty years later in [124] (while classifying EA equivalence classes of 6-variable Boolean functions according to some cryptographic properties) that every bent function in 6 variables is affinely equivalent to a function of the Maiorana-McFarland class (see below). It was later observed in [212] that any bent function of algebraic degree 3 in 6 variables is affine equivalent to a function of the form $x_1x_2x_3 \oplus x_1h_1(x_4,x_5,x_6) \oplus x_2h_2(x_4,x_5,x_6) \oplus x_3h_3(x_4,x_5,x_6) \oplus g(x_4,x_5,x_6)$ where the mapping $(x_1, x_2, x_3) \mapsto (h_1(x_4, x_5, x_6), h_2(x_4, x_5, x_6), h_3(x_4, x_5, x_6))$ is a permutation and where $h_1 \oplus h_2 \oplus h_3$ is affine (for any function of this form, this double condition is necessary and sufficient). This implies in particular that any bent function in at most 6 variables is affinely equivalent to its dual.

Bent Boolean functions in 8 variables

The (impressive) determination of all bent 8-variable functions has been completed in [743], after that Langevin and Leander enumerated them in [741] (Hou had previously classified cubic bent functions in [618]).

In [347] are constructed bent *homogeneous functions* (*i*.e. bent functions whose ANFs are the sums of monomials of the same degree) on 12 (and less) variables by using the invariant theory (which makes feasible the computer searchs).

6.1.15 Primary constructions of bent Boolean functions

Except for small values of n, there does not exist a classification of bent functions under the action of the general affine group, and the structure of the set of bent functions is not clear. In order to understand better this structure, and also to have bent functions for applications, researchers have designed constructions. We describe them below. It is not clear whether these constructions give some insight on general bent functions or if on the contrary they draw our attention to peculiar bent functions. Nevertheless, they represent some important knowledge and have practical interest. Some of the known constructions are *ex nihilo* (from scratch). We call them *primary constructions* and address them in the present subsection. The others, which use as building blocks previously constructed bent functions (often called *initial functions*), and sometimes lead to recursive constructions, are called *secondary constructions*. We shall address them in the next subsection.

1. The *Maiorana-McFarland original class* $\mathcal{M}$ (see [441, 834]) is the set of all the Boolean functions on $\mathbb{F}_2^n = \{(x, y); x, y \in \mathbb{F}_2^{n/2}\}$, of the form:

$$f(x, y) = x \cdot \pi(y) \oplus g(y) \tag{6.9}$$

where "$\cdot$" is an *inner product* in $\mathbb{F}_2^{n/2}$, π is any permutation on $\mathbb{F}_2^{n/2}$ and g any Boolean function on $\mathbb{F}_2^{n/2}$.
In *bivariate representation*, this gives $(x, y) = tr_{n/2}(x\ \pi(y)) + g(y)$, where π is any permutation polynomial over $\mathbb{F}_{2^{n/2}}$ and g any Boolean function on $\mathbb{F}_{2^{n/2}}$.

Proposition 77 *Any function of the form (6.9) is bent if and only if π is a permutation. The dual of this bent function equals then $\widetilde{f}(a, b) = b \cdot \pi^{-1}(a) \oplus g(\pi^{-1}(a))$, where π^{-1} is the inverse permutation of π.*

This is a direct consequence[8] of Proposition 53, page 189, which writes here:

$$W_f(a,b) = 2^{\frac{n}{2}} \sum_{y\in\pi^{-1}(a)} (-1)^{g(y)\oplus b\cdot y}, \tag{6.10}$$

where $\pi^{-1}(a)$ denotes the pre-image of a by π^{-1}. We see that the dual function of f also belongs to Maiorana-McFarland class but with its two inputs swapped.

As we saw already in Section 5.1, the fundamental idea of Maiorana-McFarland's construction consists in concatenating affine functions. If we order all the binary words of length n in lexicographic order, with the bit of higher weight on the right, then the truth-table of f is the concatenation of the restrictions of f obtained by setting the value of y and letting x freely range over $\mathbb{F}_2^{n/2}$. These restrictions are affine.

Of course, $\mathcal{M}$ is a particular case of the general Maiorana-McFarland construction of Boolean functions seen in Subsection 5.1.1, which has been a generalization of $\mathcal{M}$ first investigated in [181].
Note that function f above is such that, for every function $h(y)$, function $f(x,y)\oplus h(y)$ is bent. This property is characteristic of the functions of the form (6.9):

Proposition 78 *A Boolean function $f(x,y)$, $x,y \in \mathbb{F}_2^{n/2}$, belongs to class $\mathcal{M}$ if and only if, for every function $h(y)$, the function $f(x,y)\oplus h(y)$ is bent.*

Proof. The condition is necessary, according to Proposition 77. For proving that it is also sufficient, let us take $h = \delta_a$ (the indicator of $\{a\}$). For every $a,u,v \in \mathbb{F}_2^{\frac{n}{2}}$, we have that $\sum_{x,y\in\mathbb{F}_2^{\frac{n}{2}}} (-1)^{f(x,y)\oplus u\cdot x\oplus v\cdot y\oplus\delta_a(y)} = \sum_{x,y\in\mathbb{F}_2^{\frac{n}{2}}} (-1)^{f(x,y)\oplus u\cdot x\oplus v\cdot y} - 2\sum_{x\in\mathbb{F}_2^{\frac{n}{2}}} (-1)^{f(x,a)\oplus u\cdot x\oplus v\cdot a}$, and if $f(x,y)$ and $f(x,y)\oplus\delta_a(y)$ are both bent, then we have $\pm 2^{\frac{n}{2}} = \pm 2^{\frac{n}{2}} \pm 2\sum_{x\in\mathbb{F}_2^{\frac{n}{2}}} (-1)^{f(x,a)\oplus u\cdot x}$. Hence for every $a,u \in \mathbb{F}_2^{\frac{n}{2}}$, we have $\sum_{x\in\mathbb{F}_2^{\frac{n}{2}}} (-1)^{f(x,a)\oplus u\cdot x} \in \{0, \pm 2^{\frac{n}{2}}\}$. Clearly, having, for some a, that $\sum_{x\in\mathbb{F}_2^{\frac{n}{2}}} (-1)^{f(x,a)\oplus u\cdot x} = 0$ for every u is impossible because of Parseval's relation. Then, for every $a \in \mathbb{F}_2^{\frac{n}{2}}$, there exists $u \in \mathbb{F}_2^{\frac{n}{2}}$ such that $\sum_{x\in\mathbb{F}_2^{\frac{n}{2}}} (-1)^{f(x,a)\oplus u\cdot x} = \pm 2^{\frac{n}{2}}$ that is $f(x,a) = u\cdot x$ or $f(x,a) = u\cdot x\oplus 1$. □

When a new method of construction of bent functions is found, it is necessary (for showing that it does not only provide functions which could be obtained with already known methods) to prove that some constructed functions are affinely inequivalent to Maiorana-McFarland functions[9]. We know thanks to Proposition 54, page 190, that an n-variable Boolean function with n even belongs to the completed Maiorana-McFarland class (the smallest possible complete class

[8] The input is here cut in two pieces x and y of the same length; cutting it in pieces of different lengths is addressed in Proposition 79 below; bentness is then obviously not characterized by the bijectivity of π.

[9] This should ideally be checked for all known classes of bent functions; this represents much (hard) work; checking this with class $\mathcal{M}$ is usually considered as mandatory because $\mathcal{M}$ is simpler and provides the widest class of bent functions.

including $\mathcal{M}$) if and only if there exists an $\frac{n}{2}$-dimensional linear subspace E of $\mathbb{F}_2^n$ such that function D_aD_bf is identically null for every $a, b \in E$. According to Proposition 78, this is also equivalent to the fact that there exists an $\frac{n}{2}$-dimensional affine subspace A and an $\frac{n}{2}$-dimensional linear subspace E of $\mathbb{F}_2^n$ such that every element of $\mathbb{F}_2^n$ can be expressed in a unique way in the form $x+y$, where $x \in E, y \in A$, and such that $f \oplus h$ is bent for every function $h(x+y)$ depending only on y.

The completed class of $\mathcal{M}$ contains all bent functions in at most 6 variables [440] and all quadratic bent functions (according to point 4 in the characterization of quadratic bent functions of page 229, taking $\pi = id$ and g constant in (6.9)).

Derived classes $\mathcal{C}$ and $\mathcal{D}$
Two classes of bent functions have been obtained in [212] by adding to functions of Maiorana-McFarland's class the indicators of vector subspaces:

- The class, denoted by $\mathcal{D}$, whose elements are the functions of the form $f(x,y) = x \cdot \pi(y) \oplus 1_{E_1}(x)1_{E_2}(y)$, where π is any permutation on $\mathbb{F}_2^{n/2}$ and E_1, E_2 are two linear subspaces of $\mathbb{F}_2^{\frac{n}{2}}$ such that $\pi(E_2) = E_1{}^{\perp}$ (1_{E_1} and 1_{E_2} denote their indicators). The dual of f belongs to the completed class of $\mathcal{D}$.
A subclass $\mathcal{D}_0$ of $\mathcal{D}$ has for elements the functions of the form $f(x,y) = x \cdot \pi(y) \oplus \delta_0(x)$ (recall that δ_0 is the Dirac symbol). The dual of such f is the function $y \cdot \pi^{-1}(x) \oplus \delta_0(y)$. It is proved in [212] that $\mathcal{D}_0$ is not included[10] in the completed versions of classes $\mathcal{M}$ and $\mathcal{PS}$ and that every bent function in 6 variables is affinely equivalent to a function of this class, up to the addition of an affine function.

- The class $\mathcal{C}$ of all the functions of the form $f(x,y) = x \cdot \pi(y) \oplus 1_L(x)$, where L is any linear subspace of $\mathbb{F}_2^{n/2}$ and π any permutation on $\mathbb{F}_2^{n/2}$ such that, for any element a of $\mathbb{F}_2^{n/2}$, the set $\pi^{-1}(a + L^{\perp})$ is a flat. It is a simple matter to see, as shown in [198], that, under the same hypothesis on π, if g is a Boolean function whose restriction to every flat $\pi^{-1}(a + L^{\perp})$ is affine, then the function $x \cdot \pi(y) \oplus 1_L(x) \oplus g(y)$ is also bent.
The fact that any function in class $\mathcal{D}$ or class $\mathcal{C}$ is bent comes from Theorem 14, page 226. In [822], existence and nonexistence results of such π and L are given for many of the known classes of permutations, inducing generic methods of constructions. In [1154], sufficient conditions on π and L so that f is provably outside the completed Maiorana-McFarland class are found. In particular, it is shown that the $\mathcal{C}$ functions described in [822] do not belong to the completed Maiorana-McFarland class. The more difficult question whether these functions are also outside the completed $\mathcal{PS}$ class remains open.

[10] We have seen in Proposition 54 that there is a rather simple way to show that a function f does not belong to the completed class of $\mathcal{M}$; it is more difficult to show that it does not belong to the completed class of $\mathcal{PS}$.

Maiorana-McFarland construction as a secondary construction

The original Maiorana-McFarland's construction is a particular case of a more general construction of bent functions, which is a *secondary construction* for $r < \frac{n}{2}$ and a primary one for $r = \frac{n}{2}$:

Proposition 79 *[223] Let $n = r+s$ ($r \leq s$) be even. Let ϕ be any mapping from $\mathbb{F}_2^s$ to $\mathbb{F}_2^r$ such that, for every $a \in \mathbb{F}_2^r$, the set $\phi^{-1}(a)$ is an $(n-2r)$-dimensional affine subspace of $\mathbb{F}_2^s$. Let g be any Boolean function on $\mathbb{F}_2^s$ whose restriction to $\phi^{-1}(a)$ (viewed as a Boolean function on $\mathbb{F}_2^{n-2r}$ via an affine isomorphism between $\phi^{-1}(a)$ and this vector space) is bent for every $a \in \mathbb{F}_2^r$, if $n > 2r$ (no condition on g being imposed if $n = 2r$). Then the function $f_{\phi,g}(x,y) = x \cdot \phi(y) \oplus g(y)$ is bent on $\mathbb{F}_2^n$.*

Proof. This is a direct consequence of Proposition 53, page 189, which writes:

$$W_{f_{\phi,g}}(a,b) = 2^r \sum_{y \in \phi^{-1}(a)} (-1)^{g(y) \oplus b \cdot y}. \tag{6.11}$$

According to Relation (6.11), the function $f_{\phi,g}$ is bent if and only if $r \leq \frac{n}{2}$ and $\sum_{y \in \phi^{-1}(a)} (-1)^{g(y) \oplus b \cdot y} = \pm 2^{\frac{n}{2}-r}$ for every $a \in \mathbb{F}_2^r$ and every $b \in \mathbb{F}_2^s$. The hypothesis in Proposition 79 is a sufficient condition for that (but it is not a necessary one). □

This construction is pretty general: the choice of any partition of $\mathbb{F}_2^s$ in 2^r flats of dimension $s - r = n - 2r$ and of an $(n-2r)$-variable bent function on each of these flats leads to an n-variable bent function. Note that ϕ, defined so that the elements of this partition are the pre-images of the elements of $\mathbb{F}_2^r$ by ϕ, is balanced (*i.e.* has output uniformly distributed over $\mathbb{F}_2^r$). In fact, it is observed in [802] that, if a bent function has the form $f_{\phi,g}$, then ϕ is balanced. This is a direct consequence of the characterization of balanced vectorial functions by Proposition 35, page 134 and of the fact that, for every nonzero $a \in \mathbb{F}_2^r$, the Boolean function $a \cdot \phi$ is balanced, since it equals the derivative $D_{(a,0_s)} f_{\phi,g}$. Obviously, every Boolean function can be represented (in several ways) in the form $f_{\phi,g}$ for some values of $r \geq 1$ and s and for some mapping $\phi : \mathbb{F}_2^s \mapsto \mathbb{F}_2^r$ and Boolean function g on $\mathbb{F}_2^s$.

Remark. There exist $\frac{n}{2}$-dimensional vector spaces of n-variable Boolean functions whose nonzero elements are all bent. This is equivalent to the existence of bent $(n, \frac{n}{2})$-functions. Maiorana-McFarland's construction allows constructing such functions as we shall see at page 297. Dimension $\frac{n}{2}$ is maximal, since a result by K. Nyberg shows that bent (n,m)-functions cannot exist for $m > \frac{n}{2}$.□

2. The *partial spread class* $\mathcal{PS}$, introduced in [441] by J. Dillon, is the set of all the sums (modulo 2) of the indicators of $2^{\frac{n}{2}-1}$ or $2^{\frac{n}{2}-1}+1$ pairwise *supplementary subspaces* of dimension $\frac{n}{2}$ of $\mathbb{F}_2^n$ (*i.e.* such that the intersection of any two of them equals $\{0_n\}$, and given their dimension, whose sum is direct and equals $\mathbb{F}_2^n$). A set of pairwise supplementary subspaces is called a *partial spread*, and a (full) *spread* if it covers $\mathbb{F}_2^n$. Some $\mathcal{PS}$ functions are built with partial spreads which are parts of full spreads and some are built with partial spreads which can not be extended into full spreads (we shall see a quadratic example below).

Proposition 80 *Any sum (modulo 2) of the indicators of $2^{\frac{n}{2}-1}$ or $2^{\frac{n}{2}-1}+1$ pairwise supplementary subspaces of dimension $\frac{n}{2}$ of $\mathbb{F}_2^n$ is a bent function. The dual of such function has the same form, all the $\frac{n}{2}$-dimensional spaces involved in the definition being replaced by their orthogonals.*

Definition 52 *Class $\mathcal{PS}$ is the set of bent functions defined in Proposition 80. The sums of $2^{\frac{n}{2}-1}$ such indicators constitute subclass $\mathcal{PS}^-$ (whose elements have Hamming weight $2^{\frac{n}{2}-1}(2^{\frac{n}{2}-1}-1)=2^{n-1}-2^{\frac{n}{2}-1}$) and the sums of $2^{\frac{n}{2}-1}+1$ of them constitute subclass $\mathcal{PS}^+$ (whose elements have Hamming weight $(2^{\frac{n}{2}-1}+1)\,2^{\frac{n}{2}}-2^{\frac{n}{2}-1}=2^{n-1}+2^{\frac{n}{2}-1}$).*

We shall see that Proposition 80 is a particular case of a theorem (Theorem 17) that we shall state at page 267. The bentness of the functions in $\mathcal{PS}$ can also be alternatively shown: for each pair of supplementary subspaces E_i and E_j and every $a\in\mathbb{F}_2^n$, the set $E_i\cap(a+E_j)$ is a singleton; this allows proving that, for every nonzero a, the product of any function $f(x)$ in $\mathcal{PS}^-$ (resp. in $\mathcal{PS}^+$) with its shifted function $f_a(x)=f(x+a)$ has Hamming weight $2^{\frac{n}{2}-1}(2^{\frac{n}{2}-1}-1)=2^{n-2}-2^{\frac{n}{2}-1}$ if $f(a)=0$ and $(2^{\frac{n}{2}-1}-1)(2^{\frac{n}{2}-1}-2)+2^{\frac{n}{2}}-2=2^{n-2}-2^{\frac{n}{2}-1}$ if $f(a)=1$ (resp. $(2^{\frac{n}{2}-1}+1)2^{\frac{n}{2}-1}=2^{n-2}+2^{\frac{n}{2}-1}$ if $f(a)=0$ and $2^{\frac{n}{2}-1}(2^{\frac{n}{2}-1}-1)+2^{\frac{n}{2}}=2^{n-2}+2^{\frac{n}{2}-1}$ if $f(a)=1$), which implies in all cases that the derivative D_af (whose Hamming weight equals $2(w_H(f)-w_H(ff_a))$) is balanced, and thus that f is bent, according to Theorem 12, page 216.

The $\mathcal{PS}^-$ functions built with a full spread are the complements of the elements of $\mathcal{PS}^+$ built with the same full spread and *vice versa*, but the complement of a general $\mathcal{PS}$ bent function is not necessarily in $\mathcal{PS}$.

Remark The Boolean functions equal to the sums of some number of indicators of pairwise supplementary $\frac{n}{2}$-dimensional subspaces of $\mathbb{F}_2^n$ share with quadratic functions the nice and convenient property of being bent if and only if they have the Hamming weight of a bent function (which is $2^{n-1}\pm 2^{\frac{n}{2}-1}$). □

All the elements of $\mathcal{PS}^-$ have *algebraic degree* $\frac{n}{2}$ exactly (indeed, by applying a linear isomorphism of $\mathbb{F}_2^n$, we may assume that $\mathbb{F}_2^{n/2}\times\{0_{n/2}\}$ is among the $2^{\frac{n}{2}-1}$ pairwise supplementary spaces defining the function, and since the function vanishes at 0_n, Relation (2.4) page 50 shows that the monomial $x_1\dots x_{\frac{n}{2}}$ appears in its ANF).
On the contrary, the elements of $\mathcal{PS}^+$ do not all have algebraic degree $\frac{n}{2}$: Dillon observed in [441] that, when $\frac{n}{2}$ is even, all quadratic bent functions are $\mathcal{PS}^+$ functions or their complements; indeed, by affine equivalence, we can restrict

ourselves to the function $(x,\epsilon,y,\eta)\in\mathbb{F}_{2^{n/2-1}}\times\mathbb{F}_2\times\mathbb{F}_{2^{n/2-1}}\times\mathbb{F}_2\to tr_{n/2-1}(xy)\oplus\epsilon\eta\oplus 1$, where $tr_{n/2-1}$ is the trace function from $\mathbb{F}_{2^{n/2-1}}$ to $\mathbb{F}_2$; the support of this function equals the union of the $2^{n/2-1}+1$ vector spaces of dimension $n/2$ (and very much related to the Kerdock code) $S_\emptyset=\{0\}\times\{0\}\times\mathbb{F}_{2^{n/2-1}}\times\mathbb{F}_2$ and $S_a=\{(x,\epsilon,a^2x+atr_{n/2-1}(ax)+a\epsilon,tr_{n/2-1}(ax));(x,\epsilon)\in\mathbb{F}_{2^{n/2-1}}\times\mathbb{F}_2\}$ for $a\in\mathbb{F}_{2^{n/2-1}}$. Indeed, we have $tr_{n/2-1}(xy)\oplus\epsilon\eta=0$ if and only if $x=\epsilon=0$ or there exists a such that $y=a^2x+atr_{n/2-1}(ax)+a\epsilon$ and $\eta=tr_{n/2-1}(ax)$. Note that since f has algebraic degree strictly less than $\frac{n}{2}$ for $n\geq 8$, this partial spread is not extendable to a full spread.

It is an open problem to characterize the *algebraic normal forms* of the elements of class $\mathcal{PS}$ or their trace representations. It is then necessary to identify within the $\mathcal{PS}$ construction, classes of explicit bent functions[11].

Class $\mathcal{PS}_{ap}$ *in* bivariate representation

J. Dillon exhibits in [441] a subclass of $\mathcal{PS}^-$, denoted by $\mathcal{PS}_{ap}$ (where "ap" stands for "affine plane"), whose elements (that we shall call *Dillon's functions*) are defined in an explicit form, that we already addressed in Subsection 5.1.2 (more precisely at page 192).

The vector space $\mathbb{F}_2^n$ is identified with the affine plane $\mathbb{F}_{2^{n/2}}\times\mathbb{F}_{2^{n/2}}$ (an *inner product* being $(x,y)\cdot(x',y')=tr_{\frac{n}{2}}(xx'+yy')$; we know that the notion of bent function is independent of the choice of the inner product). The affine plane $\mathbb{F}_{2^{n/2}}\times\mathbb{F}_{2^{n/2}}$ is equal to the union of its $2^{n/2}+1$ lines through the origin $E_\emptyset=\{0\}\times\mathbb{F}_{2^{n/2}}$ and $E_a=\{(x,ax)\,;\,x\in\mathbb{F}_{2^{n/2}}\}$, $a\in\mathbb{F}_{2^{n/2}}$; these lines are $n/2$-dimensional $\mathbb{F}_2$-subspaces of $\mathbb{F}_2^n$ and constitute the so-called *Desarguesian spread*. Choosing any $2^{n/2-1}$ of the lines, and taking them different from E_0 and $E_\emptyset$ (of equations $x=0$ and $y=0$), leads, by definition, to an element of $\mathcal{PS}_{ap}$, of the form $f(x,y)=g\left(x\,y^{2^{n/2}-2}\right)$, *i.e.* $g\left(\frac{x}{y}\right)$ with $\frac{x}{y}=0$ if $y=0$, where g is a balanced Boolean function on $\mathbb{F}_2^{n/2}$ which vanishes at 0. *In the sequel, we shall always take this convention that* $\frac{1}{0}=0$ and write $\frac{x}{y}$ instead of $x\,y^{2^{n/2}-2}$. The bentness of the resulting function is a consequence of Relation (5.9), page 193, with $\phi=f(0)=\epsilon=a=0$.

The complements $g\left(\frac{x}{y}\right)\oplus 1$ of these functions are the functions $h(\frac{x}{y})$ where h is balanced and does not vanish at 0; they belong to class $\mathcal{PS}^+$.

For every balanced function g, the dual of the bent function $g(\frac{x}{y})$ is $g(\frac{y}{x})$ (this will be a direct consequence of Theorem 17, page 267).

Class $\mathcal{PS}_{ap}$ *in* univariate representation

We have already seen at pages 191-192, the notion of $\mathcal{PS}_{ap}$ Boolean function in univariate representation (but without studying the condition under which such function is bent). A univariate representation of the elements of Desarguesian

[11] The situation with $\mathcal{PS}$ is then similar to the situation with general bent functions: we have a nice and simple definition, but no systematic way of determining all the elements which satisfy it.

spread is $\{u\,\mathbb{F}_{2^{n/2}}, u \in U\}$, where $U = \{u \in \mathbb{F}_2^n;\, u^{2^{n/2}+1} = 1\}$ is the cyclic group of $(2^{n/2}+1)$-th roots of unity in $\mathbb{F}_2^n$ (*i.e.* the multiplicative subgroup of $\mathbb{F}_{2^n}^*$ of order $2^{n/2}+1$). Each line through the origin of the plane $\mathbb{F}_{2^n}$ over $\mathbb{F}_{2^{n/2}}$, instead of being identified by the constant value $\frac{x}{y}$ of its nonzero elements $(x,y) \in \mathbb{F}_{2^{n/2}}^2$ (which makes with the convention $\frac{1}{0} = 0$ that the two lines of equations $x = 0$ and $y = 0$ provide necessarily the same output by the $\mathcal{PS}_{ap}$ function) is identified by the unique element of U its contains. Then g is viewed as a Boolean function over U such that $g(\alpha_1) = g(\alpha_2) = 0 = f(0)$ where (α_1, α_2) is the basis chosen for the plane $\mathbb{F}_{2^n}$ over $\mathbb{F}_{2^{n/2}}$, assuming without loss of generality that α_1, α_2 both belong to U. Relation (5.5), page 192, with $m = \frac{n}{2}$, and $(-1)^{f(0)} - \sum_{\mu \in U}(-1)^{g(\mu)} = 0$ (since g is taken balanced on $U \setminus \{\alpha_i\}$, $i = 1, 2$), and $\phi = 0$ gives an alternative proof of the bentness of the $\mathcal{PS}_{ap}$ functions defined by Dillon, since $tr_m^n(z) = 0$ if and only if $z \in \mathbb{F}_{2^m}$. Moreover, for every $x \in \mathbb{F}_{2^{n/2}}^*$ and every $u \in U$, we have $(ux)^{2^{n/2}-1} = u^{-2}$ and $u \mapsto u^{-2}$ is a permutation of U, this leads to an expression of $\mathcal{PS}_{ap}$ bent functions of the form $h(z^{2^{n/2}-1})$, $z \in \mathbb{F}_{2^n}$, where h is a Boolean function over $\mathbb{F}_{2^n}$ such that $h(0) = 0$ and whose restriction to U has Hamming weight $2^{\frac{n}{2}-1}$.

Dillon shows in [442] that all bent functions of the form $tr_n(az^{2^{n/2}-1})$, $z \in \mathbb{F}_{2^n}$, are affinely inequivalent to the Maiorana-McFarland functions.

It is possible to deduce the univariate representation of $\mathcal{PS}_{ap}$ functions from their bivariate representation. We have seen at page 65 that any bivariate function $f(x,y)$ over $\mathbb{F}_{2^{n/2}}$ can be represented as a function of $z \in \mathbb{F}_{2^n}$, that we shall also denote by $f(z)$ (by abuse of notation), by posing $x = tr_{n/2}^n(az) = az + (az)^{2^{n/2}}$ and $y = tr_{n/2}^n(bz) = bz + (bz)^{2^{n/2}}$ for some elements $a, b \in \mathbb{F}_{2^n}$ which need to be $\mathbb{F}_2^{n/2}$-linearly independent (for instance, we can choose $\omega \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^{n/2}}$ and the pair $(1, \omega)$ is then a basis of the $\mathbb{F}_{2^{n/2}}$-vector space $\mathbb{F}_{2^n}$; we then take for (a, b) a basis orthonormal with $(1, \omega)$). For $f(x,y) = g\left(\frac{x}{y}\right)$, we have then the following expression valid for $z \neq 0$:

$$f(z) = g\left(\left(a + a^{2^{n/2}} z^{2^{n/2}-1}\right)\left(b + b^{2^{n/2}} z^{2^{n/2}-1}\right)^{2^{n/2}-2}\right).$$

Given a primitive element α of $\mathbb{F}_{2^n}$, we have for $i = 0, \ldots, 2^{n/2}$ and $j = 0, \ldots, 2^{n/2}-2$:

$$f\left(\alpha^{i+j(2^{n/2}+1)}\right) = g\left((a + a^{2^{n/2}}\beta^i)\,(b + b^{2^{n/2}}\beta^i)^{2^{n/2}-2}\right),$$

where $\beta = \alpha^{2^{n/2}-1}$.

Dillon [442], observing that function $tr_n(az^{2^{n/2}-1})$, $z \in \mathbb{F}_{2^n}$, $a \in \mathbb{F}_{2^n}^*$, is bent if and only if (see above) the restriction of $tr_n(az)$ to U has *Hamming weight* $2^{n/2-1}$, conjectures that such a exists for every even n. He gives the translation of this conjecture in terms of cyclic codes: let θ be a primitive element of U (*i.e.* a primitive $(2^{n/2}+1)$-th root of unity in $\mathbb{F}_{2^n}$), then the condition is that the *cyclic code* $C = \{(tr_n(a), tr_n(a\theta), tr_n(a\theta^2), tr_n(a\theta^3), \ldots, tr_n(a\theta^{2^{n/2}});\, a \in \mathbb{F}_{2^n}\}$ contains codewords of Hamming weight $2^{n/2-1}$. Since multiplying a by an element of U corresponds to a cyclic shift, he can restrict himself to $a \in \mathbb{F}_{2^{n/2}}$. Then $tr_n(a\,\theta^j) = tr_{n/2}(a\,tr^n_{n/2}(\theta^j)) = tr_{n/2}(a\,(\theta^j + \theta^{-j}))$. We know (see page 253, see also Appendix, page 533 and foll.) that when $j = 1, \ldots, 2^{n/2}$, $\theta^j + \theta^{-j}$ (*i.e.* when $z \in U \setminus \{1\}$, $z + z^{-1}$) takes twice each value in $\{x \in \mathbb{F}^*_{2^{n/2}}; tr_{n/2}(x^{-1}) = 1\}$. The condition on a is then that $\sum_{x\in\mathbb{F}^*_{2^{n/2}};\, tr_{n/2}(x^{-1})=1} (-1)^{tr_{n/2}(ax)} = 2^{n/2-1} - 2\cdot 2^{n/2-2} = 0$, which is equivalent to $\sum_{x\in\mathbb{F}^*_{2^{n/2}}} \left(1 - (-1)^{tr_{n/2}(x^{-1})}\right)(-1)^{tr_{n/2}(ax)} = 0$. The conjecture is then that a exists in $\mathbb{F}^*_{2^{n/2}}$ such that $\sum_{x\in\mathbb{F}^*_{2^{n/2}}} (-1)^{tr_{n/2}(x^{-1}+ax)} = -1$. We have already seen at page 211 that such sum added with 1 is called a Kloosterman sum. Lachaud and Wolfmann proved this conjecture in [733]; they proved that the values of such Kloosterman sums are all the numbers divisible by 4 in the range $[-2^{n/4+1} + 1; 2^{n/4+1} + 1]$, by relating such sums to elliptic curves (and this relation was exploited later in [781] for deriving an algorithm checking bentness more efficiently in such context).

It has been later observed that all these results remain valid with exponents of the form $j \cdot (2^{\frac{n}{2}} - 1)$, where $\gcd(j, 2^{\frac{n}{2}} + 1) = 1$, with the same arguments (the mapping $x \mapsto x^j$ by which function $x \mapsto x^{2^{\frac{n}{2}}-1}$ is composed being a permutation of U). These exponents are now widely called *Dillon exponents*. Leander [750] has found another proof which gives more insight; a small error in his proof has been corrected in [350].

Dillon checked that one of the functions in $\mathcal{PS}_{ap}$ does not belong to the completed $\mathcal{M}$ (Maiorana-McFarland) class: function $tr_8(x^{15})$ over $\mathbb{F}_{2^8}$, is affinely inequivalent to $\mathcal{M}$ functions because (we omit the proof) there cannot exist an $n/2$-dimensional subspace W of $\mathbb{F}_2^n$ such that D_aD_bf is null for every a and b both in W.

It may be more difficult to prove that a given function is not affinely equivalent to $\mathcal{PS}$ functions than to $\mathcal{M}$ functions; see an example in [212].

Extended $\mathcal{PS}_{ap}$ class

Class $\mathcal{PS}_{ap}$ is slightly extended into the subclass of $\mathcal{PS}^-$ denoted by $\mathcal{PS}^{\#}_{ap}$, of those Boolean functions over $\mathbb{F}_{2^n}$ which can be obtained from those of $\mathcal{PS}_{ap}$ by composition by the transformations $x \in \mathbb{F}_{2^n} \mapsto \delta x$, $\delta \neq 0$, and by addition of a constant[12]. The elements of $\mathcal{PS}^{\#}_{ap}$ are the Boolean functions f of Hamming weight $2^{n-1} \pm 2^{n/2-1}$ on $\mathbb{F}_{2^n}$ such that, denoting by α a primitive element of this field, $f(\alpha^{2^{n/2}+1}x) = f(x)$ for every $x \in \mathbb{F}_{2^n}$. We shall see in Subsection 6.1.20 that the functions in $\mathcal{PS}^{\#}_{ap}$ have a stronger property than bentness, called hyper-bentness. It is proved in [278] (by extension of the results of [441]) that they are the functions of Hamming weight $2^{n-1} \pm 2^{n/2-1}$ which can be written as $\sum_{i=1}^{r} tr_n(a_i x^{j_i})$ for $a_i \in \mathbb{F}_{2^n}$ and j_i a multiple of $2^{n/2} - 1$ with $j_i \leq 2^n - 1$.

[12] The functions of $\mathcal{PS}_{ap}$ are among them those satisfying $f(0) = f(1) = 0$.

Other classes of $\mathcal{PS}$ functions in explicit form
The functions in $\mathcal{PS}_{ap}$ are not the only $\mathcal{PS}$ bent functions which can be given with explicit trace representation (useful for applications, *e.g.* in telecommunications).
For instance, the $\mathcal{PS}$ bent functions related to André's spreads[13] have been studied in [246]. These spreads introduced by J. André in the fifties and independently by Bruck later are defined as follows: let k and m be positive integers such that k divides m, say $m = kl$. Let N_k^m be the norm map from $\mathbb{F}_{2^m}$ to $\mathbb{F}_{2^k}$:

$$N_k^m(x) = x^{\frac{2^m-1}{2^k-1}}.$$

Let ϕ be any function from $\mathbb{F}_{2^k}$ to $\mathbb{Z}/l\mathbb{Z}$. Then, denoting $\phi \circ N_k^m$ by φ (it can be any function from $\mathbb{F}_{2^m}$ to $\mathbb{Z}/l\mathbb{Z}$ which is constant on any coset of the subgroup U of order $\frac{2^m-1}{2^k-1}$ of $\mathbb{F}_{2^m}^*$), the $\mathbb{F}_2$-vector subspaces:

$$\{(0,y),\, y \in \mathbb{F}_{2^m}\} \text{ and } \{(x, x^{2^{k\varphi(z)}} z), x \in \mathbb{F}_{2^m}\}, \text{ where } z \in \mathbb{F}_{2^m}$$

form together a spread of $\mathbb{F}_{2^m}^2$. Indeed, these subspaces have trivial pairwise intersection: suppose that $x^{2^{k\varphi(y)}} y = x^{2^{k\varphi(z)}} z$ for some nonzero elements x, y, z of $\mathbb{F}_{2^m}$ (the other cases of trivial intersection are obvious), then we have $N_k^m(x^{2^{k\varphi(y)}} y) = N_k^m(x^{2^{k\varphi(z)}} z)$, that is, $N_k^m(x^{2^{k\varphi(y)}}) N_k^m(y) = N_k^m(x^{2^{k\varphi(z)}}) N_k^m(z)$; equivalently, since $x \mapsto x^{2^{k\varphi(z)}}$ is in the Galois group of $\mathbb{F}_{2^m}^2$ over $\mathbb{F}_{2^k}$, $N_k^m(x) N_k^m(y) = N_k^m(x) N_k^m(z)$ and hence $N_k^m(y) = N_k^m(z)$ and $\varphi(y) = \varphi(z)$, which together with $x^{2^{k\varphi(y)}} y = x^{2^{k\varphi(z)}} z$ implies then $y = z$.
Those spreads provide asymptotically the largest part of the known examples, due to the large number of choices for the map ϕ.
The trace representation of the $\mathcal{PS}$ bent functions associated to André's spreads is easily obtained. A pair $(x,y) \in \mathbb{F}_{2^m}^* \times \mathbb{F}_{2^m}$ belongs to $\{(x, x^{2^{k\varphi(z)}} z), x \in \mathbb{F}_{2^m}\}$ if and only if

$$y = x^{2^{k\varphi(z)}} z = x^{2^{k\phi(N_k^m(z))}} z = x^{2^{k\phi\left(\frac{N_k^m(y)}{N_k^m(x)}\right)}} z = x^{2^{k\varphi(y/x)}} z. \tag{6.12}$$

Then a Boolean function in this class has the form:

$$f(x,y) = g\left(\frac{y}{x^{2^{k\varphi(y/x)}}}\right) \tag{6.13}$$

(with the usual convention $\frac{y}{0} = 0$) where g is balanced on $\mathbb{F}_{2^m}$ and vanishes at 0. Such bent function is in $\mathcal{PS}$ and is potentially inequivalent to $\mathcal{PS}_{ap}$ functions (this needs to be further studied, though).
Let us study now the dual of f. If S is the support of g, then since $0 \notin S$, the support of f is equal to the union $\bigcup_{z \in S} \{(x, x^{2^{k\varphi(z)}} z), x \in \mathbb{F}_{2^m}\}$, less $\{0\}$. The support of the dual of f is the union of the orthogonals of these subspaces, less $\{0\}$ as well (see Proposition 80). A pair $(x', y') \in \mathbb{F}_{2^m}^2$ belongs to the orthogonal of $\{(x, x^{2^{k\varphi(z)}} z), x \in \mathbb{F}_{2^m}\}$ if and only if $tr_m(xx' + x^{2^{k\varphi(z)}} zy') =$

[13] We thank W. Kantor for mentioning these spreads which lead to numerous bent functions.

$tr_m((x'+(zy')^{2^{m-k\varphi(z)}})x)$ equals 0 for all $x \in \mathbb{F}_{2^m}$, that is, if $x'+(zy')^{2^{m-k\varphi(z)}} = 0$, that is, if $x' = y' = 0$ or $z = \frac{x'^{2^{k\varphi(z)}}}{y'}$; hence we have:

$$\widetilde{f}(x,y) = g\left(\frac{x^{2^{k\varphi(x/y)}}}{y}\right). \tag{6.14}$$

Of course, if g does not vanish at 0, the function defined by (6.13) is bent as well. We can see this by changing g into its complement $g \oplus 1$ (which changes f and its dual into their complements as well).
Note that class $\mathcal{PS}_{ap}$ corresponds to the case where φ is the null function. It also corresponds to the case $k = m$ since we have then $f(x,y) = g\left(\frac{y}{x}\right)$, because $x^{2^m} = x$. Note finally that if $k = 1$ then $N_k^m(x) = 1$ for every $x \neq 0$ and the groups of the spread are $\{(0,y), y \in \mathbb{F}_{2^m}\}$, $\{(x,0), x \in \mathbb{F}_{2^m}\}$ and $\{(x, x^{2^j}z), x \in \mathbb{F}_{2^m}\}, z \in \mathbb{F}_{2^m}^*$ for some j and $f(x,y) = g\left(\frac{y}{x^{2^j}}\right)$; the functions are in the $\mathcal{PS}_{ap}$ class up to linear equivalence.
Finite prequasifield spreads from finite geometry (see [963]) have also been investigated by Wu [1123] to give explicit forms of the related functions in $\mathcal{PS}$ and of their duals, thanks to the determination of the compositional inverses of certain parametric permutation polynomials. In particular, Wu has considered the Dempwolff-Muller prequasifields and the Knuth presemifields to obtain the expressions of the corresponding $\mathcal{PS}$ bent functions. The constructed functions and their dual functions are in a similar shape as the $\mathcal{PS}_{ap}$ functions, but are more complex. See more in [663].
Explicit constructions of bent functions derived from symplectic presemifields associated to pseudo-planar functions (see page 296) $\sum_{i<j} a_{i,j}x^{2^i+2^j}$ (whose multiplicative operation is $x \circ y = xy + \sum_{i<j} a_{i,j}^{2^{m-j}} x^{2^{m+i-j}} y^{2^{m-j}} + \sum_{i<j} a_{i,j}^{2^{m-i}} x^{2^{j-i}} y^{2^{m-i}}$) have been obtained in [4]; see also [660].

Class $\mathcal{PS}$ has been generalized into the generalized Partial Spread class $\mathcal{GPS}$, see Definition 56, page 267.

3. Class $\mathcal{H}$ and Niho functions: We have already seen in Subsection 5.1.2, at pages 191-192, the principle of Niho Boolean functions, among which we shall characterize (in Corollary 14) those which are bent. It is proved in [311, Proposition 5] that all bent functions affine on each coset of $\mathbb{F}_{2^{n/2}}^*$ are EA equivalent to $\mathcal{PS}_{ap}$ or Niho functions possibly added with the indicator of one coset of $\{0\} \cup \mathbb{F}_{2^{n/2}}^*$. As observed in [311], Niho bent functions happen to be the univariate version of bivariate bent Boolean functions that we shall introduce with class $\mathcal{H}$ below in Definition 53, and which are closely related to the functions introduced by Dillon in [441] as the elements of a family that he denoted by H. The functions of this family were defined as $f(x,y) = tr_{n/2}(y + x\,G(yx^{2^{n/2}-2}))$;

$x, y \in \mathbb{F}_{2^{n/2}}$, where G is a permutation[14] of $\mathbb{F}_{2^{n/2}}$ such that, for every $b \in \mathbb{F}^*_{2^{n/2}}$, the function $G(x) + bx$ is 2-to-1 (that is, the pre-image of any element of $\mathbb{F}_{2^{n/2}}$ by this function contains 0 or 2 elements). We shall see below why these conditions characterize bentness. New bent functions were found recently within this framework. The linear term $tr_{n/2}(y)$ being not useful in the function above, we take it off and consider those functions of the form:

$$f(x,y) = \begin{cases} tr_{n/2}\left(x\, G\left(\frac{y}{x}\right)\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0, \end{cases} \tag{6.15}$$

where G is any function from $\mathbb{F}_{2^{n/2}}$ to itself. As seen at page 193, we have:

$$\begin{aligned} W_f(a,b) &= \sum_{x\in\mathbb{F}^*_{2^{n/2}}, y\in\mathbb{F}_{2^{n/2}}} (-1)^{tr_{n/2}\left(x\, G\left(\frac{y}{x}\right)+ax+by\right)} + \sum_{y\in\mathbb{F}_{2^{n/2}}} (-1)^{tr_{n/2}(by)} \\ &= \sum_{x\in\mathbb{F}^*_{2^{n/2}}, z\in\mathbb{F}_{2^{n/2}}} (-1)^{tr_{n/2}(x(G(z)+a+bz))} + 2^{n/2}\delta_0(b) \\ &= 2^{n/2}\left(|\{z\in\mathbb{F}_{2^{n/2}};\, G(z)+a+bz=0\}| + \delta_0(b) - 1\right). \end{aligned} \tag{6.16}$$

Proposition 81 *[441, 311] Any Boolean function of the form (6.15) is bent if and only if G is a permutation of $\mathbb{F}_{2^{n/2}}$ and:*

$$\text{for every } b\in\mathbb{F}^*_{2^{n/2}},\ \text{the function } z\mapsto G(z)+bz \text{ is 2-to-1 on } \mathbb{F}_{2^{n/2}}. \tag{6.17}$$

The dual function of f in (6.15) is:

$$\widetilde{f}(a,b) = \begin{cases} 1 \text{ if the equation } G(z)+bz=a \text{ has no solution in } \mathbb{F}_{2^{n/2}} \\ 0 \text{ otherwise.} \end{cases}$$

Note that an n-variable function (6.15), or a Niho function (see below), is then bent if and only if $\sum_{u\in\mathbb{F}_2^n} W_f^3(u) = 2^{2n}$, that is, $\sum_{x,y\in\mathbb{F}_2^n}(-1)^{f(x)\oplus f(y)\oplus f(x+y)} = 2^n$ (the same characterization is valid for quadratic functions vanishing at 0_n, but for general functions, we have only a necessary condition). Indeed, $|\{z\in\mathbb{F}_{2^{n/2}};\, G(z)+a+bz=0\}|+\delta_0(b)-1 \geq -1$ implies $(|\{z\in\mathbb{F}_{2^{n/2}};\, G(z)+a+bz=0\}|+\delta_0(b)-1)^3 \geq |\{z\in\mathbb{F}_{2^{n/2}};\, G(z)+a+bz=0\}|+\delta_0(b)-1$ and then $\sum_{u\in\mathbb{F}_2^n} W_f^3(u) \geq 2^n\sum_{u\in\mathbb{F}_2^n} W_f(u) = 2^{2n}$, with equality if and only if $W_f(u)\in\{\pm 2^{n/2}, 0\}$ for all u, and therefore $W_f(u)\in\{\pm 2^{n/2}\}$ because of Parseval's relation.

Class $\mathcal{H}$

The restrictions of f to the lines through the origin of the *affine plane* are linear. More generally, any function whose restriction to each subspace in the

[14] Dillon also assumed that $G(x)+x$ does not vanish, but this condition is not necessary for bentness.

Desarguesian spread is linear has the form:

$$g(x,y) = \begin{cases} tr_{n/2}\left(x\psi\left(\frac{y}{x}\right)\right) & \text{if } x \neq 0 \\ tr_{n/2}\left(\mu y\right) & \text{if } x = 0, \end{cases} \tag{6.18}$$

where $\mu \in \mathbb{F}_{2^{n/2}}$ and ψ is a mapping from $\mathbb{F}_{2^{n/2}}$ to itself; this is a particular case of (5.8).

Definition 53 *The set of those bent functions of the form (6.18) (*i.e. *which are linear over each element of the Desarguesian spread) is denoted by* $\mathcal{H}$*.*

All the functions in class $\mathcal{H}$ being clearly *EA equivalent* to functions of the form (6.15), Proposition 81 settles the case of all Niho bent functions ([311] also settled the more general case where the restrictions are affine).
As seen in [311] (see Lemma 7 below for a proof), Condition (6.17) implies the bijectivity of G and is then necessary and sufficient for f to be bent. The set of those functions G which satisfy (6.17) is stable under some transformations, among which $G \mapsto G^{-1}$, and [311] observed that the functions corresponding to G and G^{-1} are in general EA inequivalent. Three other transformations, leading to bent functions which are in general EA inequivalent as well, have been investigated in [154].

$\mathcal{H}$ functions and o-polynomials
A connection between functions in class $\mathcal{H}$ and oval polynomials has been shown in [311]; *oval polynomials* (also called o-polynomials) are a notion in finite geometry related to hyperovals in the *projective plane* $PG(2, 2^{n/2})$. Recall that, for a given power q of 2, $PG(2,q)$ has for points all the 1-dimensional subspaces in $\mathbb{F}_q^3$ and for lines all the 2-dimensional subspaces of $\mathbb{F}_q^3$. In other words, the points of this projective plane are the equivalence classes of $\mathbb{F}_q^3 \setminus \{(0,0,0)\}$ modulo the equivalence relation of proportionality[15]. Then two distinct lines always intersect in one point. More precisely, the projective plane can be obtained from the affine plane by adding points at infinity in the following way: each set of parallel lines in the affine plane defines a point at infinity, and this gives one point at infinity corresponding to the parallel lines $x = a$, and q others corresponding to the parallel lines $y = bx + a$. The lines of the projective plane are the lines of the affine plane completed with their corresponding points at infinity and the line at infinity (made of all points at infinity). A *hyperoval* of the projective plane $PG(2,q)$ is a set of $q+2$ points no three of which are on a same line; any hyperoval is equivalent to a hyperoval containing the following 4 points: $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$; it can then be represented as $\{(1:t:G(t)); t \in \mathbb{F}_q\} \cup \{(0:1:0),(0:0:1)\}$ where $G(0)=0, G(1)=1$ and G is equivalently an o-polynomial on $\mathbb{F}_q$:

[15] The coordinates $(x:y:z)$ of a point in $PG(2,q)$, which are defined up to multiplication by a nonzero element of $\mathbb{F}_q$, are called homogeneous coordinates; we can consider that $PG(2,q)$ contains one special affine plane whose points have the form $(1:x:y)$ while points at infinity are of the form $(0:x:y)$, among which is the so-called nucleus $(0:1:0)$.

Definition 54 *Let m be any positive integer. A permutation polynomial G over $\mathbb{F}_{2^m}$ is called an o-polynomial (an oval polynomial) if, for every $c \in \mathbb{F}_{2^m}$, the function*

$$z \in \mathbb{F}_{2^m} \mapsto \begin{cases} \frac{G(z+c)+G(c)}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

is a permutation of $\mathbb{F}_{2^m}$.

As observed in [311]:

Lemma 7 *Condition (6.17) is equivalent to the fact that G is an o-polynomial on $\mathbb{F}_{2^{n/2}}$.*

Proof. For every $b, c \in \mathbb{F}_{2^m}$, $m = n/2$, the equation $G(z) + bz = G(c) + bc$ is satisfied by c. Thus, if Condition 6.17 is satisfied, then for every $b \in \mathbb{F}_{2^m}^*$ and every $c \in \mathbb{F}_{2^m}$, there exists exactly one $z \in \mathbb{F}_{2^m}^*$ such that $G(z + c) + b(z + c) = G(c) + bc$, that is, $\frac{G(z+c)+G(c)}{z} = b$. Then, for every $c \in \mathbb{F}_{2^m}$, the function $z \in \mathbb{F}_{2^m}^* \mapsto \frac{G(z+c)+G(c)}{z} \in \mathbb{F}_{2^m}^*$ is bijective, that is, G and the function $z \in \mathbb{F}_{2^m} \mapsto \begin{cases} \frac{G(z+c)+G(c)}{z} \text{ if } z \neq 0 \\ 0 \text{ if } z = 0 \end{cases}$ are permutations. Hence, G is an o-polynomial. Conversely, if G is an o-polynomial, then for every $c \in \mathbb{F}_{2^m}$, we have $\frac{G(z+c)+G(c)}{z} \neq 0$ for every $z \neq 0$ and for every $b \neq 0$ there exists exactly one nonzero z such that $G(z + c) + G(c) = bz$. Then for every $u \in \mathbb{F}_{2^m}$, either the equation $G(z) + bz = u$ has no solution, or it has at least a solution u and then exactly one second solution $z+u$ $(z \neq 0)$. This completes the proof. □

Remark. We have already observed with Lemma 5, page 214, that any Boolean function is bent if and only if its Walsh transform takes values congruent with $2^{n/2}$ modulo $2^{n/2+1}$. This property and Relation (6.16) show that a permutation polynomial is an o-polynomial if and only if any equation $G(z) + bz = c$ with $b \neq 0$ has an even number of solutions. □

The known classes of inequivalent o-polynomials are[16] (see [311, 168] and their references):

1. $G(z) = z^{2^i}$ where i is co-prime with m;
2. $G(z) = z^6$ where m is odd;
3. $G(z) = z^{3\cdot 2^k+4}$, where $m = 2k - 1$;
4. $G(z) = z^{2^k+2^{2k}}$, where $m = 4k - 1$;
5. $G(z) = z^{2^{2k+1}+2^{3k+1}}$, where $m = 4k + 1$;
6. $G(z) = z^{2^k} + z^{2^k+2} + z^{3\cdot 2^k+4}$, where $m = 2k - 1$;
7. $G(z) = z^{\frac{1}{6}} + z^{\frac{1}{2}} + z^{\frac{5}{6}}$ where m is odd; note that $G(z) = D_5\left(z^{\frac{1}{6}}\right)$, where D_5 is the Dickson polynomial of index 5 (see the definition of Dickson polynomials at page 422);

[16] Two more, given in [168], are equivalent to z^{2^i}; another in the list of [769] has a typo.

8. $G(z) = \frac{\delta^2(z^4+z)+\delta^2(1+\delta+\delta^2)(z^3+z^2)}{z^4+\delta^2z^2+1} + z^{1/2}$, where $tr_m(1/\delta) = 1$ and, if $m \equiv 2$ [mod 4], then $\delta \notin \mathbb{F}_4$;
9. $G(z) = \frac{1}{tr_m^n(v)}\left[tr_m^n(v^r)(z+1) + tr_m^n\left[(vz+v^{2^m})^r\right]\left(z+tr_m^n(v)z^{1/2}+1\right)^{1-r}\right]+ z^{1/2}$, where m is even, $r = \pm\frac{2^m-1}{3}$, $v \in \mathbb{F}_2^{2m}$, $v^{2^m+1} = 1$ and $v \neq 1$.

The two last classes are related to Subiaco and Adelaide hyperovals, whose description has been simplified in [3] thanks to a new type of homogeneous coordinates. The known o-polynomials provided a number of potentially new bent functions detailed in [311], since each class of o-polynomials gives rise to several EA inequivalent classes of bent functions, see more in [154, 942]. Continuing the work of the author and Mesnager, [2] gives geometrical characterization of Niho bent functions; it shows that they are in one-to-one correspondence with the so-called *line ovals* in the *affine plane* (which are sets of $q+1$ non-parallel lines no three of which are concurrent, where q is the order of the base field) and that their dual functions are the complements of the characteristic functions of these line ovals; it extends this to arbitrary spreads.

Remark. A new notion of equivalence between bent functions in class $\mathcal{H}$ is deduced from Lemma 7. Hyperovals being called equivalent if they are mapped to each other by *collineations* (*i.e.* permutations mapping lines to lines), it provides a notion of equivalence between o-polynomials, and between the related bent functions, called *projective equivalence*. In particular, as recalled in [414], the group $P\Gamma L(2, 2^m)$ of all $\mathbb{F}_2$-linear automorphisms of $\mathbb{F}_{2^m}$ of the form $L(x^{2^j})$ where L is an element of $GL(2, 2^m)$ (associated with a 2×2 matrix over $\mathbb{F}_{2^m}$) acts on $PG(2, 2^m)$, and then acts on o-polynomials, see more in [414]. EA equivalence classes of Niho bent functions are in one-to-one correspondence with projective equivalence classes of ovals in the projective plane $PG(2, q)$ [942, 2]. Notions of duality for bent functions and duality for projective planes are consistent for Niho bent functions (a duality of $PG(2, q)$ is a bijection from the set of points of PG(2,q) to the set of lines which preserves incidence of points and lines). □

Niho bent functions

In *univariate representation*, functions in class $\mathcal{H}$ are those functions whose restrictions to the multiplicative cosets $\mu\,\mathbb{F}_{2^{n/2}}$ of $\mathbb{F}^*_{2^{n/2}}$ are linear, *i.e.* are Niho functions (5.6). Niho bent functions have been investigated in [479] and [749, 752] without that the authors notice their relationship with class H. Relation (5.5), page 192 (in which we can take for U the multiplicative subgroup of $\mathbb{F}^*_{2^n}$ of order 2^m+1 since $n = 2m$) gives for $g = 0$ and $f(0) = 0$:

$$\forall u \in \mathbb{F}_{2^n}, \quad W_f(u) = 2^m\left(|\{\mu \in U; \phi(\mu) + tr_m^n(u\mu) = 0\}| - 1\right).$$

We deduce, denoting by U the multiplicative subgroup of $\mathbb{F}^*_{2^n}$ of order 2^m+1:

Corollary 14 *Let f be any Niho function (5.6) in n variables (n even). Then f is bent if and only if, for every $u \in \mathbb{F}_{2^n}$, we have $|\{\mu \in U; \phi(\mu) + tr_{n/2}^n(u\mu) = 0\}| \in \{0, 2\}$.*

A few examples of infinite classes of Niho bent functions are known up to affine equivalence. The simplest one is quadratic and has been already encountered in Section 5.2: $tr_{n/2}\left(ax^{2^{n/2}+1}\right)$, where $a \in \mathbb{F}^*_{2^{n/2}}$, $x \in \mathbb{F}_{2^n}$. The other examples, from [479], are binomials of the form $f(x) = tr_n(\alpha_1 x^{d_1} + \alpha_2 x^{d_2})$, $x \in \mathbb{F}_{2^n}$, $d_1, d_2 \in \mathbb{Z}/(2^n-1)\mathbb{Z}$, where $2d_1 = 2^{n/2}+1$ and $\alpha_1, \alpha_2 \in \mathbb{F}^*_{2^n}$ are such that $(\alpha_1 + \alpha_1^{2^{n/2}})^2 = \alpha_2^{2^{n/2}+1}$. Equivalently, denoting $a = (\alpha_1 + \alpha_1^{2^{n/2}})^2$ and $b = \alpha_2$, we have $a = b^{2^{n/2}+1} \in \mathbb{F}^*_{2^{n/2}}$ and $f(x) = tr_{n/2}(ax^{2^{n/2}+1}) + tr_n(bx^{d_2})$ (note that if $b = 0$ and $a \neq 0$ then f is also bent but it belongs then to the class of quadratic Niho bent functions seen above). The values of d_2 are (see [479] for the proofs):

1. $d_2 = (2^{n/2}-1)\,3 + 1$ (originally in [479] was included the condition that, if $n \equiv 4$ [mod 8], then $b = \alpha_2$ is the fifth power of an element in $\mathbb{F}_{2^n}$, but as observed in [596], the value of b can be taken arbitrary under the condition that $a = b^{2^{n/2}+1}$).
2. $4d_2 = (2^{n/2}-1) + 4$ (with the condition that $n/2$ is odd), This example has been extended by Leander and Kholosha [749, 752] into the functions: $tr_n\big(\alpha x^{2^{n/2}+1} + \sum_{i=1}^{2^{r-1}-1} x^{s_i}\big)$, $r > 1$ such that $gcd(r, n/2) = 1$, $\alpha \in \mathbb{F}_{2^n}$ such that $\alpha + \alpha^{2^{n/2}} = 1$, $s_i = (2^{n/2}-1)\frac{i}{2^r} + 1 \pmod{2^{n/2}+1}$, $i \in \{1, \ldots, 2^{r-1}-1\}$. It is shown in [763] that the functions $\sum_{i=1}^{2^r-1} tr_n(\alpha x^{(i2^{n/2-r}+1)(2^{n/2}-1)+1})$; $\alpha \in \mathbb{F}_{2^n}$, $\alpha + \alpha^{2^{n/2}} \neq 0$, enter in this class up to EA equivalence while they cover it for $\alpha + \alpha^{2^{n/2}} = 1$, with a nice original proof of their bentness.
3. $6d_2 = (2^{n/2}-1) + 6$ (with the condition that $n/2$ is even).

As observed in [479] and in [155], these functions have respectively *algebraic degree* $n/2$, 3 and $n/2$. In [475], the value distribution of the Walsh spectrum of the monomial function corresponding to the first exponent d_2 above was determined for $\frac{n}{2}$ odd, in terms of Kloosterman sums.

After [311], several works investigated the properties of the known Niho bent functions and their relation with o-polynomials (when transformed from univariate form to bivariate form); we follow here the survey [313] on bent functions:
- the dual function of the second example above (with $4d_2 = (2^{n/2}-1)+4$) has been calculated (in [311]) as well as that of the Niho bent function consisting of 2^r exponents (see [296, 155]); it has been shown in [311, 155] that the dual bent functions are not of the Niho type; this replied negatively to an open question stated in [479];
- the quadratic monomial and (as shown in [311]) the second example above belong to the completed $\mathcal{M}$ class, but (as proved in [155]), when $m = n/2 > 2$, the two others and the generalization of the second example do not; this gives a positive answer to an open question (since 1974) whether completed class H differs from completed class $\mathcal{M}$;
- it is shown in [296] that the o-polynomials associated with the Leander-Kholosha bent functions are equivalent to *Frobenius automorphisms*; the relation between the binomial Niho bent functions above with $d_2 = (2^m-1)\,3+1$ and $6d_2 = (2^m-1)+6$ and the Subiaco and Adelaide classes of hyperovals (related to

the two last o-polynomials above) was found in [596]; this allowed when $m \equiv 2 \pmod 4$ to expand the class of bent functions corresponding to Subiaco hyperovals. Later, in [168], the o-polynomials associated to all known Niho bent functions have been identified and the class of Niho bent functions consisting of 2^r terms has been extended by inserting coefficients of the power terms in the original function; it can then give any Niho bent function. Several classes of explicit Niho bent functions have been deduced (as also detailed in [769, Section 3]).

Remark We have seen in Proposition 66, page 217, a characterization of bent functions by power moments of even exponents of the Walsh transform. In the case of Niho functions, we have a characterization with odd exponents as well:

Proposition 82 *Let $n = 2m$ be any even positive integer, w any odd integer such that $w \geq 3$ and f any Niho n-variable Boolean function. Then we have:*

$$\sum_{u\in\mathbb{F}_{2^n}} W_f^w(u) \geq 2^{(w+1)m}, \text{ i.e. } \sum_{x_1,\ldots,x_{w-1}\in\mathbb{F}_{2^n}} (-1)^{\bigoplus_{i=1}^{w-1} f(x_i)\oplus f(\sum_{i=1}^{w-1} x_i)} \geq 2^{(w-1)m}$$

with equality if and only if f is bent.

Proof. We still denote by U the multiplicative subgroup of $\mathbb{F}_{2^n}^*$ of order 2^m+1, where $n = 2m$. Let $f(\mu x) = tr_m(x\,\phi(\mu))$, $\mu \in U$, $x \in \mathbb{F}_{2^m}$, where ϕ is some function from U to $\mathbb{F}_{2^m}$. We have $W_f(0) = \mathcal{F}(0) = \sum_{x\in\mathbb{F}_{2^m},\mu\in U}(-1)^{tr_m(x\phi(\mu))} - 2^m = 2^m(|\phi^{-1}(0)| - 1)$.
For every $u \in \mathbb{F}_{2^n}$, the function $f(z) + tr_n(uz)$ is Niho too since its value at $z = \mu x$ equals $tr_m(x\,\phi_u(\mu))$, where $\phi_u(\mu) = \phi(\mu) + tr_m^n(u\mu)$. We have then $W_f(u) = 2^m(|\phi_u^{-1}(0)| - 1)$ and $\sum_{u\in\mathbb{F}_{2^n}} W_f^w(u) = 2^{wm}\sum_{u\in\mathbb{F}_{2^n}}(|\{\mu \in U; \phi(\mu) = tr_m^n(u\mu)\}| - 1)^w$.
For all $u \in \mathbb{F}_{2^n}$, we have $|\{\mu \in U; \phi(\mu) = tr_m^n(u\mu)\}| - 1 \geq -1$ and therefore

$$(|\{\mu \in U; \phi(\mu) = tr_m^n(u\mu)\}| - 1)^w \geq |\{\mu \in U; \phi(\mu) = tr_m^n(u\mu)\}| - 1. \quad (6.19)$$

We deduce that $\sum_{u\in\mathbb{F}_{2^n}}(|\{\mu \in U; \phi(\mu) = tr_m^n(u\mu)\}| - 1)^w \geq \sum_{u\in\mathbb{F}_{2^n}}(|\{\mu \in U; \phi(\mu) = tr_m^n(u\mu)\}| - 1) = \sum_{\mu\in U}|\{u \in \mathbb{F}_{2^n}; \phi(\mu) = tr_m^n(u\mu)\}| - 2^n$. For each μ, since $u\mu$ ranges over $\mathbb{F}_{2^n}$ when u ranges over $\mathbb{F}_{2^n}$, and since $tr_m^n(z)$ ranges uniformly over $\mathbb{F}_{2^m}$ when z ranges over $\mathbb{F}_{2^n}$, we have $|\{u \in \mathbb{F}_{2^n}; \phi(\mu) = tr_m^n(u\mu)\}| = 2^m$. Hence $\sum_{u\in\mathbb{F}_{2^n}} W_f^w(u) \geq 2^{wm}((2^m+1)2^m - 2^n) = 2^{(w+1)m}$ with equality if and only if, for every $u \in \mathbb{F}_{2^n}$, we have equality in (6.19), that is, $|\{\mu \in U; \phi(\mu) = tr_m^n(u\mu)\}| \in \{0, 1, 2\}$, that is[17], $W_f(u) \in \{-2^m, 0, 2^m\}$. Moreover, this last condition is equivalent to $W_f(u) \in \{-2^m, 2^m\}$ for every u, that is, f is bent, because of the Parseval identity $\sum_{u\in\mathbb{F}_{2^n}} W_f^2(u) = 2^{2n}$. And we have $\sum_{u\in\mathbb{F}_2^n} W_f^w(u) = 2^n\sum_{x_1,\ldots,x_{w-1}\in\mathbb{F}_{2^n}}(-1)^{\bigoplus_{i=1}^{w-1} f(x_i)\oplus f(\sum_{i=1}^{w-1} x_i)}$. □

Proposition 82 allows proving the bentness of classes of Niho functions: a set E of Niho functions is made of bent functions if and only if $\sum_{f\in E}\sum_{a\in\mathbb{F}_{2^n}} W_f^w(a) =$

[17] Recall that w is odd; for $w \geq 4$ even there can not be equality, and we know it already from Proposition 66.

$2^{(w+1)m}|E|$. And handling $w=3$ is easier than $w=4$. Corollary 13, page 218, and the remark which follows it generalize to odd exponents.

Note that this characterization is not valid for all Boolean functions, even if their algebraic degree is bounded above by $\frac{n}{2}$ (like bent functions). For instance, it is easily seen that the function which is null when $x_1 = x_2 = \cdots = x_{\frac{n}{2}} = 0$ and has value 1 everywhere else has a value of $\sum_{a\in\mathbb{F}_{2^n}} W_f^3(a)$ negative and has algebraic degree bounded above by $\frac{n}{2}$ (since the value of the function depends in half its variables only).
However, it is interesting to see that this characterization is also valid for those *quadratic functions* which are null at 0, since for any quadratic function f, we have $\sum_{a\in\mathbb{F}_{2^n}} W_f^3(a) = (-1)^{f(0)}2^n \sum_{x,y\in\mathbb{F}_{2^n}}(-1)^{\beta_f(x,y)} = (-1)^{f(0)}2^{2n}|\mathcal{E}_f|$ where β_f is the symplectic form $\beta_f(x,y) = f(x+y)\oplus f(x)\oplus f(y)\oplus f(0)$ and $\mathcal{E}_f$ is its kernel, and we know that f is bent if and only if $\mathcal{E}_f = \{0\}$.
It has been shown in [861] that the only bent functions of the form (5.4) page 191, equivalently (5.8) are, up to translation, those corresponding to Niho-bent and $\mathcal{PS}_{ap}^{\#}$ classes.

Niho-like and $\mathcal{H}$-like bent functions
It is possible to extend to other spreads (than the Desarguesian spread) the principle of $\mathcal{H}$ and Niho functions (*i.e.* considering Boolean functions whose restrictions to the elements of a spread are linear). This has been done with André's spreads in [246] and with three spreads from prequasifields and presemifields in [335] and in [246], independently[18]. Probably many other spreads could be investigated, since many more exist, see [661, 425, 649]. But we wish to find explicit examples of such bent functions (and the associated o-like-polynomials).
Let us first study the general framework into which these four examples of spreads will fit. Consider a spread whose elements are the subspace $\{(0,y),\, y\in\mathbb{F}_{2^{n/2}}\}$ and the $2^{n/2}$ subspaces of the form $\{(x,L_z(x)), x\in\mathbb{F}_{2^{n/2}}\}$, where, for every $z\in\mathbb{F}_{2^{n/2}}$, function L_z is linear. The property of being a spread corresponds to the fact that, for every nonzero $x\in\mathbb{F}_{2^{n/2}}$, the mapping $z\mapsto L_z(x)$ is a permutation of $\mathbb{F}_{2^{n/2}}$. Let us denote by Γ_x the compositional inverse of this bijection. A Boolean function over $\mathbb{F}_{2^{n/2}}^2$ is linear over each element of the spread if and only if there exists a mapping $G:\mathbb{F}_{2^{n/2}}\mapsto\mathbb{F}_{2^{n/2}}$ and an element ν of $\mathbb{F}_{2^{n/2}}$ such that, for every $y\in\mathbb{F}_{2^{n/2}}$, $f(0,y) = tr_{n/2}(\nu y)$ and, for every $x,z\in\mathbb{F}_{2^{n/2}}$, $x\neq 0$:

$$f(x,L_z(x)) = tr_{n/2}(G(z)x). \qquad (6.20)$$

Note that, up to EA equivalence, we can assume that $\nu = 0$. Indeed, we can add the linear n-variable function $(x,y)\mapsto tr_{n/2}(\nu y)$ to f; this changes ν into

[18] Ref. [335] is a little more general: it deals with functions affine on each spread and also addresses odd characteristic. It shows that bent functions from $\mathbb{F}_p^m\times\mathbb{F}_p^m$ to $\mathbb{F}_p$ which are affine on the elements of a given spread of $\mathbb{F}_p^m\times\mathbb{F}_p^m$ either arise from partial spread bent functions, or are a generalization in characteristic 2 of class $\mathcal{H}$. Ref. [246] is slightly more general as well since it also addresses spreads not related to prequasifields.

0 and $G(z)$ into $G(z) + L_z^*(\nu)$, where L_z^* is the adjoint operator of L_z, since for $y = L_z(x)$, we have $tr_{n/2}(\nu y) = tr_{n/2}(xL_z^*(\nu))$. We take $\nu = 0$ and define $\Gamma_0(y) = 0$. By definition of Γ_x, Relation (6.20) is equivalent to:

$$\forall x, y \in \mathbb{F}_{2^{n/2}},\ f(x, y) = tr_{n/2}\left(G\left(\Gamma_x(y)\right) x\right). \tag{6.21}$$

The value of the Walsh transform $W_f(a, b) = \sum_{x,y\in\mathbb{F}_{2^{n/2}}}(-1)^{f(x,y)+tr_{n/2}(ax+by)}$ equals then, for every $(a, b) \in \mathbb{F}_{2^{n/2}}^2$:

$$\begin{aligned}
&\sum_{(x,y)\in\mathbb{F}_{2^{n/2}}^2}(-1)^{tr_{n/2}(G(\Gamma_x(y))x+ax+by)} = \\
&2^{n/2}\delta_0(b) + \sum_{x\in\mathbb{F}_{2^{n/2}}^*, z\in\mathbb{F}_{2^{n/2}}}(-1)^{tr_{n/2}(G(z)x+ax+bL_z(x))} = \\
&2^{n/2}(\delta_0(b) - 1) + \sum_{z\in\mathbb{F}_{2^{n/2}}}\sum_{x\in\mathbb{F}_{2^{n/2}}}(-1)^{tr_{n/2}((G(z)+a+L_z^*(b))x)} = \\
&2^{n/2}\left(\delta_0(b) - 1 + |\{z \in \mathbb{F}_{2^{n/2}};\ G(z) + a + L_z^*(b) = 0\}|\right).
\end{aligned}$$

Hence f is bent if and only if G is a permutation and:

$$|\{z \in \mathbb{F}_{2^{n/2}};\ G(z) + a + L_z^*(b) = 0\}| \in \{0, 2\}, \forall a, b \in \mathbb{F}_{2^{n/2}}, b \neq 0. \tag{6.22}$$

This condition on $G(z)$ is similar to the definition of o-polynomials. In the case of André's spreads, it is a generalization of the notion of o-polynomial.

Remark As explained for instance in the nice survey by W. Kantor [661], every spread has a dual in the space of linear forms. Viewing this in $\mathbb{F}_{2^{n/2}}^2$ the subspaces belonging to this spread are the orthogonals of those corresponding to the original spread. In other words, the fact that for every $x \neq 0$ and every $b \neq 0$, the function $z \mapsto tr_{n/2}(bL_z(x)) = tr_{n/2}(xL_z^*(b))$ is balanced implies that function $z \mapsto L_z^*(b)$ is also a permutation and the elements of the dual spread are the subspace $\{(x, 0),\ x \in \mathbb{F}_{2^{n/2}}\}$ and the $2^{n/2}$ subspaces $\{(L_z^*(y), y), y \in \mathbb{F}_{2^{n/2}}\}$. □

It is shown in [246] that, as in the case of o-polynomials, the condition that G is a permutation is implied by Relation (6.22).
The question which can lead to new bent functions when addressed positively is: can we build efficiently permutations G of $\mathbb{F}_{2^{n/2}}$ and linear mappings L_z : $\mathbb{F}_{2^{n/2}} \mapsto \mathbb{F}_{2^{n/2}}$, with $z \in \mathbb{F}_{2^{n/2}}$, such that function $z \mapsto L_z(x)$ is bijective for every $x \neq 0$ and that the equation $G(z) + L_z(b) = a$ has 0 or 2 solutions for every $a \in \mathbb{F}_{2^{n/2}}$ and every $b \in \mathbb{F}_{2^{n/2}}^*$? Equivalently, by denoting by H_x the permutation $z \mapsto L_z(x)$, can we find a permutation G and a set of permutations H_x, $x \in \mathbb{F}_{2^{n/2}}^*$, such that, denoting $H_0 = 0$, the set $\{H_x,\ x \in \mathbb{F}_{2^{n/2}}\}$ is a vector space and every function $G + H_x$, $x \in \mathbb{F}_{2^{n/2}}^*$, is 2-to-1? Note that finding nine classes of o-polynomials has been a hard 40 year long mathematical work and we can expect that finding such o-like-polynomials will be also difficult, except

maybe for a few simple cases like with o-polynomials.

In the case of André's spreads, we have $L_z(x) = x^{2^{k\varphi(z)}}z$. According to (6.12), we have then $\Gamma_x(y) = \frac{y}{x^{2^{k\varphi(y/x)}}}$ and $L_z^*(b) = (bz)^{2^{m-k\varphi(z)}}$, $m = n/2$. Relation (6.21) becomes:

$$\forall x, y \in \mathbb{F}_{2^{n/2}},\ f(x,y) = tr_{n/2}\left(G\left(\frac{y}{x^{2^{k\varphi(y/x)}}}\right)x\right). \tag{6.23}$$

The condition for such f to be bent is that, for every $b \in \mathbb{F}_{2^{n/2}}^*$ and every $a \in \mathbb{F}_{2^{n/2}}$, there exist two values of z or none such that $G(z) + (bz)^{2^{m-k\varphi(z)}} = a$.

As shown for instance in [425, 649] (and recalled by Kantor in [662]) a spread can be derived from any prequasifield, that is, any Abelian finite group having a second law $*$ which is left-distributive with respect to the first law and is such that the right and left multiplications by a nonzero element are bijective and that the multiplications by 0 are absorbent. The elements of this spread are the $\mathbb{F}_2$-vector subspaces $\{(0,y),\ y \in \mathbb{F}_{2^{n/2}}\}$ and $\{(x, z*x), x \in \mathbb{F}_{2^{n/2}}\}$, $z \in \mathbb{F}_{2^{n/2}}$. Wu [1123] has studied three particular examples for designing $\mathcal{PS}$ functions (many others could have been studied) and he determined explicitly the related functions Γ_x. Let us see what we obtain with them in the framework of Niho-like functions.
The Dempwolff-Müller prequasifield is defined as follows. Let k and m be co-prime odd integers. Let $e = 2^{m-1} - 2^{k-1} - 1$, $L(x) = \sum_{i=0}^{k-1} x^{2^i}$, and define $x * y = x^e L(xy)$. Then $(\mathbb{F}_{2^m}, +, *)$ is a prequasifield [431], leading to the spread of the $\mathbb{F}_2$-vector subspaces $\{(0,y),\ y \in \mathbb{F}_{2^m}\}$ and $\{(x, z*x), x \in \mathbb{F}_{2^m}\} = \{(x, z^e L(xz)), x \in \mathbb{F}_{2^m}\}$, $z \in \mathbb{F}_{2^m}$.
Then $\Gamma_x(y) = \frac{1}{xD_d\left(\frac{y^2}{x^{2^k+1}}\right)}$, where D_d is the Dickson polynomial (see definition at page 422) of index the inverse d of $2^k - 1$ modulo $2^m - 1$, and $L_z^*(b) = \sum_{i=0}^{k-1}(bz^e)^{2^{-i}}z$. Relation (6.21) becomes:

$$\forall x, y \in \mathbb{F}_{2^m},\ f(x,y) = tr_m\left(G\left(\frac{1}{xD_d\left(\frac{y^2}{x^{2^k+1}}\right)}\right)x\right), \tag{6.24}$$

and such f is bent if and only if the equation $G(z) + \sum_{i=0}^{k-1}(bz^e)^{2^{-i}}z = a$ has 0 or 2 solutions for every $b \neq 0$ and every a.
The Knuth commutative presemifield is defined as follows. Let m be an odd integer and $b \in \mathbb{F}_{2^m}^*$. Then $x * y = xy + x^2 tr_m(by) + y^2 tr_m(bx)$ defines a presemifield (a prequasifield which remains one when $a*b$ is replaced by $b*a$), leading to the spread of the $\mathbb{F}_2$-vector subspaces $\{(0,y),\ y \in \mathbb{F}_{2^m}\}$ and $\{(x, z*x), x \in \mathbb{F}_{2^m}\} = \{(x, zx + x^2 tr_m(bz) + z^2 tr_m(bx)), x \in \mathbb{F}_{2^m}\}$, $z \in \mathbb{F}_{2^m}$.

Then $\Gamma_x(y) = (1 + tr_m(bx))\frac{y}{x} + x tr_m\left(b\frac{y}{x}\right) + x tr_m(bx) C_{\frac{1}{bx}}\left(\frac{y}{x^2}\right)$, where $C_a(x) = \sum_{i=0}^{m-1} c_i x^{2^i}$, where $c_0 = \frac{1}{a^{2^i}} + \frac{1}{a^{3\cdot 2^i}} + \cdots + \frac{1}{a^{(m-3)\cdot 2^i}}$, $c_i = 1 + \frac{1}{a^{2^i}} + \frac{1}{a^{3\cdot 2^i}} + \cdots + \frac{1}{a^{(i-2)\cdot 2^i}} + \frac{1}{a^{(i+1)\cdot 2^i}} + \cdots + \frac{1}{a^{(m-1)\cdot 2^i}}$ if i is odd and $c_i = 1 + \frac{1}{a^{2\cdot 2^i}} + \frac{1}{a^{4\cdot 2^i}} + \cdots + \frac{1}{a^{(i-2)\cdot 2^i}} + \frac{1}{a^{(i+1)\cdot 2^i}} + \cdots + \frac{1}{a^{(m-2)\cdot 2^i}}$ if i is even. We have $L_z^*(b) = bz + b^{2^{m-1}} tr_m(bz) +$

$btr_m(b^{2^{m-1}}z)$. Relation (6.21) becomes:

$$tr_m\left(G\left((1+tr_m(bx))\frac{y}{x}+xtr_m\left(b\frac{y}{x}\right)+xtr_m(bx)C_{\frac{1}{bx}}\left(\frac{y}{x^2}\right)\right)x\right), \tag{6.25}$$

and such f is bent if and only if the equation $G(z)+bz+b^{2^{m-1}}tr_m(bz)+btr_m(b^{2^{m-1}}z)=a$ has 0 or 2 solutions for every $b\neq 0$ and every a.
There are more examples of semifields due to Knuth [710], which could be studied.
A third example is the dual of the symplectic version of the Knuth commutative presemifield. Assume m is an odd integer. Then $x*y=x^2y+tr_m(xy)+xtr_m(y)$ defines a presemifield [659], leading to two spreads:
- the spread of the $\mathbb{F}_2$-vector subspaces $\{(0,y),\, y\in\mathbb{F}_{2^m}\}$ and $\{(x,z*x), x\in\mathbb{F}_{2^m}\}=\{(x,z^2x+tr_m(zx)+ztr_m(x)), x\in\mathbb{F}_{2^m}\}$;
- the spread of the $\mathbb{F}_2$-vector subspaces $\{(0,y),\, y\in\mathbb{F}_{2^m}\}$ and $\{(x,x*z), x\in\mathbb{F}_{2^m}\}=\{(x,x^2z+tr_m(xz)+xtr_m(z)), x\in\mathbb{F}_{2^m}\})$, where $z\in\mathbb{F}_{2^m}$ (two such spreads are sometimes called *opposite* of each other).

In the first case, the corresponding function Γ_x has been determined in [1123] and $L_z^*(b)=bz^2+ztr_m(b)+tr_m(bz)$. Then $f(x,y)$ equals:

$$tr_m\left(G\left(\left[(xy)^{2^{m-1}}+\sum_{i=0}^{\frac{m-1}{2}}(xy)^{2^{2i}-1}+\sum_{i=0}^{\frac{m-3}{2}}x^{2^{2i}}tr_m(xy)\right]\frac{tr_m(x)}{x}\right.\right.$$
$$\left.\left.+x^{2^{m-1}-1}y^{2^{m-1}}+x^{2^{m-1}-1}tr_m(xy)\right)x\right), \tag{6.26}$$

and such f is bent if and only if the equation $G(z)+bz^2+ztr_m(b)+tr_m(bz)=a$ has 0 or 2 solutions for every $b\neq 0$ and every a.
In the second case, the relation $y=x^2z+tr_m(xz)+xtr_m(z)$ implies for $x\neq 0$ that $\begin{cases} z=\frac{y}{x^2}+\frac{tr_m(xz)}{x^2}+\frac{tr_m(z)}{x}\\ tr_m(xz)=tr_m\left(\frac{y}{x}\right)+tr_m(xz)tr_m\left(\frac{1}{x}\right)+tr_m(z)\\ tr_m(z)=tr_m\left(\frac{y}{x^2}\right)+\left(tr_m(xz)+tr_m(z)\right)tr_m\left(\frac{1}{x}\right)\end{cases}$ and is equivalent to

$$z=\frac{y}{x^2}+tr_m\left(\frac{1}{x}\right)\left(\frac{tr_m\left(\frac{y}{x^2}\right)}{x^2}+\frac{tr_m\left(\frac{y}{x}\right)}{x}\right)$$
$$+\left(tr_m\left(\frac{1}{x}\right)+1\right)\left(\frac{tr_m\left(\frac{y}{x^2}\right)+tr_m\left(\frac{y}{x}\right)}{x^2}+\frac{tr_m\left(\frac{y}{x^2}\right)}{x}\right),$$

which gives $\Gamma_x(y)$. We have $L_z^*(b)=(bz)^{2^{m-1}}+ztr_m(b)+btr_m(z)$. Then $f(x,y)$ equals:

$$tr_m\left(G\left(\frac{y}{x^2}+tr_m\left(\frac{1}{x}\right)\left(\frac{tr_m\left(\frac{y}{x^2}\right)}{x^2}+\frac{tr_m\left(\frac{y}{x}\right)}{x}\right)+\left(tr_m\left(\frac{1}{x}\right)+1\right)\right.\right.$$
$$\left.\left.\left(\frac{tr_m\left(\frac{y}{x^2}\right)+tr_m\left(\frac{y}{x}\right)}{x^2}+\frac{tr_m\left(\frac{y}{x^2}\right)}{x}\right)\right)x\right), \tag{6.27}$$

and such f is bent if and only if the equation $G(z)+(bz)^{2^{m-1}}+ztr_m(b)+btr_m(z)=a$ has 0 or 2 solutions for every $b\neq 0$ and every a.
See in [5] more constructions of bent functions linear on elements of presemifield spreads and a survey on this topic, with explicit descriptions of such functions for known commutative presemifields and related (new types of) oval polynomials.
4. Class C^+ has been introduced by Dillon [441]. Since $\mathcal{PS}_{ap}$ functions have for supports the unions of multiplicative cosets of the subgroup $\mathbb{F}^*_{2^{n/2}}$ of $\mathbb{F}^*_{2^n}$ (*i.e.* the subgroup of all $(2^{n/2}+1)$-th powers), plus possibly the 0 element, he addressed the other possible subgroup U of $(2^{n/2}-1)$-th powers in $\mathbb{F}^*_{2^n}$, and studied then the functions of the form $f(z)=g(z^{2^{n/2}+1})$, where $z\in\mathbb{F}_{2^n}$ and g is balanced over $\mathbb{F}_{2^{n/2}}$ (note that $z^{2^{n/2}+1}\in\mathbb{F}_{2^{n/2}}$) and vanishes at 0 (if not, we can apply the result to $g\oplus 1$). He showed that such function is bent if and only if the mapping $a\in\mathbb{F}_{2^{n/2}}\mapsto W_g(a^{-1})=\sum_{x\in\mathbb{F}_{2^{n/2}}}(-1)^{g(x)+tr_{n/2}(a^{-1}x)}$, with the convention $0^{-1}=0$, equals the Walsh transform of some Boolean function over $\mathbb{F}_{2^{n/2}}$.
Dillon refers to the Singer difference set in his proof. An elementary proof is as follows: using polar representation ux (with $x\in\mathbb{F}^*_{2^{n/2}}$, $u\in U$) in $\mathbb{F}^*_{2^n}$, we have for every $\lambda\in\mathbb{F}_{2^n}$ that $W_f(\lambda)=\sum_{z\in\mathbb{F}_{2^n}}(-1)^{g(z^{2^{n/2}+1})+tr_n(\lambda z)}=1+$
$$\sum_{u\in U}\sum_{x\in\mathbb{F}^*_{2^{n/2}}}(-1)^{g(x^2)+tr_{n/2}((\lambda u+\lambda^{2^{n/2}}u^{2^{n/2}})x)}=-2^{n/2}+\sum_{u\in U}W_g((\lambda u+\lambda^{2^{n/2}}u^{2^{n/2}})^2).$$
If $\lambda=0$ then $W_f(\lambda)=-2^{n/2}+(2^{n/2}+1)W_g(0)=-2^{n/2}$. Otherwise, we can assume without loss of generality that λ belongs to $\mathbb{F}^*_{2^{n/2}}$ (since $W_f(\lambda)$ is clearly invariant when multiplying λ by an element of U) and we have then $W_f(\lambda)=-2^{n/2}+\sum_{u\in U}W_g\left(\lambda^2\left(u+u^{2^{n/2}}\right)^2\right)$. It is well-known that, when u ranges over $U\backslash\{1\}$, $u+u^{2^{n/2}}$ ranges twice over the set $\{z\in\mathbb{F}^*_{2^{n/2}}, tr_{n/2}\left(z^{-1}\right)=1\}$ (indeed, we have $u^{2^{n/2}}=u^{-1}$ and the equation $u+u^{-1}=z$, *i.e.* $\left(\frac{u}{z}\right)^2+\frac{u}{z}=z^{-2}$, has solutions in $U\setminus\{1\}$ if and only if $tr_{n/2}\left(z^{-1}\right)=1$). Then, since g is balanced, $W_f(\lambda)$ equals

$$-2^{n/2}+2\sum_{\substack{z\in\mathbb{F}^*_{2^{n/2}}\\ tr_{n/2}(z^{-1})=1}}W_g\left((\lambda z)^2\right)=-2^{n/2}+\sum_{z\in\mathbb{F}_{2^{n/2}}}W_g\left((\lambda z)^2\right)\left(1-(-1)^{tr_{n/2}\left(z^{-2}\right)}\right)$$

and $W_f(\lambda)$ is equal to $-2^{n/2}+2^{n/2}(-1)^{g(0)}-\sum_{z\in\mathbb{F}_{2^{n/2}}}W_g\left(\lambda^2 z\right)(-1)^{tr_{n/2}\left(z^{-1}\right)}=-\sum_{z\in\mathbb{F}_{2^{n/2}}}W_g\left(z^{-1}\right)(-1)^{tr_{n/2}(\mu z)}$, where $\mu=\lambda^2\neq 0$.
Hence, f is bent if and only if $\sum_{z\in\mathbb{F}_{2^{n/2}}}W_g\left(z^{-1}\right)(-1)^{tr_{n/2}(\mu z)}\in\{2^{n/2},-2^{n/2}\}$ for every $\mu\neq 0$, which is then equivalent to the condition stated by Dillon, according to the inverse Fourier transform formula, since it is always verified for $\mu=0$.
Dillon mentions the example where g is the absolute trace function $tr_{n/2}(x)$ over $\mathbb{F}_{2^{n/2}}$; the resulting function is quadratic and so belongs to class $\mathcal{M}$ completed, and it also belongs to class $\mathcal{H}$, up to EA equivalence. No example is known yet

lying outside known completed classes.

5. Dobbertin's Class introduced in [466] is a class of bent functions which contains both $\mathcal{PS}_{ap}$ and $\mathcal{M}$ and is based on the so-called *triple construction*. The elements of this class are the functions f defined by $f(x,\phi(y)) = g\left(\frac{x+\psi(y)}{y}\right)$, where g is a balanced Boolean function on $\mathbb{F}_{2^{\frac{n}{2}}}$ and ϕ, ψ are two mappings from $\mathbb{F}_{2^{\frac{n}{2}}}$ to itself such that, if T denotes the affine subspace of $\mathbb{F}_{2^{\frac{n}{2}}}$ spanned by the support of function W_g, then, for any a in $\mathbb{F}_{2^{\frac{n}{2}}}$, the functions ϕ and ψ are affine on $aT = \{ax, x \in T\}$. The mapping ϕ must additionally be one to one. Dobbertin gives two explicit examples of bent functions constructed this way. In both, ϕ is a power function.

6. The class of functions γ related to almost bent functions exists when $n \equiv 2$ [mod 4]. Recall that a vectorial Boolean function $F : \mathbb{F}_{2^{n/2}} \to \mathbb{F}_{2^{n/2}}$ is called almost bent (see Definition 31, page 141) if the Walsh transforms of all component functions $v \cdot F$, $v \neq 0$ in $\mathbb{F}_{2^{n/2}}$ take values in $\{-2^{\frac{n/2+1}{2}}, 0, 2^{\frac{n/2+1}{2}}\}$. The function $\gamma_F(a,b)$, $a,b \in \mathbb{F}_{2^{n/2}}$, equal to 1 if the equation $F(x)+F(x+a) = b$ admits solutions, with $a \neq 0$ in $\mathbb{F}_{2^{n/2}}$, and equal to 0 otherwise is then bent (see Proposition 158, page 407) and the dual of γ_F is the indicator of the Walsh support of F, deprived of $(0,0)$. Several classes of AB functions are known (see Section 11.4, page 427). The bent functions γ_F associated to known AB functions have been investigated in [152]. We give them below:
- *Gold:* $F(x) = x^{2^i+1}$, $\gcd(i, n/2) = 1$, $\gamma_F(a,b) = tr_{n/2}(\frac{b}{a^{2^i+1}})$ with $\frac{1}{0} = 0$;
- *Inverse:* $F(x) = x^{2^n-2}$, $\gamma_F(a,b) = tr_n\left(\frac{1}{ab}\right) + 1 + \delta_0(a) + \delta_0(b) + \delta_0(a)\delta_0(b) + \delta_0(ab+1)$ where $\delta_0(x)$ is the Dirac (or Kronecker) function.
- *Kasami-Welch* $F(x) = x^{2^{2i}-2^i+1}$, $\gcd(i, n/2) = 1$, *Welch* $F(x) = x^{2^{\frac{n/2-1}{2}}+3}$, *Niho* $F(x) = x^{2^{\frac{n/2-1}{2}}+2^{\frac{n/2-1}{4}}-1}$ *if* $n \equiv 1$ *[mod 4]*, $F(x) = x^{2^{\frac{n/2-1}{2}}+2^{\frac{3\frac{n/2-1}{2}+1}{2}}-1}$ *if* $n \equiv 3$ *[mod 4]*:
we have $F(x+1) + F(x) = q(x^{2^s} + x)$, where $\gcd(s, n/2) = 1$ and q is in each case a permutation determined by Dobbertin (see [470]):

1. Kasami-Welch: $s = i$, $q(x) = \frac{x^{2^i+1}}{\sum_{j=1}^{i'} x^{2^{ji}} + \alpha tr_{n/2}(x)} + 1$, where $i' \equiv 1/i$ [mod $n/2$], $\alpha = \begin{cases} 0 \text{ if } i' \text{ is odd} \\ 1 \text{ otherwise;} \end{cases}$
2. Welch: $s = \frac{n/2-1}{2}$, $q(x) = x^{2^{\frac{n/2-1}{2}+1}+1} + x^3 + x + 1$;
3. Niho: $s = \frac{n/2-1}{4}$ if $n \equiv 1$ [mod 4] and $s = \frac{3\frac{n/2-1}{2}+1}{2}$ if $n \equiv 3$ [mod 4], $q(x) = \begin{cases} \frac{1}{g(x^{2^s-1})+1} + 1 & \text{if } x \notin \mathbb{F}_2 \\ 1 & \text{otherwise} \end{cases}$ where

$$g(x) = x^{2^{2s+1}+2^{s+1}+1} + x^{2^{2s+1}+2^{s+1}-1} + x^{2^{2s+1}+1} + x^{2^{2s+1}-1} + x.$$

and $F(x+1)+F(x)=b$ has solutions if and only if $tr_{n/2}(q^{-1}(b))=0$. Then:

$$\gamma_F(a,b)=\begin{cases} tr_{n/2}(q^{-1}(b/a^d))+1 & \text{if } a\neq 0,\\ 0 & \text{otherwise;}\end{cases}$$

The functions γ_F associated to Kasami-Welch, Welch and Niho functions with $n/2=7,9$, are neither in completed $\mathcal{M}$ class, nor in completed $\mathcal{PS}_{ap}$ class.

The other known infinite classes of AB functions are quadratic; their associated γ_F belong to completed $\mathcal{M}$ class.

7. Classes of *bent monomial Boolean univariate functions* (which can more simply be called *monomial bent* functions and are sometimes called *power bent* functions), that is, functions of the form $f(x)=tr_n(ax^d)$, where $x\in\mathbb{F}_{2^n}$ and a belongs to some subset[19] of $\mathbb{F}_{2^n}^*$.
Obviously, $tr_n(ax^d)$ can be bent only if the mapping $x\to x^d$ is not a permutation (otherwise, the function would be balanced, a contradiction), that is, if d is not co-prime with 2^n-1.
It has been proved in [750] that d must be co-prime either with $2^{\frac{n}{2}}-1$ or with $2^{\frac{n}{2}}+1$. Indeed, since $f(x)$ is invariant under multiplication of x by $\beta=\alpha^{\frac{2^n-1}{\gcd(d,2^n-1)}}$ where α is a primitive element of $\mathbb{F}_{2^n}$, and is then invariant under multiplication by any element of the multiplicative group of order $\gcd(d,2^n-1)$, we have $W_f(0)\equiv 1$ [mod $\gcd(d,2^n-1)$]. Hence, $\gcd(d,2^n-1)$, which equals $\gcd(d,2^{n/2}-1)\gcd(d,2^{n/2}+1)$, since $2^{\frac{n}{2}}-1$ and $2^{\frac{n}{2}}+1$ are co-prime, divides $W_f(0)-1$. If $W_f(0)=2^{\frac{n}{2}}$ then $\gcd(d,2^{\frac{n}{2}}+1)=1$ and if $W_f(0)=-2^{\frac{n}{2}}$ then $\gcd(d,2^{\frac{n}{2}}-1)=1$.

Apart from the particular case of quadratic bent function $f(x)=tr_{\frac{n}{2}}(x^{2^{n/2}+1})$, already encountered, the known values of d for which there exists at least one a such that $tr_n(ax^d)$ is bent (such values are called *bent exponents*) are the following (up to conjugacy $d\to 2^jd$ [mod 2^n-1]):

- the *Gold exponents* (already seen at page 230) $d=2^j+1$, where $\frac{n}{\gcd(j,n)}$ is even and $a\notin\{x^d,\, x\in\mathbb{F}_{2^n}\}=\{x^{\gcd(d,2^n-1)},\, x\in\mathbb{F}_{2^n}\}$; being quadratic, function $tr_n(ax^{2^j+1})$ belongs to the completed Maiorana-McFarland class; these functions have been generalized in [669, 701, 808, 355, 699, 1144, 629] to functions of the form $tr_n(\sum_{i=1}^{n/2-1}a_ix^{2^i+1})+tr_{n/2}(a_{n/2}x^{2^{n/2}+1})$, $a_i\in\mathbb{F}_2$. Being quadratic, these functions all belong to completed class $\mathcal{M}$.
A particular case of Gold exponents is when $\gcd(j,n)=1$, function $tr_n(ax^{2^j+1})$ is then bent if and only if a is not the (2^j+1)-th power of an element of $\mathbb{F}_{2^n}$, that is (since $\gcd(2^j+1,2^n-1)=3$), a is not a cube in $\mathbb{F}_{2^n}$. The same result exists with:

[19] It is impossible that $tr_n(ax^d)$ be bent for every $a\neq 0$ since this would mean that (n,n)-function x^d is bent and we shall see in Proposition 104, page 296, that this is impossible.

- the *Kasami exponents*: $2^{2i}-2^i+1$ with $\gcd(i,n)=1$: function $tr_n(ax^{2^{2i}-2^i+1})$ is bent if and only if a is not a cube (this is proved in [448, Theorem 11] for n not divisible by 3 and is true also for n divisible by 3 as seen by Leander [750]). Note that since the functions in Maiorana-McFarland's and $\mathcal{PS}^+$ classes are normal and functions in $\mathcal{PS}^-$ class have algebraic degree $\frac{n}{2}$, the Kasami bent functions, which have algebraic degree $w_2(4^k-2^k+1)=k+1$, do not belong, in general, to these classes (see page 279).

- the *Dillon exponents* [440] (already seen at page 239): $d=j\cdot(2^{\frac{n}{2}}-1)$, where $\gcd(j,2^{\frac{n}{2}}+1)=1$; function $tr_n(ax^d)$, with $a\in\mathbb{F}_{2^{\frac{n}{2}}}$ without loss of generality, is bent if and only if the Kloosterman sum $\sum_{x\in\mathbb{F}_{2^{\frac{n}{2}}}}(-1)^{tr_{\frac{n}{2}}(x^{-1}+ax)}$ is null, where $1/0=0$ (it belongs then to the PS_{ap} class); see also [750];

- two exponents that we give without proof:

- the *Leander exponent* $d=(2^{n/4}+1)^2$ where n is divisible by 4 but not by 8, see [750]; see also [352] where the set of all a's such that the corresponding function $tr_n(ax^d)$ is bent is determined: $a=a'b^i$, $a'\in w\mathbb{F}_{2^{n/4}}$, $w\in\mathbb{F}_4\setminus\mathbb{F}_2$, $b\in\mathbb{F}_{2^n}$; the function belongs to the Maiorana-McFarland class;
- the *Canteaut-Charpin-Kyureghyan exponent* [197] $d=2^{n/3}+2^{n/6}+1$, where n is divisible by 6 (the corresponding function $tr_n(ax^d)$ is bent if and only if $a=a'b^i$, $a'\in\mathbb{F}_{2^{\frac{n}{2}}}$ such that $tr_{\frac{n}{2}}^{n/6}(a')=a'+a'^{2^{n/6}}+a'^{2^{2n/6}}=0$, $b\in\mathbb{F}_{2^n}$; it belongs to the Maiorana-McFarland class).

It has been checked by Canteaut that all bent functions $tr_n(ax^i)$ are covered by these classes for $n\le 20$ and shown in [352] that there is no other cubic exponent giving infinite classes of bent functions in the Maiorana-McFarland class.

Remark. The bent sequences given in [1137] are particular cases of the constructions given above (using also some of the secondary constructions given below).□

8. Classes of bent polynomial functions in *univariate representation*
We also give them without proof. See more in [227]:

- quadratic bent functions, see page 230;
- $f(x)=tr_n\left(a[x^{2^i+1}+(x^{2^i}+x+1)tr_n(x^{2^i+1})]\right)$, where $n\ge 6$, $\frac{n}{2}$ does not divide i, $\frac{n}{\gcd(i,n)}$ even, $a\in\mathbb{F}_{2^n}\setminus\mathbb{F}_{2^i}$, $\{a,a+1\}\cap\{x^{2^i+1};\,x\in\mathbb{F}_{2^n}\}=\emptyset$; these functions found in [150] by applying CCZ equivalence to non-bent vectorial functions belong to completed $\mathcal{M}$ when $a\in\mathbb{F}_{2^{n/2}}$;
- $f(x)=tr_n\big(a\big[\big(x+tr_3^n\big(x^{2(2^i+1)}+x^{4(2^i+1)}\big)$
$$+tr_n(x)tr_3^n\big(x^{2^i+1}+x^{2^{2i}(2^i+1)}\big)\big)^{2^i+1}\big]\big),$$
where $6\,|\,n$, $\frac{n}{2}$ does not divide i, $\frac{n}{\gcd(i,n)}$ even, $b+d+d^2\notin\{x^{2^i+1};\,x\in\mathbb{F}_{2^n}\}$ for every $d\in\mathbb{F}_{2^3}$; these functions found in [150]) belong to completed $\mathcal{M}$;
- the 4 known classes of Niho bent functions studied above;

- classes of bent functions via Dillon exponents and their generalizations [441, 448, 478, 749, 752, 1144, 479, 629, 350, 851, 852, 853, 871, 764] (we develop some of them in other subsections of this book and do not have the room for detailing each);
- the trace function of the *multinomial APN functions* that we shall describe at page 439 [116];
- sums of some known bent functions and products of linear functions [1131].

9. Classes of bent polynomial functions in *bivariate representation* Except for Maiorana-McFarland functions, $\mathcal{PS}_{ap}$ functions and functions in class $\mathcal{H}$ in bivariate form, there is the isolated class seen at page 231 $f(x,y) = tr_m(x^{2^i+1} + y^{2^i+1} + xy)$, $x, y \in \mathbb{F}_{2^m}$, where $\gcd(3,n) = \gcd(i,n) = 1$.

10. Bent functions obtained as restrictions and extensions In [734] is studied if the restrictions to hyperplanes of Gold functions $tr_n(x^{2^i+1})$ (see page 230) on $\mathbb{F}_{2^n}$, for n odd, $gcd(i,n) = 1$, could be bent. It is shown that this happens with any linear hyperplane not containing element 1. It was already known[20] from [57] that, for any (n,n)-function F satisfying $F(0) = 0$ and such that, for every $a \in \mathbb{F}_{2^n}^*$, the set $H_a = \{D_aF(x); x \in \mathbb{F}_2^n\}$ is the complement of a linear hyperplane[21], the restriction of the Boolean function $1_{H_a} \circ F$ to any linear hyperplane not containing a is bent. Note that the restriction to its complement $a + H_a$ is bent too, since $1_{H_a} \circ F(x) + 1_{H_a} \circ F(x+a)$ equals constant function 1, because $F(x) + F(x+a) \in H_a$ implies that $F(x) \in H_a$ is equivalent to $F(x+a) \notin H_a$.
For $F(x) = x^{2^i+1}$, we have $H_a = \{a^{2^i+1}(x^{2^i} + x + 1); x \in \mathbb{F}_{2^n}\}$ and $1_{H_a} \circ F(x) = tr_n\left(\left(\frac{x}{a}\right)^{2^i+1}\right)$. In [548] is proved that the restriction of any Gold AB function to any linear hyperplane is bent.
Dillon and McGuire studied in [449] the more difficult case of Kasami functions $tr_n(x^{4^i-2^i+1})$ (see page 256) on $\mathbb{F}_{2^n}$, for n odd, $gcd(i,n) = 1$. They showed that for n not divisible by 3, there is one Kasami exponent with $n = 3k \pm 1$ for which the function is bent when restricted to one particular hyperplane (of equation $tr_n(x) = 0$). This function is not bent when restricted to any other hyperplane. They also presented a criterion for the restriction of a near-bent function (see Subsection 6.2.4) to a hyperplane to be bent. More investigations between bent restrictions and near-bent extensions were made in [754], see also some results in [191]. In [755], Leander and McGuire have considered, in the other sense, the problem of going from a near-bent n-variable function to a bent $(n+1)$-variable function; using the construction of bent functions by the concatenation (f,g) of two near-bent functions f, g whose Walsh spectra are complementary (this condition is straightforwardly necessary and sufficient), that is, disjoint, and consequently that function $(f, f \oplus tr_n)$ is bent if and only if the indicator h of

[20] But [734] also determines the dual function.
[21] Ref. [57] calls these functions *crooked*, but we shall use this term at page 306 for a slightly more general notion.

the support of W_f satisfies $D_1 h = 1$ where $1 \in \mathbb{F}_{2^n}$ (which is also easily seen and is equivalent to the bentness of the restrictions of f to $\{x \in \mathbb{F}_{2^n}; tr_n(x) = 0\}$ and its complement), they deduced from the Kasami near-bent functions, the first examples of non-weakly-normal bent functions in dimensions 10 and 12.

6.1.16 Secondary constructions of bent Boolean functions

Since very few bent functions are known from primary constructions, it seems useful to derive secondary constructions[22]. We have already seen in Proposition 79, page 236, a *secondary construction* based on the Maiorana-McFarland construction. We describe now the others (which have been found so far).

1. The *direct sum* is the first secondary construction given by J. Dillon and O. Rothaus in [441, 1005]: let f be a bent function on $\mathbb{F}_2^n$ (n even) and g a bent function on $\mathbb{F}_2^m$ (m even) then the function h defined on $\mathbb{F}_2^{n+m}$ by $h(x, y) = f(x) \oplus g(y)$ is bent. Indeed, a straightforward calculation gives

$$W_h(a, b) = W_f(a) \times W_g(b). \tag{6.28}$$

This construction provides *decomposable functions* only (a Boolean function is called decomposable if it is equivalent to the sum of two functions that depend on two disjoint subsets of coordinates). Such peculiarity is easy to detect and can be used for designing divide-and-conquer attacks, as pointed out by J. Dillon in [442]. However, in some cases (see an example in [839]), this construction provides nice solutions to specific problems. Anyway, if the direct sum provides weak functions in a given framework, the indirect sum (see below) is an alternative, since it has almost the same property with respect to the Walsh transform and does not have the drawback of direct sum.

2. The *Rothaus construction* was introduced by the same authors: if g, h, k and $g \oplus h \oplus k$ are bent on $\mathbb{F}_2^n$ (n even), then the function defined at every element (x, y_1, y_2) of $\mathbb{F}_2^{n+2}$ ($x \in \mathbb{F}_2^n$, $y_1, y_2 \in \mathbb{F}_2$) by $f(x, y_1, y_2) =$

$$g(x)h(x) \oplus g(x)k(x) \oplus h(x)k(x) \oplus [g(x) \oplus h(x)]y_1 \oplus [g(x) \oplus k(x)]y_2 \oplus y_1 y_2$$

is bent. We do not give a proof since this construction will be a particular case of Theorem 15 below (see also [876]). A method is proposed in [329] to construct three bent functions which can be used as initial functions in the Rothaus construction.

3. The *indirect sum* generalizes the direct sum. It has first been found as a construction of resilient functions, which generalized and unified several previous constructions, see Theorem 21, page 329. The same principle allows constructing bent functions:

[22] However, as Dobbertin and Leander write in [477], "most bent functions appear without any roots to bent functions in lower dimensions which could explain their existence".

Proposition 83 *[225] Let f_1 and f_2 be two n-variable bent functions (n even) and let g_1 and g_2 be two m-variable bent functions (m even). Define*[23]

$$h(x,y) = f_1(x) \oplus g_1(y) \oplus (f_1 \oplus f_2)(x)\,(g_1 \oplus g_2)(y);\ x \in \mathbb{F}_2^n,\ y \in \mathbb{F}_2^m.$$

Then h is bent and its dual is obtained from $\widetilde{f_1}, \widetilde{f_2}, \widetilde{g_1}$ and $\widetilde{g_2}$ by the same formula as h is obtained from f_1, f_2, g_1 and g_2.

We do not give a proof of this result either, since we shall see that it is also a particular case of Theorem 15 below.
Similarly to the direct sum and contrary to the Rothaus construction above and to the bent concatenation construction below, the indirect sum requires no condition on the bent functions f_1, f_2, g_1 and g_2 used.
An interest of this construction, compared to the direct sum, is that it allows designing functions h which are more complex (in particular, which may have larger algebraic degree and algebraic immunity) than the functions f_1, f_2, g_1 and g_2 used.

The indirect sum has been modified and generalized in several ways. These generalizations often require conditions on the functions used. In [329] are introduced the constructions:

1. $f(x,y) = f_1(x) \oplus g_1(y) \oplus (f_1 \oplus f_2)(x)(g_1 \oplus g_2)(y) \oplus (f_2 \oplus f_3)(x)(g_2 \oplus g_3)(y),$

where f_1, f_2 and f_3 are bent functions in n variables such that $f_1 \oplus f_2 \oplus f_3$ is bent and has $\widetilde{f_1} \oplus \widetilde{f_2} \oplus \widetilde{f_3}$ for dual, g_1, g_2 and g_3 are bent functions in m variables such that $g_1 \oplus g_2 \oplus g_3$ is bent,

2. $f(x,y) = f_0(x) \oplus g_0(y) \oplus (f_0 \oplus f_1)(x)(g_0 \oplus g_1)(y) \oplus (f_1 \oplus f_2)(x)(g_1 \oplus g_2)(y) \oplus$

$$(f_2 \oplus f_3)(x)(g_2 \oplus g_3)(y),$$

with a slightly more complex condition on functions $f_0, \ldots, f_3, g_0, \ldots, g_3$.

A modified indirect sum is also introduced in [1153], in which functions f_1 and f_2 (resp. g_1 and g_2) are the restrictions of a bent function f (resp. g) to two hyperplanes, complementary of each other.

It is also shown in [245] (where more details on secondary constructions can be found) that if f, g are n-variable Boolean functions, g being bent, and if ϕ is a mapping from $\mathbb{F}_2^n$ to itself, then the $2n$-variable function $f(x) \oplus \widetilde{g}(y) \oplus \phi(x) \cdot y$ is bent if and only if $f(x) \oplus g(\phi(x) + b)$ is bent for every b. Three cases of application are exhibited; all three use two bent functions g and h, and we have:

- if g and h differ by a quadratic function then the $2n$-variable function $(g \oplus h)(x) \oplus \widetilde{g}(y) \oplus x \cdot y$ is bent,

[23] h can be seen as the concatenation of the four functions f_1, $f_1 \oplus 1$, f_2 and $f_2 \oplus 1$, in an order controlled by $g_1(y)$ and $g_2(y)$.

- if g is quadratic and ϕ is an affine permutation, then the $2n$-variable function $g(\phi(x)) \oplus h(x) \oplus \widetilde{g}(y) \oplus \phi(x) \cdot y$ is bent,
- if $Im(\phi) = \{\phi(x);\ x \in \mathbb{F}_2^n\}$ is either included in or disjoint from any translate of $supp(g)$, then the $2n$-variable function $f(x) \oplus \widetilde{g}(y) \oplus \phi(x) \cdot y$ is bent.

4. The *semi-direct sum* [336] $f(x) \oplus g(y + H(x))$, where f and g are bent and H is such that $f \oplus u \cdot H$ is bent for every u.

5. The *bent concatenation construction* generalizes the direct sum, the Rothaus construction, the indirect sum and the semi-direct sum (but as this latter, it needs to find initial bent functions satisfying additional conditions):

Theorem 15 *[215] Let n and m be two even positive integers. Let f be a Boolean function on $\mathbb{F}_2^{n+m} = \mathbb{F}_2^n \times \mathbb{F}_2^m$ such that, for any element y of $\mathbb{F}_2^m$, the function $f_y : x \in \mathbb{F}_2^n \mapsto f(x, y)$ is bent. Then f is bent if and only if, for any element s of $\mathbb{F}_2^n$, the function*

$$\varphi_s : y \mapsto \widetilde{f_y}(s)$$

is bent on $\mathbb{F}_2^m$. If this condition is satisfied, then the dual of f is the function $\widetilde{f}(s, t) = \widetilde{\varphi_s}(t)$ (taking as inner product in $\mathbb{F}_2^n \times \mathbb{F}_2^m$: $(x, y) \cdot (s, t) = x \cdot s \oplus y \cdot t$).

This very general result is easy to prove, using that, for every $s \in \mathbb{F}_2^n$,

$$\sum_{x \in \mathbb{F}_2^n} (-1)^{f(x,y) \oplus x \cdot s} = 2^{\frac{n}{2}} (-1)^{\widetilde{f_y}(s)} = 2^{\frac{n}{2}} (-1)^{\varphi_s(y)},$$

and thus that $W_f(s, t) = 2^{\frac{n}{2}} \sum_{y \in \mathbb{F}_2^m} (-1)^{\varphi_s(y) \oplus y \cdot t}$.

A very particular case of this construction had been previously considered by Adams and Tavares [7] under the name of bent-based functions, and later studied by J. Seberry and X.-M. Zhang in [1025]. The direct sum and Rothaus' constructions are particular cases of Theorem 15 (the latter covers the case $m = 2$). Several classes of bent functions have been deduced in [215], and later in [620]. It is also deduced in [245], with corollaries, that if f, g are n-variable Boolean functions, with g bent, and ϕ a mapping from $\mathbb{F}_2^n$ to itself, then $f(y) \oplus \widetilde{g}(x) \oplus \phi(y) \cdot x$ is bent if and only if $f(y) \oplus g(\phi(y) + b)$ is bent for every $b \in \mathbb{F}_2^n$.

The *indirect sum* is a particular case of the bent concatenation construction of Theorem 15: let h be defined as in Proposition 83, then for every y, the function $h_y(x)$ of Theorem 15 (with h instead of f) equals $f_1(x)$ plus the constant $g_1(y)$ if $g_1(y) = g_2(y)$ and $f_2(x)$ plus the constant $g_1(y)$ if $g_1(y) \neq g_2(y)$; thus it is bent and function $\varphi_s(y)$ equals $\widetilde{f_1}(s) \oplus g_1(y)$ if $g_1(y) = g_2(y)$ and $\widetilde{f_2}(s) \oplus g_1(y)$ if $g_1(y) \neq g_2(y)$, that is, equals $\widetilde{f_1}(s) \oplus g_1(y) \oplus (\widetilde{f_1} \oplus \widetilde{f_2})(s)\,(g_1 \oplus g_2)(y)$; hence, $\varphi_s(y)$ is bent too since it equals $\widetilde{f_1}(s) \oplus g_1(y)$ or $\widetilde{f_1}(s) \oplus g_2(y)$ according to whether $(\widetilde{f_1} \oplus \widetilde{f_2})(s)$ vanishes or not, and according to Theorem 15, h is then bent and its dual equals:

$$\widetilde{h}(s, t) = \widetilde{f_1}(s) \oplus \widetilde{g_1}(t) \oplus (\widetilde{f_1} \oplus \widetilde{f_2})(s)(\widetilde{g_1} \oplus \widetilde{g_2})(t).$$

The semi-direct sum is also a direct consequence thanks to $\widetilde{g \circ t_a} = \widetilde{g} \oplus \ell_a$, $t_a(y) = y + a$, $\ell_a(s) = a \cdot s$.
Another simple application of Theorem 15, called *extension of Maiorana-Mc-Farland type* is given in [270]: let m be even and π be a permutation of $\mathbb{F}_2^{m/2}$ and g an $m/2$-variable Boolean function, and let $f_{\pi,g}(z,y) = z \cdot \pi(y) \oplus g(y)$ be the related Maiorana-McFarland bent function; let $(h_y)_{y \in \mathbb{F}_2^{m/2}}$ be a collection of bent functions on $\mathbb{F}_2^n$ for some even integer n, then the function:

$$(x,y,z) \in \mathbb{F}_2^n \times \mathbb{F}_2^{m/2} \times \mathbb{F}_2^{m/2} \to h_y(x) \oplus f_{\pi,g}(z,y) \tag{6.29}$$

is bent. Indeed, Theorem 15 with (z,y) in the place of y applies with $x \mapsto h_y(x) \oplus f_{\pi,g}(z,y)$ in the place of f_y, and with $\varphi_s(z,y) = \widetilde{h_y}(s) \oplus f_{\pi,g}(z,y)$, which is a bent Maiorana-McFarland function.
This generalizes a construction due to Davis and Jedwab [415] which was slightly posterior to [215] but anterior to [270]: let n and m be two positive even integers; let $h_y(x)$ be a collection of bent functions on $\mathbb{F}_2^n$ for $y \in \mathbb{F}_2^{m/2}$, then the function $(x,y,z) \in \mathbb{F}_2^n \times \mathbb{F}_2^{m/2} \times \mathbb{F}_2^{m/2} \mapsto h_y(x) \oplus y \cdot z$ is bent.
Note that in (6.29), no term involves both x and z, so the structure of the bent function is peculiar (to a lesser extent than for a direct sum, though); instead can be tried $(x,y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n \to h_y(x) \oplus f_{\pi,g}(x,y)$. The restriction of such function when fixing y is bent since that of $f_{\pi,g}(x,y)$ is affine. Then for the global function to be bent, it is necessary and sufficient that $\varphi_s(y) = \widetilde{h_y}(s+\pi(y)) \oplus g(y)$ be bent for all s. Note that the semi-direct sum is a particular case of its dual.
Of course, if $f(x,y)$ is an $(n+s)$-variable function such that, for any $y \in \mathbb{F}_2^s$, the n-variable function $f_y : x \mapsto f(x,y)$ is s-plateaued (see the definition at page 285) and the supports of the Walsh transforms of these functions f_y are pairwise disjoint, then these supports constitute a partition of $\mathbb{F}_2^n$ and f is bent.

6. A permutation based construction due to X.-D. Hou and P. Langevin is built on a very simple observation which leads to potentially new bent functions:

Proposition 84 *[627] Let f be a Boolean function on $\mathbb{F}_2^n$, n even. Let σ be a permutation of $\mathbb{F}_2^n$. We denote its coordinate functions by $\sigma_1, \dots, \sigma_n$ and we assume that, for every $a \in \mathbb{F}_2^n$, we have:*

$$d_H(f, \bigoplus_{i=1}^{n} a_i\, \sigma_i) = 2^{n-1} \pm 2^{\frac{n}{2}-1}.$$

Then $f \circ \sigma^{-1}$ is bent.

Indeed, the Hamming distance between $f \circ \sigma^{-1}$ and the linear function $\ell_a(x) = a \cdot x$ equals $d_H(f, \bigoplus_{i=1}^n a_i\, \sigma_i)$.
Hou and Langevin proposed two frameworks for applying Proposition 84:
- if h is an affine function on $\mathbb{F}_2^n$ and f_1, f_2 and g are Boolean functions on $\mathbb{F}_2^n$ such that the following function is bent:

$$f(x_1,x_2,x) = x_1\, x_2\, h(x) \oplus x_1\, f_1(x) \oplus x_2\, f_2(x) \oplus g(x);\ x \in \mathbb{F}_2^n,\ x_1,\, x_2 \in \mathbb{F}_2,$$

then the function

$$f(x_1, x_2, x) \oplus (h(x) \oplus 1)\, f_1(x) f_2(x) \oplus f_1(x) \oplus (x_1 \oplus h(x) \oplus 1)\, f_2(x) \oplus x_2\, h(x)$$

is bent; in [932] are given cases of application by taking f as the indirect sum of bent functions and using semi-bent 4-decomposition of bent functions;
- if f is a bent function on $\mathbb{F}_2^n$ whose algebraic degree is at most 3, and if σ is a permutation of $\mathbb{F}_2^n$ such that, for every $i = 1, \ldots, n$, there exists a subset U_i of $\mathbb{F}_2^n$ and an affine function h_i such that:

$$\sigma_i(x) = \bigoplus_{u \in U_i} (f(x) \oplus f(x+u)) \oplus h_i(x),$$

then $f \circ \sigma^{-1}$ is bent.
X.-D. Hou in [620] deduced that if $f(x, y)$ $(x, y \in \mathbb{F}_2^{n/2})$ is a Maiorana-McFarland's function of the particular form $x \cdot y \oplus g(y)$ and if $\sigma_1, \ldots, \sigma_n$ are all of the form $\bigoplus_{1 \leq i < j \leq \frac{n}{2}} a_{i,j} x_i\, y_j \oplus b \cdot x \oplus c \cdot y \oplus h(y)$, then $f \circ \sigma^{-1}$ is bent. He gave several examples of application of this result.

7. A construction without extension of the number of variables[24] has been introduced in [227] and is based on the following result:

Proposition 85 *Let f_1, f_2 and f_3 be three Boolean functions on $\mathbb{F}_2^n$. Denote by s_1 the Boolean function equal to $f_1 \oplus f_2 \oplus f_3$ and by s_2 the Boolean function equal to $f_1 f_2 \oplus f_1 f_3 \oplus f_2 f_3$. Then we have $f_1 + f_2 + f_3 = s_1 + 2s_2$. This implies the following equality between the Fourier-Hadamard transforms: $\widehat{f_1} + \widehat{f_2} + \widehat{f_3} = \widehat{s_1} + 2\widehat{s_2}$ and the similar equality between the Walsh transforms:*

$$W_{f_1} + W_{f_2} + W_{f_3} = W_{s_1} + 2\, W_{s_2}. \tag{6.30}$$

Proof. The fact that $f_1 + f_2 + f_3 = s_1 + 2s_2$ (the sums being computed in $\mathbb{Z}$ and not modulo 2) can be checked easily. The $\mathbb{R}$-linearity of the Fourier-Hadamard transform implies then $\widehat{f_1} + \widehat{f_2} + \widehat{f_3} = \widehat{s_1} + 2\widehat{s_2}$. The equality $f_1 + f_2 + f_3 = s_1 + 2s_2$ also directly implies $\mathrm{f_1}_\chi + \mathrm{f_2}_\chi + \mathrm{f_3}_\chi = \mathrm{s_1}_\chi + 2\mathrm{s_2}_\chi$, thanks to the equality $f_\chi = 1 - 2f$ valid for every Boolean function, which implies Relation (6.30). □

Remark. It is observed in [8, Lemma 1] that, given four Boolean functions f_1, f_2, f_3, f_4, the pseudo-Boolean function $\frac{1}{2}(W_{f_1} + W_{f_2} + W_{f_3} + W_{f_4})$ is the Walsh transform of a Boolean function, say g, if and only if $f_1 \oplus f_2 \oplus f_3 \oplus f_4$ equals constant function 1 (this is easily deduced from the fact that, by the $\mathbb{R}$-linearity of the Fourier-Hadamard transform on pseudo-Boolean functions and its bijectivity, we have equivalently $(-1)^g = \frac{1}{2}((-1)^{f_1} + (-1)^{f_2} + (-1)^{f_3} + (-1)^{f_4})$. This means that $f_4 = s_1 \oplus 1$ and then according to (6.30), we have $g = f_1 f_2 \oplus f_1 f_3 \oplus f_2 f_3$, as also observed in [8]. □

Proposition 85 leads then to a double construction of bent functions:

[24] Note that Hou-Langevin's permutation based construction above does not increase either the number of variables, contrary to most other secondary constructions.

Corollary 15 *[227] Let f_1, f_2 and f_3 be three n-variable bent functions, n even. Let $s_1 = f_1 \oplus f_2 \oplus f_3$ and $s_2 = f_1 f_2 \oplus f_1 f_3 \oplus f_2 f_3$. Then:*
- if s_1 is bent and if $\tilde{s_1} = \tilde{f}_1 \oplus \tilde{f}_2 \oplus \tilde{f}_3$, then s_2 is bent, and $\tilde{s_2} = \tilde{f}_1\tilde{f}_2 \oplus \tilde{f}_1\tilde{f}_3 \oplus \tilde{f}_2\tilde{f}_3$;
- if $W_{s_2}(a)$ is divisible by $2^{\frac{n}{2}}$ for every a (e.g. if s_2 is bent, or quadratic, or more generally if it is plateaued; see the definition in Section 6.2), then s_1 is bent.

Proof. - If s_1 is bent and if $\tilde{s_1} = \tilde{f}_1 \oplus \tilde{f}_2 \oplus \tilde{f}_3$, then, for every a, Relation (6.30) implies:

$$\begin{aligned} W_{s_2}(a) &= \left[(-1)^{\tilde{f}_1(a)} + (-1)^{\tilde{f}_2(a)} + (-1)^{\tilde{f}_3(a)} - (-1)^{\tilde{f}_1(a)\oplus\tilde{f}_2(a)\oplus\tilde{f}_3(a)}\right] 2^{\frac{n-2}{2}} \\ &= (-1)^{\tilde{f}_1(a)\tilde{f}_2(a)\oplus\tilde{f}_1(a)\tilde{f}_3(a)\oplus\tilde{f}_2(a)\tilde{f}_3(a)}\, 2^{\frac{n}{2}}. \end{aligned}$$

Indeed, as we already saw above with the relation $\mathrm{f_1}_\chi + \mathrm{f_2}_\chi + \mathrm{f_3}_\chi = \mathrm{s_1}_\chi + 2\mathrm{s_2}_\chi$, for every bits ϵ, η and τ, we have $(-1)^\epsilon + (-1)^\eta + (-1)^\tau - (-1)^{\epsilon\oplus\eta\oplus\tau} = 2\,(-1)^{\epsilon\eta\oplus\epsilon\tau\oplus\eta\tau}$.
- If $W_{s_2}(a)$ is divisible by $2^{\frac{n}{2}}$ for every a, then the number $W_{s_1}(a)$, which is equal to $\left[(-1)^{\tilde{f}_1(a)} + (-1)^{\tilde{f}_2(a)} + (-1)^{\tilde{f}_3(a)}\right] 2^{\frac{n}{2}} - 2\,W_{s_2}(a)$, according to Relation (6.30), is congruent with $2^{\frac{n}{2}}$ modulo $2^{\frac{n}{2}+1}$ for every a. This is sufficient to imply that s_1 is bent, according to Lemma 5, page 214. □
Corollaries are deduced in [227] which revisit results from [327] (this latter reference also includes constructions of plateaued functions).

This construction has been used in [860, 864] (where is observed that, conversely, if f_1, f_2, f_3, s_1 and s_2 are bent, then $\tilde{s}_1 = \tilde{f}_1 \oplus \tilde{f}_2 \oplus \tilde{f}_3$) and is called Carlet's secondary construction in [386, 873]. It is used in [711, 873] with linear structures. In the continuation of [863], it it is shown in [386] that using Corollary 15, three involutions whose sum is an involution give rise through the Maiorana-McFarland construction to bent functions in bivariate representation.

The construction of Corollary 15 was extended to more than three functions:

Proposition 86 *[227] Let $f_1, \ldots, f_m$ be Boolean functions on $\mathbb{F}_2^n$. For every positive integer l, let s_l be the Boolean function defined by*

$$s_l = \bigoplus_{1\le i_1<\ldots<i_l\le m} \prod_{j=1}^{l} f_{i_j} \quad \text{if } l \le m \text{ and } s_l = 0 \text{ otherwise.}$$

Then we have $f_1+\ldots+f_m = \sum_{i\ge 0} 2^i\, s_{2^i}$ (sums in $\mathbb{Z}$). This implies $\widehat{f_1}+\ldots+\widehat{f_m} = \sum_{i\ge 0} 2^i\, \widehat{s_{2^i}}$. Moreover, if m is primitive, say $m = 2^r - 1$, then

$$W_{f_1} + \ldots + W_{f_m} = \sum_{i=0}^{r-1} 2^i\, W_{s_{2^i}}. \tag{6.31}$$

Proof. Let $x \in \mathbb{F}_2^n$ and $j_x = \sum_{k=1}^m f_k(x)$. According to Lucas' Theorem (see page 528), the binary expansion of j_x is $\sum_{i\ge 0}\left[2^i\left(\binom{j_x}{2^i}\ [\mathrm{mod}\ 2]\right)\right]$. It is a simple matter to check that $\binom{j_x}{2^i}\ [\mathrm{mod}\ 2] = s_{2^i}(x)$. Thus, $f_1 + \ldots + f_m = \sum_{i\ge 0} 2^i\, s_{2^i}$. The linearity of the Walsh transform with respect to the addition in $\mathbb{R}$ implies

then directly $\widehat{f_1} + \ldots + \widehat{f_m} = \sum_{i\geq 0} 2^i\, \widehat{s_{2^i}}$.
If $m = 2^r - 1$ (recall that in coding theory, such number is called primitive), then we have $m = \sum_{i=0}^{r-1} 2^i$. Thus, we deduce $(-1)^{f_1} + \ldots + (-1)^{f_m} = \sum_{i=0}^{r-1} 2^i\,(-1)^{s_{2^i}}$ from $f_1 + \ldots + f_m = \sum_{i=0}^{r-1} 2^i\, s_{2^i}$. The linearity of the Walsh transform implies then Relation (6.31). □

Corollary 16 *[227] Let n be any positive even integer and $f_1, \ldots, f_m$ ($m \leq 7$) be bent functions on $\mathbb{F}_2^n$.*

- *Assume that s_1 is bent, and that, for every $a \in \mathbb{F}_2^n$, the number $W_{s_4}(a)$ is divisible by $2^{n/2}$. Then:*
  - *if $m = 5$ and $\widetilde{s_1} = \widetilde{f_1} \oplus \ldots \oplus \widetilde{f_5} \oplus 1$ then s_2 is bent;*
  - *if $m = 7$ and $\widetilde{s_1} = \widetilde{f_1} \oplus \ldots \oplus \widetilde{f_7}$, then s_2 is bent.*
- *Assume that $m \in \{5, 7\}$ and, for every $a \in \mathbb{F}_2^n$, the number $W_{s_4}(a)$ is divisible by $2^{n/2-1}$ and the number $W_{s_2}(a)$ is divisible by $2^{n/2}$, then s_1 is bent.*

Proof. We have for $i = 1, \ldots, m$ and for every vector $a \neq 0$: $W_{f_i}(a) = -2\widehat{f_i}(a) = (-1)^{\widetilde{f_i}(a)}\, 2^{n/2}$ and $\widehat{f_1}(a) + \ldots + \widehat{f_m}(a) = \sum_{i\geq 0} 2^i\, \widehat{s_{2^i}}(a)$.
- If s_1 is bent and, for every $a \in \mathbb{F}_2^n$, the number $W_{s_4}(a)$ is divisible by $2^{n/2}$, then $W_{s_2}(a)$ is congruent with $\left[(-1)^{\widetilde{f_1}(a)} + \ldots + (-1)^{\widetilde{f_m}(a)} - (-1)^{\widetilde{s_1}(a)}\right] 2^{n/2-1}$ modulo $2^{n/2+1}$, for every $a \neq 0_n$.
If $m = 5$ and $\widetilde{s_1} = \widetilde{f_1} \oplus \ldots \oplus \widetilde{f_5} \oplus 1$ then, denoting by k the Hamming weight of the word $(\widetilde{f_1}(a), \ldots, \widetilde{f_5}(a))$, the number $W_{s_2}(a)$ is congruent with $[5 - 2k + (-1)^k]\, 2^{n/2-1}$ modulo $2^{n/2+1}$.
If $m = 7$ and $\widetilde{s_1} = \widetilde{f_1} \oplus \ldots \oplus \widetilde{f_7}$ then, denoting by k the Hamming weight of the word $(\widetilde{f_1}(a), \ldots, \widetilde{f_7}(a))$, the number $W_{s_2}(a)$ is congruent with $[7 - 2k - (-1)^k]\, 2^{n/2-1}$ modulo $2^{n/2+1}$. So, in both cases, we have $W_{s_2}(a) \equiv 2^{n/2}$ [mod $2^{n/2+1}$], and s_2 is bent, according to Lemma 5, page 214.
- if, for every $a \in \mathbb{F}_2^n$, the number $W_{s_4}(a)$ is divisible by $2^{n/2-1}$ and the number $W_{s_2}(a)$ is divisible by $2^{n/2}$, then, for every $a \neq 0_n$, the number $W_{s_1}(a)$ is congruent with $\left[(-1)^{\widetilde{f_1}(a)} + \ldots + (-1)^{\widetilde{f_m}(a)}\right] 2^{n/2} \bmod 2^{n/2+1}$. Since $m \in \{5, 7\}$, it is then congruent with $2^{n/2} \bmod 2^{n/2+1}$ and s_1 is bent, according to Lemma 5 again.

8. A construction related to the notion of normal extension of bent function can be found in Proposition 93, page 280.

9. A construction related to bent rectangles. In [8, 10] are represented n-variable Boolean functions f by matrices called *rectangles* (among which squares, when n is even). The rows of such matrices are the Walsh transforms of the restrictions of f obtained by fixing m coordinates at fixed positions (say, at the first m positions), where $1 \leq m \leq n-1$: denoting by f_u the restriction of f obtained by fixing $x_i = u_i$ for $i = 1, \ldots, m$, the term at row indexed by $u \in \mathbb{F}_2^m$

and column indexed by $v \in \mathbb{F}^{n-m}$ equals[25] $W_{f_u}(v) = \sum_{y\in\mathbb{F}_2^{n-m}}(-1)^{f(u,y)\oplus v\cdot y}$ (i.e. the row equals the Walsh transform vector of f_u). It is proved in [10] that f is bent if and only if the columns, when multiplied by $2^{m-\frac{n}{2}}$, are also the Walsh transforms of Boolean functions. This is, in a way, a generalization of Theorem 15, page 260, since m does not need to be even and the restrictions do not need to be bent. The condition is necessary since, for every $a \in \mathbb{F}_2^m$ and $v \in \mathbb{F}_2^{n-m}$, we have $\sum_{u\in\mathbb{F}_2^m} W_{f_u}(v)(-1)^{u\cdot a} = \sum_{u\in\mathbb{F}_2^m, y\in\mathbb{F}_2^{n-m}}(-1)^{f(u,y)\oplus v\cdot y\oplus u\cdot a} = W_f(a,v) = 2^{\frac{n}{2}}(-1)^{\widetilde{f}(a,v)}$, and denoting by $\widetilde{f}_v$ the restriction of $\widetilde{f}$ obtained by fixing its $n-m$ last input coordinates to the corresponding values of v, and by applying the inverse Walsh transform formula to $\widetilde{f}_v$, we see that the column indexed by v and multiplied by $2^{m-\frac{n}{2}}$ equals the Walsh transform of $\widetilde{f}_v$, because $\sum_{a\in\mathbb{F}_2^m}(-1)^{\widetilde{f}_v(a)\oplus a\cdot u} = 2^{m-\frac{n}{2}}W_{f_u}(v)$. It is also easily seen that the condition is sufficient. Constructions of bent squares are deduced in [10] by using so-called biaffine transformations and partitions of $\mathbb{F}_2^n$ into affine planes of equal dimension (but it is not checked whether such constructions can provide new bent functions nor whether the constructions themselves fall within known ones or not).

10. A general construction in the framework of the so-called $\mathbb{Z}$-*bent functions*. Most constructions above build bent functions from bent functions. The idea of $\mathbb{Z}$-bent functions is to extend the corpus in order to embed bent functions into a recursive context. This has been initiated by Dobbertin in 2005 and G. Leander has presented the results and given guidelines for further research in a paper posthumously co-authored by Hans Dobbertin [477] (see also [476]). $\mathbb{Z}$-bent functions are integer-valued functions φ on $\mathbb{F}_2^n$ whose normalized Fourier transform $\widehat{\varphi}_{norm} = 2^{-n/2}\widehat{\varphi}$, is also integer-valued. Bent Boolean functions (or more precisely their sign functions) will be among $\mathbb{Z}$-bent functions those which are ± 1-valued.
The following nested subsets of $\mathbb{Z}$ are defined: $W_0 = \{\pm 1\}$ and for $r \neq 0$, $W_r = \{w \in \mathbb{Z} \mid -2^{r-1} \leq w \leq 2^{r-1}\}$. They satisfy $W_r \pm W_r = W_{r+1}$ for $r > 0$ and lead to a hierarchy on $\mathbb{Z}$-bent functions:

Definition 55 *[477] A function $\varphi : \mathbb{F}_2^n \to W_r$ is called a $\mathbb{Z}$-bent function of size $\frac{n}{2}$ and level r if $\widehat{\varphi}_{norm}$ is also valued in W_r.*

In this hierarchy, the (sign functions of) usual bent functions are the zero level $\mathbb{Z}$-bent functions. Since the normalized Fourier transform is self-inverse, $\widehat{\varphi}_{norm}$ is then also a $\mathbb{Z}$-bent function of size $\frac{n}{2}$ and level r which is called the dual of φ.
$\mathbb{Z}$-bent functions of level r on n variables can be used to construct all $\mathbb{Z}$-bent functions of level $r-1$ on $(n+2)$ variables. This is referred to as "gluing" technique. All bent functions in $(n+2r)$ variables (*i.e.* all $\mathbb{Z}$-bent functions of level 0 in $(n+2r)$ variables) are eventually reached this way.
The construction of partial spread ($\mathcal{PS}$) bent functions has been generalized to

[25] There seems to be a slight confusion between rows and columns in the description given at the bottom of page 5 in [10].

partial spread $\mathbb{Z}$-bent functions of arbitrary level in [526]. This led to a new construction of bent Boolean functions. A bent function in 8 variables outside the completed $\mathcal{M}$ and $\mathcal{PS}_{ap}$ classes was deduced; all bent functions in 6 variables can be obtained, up to equivalence, by this construction.
Secondary construction of bent functions from near-bent functions have been also proposed, for instance in [1122].

6.1.17 Decompositions of bent functions

The following theorem is a direct consequence of the second-order Poisson formula (2.57), page 81, applied to $f \oplus \ell$ where ℓ is linear, and to a linear hyperplane E of $\mathbb{F}_2^n$, and of the well-known (easy to prove) fact that, for every even integer n, the sum of the squares of two integers equals 2^n (resp. 2^{n+1}) if and only if one of these squares is null and the other one equals 2^n (resp. both squares equal 2^n):

Theorem 16 *[191] Let $n \geq 4$ be an even integer and let f be an n-variable Boolean function. Then the following properties are equivalent.*

1. *f is bent,*
2. *For every (or some) hyperplane E of $\mathbb{F}_2^n$, the restrictions of f to E and $\mathbb{F}_2^n \setminus E$ (viewed as Boolean functions on $\mathbb{F}_2^{n-1}$) are plateaued (see the definition at page 285) with amplitude $2^{\frac{n}{2}}$ (*i.e. *are near-bent), and their Walsh supports partition the whole space $\mathbb{F}_2^{n-1}$,*
3. *For every (or some) linear hyperplane E of $\mathbb{F}_2^n$, every derivative $D_e f$, $e \in E \setminus \{0_n\}$ is balanced.*

The fact that Property 3 is enough comes from Relation (2.56), page 81. Note that we have also (see [191]) that, if a function in an odd number of variables is such that, for some nonzero $a \in \mathbb{F}_2^n$, every derivative $D_u f$, $u \neq 0_n$, $u \in a^\perp$, is balanced, then its restriction to the linear hyperplane $a^\perp$ or to its complement is bent.
It is also proved in [191] that the Walsh transforms of the four restrictions of a bent function to an $(n-2)$-dimensional vector subspace E of $\mathbb{F}_2^n$ and its cosets have the same sets of absolute values. It is a simple matter to see that, denoting by a and b two vectors such that $E^\perp$ is the linear space spanned by a and b, these four restrictions are bent if and only if $D_a D_b \widetilde{f}$ takes on constant value 1, and as observed in [193] that[26] $f \oplus 1_E$ is bent if and only if $D_a D_b \widetilde{f}$ takes on constant value 0 (see examples in [198, Corollary 15]). More on decomposing bent functions can be found in [191, 193, 349].

[26] We have seen in the second remark of page 227 that this is a direct consequence of Th. 14.

6.1.18 Class $\mathcal{GPS}$ and a geometric characterization of bent Boolean functions

Class $\mathcal{PS}$ generalizes to a class introduced in [213] and called $\mathcal{GPS}$ (for generalized partial spreads), which led to a characterization of bent functions that we call geometric characterization. This characterization, given below in Theorem 17, can be proved rather simply by using Proposition 67, page 218, which is posterior to the introduction of $\mathcal{GPS}$ and to Theorem 17:

Theorem 17 *[290] Let f be a Boolean function on $\mathbb{F}_2^n$. Then f is bent if and only if there exist $\frac{n}{2}$-dimensional subspaces $E_1, \dots, E_k$ of $\mathbb{F}_2^n$ (with no constraint on number k) and integers $m_1, \dots, m_k$ (positive or negative) such that, for any element x of $\mathbb{F}_2^n$:*

$$f(x) \equiv \sum_{i=1}^{k} m_i 1_{E_i}(x) - 2^{\frac{n}{2}-1}\delta_0(x) \quad \left[\ mod\ 2^{\frac{n}{2}}\right]. \tag{6.32}$$

If we have $f(x) = \sum_{i=1}^{k} m_i 1_{E_i}(x) - 2^{\frac{n}{2}-1}\delta_0(x)$ then the dual of f equals $\widetilde{f}(x) = \sum_{i=1}^{k} m_i 1_{E_i^\perp}(x) - 2^{\frac{n}{2}-1}\delta_0(x)$.

Proof (sketch of). Relation (6.32) is a sufficient condition for f being bent, according to Lemma 5 and to Relation (2.38), page 77. This same Relation (2.38) also implies the last sentence of Theorem 17. Conversely, if f is bent, then Proposition 67 allows to deduce Relation (6.32), by expressing all the monomials x^I by means of the indicators of subspaces of dimension at least $n - |I|$ (indeed, the NNF of the indicator of the subspace $\{x \in \mathbb{F}_2^n;\ x_i = 0,\ \forall i \in I\}$ being equal to $\prod_{i \in I}(1 - x_i) = \sum_{J \subseteq I}(-1)^{|J|}x^J$, the monomial x^I can be expressed by means of this indicator and of the monomials x^J, where J is strictly included in I) and by using Lemma 8 below. □

Lemma 8 *Let F be any d-dimensional subspace of $\mathbb{F}_2^n$. There exist $\frac{n}{2}$-dimensional subspaces $E_1, \dots, E_k$ of $\mathbb{F}_2^n$ and integers $m, m_1, \dots, m_k$ such that, for any element x of $\mathbb{F}_2^n$:*

$$2^{\frac{n}{2}-d}\, 1_F(x) \equiv m + \sum_{i=1}^{k} m_i 1_{E_i}(x) \left[\ mod\ 2^{\frac{n}{2}}\right]\ \text{ if } d < \frac{n}{2}, \text{ and}$$

$$1_F(x) \equiv \sum_{i=1}^{k} m_i 1_{E_i}(x) \left[\ mod\ 2^{\frac{n}{2}}\right]\ \text{ if } d > \frac{n}{2}.$$

This lemma completes the proof of Theorem 17 since $d \geq n - |I|$ implies $|I| - \frac{n}{2} \geq \frac{n}{2} - d$.

Definition 56 *The class of those functions f which satisfy the relation obtained from (6.32) by withdrawing "$[mod\ 2^{\frac{n}{2}}]$" is called* generalized partial spread *class and denoted by $\mathcal{GPS}$.*

Class $\mathcal{GPS}$ includes $\mathcal{PS}$, see [213]. The dual $\widetilde{f}$ of such function f of $\mathcal{GPS}$ equaling

$\widetilde{f}(x) = \sum_{i=1}^{k} m_i 1_{E_i^\perp}(x) - 2^{\frac{n}{2}-1}\delta_0(x)$, it belongs to $\mathcal{GPS}$ too.

There is no uniqueness of the representation of a given bent function in the form (6.32). There exists another characterization, shown in [291], in the form $f(x) = \sum_{i=1}^{k} m_i 1_{E_i}(x) \pm 2^{\frac{n}{2}-1}\delta_0(x)$, where $E_1, \dots, E_k$ are vector subspaces of $\mathbb{F}_2^n$ of dimensions $\frac{n}{2}$ or $\frac{n}{2}+1$ and where $m_1, \dots, m_k$ are integers (positive or negative). There is not a unique way, either, to choose these spaces E_i. But it is possible to define some subclass of $\frac{n}{2}$-dimensional and $(\frac{n}{2}+1)$-dimensional spaces such that there is uniqueness, if the spaces E_i are chosen in this subclass.
P. Guillot has proved subsequently in [579] that, *up to the composition by a translation $x \mapsto x + a$, every bent function belongs to $\mathcal{GPS}$.* The proof is a little too technical for being included here.

6.1.19 On the number of bent Boolean functions

Nonexistence of efficient lower bounds

The original Maiorana-McFarland's class is one of the the widest classes. The number of bent functions of the form (6.9), page 233, equals $(2^{\frac{n}{2}})! \times 2^{2^{\frac{n}{2}}}$, which is asymptotically equivalent to $\left(\frac{2^{\frac{n}{2}+1}}{e}\right)^{2^{\frac{n}{2}}} \sqrt{2^{\frac{n}{2}+1}\pi}$ (according to Stirling's formula) while the other important straightforward construction of bent functions, $\mathcal{PS}_{ap}$, leads only to $\binom{2^{\frac{n}{2}}}{2^{\frac{n}{2}-1}} \approx \frac{2^{2^{\frac{n}{2}}+\frac{1}{2}}}{\sqrt{\pi 2^{\frac{n}{2}}}}$ functions[27]. However, the number of bent Maiorana-McFarland's functions seems negligible with respect to the total number of bent functions. The size of the completed Maiorana-McFarland's class is unknown; it is at most equal to the number of Maiorana-McFarland's functions times the number of affine automorphisms, which equals $2^n(2^n-1)(2^n-2)\dots(2^n-2^{n-1})$. It seems also negligible with respect to the total number of bent functions. In fact, the lower bounds which can be deduced from all known constructions of bent functions seem very far from the actual number. For instance, in 8 variables, there are approximately 2^{106} different bent functions[28], see below, and about 2^{77} correspond to the known constructions. The problem of determining an efficient lower bound on the number of n-variable bent functions is then open.
There exists a related open question by N. Tokareva in [1087] (that she calls the bent sum decomposition problem) whether all Boolean functions of *algebraic degree* at most $\frac{n}{2}$ are equal to the sums of two n-variable bent functions (which is equivalent to asking whether the set of such sums is stable under addition [977]). The reply to this question seems probably negative, but there is no proof that it is, and it is shown in [977] that the reply is positive when restricting ourselves to a number of subclasses (Boolean functions in at most 6 variables,

[27] Its extension with André's spreads, see page 241, has nevertheless more elements.
[28] Among which probably many could lead to new infinite classes; this shows how limited is our knowledge.

quadratic Boolean functions, Maiorana-McFarland bent functions, partial spread functions). And the usual parameters and properties of Boolean functions (ANF, NNF and numerical degree, generalized degree, divisibility of the Fourier transform or of the coefficients of the NNF, other properties of the Fourier or Walsh transform values) do not seem to allow discriminating sums of two bent functions from other Boolean functions of degrees at most $n/2$. If the reply to Tokareva's question was finally positive, this would give a straightforward lower bound on the number of bent functions which would be much better than what is known.

Upper bounds

Rothaus' inequality recalled in Section 6.1.8 (Theorem 13, page 224) states that any bent function has algebraic degree at most $\frac{n}{2}$. Thus, the number of bent functions is at most

$$2^{1+n+\dots+\binom{n}{n/2}} = 2^{2^{n-1}+\frac{1}{2}\binom{n}{n/2}}.$$

We shall call this upper bound the *naive bound*. For $n = 6$, the number of bent functions is known and is approximately equal to $2^{32.3}$ (see [968]), which is much less than the naive bound gives: 2^{42}. For $n = 8$, the number is also known: it has been first shown in [744] that it is inferior to $2^{129.2}$; it has been later calculated by Langevin, Leander et al. [743] and equals approximately $2^{106.3}$ (the naive bound gives 2^{163}). Hence picking at random an 8-variable Boolean function of algebraic degree bounded above by 4 does not allow obtaining bent functions (but more clever methods exist, see [413, 278]). An upper bound improving upon the naive bound has been found in [301]. It is exponentially better than the naive bound since it divides it by approximately $2^{2^{\frac{n}{2}}-\frac{n}{2}-1}$. But it seems to be still far from the exact number of bent functions: for $n = 6$ it gives roughly 2^{38} (to be compared with $2^{32.3}$) and for $n = 8$ it gives roughly 2^{152} (to be compared with $2^{106.3}$). But the bound of [301] could not be improved since it was obtained.

Number of bent components of a vectorial function

It is shown in [962] that the number of bent components of any (n, n)-function is at most $2^n - 2^{\frac{n}{2}}$, with equality if and only if the set of those v such that $v \cdot F$ is not bent is an $\frac{n}{2}$-dimensional vector space, and that this upper bound is achieved with equality by the Niho power function $x^{2^{\frac{n}{2}}+1}$, and the function $x^{2^i}(x + x^{2^{\frac{n}{2}}})$ for all $i = 0, \dots, n-1$ (these latter functions are pairwise EA/CCZ inequivalent for $i \neq 0, n/2$). In [877] is shown that the set of those (n, n)-functions having maximum number of bent components is preserved by CCZ equivalence and does not contain any APN plateaued function.

6.1.20 Hyper-bent, homogeneous, symmetric/rotation symmetric bent Boolean functions

Hyper-bent Boolean functions

Hyper-bent functions were initially proposed by Golomb and Gong [554] in relation with the security of symmetric cryptosystems, for the reason that when $\gcd(i, 2^n - 1) = 1$, both functions $tr_n(ax)$ and $tr(ax^i)$ provide m-sequences. But no explicit attack was proposed. In [202], Canteaut and Rotella showed that, in the context of filtered LFSR, a relevant criterion is the minimum distance between the function and the Boolean functions of the form $tr_n(ax^i) \oplus \epsilon$, where $\gcd(i, 2^n - 1) = 1$, $a \in \mathbb{F}_{2^n}$ and $\epsilon \in \mathbb{F}_2$: they showed that if $f(x) \oplus tr_n(ax^i)$ is biased, then a *fast correlation attack* can be performed to recover the initial state. Even the case when i is not co-prime with $2^n - 1$ leads to an attack, and this provides a new criterion to evaluate the security of filtered LFSR. Nevertheless, these new considerations confirm the interest of the definition introduced by Golomb and Gong.

Definition 57 *Let n be any even positive integer. An n-variable Boolean function f on the field $\mathbb{F}_{2^n}$ is a* hyper-bent function *if, for every positive integer i co-prime with $2^n - 1$, function $f(x^i)$ is bent (or equivalently, since the compositional inverse of a generic power permutation x^i is a generic power permutation, if for any such i, we have $\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+tr_n(a\,x^i)} = \pm 2^{n/2}$ for every $a \in \mathbb{F}_{2^n}$).*

Remark. In [214] have been determined those Boolean functions on $\mathbb{F}_2^n$ such that, for a given even integer k $(2 \le k \le n - 2)$, any of the Boolean functions on $\mathbb{F}_2^{n-k}$, obtained by keeping constant k coordinates among $x_1, \dots, x_n$, is bent (*i.e.* those functions which satisfy the propagation criterion of degree $n - k$ and order k). These are the four bent *symmetric Boolean functions* (see Section 10.1). They were called hyper-bent in [214] but we keep this term for the notion introduced by Golomb and Gong. □

Hyper-bent functions can be characterized in terms of the *extended Walsh transform* [554]:

$$W_f(a, i) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+tr_n(ax^i)}, \forall a \in \mathbb{F}_{2^n}, \text{ with } \gcd(i, 2^n - 1) = 1,$$

as those functions whose extended Walsh transform takes only the values $\pm 2^{\frac{n}{2}}$. The condition seems difficult to satisfy. However, A. Youssef and G. Gong, who introduced the term in [1143], showed that hyper-bent functions exist. Recall that class $\mathcal{PS}_{ap}^{\#}$, defined at page 240, is the set of those bent functions over $\mathbb{F}_{2^n}$ which can be obtained from those of $\mathcal{PS}_{ap}$ by composition by the transformations $x \in \mathbb{F}_{2^n} \mapsto \delta x$, $\delta \neq 0$, and by addition of a constant. We have:

Proposition 87 *[278] All the functions of class $\mathcal{PS}_{ap}^{\#}$ are hyper-bent.*

Let us give here a direct proof of this fact.

Proof. We can restrict ourselves without loss of generality to the functions of class $\mathcal{PS}_{ap}$. Let ω be any element in $\mathbb{F}_{2^n} \setminus \mathbb{F}_{2^{n/2}}$. The pair $(1, \omega)$ is a basis of the $\mathbb{F}_{2^{n/2}}$-vector space $\mathbb{F}_{2^n}$. Hence, we have $\mathbb{F}_{2^n} = \mathbb{F}_{2^{n/2}} + \omega \mathbb{F}_{2^{n/2}}$ and the elements of class $\mathcal{PS}_{ap}$ are the functions $f(y' + \omega\, y) = g\left(\frac{y'}{y}\right)$, with $\frac{y'}{y} = 0$ if $y = 0$, where g is balanced on $\mathbb{F}_2^{n/2}$ and vanishes at 0. Note that every element y of $\mathbb{F}_{2^{n/2}}$ satisfies $y^{2^{n/2}} = y$ and therefore $tr_n(y) = y + y^2 + \cdots + y^{2^{n/2-1}} + y + y^2 + \cdots + y^{2^{n/2-1}} = 0$. Consider the inner product in $\mathbb{F}_{2^n}$ defined by: $y \cdot y' = tr_n(y\, y')$; the subspace $\mathbb{F}_{2^{n/2}}$ is then its own orthogonal; hence, according to Relation (2.38), page 77, any sum of the form $\sum_{y \in \mathbb{F}_{2^{n/2}}} (-1)^{tr_n(\lambda y)}$ is null if $\lambda \notin \mathbb{F}_{2^{n/2}}$ and equals $2^{n/2}$ if $\lambda \in \mathbb{F}_{2^{n/2}}$. For every $a \in \mathbb{F}_{2^n}$, we have:

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + tr_n(a\, x^i)} = \sum_{y, y' \in \mathbb{F}_{2^{n/2}}} (-1)^{g\left(\frac{y'}{y}\right) + tr_n(a\,(y' + \omega y)^i)}.$$

Denoting $\frac{y'}{y}$ by z, we see that:

$$\sum_{y \in \mathbb{F}^*_{2^{n/2}}, y' \in \mathbb{F}_{2^{n/2}}} (-1)^{g\left(\frac{y'}{y}\right) + tr_n(a\,(y' + \omega y)^i)} = \sum_{z \in \mathbb{F}_{2^{n/2}}, y \in \mathbb{F}^*_{2^{n/2}}} (-1)^{g(z) + tr_n(a\, y^i (z + \omega)^i)}.$$

The remaining sum $\displaystyle\sum_{y' \in \mathbb{F}_{2^{n/2}}} (-1)^{g(0) + tr_n(a\, y'^i)} = \sum_{y' \in \mathbb{F}_{2^{n/2}}} (-1)^{tr_n(a\, y')}$ equals $2^{n/2}$ if $a \in \mathbb{F}_{2^{n/2}}$ and is null otherwise.
Thus, $\displaystyle\sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + tr_n(a\, x^i)}$ equals:

$$\sum_{z \in \mathbb{F}_{2^{n/2}}} \left((-1)^{g(z)} \sum_{y \in \mathbb{F}_{2^{n/2}}} (-1)^{tr_n(a (z+\omega)^i\, y)} \right) - \sum_{z \in \mathbb{F}_{2^{n/2}}} (-1)^{g(z)} + 2^{n/2} 1_{\mathbb{F}_{2^{n/2}}}(a).$$

The sum $\sum_{z \in \mathbb{F}_{2^{n/2}}} (-1)^{g(z)}$ is null since g is balanced.
The sum $\sum_{z \in \mathbb{F}_{2^{n/2}}} \left((-1)^{g(z)} \sum_{y \in \mathbb{F}_{2^{n/2}}} (-1)^{tr_n(a (z+\omega)^i\, y)} \right)$ equals $\pm 2^{n/2}$ if $a \notin \mathbb{F}_{2^{n/2}}$, since we prove in the next Lemma that there exists then exactly one $z \in \mathbb{F}_{2^{n/2}}$ such that $a(z + \omega)^i \in \mathbb{F}_{2^{n/2}}$; and this sum is null if $a \in \mathbb{F}_{2^{n/2}}$ (this can be checked, if $a = 0$ thanks to the balancedness of g, and if $a \neq 0$ because y ranges over $\mathbb{F}_{2^{n/2}}$ and $a(z + \omega)^i \notin \mathbb{F}_{2^{n/2}}$). This completes the proof. □

Lemma 9 *Let n be any positive integer. Let a and ω be two elements of the set $\mathbb{F}_{2^n} \setminus \mathbb{F}_{2^{n/2}}$ and let i be co-prime with $2^n - 1$. There exists a unique element $z \in \mathbb{F}_2^{n/2}$ such that $a(z + \omega)^i \in \mathbb{F}_2^{n/2}$.*

Proof. Let j be the inverse of i modulo $2^n - 1$. We have $a(z + \omega)^i \in \mathbb{F}_2^{n/2}$ if and only if $z \in \omega + a^{-j} \times \mathbb{F}_2^{n/2}$. The sets $\omega + a^{-j} \times \mathbb{F}_2^{n/2}$ and $\mathbb{F}_2^{n/2}$ are two flats whose directions $a^{-j} \times \mathbb{F}_2^{n/2}$ and $\mathbb{F}_2^{n/2}$ are subspaces whose sum is direct and equals $\mathbb{F}_{2^n}$. Hence, they have a unique vector in their intersection. □

The duals of hyper-bent functions in $\mathcal{PS}_{ap}^{\#}$ are also in $\mathcal{PS}_{ap}^{\#}$ and then are hyper-bent.

Relationships between the notion of hyper-bent function and *cyclic codes* are studied in [278] and it is deduced that:

Proposition 88 *[278] Every hyper-bent function $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$ can be represented as: $f(x) = \sum_{i=1}^{r} tr_n(a_i x^{t_i}) + \epsilon$, where $a_i \in \mathbb{F}_{2^n}, \epsilon \in \mathbb{F}_2$ and $w_2(t_i) = n/2$, where w_2 denotes the 2-weight (see page 62). Consequently, all hyper-bent functions have algebraic degree $n/2$.*

It is also shown in [278] that the elements in $\mathcal{PS}_{ap}^{\#}$ are the functions of Hamming weight $2^{n-1} \pm 2^{n/2-1}$ which can be written in the form $\sum_{i=1}^{r} tr_n(a_i x^{j_i})$, where $a_i \in \mathbb{F}_{2^n}$ and j_i is a multiple of $2^{n/2} - 1$. Hence, $\mathcal{PS}_{ap}^{\#}$ coincides with the set of bent functions whose trace form involves Dillon-like exponents $r(2^{n/2} - 1)$ only.

In [350] is proved that, for every n even, $\lambda \in \mathbb{F}_{2^{n/2}}^*$ and $r \in]0; \frac{n}{2}[$ such that the cyclotomic cosets of 2 modulo $2^{n/2} + 1$ containing respectively $2^r - 1$ and $2^r + 1$ have size n and such that the function $tr_{\frac{n}{2}}\left(\lambda x^{2^r+1}\right)$ is balanced on $\mathbb{F}_{2^{n/2}}$, the function $tr_n\left(\lambda\left(x^{(2^r-1)(2^{n/2}-1)} + x^{(2^r+1)(2^{n/2}-1)}\right)\right)$ is bent (*i.e.* hyper-bent) if and only if the function $tr_{\frac{n}{2}}\left(x^{-1} + \lambda x^{2^r+1}\right)$ is also balanced on $\mathbb{F}_{2^{n/2}}$.

Computer experiments have been reported in [278]. For $n = 4$, there exist hyper-bent functions which are not in $\mathcal{PS}_{ap}^{\#}$. Hence, *stricto-sensu*, the set of hyper-bent functions contains strictly $\mathcal{PS}_{ap}^{\#}$, but no other example was found for $n > 4$. See more in [725].

Constructions of hyper-bent functions in univariate trace form and characterizations

The simplest examples of hyper-bent functions (belonging to $\mathcal{PS}_{ap}^{\#}$) in trace form are the (generalized) Dillon monomial functions $tr_n(ax^{r(2^{n/2}-1)})$, $x \in \mathbb{F}_{2^n}$, $a \in \mathbb{F}_{2^n}^*$, $\gcd(r, 2^{n/2}+1) = 1$, where the restriction of $tr_n(ax)$ to U has Hamming weight $2^{n/2-1}$ (see page 240). The bentness (hyper-bentness) of such functions has been studied by several authors: in the case $r = 1$ by Dillon [441], next by Leander [750] and when r is co-prime with $2^{n/2}+1$, by Charpin and Gong [350]:

1. the bentness of $tr_n(ax^{r(2^{n/2}-1)})$ does not depend on the choice of r,
2. it is bent if and only if the Kloosterman sum $\sum_{x \in \mathbb{F}_{2^{n/2}}} (-1)^{tr_{n/2}(a^{2^{n/2}+1}x + \frac{1}{x})}$ equals 0,
3. When bent, $tr_n(ax^{r(2^{n/2}-1)})$ is self-dual.

The other known examples are:

- *Binomial hyper-bent functions* mainly due to S. Mesnager [851, 853] who made deep work on this subject; these functions are the sums of a Dillon monomial function and of a function expressed by means of the trace function over the subfield $\mathbb{F}_4$ of $\mathbb{F}_{2^n}$:

- $tr_n\left(ax^{r(2^{n/2}-1)}\right) + tr_2\left(bx^{\frac{2^n-1}{3}}\right)$, where $a \in \mathbb{F}_{2^n}^*$, $b \in \mathbb{F}_4{}^*$, $\gcd(r, 2^{n/2}+1) = 1$.

When $n/2$ is odd larger than 3, such function is hyper-bent if and only if $\sum_{x\in\mathbb{F}_{2^{n/2}}}(-1)^{tr_{n/2}(a^{2^{n/2}+1}x+\frac{1}{x})}=4$ (and this implies $tr_{n/2}(a^{\frac{2^{n/2}+1}{3}})=0$); the function belongs then to class $\mathcal{PS}^{\#}_{ap}$ (it belongs to $\mathcal{PS}_{ap}$ if $b\in\mathbb{F}_2$). The dual has the same form.
When $n/2$ is even, the characterization of the bentness of this function is an open problem (but we know that $\sum_{x\in\mathbb{F}_{2^{n/2}}}(-1)^{tr_{n/2}(a^{2^{n/2}+1}x+\frac{1}{x})}=4$ is necessary) and it is not known whether the function, when bent, belongs to the class $\mathcal{PS}^-$ or not.

- $tr_n\left(a\zeta^i x^{3(2^{n/2}-1)}\right)+tr_2\left(\beta^j x^{\frac{2^n-1}{3}}\right)$, where $a\in\mathbb{F}^*_{2^{n/2}}$, β is a primitive element of $\mathbb{F}_4$, ζ a generator of the cyclic group of (2^m+1)-th of unity and with $n/2$ odd and not congruent with 3 mod 6, is a *hyper-bent function* if and only if we are in one of the following cases:
  - $tr_{n/2}(a^{1/3})=0$ and $\sum_{x\in\mathbb{F}_{2^{n/2}}}(-1)^{tr_{n/2}(ax+\frac{1}{x})}=4$,
  - $tr_{n/2}(a^{1/3})=1$, $i\in\{1,2\}$, and
$$\sum_{x\in\mathbb{F}_{2^{n/2}}}(-1)^{tr_{n/2}(ax+\frac{1}{x})}+\sum_{x\in\mathbb{F}_{2^{n/2}}}(-1)^{tr_{n/2}(a(x+x^3))}=4.$$

  When these functions are bent, they belong to class $\mathcal{PS}^{\#}$ (and to $\mathcal{PS}_{ap}$ if $b\in\mathbb{F}_2$) and the dual function has the same form.
  Note that $n/2$ being odd, $3(2^{n/2}-1)$ is not a Dillon exponent because 3 divides $2^{n/2}+1$, contrary to when $n/2$ is even; hence this second class is not included in the first class.

In [871] is studied the hyper-bentness of more general binomial functions and obtain a long list of (potentially new) hyper-bent functions.

- *Polynomial hyper-bent functions*:

- in [350] in the form $\sum_{r\in R}tr_n(\beta_r x^{r(2^{n/2}-1)})$, $\beta_r\in\mathbb{F}_{2^n}$, where R is a set of representatives of full size[29] cyclotomic cosets modulo $2^{n/2}+1$, with a characterization of hyper-bentness by means of Dickson polynomials (see also [475, 782]). When r is co-prime with $2^{n/2}+1$, the functions are the sums of several Dillon monomial functions.

- In [546] in the form:
  - $\sum_{i=1}^{2^{n/2-1}-1}tr_n\left(\beta x^{i(2^{n/2}-1)}\right)$, where $\beta\in\mathbb{F}_{2^{n/2}}\setminus\mathbb{F}_2$,
  - $\sum_{i=1}^{2^{n/2-2}-1}tr_n\left(\beta x^{i(2^{n/2}-1)}\right)$, where $n/2$ is odd, and $\beta^{(2^{n/2}-4)^{-1}}\in\{x\in\mathbb{F}^*_{2^{n/2}};tr_{n/2}(x)=0\}$.
- in [852] (in [350] for $b=0$) in the form $\sum_{r\in R}tr_n(a_r x^{r(2^{n/2}-1)})+tr_2(bx^{\frac{2^n-1}{3}})$, $x\in\mathbb{F}_{2^n}$, $b\in\mathbb{F}_4$;

[29] It has been shown later in [871] that it is enough to assume that the size does not divide $n/2$.

hyper-bentness can be characterized by means of exponential sums involving Dickson polynomials (see also [511]);
when b is a primitive element of $\mathbb{F}_4$, the condition reduces to the evaluation of the Hamming weight of some Boolean functions.

- in [871] in the form $\sum_{r\in R} tr_n(a_r x^{r(2^{n/2}-1)}) + tr_t(bx^{s(2^{n/2}-1)})$, where
  - R is a set of representatives of the cyclotomic classes modulo $2^{n/2}+1$ (not necessarily of maximal size)
  - the coefficients a_r are in $\mathbb{F}_{2^{n/2}}$,
  - s divides $2^{n/2}+1$, i.e $s(2^{n/2}-1)$ is a Dillon-like exponent; we set $\tau = \frac{2^{n/2}+1}{s}$,
  - t is the size of the cyclotomic coset of s modulo $2^{n/2}+1$,
  - $b \in \mathbb{F}_{2^t}$,

  but the characterization of hyper-bentness in terms of exponential sums is so complex that no new hyper-bent function could be deduced except in some particular cases.
- in [1062], more Dillon exponent hyper-bent functions (see also [764]), with coefficients in $\mathbb{F}_{2^n}$ (with a general result unifying results from the references above), and generalized exponents in [1063].

See also [512].

Homogeneous bent functions

Definition 58 *[975] A Boolean function is called a* homogeneous function *if all the monomials of its* algebraic normal form *have the same degree.*

In [347], Charnes, Rötteler and Beth showed how to use invariant theory to construct homogeneous bent functions. They showed connections between homogeneous *cubic functions* and 1-designs and certain graphs and proved that there exist cubic homogeneous bent functions in each even number of variables $n \geq 6$. They studied the equivalence between the constructed bent functions and the properties of the associated elementary Abelian difference sets. It is proved in [1126] that no homogeneous bent function of degree $\frac{n}{2}$ exists in n variables for $n > 6$, and in [848] that, for any non-negative integer k, if n is large enough, there exists no homogeneous bent function in n variables having degree $\frac{n}{2} - k$ at least. Partial results towards a conjectured nonexistence of homogeneous rotation symmetric bent functions (see below) having algebraic degree larger than 2 have been obtained in [847].

Rotation symmetric bent functions and idempotent bent functions

Symmetry, that is, invariance under any permutation of input variables, simplifies the study of Boolean functions, but all symmetric Boolean bent functions (see Section 10.1, page 383) are quadratic and belong then to one EA equivalence class of Boolean functions. The super-class of *rotation symmetric* (RS) Boolean functions has then been introduced by Pieprzyk and Qu in [954].

Definition 59 *Let n be any positive integer. A Boolean function over $\mathbb{F}_2^n$ is called rotation symmetric (RS) if it is invariant under any cyclic shift of input coordinates, which is equivalent to saying that it is invariant under a primitive cyclic shift, for instance: $(x_0, x_1, \ldots, x_{n-1}) \to (x_{n-1}, x_0, x_1, \ldots, x_{n-2})$.*

RS functions are in fact linked to a notion which had been anteriorly introduced by Filiol and Fontaine in [503, 515] as observed by them:

Definition 60 *Let n be any positive integer. A Boolean function f on $\mathbb{F}_{2^n}$ is called an* idempotent function *(or briefly an idempotent) if it satisfies $f(x) = f(x^2)$, for all $x \in \mathbb{F}_{2^n}$.*

Note that a Boolean function given in univariate form $f(x) = \sum_{j=0}^{2^n-1} \delta_j \, x^j$ (or in subfield trace representation, see page 61) is an idempotent if and only if every coefficient δ_j belongs to $\mathbb{F}_2$. The link between RS functions and idempotents is through normal bases. Recall that for every n, there exists a primitive element α in $\mathbb{F}_{2^n}$ such that $(\alpha, \alpha^2, \alpha^{2^2}, \ldots, \alpha^{2^{n/2-1}})$ is a basis of the vector space $\mathbb{F}_{2^n}$ (see [775, 890]). Such basis is called a *normal basis*.

Proposition 89 *For any Boolean function $f(x)$ over $\mathbb{F}_{2^n}$, and every normal basis $(\alpha, \alpha^2, \ldots, \alpha^{2^{n-1}})$ of $\mathbb{F}_{2^n}$, the function*

$$(x_0, \ldots, x_{n-1}) \in \mathbb{F}_2^n \mapsto f\left(\sum_{i=0}^{n-1} x_i \alpha^{2^i}\right)$$

is RS if and only if f is an idempotent.

This is easily proved. Hence the two notions are theoretically equivalent (but knowing infinite classes for each notion is not equivalent). Proposition 89 leads to a notion of circulant equivalence of RS functions, see *e.g.* [245].
The *bivariate representation* and more general k-variate representation of RS functions and of idempotent functions is studied in [281], where the link between these notions is studied further, see Section 10.2, page 392.

Quadratic RS functions and idempotents:
The purely quadratic part of any quadratic RS Boolean function has the form:

$$\bigoplus_{i=1}^{n/2-1} c_i \left(\bigoplus_{j=0}^{n-1} x_j x_{i+j}\right) \oplus c_{n/2} \left(\bigoplus_{j=0}^{n/2-1} x_j x_{n/2+j}\right), \tag{6.33}$$

where $c_1, \ldots, c_{n/2} \in \mathbb{F}_2$ and where the indices of x are modulo n. We have:

Proposition 90 *[531] Let n be any even integer. Any RS quadratic function (6.33) is bent if and only if the polynomial $P(X) = \sum_{i=1}^{n/2-1} c_i(X^i + X^{n-i}) + c_{n/2}X^{n/2}$ is co-prime with $X^n + 1$, that is, the linearized polynomial $L(X) = \sum_{i=1}^{n/2-1} c_i(X^{2^i} + X^{2^{n-i}}) + c_{n/2}X^{2^{n/2}}$ is a permutation polynomial.*

Indeed, according to the characterization of quadratic bent functions recalled at page 229, the function is bent if and only if the matrix of its associated symplectic form is non-singular, that is, the *cyclic code* generated by the rows of this matrix equals $\mathbb{F}_2^n$, and the generator polynomial of this code equals $\gcd(\sum_{i=1}^{n/2-1} c_i(X^i + X^{n-i}) + c_{n/2}X^{n/2}, X^n+1)$.

Infinite classes of bent quadratic RS functions have been deduced:

- $\bigoplus_{j=0}^{n/2-1} x_j x_{n/2+j}$ (and we can add $h(x_0 \oplus x_{n/2}, \dots, x_{n/2-1} \oplus x_{n-1})$ to this Maiorana-McFarland function, where h is any RS function, as observed in [1065]),
- $\bigoplus_{i=1}^{n/2-1}(\bigoplus_{j=0}^{n-1} x_j x_{i+j}) \oplus (\bigoplus_{j=0}^{n/2-1} x_j x_{n/2+j})$.

These two examples correspond to $c_{n/2} = 1$ and $c_i = 0$ for $i \neq n/2$ in the former case and $c_i = 1$ for $i = 1, \dots, n/2-1$ in the latter case. Note that $L(X)$ equals $X^{2^{n/2}}$ is the former case and $X + tr_n(X)$ in the latter case, and these are permutation polynomials since n is even; equivalently $P(X)$ equals $X^{n/2}$ in the former case and $\sum_{i=1}^{n-1} X^i$ in the latter case, and these are co-prime with X^n+1. More examples can be found as observed in [245]. For instance, let k be such that 2^k-2 divides n (and 2^k-1 is co-prime with n). Then $\left(\frac{X^{2^k-1}+1}{X+1}\right)^{\frac{n}{2^k-2}} + X^n + 1$ has the form $\sum_{i=1}^{n/2-1} c_i(X^i + X^{n-i}) + c_{n/2}X^{n/2}$ (indeed, $\left(\frac{X^{2^k-1}+1}{X+1}\right)^{\frac{n}{2^k-2}}$ is self-reciprocal, has degree n and is normalized) and is co-prime with X^n+1 (indeed, the zeros of $\left(\frac{X^{2^k-1}+1}{X+1}\right)^{\frac{n}{2^k-2}}$ in the algebraic closure of $\mathbb{F}_2$ are the elements of $\mathbb{F}_{2^k} \setminus \mathbb{F}_2$ and for any $\xi \in \mathbb{F}_{2^k} \setminus \mathbb{F}_2$ we have $\xi^n + 1 \neq 0$, since $\xi \mapsto \xi^n$ is a permutation of $\mathbb{F}_{2^k}^*$). Taking for example $k=2$ we have $\left(X^2+X+1\right)^{n/2} + X^n + 1 = \sum_{\substack{0 \le u,v,w \le n/2 \\ u+v+w=n/2, 2u+v \notin \{0,n\}}} \frac{(n/2)!}{u!v!w!} X^{2u+v}$, and for n not divisible by 3, the following function is RS bent:

$$\bigoplus_{\substack{0 \le u,v,w \le n/2 \\ u+v+w=n/2, 2u+v \in \{1,\dots,n/2-1\}}} \frac{(n/2)!}{u!v!w!} (\bigoplus_{j=0}^{n-1} x_j x_{2u+v+j}) \oplus (\bigoplus_{j=0}^{n/2-1} x_j x_{n/2+j}),$$

where the coefficients are taken modulo 2.

Another example is as follows. If n is a power of 2, then according to [1043, Proposition 3.1], the function $\bigoplus_{i=1}^{n/2-1} c_i(\bigoplus_{j=0}^{n-1} x_j x_{i+j}) \oplus c_{n/2}(\bigoplus_{j=0}^{n/2-1} x_j x_{n/2+j})$ is bent if and only if $\bigoplus_{i=0}^{n-1} c_i = 1$ (with $c_{n-i} = c_i$), that is, $c_{n/2} = 1$. See more in [245].

Quadratic bent idempotents have been also characterized: as shown in [808], for $c_1, \dots, c_{n/2} \in \mathbb{F}_2$, the function equal to $\sum_{i=1}^{n/2-1} c_i tr_n(x^{2^i+1}) + c_{n/2} tr_{n/2}(x^{2^{n/2}+1})$ is bent if and only if $gcd(\sum_{i=1}^{n/2-1} c_i(X^i + X^{n-i}) + c_{n/2}X^{n/2}, X^n+1) = 1$ (and necessarily, $c_{n/2} = 1$). This condition is the same as that obtained for quadratic

RS bent functions above. The infinite classes of RS bent functions seen above provide the following bent idempotents:

- the bent quadratic monomial idempotent $f'(x) = tr_{n/2}(x^{2^{n/2}+1})$,
- functions $f'(x) = \sum_{i=1}^{n/2-1} tr_n(x^{2^i+1}) + tr_{n/2}(x^{2^{n/2}+1})$,
- for n a power of 2, all nonzero quadratic idempotents,
- for n not divisible by 3, functions:
$$tr_{n/2}(z^{2^{n/2}+1}) + \sum_{\substack{0\le u,v,w\le n/2 \\ u+v+w=n/2,2u+v\in\{1,\dots,n/2-1\}}} \frac{(n/2)!}{u!v!w!} tr_n(z^{2^{2u+v}+1}),$$
where the coefficients are taken modulo 2 [245]. Of course, what is written above for RS functions when n is a power of 2 is valid here.
- More results can be found in [1144].

The similarities between the quadratic RS bent functions and the quadratic bent idempotents seen above leads to considering below a transformation of RS functions into idempotents. Before that, let us recall what is known for non-quadratic functions.

Non-quadratic RS functions and idempotents:
Two infinite classes of cubic RS bent functions (belonging to the completed Maiorana-McFarland class) are:

- $\bigoplus_{i=0}^{n-1}(x_i x_{t+i} x_{n/2+i} \oplus x_i x_{t+i}) \oplus \bigoplus_{i=0}^{n/2-1} x_i x_{n/2+i}$, where $\frac{n/2}{\gcd(n/2,t)}$ is odd [531] (and here also we can of course add $h(x_0 \oplus x_{n/2}, \dots, x_{n/2-1} \oplus x_{n-1})$ to this MM function, where h is any RS function, [1065]);
- $\bigoplus_{i=0}^{n-1} x_i x_{i+r} x_{i+2r} \oplus \bigoplus_{i=0}^{2r-1} x_i x_{i+2r} x_{i+4r} \oplus \bigoplus_{i=0}^{n/2-1} x_i x_{i+n/2}$, where $n/2 = 3r$ [282].

The Dillon and Kasami power functions with coefficient 1, and the Niho bent functions $tr_{n/2}(z^{2^{n/2}+1}) + tr_n(z^{d_2})$ (see page 246) are bent idempotents. The extension of the second class of Niho bent functions by Leander and Kholosha gives also a bent idempotent.
For $n = 6r$, $r \ge 1$, $tr_n(z^{1+2^r+2^{2r}}) + tr_{2r}(z^{1+2^{2r}+2^{4r}}) + tr_{3r}(z^{1+2^t}) = tr_r((z + z^{2^{3r}})^{1+2^r+2^{2r}}) + tr_{3r}(z^{1+2^t})$ is a bent idempotent [282].
More bent idempotents of any algebraic degrees between 2 and $n/2$ are given in [1066] in the form $g(x) \oplus h(tr_n(\alpha x), tr_n(\alpha^2 x), \dots, tr_n(\alpha^{2^{n/2-1}} x))$, where g is an n-variable bent function satisfying a strong condition and h is an $n/2$-variable rotation symmetric function.

Remark. The generalized Dillon and Mesnager functions could be viewed as bent idempotent candidates, but the conditions happen not to be satisfiable: it is known that

- for every $m = n/2$ such that $K_m(1)$ is null, $g_1(x) = tr_n(x^{r(2^m-1)})$ is bent when $\gcd(r, 2^m+1) = 1$,
- for every $m = n/2$ odd such that $K_m(1) = 4$, $g_2(x) = tr_n(x^{r(2^m-1)}) + tr_2(x^{\frac{2^n-1}{3}})$ is bent when $\gcd(r, 2^m+1) = 1$;

but the condition $K_m(1) = 0$ never happens as shown in [783, Theorem 2.2] and it can be checked by computer that the condition $K_m(1) = 4$ never happens as well for $5 \leq m \leq 20$. □

Other non-quadratic functions:
A *secondary construction* of rotation symmetric functions (and equivalently of idempotent bent functions) from near-bent RS functions (the definition of near-bent functions is given in Subsection 6.2.4, page 289) based on the indirect sum (see page 259) is given in [281] (see also [245]): let f_1 and f_2 be two m-variable RS near-bent functions (m odd); if the Walsh supports of f_1 and f_2 are complementary, then function

$$h(x_0, y_1, x_2, y_3, \ldots, x_{n-2}, y_{n-1}) = f_1(x_0, x_1, \ldots, x_{m-1}) \oplus f_1(y_0, y_1, \ldots, y_{m-1}) \oplus (f_1 \oplus f_2)(x_0, x_1, \ldots, x_{m-1})(f_1 \oplus f_2)(y_0, y_1, \ldots, y_{m-1})$$

is bent RS. This provides constructions of RS functions and idempotent bent functions of *algebraic degree* 4, for m odd: given the two RS functions $f_1(x) = \bigoplus_{i=0}^{m-1}(x_i \oplus x_i x_{(m-1)/2+i})$ and $f_2(x) = \bigoplus_{i=0}^{m-1} x_i x_{1+i}$, where the subscripts are taken modulo m, function $h(x_0, y_1, x_2, y_3, \ldots, x_{n-2}, y_{n-1}) = f_1(x_0, \ldots, x_{m-1}) \oplus f_1(y_0, \ldots, y_{m-1}) \oplus (f_1 \oplus f_2)(x_0, \ldots, x_{m-1})(f_1 \oplus f_2)(y_0, \ldots, y_{m-1})$ is an RS bent function. Similarly, given the m-variable idempotent functions $f_1(x) = tr_m(x) + tr_m(x^{2^{(m-1)/2}+1})$ and $f_2(x) = tr_m(x^3)$, function $h(x, y) = f_1(x) \oplus f_1(y) \oplus (f_1 \oplus f_2)(x)\,(f_1 \oplus f_2)(y)$ is a bent idempotent.
Su and Tang [1054] have proposed, for any even n, constructions of rotation symmetric bent functions with any possible algebraic degree ranging from 2 to $n/2$, obtained by the modification of quadratic symmetric bent functions, and of bent idempotent functions of algebraic degree $n/2$, obtained by the modification of the bent quadratic monomial idempotent (see page 277).

A transformation:
As observed with quadratic RS functions and idempotents, there is a natural way of transforming a RS function into an idempotent: let $f(x_0, \cdots, x_{n-1}) = \sum_{u \in \mathbb{F}_2^n} a_u \prod_{i=0}^{n-1} x_i^{u_i}$, $a_u \in \mathbb{F}_2$, be any Boolean RS function over $\mathbb{F}_2^n$, then $f'(x) = f(x, x^2, \cdots, x^{2^{n-1}}) = \sum_{u \in \mathbb{F}_2^n} a_u x^{\sum_{i=0}^{n-1} u_i 2^i}$ is a Boolean idempotent, and any idempotent Boolean function can be obtained this way. We have seen that if f is a quadratic RS function, then f is bent if and only if f' is bent[30]. But for non-quadratic functions, it is shown in [281, 282] that all cases can happen: examples

[30] Note that if $n \equiv 2$ [mod 4], then there exists a self-dual normal basis of $\mathbb{F}_{2^n}$ and that f' expressed over $\mathbb{F}_2^n$ by means of such basis is then the same function as f; this is also the case if n is odd.

are given of an infinite class of cubic bent RS functions f such that f' is not bent, of an infinite class of cubic bent idempotents f' such that f is not bent, and of infinite classes of bent RS functions f such that f' is bent.

6.1.21 Normal and non-normal bent Boolean functions

We have seen the definition of normal functions in Definition 28, page 126.
As observed in [212] (see Theorem 14, page 226), if a bent function f is normal (resp. weakly-normal), that is, constant (resp. affine) on an $\frac{n}{2}$-dimensional flat $b+E$, where E is a subspace of $\mathbb{F}_2^n$, then its dual $\widetilde{f}$ is such that $\widetilde{f}(u) \oplus b \cdot u$ is constant on $E^\perp$ (resp. on $a + E^\perp$, where a is a vector such that $f(x) \oplus a \cdot x$ is constant on E). Thus, $\widetilde{f}$ is weakly-normal. Moreover, we have already seen that f (resp. $f(x) \oplus a \cdot x$) is balanced on each of the other cosets of the flat.
H. Dobbertin used normal bent functions to construct *balanced* functions with high nonlinearities: take a bent function f in n variables which is constant on an $n/2$-dimensional flat A of $\mathbb{F}_2^n$; replace the values of f on A by the values of a highly nonlinear balanced function on A (identified to a function g on $\mathbb{F}_2^{n/2}$); note that this process is recursive since such $n/2$-variable Boolean function g can be obtained by the same process (as long as $n/2$ is even) with n replaced by $n/2$; when n becomes odd (say $n = 2k + 1$), replace the constant value by a balanced function of best known nonlinearity nl_{2k+1} (larger than or equal to $2^{2k} - 2^k$); this provides a balanced function (as we shall see in Proposition 121, page 325) whose nonlinearity equals $2^{n-1} - 2^{n/2-1} - \cdots - 2^{2k} + (nl_{2k+1} - 2^{2k}) \geq 2^{n-1} - 2^{n/2-1} - \cdots - 2^{2k} - 2^k$.
The existence of non-normal (and even non-weakly-normal) bent functions, *i.e.* bent functions which are non-constant (resp. non-affine) on every $\frac{n}{2}$-dimensional flat, has been shown, contradicting a conjecture made by several authors that such bent function did not exist. It is proved in [448] that the so-called Kasami function defined over $\mathbb{F}_{2^n}$ by $f(x) = tr_n\left(ax^{2^{2k}-2^k+1}\right)$, with $\gcd(k, n) = 1$, is bent if n is not divisible by 3 and if $a \in \mathbb{F}_{2^n}$ is not a cube. As shown in [198] (thanks to [412]), if $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$ and $k = 3$, then for $n = 10$, the function $f(x) \oplus tr_n(bx)$ is non-normal for some b, and for $n = 14$, the function $f(x)$ is not weakly normal (while the Kasami function is normal for n divisible by 4 or $k = 1$). A non-normal bent function in 12 variables is given in [278]. Cubic bent functions on 8 variables are all normal, as shown in [349].
The direct sum (see definition in Subsection 6.1.16) of two normal functions is obviously a normal function, while the direct sum of two non-normal functions can be normal. What about the sum of a normal bent function and of a non-normal bent function? This question has been studied in [270]. To this aim, a notion more general than normality has been introduced as follows:

Definition 61 *Let $U \subseteq V$ be two vector spaces over $\mathbb{F}_2$. Let $\beta : U \to \mathbb{F}_2$ and $f : V \to \mathbb{F}_2$ be bent functions. Then we say that f is a* normal extension *of β,* *in symbols* $\beta \preceq f$*, if there is a direct decomposition* $V = U \oplus W_1 \oplus W_2$ *such that (i)* $\beta(u) = f(u + w_1)$ *for all* $u \in U$*,* $w_1 \in W_1$*, and (ii)* $\dim W_1 = \dim W_2$*.*

Obviously, we get a normal extension of any β by taking any normal bent function g and making its direct sum with β. The relation $\preceq$ is transitive and if $\beta \preceq f$ then the same relation exists between the duals: $\widetilde{\beta} \preceq \widetilde{f}$.
A bent function is normal if and only if $\epsilon \preceq f$, where $\epsilon \in \mathbb{F}_2$ is viewed as a Boolean function over the vector space $\mathbb{F}_2^0 = \{0\}$.
Examples of normal extensions are given in [270] (some by the construction of Theorem 15, page 260, and its particular cases, the indirect sum and the extension of Maiorana-McFarland type).
The clarification about the sum of a normal bent function and of a non-normal bent function comes from the two following propositions (see the proofs in [270]):

Proposition 91 *Let* $f_i : V_i \to \mathbb{F}_2$*,* $i = 1, 2$*, be bent functions. The direct sum* $f_1 \oplus f_2$ *is normal if and only if bent functions* β_1 *and* β_2 *exist such that* f_i *is a normal extension of* β_i $(i = 1, 2)$ *and either* β_1 *and* β_2 *or* β_1 *and* $\beta_2 \oplus 1$ *are linearly equivalent.*

Proposition 92 *Suppose that* $\beta \preceq f$ *for bent functions* β *and* f*. If* f *is normal, then also* β *is normal.*

Hence, since the direct sum of a bent function β and of a normal bent function g is a normal extension of β, the direct sum of a normal and a non-normal bent function is always non-normal.
Normal extension leads to a secondary construction of bent functions:

Proposition 93 *Let* β *be a bent function on* U *and* f *a bent function on* $V = U \times W \times W$*. Assume that* $\beta \preceq f$*. Let*

$$\beta' : U \to \mathbb{F}_2$$

be any bent function. Modify f *by setting for all* $x \in U$*,* $y \in W$

$$f'(x, y, 0) = \beta'(x),$$

while $f'(x, y, z) = f(x, y, z)$ *for all* $x \in U$*,* $y, z \in W$*,* $z \neq 0$*. Then* f' *is bent and we have* $\beta' \preceq f'$*.*

Hence, we can replace β by any other bent function on U and get again a normal extension.

6.1.22 Kerdock codes

For every even n, the *Kerdock code* $\mathcal{K}_n$ [689] is a supercode of $RM(1, n)$ (*i.e.* contains $RM(1, n)$ as a subset) and is a subcode of $RM(2, n)$. More precisely $\mathcal{K}_n$ is a union of cosets $f_u \oplus RM(1, n)$ of $RM(1, n)$, where the functions f_u are quadratic (one of them is null and all the others have algebraic degree 2). The difference $f_u \oplus f_v$ between two distinct functions f_u and f_v being bent, $\mathcal{K}_n$ has minimum distance $2^{n-1} - 2^{\frac{n}{2}-1}$ (n even), which is the best possible minimum distance for a code equal to a union of cosets of $RM(1,n)$, according to the covering radius bound. The size of $\mathcal{K}_n$ equals 2^{2n}. This is the best possible size for such length and minimum distance (see [422, 177]). The Kerdock code of length 16 is called the *Nordstrom-Robinson code*. We describe now how the construction of Kerdock codes can be simply presented.

Construction of the Kerdock code

We revisit Kerdock's construction, which was presented by means of idempotents, that we shall not need here. The function already seen at page 230:

$$f(x) = \sigma_2(x) = \binom{w_H(x)}{2} \text{ [mod 2]} = \bigoplus_{1 \le i < j \le n} x_i x_j \tag{6.34}$$

is bent. Thus, the linear code $RM(1,n) \cup (f \oplus RM(1,n))$ has minimum distance $2^{n-1} - 2^{\frac{n}{2}-1}$.

We have recalled at page 59 and foll. and at page 275 some properties of the field $\mathbb{F}_{2^m}$ (where m is any positive integer). In particular, we have seen that $\mathbb{F}_{2^m}$ admits normal bases $(\alpha, \alpha^2, \dots, \alpha^{2^{m-1}})$. If m is odd, there exists a self-dual normal basis, that is, a normal basis such that $tr_m(\alpha^{2^i+2^j}) = 1$ if $i = j$ (that is, $tr_m(\alpha) = 1$) and $tr_m(\alpha^{2^i+2^j}) = 0$ otherwise (see [775, 890]). As a consequence, for all $x = x_1\alpha + \cdots + x_m\alpha^{2^{m-1}}$ in $\mathbb{F}_{2^m}$, we have

$$tr_m(x) = \bigoplus_{i=1}^{m} x_i \qquad tr_m(x^{2^j+1}) = \bigoplus_{i=1}^{m} x_i x_{i+j},$$

(where $i + j$ is taken mod m).

The function f of Relation (6.34), viewed as a function $f(x, x_n)$ on $\mathbb{F}_{2^m} \times \mathbb{F}_2$, where $m = n - 1$ is odd – say $m = 2t + 1$ – can now be written as:

$$f(x, x_n) = tr_m\Big(\sum_{j=1}^{t} x^{2^j+1}\Big) + x_n tr_m(x),$$

and this expression can be taken as the definition of f. Notice that the associated symplectic form $\beta_f((x, x_n), (y, y_n))$ associated to f equals $tr_m(x)tr_m(y) + tr_m(xy) + x_n tr_m(y) + y_n tr_m(x)$.

Let us denote $f(ux, x_n)$ by $f_u(x, x_n)$ ($u \in \mathbb{F}_{2^m}$), then $\mathcal{K}_n$ is defined as the union, when u ranges over $\mathbb{F}_{2^m}$, of the cosets $f_u + RM(1,n)$.

$\mathcal{K}_n$ contains all 2^{n+1} affine functions (since for $u = 0$, we have $f_u = 0$) and $2^{2n} - 2^{n+1}$ quadratic bent functions. Its minimum distance equals $2^{n-1} - 2^{\frac{n}{2}-1}$ since the sum of two distinct functions f_u and f_v is bent. Indeed, the kernel of the associated symplectic form equals the set of all ordered pairs (x, x_n) such that $tr_m(ux)tr_m(uy) + tr_m(u^2xy) + x_n tr_m(uy) + y_n tr_m(ux) = tr_m(vx)tr_m(vy) + tr_m(v^2xy) + x_n tr_m(vy) + y_n tr_m(vx)$ for every $(y, y_n) \in \mathbb{F}_{2^m} \times \mathbb{F}_2$, which is equivalent to $u tr_m(ux) + u^2x + x_n u = v tr_m(vx) + v^2x + x_n v$ and $tr_m(ux) = tr_m(vx)$;

it is a simple matter to show that it equals $\{(0,0)\}$.
A more general approach to the construction of Kerdock codes is developed in [327].

Open problem: Other examples of codes having the same parameters exist, see [657] (see also [658] and observations in [72, 208, 217]). All are equal to subcodes of the Reed-Muller code of order 2, up to affine equivalence. We do not know how to obtain the same parameters with non-quadratic functions (up to code equivalence). This would be useful for cryptographic purposes and for the design of sequences for code division multiple access (CDMA) in telecommunications.

Remark.
The Kerdock codes are not linear. However, they share some nice properties with linear codes: the distance distribution between any codeword and all the other codewords does not depend on the choice of the codeword (we say that the Kerdock codes are *distance-invariant*; this results in the fact that their *distance enumerators* are equal to their weight enumerators); and, as proved by Semakov and Zinoviev [1029], the weight enumerators of the Kerdock codes satisfy a MacWilliams-like relation, similar to Relation (1.1), page 30, in which C is replaced by $\mathcal{K}_n$ and $C^\perp$ is replaced by the so-called Preparata code [43] of the same length (we say that the Kerdock codes and the Preparata codes are formally dual). An explanation of this astonishing property has been given in [586]: the Kerdock code is stable under an addition inherited of the addition in $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ (we say it is $\mathbb{Z}_4$-linear) and the Mac Williams identity still holds in this different framework. Such an explanation had been an open problem for two decades. $\square$

6.2 Partially-bent and plateaued Boolean functions

We have seen that bent Boolean functions can never be balanced, which makes them improper for a direct cryptographic use. This has led to a research on super-classes of the class of bent functions, whose elements can have high nonlinearities, but can also be balanced[31] (and possibly, be resilient).

6.2.1 Partially-bent functions

A first super-class of possibly balanced functions with high nonlinearity has been obtained as the set of those functions which achieve a bound conjectured by B. Preneel in [969] and expressing some trade-off between the number of unbalanced derivatives (*i*.e. of nonzero autocorrelation coefficients) of a Boolean function and the number of nonzero values of its Walsh transform.

[31] The functions found will however still have bounded algebraic degree, which is cryptographically crippling in many situations.

Proposition 94 *[211] Let n be any positive integer. Let f be any Boolean function on $\mathbb{F}_2^n$. Let us denote the cardinalities of the sets $\{b \in \mathbb{F}_2^n \mid \mathcal{F}(D_b f) \neq 0\}$ and $\{a \in \mathbb{F}_2^n \mid W_f(a) \neq 0\}$ by N_{Δ_f} and N_{W_f}, respectively. Then:*

$$N_{\Delta_f} \times N_{W_f} \geq 2^n. \tag{6.35}$$

Moreover, $N_{\Delta_f} \times N_{W_f} = 2^n$ if and only if, for every $b \in \mathbb{F}_2^n$, the derivative $D_b f$ is either balanced or constant. This property is also equivalent to the fact that there exist two linear subspaces E (of even dimension) and E' of $\mathbb{F}_2^n$, whose direct sum equals $\mathbb{F}_2^n$, and Boolean functions g, bent on E, and h, affine on E', such that:

$$\forall x \in E,\, \forall y \in E',\, f(x+y) = g(x) \oplus h(y). \tag{6.36}$$

Inequality (6.35) comes directly from the Wiener-Khintchine Relation (2.53), page 80: since the value of the autocorrelation coefficient $\mathcal{F}(D_b f)$ lies between -2^n and 2^n for every $b \in \mathbb{F}_2^n$, the arithmetic mean of $(-1)^{u\cdot b}\mathcal{F}(D_b f)$ when b ranges over the set $\{b \in \mathbb{F}_2^n \mid \mathcal{F}(D_b f) \neq 0\}$ is at most 2^n, for every $u \in \mathbb{F}_2^n$, and we have then $N_{\Delta_f} \geq 2^{-n} \sum_{b\in\mathbb{F}_2^n} (-1)^{u\cdot b}\mathcal{F}(D_b f) = 2^{-n}\, W_f^2(u)$ and thus $N_{\Delta_f} \geq 2^{-n} \max_{u\in\mathbb{F}_2^n} W_f^2(u)$. Moreover, we have $N_{W_f} \geq \frac{\sum_{u\in\mathbb{F}_2^n} W_f^2(u)}{\max_{u\in\mathbb{F}_2^n} W_f^2(u)} = \frac{2^{2n}}{\max_{u\in\mathbb{F}_2^n} W_f^2(u)}$. This proves Inequality (6.35).
This inequality is an equality if and only if both inequalities above are equalities, that is, for every $b \in \mathbb{F}_2^n$, the autocorrelation coefficient $\mathcal{F}(D_b f)$ equals 0 or $2^n(-1)^{u_0\cdot b}$, where $\max_{u\in\mathbb{F}_2^n} W_f^2(u) = W_f^2(u_0)$ (and this implies that, for every $b \in \mathbb{F}_2^n$, $D_b f$ is either balanced or constant) and f is plateaued (see page 285).
The single condition that $D_b f$ is either balanced or constant for every b implies that f has the form (6.36). Indeed, let E be any supplementary space of the linear kernel $\mathcal{E}_f$, then E having trivial intersection with $\mathcal{E}_f$, the restriction of f to E has balanced derivatives (their balancedness over E being equivalent to their balancedness over $\mathbb{F}_2^n$) and is then bent and f has the form (6.36) with $E' = \mathcal{E}_f$. Then it is easily seen that (6.35) is an equality. This completes the proof. $\square$
See some more properties in [338].

A generalization of Relation (6.35) to *pseudo-Boolean* functions has been obtained in [986].

Definition 62 *The n-variable Boolean functions such that (6.35) is an equality, that is, whose derivatives are all either balanced or constant, that is, the functions of the form (6.36), are called* partially-bent function*s.*

Bounds similar to Relation (6.35) but different are obtained in [1178] and lead to other characterizations of partially-bent functions.
Every quadratic function is partially-bent. Partially-bent functions share with *quadratic functions* almost all of their nice properties (Walsh spectrum easier to calculate, potential good nonlinearity and good resiliency order), see [211] where the cryptographic properties of partially-bent functions are characterized. In particular, the values of the Walsh transform equal 0 or $\pm 2^{dim(E')+dim(E)/2}$.

The support of such plateaued function is a coset (*i.e.* a translate) of E. Note that, viewing a function of the form (6.36) as a bivariate function, its Walsh transform equals $W_f(u,v) = W_g(u)W_h(v)$.

Instead of using Relation (2.53), we can use Relations (3.7), page 118 and (3.10), page 119. We have then $N_{\Delta_f} \geq 2^{-2n}\sum_{b\in\mathbb{F}_2^n}\mathcal{F}^2(D_bf) = 2^{-2n}\mathcal{V}(f)$ and $N_{W_f} \leq \frac{\sum_{u\in\mathbb{F}_2^n} W_f^4(u)}{\min\{W_f^4(u);\, u\in\mathbb{F}_2^n,\, W_f(u)\neq 0\}} = \frac{2^n\,\mathcal{V}(f)}{\min\{W_f^4(u);\, u\in\mathbb{F}_2^n,\, W_f(u)\neq 0\}}$, and therefore:

Proposition 95 *Let n be any positive integer. Let f be any Boolean function on $\mathbb{F}_2^n$. With the same notation as in Proposition 94, we have:*

$$\frac{N_{\Delta_f}}{N_{W_f}} \geq 2^{-3n}\min\{W_f^4(u);\, u\in\mathbb{F}_2^n,\, W_f(u)\neq 0\},$$

with equality if and only if f is partially-bent.

We can also use Relations (3.9) and (3.10). Denoting by $N_{\Delta_f^{(2)}}$ the size of the set $\{(a,b)\in(\mathbb{F}_2^n)^2 \mid \mathcal{F}(D_aD_bf)\neq 0\}$, we have then $N_{\Delta_f^{(2)}} \geq 2^{-n}\sum_{a,b\in\mathbb{F}_2^n}\mathcal{F}(D_aD_bf) = 2^{-n}\mathcal{V}(f)$ and $N_{W_f} \leq \frac{2^n\,\mathcal{V}(f)}{\min\{W_f^4(u);\, u\in\mathbb{F}_2^n,\, W_f(u)\neq 0\}}$, and therefore:

$$\frac{N_{\Delta_f^{(2)}}}{N_{W_f}} \geq 2^{-2n}\min\{W_f^4(u);\, u\in\mathbb{F}_2^n,\, W_f(u)\neq 0\}, \tag{6.37}$$

with equality if and only if both inequalities are equalities, which is equivalent to the fact that all second-order derivatives of f are either balanced or equal to the constant function 0 and that f is plateaued. We leave open the determination of such functions.

The functions achieving (6.37) with equality seem somewhat related to the so-called *second-order bent functions* introduced in [275], which are by definition those Boolean functions such that, for every $\mathbb{F}_2$-linearly independent elements $a, b\in\mathbb{F}_2^n$ (*i.e.* $a\neq 0_n, b\neq 0_n, a\neq b$), D_aD_bf is balanced (which is a more demanding condition on the second-order derivatives but does not require that f be plateaued). In fact, there is no intersection between the two sets of functions, because no second-order bent function can be plateaued. Indeed, it is shown in [275] that f is second-order bent if and only if, for all $b, c\in\mathbb{F}_2^n$, we have:

$$\sum_{u\in\mathbb{F}_2^n} W_f(u+b+c)W_f(u+b)W_f(u+c)W_f(u) = \begin{cases} -2^{2n+1} & \text{if } b\neq 0_n, c\neq 0_n, b\neq c,\\ 3\cdot 2^{3n}-2^{2n+1} & \text{if } b=c=0_n,\\ 2^{3n}-2^{2n+1} & \text{otherwise}.\end{cases}$$

Then taking $b=c=0_n$, we see that if f is plateaued, its amplitude (see Definition 63) must divide $2^{\lfloor\frac{2n+1}{4}\rfloor}$ and therefore must divide $2^{\frac{n-1}{2}}$ (since n is odd, see below) and the size of the support of the Walsh transform of f is then a multiple of $3\cdot 2^{n+2}-2^3$ which is impossible since it cannot be larger than 2^n.

The only known second-order bent functions are the 3-variable functions equal to $x_1x_2x_3$ plus a quadratic function. It is shown in [275] that second-order bent n-variable functions can exist only if $n \equiv 3$ [mod 4] and the existence of such functions in more than 3 variables is an open question.

Remark. Partially-bent functions must not be mistaken for *partial bent functions*, studied by P. Guillot in [578]. By definition, the Fourier-Hadamard transforms of partial bent functions take exactly two values[32] λ and $\lambda+2^{\frac{n}{2}}$ on $\mathbb{F}_2^n\setminus\{0_n\}$ (n even). Rothaus' bound on the degree generalizes to partial bent functions. The dual $\widetilde{f}$ of f, defined by $\widetilde{f}(u) = 0$ if $\widehat{f}(u) = \lambda$ and $\widetilde{f}(u) = 1$ if $\widehat{f}(u) = \lambda + 2^{\frac{n}{2}}$, is also partial bent; and its dual is f. Two kinds of partial bent functions f exist: those such that $\widehat{f}(0_n) - f(0_n) = -\lambda(2^{\frac{n}{2}} - 1)$ and those such that $\widehat{f}(0_n) - f(0_n) = (2^{\frac{n}{2}} - \lambda)(2^{\frac{n}{2}} + 1)$. This can be deduced from Parseval's Relation (2.47). The sum of two partial bent functions of the same kind, whose supports share at most the zero vector, is partial bent. An interest of partial bent functions is in the possibility of using them as building blocks for constructing bent functions. □

6.2.2 Plateaued Boolean functions

In spite of their good properties, partially-bent functions, when they are not bent, have by definition nonzero linear structures and so do not give full satisfaction. The class of plateaued functions, already encountered above in Section 3.1, (and sometimes called *three-valued functions*) is a natural extension of that of partially-bent functions. They have been first studied by Zheng and Zhang in [1173, 1174, 1176] and more recently in [317, 1178, 858, 247].

Definition 63 *A function is called* plateaued *if its Walsh transform takes at most one nonzero absolute value λ, that is, takes at most three values 0 and $\pm\lambda$ (where λ is some positive integer, that we call the* amplitude *of the plateaued function).*

Because of *Parseval's relation* (2.47), the amplitude λ of any plateaued function must be of the form 2^j where $j \geq \frac{n}{2}$ (since $N_{W_f} \leq 2^n$). Then some authors call f a $(2j - n)$-plateaued function (*i.e.* call r-plateaued the plateaued functions of amplitude $2^{\frac{n+r}{2}}$), and bent functions are 0-plateaued, near-bent functions are 1-plateaued and semi-bent functions in even dimension are 2-plateaued. According to Parseval's relation, a plateaued function is bent if and only if its Walsh transform never takes the value 0. The Walsh spectrum of a plateaued function of amplitude λ is (thanks to Parseval's and inverse Walsh transform formulae):

Walsh Transform Value	Frequency
0	$2^n - 2^{2n-2j}$
2^j	$2^{2n-2j-1} + (-1)^{f(0_n)}\, 2^{n-j-1}$
-2^j	$2^{2n-2j-1} - (-1)^{f(0_n)}\, 2^{n-j-1}$

[32] Partial bent functions are the indicators of partial difference sets.

and we have $\sum_{a\in\mathbb{F}_{2^n}} W_f^3(a) = (-1)^{f(0_n)}\, 2^{n+2j}$ and $\sum_{a\in\mathbb{F}_{2^n}} W_f^4(a) = 2^{2n+2j}$.
The characterization of bent functions by difference sets has been extended in [918] to a characterization of plateaued functions by so-called one-and-half difference sets.
Of course, an n-variable Boolean function f is plateaued with amplitude λ if and only if its Walsh transform satisfies $W_f^2 = \lambda^2\, 1_{supp(W_f)}$, where $supp(W_f)$ is the Walsh support of f and $1_{supp(W_f)}$ is its indicator. Since the autocorrelation function Δ_f has W_f^2 for Fourier transform, partially-bent functions are then those plateaued functions whose Walsh support is an affine subspace of $\mathbb{F}_2^n$. Indeed, this condition is necessary and it is also sufficient since Relation (6.35) is then an equality because N_{Δ_f} equals then the size of the dual of the vector space equal to the direction of $supp(W_f)$, and it equals then $\frac{2^n}{N_{W_f}}$.
Note that, according to Parseval's relation, for every n-variable Boolean function f, we have $N_{W_f} \times \max_{a\in\mathbb{F}_2^n} W_f^2(a) \geq 2^{2n}$ and therefore, according to Relation (3.1), page 99: $nl(f) \leq 2^{n-1}\left(1 - \frac{1}{\sqrt{N_{W_f}}}\right)$. Equality is achieved if and only if f is *plateaued*.
According to Theorem 2, page 82, we have:

Proposition 96 *The* algebraic degree *of any n-variable plateaued function is bounded above by $n - j + 1$ where $\lambda = 2^j$ is the amplitude of f, and therefore by $\frac{n}{2} + 1$ if n is even (and by $\frac{n}{2}$ in the particular case of bent functions), and by $\frac{n+1}{2}$ if n is odd.*

Note that the second part of the remark at page 86 gives additional information on the ANF of plateaued functions.
Proposition 96 makes all plateaued functions weak against fast algebraic and Rønjom-Helleseth attacks on stream ciphers. The class of plateaued functions contains those functions which achieve the best possible trade-offs between resiliency, nonlinearity and algebraic degree: the order of resiliency and the nonlinearity of any Boolean function are bounded by Sarkar et al.'s bound (see Chapter 7 below) and the best compromise between those two criteria is achieved by plateaued functions only; the third criterion – the algebraic degree – is then also optimal. Other properties of plateaued functions can be found in [191, 692].

6.2.3 Characterizations of plateaued Boolean functions

A few characterizations of plateaued functions are given in [1173] for Boolean functions, which are direct consequences of the definition. Plateaued functions have been more recently characterized by their derivatives, their autocorrelation functions, and power moments of their Walsh transforms.

Characterization by means of the derivatives

Proposition 97 *[317] A Boolean function f on $\mathbb{F}_2^n$ is plateaued if and only if there exists $\lambda \in \mathbb{N}$ such that, for every $x \in \mathbb{F}_2^n$:*

$$\sum_{a,b\in\mathbb{F}_2^n} (-1)^{D_aD_bf(x)} = \lambda^2. \tag{6.38}$$

λ is then the amplitude of the plateaued function.

The proof is very similar to that of Proposition 6.1, page 216. A function f is plateaued with amplitude λ if and only if, for every $u \in \mathbb{F}_2^n$, we have $W_f(u)\left(W_f^2(u) - \lambda^2\right) = 0$, that is, $W_f^3(u) = \lambda^2 W_f(u)$. Applying the Fourier-Hadamard transform to both terms of this equality and using Relations (2.42), page 78, and (2.44) iterated (with three functions), page 79, we see that this is equivalent to the fact that, for every $a \in \mathbb{F}_2^n$, we have:

$$\sum_{x,y\in\mathbb{F}_2^n} (-1)^{f(x)\oplus f(y)\oplus f(x+y+a)} = \lambda^2(-1)^{f(a)},$$

and this completes the proof (after moving $(-1)^{f(a)}$ to the other hand side and changing x, y, a into $x+a, x+b, x$).

The fact that quadratic functions are *plateaued* is a direct consequence of Proposition 97, since their second-order derivatives are constant; and Proposition 97 gives more insight on the relationship between the nonlinearity of a quadratic function and the number of its nonzero second-order derivatives.

Characterization by means of the autocorrelation function

A Boolean function f being plateaued of amplitude λ if and only if the functions $W_f^2 \times W_f^2$ and $\lambda^2 W_f^2$ are equal, applying the Fourier transform to both functions, and using the formula $\widehat{\varphi \times \psi} = 2^{-n}\,\widehat{\varphi} \otimes \widehat{\psi}$ with $\varphi = \psi = W_f^2$, where $\otimes$ denotes the convolutional product, gives:

Proposition 98 *[247] Let n be any positive integer and f any Boolean function. Let $\Delta_f(a) = \sum_{x\in\mathbb{F}_2^n}(-1)^{f(x)\oplus f(x+a)}$ be the autocorrelation function of f. Then f is plateaued of amplitude λ if and only, for every $x \in \mathbb{F}_2^n$:*

$$\sum_{a\in\mathbb{F}_2^n} \Delta_f(a)\Delta_f(a+x) = \lambda^2\Delta_f(x).$$

Characterization by means of power moments of the Walsh transform

The sum $\sum_{a,b\in\mathbb{F}_2^n}(-1)^{D_aD_bf(x)}$ in Proposition 97, equals

$$2^{-n}\sum_{a,b,c,w\in\mathbb{F}_2^n} (-1)^{f(x)\oplus f(a)\oplus f(b)\oplus f(c)\oplus w\cdot(x+a+b+c)}.$$

Let us apply the Fourier transform to this real-valued function of x and use that any function of x is constant if and only if its Fourier transform is null at every nonzero vector α. We deduce that f is plateaued if and only if, for every nonzero $\alpha \in \mathbb{F}_2^n$, the sum $\sum_{x,a,b,c,w\in\mathbb{F}_2^n}(-1)^{f(x)\oplus f(a)\oplus f(b)\oplus f(c)\oplus w\cdot(x+a+b+c)\oplus\alpha\cdot x}$ is null. This latter sum equals:

$$\sum_{w\in\mathbb{F}_2^n}\sum_{x\in\mathbb{F}_2^n}(-1)^{f(x)\oplus(w+\alpha)\cdot x}\sum_{a\in\mathbb{F}_2^n}(-1)^{f(a)\oplus w\cdot a}\sum_{b\in\mathbb{F}_2^n}(-1)^{f(b)\oplus w\cdot b}\sum_{c\in\mathbb{F}_2^n}(-1)^{f(c)\oplus w\cdot c}.$$

We deduce:

Proposition 99 *[247] Any n-variable Boolean function f is* plateaued *if and only if, for every nonzero $\alpha \in \mathbb{F}_2^n$, we have*

$$\sum_{w\in\mathbb{F}_2^n} W_f(w+\alpha)\, W_f^3(w) = 0.$$

Another characterization of plateaued functions by means of the Walsh transform exists. For a plateaued Boolean function of amplitude λ, we have, using Parseval's relation, that $\sum_{a\in\mathbb{F}_2^n} W_f^4(a) = 2^{2n}\lambda^2$. We also have, for every $b \in \mathbb{F}_2^n$, that $\sum_{a\in\mathbb{F}_2^n}(-1)^{a\cdot b}W_f^3(a) = \lambda^2\sum_{a\in\mathbb{F}_2^n}(-1)^{a\cdot b}W_f(a) = \lambda^2\, 2^n(-1)^{f(b)}$. A necessary condition for f to be plateaued is then that, for every $b\in\mathbb{F}_2^n$, $\sum_{a\in\mathbb{F}_2^n} W_f^4(a) = 2^n(-1)^{f(b)}\sum_{a\in\mathbb{F}_2^n}(-1)^{a\cdot b}W_f^3(a)$. Conversely, if this property is satisfied by f, then the function $b\in\mathbb{F}_2^n \mapsto (-1)^{f(b)}\sum_{a\in\mathbb{F}_2^n}(-1)^{a\cdot b}W_f^3(a)$ is constant. Then the Fourier transform of this function, that is, the function which maps every $\alpha\in\mathbb{F}_2^n$ to the sum $\sum_{b\in\mathbb{F}_2^n}\sum_{a\in\mathbb{F}_2^n}(-1)^{(a+\alpha)\cdot b\oplus f(b)}W_f^3(a) = \sum_{a\in\mathbb{F}_2^n} W_f(a+\alpha)W_f^3(a)$ is null at every nonzero α, and f is plateaued, according to Proposition 99:

Corollary 17 *[247] Any n-variable Boolean function f is plateaued if and only if, for every $b\in\mathbb{F}_2^n$:*

$$\sum_{a\in\mathbb{F}_2^n} W_f^4(a) = 2^n(-1)^{f(b)}\sum_{a\in\mathbb{F}_2^n}(-1)^{a\cdot b}W_f^3(a).$$

More characterizations exist. An obvious one is that, for every positive integer k, an n-variable Boolean function f is plateaued if and only if there exists $\nu\in\mathbb{Z}$ such that we have $\sum_{a\in\mathbb{F}_2^n} W_f^{2k}(a)\left(W_f^2(a)-\nu\right)^2 = 0$ (ν equals then the square of the amplitude of the plateaued function). This non-negative expression of degree 2 in ν writes $\sum_{a\in\mathbb{F}_2^n} W_f^{2k+4}(a) - 2\nu\sum_{a\in\mathbb{F}_2^n} W_f^{2k+2}(a) + \nu^2\sum_{a\in\mathbb{F}_2^n} W_f^{2k}(a)$; hence, the reduced discriminant $\big(\sum_{a\in\mathbb{F}_2^n} W_f^{2k+2}(a)\big)^2 - \big(\sum_{a\in\mathbb{F}_2^n} W_f^{2k+4}(a)\big)\big(\sum_{a\in\mathbb{F}_2^n} W_f^{2k}(a)\big)$ is non-positive and is null if and only if f is plateaued. We deduce (see more in [247]):

Proposition 100 *[858, 247] For every n-variable Boolean function f and every $k\in\mathbb{N}^*$, we have:*

$$\left(\sum_{a\in\mathbb{F}_2^n} W_f^{2k+2}(a)\right)^2 \le \left(\sum_{a\in\mathbb{F}_2^n} W_f^{2k}(a)\right)\left(\sum_{a\in\mathbb{F}_2^n} W_f^{2k+4}(a)\right),$$

with equality if and only if f is plateaued.

Characterization by means of codes

Any plateaued Boolean function f, viewed as a (vectorial) $(n,1)$-function, can be related to the code C'_f seen at page 183, which has then weight distribution given by Table 6.2. See in [874] examples of such codes.

Hamming weight w	Multiplicity A_w
0	1
2^{n-1}	$2^{n+1} - \frac{2^{2n}}{\lambda^2} - 1$
$2^{n-1} - \frac{\lambda}{2}$	$\frac{2^{2n-1}}{\lambda^2} + (-1)^{f(0_n)}\frac{2^{n-1}}{\lambda}$
$2^{n-1} + \frac{\lambda}{2}$	$\frac{2^{2n-1}}{\lambda^2} - (-1)^{f(0_n)}\frac{2^{n-1}}{\lambda}$

Table 6.2 Weight distribution of C'_f for f plateaued of amplitude λ.

Langevin proved in [738] that, if f is a plateaued function, then the coset $f \oplus RM(1,n)$ of the Reed-Muller code of order 1, is an *orphan of* $RM(1,n)$. The notion of orphan has been introduced in [599] (with the term "urcoset" instead of orphan) and studied in [137]. A coset of $RM(1,n)$ is an orphan if it is maximum with respect to the following partial order relation: $g \oplus RM(1,n)$ is smaller than $f \oplus RM(1,n)$ if there exists in $g \oplus RM(1,n)$ an element g_1 of Hamming weight $nl(g)$ (that is, of minimum Hamming weight in $g \oplus RM(1,n)$), and in $f \oplus RM(1,n)$ an element f_1 of Hamming weight $nl(f)$, such that $supp(g_1) \subseteq supp(f_1)$. Clearly, if f is a function of maximum nonlinearity, then $f \oplus RM(1,n)$ is an orphan of $RM(1,n)$ (the converse is false, since plateaued functions with non-optimal nonlinearity exist). The notion of orphan can be used in algorithms searching for functions with high nonlinearities.

6.2.4 The subclasses of semi-bent and near-bent functions

- Recall that for n odd, *near-bent* functions (also called *semi-bent functions*) are those plateaued functions of amplitude $2^{\frac{n+1}{2}}$. In [191] is observed that the class of so-called *three-valued almost optimal* functions, such that the coset $f \oplus RM(1,n)$ takes exactly three weights and whose nonlinearity is at least $2^{n-1} - 2^{\frac{n-1}{2}}$, coincides with that of near-bent functions (such functions are plateaued because there are three weights and because the coset is stable under complementation, and the amplitude of such plateaued functions is minimal). Parseval's identity shows that the support of their Walsh transform has cardinality 2^{n-1}. Other properties have been shown in [1121] in connection with the theory of *cyclic codes* and in [427] in connection with that of designs.

According to the properties seen in Section 5.2, page 193, quadratic Boolean functions are near-bent if and only if their linear kernel has dimension 1, that is, their rank equals $n-1$.

Several constructions of quadratic near-bent functions exist, see a survey in [859].

All the component functions of almost bent (n,n)-functions (see Subsection 11.3, page 403) are near-bent, by definition, and the restriction of any bent Boolean

Hamming weight	Multiplicity
0	1
$\frac{w_H(f)-2^{(n-1)/2}}{2}$	$w_H(f)[1-2^{-n}w_H(f)-2^{-(n+1)/2}]$
$\frac{w_H(f)}{2}$	$2^n-1-w_H(f)(2^n-w_H(f))2^{-(n-1)}$
$\frac{w_H(f)+2^{(n-1)/2}}{2}$	$w_H(f)[1-2^{-n}w_H(f)+2^{-(n+1)/2}]$

Table 6.3 Weight distribution of the code $C_{supp(f)}$ for f near-bent such that $f(0_n)=0$

function to an affine hyperplane is near-bent (the restrictions to an affine hyperplane and to its complement have complementary Walsh supports and conversely such pair of near-bent functions arises from a bent function).
In [453] is extended Proposition 68, page 219, to near-bent functions: any n-variable Boolean function f such that $f(0_n)=0$ is near-bent if and only if the linear code $C_{supp(f)}$ whose generator matrix has for columns the vectors of $supp(f)$ has dimension n, and has weight distribution given by Table 6.3. This provides codes with three weights.
- For n even, as also seen at page 201, *semi-bent functions* are those plateaued functions of amplitude $2^{\frac{n+2}{2}}$. The term of semi-bent has been introduced in [357], but as for n odd, these functions had been anteriorly studied under the name of three-valued almost optimal Boolean functions in [191], where is observed that the class of such functions whose nonlinearity is at least $2^{n-1}-2^{\frac{n}{2}}$ coincides with that of semi-bent functions. In [312] is shown that the sum of a Boolean function g equal to the linear combination of the indicators of the elements of a spread and of a Boolean function h whose restrictions to these elements are linear, is semi-bent if and only if g and h are both bent; related infinite classes are specified and a version with partial spreads is also given. Other recent works on semi-bent functions are [206, 282, 711, 855, 856, 857, 868, 876, 1131]. Up to recently, the known semi-bent functions were often quadratic or the component functions of *power functions* (see *e.g.* [355]). More constructions have been proposed in [376] to derive semi-bent functions from bent functions. See a survey in [859].

6.2.5 Primary constructions of plateaued Boolean functions

All quadratic Boolean functions and all bent and semi-bent Boolean functions are plateaued. We recall from [247] the other *primary constructions*. Most of them have been already presented above for constructing bent functions; they are extended here to more general plateaued functions.

Maiorana-McFarland (MM) functions

Any function $f_{\phi,h}(x,y)=x\cdot\phi(y)\oplus h(y)$; $x\in\mathbb{F}_2^r$, $y\in\mathbb{F}_2^s$, is plateaued if and only if $|\sum_{y\in\phi^{-1}(a)}(-1)^{b\cdot y\oplus h(y)}|$ can take two values, one of which is 0, when $(a,b)\in\mathbb{F}_2^r\times\mathbb{F}_2^s$, since $W_{f_{\phi,h}}(a,b)=2^r\sum_{y\in\phi^{-1}(a)}(-1)^{b\cdot y\oplus h(y)}$. If ϕ is injective (resp.

takes exactly 2 times each value of $Im\,(\phi)$), then $f_{\phi,h}$ is plateaued of amplitude 2^r (resp. 2^{r+1}). Note that the *address function* (see page 87) is plateaued as observed in [692] and easily checked.

Zheng-Zhang's functions

In [1173], Zheng and Zhang introduce a class of *plateaued* functions and prove that some of them are not partially-bent. These functions are defined as follows: let t and k be two integers such that $k < 2^t < 2^k$ and let $E \subseteq \mathbb{F}_2^k$ be a subset of 2^t elements such that any linear non null function on $\mathbb{F}_2^k$ is not constant on E. For every element e_i of E, let ξ_i denote the truth table of the linear function $x \mapsto x \cdot e_i$ on $\mathbb{F}_2^k$. Then, the Boolean function f on $\mathbb{F}_2^{k+t}$ having for truth table the concatenation $\xi_0\xi_1 \cdots \xi_{2^t-1}$ of these truth tables is plateaued on $\mathbb{F}_2^{k+t}$ and its amplitude equals 2^k. Such function is the concatenation of distinct linear functions. Then, as already observed in [317], it belongs to the Maiorana-McFarland class and satisfies the first hypothesis above.

Generalizations of Maiorana-McFarland functions

Concatenations of quadratic functions in Dickson form

Let n and r be positive integers such that $r \le n$. As proved in [223] and recalled at page 203, the function:

$$f_{\psi,\phi,g}(x,y) = \bigoplus_{i=1}^{t} x_{2i-1}x_{2i}\psi_i(y) \oplus x \cdot \phi(y) \oplus g(y) =$$

$$\bigoplus_{i=1}^{t} x_{2i-1}x_{2i}\psi_i(y) \oplus \bigoplus_{j=1}^{r} x_i\,\phi_i(y) \oplus g(y);\ x \in \mathbb{F}_2^r,\ y \in \mathbb{F}_2^s,$$

where $t = \left\lfloor \frac{r}{2} \right\rfloor$, satisfies $W_{f_{\psi,\phi,g}}(a,b) =$

$$\sum_{y \in E_a} 2^{r-w(\psi(y))}(-1)^{\bigoplus_{i=1}^{t}(\phi_{2i-1}(y)\oplus a_{2i-1})(\phi_{2i}(y)\oplus a_{2i})\oplus g(y)\oplus y\cdot b},$$

where $w(\psi(y))$ denotes the Hamming weight and E_a is the superset of $\phi^{-1}(a)$ equal if r is even to

$$\left\{ y \in \mathbb{F}_2^s;\ \forall i \le t,\ \begin{array}{l} \psi_i(y) = 0 \Rightarrow \\ (\phi_{2i-1}(y) = a_{2i-1} \text{ and } \phi_{2i}(y) = a_{2i}) \end{array} \right\},$$

and if r is odd to

$$\left\{ y \in \mathbb{F}_2^s;\ \left\{ \begin{array}{l} \forall i \le t,\ \psi_i(y) = 0 \Rightarrow \\ \quad (\phi_{2i-1}(y) = a_{2i-1} \text{ and } \phi_{2i}(y) = a_{2i}) \\ \phi_r(y) = a_r \end{array} \right. \right\}.$$

As observed in [317], if E_a has size 0 or 1 (respectively 0 or 2) for every a and if ψ has constant weight, then $f_{\psi,\phi,g}$ is plateaued.

Concatenations of quadratic functions of rank 2

As seen at page 203, assuming that $\phi_2(y) \neq 0_r$ for every $y \in \mathbb{F}_2^s$ and denoting by E the set of $y \in \mathbb{F}_2^s$ such that $\phi_1(y)$ and $\phi_2(y)$ are linearly independent, function $f_{\phi_1,\phi_2,\phi_3,g}(x, y) = (x \cdot \phi_1(y))\,(x \cdot \phi_2(y)) \oplus x \cdot \phi_3(y) \oplus g(y)$ satisfies $W_{f_{\phi_1,\phi_2,\phi_3,g}}(a, b) =$

$$2^{r-1} \sum_{\substack{y\in E;\\ \phi_3(y)+a\in\{0_r,\phi_1(y),\phi_2(y)\}}} (-1)^{g(y)\oplus b\cdot y} - 2^{r-1} \sum_{\substack{y\in E;\\ \phi_3(y)+a=\phi_1(y)+\phi_2(y)}} (-1)^{g(y)\oplus b\cdot y} +$$
$$2^r \sum_{\substack{y\in \mathbb{F}_2^s\setminus E;\\ \phi_3(y)+a=\phi_1(y)}} (-1)^{g(y)\oplus b\cdot y},$$

for every $a \in \mathbb{F}_2^r$ and $b \in \mathbb{F}_2^s$. As shown in [317], if $E = \mathbb{F}_2^s$ and the 2-dimensional flats $\phi_3(y) + \langle \phi_1(y), \phi_2(y)\rangle$; $y \in \mathbb{F}_2^s$, are pairwise disjoint, then $f_{\phi_1,\phi_2,\phi_3,g}$ is plateaued of amplitude 2^{r-1}. And assuming that $\phi_2(y)$ is nonzero for every $y \in \mathbb{F}_2^s$ and denoting by F'_a (resp. F''_a) the set of all $y \in \mathbb{F}_2^s$ such that $\phi_1(y)$ and $\phi_2(y)$ are linearly independent (resp. dependent) and such that a belongs to the flat $\phi_3(y) + \langle \phi_1(y), \phi_2(y)\rangle$ (resp. $a = \phi_3(y) + \phi_1(y)$), we have that if, for every $a \in \mathbb{F}_2^r$, the number $|F'_a| + 2|F''_a|$ equals 0 or 2, then $f_{\phi_1,\phi_2,\phi_3,g}$ is plateaued of amplitude 2^r. See a little more in [1058].

6.2.6 Secondary constructions of plateaued Boolean functions

The *direct sum* preserves plateauedness since $h(x, y) = f(x) \oplus g(y)$, $x \in \mathbb{F}_2^r, y \in \mathbb{F}_2^s$ satisfies $W_h(a, b) = W_f(a)W_g(b)$ (and we have then $nl(h) = 2^s nl(f) + 2^r nl(g) - 2nl(f)nl(g)$). The *indirect sum* does too, under some conditions:

Proposition 101 *[247] Let $h(x, y) = f_1(x) \oplus g_1(y) \oplus (f_1 \oplus f_2)(x)\,(g_1 \oplus g_2)(y)$, then if f_1 and f_2 are plateaued with the same amplitude, g_1 and g_2 are plateaued with the same amplitude, and*

- *f_1 and f_2 have the same Walsh support (*i.e. *the same extended Walsh spectrum),*
- *or g_1 and g_2 have the same Walsh support (idem)*
- *or f_1 and f_2 have disjoint Walsh supports and g_1 and g_2 have disjoint Walsh supports,*

then h is plateaued.

Proof. We have seen already that:

$$W_h(a, b) = \frac{1}{2}W_{f_1}(a)\,[W_{g_1}(b) + W_{g_2}(b)] + \frac{1}{2}W_{f_2}(a)\,[W_{g_1}(b) - W_{g_2}(b)]\,. \quad (6.39)$$

Moreover, if f_1 and f_2 have both amplitude λ, and if g_1 and g_2 have both amplitude μ, then according to Relation (6.39), we have that:

- if g_1 and g_2 have the same Walsh support, then $W_h(a, b) \in \{0, \pm\lambda\mu\}$ (indeed, at most one of the two values $W_{g_1}(b) + W_{g_2}(b)$ and $W_{g_1}(b) - W_{g_2}(b)$ is then nonzero, and this value equals $\pm 2\mu$),

- if f_1 and f_2 have the same Walsh support, then $W_h(a,b) \in \{0, \pm\lambda\mu\}$ (same argument, after exchanging the roles of the f_i's and the g_i's in (6.39)),
- if f_1 and f_2 have disjoint Walsh supports and g_1 and g_2 have disjoint Walsh supports, then $W_h(a,b) \in \{0, \pm\frac{\lambda\mu}{2}\}$.

Hence, h is plateaued. □

In [1152] is given a *secondary construction* of plateaued functions (with disjoint supports) from 3 bent functions and 3 plateaued functions, under some conditions.

The *construction without extension of the number of variables* viewed at page 262 can be easily adapted to plateaued functions:

Proposition 102 *[247] Let f_1, f_2 and f_3 be three n-variable Boolean functions. Denote by s_1 the Boolean function $f_1 + f_2 + f_3$ and by s_2 the Boolean function $f_1f_2 + f_1f_3 + f_2f_3$. We have:*

$$W_{f_1} + W_{f_2} + W_{f_3} = W_{s_1} + 2\,W_{s_2}.$$

Moreover,

- *if f_1, f_2, f_3 and s_1 are plateaued with the same amplitude λ and with disjoint Walsh supports, then s_2 is plateaued with amplitude $\frac{\lambda}{2}$.*
- *if f_1, f_2, f_3 and s_1 are plateaued with the same amplitude λ and with Walsh supports whose multi-set equals twice some subset of $\mathbb{F}_2^n$, then s_2 is plateaued with amplitude λ.*
- *if f_1, f_2, f_3 are plateaued with the same amplitude λ and with disjoint Walsh supports, and s_2 is plateaued with amplitude $\frac{\lambda}{2}$ and Walsh support disjoint from those of f_1, f_2, f_3, then s_1 is plateaued with amplitude λ.*
- *if f_1, f_2, f_3 are plateaued with the same amplitude λ, s_2 is plateaued with amplitude $\frac{\lambda}{2}$ and the Walsh supports of f_1, f_2, f_3 and s_2 make a multi-set equal to twice some subset of $\mathbb{F}_2^n$, then s_1 is plateaued with amplitude 2λ.*

6.3 Bent$_4$ and partially-bent$_4$ functions

There exist several generalizations of the notion of bent function, see *e.g.* [313]. We shall not address them here since we focus on Boolean functions. But bent$_4$ functions [993, 927, 529, 17, 18] are Boolean functions (whose definition is a modification of that of bent function); we need then to give the main definitions and results on them, even if their use in cryptography and coding is not so clear[33]. In even dimension, bent$_4$ functions are defined as bent functions, but with respect to a transformation called *unitary transformation* that we recall

[33] The motivation given in [993] comes from the quantum domain; another motivation comes from the relation to the notion of modified planar functions, see [18], where is proved that bent$_4$ functions describe the components of modified planar functions.

below, and which generalizes the Walsh transform. They can also be defined by the balancedness of so-called *modified derivatives*. In odd dimension there is a one-to-one correspondence between the set of bent$_4$ functions and the set of semi-bent functions satisfying additional properties that we shall describe as well.

Definition 64 *[993, 18] Let n be any positive integer and f a Boolean function over $\mathbb{F}_{2^n}$. For any element $c \in \mathbb{F}_{2^n}$, the unitary transformation $\mathcal{V}_f^c : \mathbb{F}_{2^n} \to \mathbb{C}$ is defined as*

$$\mathcal{V}_f^c(u) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+\sigma^c(x)} i^{tr_n(cx)} (-1)^{tr_n(ux)},$$

where $\sigma^c(x)$ is the Boolean function whose univariate representation equals:

$$\sigma^c(x) = \sum_{0 \le i < j \le n-1} (cx)^{2^i} (cx)^{2^j}.$$

For $c = 0$, the transformation $\mathcal{V}_f^c$ is simply the well-known Walsh transform. For $c = 1$, $\mathcal{V}_f^c$ is the nega-Hadamard transform (see. [927]).

In even dimension, the class of *bent*$_4$ functions can be defined as follows in terms of the unitary transformation,

Definition 65 *Let n be an even integer. A Boolean function f is called a c-bent$_4$ function, for some $c \in \mathbb{F}_{2^n}$, if the unitary transformation $\mathcal{V}_f^c$ satisfies $|\mathcal{V}_f^c(u)| = 2^{n/2}$ for all $u \in \mathbb{F}_{2^n}$. A function is bent$_4$ if it is c-bent$_4$ for some $c \in \mathbb{F}_{2^n}$.*

In other words, a Boolean function is c-bent$_4$ if it has a flat spectrum with respect to at least one of the transforms $\mathcal{V}_f^c$. Note that when $c = 0$, a c-bent$_4$ function is a classical bent and when $c = 1$, a c-bent is so-called *nega-bent*.

Proposition 103 *[18] Let n be an even integer. A Boolean function $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$ is c-bent$_4$ if and only if $f \oplus \sigma^c$ is bent.*

Proof. We will employ (again) Jacobi's two-square theorem stating that for an even integer n, the integer solutions of the Diophantine equation $R^2 + I^2 = 2^n$ are $(R, I) = (0, \pm 2^{n/2})$ or $(\pm 2^{n/2}, 0)$. One has:

$$\begin{aligned}
\mathcal{V}_f^c(u) &= \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+\sigma^c(x)} i^{tr_n(cx)} (-1)^{tr_n(ux)} \\
&= \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+\sigma^c(x)+tr_n(ux)} \left(\frac{1 + (-1)^{tr_n(cx)}}{2} + i \frac{1 - (-1)^{tr_n(cx)}}{2} \right) \\
&= \frac{W_{f \oplus \sigma^c}(u) + W_{f \oplus \sigma^c}(u+c)}{2} + i \frac{W_{f \oplus \sigma^c}(u) - W_{f \oplus \sigma^c}(u+c)}{2}
\end{aligned}$$

If f is c-bent$_4$ then $|\mathcal{V}_f^{(c)}(u)| = 2^{n/2}$, that is,

$$\Big(W_{f \oplus \sigma^c}(u) + W_{f \oplus \sigma^c}(u+c) \Big)^2 + \Big(W_{f \oplus \sigma^c}(u) - W_{f \oplus \sigma^c}(u+c) \Big)^2 = 2^{n+2}. \quad (6.40)$$

Now, by Jacobi's two-square theorem , one has $|W_{f\oplus\sigma^c}(u)| = |W_{f\oplus\sigma^c}(u+c)| = 2^{n/2}$, which proves that $f \oplus \sigma^c$ is bent. The converse of the statement comes immediately from Equation (6.40). □

Some authors call such f a *shifted bent* function (*i.e.* f is the shifted version of the bent function $f \oplus \sigma^c$).

Remark. $\frac{W_{f\oplus\sigma^c}(u)+W_{f\oplus\sigma^c}(u+c)}{2}$ (resp. $\frac{W_{f\oplus\sigma^c}(u)-W_{f\oplus\sigma^c}(u+c)}{2}$) ranges (twice, when u ranges over $\mathbb{F}_{2^n}$) over the Walsh spectrum of the restriction of $f \oplus \sigma^c$ to the linear hyperplane of equation $tr_n(cx) = 0$ (resp. its complement) and we know from Theorem 16, page 266, that $f \oplus \sigma^c$ is bent if and only if these two restrictions are semi-bent (*i.e.* near-bent) with complementary Walsh supports. □

An alternative definition of a c-bent$_4$ function f can be given in relation to the so-called *modified derivative* of f. More specifically, it has been proved in [18] that f is c-bent$_4$ if and only if the modified derivative $f(x+a)\oplus f(x)\oplus tr_n(c^2ax)$ is balanced for all nonzero $a \in \mathbb{F}_{2^n}$. This corresponds to the characterization of bent functions via derivatives when $c = 0$.

Bent$_4$ functions exist also in odd dimension. More precisely, let n be an odd integer. Then a function $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$ is c-bent$_4$ if and only if $f \oplus \sigma^c(x)$ is a *semi-bent function* and $|W_{f\oplus\sigma^c}(u)| \neq |W_{f\oplus\sigma^c}(u+c)|$ for all $u \in \mathbb{F}_{2^n}$.

In [17], the authors have introduced the notion of partially-bent$_4$ functions which are functions whose modified derivative is either constant or balanced for every element of the input set. It is known that every quadratic function is partially c-bent$_4$.

6.4 Bent vectorial functions

Definition 66 *An (n,m) function is called bent if all its* component functions *$v \cdot F$, $v \in \mathbb{F}_2^m \setminus \{0_m\}$ (where "$\cdot$" is an inner product in $\mathbb{F}_2^m$), are bent Boolean functions, that is, if $W_F^2(u,v) = 2^n$ for every $v \in \mathbb{F}_2^m \setminus \{0_m\}$ and every $u \in \mathbb{F}_2^n$. Equivalently, all the derivatives D_aF, $a \in \mathbb{F}_2^n \setminus \{0_n\}$, are balanced.*

The equivalence between these two characteristic properties, called respectively bentness and perfect nonlinearity[34], is a direct consequence of Theorem 12, page 216, which implies that F is bent if and only if, for every $v \in \mathbb{F}_2^m \setminus \{0_m\}$ and every $a \in \mathbb{F}_2^n \setminus \{0_n\}$, the function $v \cdot D_aF$ is balanced, and of Proposition 35, page 134 applied to D_aF.
Up to linear equivalence (precisely, up to the composition on the left by a linear automorphism), the knowledge of a bent (n,m)-*function* is equivalent to that of an m-dimensional $\mathbb{F}_2$-vector space of Boolean functions, all being bent except the zero one (the vector space is made of all component functions and of

[34] There are then (over $\mathbb{F}_2^n$) two different terminologies for the same class of functions.

the zero function); from such an m-dimensional space E, we can build a bent (n,m)-function by choosing its coordinate functions as m linearly independent functions in E.
Bent vectorial functions are never balanced since their component functions are not balanced. More precisely, we saw at page 135 that their *imbalance* $Nb_F = \sum_{b\in\mathbb{F}_2^m}\left(|F^{-1}(b)| - 2^{n-m}\right)^2$ satisfies $Nb_F = \sum_{a\in\mathbb{F}_2^n}|(D_aF)^{-1}(0_m)| - 2^{2n-m} = 2^n - 2^{n-m}$ (and that $Nb_{F+L} = 2^n - 2^{n-m}$ for every linear function L). We have also seen that $NB_F = \sum_{a\in\mathbb{F}_2^n\setminus\{0_n\}} Nb_{D_aF}$ equals 0 if and only if F is bent.
The algebraic degree of any bent (n,m)-function is at most $\frac{n}{2}$, since this bound is true for any component function.

Remark. We have seen with Proposition 37, page 142, that it is possible, as for Boolean functions, to characterize the bentness of (n,m)-functions F by a property of the functions $F+L$ where L is a linear (n,m)-function expressing that $F+L$ is not far from a balanced function. □.

Bent vectorial functions have been initially considered by Nyberg who proved:

Proposition 104 *[906] Bent (n,m)-functions exist if and only if n is even and $m \le \frac{n}{2}$.*

Proof. It is easily seen that the condition is sufficient, thanks to the constructions of bent functions that we shall see in Subsection 6.4.1, page 297. Let us prove that it is necessary. We have seen in Relation (3.17), page 134, that, for every (n,m)-function F and any element $b\in\mathbb{F}_2^m$, the size of $F^{-1}(b)$ is equal to $2^{-m}\sum_{x\in\mathbb{F}_2^n;v\in\mathbb{F}_2^m}(-1)^{v\cdot(F(x)+b)}$. Assuming that F is bent and denoting, for every $v\in\mathbb{F}_2^n\setminus\{0_n\}$, by $\widetilde{v\cdot F}$ the dual of the bent Boolean function $x\mapsto v\cdot F(x)$, we have, by definition: $\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)} = 2^{\frac{n}{2}}(-1)^{\widetilde{v\cdot F}(0_n)}$. The size of $F^{-1}(b)$ equals then $2^{n-m} + 2^{\frac{n}{2}-m}\sum_{v\in\mathbb{F}_2^n\setminus\{0_n\}}(-1)^{\widetilde{v\cdot F}(0_n)\oplus v\cdot b}$. Since the sum $\sum_{v\in\mathbb{F}_2^n\setminus\{0_n\}}(-1)^{\widetilde{v\cdot F}(0_n)\oplus v\cdot b}$ has an odd value ($\mathbb{F}_2^n\setminus\{0_n\}$ having an odd size), we deduce that, if $m\le n$ then $2^{\frac{n}{2}-m}$ must be an integer. And it is also easily shown that $m>n$ is impossible. □

Remark. The situation with PN functions is different for odd characteristic, in which PN (n,n)-functions (defined similarly) do exist for every n (they are also called *planar*). A notion of planar function in characteristic 2 (stating that $x\in\mathbb{F}_{2^n}\mapsto D_aF(x)+ax$ is bijective for every $a\ne 0$) sometimes called *pseudo-planar* or *modified planar* has been proposed in [1023] (see also [963]). Such functions share many of the properties of planar functions in odd characteristic, in relation with relative difference sets and finite geometries. □

A survey on bent vectorial functions can be found in [310].
In [337] are called *dual-bent vectorial functions* the bent (n,m)-functions hav-

ing the property that the duals of their component functions form, together with the zero function, a vector space of dimension m, and are then the component functions of some vectorial bent function, called a vectorial dual of F; classical classes are then studied from this viewpoint.

CCZ equivalence and EA equivalence coincide for bent functions [148, 150]: let F be a bent (n,m)-function (n even, $m \leq \frac{n}{2}$) and let (without loss of generality) L_1 and L_2 be two linear functions from $\mathbb{F}_2^n \times \mathbb{F}_2^m$ to (respectively) $\mathbb{F}_2^n$ and $\mathbb{F}_2^m$, such that (L_1, L_2) is a permutation of $\mathbb{F}_2^n \times \mathbb{F}_2^m$ and $F_1(x) = L_1(x, F(x))$ is a permutation of $\mathbb{F}_2^n$. For every vector v in $\mathbb{F}_2^n$, the function $v \cdot F_1$ is necessarily non-bent since, if $v = 0_m$ then it is null and if $v \neq 0_m$ then it is balanced. Let us denote $L_1(x,y) = L'(x) + L''(y)$. We have then $F_1(x) = L'(x) + L'' \circ F(x)$. The adjoint operator L''' of L'' (satisfying by definition $v \cdot L''(y) = L'''(v) \cdot y$) is then the null function, since if $L'''(v) \neq 0_m$ then $v \cdot F_1(x) = v \cdot L'(x) \oplus L'''(v) \cdot F(x)$ is bent. This means that L'' is null and L_1 depends then only on x, which corresponds to EA equivalence.

We have seen in Proposition 104 that bent (n,m)-functions exist if and only if n is even and $m \leq n/2$. Better bounds than the *covering radius bound* are open problems for:
- n odd and $m < n$ (for $m \geq n$, the Sidelnikov-Chabaud-Vaudenay bound, and other bounds if m is large enough, are better);
- n even and $\frac{n}{2} < m < n$.

In [459], the authors have provided a coding-theoretic characterization of bent vectorial functions and used them for the construction of a two-parameter family of binary linear codes that do not satisfy the conditions of the Assmus-Mattson theorem [36], but nevertheless hold 2-designs.

6.4.1 Primary constructions of bent vectorial functions

Recall that bent (n,m)-functions can exist only for n even and $m \leq n/2$, that we shall assume satisfied. The main classes of bent Boolean functions lead to classes of bent (n,m)-functions (this was first observed in [906] by Nyberg, who proposed constructions within the Maiorana-McFarland and $\mathcal{PS}_{ap}$ constructions).

Constructions in *bivariate representation*

The three first *primary constructions* below are by increasing order of generality. We follow [310, 313] for the description. When necessary (*i.e.* when we need to make multiplications or divisions), we endow $\mathbb{F}_2^{\frac{n}{2}}$ with the structure of the field $\mathbb{F}_{2^{\frac{n}{2}}}$ and we identify $\mathbb{F}_2^n$ with $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$.

- Bent (n,m)-functions in the strict class of Maiorana-McFarland are defined as: $F(x,y) = L(x\,\pi(y)) + G(y)$, $x, y \in \mathbb{F}_{2^{n/2}}$, where π is a permutation of $\mathbb{F}_{2^{n/2}}$, $L : \mathbb{F}_{2^{n/2}} \mapsto \mathbb{F}_2^m$ is linear surjective and G is any $(n/2, m)$-function. An example is given in [1018] (which achieves optimal algebraic degree $n/2$): the i-th coordinate of this function is defined as $f_i(x,y) = tr_{\frac{n}{2}}(x\,\phi_i(y)) \oplus g_i(y)$, $x,y \in \mathbb{F}_{2^{\frac{n}{2}}}$, where g_i is any Boolean function on $\mathbb{F}_{2^{\frac{n}{2}}}$ and where $\phi_i(y) = \begin{cases} 0 \text{ if } y=0 \\ \alpha^{dec(y)+i-1} \text{ otherwise} \end{cases}$, where α is a primitive element of $\mathbb{F}_{2^{\frac{n}{2}}}$ and $dec(y) = 2^{\frac{n}{2}-1}y_1 + 2^{\frac{n}{2}-2}y_2 + \cdots + y_{\frac{n}{2}}$. This function belongs to the strict Maiorana-McFarland class because the mapping $y \to \begin{cases} 0 \text{ if } y=0 \\ \alpha^{dec(y)} \text{ otherwise} \end{cases}$ is a permutation from $\mathbb{F}_2^{\frac{n}{2}}$ to $\mathbb{F}_{2^{\frac{n}{2}}}$, and the function $L : x \in \mathbb{F}_{2^{\frac{n}{2}}} \to (tr_{\frac{n}{2}}(x), tr_{\frac{n}{2}}(\alpha x), \ldots, tr_{\frac{n}{2}}(\alpha^{\frac{n}{2}-1}x)) \in \mathbb{F}_2^{\frac{n}{2}}$ is an isomorphism.

- Bent (n,m)-functions in the extended class of Maiorana-McFarland are defined as: $F(x,y) = \psi(x,y) + G(y)$, where G is any $(n/2,m)$-function and ψ is such that, for all $y \in \mathbb{F}_{2^{n/2}}$, the function $x \mapsto \psi(x,y)$ is linear and for all $x \in \mathbb{F}_{2^{n/2}} \setminus \{0\}$, the function $y \mapsto \psi(x,y)$ is balanced. Such function is bent since, for every nonzero $v \in \mathbb{F}_2^m$ and every $y \in \mathbb{F}_2^{n/2}$, there exists a unique vector v_y such that $v \cdot \psi(x,y) = x \cdot v_y$ and the function $y \mapsto v_y$ is bijective.
- Bent (n,m)-functions in the general class of Maiorana-McFarland are defined such that, for all $v \in \mathbb{F}_{2^m}^*$, function $v \cdot F$ belongs, up to affine equivalence, to the Maiorana-McFarland class of Boolean bent functions. Some bent quadratic functions, elements of the general class, may not belong to the strict class.
- Modifications of the Maiorana-McFarland bent functions have been proposed in [909], using the classes $\mathcal{C}$ and $\mathcal{D}$ of bent Boolean functions.
- Bent (n,m)-functions in the $\mathcal{PS}_{ap}$ class of vectorial functions are defined as: $F(x,y) = G(xy^{2^n-2}) = G\left(\frac{x}{y}\right)$, with the convention $\frac{1}{0} = 0$, where G is a balanced $(n/2,m)$-function. These functions are hyper-bent in the sense that their component functions are hyper-bent. In [804] is given their expression in the *polar representation* that we saw at page 191.
- Bent $(n+n',m)$-functions (where n' is also even) can be defined in the form: $F(x,y) = K(\frac{x}{y}, \frac{z}{t})$ where K is a $(\frac{n+n'}{2}, m)$-function such that, for all $x \in \mathbb{F}_{2^{n/2}}$, the function $y \in \mathbb{F}_{2^{n'/2}} \mapsto K(x,y)$ is balanced and for all $y \in \mathbb{F}_{2^{n'/2}}$, the function $x \in \mathbb{F}_{2^{n/2}} \mapsto K(x,y)$ is balanced.
- Bent $(n,n/2)$-functions from class $\mathcal{H}$ of bent Boolean functions (see page 244) are defined as: $F(x,y) = xG(yx^{2^{n/2}-2})$, where G is an o-polynomial on $\mathbb{F}_{2^{n/2}}$, see [862]. A version in univariate form can be found in [479], see also [310].
- Bent (n,m)-functions are built from m-dimensional vector spaces of functions whose nonzero elements are all bent. Examples are $(n,2)$-functions derived from the Kerdock codes, see [310]. Another example (found by the author in common with G. Leander) takes $n \equiv 2$ [mod 4]; then $\mathbb{F}_{2^{\frac{n}{2}}}$ consists of cubes only (since $\gcd(3, 2^{\frac{n}{2}}-1) = 1$). If $w \in \mathbb{F}_{2^n}$ is not a cube, then all the nonzero elements of the vector space $E = w\mathbb{F}_{2^{\frac{n}{2}}}$

are non-cubes. Then if $F(z) = z^d$ where $d = 2^i + 1$ (Gold exponent) or $2^{2i} - 2^i + 1$ (Kasami exponent) and $\gcd(n, i) = 1$, all the functions $tr_n(vF(z))$, where $v \in E^*$, are bent. This leads to the bent $(n, \frac{n}{2})$-functions $z \in \mathbb{F}_{2^n} \to (tr_n(\beta_1 wz^d), \ldots, tr_n(\beta_{\frac{n}{2}} wz^d) \in \mathbb{F}_2^{\frac{n}{2}}$, where $(\beta_1, \ldots, \beta_{\frac{n}{2}})$ is a basis of $\mathbb{F}_{2^{\frac{n}{2}}}$ over $\mathbb{F}_2$. To make such function valued in $\mathbb{F}_{2^{\frac{n}{2}}}$, we choose a basis $(\alpha_1, \ldots, \alpha_{\frac{n}{2}})$ of $\mathbb{F}_{2^{\frac{n}{2}}}$ orthogonal to $(\beta_1, \ldots, \beta_{\frac{n}{2}})$, that is, such that $tr_{\frac{n}{2}}(\alpha_i \beta_j) = \delta_{i,j}$ (the Kronecker symbol). For every $y \in \mathbb{F}_{2^{\frac{n}{2}}}$, we have then $y = \sum_{j=1}^{\frac{n}{2}} \alpha_j tr_{\frac{n}{2}}(\beta_j y)$. The image of every $z \in \mathbb{F}_{2^n}$ by the function equals $\sum_{j=1}^{\frac{n}{2}} \alpha_j tr_n(\beta_j wz^d) = \sum_{j=1}^{\frac{n}{2}} \alpha_j tr_{\frac{n}{2}}(\beta_j(wz^d + (wz^d)^{2^{\frac{n}{2}}})) = wz^d + (wz^d)^{2^{\frac{n}{2}}}$. In the case of the Gold exponent, it can be made a function from $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$ to $\mathbb{F}_{2^{\frac{n}{2}}}$: we express z in the form $x + wy$ where $x, y \in \mathbb{F}_{2^{\frac{n}{2}}}$ and if n is not a multiple of 3, we can take w primitive in $\mathbb{F}_4$ (otherwise, all elements of $\mathbb{F}_4$ are cubes and we have then to take w outside $\mathbb{F}_4$), for which we have then $w^2 = w + 1$, $w^{2^i} = w^2$ (since i is necessarily odd) and $w^{2^i+1} = w^3 = 1$. We have then $z^d = x^{2^i+1} + wx^{2^i}y + w^2xy^{2^i} + y^{2^i+1}$ and $wz^d + (wz^d)^{2^{\frac{n}{2}}} = (w + w^2)x^{2^i+1} + (w^2 + w)x^{2^i}y + (w^3 + w^3)xy^{2^i} + (w + w^2)y^{2^i+1} = x^{2^i+1} + x^{2^i}y + y^{2^i+1}$. We can extend the construction to $\gcd(i, n) \neq 1$; the exact condition is that $\frac{n}{\gcd(i,n)}$ is even and $v \notin \{x^d, x \in \mathbb{F}_{2^n}\}$.

Constructions of bent vectorial functions in *univariate representation*

The bent (n, m)-functions built from m-dimensional vector spaces of functions above provide first examples, like $tr^n_{n/2}(wx^d)$, where w is not a cube and $d = 2^i+1$ or 4^i-2^i+1, $\gcd(i, n) = 1$. The other functions above, which are defined in bivariate representation (over $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$ and valued in $\mathbb{F}_{2^{\frac{n}{2}}}$), can be seen in univariate representation, from $\mathbb{F}_{2^n}$ to itself. If $\frac{n}{2}$ is odd, this is quite easy: we have then $\mathbb{F}_{2^{\frac{n}{2}}} \cap \mathbb{F}_4 = \mathbb{F}_2$ and we can choose the basis $(1, w)$ of the 2-dimensional vector space $\mathbb{F}_{2^n}$ over $\mathbb{F}_{2^{\frac{n}{2}}}$, where w is a primitive element of $\mathbb{F}_4$. Then $w^2 = w + 1$ and $w^{2^{\frac{n}{2}}} = w^2$ since $\frac{n}{2}$ is odd. A general element of $\mathbb{F}_{2^n}$ has the form $z = x + wy$ where $x, y \in \mathbb{F}_{2^{\frac{n}{2}}}$ and we have $z^{2^{\frac{n}{2}}} = x + w^2y = z + y$ and therefore $y = z + z^{2^{\frac{n}{2}}}$, and $x = z^{2^{\frac{n}{2}}} + w^2y = w^2z + wz^{2^{\frac{n}{2}}}$. For instance, the univariate representation of the simplest Maiorana-McFarland function, that is the function $(x, y) \to xy$, is $(z + z^{2^{\frac{n}{2}}})(w^2z + wz^{2^{\frac{n}{2}}})$, that is, up to linear terms: $z^{1+2^{\frac{n}{2}}}$.
We describe now the constructions which are given directly in univariate form. In [935] is observed that if $tr_n(ax^d)$ is a bent Boolean function and x^d permutes $\mathbb{F}_{2^m}$ for some divisor m of $n \geq 4$, then $tr^n_m(ax^d)$ is bent (the double condition is necessary if $m = n/2$, see [1133]); more is obtained in [892] for multiple trace term functions with Dillon-like exponents. In [483, 935, 1064] are studied (further) bent vectorial functions of the form $tr^n_{n/2}(ax^d)$. All functions $tr^n_m(ax^d)$ where m divides n are addressed in the recent paper [1133], where is proved that if $m \mid n$ and $\gcd\left(2^m - 1, \frac{2^n-1}{2^m-1}\right) = 1$, and if the (n, m)-function $tr^n_m(ax^d)$ is bent, then $\gcd\left(d, \frac{2^n-1}{2^m-1}\right) \neq 1$. Characterizations are given when d is a Gold 2^i+1 (with

any i), a Kasami $2^{2i}-2^i+1$ (idem), a Leander $(2^{n/4}+1)^2$, a Canteaut-Charpin-Khyureghyan $2^{n/3}+2^{n/6}+1$ and a Dillon $j\cdot(2^{\frac{n}{2}}-1)$ exponent (with precisions and corrections of errors from previous papers) as well as functions with multiple terms with Niho and Dillon exponents. The authors of [483] also propose a method to construct bent vectorial functions based on $\mathcal{PS}^-$ and $\mathcal{PS}^+$ bent functions. In [892] are derived three necessary and sufficient conditions for a function of the form $F(x)=tr^n_{n/2}(\sum_{i=1}^r a_i x^{r_i(2^{n/2}-1)})$ to be bent. The first characterization is a direct consequence of a result in [854]. The second characterization provides an interesting link between the bentness of F and its evaluation on the cyclic group U. The third characterization is stated in terms of the evaluation of certain elementary symmetric polynomials, and can be transformed into some explicit conditions regarding the choice of some coefficients. In [961] are studied the quadratic vectorial functions of the form $F(x)=tr^n_{n/2}(ax^{2^i}(x^{2^j}+(x^{2^j})^{2^{n/2}}))$, where $n\geq 4$ is even and $a\notin\mathbb{F}_{2^{n/2}}$, which are all bent.

The existence, and the constructions in case of existence, of bent vectorial functions of the form $tr^n_{n/2}(P(x))$ where $P(x)\in\mathbb{F}_{2^n}[x]$ has been studied on the basis of known Boolean bent functions of the form $tr_n(P(x))$. For instance, the nonexistence of some bent vectorial functions with binomial trace representation in $\mathcal{PS}^-$ has been proved in [930, 931]: for $n\equiv 0 \pmod 4$, there is no bent vectorial function of the form $F(x)=tr^n_{n/2}(x^{2^{n/2}-1}+ax^{r(2^{n/2}-1)})$ where $1\leq r\leq 2^{n/2}$ and $a\in\mathbb{F}_{2^n}$.

We have seen at pages 47 and 297 that CCZ equivalence on bent functions coincides with EA equivalence and then does not provide new (bent) functions. However, applied to non-bent functions, it can give functions having some bent components and lead to bent vectorial functions with less output bits (but possibly larger algebraic degree). Examples like $F(x)=x^{2^i+1}+(x^{2^i}+x+1)tr_n(x^{2^i+1})$, for $n\geq 6$ even and $F(x)=\big(x+tr^n_3(x^{2(2^i+1)}+x^{4(2^i+1)})+tr_n(x)tr^n_3(x^{2^i+1}+x^{2^{2i}(2^i+1)})\big)^{2^i+1}$, where $6\,|\,n$ and in both cases $\frac{n}{\gcd(i,n)}$ even, are given in [150] (deduced from functions in [163]). Ideas for deriving bent vectorial functions from AB functions are given in [248, Subsection 4.3].

In [1143], Youssef and Gong have extended the notion of *hyper-bent function* to vectorial functions: such F is called hyper-bent if all its *component functions* are hyper-bent. Muratović-Ribić, Pasalic, and Bajrić [893] have characterized a class of vectorial hyper-bent functions of the form $F(x)=tr^n_{n/2}(\sum_{i=0}^{2^{n/2}} a_i x^{i(2^{n/2}-1)})$ from the class $\mathcal{PS}_{ap}$, and determined the number of such hyper-bent functions.

6.4.2 Secondary constructions of bent vectorial functions

Given any bent (n,m)-function F, any chopped function obtained by deleting some coordinates of F (or more generally by composing it on the left with any surjective affine mapping) is obviously still bent. But there exist other more useful *secondary constructions* (that is, constructions of new bent functions from known ones). The secondary construction of Boolean bent functions of Proposition 79, page 236, generalizes directly to vectorial functions [234]:

Proposition 105 *Let r and s be two positive integers with the same parity and such that $r \leq \frac{s}{3}$. Let ψ be any (balanced) mapping from $\mathbb{F}_2^s$ to $\mathbb{F}_{2^r}$ such that, for every $a \in \mathbb{F}_{2^r}$, the set $\psi^{-1}(a)$ is an $(s-r)$-dimensional affine subspace of $\mathbb{F}_2^s$. Let H be any (s,r)-function whose restriction to $\psi^{-1}(a)$ (viewed as an $(s-r,r)$-function via an affine isomorphism between $\psi^{-1}(a)$ and $\mathbb{F}_2^{s-r}$) is bent for every $a \in \mathbb{F}_{2^r}$. Then the function $F_{\psi,H}(x,y) = x\,\psi(y) + H(y)$, $x \in \mathbb{F}_{2^r}, y \in \mathbb{F}_2^s$, is a bent function from $\mathbb{F}_2^{r+s}$ to $\mathbb{F}_{2^r}$.*

Indeed, taking $x \cdot y = tr_r(xy)$ for inner product in $\mathbb{F}_{2^r}$, for every $v \in \mathbb{F}_{2^r}^*$, the function $tr_r(v\,F_{\psi,H}(x,y))$ is bent, according to Proposition 79, with $\phi(y) = v\,\psi(y)$ and $g(y) = tr_r(v\,H(y))$ (the more restrictive condition $r \leq \frac{s}{3}$ is meant so that $r \leq \frac{s-r}{2}$, which is necessary, according to Proposition 104, for allowing the restrictions of H to be bent). The condition on ψ being easily satisfied[35], it is then a simple matter to choose H. Hence, this construction is quite effective (but only for designing bent (n,m)-functions such that $m \leq n/4$, since $r \leq \frac{s}{3}$ is equivalent to $r \leq \frac{r+s}{4}$).

The construction of Theorem 15, page 260, can also be adapted to vectorial functions as follows [234]:

Proposition 106 *Let r and s be two positive even integers and m a positive integer such that $m \leq r/2$. Let H be a function from $\mathbb{F}_2^n = \mathbb{F}_2^r \times \mathbb{F}_2^s$ to $\mathbb{F}_2^m$. Assume that, for every $y \in \mathbb{F}_2^s$, the function $H_y : x \in \mathbb{F}_2^r \to H(x,y)$ is a bent (r,m)-function. For every nonzero $v \in \mathbb{F}_2^m$ and every $a \in \mathbb{F}_2^r$ and $y \in \mathbb{F}_2^s$, let us denote by $f_{a,v}(y)$ the value at a of the dual of the Boolean function $v \cdot H_y$, defined by $\sum_{x \in \mathbb{F}_2^r} (-1)^{v \cdot H(x,y) \oplus a \cdot x} = 2^{r/2}(-1)^{f_{a,v}(y)}$. Then H is bent if and only if, for every nonzero $v \in \mathbb{F}_2^m$ and every $a \in \mathbb{F}_2^r$, the Boolean function $f_{a,v}$ is bent.*

Indeed, we have, for every nonzero $v \in \mathbb{F}_2^m$ and every $a \in \mathbb{F}_2^r$ and $b \in \mathbb{F}_2^s$:

$$\sum_{\substack{x \in \mathbb{F}_2^r \\ y \in \mathbb{F}_2^s}} (-1)^{v \cdot H(x,y) \oplus a \cdot x \oplus b \cdot y} = 2^{r/2} \sum_{y \in \mathbb{F}_2^s} (-1)^{f_{a,v}(y) \oplus b \cdot y}.$$

An example of application of Proposition 106 is when we choose every H_y in the Maiorana-McFarland's class: $H_y(x,x') = x\,\pi_y(x') + G_y(x')$, $x,x' \in \mathbb{F}_{2^{r/2}}$, where π_y is bijective for every $y \in \mathbb{F}_2^s$. According to the results on the duals of Maiorana-McFarland's functions, for every $v \in \mathbb{F}_{2^{r/2}}^*$ and every $a,a' \in \mathbb{F}_{2^{r/2}}$, we have then $f_{(a,a'),v}(y) = tr_{\frac{r}{2}}\left(a'\,\pi_y^{-1}\left(\frac{a}{v}\right) + v\,G_y\left(\pi_y^{-1}\left(\frac{a}{v}\right)\right)\right)$, where $tr_{\frac{r}{2}}$ is the trace function from $\mathbb{F}_{2^{r/2}}$ to $\mathbb{F}_2$. Then H is bent if and only if, for every $v \in \mathbb{F}_{2^{r/2}}^*$ and every $a,a' \in \mathbb{F}_{2^{r/2}}$, the function $y \to tr_{\frac{r}{2}}\left(a'\,\pi_y^{-1}(a) + v\,G_y(\pi_y^{-1}(a))\right)$ is bent on $\mathbb{F}_2^s$. A simple possibility for achieving this is for $s = r/2$ to choose π_y^{-1} such that, for every a, the mapping $y \to \pi_y^{-1}(a)$ is an affine automorphism of $\mathbb{F}_{2^{r/2}}$ (*e.g.*

[35] Note that it does not make ψ necessarily affine.

$\pi_y^{-1}(a) = \pi_y(a) = a + y$) and to choose G_y such that, for every a, the function $y \to G_y(a)$ is bent.

An obvious corollary of Proposition 106 is that the so-called *direct sum of bent functions* gives bent functions: we define $H(x, y) = F(x) + G(y)$, where F is any bent (r, m)-function and G any bent (s, m)-function, and we have then $f_{a,v}(y) = v \cdot F(a) \oplus v \cdot G(y)$, which is a bent Boolean function for every a and every $v \neq 0_m$. Hence, H is bent.

Remark. Identifying $\mathbb{F}_2^m$ with $\mathbb{F}_{2^m}$ and defining $H(x, y) = F_1(x) + G_1(y) + (F_1(x)+F_2(x))\,(G_1(y)+G_2(y))$, a component function $v \cdot H_y(x) = tr_m(v\,F_1(x)) + tr_m(v\,G_1(y)) + tr_m\,(v\,(F_1(x) + F_2(x))\,(G_1(y) + G_2(y)))$ does not enter, in general, in the framework of Proposition 83 nor of Proposition 106. Note that the function $f_{a,v}$ exists under the sufficient condition that, for every nonzero ordered pair $(v, w) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$, the function $tr_m(v\,F_1(x)) + tr_m(w\,F_2(x))$ is bent (which is equivalent to saying that the $(r, 2m)$-function (F_1, F_2) is bent).
There are particular cases where the construction works, as shown in [310]: let F_1 and F_2 be two bent (n, r)-functions and $G = (g_1, \ldots, g_{r+1})$ an $(m, r + 1)$-function such that for every nonzero v in $\mathbb{F}_2^{r+1}$ different from $(1, 0, \ldots, 0)$, the component function $v \cdot G$ is bent, then the function $H(x, y) = F_1(x) + G_1(y) + g_1(y)(F_1(x) + F_2(x))$, where G_1 is the (m, r)-function $(g_2, \ldots, g_{r+1})$, is a bent $(n + m, r)$-function. This indirect sum has been generalized in [310]. □

Remark. In [18], bent$_4$ functions have been extended to *vectorial bent$_4$ functions* (over finite fields), which correspond to relative difference sets in certain groups. The authors have provided conditions under which Maiorana-McFarland functions are bent$_4$. □

6.5 Plateaued vectorial functions

There exist three notions of plateauedness for vectorial functions:

Definition 67 *An (n, m)-function is called* strongly plateaued *if all its* component functions $v \cdot F$*; $v \in \mathbb{F}_2^m$, $v \neq 0_m$, where "$\cdot$" is an inner product in $\mathbb{F}_2^m$, are partially-bent (see Definition 62, page 283).*
An (n, m)-function is called plateaued with single amplitude *if all its component functions are plateaued with the same amplitude (see Definition 63, page 285).*
An (n, m)-function is called plateaued *if all its component functions are plateaued, with possibly different amplitudes.*

The reason why the first notion is called strongly plateaued will be made clear with Corollary 18 below. The two first notions are independent in the sense that none is a particular case of the other (there exist indeed strongly plateaued vectorial functions with different amplitudes and plateaued functions with single amplitude which are not strongly plateaued). Both are a particular case of the third. Quadratic functions (which are all strongly plateaued) can have components with different amplitudes (this is the case for instance of the Gold functions x^{2^i+1}, $gcd(i,n) = 1$, for n even). They can also have single amplitude (this is the case of Gold functions for n odd). Of course, the two definitions of plateaued functions and of plateaued with single amplitude functions coincide for Boolean functions.

Note that, since the Walsh transform values of plateaued (n,m)-functions are divisible by $2^{\lceil \frac{n}{2} \rceil}$ and the Walsh transform of F equals the Fourier transform of the *indicator* $1_{\mathcal{G}_F}$ of its *graph* $\mathcal{G}_F$, the algebraic degree of $1_{\mathcal{G}_F}$ is at most $n+m-\lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor + m$, according to Theorem 2, page 82. Applying Relation (2.7), page 57, we have then that, for every subset J of $\{1,\ldots,m\}$, we have $d_{alg}\big(\prod_{j\in\{1,\ldots,m\}\setminus J}(f_j \oplus 1)\big) \leq \lfloor \frac{n}{2} \rfloor + m - |J|$, where the f_j's are the coordinate functions of F. And if F is plateaued with single amplitude 2^r, then we have $d_{alg}\big(\prod_{j\in\{1,\ldots,m\}\setminus J}(f_j \oplus 1)\big) \leq n+m-r-|J|$. This gives much more information than the single inequality $d_{alg}(F) \leq \lfloor \frac{n}{2} \rfloor + 1$ (resp. $\leq n-r+1$) provided by Proposition 96, page 286.

It has been proved in [174] that, when n is a power of 2, no power plateaued (n,n)-permutation exists[36] and in [835] that, when n is divisible by 4, no such function exists with Walsh spectrum $\{0, \pm 2^{\frac{n}{2}+1}\}$.

The set of plateaued vectorial functions with single amplitude is *CCZ invariant*: if the graphs $\{(x,F(x));\ x\in \mathbb{F}_2^n\}$ and $\{(x,G(x));\ x\in\mathbb{F}_2^n\}$ of two (n,m)-functions F,G correspond to each other by an affine permutation of $\mathbb{F}_2^n \times \mathbb{F}_2^m$, then one is plateaued with single amplitude if and only if the other is. The larger set of plateaued vectorial functions is (only) *EA invariant*: it is indeed invariant under composition on the right by affine automorphisms and under addition of an affine function, and it is also invariant under composition on the left by a linear automorphism L since $W_{L\circ F}(u,v) = W_F(L^*(v),u)$, where L^* is the adjoint operator of L.

6.5.1 Characterizations of plateaued vectorial functions

The characterization of plateaued Boolean functions by Proposition 97, page 287, has been generalized to vectorial functions for each notion, by means of the value distributions of their *derivative*s. This allowed to derive several characterizations of APN functions in this framework. Characterizations of plateaued vectorial functions have been also obtained by means of their autocorrelation functions and of the power moments of their Walsh transforms. We survey below all these results from [247].

[36] A conjecture by T. Helleseth states that there is no power permutation having 3 Walsh transform values when n is a power of 2.

Characterization by means of the derivatives

Applying Proposition 97, page 287, an (n,m)-function F is plateaued if and only if, for every $v \in \mathbb{F}_2^m$, the expression $\sum_{a,b\in\mathbb{F}_2^n}(-1)^{v\cdot D_aD_bF(x)}$ does not depend on $x \in \mathbb{F}_2^n$ and F is plateaued with single amplitude if and only if this sum does not depend on x nor on $v \neq 0_m$.

Theorem 18 *[247] Let F be an (n,m)-function. Then:*

- *F is plateaued if and only if, for every $w \in \mathbb{F}_2^m$, the size of the set*
$$\{(a,b) \in (\mathbb{F}_2^n)^2\,;\, D_aD_bF(x) = w\} \tag{6.41}$$
does not depend on $x \in \mathbb{F}_2^n$ (in other words, the value distribution of $D_aD_bF(x)$ when (a,b) ranges over $(\mathbb{F}_2^n)^2$ is independent of $x \in \mathbb{F}_2^n$).
- *F is plateaued with single amplitude if and only if the size of the set in (6.41) does not depend on $x \in \mathbb{F}_2^n$, nor on $w \in \mathbb{F}_2^m$ when $w \neq 0_m$.*

Moreover:

- *for every (n,m)-function F, the value distribution of $D_aD_bF(x)$ when (a,b) ranges over $(\mathbb{F}_2^n)^2$ equals the value distribution of $D_aF(b) + D_aF(x)$,*
- *if two plateaued functions F, G have the same such distribution, then for every v, their component functions $v \cdot F$ and $v \cdot G$ have the same amplitude.*

Proof. Recall that any two integer-valued functions over $\mathbb{F}_2^m$ are equal if and only if their Fourier transforms are equal, and that any integer-valued function is constant except at 0_m if and only if its Fourier transform is constant except at 0_m as well. Applying this to the functions $v \mapsto \sum_{a,b\in\mathbb{F}_2^n}(-1)^{v\cdot D_aD_bF(x)}$ for different values of x, we deduce that F is plateaued if and only if, for every $w \in \mathbb{F}_2^m$, the sum $\sum_{v\in\mathbb{F}_2^m}\sum_{a,b\in\mathbb{F}_2^n}(-1)^{v\cdot D_aD_bF(x)\oplus v\cdot w}$, which is equal to $\sum_{a,b\in\mathbb{F}_2^n}\sum_{v\in\mathbb{F}_2^m}(-1)^{v\cdot(D_aD_bF(x)+w)} = 2^m|\{(a,b) \in (\mathbb{F}_2^n)^2\,;\, D_aD_bF(x) = w\}|$ does not depend on $x \in \mathbb{F}_2^n$, and F is plateaued with single amplitude if and only if this size does not depend on x nor on $w \neq 0_m$. This proves the first part.
By the change of variable $b \mapsto b+x$, we have that $|\{(a,b) \in (\mathbb{F}_2^n)^2\,;\, D_aD_bF(x) = w\}|$ equals $|\{(a,b) \in (\mathbb{F}_2^n)^2\,;\, D_aD_{b+x}F(x) = w\}|$, that is, $|\{(a,b) \in (\mathbb{F}_2^n)^2\,;\, F(x)+F(x+a)+F(b)+F(b+a) = w\}|$. This proves the first item of the second part.
The last item is a direct consequence of the fact that, for a plateaued function F, the sum $\sum_{a,b\in\mathbb{F}_2^n}(-1)^{v\cdot D_aD_bF(x)}$ equals the square of the amplitude of $v\cdot F$. □

It is observed in [194, 196] that $|\{(a,b,x) \in (\mathbb{F}_2^n)^3\,;\, D_aD_bF(x) = 0_n, a \neq 0_n, b \neq 0_n, a \neq b\}| \leq (2^n-1)(\max_{u,v\in\mathbb{F}_2^n, v\neq 0_n} W_F(u,v)^2 - 2^{n+1})$, for every (n,n)-function F, with equality if and only if F is plateaued with single amplitude (this is a straightforward consequence of the calculations made in the proof of the SCV bound, see Theorem 6, page 140).

Note that the algebraic degree $d = 2$ (for which the first item of Theorem 18 is straightforwardly satisfied since the second-order derivatives are then constant) is the only one for which all functions of algebraic degree at most d are plateaued, since we know that cubic Boolean functions can have values of the Walsh transform at 0_n different from 0 and from powers of 2, see Section 5.3, page 204, and therefore be non-plateaued.

Examples. 1. Almost bent (AB) functions, see Definition 31, page 141, are an example of plateaued functions with single amplitude. The distribution of values of the second-order derivatives in Relation (6.41) is as follows: the equation $D_aD_bF(x) = w$ has $3 \cdot 2^n - 2$ solutions (a, b) for any x if $w = 0_n$ and $2^n - 2$ solutions if $w \neq 0_n$ (see Corollary 27, page 409). Conversely, any function having this property is AB.
2. Let n now be even and $F(x) = x^{2^i+1}$ be a Gold APN function, $(i, n) = 1$. We have $D_aD_bF(x) = a^{2^i}b + ab^{2^i}$. The number of solutions (a, b) of $D_aD_bF(x) = 0$ equals again $3 \cdot 2^n - 2$ (as for any APN function), and for $w \neq 0$ the number of solutions (a, b) of $D_aD_bF(x) = w$ is constant when w ranges over a coset of the multiplicative group of all cubes in $\mathbb{F}_{2^n}^*$, since for every $\lambda \in \mathbb{F}_{2^n}^*$, $(\lambda a)^{2^i}(\lambda b) + (\lambda a)(\lambda b)^{2^i} = \lambda^{2^i+1}(a^{2^i}b + ab^{2^i})$ and $\lambda \mapsto \lambda^{2^i+1}$ is 3-to-1 over $\mathbb{F}_{2^n}^*$ and has the group of cubes for range. This allows taking $w = 1$ without loss of generality when w is a cube, and $a^{2^i}b + ab^{2^i} = 1$ is equivalent when $b \neq 0$ to $\left(\frac{a}{b}\right)^{2^i} + \frac{a}{b} = \frac{1}{b^{2^i+1}}$ and has two solutions a for every b such that $\frac{1}{b^{2^i+1}}$ has null trace (and none otherwise). The number of such nonzero b equals $2^{n-1} \pm 2^{\frac{n}{2}} - 1$ since $f(x) = tr_n(x^{2^i+1})$ has the same Hamming weight as $tr_n(x^3)$, which is $2^{n-1} \pm 2^{\frac{n}{2}}$ according to Carlitz' result recalled at page 201. When w is not a cube, $a^{2^i}b + ab^{2^i} = w$ is equivalent when $b \neq 0$ to $\left(\frac{a}{b}\right)^{2^i} + \frac{a}{b} = \frac{w}{b^{2^i+1}}$ and has two solutions a for every b such that $\frac{w}{b^{2^i+1}}$ has null trace. The number of such nonzero b equals $2^{n-1} \pm 2^{\frac{n}{2}-1} - 1$ since $tr_n(wb^{2^i+1})$ is bent (see page 230). Hence the number of solutions (a, b) of $D_aD_bF(x) = w$ equals:

$$\begin{cases} 3 \cdot 2^n - 2 & \text{for } w = 0, \\ 2^n \pm 2^{\frac{n}{2}+1} - 2 & \text{for } w \text{ a nonzero cube } (\frac{2^n-1}{3} \text{ cases}) \\ 2^n \pm 2^{\frac{n}{2}} - 2 & \text{for } w \text{ a non-cube } (2 \cdot \frac{2^n-1}{3} \text{ cases}), \end{cases}$$

where, among the two "$\pm$" above, one is a "$+$" and one is a "$-$". We shall see below that the Kasami APN functions (see page 433) have the same distribution.
3. The case of functions $F(x, y) = (x\pi(y) + \phi(y), x(\pi(y))^{2^i} + \psi(y))$ (which are plateaued (n, n)-functions when π is a permutation, as we shall see in Proposition 115, page 311) is studied in [247]. □

A particular case where the condition of Theorem 18 is satisfied is when, for each fixed value of a, the value distribution of the function $b \mapsto D_aD_bF(x)$ is independent of x. It is easily seen, as in the proof of Theorem 18, that an (n, m)-function F has this property if and only if all of its component functions have it, and that, for every Boolean function f, the size, for every $a \in \mathbb{F}_2^n, w \in \mathbb{F}_2$, of the set $\{b \in \mathbb{F}_2^n\,;\, D_af(b) = D_af(x) + w\}$ does not depend on x if and only if the derivatives of f are either constant or balanced, that is, f is partially-bent. The condition is indeed sufficient, and it is necessary because if D_af is not constant then it means that $\{b \in \mathbb{F}_2^n\,;\, D_af(b) = 0\}| = \{b \in \mathbb{F}_2^n\,;\, D_af(b) = 1\}$. Hence:

Corollary 18 *[247] A vectorial function F is strongly plateaued if and only if, for every a in $\mathbb{F}_2^n$ and every w, the size of the set $\{b \in \mathbb{F}_2^n\,;\, D_aD_bF(x) = w\}$ does not depend on $x \in \mathbb{F}_2^n$, or equivalently the size of the set $\{b \in \mathbb{F}_2^n\,;\, D_aF(b) = D_aF(x) + w\}$ does not depend on $x \in \mathbb{F}_2^n$.*

Proposition 107 *[247] For every* strongly plateaued *(n, m)-function F, the image set $Im(D_aF) = (D_aF)(\mathbb{F}_2^n)$ of any derivative D_aF is an affine space.*

Proof. By hypothesis, every derivative D_aF of F matches the same number of times any two values $D_aF(x) + w$ and $D_aF(y) + w$. Hence, it matches at least once $D_aF(x)+w$ (*i.e.*, we have $w \in D_aF(x)+Im(D_aF)$) if and only if it matches at least once $D_aF(y)+w$ (*i.e.*, we have $w \in D_aF(y)+Im(D_aF)$). Hence, the set $Im(D_aF)$ is invariant under translation by any element of $Im(D_aF)+Im(D_aF)$ and is then an affine space. □

Crooked functions

According to Proposition 107, if F is a *strongly plateaued* APN (n, n)-function, then it is a so-called *crooked* function, in the sense[37] of [727, 80, 729]:

Definition 68 *An (n, n)-function F is called crooked if, for every nonzero a, the set $\{D_aF(x); x \in \mathbb{F}_2^n\}$ is an affine hyperplane (*i.e. *a linear hyperplane or its complement).*

Conversely, crooked functions are strongly plateaued (and APN), *i.e.* their component functions are partially-bent [730, 247, 252], because the affine hyperplane $\{D_aF(x), x \in F_2^n\}$ is matched twice and the function $y \to v \cdot y$ restricted to an affine hyperplane is either constant or balanced for every v. This allows to show more directly some results which were first obtained in [729, 730]): crooked functions are plateaued, and for n odd, they are then AB (since we know that "plateaued APN" implies AB for n odd; see Proposition 163, page 414). Their *component functions* being partially-bent, they all satisfy $N_{\Delta_{v\cdot F}} \times N_{W_{v\cdot F}} = 2^n$ (see page 283); therefore, in the case of n odd, we have $N_{\Delta_{v\cdot F}} = 2$ for every $v \neq 0_n$, that is, there exists a unique $a \neq 0_n$ such that $\Delta_{v\cdot F}(a) \neq 0$, *i.e.* $\{D_aF(x), x \in \mathbb{F}_2^n\} = \{0_n, v\}^\perp$ or $\{D_aF(x), x \in \mathbb{F}_2^n\} = \mathbb{F}_2^n \setminus \{0_n, v\}^\perp$. And for every n, a function F is crooked if and only if, for every $a \neq 0_n$, there exists a unique $v \neq 0_n$ such that $W_{D_aF}(0_n, v) = \Delta_{v\cdot F}(a) \neq 0$ and

[37] This is nowadays the most used definition of crooked functions, but originally in [57], they were defined such that, for every nonzero a, the set $\{D_aF(x); x \in \mathbb{F}_2^n\}$ is the complement of a linear hyperplane; this restricted definition required that crooked functions be bijective; they were also AB; some authors call "generalized crooked" the functions we call crooked here.

then $\Delta_{v\cdot F}(a) = 2^n$ and $\{D_aF(x), x \in \mathbb{F}_2^n\} = \{0_n, v\}^\perp$ or $\Delta_{v\cdot F}(a) = -2^n$ and $\{D_aF(x), x \in \mathbb{F}_2^n\} = \mathbb{F}_2^n \setminus \{0_n, v\}^\perp$ (there is then a function mapping any $a \neq 0$ to $v \neq 0$ such that $\Delta_{v\cdot F}(a) \neq 0$; for n odd, this mapping is bijective). Indeed, a set E is an affine hyperplane if and only if (1) there exists a unique $v \neq 0_n$ such that $\sum_{y\in E}(-1)^{v\cdot y}$ equals $\pm|E|$ and (2) such sum is null for any other v. This characterization can be expressed by means of the Walsh transform of F since $W_{D_aF}(0_n, v) = \Delta_{v\cdot F}(a) = 2^{-n}\sum_{u\in\mathbb{F}_2^n} W_F^2(u, v)(-1)^{u\cdot a}$.
Of course, all quadratic APN functions are crooked; the question of knowing whether non-quadratic crooked functions exist is open. It is proved in [728, 729] that the reply is no for *power functions* (monomials) and in [80] that it is no for binomials.
Assuming that $F(0_n) = 0_n$, it is proved in [57] that the set $H_a = \{D_aF(x); x \in \mathbb{F}_2^n\}$ is the complement of a linear hyperplane for every nonzero a (*i.e.* F is crooked in the original restricted sense of [57]) if and only if F is APN and for every nonzero a, we have $D_aF(x) + D_aF(y) + D_aF(z) \neq 0_n$ for every x, y, z; n is then necessarily odd. Then, F is bijective (take $x = y = z$) and AB, and we have seen that all the sets H_a, for $a \neq 0_n$, are distinct (and therefore every complement of a linear hyperplane equals H_a for some unique $a \neq 0_n$). More characterizations are given in [539], in relation with nonlinear codes. Note that crookedness may represent a weakness, see [200].

The case of power functions
It is often simpler to consider power functions than general functions. In the case of plateaued functions, we have:

Corollary 19 *[247] Let $F(x) = x^d$ be any power function. Then, for every $w \in \mathbb{F}_{2^n}$, every $x \in \mathbb{F}_{2^n}$, and every $\lambda \in \mathbb{F}_{2^n}^*$, $|\{(a, b) \in \mathbb{F}_{2^n}^2;\ D_aF(b) + D_aF(x) = w\}|$ equals $|\{(a, b) \in \mathbb{F}_{2^n}^2;\ D_aF(b)+D_aF(x/\lambda) = w/\lambda^d\}|$ and $|\{(a, b) \in \mathbb{F}_{2^n}^2;\ D_aF(b)+D_aF(0) = w\}|$ is invariant when w is multiplied by any d-th power in $\mathbb{F}_{2^n}^*$. Then:*

- *F is* plateaued *if and only if, for every $w \in \mathbb{F}_{2^n}$:*

$$|\{(a, b) \in \mathbb{F}_{2^n}^2;\ D_aF(b)+D_aF(1) = w\}| = |\{(a, b) \in \mathbb{F}_{2^n}^2;\ D_aF(b)+D_aF(0) = w\}|;$$

- *F is plateaued with single amplitude if and only if additionally this common size does not depend on $w \neq 0$.*

If d is co-prime with $2^n - 1$, then F is plateaued if and only if it is plateaued with single amplitude.

This is a more or less direct consequence of the fact that, for every $\lambda \neq 0$, we have $D_{\lambda a}F(\lambda x) = \lambda^d D_aF(x)$.

The case of unbalanced components
In the particular case where all the *component functions* of a function are unbalanced (we shall see that this is for instance the case of all APN power functions x^d when n is even, since they satisfy, as proved by Dobbertin, see Proposition

165, page 417, that $\gcd(d, 2^n - 1) = 3$), plateauedness is simpler to study because, for each v, the value of $|W_F(0_n, v)|$ being nonzero, equals the amplitude of the component function $v \cdot F$. Hence, according to Proposition 97, page 287, if F is plateaued with unbalanced components then, for every v, x, the sum $\sum_{a,b \in \mathbb{F}_2^n} (-1)^{v \cdot D_a D_b F(x)}$ equals $W_F^2(0_n, v) = \sum_{a,b \in \mathbb{F}_2^n} (-1)^{v \cdot (F(a)+F(b))}$. The converse is straightforward too since, when constant, $\sum_{a,b \in \mathbb{F}_2^n} (-1)^{v \cdot D_a D_b F(x)}$ is equal to the squared amplitude and cannot then be null, and this gives by the same method as in the proof of Theorem 18:

Theorem 19 *[247] Let F be any (n, m)-function. Then F is plateaued with component functions all unbalanced if and only if, for every $w, x \in \mathbb{F}_2^n$, we have:*

$$\left|\{(a, b) \in (\mathbb{F}_2^n)^2\,;\, D_a D_b F(x) = w\}\right| = \left|\{(a, b) \in (\mathbb{F}_2^n)^2\,;\, F(a) + F(b) = w\}\right|.$$

Moreover, F is then plateaued with single amplitude if and only if, additionally, this common value does not depend on w for $w \neq 0_n$.

This theorem will have interesting consequences in Subsection 11.3, page 403.

Characterization by means of the autocorrelation functions and related value distributions

We have seen in Proposition 98, page 287, that a Boolean function f is plateaued of amplitude λ if and only if $\sum_{a \in \mathbb{F}_2^n} \Delta_f(a) \Delta_f(a + x) = \lambda^2 \Delta_f(x)$. To be able to deduce a characterization of plateaued vectorial functions, we need to eliminate λ^2 from this relation. The value of λ can be obtained from this same relation, with $x = 0_n$: $\sum_{a \in \mathbb{F}_2^n} \Delta_f^2(a) = \lambda^2 \Delta_f(0_n) = \lambda^2 2^n$. Hence, if f is plateaued then $2^n \Delta_f \otimes \Delta_f = [\sum_{a \in \mathbb{F}_2^n} \Delta_f^2(a)]\, \Delta_f$. Conversely, if $2^n \Delta_f \otimes \Delta_f = (\sum_{a \in \mathbb{F}_2^n} \Delta_f^2(a))\, \Delta_f$ then $\Delta_f \otimes \Delta_f = \lambda^2 \Delta_f$ where $\sum_{a \in \mathbb{F}_2^n} \Delta_f^2(a) = \lambda^2\, 2^n$ and f is plateaued of amplitude λ. We deduce:

Proposition 108 *Any (n, m)-function F is plateaued if and only if, for every $x \in \mathbb{F}_2^n$ and every $v \in \mathbb{F}_2^m$, we have*

$$2^n \sum_{a \in \mathbb{F}_2^n} \Delta_{v \cdot F}(a) \Delta_{v \cdot F}(a + x) = [\sum_{a \in \mathbb{F}_2^n} \Delta_{v \cdot F}^2(a)]\, \Delta_{v \cdot F}(x).$$

It is plateaued with single amplitude λ if and only if, for every $x \in \mathbb{F}_2^n$ and every $v \in \mathbb{F}_2^m$, we have

$$\sum_{a \in \mathbb{F}_2^n} \Delta_{v \cdot F}(a) \Delta_{v \cdot F}(a + x) = \lambda^2 \Delta_{v \cdot F}(x).$$

Characterization by means of power moments of the Walsh transform

We have seen in Proposition 99, page 288, that any n-variable Boolean function f is plateaued if and only if, for every nonzero $\alpha \in \mathbb{F}_2^n$, we have $\sum_{u \in \mathbb{F}_2^n} W_f(u + \alpha)\, W_f^3(u) = 0$. We deduce:

Proposition 109 *[247] Any (n,m)-function F is plateaued if and only if:*

$$\forall v \in \mathbb{F}_2^m, \forall \alpha \in \mathbb{F}_2^n, \alpha \neq 0_n, \sum_{u \in \mathbb{F}_2^n} W_F(u+\alpha, v)\, W_F^3(u,v) = 0.$$

F is plateaued with single amplitude if and only if, additionally, $\sum_{u\in\mathbb{F}_2^n} W_F^4(u,v)$ does not depend on v for $v \neq 0_m$.

We deduce also from Corollary 17, page 288:

Corollary 20 *[247] Any (n,m)-function F is plateaued if and only if, for every $b \in \mathbb{F}_2^n$ and every $v \in \mathbb{F}_2^m$:*

$$\sum_{a\in\mathbb{F}_2^n} W_F^4(a,v) = 2^n(-1)^{v\cdot F(b)} \sum_{a\in\mathbb{F}_2^n} (-1)^{a\cdot b} W_F^3(a,v).$$

And F is plateaued with single amplitude if and only if, additionally, these sums do not depend on v, for $v \neq 0_m$.

We have seen that plateaued functions can be characterized by the constance of the ratio of two consecutive Walsh power moments of even orders [858].
We deduce from Proposition 100, page 288:

Proposition 110 *[858, 247] For every (n,m)-function F, and every $k \in \mathbb{N}^*$, we have:*

$$\sum_{v\in\mathbb{F}_2^m}\left(\sum_{a\in\mathbb{F}_2^n} W_F^{2k+2}(a,v)\right)^2 \leq \sum_{v\in\mathbb{F}_2^m}\left(\sum_{a\in\mathbb{F}_2^n} W_F^{2k}(a,v)\right)\left(\sum_{a\in\mathbb{F}_2^n} W_F^{2k+4}(a,v)\right),$$

with equality if and only if F is plateaued.

See more in [858, 247].

6.5.2 CCZ and EA equivalence of plateaued functions

In [247] is deduced from Theorem 18, page 304, the following:

Corollary 21 *Let n be any even integer, $n \geq 4$. Let F be an (n,n)-function* CCZ equivalent *to a Gold APN function $G(x) = x^{2^i+1}$ or to a Kasami APN function $G(x) = x^{4^i-2^i+1}$, $(i,n)=1$. Then F is plateaued if and only if it is* EA equivalent *to $G(x)$.*

This result has been later generalized in [1141] by S. Yoshiara:

Proposition 111 *Let F and G be plateaued APN functions on $\mathbb{F}_{2^n}$ with n even. Assume that F is a power function, then it is CCZ equivalent to G if and only if F is EA equivalent to G.*

This same author had proved in [1139]:

Proposition 112 *Two quadratic APN functions are CCZ equivalent if and only if they are EA equivalent.*

and in [1140]:

Proposition 113 *For any $n \geq 3$, two power APN functions x^d and x^e over $\mathbb{F}_{2^n}$ are CCZ equivalent if and only if there is an integer a such that $0 \leq a \leq n-1$ and either $e = 2^a d$ [mod $2^n - 1$] or $de = 2^a$ [mod $2^n - 1$], where the latter case occurs only when n is odd.*

Proposition 114 *Any quadratic APN function is CCZ equivalent to a power APN function if and only if it is EA equivalent to one of the Gold APN functions.*

From Proposition 113 are deduced all cases of CCZ equivalence/inequivalence between the known APN functions, see Proposition 177.

6.5.3 Constructions of plateaued vectorial functions

Primary constructions

All quadratic functions are plateaued. The Maiorana-McFarland construction $F(x, y) = x\pi(y) + \phi(y)$; $x, y \in \mathbb{F}_{2^m}$, allows constructing non-quadratic ones; it gives a plateaued $(2m, m)$-function when π is a permutation (F is then bent) and when π is 2-to-1, ϕ being any (m, m)-function in both cases. Erasing some coordinates from their output provides plateaued (n, m)-functions with $m \leq n/2$.
We recall from [247] an example of *primary construction* of plateaued (n, n)-functions also based on the Maiorana-McFarland construction. Let π be a permutation of $\mathbb{F}_{2^m}$ and ϕ, ψ two functions from $\mathbb{F}_{2^m}$ to $\mathbb{F}_{2^m}$. Let i be an integer co-prime with m. We define the $(2m, 2m)$-function $F(x, y) = (x\pi(y) + \phi(y), x(\pi(y))^{2^i} + \psi(y)) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$. For every element $(a, b) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$, the Walsh transform at (a, b) of $(u, v) \cdot F(x, y)$ equals

$$\sum_{x,y \in \mathbb{F}_{2^m}} (-1)^{tr_m(ux\pi(y) + vx(\pi(y))^{2^i} + u\phi(y) + v\psi(y) + ax + by)} =$$

$$\sum_{y \in \mathbb{F}_{2^m}} (-1)^{tr_m(u\phi(y) + v\psi(y) + by)} \left(\sum_{x \in \mathbb{F}_{2^m}} (-1)^{tr_m((u\pi(y) + v(\pi(y))^{2^i} + a)x)} \right) =$$

$$2^m \sum_{\substack{y \in \mathbb{F}_{2^m} \\ u\pi(y) + v(\pi(y))^{2^i} = a}} (-1)^{tr_m(u\phi(y) + v\psi(y) + by)}.$$

The number of solutions of the equation $u\pi(y) + v(\pi(y))^{2^i} = a$ equals the number of solutions of the linear equation $uy + vy^{2^i} = a$. If $u = 0$ and $v \neq 0$, or if $u \neq 0$ and $v = 0$, the number of solutions of this equation equals 1; hence, $(u, v) \cdot F$ is plateaued of amplitude 2^m (*i.e.* is bent). If $u \neq 0$ and $v \neq 0$, this number either equals 0 or equals the number of solutions of the associated homogeneous equation $uy + vy^{2^i} = 0$, that is 2 (indeed, $uy + vy^{2^i} = 0$ is equivalent to $y = 0$ or $y^{2^i - 1} = \frac{u}{v} \neq 0$ and i being co-prime with m, $2^i - 1$ is co-prime with $2^m - 1$); hence, $(u, v) \cdot F(x, y)$ is plateaued of amplitude 2^{m+1} (*i.e.* is semi-bent). Then:

Proposition 115 *[247] Let m be a positive integer, π a permutation of $\mathbb{F}_{2^m}$ and ϕ, ψ two functions from $\mathbb{F}_{2^m}$ to $\mathbb{F}_{2^m}$. Let i be an integer co-prime with m. Then, function $F(x,y) = (x\pi(y) + \phi(y), x(\pi(y))^{2^i} + \psi(y))$ is plateaued (but does not have single amplitude).*

Of course, this observation more generally applies when $(\pi(y))^{2^i}$ is replaced by any other permutation $\pi'(y)$ such that, for every $u \neq 0$, $v \neq 0$, the equation $u\pi(y) + v\pi'(y) = a$ has 0 or a fixed number (depending on u and v only) of solutions.
There exist other examples of non-quadratic plateaued (n,n)-functions: AB functions, see page 428, Kasami APN functions in even dimension, see page 433; erasing some coordinates gives plateaued (n,m)-functions with $n/2 < m \leq n$.

Secondary constructions

Let r, s, t, p be positive integers. Let F be a *plateaued* (r,t)-function and G a plateaued (s,p)-function, then function: $H(x,y) = (F(x), G(y))$; $x \in \mathbb{F}_2^r, y \in \mathbb{F}_2^s$ is a plateaued $(r+s, t+p)$-function. Indeed, for every $(a,b) \in \mathbb{F}_2^r \times \mathbb{F}_2^s$ and every $(u,v) \in \mathbb{F}_2^t \times \mathbb{F}_2^p$, we have: $W_H((a,b),(u,v)) = W_F(a,u)W_G(b,v)$. Note that this works even if u or v is null, but such function is never with single amplitude, except when F and G are affine.

7 Correlation immune and resilient functions

The notion of correlation immune Boolean function is due to Siegenthaler [1041] as a criterion for resistance to his correlation attack on the *combiner model* of stream cipher, as we saw at page 105. Balanced correlation immune functions have soon been called resilient after [370] which dealt with another cryptographic issue: the bit extraction problem. It has been later observed in [181] that the notion of correlation immune Boolean function already existed in combinatorics (in a wider framework) under another name, since the support of a correlation immune function is an orthogonal array (see Definition 22, page 106). Resilient functions have been extensively studied in the nineties in relation with nonlinearity. But in 2003 were invented the *fast algebraic attack* [388] and the Rønjom-Helleseth attack [1003], which are very efficient against stream ciphers using nonlinear functions whose algebraic degrees are not large. Since correlation immune and resilient functions have algebraic degree bounded from above, this made them weak. But, as we already recalled at page 171, correlation immune and resilient Boolean functions can be employed for secret sharing, as shown in [461]. Recently, the interest of correlation immune functions has been also renewed in the framework of side channel attacks (see [286] and Section 12.1). The functions need then to have low Hamming weight (and this excludes resilient functions).

7.1 Correlation immune and resilient Boolean functions

For the convenience of the reader, we recall in the next definition what we have seen in Section 3.1, page 105, on correlation immune and resilient functions.

Definition 69 *Let n be a positive integer and $t \leq n$ a non-negative integer. An n-variable Boolean function f is called a t-th order correlation immune (t-CI) function if its output distribution probability (*i.e.* the density of the support) is unaltered when at most t (or, equivalently, exactly t) of its input bits are kept constant, that is, if the code equal to its support has* dual distance *(see Definition 4, page 32) at least $t+1$. It is called a t-*resilient *function if it is balanced and t-th order correlation immune. Equivalently, f is t-th order correlation immune if $W_f(u) = 0$,* i.e. *$\widehat{f}(u) = 0$, for all $u \in \mathbb{F}_2^n$ such that $1 \leq w_H(u) \leq t$, and it is t-resilient if $W_f(u) = 0$ for all $u \in \mathbb{F}_2^n$ such that $w_H(u) \leq t$.*

This generalizes to other alphabets [178]. Note that thanks to the $\mathbb{R}$-linearity of the Fourier-Hadamard transform, the sum of t-th order correlation immune functions with disjoint supports is a t-th order correlation immune function.
The combining functions in stream ciphers must be t-resilient with large t. As any cryptographic functions, they must also have high algebraic degree (which is partially contradictory with correlation immunity, but trade-offs can be found) and high nonlinearity (idem), and we know since 2003 that they must have high resistance to algebraic attacks and fast algebraic attacks (which is problematic).
Notation: by an $(n, t, d, \mathcal{N})$- function, we mean an n-variable, t-resilient function having algebraic degree at least d and nonlinearity at least $\mathcal{N}$.

7.1.1 Bound on the correlation immunity order

The correlation immunity order of n-variable functions (*i.e.* the maximum t such that they are t-CI) is unbounded (that is, it can be as high as n, since constant functions are n-th order correlation immune), and their resiliency order is only bounded above by $n-1$, since the Boolean function $\bigoplus_{i=1}^n x_i$ is $(n-1)$-resilient. In the case of unbalanced and non-constant correlation immune functions, the situation is different:

Proposition 116 *[514] Let f be an unbalanced non-constant t-th order correlation immune Boolean function. Then $t \leq \frac{2n}{3} - 1$.*

Proof. Let f be an unbalanced non-constant t-CI Boolean function. Since f is unbalanced, we have $W_f(0_n) \neq 0$ and since f is non-constant, there exists $a \in \mathbb{F}_2^n$ nonzero such that $W_f(a) \neq 0$. The Golomb-Xiao-Massey characterization (Theorem 5, page 107) gives that $w_H(a) \geq t+1$.
Suppose that $t > \frac{2n}{3} - 1$. By the Titsworth relation (2.51), page 80, we have:

$$\sum_{u \in \mathbb{F}_2^n} W_f(u) W_f(u+a) = 0 \quad . \tag{7.1}$$

For $u = 0_n$, the summand in the left part of (7.1) equals 2^{2n}, according to Parseval's identity. If $1 \leq w_H(u) \leq \frac{2}{3}n < t+1$, then $W_f(u) = 0$. If $w_H(u) > \frac{2}{3}n$, then the vectors u and a have more than $\frac{n}{3}$ common 1's, therefore $w_H(u+a) < \frac{2}{3}n$. Thus the left hand side of Eqn. (7.1) has exactly two equal non-zero summands (for $u = 0_n$ and $u = a$), therefore the equality in Eqn. (7.1) cannot be achieved. $\square$

7.1.2 Bounds on algebraic degree

The *Siegenthaler bound* states:

Proposition 117 *[1041] Let n be any positive integer and let $0 \leq t \leq n$. Any t-th order correlation immune n-variable Boolean function has* Hamming weight *divisible by 2^t and algebraic degree smaller than or equal to $n-t$. Any t-resilient* *function has algebraic degree smaller than or equal to $n-t-1$ if $t \leq n-2$ and to 1 (*i.e. *is affine) if $t = n-1$. Moreover, if a t-th order correlation immune function has Hamming weight divisible by 2^{t+1}, then it satisfies the same bound as t-resilient functions.*

Siegenthaler's bound gives an example of the trade-offs which must be accepted in the design of combiner generators[1].
The first assertion in Proposition 117 comes directly from the fact that all the restrictions obtained by fixing t coordinates of the input have the same Hamming weight. The other results can be proved directly by using Relation (2.4), page 50, since the bit $\bigoplus_{x\in\mathbb{F}_2^n;\ supp(x)\subseteq I} f(x)$ equals the parity of the Hamming weight of the restriction of f obtained by setting to 0 the coordinates of x which lie outside I. It is then null if $|I| > n-t$. In the case that the restriction by fixing t input coordinates has even Hamming weight, that is, when $w_H(f)$ is divisible by 2^{t+1}, this bit is null if $|I| \geq n-t$. Note that we can also use the *Golomb-Xiao-Massey characterization* (Theorem 5, page 107, resulting in Definition 69, page 312) together with the *Poisson summation formula* (2.40), page 77, applied to $\varphi = f$ and with $E^\perp = \{x \in \mathbb{F}_2^n;\ supp(x) \subseteq I\}$, where I has size strictly larger than $n-t-1$. But this gives a less simple proof.

7.1.3 Characterization by the NNF

Siegenthaler's bound is also a direct consequence of a characterization of correlation immune and of resilient functions through their NNFs and of the facts that the ANF equals the NNF mod 2 and that the Hamming weight of a t-th order correlation immune function with $t \geq 1$ is even (the Walsh transform at 0_n is then divisible by 4).

Proposition 118 *[293, 220] Let n be any positive integer and $t < n$ a non-negative integer. A Boolean function f on $\mathbb{F}_2^n$ is t-th order correlation immune if and only if the* numerical normal form $\sum_{I\subseteq\{1,\dots,n\}} \lambda_I\, x^I$ *of the function $g(x) = f(x)\oplus x_1 \oplus \cdots \oplus x_n$ satisfies that, for every I of size larger than or equal to $n-t$, $(-2)^{n-|I|}\lambda_I$ is independent of the choice of I. And f is t-resilient if and only if the numerical normal form $\sum_{I\subseteq\{1,\dots,n\}} \lambda_I\, x^I$ of g has degree at most $n-t-1$.*

Proof. For each vector $a \in \mathbb{F}_2^n$, we denote by $\overline{a}$ the componentwise complement of a equal to $a + 1_n$. We have $W_f(a) = W_g(\overline{a})$. Thus, f is t-th order correlation immune (resp. t-resilient) if and only if, for every vector $u \neq 1_n$ of Hamming weight larger than or equal to $n-t$ (resp. for every vector u of Hamming weight larger than or equal to $n-t$), the number $W_g(u)$ is null. According to Relations (2.61), (2.62), page 86, and (2.32), page 74, applied to g, we have for nonzero u:

$$W_g(u) = (-1)^{w_H(u)+1} \sum_{I\subseteq\{1,\dots,n\};\ supp(u)\subseteq I} 2^{n-|I|+1}\lambda_I,$$

[1] One approach to avoid such trade-off is to allow memory in the nonlinear combination generator, that is, to replace the combining function by a finite state machine, see [845].

and for nonempty I:

$$\lambda_I = 2^{-n}(-2)^{|I|-1} \sum_{u\in\mathbb{F}_2^n;\ I\subseteq supp(u)} W_g(u).$$

This completes the proof. $\square$

Proposition 118 proves, by applying Relation (2.64), page 86, to $g(x) = f(x) \oplus x_1 \oplus \cdots \oplus x_n$, that if t is the resiliency order of an n-variable function f of algebraic degree at least 2, and each variable x_i is effective in $g(x)$, then $n - t - 1 \geq n\, 2^{-d_{alg}(f)+1}$, that is:

$$d_{alg}(f) \geq \log_2\left(\frac{2n}{n-t-1}\right).$$

Remark. According to Proposition 118, a non-affine balanced n-variable Boolean function g has its *algebraic degree* and *numerical degree* equal to each other if and only if, given Boolean function $f(x) = g(x) \oplus x_1 \oplus \cdots \oplus x_n$ and its resiliency order, Siegenthaler's bound is an equality. $\square$

Proposition 118 has been used by X.-D. Hou in [621] for constructing resilient functions.

7.1.4 Bounds on the nonlinearity

Sarkar and Maitra showed that:

Proposition 119 *[1012] The values of the Walsh transform of an n-variable, t-resilient (resp. t-th order correlation immune) function are divisible by 2^{t+2} (resp. 2^{t+1}) if $0 \leq t \leq n-3$ (resp. $1 \leq t \leq n-2$).*

A more precise result being given in Proposition 120 below, we skip the proof of Proposition 119. A little more is proved in [220, 322]; in particular: if the Hamming weight of a t-th order correlation immune function is divisible by 2^{t+1}, then the values of its Walsh transform are divisible by 2^{t+2}. This *Sarkar-Maitra's divisibility bound* and its extension have provided nontrivial upper bounds on the nonlinearity of resilient functions, independently obtained by Tarannikov [1080] and by Zheng and Zhang [1177]:

Theorem 20 *[1012, 1080, 1177] For every n and $t \leq n-2$, the nonlinearity of any t-th order correlation immune (resp. t-resilient) function is bounded above by $2^{n-1} - 2^t$ (resp. $2^{n-1} - 2^{t+1}$).*

Of course, this brings information only if $2^{n-1} - 2^t$ (resp. $2^{n-1} - 2^{t+1}$) is smaller than $2^{n-1} - 2^{\frac{n}{2}-1}$. Zheng and Zhang [1177] showed that correlation immune functions of high orders satisfy the same upper bound on the nonlinearity as resilient functions of the same orders. In [1083] (where is also obtained a bound on Δ_f for f resilient and studied the resiliency order of all quadratic functions), Tarannikov-Korolev-Botev showed for each $i \in \{1,2\}$ that if t is larger than some rather complex expression of i and n, then for every unbalanced nonconstant t-CI function, we have $nl(f) \leq 2^{n-1} - 2^{t+i}$. The maximal higher-order nonlinearity of resilient functions has also been studied in [719, 101] and determined for low order (≤ 2) or low number of variables (≤ 7).

The bound of Theorem 20 for resilient functions is tight when $t \geq 0.6\ n$, see [1080, 1081]. We shall call it *Sarkar et al.'s bound*. Notice that, if a t-resilient function f achieves nonlinearity $2^{n-1}-2^{t+1}$, then f is plateaued. Indeed, the distances between f and affine functions lie then between $2^{n-1}-2^{t+1}$ and $2^{n-1}+2^{t+1}$ and must be therefore equal to $2^{n-1} - 2^{t+1}$, 2^{n-1} and $2^{n-1} + 2^{t+1}$ because of the divisibility result of Sarkar and Maitra. Thus, the Walsh transform of f takes three values 0 and $\pm 2^{t+2}$. Moreover, it is proved in [1080] (and is a direct consequence of Proposition 120 below) that such function f also achieves Siegenthaler's bound (and as proved in [814], achieves minimum *sum-of-squares indicator*).

If $2^{n-1} - 2^{t+1}$ is larger than the best possible nonlinearity of all balanced functions (and in particular if it is larger than the covering radius bound) then, obviously, a better bound than in Theorem 20 exists. In the case of n even, the best possible nonlinearity of all balanced functions being strictly smaller than $2^{n-1} - 2^{\frac{n}{2}-1}$, Sarkar and Maitra deduce that $nl(f) \leq 2^{n-1} - 2^{\frac{n}{2}-1} - 2^{t+1}$ for every t-resilient function f with $t \leq \frac{n}{2} - 2$. In the case of n odd, they state that $nl(f)$ is smaller than or equal to the highest multiple of 2^{t+1}, which is less than or equal to the best possible nonlinearity of all Boolean functions. But a potentially better upper bound can be given, whatever is the parity of n. Indeed, Sarkar-Maitra's divisibility bound shows that $W_f(a) = \omega(a) \times 2^{t+2}$ where $\omega(a)$ is integer-valued. Parseval's Relation (2.48), page 79, and the fact that $W_f(a)$ is null for every vector a of Hamming weight $\leq t$ imply

$$\sum_{a \in \mathbb{F}_2^n;\ w_H(a)>t} \omega^2(a) = 2^{2n-2t-4}$$

and, thus,

$$\max_{a \in \mathbb{F}_2^n} |\omega(a)| \geq \sqrt{\frac{2^{2n-2t-4}}{2^n - \sum_{i=0}^{t} \binom{n}{i}}} = \frac{2^{n-t-2}}{\sqrt{2^n - \sum_{i=0}^{t} \binom{n}{i}}}.$$

Hence, we have $\max_{a\in\mathbb{F}_2^n} |\omega(a)| \geq \left\lceil \frac{2^{n-t-2}}{\sqrt{2^n - \sum_{i=0}^{t} \binom{n}{i}}} \right\rceil$, and this implies:

$$nl(f) \leq 2^{n-1} - 2^{t+1} \left\lceil \frac{2^{n-t-2}}{\sqrt{2^n - \sum_{i=0}^{t} \binom{n}{i}}} \right\rceil. \tag{7.2}$$

When n is even and $t \leq \frac{n}{2} - 2$, this number is always less than or equal to the number $2^{n-1} - 2^{\frac{n}{2}-1} - 2^{t+1}$ (given by Sarkar and Maitra), because $\frac{2^{n-t-2}}{\sqrt{2^n - \sum_{i=0}^{t} \binom{n}{i}}}$ is strictly larger than $2^{\frac{n}{2}-t-2}$ and $2^{\frac{n}{2}-t-2}$ is an integer, and, thus, $\left\lceil \frac{2^{n-t-2}}{\sqrt{2^n-\sum_{i=0}^t \binom{n}{i}}} \right\rceil$ is at least $2^{\frac{n}{2}-t-2}+1$. And when n increases, the right-hand side of Relation (7.2) is strictly smaller than $2^{n-1} - 2^{\frac{n}{2}-1} - 2^{t+1}$ for an increasing number of values of $t \leq \frac{n}{2} - 2$ (but this improvement does not appear when we compare the values we obtain with this bound to the values indicated in the table given by Sarkar and Maitra in [1012], because the values of n they consider in this table are small).

When n is odd, it is difficult to say if Inequality (7.2) is better than the bound given by Sarkar and Maitra, because their bound involves a value which is unknown for $n \geq 9$ (the best possible nonlinearity of all balanced Boolean functions). In any case, this makes (7.2) better usable.

We know (see [809, page 310]) that $\sum_{i=0}^t \binom{n}{i} \geq \frac{2^{nH_2(t/n)}}{\sqrt{8t(1-t/n)}}$, where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$, the so-called *binary entropy function*, satisfies $H_2(\frac{1}{2} - x) = 1 - 2x^2 \log_2 e + o(x^2)$. Thus, we have

$$nl(f) \leq 2^{n-1} - 2^{t+1} \left\lceil \frac{2^{n-t-2}}{\sqrt{2^n - \frac{2^{nH_2(t/n)}}{\sqrt{8t(1-t/n)}}}} \right\rceil . \tag{7.3}$$

Remark. If a Boolean function f is t-th order correlation immune (resp. t-resilient), then for every $1 \leq e \leq t$ and every set $\{i_1, \ldots, i_e\}$ of size e, its restriction obtained by fixing coordinates $x_{i_1}, \ldots, x_{i_e}$ is a $(t-e)$-th order correlation immune (resp. $(t-e)$-resilient) $(n-e)$-variable function. But the n-variable function equal to the product of f with the monomial function $m(x) = \prod_{j=1}^e x_{i_j}$ of degree e is not $(t-e)$-th order correlation immune, although the support of fm equals the intersection of the support of f with the set $\{i_1, \ldots, i_e\}$: fixing $(t-e)$ coordinates of x preserves the output distribution probability only if these coordinates are outside $\{i_1, \ldots, i_e\}$. Nevertheless, it is possible to prove that such fm has same Walsh divisibility property as a $(t-e)$-th order correlation immune (resp. $(t-e)$-resilient) function. $\square$

Proposition 119 has been improved:

Proposition 120 *[220, 322] Let n be any positive integer and let $t \leq n-2$ be a non-negative integer. Let f be any n-variable t-th order correlation immune function (resp. any t-resilient function or any t-th order correlation immune function whose Hamming weight is divisible by $2^{t+1+\lfloor \frac{n-t-2}{d} \rfloor}$) and let d be its algebraic degree. The values of the Walsh transform of f are divisible by $2^{t+1+\lfloor \frac{n-t-1}{d} \rfloor}$ (resp. by $2^{t+2+\lfloor \frac{n-t-2}{d} \rfloor}$). Hence the nonlinearity of f is divisible by $2^{t+\lfloor \frac{n-t-1}{d} \rfloor}$ (resp. by $2^{t+1+\lfloor \frac{n-t-2}{d} \rfloor}$).*

A little more can be said in the former case, see [322].

The approach for proving this tight bound was first to use the numerical normal form (we refer the reader to [220] for this proof, for the tightness, and for an improvement when the number of terms of highest degree in the ANF is small enough). Later, a second proof using only the properties of the Fourier-Hadamard transform was given in [322]:

Proof. The Poisson summation formula (2.40), page 77, applied to $\varphi = f_\chi$ and to the vector space $E = \{u \in \mathbb{F}_2^n;\ \forall i \in \{1, \ldots, n\},\ u_i \leq v_i\}$ where v is some vector of $\mathbb{F}_2^n$, whose orthogonal equals $E^\perp = \{u \in \mathbb{F}_2^n;\ \forall i \in \{1, \ldots, n\},\ u_i \leq v_i \oplus 1\}$, gives $\sum_{u \in E} W_f(u) = 2^{w_H(v)} \sum_{x \in E^\perp} f_\chi(x)$. It is then a simple matter to prove the result by induction on the Hamming weight of v, starting with the vectors of weight t (resp. $t+1$), and using McEliece's divisibility property (see Subsection 4.1.5, page 179). □

Proposition 120 gives directly more precise upper bounds on the nonlinearity of any t-resilient function of degree d: for instance, this nonlinearity is bounded above by $2^{n-1} - 2^{t+1+\lfloor \frac{n-t-2}{d} \rfloor}$. This gives a simpler proof that it can be equal to $2^{n-1} - 2^{t+1}$ only if $d = n - t - 1$, *i.e.* if Siegenthaler's bound is achieved with equality. Moreover, the proof above also shows that the nonlinearity of any t-resilient n-variable Boolean function is bounded above by $2^{n-1} - 2^{t+1+\lfloor \frac{n-t-2}{d} \rfloor}$ where d is the minimum algebraic degree of the restrictions of f to the subspaces $\{u \in \mathbb{F}_2^n;\ \forall i \in \{1, \ldots, n\},\ u_i \leq v_i \oplus 1\}$ such that v has Hamming weight $t+1$ and $W_f(v) \neq 0$. See more in [322].

7.1.5 Bound on the maximum correlation with index subsets

An upper bound on the maximum correlation of t-resilient functions with respect to subsets I of $\{1, \ldots, n\}$ can be directly deduced from Relation (3.14), page 123, and from Sarkar et al.'s bound. Note that we get an improvement by using that the support of W_f, restricted to the set of vectors $u \in \mathbb{F}_2^n$ such that $u_i = 0,\ \forall i \notin I$, contains at most $\sum_{i=t+1}^{|I|} \binom{|I|}{i}$ vectors. In particular, if $|I| = t+1$, the maximum correlation of f with respect to I equals $2^{-n}\,|W_f(u)|$, where u is the vector of support I, see [187, 203, 1155]. The optimal number of LFSRs which should be considered together in a correlation attack on a cryptosystem using a t-resilient combining function is $t+1$, see [187].

7.1.6 Relationship with other criteria

The relationships between resiliency and other criteria have been studied in [354, 814, 1083, 1175]. For instance, t-resilient $PC(l)$ functions can exist only if $t + l \leq n - 1$. This is a direct consequence of Relation (2.56), page 81, applied with $a = b = 0_n$, $E = \{x \in \mathbb{F}_2^n;\ x_i = 0,\ \forall i \in I\}$ and $E^\perp = \{x \in \mathbb{F}_2^n;\ x_i = 0,\ \forall i \notin I\}$, where I has size $n-t$: if $l \geq n-t$ then the right-hand side term of Relation (2.56) is nonzero while the left-hand side term is null. Equality $t + l = n - 1$ is possible only if $l = n - 1$, n is odd and $t = 0$ [1175, 354]. The known upper bounds on the nonlinearity can then be improved for such functions.

The definition of resiliency has been weakened in [126, 294, 720, 721] in order to relax some of the trade-offs recalled above, without weakening the cryptosystem against the correlation attack.

Resiliency is related to the notion of corrector (useful for the generation of random sequences having good statistical properties) introduced by Lacharme in [732].

7.1.7 Relationship with covering sequences

According to Proposition 60, page 206, knowing a covering sequence $\lambda = (\lambda_a)_{a\in\mathbb{F}_2^n}$ (trivial or not) of a function f allows knowing that $supp(W_f) \subseteq \widehat{\lambda}^{-1}\big(\widehat{\lambda}(0_n)-2\rho\big)$, where ρ is the level of the sequence. Hence, as observed in [326], f is t-th order correlation immune where $t+1$ is the minimum Hamming weight of nonzero $b \in \mathbb{F}_2^n$ such that $\widehat{\lambda}(b) = \widehat{\lambda}(0_n)-2\rho$, and if $\rho \neq 0$, it is then t-resilient. Conversely, if f is t-th order correlation immune (resp. t-resilient) and if it is not $(t+1)$-th order correlation immune (resp. $(t+1)$-resilient), then there exists at least one (non-trivial) covering sequence $\lambda = (\lambda_a)_{a\in\mathbb{F}_2^n}$ with level ρ such that $t+1$ is the minimum Hamming weight of $b \in \mathbb{F}_2^n$ satisfying $\widehat{\lambda}(b) = \widehat{\lambda}(0_n) - 2\rho$.
A particularly simple covering sequence is the indicator of the set of vectors of Hamming weight one. The functions which admit this covering sequence are called regular; they are $(\rho - 1)$-resilient, where ρ is the level. More generally, any function admitting as covering sequence the indicator of a set of vectors of weight 1 has this same property (this generalizes to any vectors with disjoint supports). We speak then of a simple covering sequence, see [326], where the algebraic degree and the nonlinearity of regular functions are studied, and where constructions are given as well as bounds on the number of variables.

7.1.8 Primary constructions of correlation immune and resilient functions

In the 90's, high-order resilient functions with the best possible algebraic degree and nonlinearity were needed for applications in stream ciphers using the combiner model. But *fast algebraic attack*s (*FAA*) have changed the situation. The combiner model is now considered as problematic, because of Siegenthaler's bound and the fact that combiner or filter functions need to have very high algebraic degree for resisting FAA. For the sake of completeness and also because building correlation immune functions means building orthogonal arrays (see Definition 22, page 106) which are of interest in combinatorics and statistics, and because a new way of using low weight correlation immune functions exists (see Section 12.1), and new ways of using resilient functions may be found in the future, we report the state of the art for constructing highly nonlinear correlation immune and resilient functions. As we shall see, most constructions build in fact resilient functions and these constructions unfortunately do not

allow to construct low weight correlation immune functions. More work is then needed to build such functions. Such work, that we shall report at the end of this subsection, has been initiated in [258] and continued in [1103].
The *primary constructions* (which allow designing resilient functions without using known ones) are supposed to lead potentially to wider classes of functions than secondary constructions (recall that the number of Boolean functions on $n-1$ variables is only equal to the square root of the number of n-variable Boolean functions). But the known primary constructions of resilient Boolean functions do not lead to very large classes of functions. In fact, only one reasonably large class of Boolean functions is known, whose elements can be analyzed with respect to the cryptographic criteria recalled in Section 3.1. So we observe some imbalance in the knowledge on cryptographic functions for stream ciphers: much is known on the properties of resilient functions, but little is known on how constructing them. Examples of t-resilient functions achieving the best possible nonlinearity $2^{n-1}-2^{t+1}$ (and thus the best algebraic degree) have been obtained for $n \le 10$ in [934, 1011, 1012] and for every $t \ge 0.6\ n$ [1080, 1081] (n being then not limited). But $n \le 10$ is too small for applications and $t \ge 0.6\ n$ is too large (because of Siegenthaler's bound)[2]. Moreover, these examples give very limited numbers of functions (they are often defined recursively or obtained after a computer search) and many of these functions have cryptographic weaknesses such as linear structures (see [354, 814]). Balanced Boolean functions with high nonlinearities have been obtained by Fontaine in [515] and by Filiol and Fontaine in [503], who made a computer investigation - but for $n = 7, 9$ which is too small - on the corpus of idempotent functions (see definition at page 275). These functions, whose *ANF*s are invariant under the cyclic shifts of the coordinates x_i, have been called later *rotation symmetric* (see Section 10.2, page 392). Other ad hoc constructions can be found in [819, 1011].

A construction derived from the characterization of correlation immunity by the *dual distance*

It has been observed in [417] that the characterization of Corollary 6, page 108, can be straightforwardly applied to build correlation immune functions from linear codes. In fact, this was already known from [58].

Corollary 22 *Let C be any (linear) $[n,k,d]$-code and G a generator matrix of C. Then for every k-variable function g, the n-variable function $f(x) = g(x \times G^t)$ is $(d-1)$-th order correlation immune (and it is $(d-1)$-resilient if g is balanced).*

Proof. If g is the indicator δ_0 of the singleton $\{0_k\}$, the result is a direct consequence of Corollary 6, page 108, since we have $f(x) = 0$ if and only if $x \times G^t = 0_k$, that is, $x \in C^\perp$. It is easily seen that if $g = \delta_a$, we have then that $f(x) = 0$ if and only if x belongs either to the empty set or to a coset of $C^\perp$. Then f is $(d-1)$-th

[2] And almost nothing is known on the immunity of these functions to algebraic attacks; anyway, their resistance to FAA is bad.

order correlation immune according to Corollary 6, since the dual distance is invariant by translation. And if g is any sum of such atomic functions, that is, any Boolean function, we have the same result since the sum of t-th order correlation immune functions with disjoint supports is a t-th order correlation immune function. Finally, G being a generator matrix, function $x \in \mathbb{F}_2^n \mapsto x \times G^t \in \mathbb{F}_2^k$ is balanced and then f is balanced if and only if g is balanced. $\square$

Such correlation immune function can have at most algebraic degree $d_{alg}(g) \leq k$ (and $\leq k-1$ if it is resilient).

Remark. Given $k < n$, a k-variable function g, a surjective linear mapping $L : \mathbb{F}_2^n \to \mathbb{F}_2^k$ and an element u of $\mathbb{F}_2^n$, the function $f(x) = g \circ L(x) \oplus u \cdot x$ is $(d-1)$-resilient, where d is the Hamming distance between u and the linear code C whose generator matrix equals the matrix of L. Indeed, for any vector $a \in \mathbb{F}_2^n$ of Hamming weight at most $d-1$, the vector $u+a$ does not belong to C. This implies that the Boolean function $f(x) \oplus a \cdot x$ is linearly equivalent to the function $g(x_1, \ldots, x_k) \oplus x_{k+1}$, since we may assume without loss of generality that L is systematic (*i.e.* has the form $[Id_k|N]$). Boolean function $f(x) \oplus a \cdot x$ is therefore balanced. This construction is similar to that of Corollary 22 but different (note that g does not need to be balanced for f to be balanced).

In both constructions, f has nonzero linear structures since it is EA equivalent to $g(x_1, \ldots, x_k)$; then it does not give full satisfaction. $\square$

Maiorana-McFarland's construction

An extension of the class of bent functions that we called above the *Maiorana-McFarland* original class has been given in [181] (where are also characterized the quadratic n-variable correlation immune functions of order $n-3$), based on the same principle of concatenating affine functions[3] (we have already met in Section 5.1 this generalization): let r be a positive integer smaller than n; we denote $n-r$ by s; let g be any Boolean function on $\mathbb{F}_2^s$ and let ϕ be a mapping from $\mathbb{F}_2^s$ to $\mathbb{F}_2^r$. Then, we define the function:

$$f_{\phi,g}(x,y) = x \cdot \phi(y) \oplus g(y) = \bigoplus_{i=1}^{r} x_i \phi_i(y) \oplus g(y),\ x \in \mathbb{F}_2^r,\ y \in \mathbb{F}_2^s \tag{7.4}$$

where $\phi_i(y)$ is the i-th coordinate function of $\phi(y)$.

For every $a \in \mathbb{F}_2^r$ and every $b \in \mathbb{F}_2^s$, we have seen in Section 6.1.15 that

$$W_{f_{\phi,g}}(a,b) = 2^r \sum_{y \in \phi^{-1}(a)} (-1)^{g(y) \oplus b \cdot y}. \tag{7.5}$$

This can be used to design resilient functions: if every element in $\phi(\mathbb{F}_2^s)$ has Hamming weight strictly larger than t, then $f_{\phi,g}$ is t-resilient (in particular,

[3] These functions have also been studied under the name of linear-based functions in [7, 1137].

if $\phi(\mathbb{F}_2^s)$ does not contain the null vector, then $f_{\phi,g}$ is balanced). Indeed, if $w_H(a) \leq t$ then $\phi^{-1}(a)$ is empty in Relation (7.5); hence, if $w_H(a) + w_H(b) \leq t$ then $W_{f_{\phi,g}}(a,b)$ is null. The t-resiliency of $f_{\phi,g}$ under this hypothesis can also be deduced from the facts that any affine function $x \in \mathbb{F}_2^r \mapsto c \cdot x \oplus \epsilon$ ($c \in \mathbb{F}_2^r$ nonzero, $\epsilon \in \mathbb{F}_2$) is $(w_H(c)-1)$-resilient, and that any Boolean function equal to the concatenation of t-resilient functions is a t-resilient function (see secondary construction 3 below).

It is possible (see [398, 221, 223]) to obtain a t-resilient function with (7.4) when every element in $\phi(\mathbb{F}_2^s)$ has Hamming weight larger than or equal to t (instead of strictly larger): we know that such function is $(t-1)$-resilient by the observation above, and it is moreover t-resilient if, for every $a \in \mathbb{F}_2^r$ of Hamming weight t, we have $\sum_{y \in \phi^{-1}(a)} (-1)^{g(y)} = 0$. We just need then that, for every $a \in \mathbb{F}_2^r$ of Hamming weight t, if $\phi^{-1}(a) \neq \emptyset$, then $\phi^{-1}(a)$ has even size and the restriction of g to $\phi^{-1}(a)$ is balanced.

It is more difficult to construct unbalanced correlation immune functions with this method: in practice, we need that every nonzero element in $\phi(\mathbb{F}_2^s)$ has Hamming weight strictly larger than t and that, for every $b \in \mathbb{F}_2^s$ such that $1 \leq w_H(b) \leq t$, we have $\sum_{y \in \phi^{-1}(0_r)} (-1)^{g(y) \oplus b \cdot y} = 0$. If $\phi^{-1}(0_r)$ is an affine space, then this results in a condition on the restriction of g to $\phi^{-1}(0_r)$ which is similar to t-th order correlation immunity (this gives a construction which is more secondary than primary) and if $\phi^{-1}(0_r)$ has no such structure, then g needs to be built from scratch (very little work has been done on that).

Degree: The algebraic degree of $f_{\phi,g}$ is at most $s + 1 = n - r + 1$. It equals $s + 1$ if and only if ϕ has algebraic degree s (*i.e.* if at least one of its coordinate functions has algebraic degree s, that is, has odd Hamming weight, which is equivalent to $\sum_{y \in \mathbb{F}_2^s} \phi(y) \neq 0_r$). If we assume that every element in $\phi(\mathbb{F}_2^s)$ has Hamming weight strictly larger than t, then ϕ can have algebraic degree s only if $t \leq r - 2$, since if $t = r - 1$ then ϕ is constant. Thus, the algebraic degree of $f_{\phi,g}$ reaches Siegenthaler's bound $n - t - 1$ if and only if either $t = r - 2$ and ϕ has algebraic degree $s = n - t - 2$ or $t = r - 1$ and g has algebraic degree $s = n - t - 1$.

Nonlinearity: Relations (3.1), page 99, relating the nonlinearity to the Walsh transform, and (7.5) above lead straightforwardly to a general lower bound on the nonlinearity of Maiorana-McFarland's functions (first observed in [1026]):

$$nl(f_{\phi,g}) \geq 2^{n-1} - 2^{r-1} \max_{a \in \mathbb{F}_2^r} |\phi^{-1}(a)| \tag{7.6}$$

(where $|\phi^{-1}(a)|$ denotes the size of $\phi^{-1}(a)$). An upper bound obtained in [221] strengthens a bound previously obtained in [358, 359] which stated $nl(f_{\phi,g}) \leq 2^{n-1} - 2^{r-1}$:

$$nl(f_{\phi,g}) \leq 2^{n-1} - 2^{r-1} \left\lceil \sqrt{\max_{a \in \mathbb{F}_2^r} |\phi^{-1}(a)|} \right\rceil . \tag{7.7}$$

Proof of (7.7): The sum

$$\sum_{b\in\mathbb{F}_2^s}\left(\sum_{y\in\phi^{-1}(a)}(-1)^{g(y)\oplus b\cdot y}\right)^2=\sum_{y,z\in\phi^{-1}(a);\,b\in\mathbb{F}_2^s}(-1)^{g(y)\oplus g(z)\oplus b\cdot(y+z)}$$

equals $2^s|\phi^{-1}(a)|$ (since the sum $\sum_{b\in\mathbb{F}_2^s}(-1)^{b\cdot(y+z)}$ is null if $y\neq z$). The maximum of a set of values being always larger than or equal to its arithmetic mean, we deduce:

$$\max_{b\in\mathbb{F}_2^s}\left|\sum_{y\in\phi^{-1}(a)}(-1)^{g(y)\oplus b\cdot y}\right|\geq\sqrt{|\phi^{-1}(a)|}$$

and thus, according to Relation (7.5):

$$\max_{a\in\mathbb{F}_2^r;b\in\mathbb{F}_2^s}|W_{f_{\phi,g}}(a,b)|\geq 2^r\left\lceil\sqrt{\max_{a\in\mathbb{F}_2^r}|\phi^{-1}(a)|}\right\rceil.$$

Relation (3.1) completes the proof. □

This bound allowed characterizing the Maiorana-McFarland's functions $f_{\phi,g}$ such that $w_H(\phi(y))>k$ for every y and achieving nonlinearity $2^{n-1}-2^{k+1}$: Relation (7.7) implies $\sqrt{\max_{a\in F_2^r}|\phi^{-1}(a)|}\leq 2^{k-r+2}$ and thus $k+1\leq r\leq k+2$ since $\max_{a\in F_2^r}|\phi^{-1}(a)|\geq 1$ and it also implies the inequality $nl(f_{\phi,g})\leq 2^{n-1}-\frac{2^{r+\frac{s}{2}-1}}{\sqrt{\sum_{i=k+1}^{r}\binom{r}{i}}}$.
If $r=k+1$, then ϕ is the constant 1_s and $\max_{a\in F_2^r}|\phi^{-1}(a)|=2^s$, thus $s\leq 2(k-r+2)=2$ and $n\leq k+3$. Either $s=1$ and $g(y)$ is then any function in one variable, or $s=2$ and g (which is then bent) is any function of the form $y_1y_2\oplus\ell(y)$ where ℓ is affine.
If $r=k+2$, then ϕ is injective, therefore $2^s\leq\binom{r}{r-1}+\binom{r}{r}=r+1$ and thus $n\leq k+2+\log_2(k+3)$, g is any function on $n-k-2$ variables and $d_{alg}(f_{\phi,g})\leq 1+\log_2(k+3)$. See more in [221] on how optimizing the nonlinearity.

A simple example of k-resilient Maiorana-McFarland's functions such that $nl(f_{\phi,g})=2^{n-1}-2^{k+1}$ (and thus achieving Sarkar et al.'s bound) can be given for any $r\geq 2^s-1$ and for $k=r-2$ (see [221]). And, for every even $n\leq 10$, Sarkar et al.'s bound with $t=\frac{n}{2}-2$ can be achieved by Maiorana-McFarland's functions. Also, functions with high nonlinearities but not achieving Sarkar et al.'s bound with equality exist in Maiorana-McFarland's class (for every $n\equiv 1\,[\bmod 4]$, there exist such $\frac{n-1}{4}$-resilient functions on $\mathbb{F}_2^n$ with nonlinearity $2^{n-1}-2^{\frac{n-1}{2}}$).

Generalizations of Maiorana-McFarland's construction

Such generalizations, whose general frameworks have been seen in the present book in Subsections 5.2.2 and 5.4.1, have been introduced in [221] and [317]; the latter generalization has been further generalized into a class introduced in [226]. A motivation for introducing such generalizations is that Maiorana-McFarland's functions have the weakness that $x \mapsto f_{\phi,g}(x,y)$ is affine for every $y \in \mathbb{F}_2^s$ and have high divisibilities of their Fourier-Hadamard spectra (indeed, if we want to ensure that f is t-resilient with a large value of t, then we need to choose r large; then the Walsh spectrum of f is divisible by 2^r according to Relation (7.5); there is a risk that this property can be used in attacks, as it is used in [204] to attack block ciphers). The functions constructed in [221, 317] are concatenations of quadratic functions and those of [226] concatenations of indicators of flats. We have seen already in Subsections 5.2.2 and 5.4.1 the two classes:

1. $$f_{\psi,\phi,g}(x,y) = \bigoplus_{i=1}^{k} x_{2i-1}x_{2i}\,\psi_i(y) \oplus x \cdot \phi(y) \oplus g(y),$$

with $x \in \mathbb{F}_2^r$, $y \in \mathbb{F}_2^s$, where $n = r + s$, $k = \lfloor \frac{r}{2} \rfloor$, and where $\psi : \mathbb{F}_2^s \to \mathbb{F}_2^k$, $\phi : \mathbb{F}_2^s \to \mathbb{F}_2^r$ and $g : \mathbb{F}_2^s \to \mathbb{F}_2$ can be chosen arbitrarily;

2. $$\forall (x,y) \in \mathbb{F}_2^r \times \mathbb{F}_2^s, f(x,y) = \prod_{i=1}^{\varphi(y)} (x \cdot \phi_i(y) \oplus g_i(y) \oplus 1) \oplus x \cdot \phi(y) \oplus g(y),$$

where φ is a function from $\mathbb{F}_2^s$ into $\{0, 1, \ldots, r\}$, $\phi_1, \ldots, \phi_r$ and ϕ are functions from $\mathbb{F}_2^s$ into $\mathbb{F}_2^r$ such that, for every $y \in \mathbb{F}_2^s$, the vectors $\phi_1(y), \ldots, \phi_{\varphi(y)}(y)$ are linearly independent, and $g_1, \ldots, g_r$ and g are Boolean functions on $\mathbb{F}_2^s$.

We have seen at pages 203 and 205 the formulae for the Walsh transforms of the functions of these classes, which result in sufficient conditions for their resiliency and in bounds on their nonlinearities, see [221, 226] where is also studied how optimizing these parameters.

More complex ways of adapting the Maiorana-McFarland construction and other constructions can be found in [817, 934, 928, 1013, 1161, 1164], where some better parameters can be found but trade-offs are less clear.

Other constructions

A construction derived from $\mathcal{PS}_{ap}$ construction is introduced in [216] to obtain resilient functions: let k and r be positive integers and $n \geq r$; we denote $n - r$ by s; the vector space $\mathbb{F}_2^r$ is identified to the Galois field $\mathbb{F}_{2^r}$. Let g be any Boolean function on $\mathbb{F}_{2^r}$ and ϕ an $\mathbb{F}_2$-linear mapping from $\mathbb{F}_2^s$ to $\mathbb{F}_{2^r}$; set $a \in \mathbb{F}_{2^r}$ and $b \in \mathbb{F}_2^s$ such that, for every y in $\mathbb{F}_2^s$ and every z in $\mathbb{F}_{2^r}$, $a + \phi(y)$ is nonzero and $\phi^*(z) + b$ has Hamming weight larger than k, where ϕ^* is the adjoint of ϕ (satisfying $u \cdot \phi(x) = \phi^*(u) \cdot x$ for every x and u). Then, the function

$$f(x,y) = g\left(\frac{x}{a + \phi(y)}\right) \oplus b \cdot y, \text{ where } x \in \mathbb{F}_{2^r}, y \in \mathbb{F}_2^s, \qquad (7.8)$$

is t-resilient with $t \geq k$. There exist bounds on the nonlinearities of these functions (see [223]), similar to those existing for Maiorana-McFarland's functions.

But this class has much fewer elements than Maiorana-McFarland's class, because ϕ is linear.

Dobbertin's construction: We have seen at page 279 this method for modifying bent functions into balanced functions with high nonlinearities. Up to affine equivalence, we can assume that the bent function with which starts the method, say $f(x,y)$, $x \in \mathbb{F}_2^{n/2}$, $y \in \mathbb{F}_2^{n/2}$, is such that $f(x,0_{n/2}) = \epsilon$ ($\epsilon \in \mathbb{F}_2$) for every $x \in \mathbb{F}_2^{n/2}$ and that $\epsilon = 0$ (otherwise, consider $f \oplus 1$).

Proposition 121 *Let $f(x,y)$, $x \in \mathbb{F}_2^{n/2}$, $y \in \mathbb{F}_2^{n/2}$ be any bent function such that $f(x,0_{n/2}) = 0$ for every $x \in \mathbb{F}_2^{n/2}$ and let g be any balanced function on $\mathbb{F}_2^{n/2}$. Then the Walsh transform of the function $h(x,y) = f(x,y) \oplus \delta_0(y)g(x)$, where δ_0 is the Dirac (or Kronecker) symbol, satisfies:*

$$W_h(u,v) = 0 \text{ if } u = 0_{n/2} \text{ and } W_h(u,v) = W_f(u,v) + W_g(u) \text{ otherwise.} \tag{7.9}$$

Proof. We have $W_h(u,v) = W_f(u,v) - \sum_{x\in\mathbb{F}_2^{n/2}}(-1)^{u\cdot x} + \sum_{x\in\mathbb{F}_2^{n/2}}(-1)^{g(x)\oplus u\cdot x} = W_f(u,v) - 2^{\frac{n}{2}}\delta_0(u) + W_g(u)$. Function g being balanced, we have $W_g(0_{n/2}) = 0$. And $W_f(0_{n/2},v)$ equals $2^{\frac{n}{2}}$ for every v, since f is null on $\mathbb{F}_2^{n/2}\times\{0_{n/2}\}$ and according to Relation (6.7), page 223, applied to $E = \{0_{n/2}\} \times \mathbb{F}_2^{n/2}$ and $a = b = 0_{n/2}$ (or see the remark after Theorem 14, page 227). $\square$

We deduce that:

$$\max_{u,v\in\mathbb{F}_2^{n/2}} |W_h(u,v)| \leq \max_{u,v\in\mathbb{F}_2^{n/2}} |W_f(u,v)| + \max_{u\in\mathbb{F}_2^{n/2}} |W_g(u)|,$$

i.e. that $2^n - 2nl(h) \leq 2^n - 2nl(f) + 2^{\frac{n}{2}} - 2nl(g)$, that is:

$$nl(h) \geq nl(f) + nl(g) - 2^{\frac{n}{2}-1} = 2^{n-1} - 2^{\frac{n}{2}} + nl(g).$$

Applying recursively this principle (if $\frac{n}{2}$ is even, g can be constructed in the same way), we see that if $n = 2^k\, n'$ (n' odd), Dobbertin's method allows reaching the nonlinearity $2^{n-1} - 2^{\frac{n}{2}-1} - 2^{\frac{n}{4}-1} - \cdots - 2^{n'-1} - 2^{\frac{n'-1}{2}}$ since we know that, for every odd n', the nonlinearity of functions on $\mathbb{F}_2^{n'}$ can be as high as $2^{n'-1} - 2^{\frac{n'-1}{2}}$, and that balanced (quadratic) functions can achieve this value. If $n' \leq 7$ then this value is the best possible and $2^{n-1} - 2^{\frac{n}{2}-1} - 2^{\frac{n}{4}-1} - \cdots - 2^{n'-1} - 2^{\frac{n'-1}{2}}$ is therefore the best known nonlinearity of balanced functions in general. For $n' > 7$, the best nonlinearity of balanced n'-variable functions is larger than $2^{n'-1} - 2^{\frac{n'-1}{2}}$ (see the paragraph devoted to nonlinearity in Section 3.1) and $2^{n-1} - 2^{\frac{n}{2}-1} - 2^{\frac{n}{4}-1} - \cdots - 2^{2n'-1} - 2^{n'} + nl(g)$, where g is an n'-variable balanced function, can therefore reach higher values.
Dobbertin's conjecture on balanced functions is that his construction allows reaching the best nonlinearities of balanced functions in even numbers of variables. This question is still open and it is, in particular, an open problem to find an 8-variable balanced Boolean function with nonlinearity 118.

Unfortunately, according to Relation (7.9), Dobbertin's construction cannot produce t-resilient functions with $t > 0$ since, g being a function defined on $\mathbb{F}_2^{n/2}$, there cannot exist more than one vector a such that $W_g(a)$ equals $\pm 2^{\frac{n}{2}}$. Modifying bent functions into resilient functions has been studied in [821].

7.1.9 Secondary constructions of correlation immune and resilient functions

There exist several simple *secondary constructions*, which can be combined to obtain resilient functions achieving the bounds of Sarkar et al. and Siegenthaler. We list them below in chronological order.

I The *direct sum* of functions

A. *Adding a variable*

Let f be an r-variable t-resilient function. The Boolean function on $\mathbb{F}_2^{r+1}$:

$$h(x_1, \ldots, x_r, x_{r+1}) = f(x_1, \ldots, x_r) \oplus x_{r+1}$$

is $(t+1)$-resilient [1041], since, for $a \in \mathbb{F}_2^r$ and $a_{r+1} \in \mathbb{F}_2$, we have $W_h(a, a_{r+1}) = 2\, W_f(a)\, \delta_1(a_{r+1})$. If f is an $(r, t, r-t-1, 2^{r-1} - 2^{t+1})$ function[4], then h is an $(r+1, t+1, r-t-1, 2^r - 2^{t+2})$ function, and thus achieves Siegenthaler's and Sarkar et al.'s bounds. But h has the linear structure $(0, \ldots, 0, 1)$.

B. *Generalization*

If f is an r-variable t-resilient function ($t \geq 0$) and if g is an s-variable m-resilient function ($m \geq 0$), then the function:

$$h(x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}) = f(x_1, \ldots, x_r) \oplus g(x_{r+1}, \ldots, x_{r+s})$$

is $(t+m+1)$-resilient, since:

$$W_h(a, b) = W_f(a) \times W_g(b), \quad a \in \mathbb{F}_2^r,\ b \in \mathbb{F}_2^s. \tag{7.10}$$

We have also $d_{alg}(h) = \max(d_{alg}(f), d_{alg}(g))$ and, thanks to Relation (3.1), page 99, relating the nonlinearity to the Walsh transform, $nl(h) = 2^{r+s-1} - \frac{1}{2}(2^r - 2nl(f))(2^s - 2nl(g)) = 2^r nl(g) + 2^s nl(f) - 2nl(f)nl(g)$. Such function, called decomposable, does not give full satisfaction since such particular structure may be used in attacks. Moreover, h has a low algebraic degree, in general. And if $nl(f) = 2^{r-1} - 2^{t+1}$ ($t \leq r-2$) and $nl(g) = 2^{s-1} - 2^{m+1}$ ($m \leq s-2$, which is not the case when adding one variable), *i.e.* if $nl(f)$ and $nl(g)$ have maximum possible values, then $nl(h) = 2^{r+s-1} - 2^{t+m+3}$ and h does not achieve Sarkar's and Maitra's bound. Function h has no nonzero linear structure if and only if f and g both have no nonzero linear structure (we see then that having no linear structure is not a sufficient criterion).

Note that the result does not work with unbalanced functions.

[4] Recall that, by an $(n, m, d, \mathcal{N})$- function, we mean an n-variable, t-resilient function having algebraic degree at least d and nonlinearity at least $\mathcal{N}$.

II. Siegenthaler's construction

Let f and g be two Boolean functions on $\mathbb{F}_2^r$. Let us consider the function

$$h(x_1, \ldots, x_r, x_{r+1}) = (x_{r+1} \oplus 1) f(x_1, \ldots, x_r) \oplus x_{r+1} g(x_1, \ldots, x_r)$$

on $\mathbb{F}_2^{r+1}$. Note that the *truth-table* of h can be obtained by concatenating the truth-tables of f and g. Then:

$$W_h(a_1, \ldots, a_r, a_{r+1}) = W_f(a_1, \ldots, a_r) + (-1)^{a_{r+1}} W_g(a_1, \ldots, a_r). \tag{7.11}$$

Thus:

1. If f and g are t-resilient, then h is t-resilient [1041]; moreover, if for every $a \in \mathbb{F}_2^r$ of Hamming weight $t+1$, we have $W_f(a) + W_g(a) = 0$, then h is $(t+1)$-resilient. Note that the construction recalled in **I.A** corresponds to $g = f \oplus 1$ and satisfies this condition. Another possible choice of a function g satisfying this condition (first pointed out in [181]) is $g(x) = f(x_1 \oplus 1, \ldots, x_r \oplus 1) \oplus \epsilon$, where $\epsilon = t\,[\bmod 2]$, since $W_g(a) = \sum_{x \in \mathbb{F}_2^r} (-1)^{f(x) \oplus \epsilon \oplus (x \oplus 1_r) \cdot a} = (-1)^{\epsilon + w_H(a)} W_f(a)$. It leads to a function h having also a nonzero linear structure;

2. The value $\max\limits_{a_1, \ldots, a_{r+1} \in \mathbb{F}_2} |W_h(a_1, \ldots, a_r, a_{r+1})|$ is bounded above by the number $\max\limits_{a_1, \ldots, a_r \in \mathbb{F}_2} |W_f(a_1, \ldots, a_r)| + \max\limits_{a_1, \ldots, a_r \in \mathbb{F}_2} |W_g(a_1, \ldots, a_r)|$; this implies $2^{r+1} - 2nl(h) \le 2^{r+1} - 2nl(f) - 2nl(g)$, that is $nl(h) \ge nl(f) + nl(g)$;

a. if f and g achieve maximum possible nonlinearity $2^{r-1} - 2^{t+1}$ and if h is $(t+1)$-resilient, then the nonlinearity $2^r - 2^{t+2}$ of h is the best possible;

b. if f and g are such that, for every vector a, at least one of the numbers $W_f(a)$, $W_g(a)$ is null (in other words, if the supports of the Walsh transforms of f and g are disjoint), then we have $\max_{a_1, \ldots, a_{r+1} \in \mathbb{F}_2} |W_h(a_1, \ldots, a_r, a_{r+1})| = \max\left(\max_{a_1, \ldots, a_r \in \mathbb{F}_2} |W_f(a_1, \ldots, a_r)|; \max_{a_1, \ldots, a_r \in \mathbb{F}_2} |W_g(a_1, \ldots, a_r)|\right)$. Hence we have $2^{r+1} - 2nl(h) = 2^r - 2\min(nl(f), nl(g))$ and $nl(h)$ equals therefore $2^{r-1} + \min(nl(f), nl(g))$; thus, if f and g achieve best possible nonlinearity $2^{r-1} - 2^{t+1}$, then h achieves best possible nonlinearity $2^r - 2^{t+1}$;

3. If the monomials of highest degree in the algebraic normal forms of f and g are not all the same, then $d_{alg}(h) = 1 + \max(d_{alg}(f), d_{alg}(g))$. Note that this condition is not satisfied in the two cases indicated above in **1**, for which h is $(t+1)$-resilient.

4. For every $a = (a_1, \ldots, a_r) \in \mathbb{F}_2^r$ and every $a_{r+1} \in \mathbb{F}_2$, we have, denoting $(x_1, \ldots, x_r)$ by x: $D_{(a, a_{r+1})} h(x, x_{r+1}) = D_a f(x) \oplus a_{r+1} (f \oplus g)(x) \oplus x_{r+1} D_a (f \oplus g)(x) \oplus a_{r+1} D_a (f \oplus g)(x)$. If $d_{alg}(f \oplus g) \ge d_{alg}(f)$, then $D_{(a,1)} h$ is non-constant, for every a. And if, additionally, there does not exist $a \ne 0_r$ such that $D_a f$ and $D_a g$ are constant and equal to each other, then h admits no nonzero linear structure.

This construction allows obtaining from any two t-resilient functions f and g having disjoint Walsh spectra, achieving nonlinearity $2^{r-1} - 2^{t+1}$ and such that $d_{alg}(f \oplus g) = r - t - 1$, a t-resilient function h having algebraic degree $r - t$ and having nonlinearity $2^r - 2^{t+1}$, that is, achieving Siegenthaler's and Sarkar et al.'s bounds; note that this construction increases (by 1) the algebraic degrees of f

and g. And since, from any t-resilient function f having algebraic degree $r-t-1$ and nonlinearity $2^{r-1}-2^{t+1}$, we can deduce a function h having resiliency order $t+1$ and nonlinearity 2^r-2^{t+2}, that is, achieving Siegenthaler's and Sarkar et al.'s bounds and having same algebraic degree as f (but having nonzero linear structures), we can by combining these two methods, keep best trade-offs between resiliency order, algebraic degree and nonlinearity, and increase by 1 the degree and the resiliency order.
Generalization: let $(f_y)_{y\in\mathbb{F}_2^s}$ be a family of r-variable t-resilient functions; then the function on $\mathbb{F}_2^{r+s}$ defined by $f(x,y)=f_y(x)$ ($x\in\mathbb{F}_2^r$, $y\in\mathbb{F}_2^s$) is t-resilient. Indeed, we have $W_f(a,b)=\sum_{y\in\mathbb{F}_2^s}(-1)^{b\cdot y}\, W_{f_y}(a)$. Function f corresponds to the concatenation of the functions f_y; hence, this secondary construction can be viewed as a generalization of Maiorana-McFarland's construction (in which the functions f_y are t-resilient affine functions).

More on the resilient functions achieving high nonlinearities and constructed by using, among others, the secondary constructions above (as well as algorithmic methods) can be found in [696].

III. Tarannikov's elementary construction
Let g be any Boolean function on $\mathbb{F}_2^r$. We define the Boolean function h on $\mathbb{F}_2^{r+1}$ by $h(x_1,\ldots,x_r,x_{r+1})=x_{r+1}\oplus g(x_1,\ldots,x_{r-1},x_r\oplus x_{r+1})$. By the change of variable $x_r\leftarrow x_r\oplus x_{r+1}$, we see that the Walsh transform $W_h(a_1,\ldots,a_{r+1})$ is equal to $\displaystyle\sum_{x_1,\ldots,x_{r+1}\in\mathbb{F}_2}(-1)^{a\cdot x\oplus g(x_1,\ldots,x_r)\oplus(a_r\oplus a_{r+1}\oplus 1)x_{r+1}}$, where $a=(a_1,\ldots,a_r)$ and $x=(x_1,\ldots,x_r)$; if $a_r\oplus a_{r+1}=0$ then this value is null and if $a_r\oplus a_{r+1}=1$ then it equals $2\,W_g(a_1,\ldots,a_{r-1},a_r)$. Thus:
1. $nl(h)=2\,nl(g)$;
2. If g is t-resilient, then h is t-resilient, since $w_H(a_1,\ldots,a_r)\le w_H(a_1,\ldots,a_{r+1})$. And h is $(t+1)$-resilient if and only if, for every vector $(a_1,\ldots,a_{r+1})$ of Hamming weight $t+1$ such that $a_r\oplus a_{r+1}=1$, we have $W_g(a_1,\ldots,a_r)=0$ and the only case not implied by the t-resiliency of g is when $a_r=1$ and $a_{r+1}=0$; hence, h is $(t+1)$-resilient if and only if $W_g(a_1,\ldots,a_{r-1},1)$ is null for every vector $(a_1,\ldots,a_{r-1})$ of Hamming weight t; note that, in such case, if g has nonlinearity $2^{r-1}-2^{t+1}$ then the nonlinearity of h, which equals 2^r-2^{t+2} achieves then Sarkar et al.'s bound too. The condition that $W_g(a_1,\ldots,a_{r-1},1)$ is null for every vector $(a_1,\ldots,a_{r-1})$ of Hamming weight at most t is achieved if g does not actually depend on its last input bit; but the construction is then a particular case of the construction recalled in **I.A**. The condition is also achieved if g is obtained from two t-resilient functions, by using Siegenthaler's construction (recalled in **II**), according to Relation (7.11).
3. $d_{alg}(h)=d_{alg}(g)$ if $d_{alg}(g)\ge 1$.
4. h has the nonzero linear structure $(0,\ldots,0,1,1)$.

Tarannikov combined in [1080] this construction with the direct sum and Siegenthaler constructions recalled in **I** and **II**, to build a more complex secondary construction, which allows increasing at the same time the resiliency order and the algebraic degree of the functions and which leads to an infinite sequence of functions achieving Siegenthaler's and Sarkar et al.'s bounds. Increasing then, by using the construction recalled in **I.A**, the set of ordered pairs (r, m) for which such functions can be constructed, he deduced the existence of r-variable t-resilient functions achieving Siegenthaler's and Sarkar et al.'s bounds for any number of variables r and any resiliency order t such that $t \geq \frac{2r-7}{3}$ and $t > \frac{r}{2} - 2$ (but these functions have nonzero linear structures). In [934], Pasalic et al. slightly modified this more complex Tarannikov's construction into a construction that we shall call *Tarannikov et al.'s construction*, which allowed, when iterating it together with the construction recalled in **I.A**, to relax slightly the condition on t into $t \geq \frac{2r-10}{3}$ and $t > \frac{r}{2} - 2$.

IV. Indirect sum of functions

Tarannikov et al.'s construction has been in its turn generalized into a construction which has been named *indirect sum* a few years after it was introduced, and that we already encountered at page 259 as a construction of bent functions. Indirect sum builds a function h from 4 functions, while the previous constructions used at most 2 functions. All the secondary constructions listed above are particular cases of it: they correspond to fixing 2 or 3 of the 4 functions.

Theorem 21 *[225] Let r and s be positive integers and let t and m be nonnegative integers such that $t < r$ and $m < s$. Let f_1 and f_2 be two r-variable functions. Let g_1 and g_2 be two s-variable functions. We define the $(r+s)$-variable function:*

$$h(x, y) = f_1(x) \oplus g_1(y) \oplus (f_1 \oplus f_2)(x)\,(g_1 \oplus g_2)(y); \quad x \in \mathbb{F}_2^r, y \in \mathbb{F}_2^s.$$

If f_1 and f_2 are distinct and if g_1 and g_2 are distinct, then the algebraic degree of h equals $\max(d_{alg}(f_1), d_{alg}(g_1), d_{alg}(f_1 \oplus f_2) + d_{alg}(g_1 \oplus g_2))$; otherwise, it equals $\max(d_{alg}(f_1), d_{alg}(g_1))$. The Walsh transform of h takes value at (a, b), where $a \in \mathbb{F}_2^r, b \in \mathbb{F}_2^s$:

$$W_h(a, b) = \frac{1}{2} W_{f_1}(a) \left[W_{g_1}(b) + W_{g_2}(b)\right] + \frac{1}{2} W_{f_2}(a) \left[W_{g_1}(b) - W_{g_2}(b)\right]. \quad (7.12)$$

If f_1 and f_2 are t-resilient and g_1 and g_2 are m-resilient then h is $(t+m+1)$-resilient.

If the Walsh transforms of f_1 and f_2 have disjoint supports and if the Walsh transforms of g_1 and g_2 have disjoint supports, then

$$nl(h) = \min_{i,j \in \{1,2\}} \left(2^{r+s-2} + 2^{r-1} nl(g_j) + 2^{s-1} nl(f_i) - nl(f_i) nl(g_j)\right). \quad (7.13)$$

In particular, if f_1 and f_2 are two $(r, t, -, 2^{r-1} - 2^{t+1})$ functions with disjoint Walsh supports, if g_1 and g_2 are two $(s, m, -, 2^{s-1} - 2^{m+1})$ functions with disjoint

Walsh supports, and if $f_1 \oplus f_2$ has degree $r-t-1$ and $g_1 \oplus g_2$ has algebraic degree $s-m-1$, then h is a $(r+s, t+m+1, r+s-t-m-2, 2^{r+s-1}-2^{t+m+2})$ function, and thus achieves Siegenthaler's and Sarkar et al.'s bounds.

Proof. For every $a \in \mathbb{F}_2^r, b \in \mathbb{F}_2^s$, we have:

$$\begin{aligned}
W_h(a,b) = & \sum_{y\in\mathbb{F}_2^s;\, g_1\oplus g_2(y)=0} \left(\sum_{x\in\mathbb{F}_2^r}(-1)^{f_1(x)\oplus a\cdot x}\right)(-1)^{g_1(y)\oplus b\cdot y} \\
& + \sum_{y\in\mathbb{F}_2^s;\, g_1\oplus g_2(y)=1} \left(\sum_{x\in\mathbb{F}_2^r}(-1)^{f_2(x)\oplus a\cdot x}\right)(-1)^{g_1(y)\oplus b\cdot y} \\
& = W_{f_1}(a) \sum_{\substack{y\in\mathbb{F}_2^s;\\ g_1\oplus g_2(y)=0}} (-1)^{g_1(y)\oplus b\cdot y} + W_{f_2}(a) \sum_{\substack{y\in\mathbb{F}_2^s;\\ g_1\oplus g_2(y)=1}} (-1)^{g_1(y)\oplus b\cdot y} \\
& = W_{f_1}(a) \sum_{y\in\mathbb{F}_2^s} (-1)^{g_1(y)\oplus b\cdot y}\left(\frac{1+(-1)^{(g_1\oplus g_2)(y)}}{2}\right) \\
& + W_{f_2}(a) \sum_{y\in\mathbb{F}_2^s} (-1)^{g_1(y)\oplus b\cdot y}\left(\frac{1-(-1)^{(g_1\oplus g_2)(y)}}{2}\right).
\end{aligned}$$

We deduce Relation (7.12). If (a,b) has Hamming weight at most $t+m+1$ then a has Hamming weight at most t or b has Hamming weight at most t; hence we have $W_h(a,b)=0$. Thus, h is $t+m+1$-resilient.
If $f_1 \oplus f_2$ and $g_1 \oplus g_2$ are non-constant, then the algebraic degree of h equals $\max(d_{alg}(f_1), d_{alg}(g_1), d_{alg}(f_1\oplus f_2)+d_{alg}(g_1\oplus g_2))$ because the terms of highest degrees in $(g_1\oplus g_2)(y)\,(f_1\oplus f_2)(x)$, in $f_1(x)$ and in $g_1(y)$ cannot cancel each others.
We deduce from Relation (7.12) that if the supports of the Walsh transforms of f_1 and f_2 are disjoint, as well as those of g_1 and g_2, then:

$$\max_{(a,b)\in\mathbb{F}_2^r\times\mathbb{F}_2^s} |W_h(a,b)| = \frac{1}{2}\max_{i,j\in\{1,2\}}\left(\max_{a\in\mathbb{F}_2^r}|W_{f_i}(a)|\max_{b\in\mathbb{F}_2^s}|W_{g_j}(b)|\right)$$

and according to Relation (3.1) relating the nonlinearity to the Walsh transform, this implies:

$$2^{r+s}-2nl(h) = \frac{1}{2}\max_{i,j\in\{1,2\}}\left((2^r-2nl(f_i))(2^s-2nl(g_j))\right),$$

which is equivalent to Relation (7.13). □

Note that function h, defined this way, is the concatenation of the four functions f_1, $f_1\oplus 1$, f_2 and $f_2\oplus 1$, in an order controlled by $g_1(y)$ and $g_2(y)$.
This construction is nicely general and does not need the initial functions f_1, f_2 and g_1, g_2 to satisfy complex conditions, contrary to other constructions which have been derived later for building bent functions (see pages 259 and foll.) and could be adapted for designing resilient functions.
Examples of pairs (f_1, f_2) (or (g_1, g_2)) satisfying the hypotheses of Theorem 21 can be found in [225]. The interest of the indirect sum compared to the direct

sum is that it allows designing functions h which are more complex (have larger algebraic degree and possibly larger algebraic immunity and fast algebraic immunity).

Remark. The indirect sum (as well as all its particular cases viewed above) is less well adapted to constructing correlation immune functions: Relation (7.12) shows that if $W_{f_1}(a) = W_{f_2}(a) = W_{g_1}(b) = W_{g_2}(b) = 0$ then $W_h(a,b) = 0$ but when for instance $a = 0_r$ and $b \neq 0_s$, we have $W_h(a,b) = \frac{1}{2} W_{f_1}(0_r)\,[W_{g_1}(b) + W_{g_2}(b)] + \frac{1}{2} W_{f_2}(0_r)\,[W_{g_1}(b) - W_{g_2}(b)]$ and there are additional conditions on the values of $W_{f_1}(0_r), W_{f_2}(0_r), W_{g_1}(b)$ and $W_{g_2}(b)$ when $w_H(b) \geq m+1$ (and on the values of $W_{g_1}(0_s), W_{g_2}(0_s), W_{f_1}(a), W_{f_2}(a)$ when $w_H(a) \geq t+1$) for allowing h to be more than $(\min(t,m))$-th order correlation immune. □

V. Constructions without extension of the number of variables

Proposition 85, page 262, leads to the following construction:

Proposition 122 *[227] Let n be any positive integer and t any non-negative integer such that $t \leq n$. Let f_1, f_2 and f_3 be three t-th order correlation immune (resp. t-resilient) functions. Then the function $s_1 = f_1 \oplus f_2 \oplus f_3$ is t-th order correlation immune (resp. t-resilient) if and only if the function $s_2 = f_1 f_2 \oplus f_1 f_3 \oplus f_2 f_3$ is t-th order correlation immune (resp. t-resilient). Moreover:*

$$nl(s_2) \geq \frac{1}{2}\left(nl(s_1) + \sum_{i=1}^{3} nl(f_i)\right) - 2^{n-1} \tag{7.14}$$

and if the Walsh supports of f_1, f_2 and f_3 are pairwise disjoint (that is, if at most one value $W_{f_i}(s)$, $i = 1,2,3$ is nonzero, for every vector s), then

$$nl(s_2) \geq \frac{1}{2}\left(nl(s_1) + \min_{1 \leq i \leq 3} nl(f_i)\right). \tag{7.15}$$

Proof. Relation (6.30), page 262, and the fact that, for every nonzero vector (resp. any vector) a of Hamming weight at most t, we have $W_{f_i}(a) = 0$ for $i = 1, 2, 3$ imply that $W_{s_1}(a) = 0$ if and only if $W_{s_2}(a) = 0$. Relations (7.14) and (7.15) are also direct consequences of Relation (6.30) and of Relation (3.1), page 99, relating the nonlinearity to the Walsh transform. □

Note that this secondary construction is proper to allow achieving high algebraic immunity with s_2, given functions with lower algebraic immunities f_1, f_2, f_3 and s_1, since the support of s_2 can be made more complex than those of these functions. This is done without changing the number of variables and keeping similar resiliency order and nonlinearity.

Remark. Let g and h be two Boolean functions on $\mathbb{F}_2^n$ with disjoint supports and let f be equal to $g \oplus h = g + h$. Then, f is balanced if and only if $w_H(g) + w_H(h) = 2^{n-1}$. By linearity of the Fourier-Hadamard transform, we have: $\widehat{f} = \widehat{g} + \widehat{h}$. Thus, if g and h are t-th order correlation immune, then f is t-resilient. For every nonzero $a \in \mathbb{F}_2^n$, we have $|W_f(a)| = 2\,|\widehat{f}(a)| \leq 2\,|\widehat{g}(a)| + 2\,|\widehat{h}(a)| = |W_g(a)| + |W_h(a)|$. Thus, assuming that f is balanced, we have $nl(f) \geq nl(g) + nl(h) - 2^{n-1}$. The algebraic degree of f is bounded above by (and can be equal to) the maximum of the algebraic degrees of g and h. □

The most part of the secondary constructions of bent functions described in Subsection 6.1.16 can be altered into constructions of correlation immune and resilient functions, see [216].

The generalization of Proposition 85 given by Proposition 86, page 263, leads to:

Proposition 123 *[227] Let n be any positive integer and k any non-negative integer such that $k \leq n$. Let $f_1, \ldots, f_7$ be k-th order correlation immune (resp. k-resilient) functions. If two among the functions $s_1 = f_1 \oplus \ldots \oplus f_7$, $s_2 = f_1 f_2 \oplus f_1 f_3 \oplus \ldots \oplus f_6 f_7$ and $s_4 = \bigoplus_{1 \leq i_1 < \ldots < i_4 \leq 7} \prod_{j=1}^{l} f_{i_j}$ is k-th order correlation immune (resp. k-resilient) then the third one is k-th order correlation immune (resp. k-resilient).*

Low *Hamming weight* correlation immune functions

Except for the secondary construction without extension of the number of variables, the primary and secondary constructions of resilient functions recalled above do not work well for building unbalanced correlation immune functions, as we observed for the indirect sum in the remark at the head of page 331. We shall see in Section 12.1, page 460, that low Hamming weight correlation immune functions are useful for countermeasures to side channel attacks. More constructions are then needed.

We denote by $CI_{n,t}$ the set of n-variable t-th order correlation immune Boolean functions and by $\omega_{n,t}$ the minimal Hamming weight of nonzero functions in $CI_{n,t}$. According to Proposition 120, page 317, the Hamming weight of a t-th order correlation immune function is divisible by $2^{t+\left\lfloor \frac{n-t-1}{d_{alg}(f)} \right\rfloor}$.

The only n-variable n-th order correlation immune Boolean functions are the two constant functions. The only $(n-1)$-th order correlation immune non-constant Boolean functions are the $(n-1)$-resilient functions $\bigoplus_{i=1}^n x_i$ and $\bigoplus_{i=1}^n x_i \oplus 1$. Then $\omega_{n,n} = 2^n$ and $\omega_{n,n-1} = 2^{n-1}$.

We have of course $\omega_{n,t} \leq \omega_{n,t+1}$ and more precisely:

Lemma 10 *Let $1 \leq t \leq n$ be integers. Then*

$$\omega_{n+1,t} \leq 2\omega_{n,t} \leq \omega_{n+1,t+1}.$$

Proof. For every $f \in CI_{n,t}$, the $(n+1)$-variable function $g(x, x_{n+1}) = f(x)$ belongs to $CI_{n+1,t}$, since, for every a, we have $\widehat{g}(a,0) = 2\widehat{f}(a)$ and $\widehat{g}(a,1) = 0$.

Table 7.1 Lower bound on $\omega_{n,t}$ from Delsarte's Linear Programming bound [74]

n \ t	1	2	3	4	5	6	7	8	9	10	11	12	13
1	2												
2	2	4											
3	2	4	8										
4	2	6	8	16									
5	2	8	12	16	32								
6	2	8	16	32	32	64							
7	2	8	16	48	64	64	128						
8	2	10	16	64	88	112	128	256					
9	2	12	20	96	128	192	224	256	512				
10	2	12	24	96	192	320	384	512	512	1024			
11	2	12	24	96	192	512	640	1024	1024	1024	2048		
12	2	14	24	112	176	768	1024	1536	1792	2048	2048	4096	
13	2	16	28	128	224	1024	1536	2560	3072	3584	4096	4096	8192

Moreover, g has Hamming weight $2w_H(f)$. This proves the left-hand side inequality. For every $f \in CI_{n+1,t+1}$, the restriction of f to the hyperplane of equation $x_{n+1} = 0$ is a t-th order correlation immune Boolean function with half weight. This proves the right-hand side inequality. □

As observed in [591], the largest dimension $k_{\max}(n, t+1)$ of a binary linear code $[n, k, t+1]$ provides the upper bound $\omega_{n,t} \leq 2^{n-k_{\max}(n,t+1)}$, according to Corollary 6, page 108 and to the fact that the dual of a linear code of dimension k has dimension $n-k$. Since a binary MDS code of parameters $[n, n-1, 2]$ exists, we have then $\omega_{n,1} = 2$ for every n.

As also observed in [591], $\omega_{n,t}$ being equal to the minimal number of rows in a simple binary orthogonal array of strength t, Delsarte's linear programming bound [422] provides a lower bound on $\omega_{n,t}$ that we give in Table 7.1.

The Satisfiability Modulo Theory (SMT) tool has been used to search for correlation immune Boolean functions in [74] together with the upper bound deduced from known constructions of binary codes, the lower bound of Table 7.1 and the divisibility of $\omega_{n,t}$ by 2^t.

Table 7.2 displaying the known values of $\omega_{n,t}$ for $n \leq 13$ is taken from [74, 287, 258, 1103]. The entries in light gray follow from $\omega_{n,1} = 2$ and $\omega_{n,n} = 2^n$. The entries in dark gray follow from $\omega_{n,n-1} = 2^{n-1}$ and from Lemma 10 above, which imply $\omega_{n,t} \leq \omega_{n,n-1} = 2^{n-1}$, and from Theorem 116, page 313, which implies that $\omega_{n,t} = 2^{n-1}$ for $\lceil \frac{2n-2}{3} \rceil \leq t \leq n-1$. The entry $n = 11, t = 4$ is obtained in [74, 1103] and the entries $n = 11, t = 5$; $n = 12, t = 5$ and $n = 12, t = 7$ follow from Proposition 124 below. The entries in bold have been obtained by SMT tool. A triple question mark in this table indicates that the value is unknown. Note however that upper bounds are known for these entries: in [946] is made a detailed exploration of several evolutionary algorithms for finding Boolean functions that have various orders of correlation immunity and minimal Hamming weight. These investigations show that $\omega_{11,4} \leq 128$, $\omega_{11,5} \leq 256$, $\omega_{12,5} \leq 256$, $\omega_{12,6} \leq 1024$, $\omega_{13,7} \leq 2048$.

Table 7.2 Minimum weight of t-th order correlation immune nonzero n-variable functions

n \ t	1	2	3	4	5	6	7	8	9	10	11	12	13
1	2												
2	2	4											
3	2	4	8										
4	2	8	8	16									
5	2	8	16	16	32								
6	2	8	16	32	32	64							
7	2	8	16	64	64	64	128						
8	2	**12**	16	64	128	128	128	256					
9	2	**12**	**24**	**128**	128	256	256	256	512				
10	2	**12**	**24**	**128**	**256**	512	512	512	512	1024			
11	2	**12**	**24**	128	256	512	1024	1024	1024	1024	2048		
12	2	16	**24**	???	256	512	1024	2048	2048	2048	2048	4096	
13	2	16	32	???	???	???	1024	4096	4096	4096	4096	4096	8192

It is an open question to determine whether the columns in this table (and more generally for every value of n and t) are non-decreasing, that is, $\omega_{n,t} \leq \omega_{n+1,t}$ for every n and t. If the reply to this question is positive, then these values are optimal. It is also shown in [946] that $\omega_{12,4} \leq 256$, $\omega_{13,4} \leq 256$, $\omega_{13,5} \leq 512$, $\omega_{13,6} \leq 1024$. See also [112] where non-existence results are proved.

Remark. The indicator $1_{\mathcal{K}_n}$ of the Kerdock code $\mathcal{K}_n$, seen at page 281, provides for every even $n \geq 4$, a 5-CI Boolean function in 2^n variables and of Hamming weight 2^{2n}, since we know that $\mathcal{K}_n$ has dual distance 6. Let us determine its ANF. Denoting by $X = (X_{x,x_n})_{\substack{x\in\mathbb{F}_{2^{n-1}}\\ x_n\in\mathbb{F}_2}}$ the elements of $\mathbb{F}_2^{2^n}$, the dual of $RM(1,n)$ being equal to $RM(n-2,n)$, the ANF of the indicator of $RM(1,n)$ has the form $\prod_{\substack{I\subset\{1,\dots,n\}\\ |I|\leq n-2}} \left(1+\sum_{\substack{x\in\mathbb{F}_{2^{n-1}}\\ x_n\in\mathbb{F}_2}} X_{x,x_n}\prod_{i\in I}\ell_i(x,x_n)\right)$, where $(\ell_1,\dots,\ell_{n-1})$ are (any) linearly independent $\mathbb{F}_2$-linear forms over $\mathbb{F}_{2^{n-1}}$, viewed as functions over $\mathbb{F}_{2^{n-1}}\times\mathbb{F}_2$, and $\ell_n(x,x_n)=x_n$. The ANF of the indicator of any coset $f+RM(1,n)$ has then the form $\prod_{\substack{I\subset\{1,\dots,n\}\\ |I|\leq n-2}} \left(1+\sum_{\substack{x\in\mathbb{F}_{2^{n-1}}\\ x_n\in\mathbb{F}_2}} \left(X_{x,x_n}+f(x,x_n)\right)\prod_{i\in I}\ell_i(x,x_n)\right)$. According to the definition of the Kerdock code, the ANF of $1_{\mathcal{K}_n}$ equals then the sum, when u ranges over $\mathbb{F}_{2^{n-1}}$, of:

$$\prod_{\substack{I\subset\{1,\dots,n\}\\ |I|\leq n-2}} \left(1+\sum_{\substack{x\in\mathbb{F}_{2^{n-1}}\\ x_n\in\mathbb{F}_2}} \left(X_{x,x_n}+tr_m\Big(\sum_{j=1}^{t}(ux)^{2^j+1}\Big)+x_n tr_m(ux)\right)\prod_{i\in I}\ell_i(x,x_n)\right),$$

where $m=n-1$, $t=\frac{n}{2}-1$. We can then see that $1_{\mathcal{K}_n}$ has algebraic degree at most 2^n-1-n and does not achieve Siegenthaler's bound with equality, except maybe for $n=4$. $\square$

It is shown in [1103] that $\omega_{n,2} \geq 4\left\lceil\frac{n+1}{4}\right\rceil$ for $n \geq 2$ (and the proof can be slightly simplified): the Golomb-Xiao-Massey characterization of correlation immune functions (Theorem 5, page 107) directly gives that a Boolean function f

is in $CI_{n,2}$ if and only if the matrix $H = ((-1)^{e_i \cdot x})_{\substack{x\in supp(f)\\ i=0,1,\ldots,n}}$, where $e_0 = 0_n$ and $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{F}_2^n$ over $\mathbb{F}_2$, satisfies $H^t \times H = w_H(f)\, I_{n+1}$ where I_{n+1} is the identity matrix (*i.e.* H is a Hadamard matrix); this shows that $\omega_{n,2} \geq 4\left\lceil \frac{n+1}{4} \right\rceil$ since we know that 4 divides $\omega_{n,2}$ and that matrix I_{n+1} has rank $n+1$, while matrix H has rank at most $w_H(f)$ and $H^t \times H$ has necessarily rank smaller than or equal to that of H.
It is deduced in [1103] that for each known Hadamard $4k \times 4k$ matrix, a function in $CI_{4k-1,2}$ of (minimum) Hamming weight $4k$ (and functions in $CI_{4k+i,2}$ of Hamming weight $4k$ for every $i = 0, 1, 2$) can be deduced. It has been conjectured by J. Hadamard that there exists a $4k \times 4k$ Hadamard matrix for every k. According to the observations above, this conjecture is equivalent to conjecturing that $\omega_{n,2} = 4\left\lceil \frac{n+1}{4} \right\rceil$ for every n.

Proposition 124 *[258] Let t be any even integer such that $2 \leq t \leq n$. Then:*

$$\omega_{n+1,t+1} = 2\,\omega_{n,t}.$$

Proof. For every $f \in CI_{n,t}$, the $(n+1)$-variable function:

$$g(x, x_{n+1}) = \begin{cases} f(x), & \text{when} \quad x_{n+1} = 0 \\ f(x + 1_n), & \text{when} \quad x_{n+1} = 1, \end{cases}$$

has Hamming weight $2\,w_H(f)$ and is a $(t+1)$-th order correlation immune Boolean function. Indeed, for any $u \in \mathbb{F}_2^n$ and any $u_{n+1} \in \mathbb{F}_2$, we have:

$$\begin{aligned}
\widehat{g}(u, u_{n+1}) &= \sum_{(x,x_{n+1})\in\mathbb{F}_2^{n+1}} g(x, x_{n+1})(-1)^{(u,u_{n+1})\cdot(x,x_{n+1})} \\
&= \sum_{x\in\mathbb{F}_2^n} f(x)(-1)^{u\cdot x} + \sum_{x\in\mathbb{F}_2^n} f(x+1_n)(-1)^{(u,u_{n+1})\cdot(x,1)} \\
&= \widehat{f}(u) + \sum_{x\in\mathbb{F}_2^n} f(x)(-1)^{(u,u_{n+1})\cdot(x+1_n,1)} \\
&= \left(1 + (-1)^{w_H(u,u_{n+1})}\right) \widehat{f}(u).
\end{aligned}$$

If $w_H(u, u_{n+1}) = t+1$, then since t is an even integer, we have $1+(-1)^{w_H(u,u_{n+1})} = 1 + (-1)^{t+1} = 0$, thus $\widehat{g}(u, u_{n+1}) = 0$.
If $u = 0_n$ and $u_{n+1} = 1$, then $1 + (-1)^{w_H(u,u_{n+1})} = 0$, and $\widehat{g}(u, u_{n+1}) = 0$.
If $1 \leq w_H(u, u_{n+1}) \leq t$ and $u \neq 0_n$, we have that $1 \leq w_H(u) \leq t$, and since $f(x) \in CI_{n,t}$, we have $\widehat{f}(u) = 0$, then $\widehat{g}(u, u_{n+1}) = 0$.
Hence, if $1 \leq w_H(u, u_{n+1}) \leq t + 1$, then $\widehat{g}(u, u_{n+1}) = 0$ and $g(x, x_{n+1})$ is a $(t+1)$-th order correlation immune Boolean function. Thus, $\omega_{n+1,t+1} \leq 2\omega_{n,t}$ when t is even, and since $2\omega_{n,t} \leq \omega_{n+1,t+1}$ for any $1 \leq t \leq n$ according to Lemma 10, this completes the proof. □

This leads to the bound $\omega_{n,3} \geq 8\left\lceil \frac{n}{4} \right\rceil$ for $n \geq 3$. It is conjectured in [258] that for $n \geq 3$, we have $\omega_{n,3} = 8\left\lceil \frac{n}{4} \right\rceil$. Using the characterization of functions in

$CI_{n,2}$ given above Proposition 124 by means of Hadamard matrices and the known existence of infinitely many $4k \times 4k$ Hadamard matrices, Wang deduced in [1103] that infinitely many values $n \equiv i$ [mod 4] satisfy the conjecture for each $i = -1, 0, 1, 2$. He observed that the conjecture (which is still open) is equivalent to that of Hadamard, which is more than one hundred years old.
A construction of functions of weight 2^m in $CI_{n,t}$ has been given in [1103], which defines their support as made of the 2^m vectors of the form $(v \cdot u_1, \ldots, v \cdot u_n)$, where v ranges over $\mathbb{F}_2^m$ and the u_j's are such that none of them depends linearly on at most $t-1$ others. This construction is nothing more than Corollary 6, page 108, with a linear code whose generator matrix is made of the u_j's by columns (recall that the dual distance of such code is the minimum number of linearly dependent columns), or Corollary 22 page 320. The construction however allowed completing some entries in the table which was given in [74, 287] (Table 7.2 is the completed table).

Using the *Fourier-Hadamard transform* instead of the Walsh transform to construct correlation immune functions

We have seen that correlation immune functions are characterized by both the Fourier-Hadamard transform and the Walsh transform. We have also seen that most known constructions of correlation immune functions were based on the properties of the Walsh transform and that they built in fact resilient functions, mostly. The Fourier-Hadamard transform and the Walsh transform are closely related through Relation (2.32), page 74. However, they behave differently with respect to the operations in $\mathcal{BF}_n$: while the Walsh transform behaves well with respect to the addition of Boolean functions (for instance, the Walsh transform of a direct sum equals the product of the Walsh transforms, see Relation (7.10), page 326), the Fourier-Hadamard transform behaves well with respect to the multiplication of functions; in particular, the Fourier-Hadamard transform of a direct product equals the product of the Fourier-Hadamard transforms, since:

$$\sum_{x\in\mathbb{F}_2^n, y\in\mathbb{F}_2^m} f(x)g(y)(-1)^{a\cdot x\oplus b\cdot y} = \left(\sum_{x\in\mathbb{F}_2^n} f(x)(-1)^{a\cdot x}\right)\left(\sum_{y\in\mathbb{F}_2^m} g(y)(-1)^{b\cdot y}\right).$$

Multiplying Boolean functions produces unbalanced functions, and if the functions have low Hamming weights, the product has low Hamming weight as well. A related general construction of correlation immune functions by multiplication is deduced in [258] that we report now. In the next proposition, given a matrix $M \in \mathbb{F}_2^{ns\times ns}$ and given $i, j = 1, \ldots, s$, we denote by $M^{(i,j)}$ the $n \times n$ matrix (called a block of M) obtained from M by selecting its rows of indices between $n(i-1)+1$ and ni and its columns of indices between $n(j-1)+1$ and nj. Assuming that M is non-singular, denoting the inverse matrix of M by M^{-1} and the transposed matrix of M^{-1} by M', we denote by $M^{-1^{(i,j)}}$ and $M'^{(i,j)}$ the matrices obtained similarly from M^{-1} and M'. Since $M'^{(j,i)}$ is the transposed matrix of $M^{-1^{(i,j)}}$, we have, for any $x, y \in \mathbb{F}_2^n$:

$$x \cdot (y \times M^{-1^{(i,j)}}) = y \cdot (x \times M'^{(j,i)}), \tag{7.16}$$

where "$\cdot$" is the *usual inner product*.

Proposition 125 *[258] Let s be a positive integer and M be an $ns \times ns$ non-singular matrix over $\mathbb{F}_2$. Let $f_j \in CI_{n,t_j}$ for some non-negative integers t_j, $1 \le j \le s$. Define the following ns-variable function h, whose input is written in the form $(x^{(1)}, x^{(2)}, \dots, x^{(s)})$, where $x^{(1)}, x^{(2)}, \dots, x^{(s)} \in \mathbb{F}_2^n$:*

$$h(x^{(1)}, x^{(2)}, \dots, x^{(s)}) = \prod_{j=1}^{s} f_j\left(\sum_{i=1}^{s} x^{(i)} \times M^{(i,j)}\right).$$

Assume that if $1 \le w_H(u^{(1)}, u^{(2)}, \dots, u^{(s)}) \le t$, then $1 \le j \le s$, exists such that:

$$1 \le w_H\left(\sum_{i=1}^{s} u^{(i)} \times M'^{(i,j)}\right) \le t_j.$$

Then h belongs to $CI_{ns,t}$ and has Hamming weight $\prod_{j=1}^{s} w_H(f_j)$.

Proof. For any $(u^{(1)}, u^{(2)}, \dots, u^{(s)}) \in (\mathbb{F}_2^n)^s$, we have: $\widehat{h}(u^{(1)}, u^{(2)}, \dots, u^{(s)})$

$$= \sum_{x^{(1)},\dots,x^{(s)} \in \mathbb{F}_2^n} \left(\prod_{j=1}^{s} f_j\left(\sum_{i=1}^{s} x^{(i)} \times M^{(i,j)}\right)\right)(-1)^{\bigoplus_{j=1}^{s} u^{(j)} \cdot x^{(j)}}.$$

Replace $\sum_{i=1}^{s} x^{(i)} \times M^{(i,j)}$ by $y^{(j)}$ for $1 \le j \le s$, then $(y^{(1)}, y^{(2)}, \dots, y^{(s)}) = (x^{(1)}, x^{(2)}, \dots, x^{(s)}) \times M$, according to the well-known method of multiplication of matrices by blocks. Thus:

$$(x^{(1)}, x^{(2)}, \dots, x^{(s)}) = (y^{(1)}, y^{(2)}, \dots, y^{(s)}) \times M^{-1},$$

which means $x^{(j)} = \sum\limits_{i=1}^{s} y^{(i)} \times M^{-1(i,j)}$ for $1 \leq j \leq s$. Using (7.16), we have:

$$\begin{aligned}
&\widehat{h}(u^{(1)}, u^{(2)}, \ldots, u^{(s)}) \\
&= \sum_{y^{(1)},\ldots,y^{(s)} \in \mathbb{F}_2^n} \left(\prod_{j=1}^{s} f_j(y^{(j)}) \right) (-1)^{\bigoplus_{j=1}^{s} u^{(j)} \cdot \left(\sum\limits_{i=1}^{s} y^{(i)} \times M^{-1(i,j)} \right)} \\
&= \sum_{y^{(1)},\ldots,y^{(s)} \in \mathbb{F}_2^n} \left(\prod_{j=1}^{s} f_j(y^{(j)}) \right) (-1)^{\bigoplus_{i=1}^{s} y^{(i)} \cdot \left(\sum\limits_{j=1}^{s} u^{(j)} \times M'^{(j,i)} \right)} \\
&= \sum_{y^{(1)},\ldots,y^{(s)} \in \mathbb{F}_2^n} \left(\prod_{j=1}^{s} f_j(y^{(j)}) \right) (-1)^{\bigoplus_{j=1}^{s} y^{(j)} \cdot \left(\sum\limits_{i=1}^{s} u^{(i)} \times M'^{(i,j)} \right)} \\
&= \prod_{j=1}^{s} \left(\sum_{y^{(j)} \in \mathbb{F}_2^n} f_j(y^{(j)}) (-1)^{y^{(j)} \cdot \left(\sum\limits_{i=1}^{s} u^{(i)} \times M'^{(i,j)} \right)} \right) \\
&= \prod_{j=1}^{s} \widehat{f_j} \left(\sum_{i=1}^{s} u^{(i)} \times M'^{(i,j)} \right).
\end{aligned}$$

According to the hypothesis, for any $(u^{(1)}, u^{(2)}, \ldots, u^{(s)}) \in (\mathbb{F}_2^n)^s$ satisfying $1 \leq w_H(u^{(1)}, u^{(2)}, \ldots, u^{(s)}) \leq t$, there exists $1 \leq j \leq s$ such that

$$1 \leq w_H \left(\sum_{i=1}^{s} u^{(i)} \times M'^{(i,j)} \right) \leq t_j.$$

Since f_j is a t_j-th order correlation immune Boolean function for any $1 \leq j \leq s$, h is a t-th order correlation immune Boolean function. And h being affine equivalent to the direct product of f_j , we have $w_H(h) = \prod\limits_{j=1}^{s} w_H(f_j)$. □

Corollary 23 *[258] Let n, t, s be positive integers satisfying $t \leq n$ and $s \geq 2$. Assume that $f_1 \in CI_{n,t}$ and $f_j \in CI_{n,\lfloor \frac{t}{2} \rfloor}$ for any $2 \leq j \leq s$. Define*

$$h(x^{(1)}, x^{(2)}, \ldots, x^{(s)}) = f_1(x^{(1)}) \prod_{j=2}^{s} f_j(x^{(1)} + x^{(j)}),$$

where $x^{(1)}, x^{(2)}, \ldots, x^{(s)} \in \mathbb{F}_2^n$. Then h belongs to $CI_{ns,t}$ and has Hamming weight $\prod\limits_{j=1}^{s} w_H(f_j)$.

Proof. Let M be the $ns \times ns$ nonsingular matrix whose representation by $n \times n$

blocks equals:

$$M = \begin{bmatrix} I & I & I & \cdots & I \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & I \end{bmatrix},$$

where I is the identity $n \times n$ matrix and 0 is the all-0 $n \times n$ matrix. Then:

$$M' = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \\ I & I & 0 & \cdots & 0 \\ I & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ I & 0 & 0 & \cdots & I \end{bmatrix}.$$

We have:

$$h(x^{(1)}, x^{(2)}, \dots, x^{(s)}) = \prod_{j=1}^{s} f_j\left(\sum_{i=1}^{s} x^{(i)} \times M^{(i,j)}\right).$$

For any $(u^{(1)}, u^{(2)}, \dots, u^{(s)}) \in (\mathbb{F}_2^n)^s$ satisfying $1 \le w_H(u^{(1)}, u^{(2)}, \dots, u^{(s)}) \le t$, we have either

$$1 \le w_H\left(\sum_{i=1}^{s} u^{(i)} \times M'^{(i,1)}\right) = w_H\left(\sum_{i=1}^{s} u^{(i)}\right) \le t,$$

or $\sum\limits_{i=1}^{s} u^{(i)} = 0_n$, in which case there exists $2 \le j \le s$ such that $u^{(j)} \neq 0_n$ and $w_H(u^{(j)}) = w_H\left(\sum\limits_{i=1, i\neq j}^{s} u^{(i)}\right) = \frac{w_H(u^{(j)}) + w_H\left(\sum\limits_{i=1, i\neq j}^{s} u^{(i)}\right)}{2} \le \frac{\sum\limits_{i=1}^{s} w_H(u^{(i)})}{2} \le \frac{t}{2}$.
Proposition 125 completes the proof. □

Corollary 24 *[258] Let n, s, t be positive integers satisfying $t \le n$ and $s \ge 2$. We have:*

$$\omega_{ns,t} \le \left(\omega_{n,\lfloor \frac{t}{2} \rfloor}\right)^{s-1} \omega_{n,t}.$$

A construction of low-weight t-th order correlation immune Boolean functions through Kronecker sum

The *Kronecker sum* of two vectors:

$$(x^{(1)}, x^{(2)}) = ((x_1^{(1)}, \dots, x_{n_2}^{(1)}), (x_1^{(2)}, \dots, x_{n_1}^{(2)})) \in \mathbb{F}_2^{n_2} \times \mathbb{F}_2^{n_1} \to$$

$$x^{(1)} \boxplus x^{(2)} = (x_{i_2}^{(1)} \oplus x_{i_1}^{(2)})_{\substack{1 \le i_1 \le n_1 \\ 1 \le i_2 \le n_2}} \in \mathbb{F}_2^{n_1 n_2}$$

generalizes to s variables as follows: let $n_1, \ldots, n_s$ be positive integers and $\mathcal{I} = \{1, \ldots, n_1\} \times \cdots \times \{1, \ldots, n_s\}$, then for every $I = (i_1, \ldots, i_s) \in \mathcal{I}$ and every $1 \leq r \leq s$, we denote by $I^{(r)}$ the vector $(i_1, \cdots, i_{r-1}, i_{r+1}, \cdots, i_s)$. Writing

$$x^{(r)} = (x^{(r)}_{i_1,\cdots,i_{r-1},i_{r+1},\cdots,i_s})_{\substack{i_1 \in \{1,\ldots,n_1\} \\ \cdots \\ i_s \in \{1,\ldots,n_s\}}} \in \mathbb{F}_2^{n_1\cdots n_{r-1}n_{r+1}\cdots n_s},$$

the *s-th order Kronecker sum* is defined as:

$$(x^{(1)}, x^{(2)}, \ldots, x^{(s)}) \to x^{(1)} \boxplus \cdots \boxplus x^{(s)} = \left(\bigoplus_{r=1}^{s} x^{(r)}_{I^{(r)}}\right)_{I\in\mathcal{I}} \in \mathbb{F}_2^{n_1\cdots n_s}.$$

Proposition 126 *[258] Let s, t be positive integers such that $2^s > t$. Let $f_1(x^{(1)})$ be an $(n_2 \cdots n_s)$-variable t-th order correlation immune Boolean function and $f_2(x^{(2)})$ an $(n_1 n_3 \cdots n_s)$-variable $2\lfloor \frac{t}{2} \rfloor$-th order correlation immune Boolean function. For every $r = 3, 4, \ldots, s$, let $f_r(x^{(r)})$ be an $(n_1 \cdots n_{r-1} n_{r+1} \cdots n_s)$-variable Boolean function such that, for every $w \in \mathbb{F}_2^{n_r}$ satisfying $1 \leq w_H(w) \leq t$ with $w_H(w)$ even, we have $\widehat{f_r}(w) = 0$. We define the $\big((n_1+1)n_2n_3\cdots n_s\big)$-variable function h by its support as follows:* $\mathrm{Supp}(h) =$

$$\left\{\left(x^{(1)} \boxplus \cdots \boxplus x^{(s)}, x^{(1)}\right);\ x^{(1)} \in \mathrm{Supp}(f_1), x^{(2)} \in \mathrm{Supp}(f_2), \ldots, x^{(s)} \in \mathrm{Supp}(f_s)\right\},$$

then h is t-th order correlation immune of Hamming weight $\prod_{r=1}^{s} w_H(f_r)$.
In particular, if f_1 is a t-th order correlation immune Boolean function and if each function f_r is $2\lfloor \frac{t}{2} \rfloor$-th order correlation immune for $r = 2, \ldots, s$, then h is a t-th order correlation immune Boolean function of Hamming weight $\prod_{r=1}^{s} w_H(f_r)$.

Proof. Let us calculate the Fourier-Hadamard transform of h. Its input is any pair (u, v) where u is a binary vector of the same length as $x^{(1)} \boxplus \cdots \boxplus x^{(s)}$ and v is a binary vector of the same length as $x^{(1)}$, that is, $u = (u_I)_{I\in\mathcal{I}} \in \mathbb{F}_2^{n_1n_2\cdots n_s}$ and $v = (v_J)_{J\in\mathcal{J}} \in \mathbb{F}_2^{n_2\cdots n_s}$, $\mathcal{J} = \{1, \ldots, n_2\} \times \cdots \times \{1, \ldots, n_s\}$. We have:

$$\widehat{h}(u, v) = \sum_{\substack{x^{(1)}\in\mathrm{Supp}(f_1), \\ \ldots, x^{(s)}\in\mathrm{Supp}(f_s)}} (-1)^{\bigoplus_{I\in\mathcal{I}} u_I \left(\bigoplus_{r=1}^{s} x^{(r)}_{I^{(r)}}\right) \oplus v\cdot x^{(1)}}.$$

Let us write $u_{0,i_2,\ldots,i_s} = v_{i_2,\ldots,i_s}$; $\overrightarrow{u_1} = \left(\bigoplus_{i_1=0}^{n_1} u_{i_1,1,\ldots,1}, \cdots, \bigoplus_{i_1=0}^{n_1} u_{i_1,n_2,\ldots,n_s}\right)$ and $\overrightarrow{u_r} = \left(\bigoplus_{i_r=1}^{n_r} u_{1,\ldots,1,i_r,1,\ldots,1}, \cdots, \bigoplus_{i_r=1}^{n_r} u_{n_1,\ldots,n_{r-1},i_r,n_{r+1},\ldots,n_s}\right)$, for every $r = 2, \ldots s$. We have then:

$$\widehat{h}(u, v) = \left(\sum_{x^{(1)}\in\mathrm{Supp}(f_1)} (-1)^{\bigoplus_{I\in\mathcal{I}} u_I x^{(1)}_{I^{(1)}} \oplus v\cdot x^{(1)}}\right) \times$$
$$\prod_{r=2}^{s} \left(\sum_{x^{(r)}\in\mathrm{Supp}(f_r)} (-1)^{\bigoplus_{I\in\mathcal{I}} u_I x^{(r)}_{I^{(r)}}}\right) = \widehat{f_1}(\overrightarrow{u_1}) \times \prod_{r=2}^{s} \widehat{f_r}(\overrightarrow{u_r}).$$

For $1 \leq w_H(u,v) \leq t$, we have $w_H(\overrightarrow{u_1}) \leq w_H(u,v) \leq t$.
- If $w_H(\overrightarrow{u_1}) \neq 0$, then since f_1 is t-th order correlation immune, we have $\widehat{h}(u,v) = 0$.
- If $w_H(\overrightarrow{u_1}) = 0$, then $w_H(u,v)$ is even since $w_H(u,v) \pmod 2 = w_H(\overrightarrow{u_1}) \pmod 2$, and then $w_H(u,v) \leq 2\lfloor \frac{t}{2} \rfloor$. We have then that:
– If $w_H(\overrightarrow{u_2}) \neq 0$, then we have $1 \leq w_H(\overrightarrow{u_2}) \leq w_H(u) \leq w_H(u,v) \leq 2\lfloor \frac{t}{2} \rfloor$, and since f_2 is $2\lfloor \frac{t}{2} \rfloor$-th order correlation immune, we deduce $\widehat{h}(u,v) = 0$.
– If $w_H(\overrightarrow{u_2}) = \ldots = w_H(\overrightarrow{u_{j-1}}) = 0$ and $w_H(\overrightarrow{u_j}) \neq 0$, where $3 \leq j \leq s$, then $w_H(u)$ and $w_H(\overrightarrow{u_j})$ are even since $w_H(u) \pmod 2 = w_H(\overrightarrow{u_2}) \pmod 2 = w_H(\overrightarrow{u_j}) \pmod 2$ and we have $2 \leq w_H(\overrightarrow{u_j}) \leq 2\lfloor \frac{t}{2} \rfloor$, and the hypothesis on f_j implies $\widehat{h}(u,v) = 0$.
– If $w_H(\overrightarrow{u_2}) = \ldots = w_H(\overrightarrow{u_s}) = 0$, then since $w_H(u,v) \neq 0$ and $w_H(\overrightarrow{u_1}) = 0$, there exist $0 \leq i_1'' < i_1' \leq n_1, 1 \leq i_2 \leq n_2, \ldots, 1 \leq i_s \leq n_s$ such that $u_{i_1',i_2,\ldots,i_s} = u_{i_1'',i_2,\ldots,i_s} = 1$. Since $w_H(\overrightarrow{u_s}) = 0$, there exist in fact 2 values of i_s such that $u_{i_1',i_2,\ldots,i_s} = 1$. Since $w_H(\overrightarrow{u_{s-1}}) = 0$, there exist then 4 values of (i_{s-1}, i_s) such that $u_{i_1',i_2,\ldots,i_s} = 1$. By induction, we have then $w_H(u,v) \geq 2^s$. But we have $1 \leq w_H(u,v) \leq t < 2^s$ by hypothesis, a contradiction. Hence, $w_H(\overrightarrow{u_1}) = w_H(\overrightarrow{u_2}) = \ldots = w_H(\overrightarrow{u_s}) = 0$ cannot happen. This completes the proof. □

A corollary can be found in [258], as well as variants of the construction of Proposition 126, one of which needs weaker hypotheses but does not include the term in $x^{(1)}$ in the support of h (and provides then functions in less variables) and the other deals with 3-rd order correlation immune functions.

7.1.10 On the number of correlation immune and resilient functions

It is important to ensure that the selected criteria for the Boolean functions, supposed to be used in some cryptosystems, do not restrict the choice of the functions too severely. Hence, the set of functions should be enumerated. But this enumeration is unknown for most criteria, and the cases of correlation immune and resilient functions make no exception. We recall below what is known. More than for bent functions, the class of resilient functions produced by Maiorana-McFarland's construction[5] is by far the widest class, compared to the classes obtained from the other usual constructions, and the number of provably resilient Maiorana-McFarland's functions seems negligible with respect to the total number of functions with the same properties. For balanced (*i.e.* 0-resilient) functions, this can be checked: for every positive r, the number of balanced Maiorana-McFarland's functions (7.4) obtained by choosing ϕ such that $\phi(y) \neq 0_r$, for every y, equals $(2^r-1)^{2^s}\, 2^{2^s}$, and is smaller than or equal to $2^{2^{n-1}}$ (since $r \geq 1, s = n-r$). It is negligible with respect to the number $\binom{2^n}{2^{n-1}} \approx \frac{2^{2^n+\frac{1}{2}}}{\sqrt{\pi 2^n}}$ of all balanced functions on $\mathbb{F}_2^n$. The number of t-resilient Maiorana-McFarland's

[5] We have seen that this construction hardly allows building unbalanced correlation immune functions.

functions obtained by choosing ϕ such that $w_H(\phi(y)) > t$ for every y equals $\left[2\sum_{i=t+1}^{r}\binom{r}{i}\right]^{2^{n-r}}$, and is probably also very small compared to the number of all t-resilient functions. But this number is unknown.

The exact number of t-resilient functions is known for $t \geq n-3$ (see [181], where $(n-3)$-resilient functions are characterized) and $(n-4)$-resilient functions have been characterized [256, 125].

As for bent function, upper bounds on the numbers of correlation immune and resilient functions come directly from the Siegenthaler bound on the algebraic degree: the number of t-th order correlation immune (resp. t-resilient) n-variable functions is bounded above by $2^{\sum_{i=0}^{n-m}\binom{n}{i}}$ (resp. $2^{\sum_{i=0}^{n-m-1}\binom{n}{i}}$). These bounds are the so-called naive bounds. In 1990, Yang and Guo published an upper bound on the number of first-order correlation immune functions. At the same time, Denisov obtained a rather strong result (see below) but his result being published in Russian, it was not known internationally. His paper was translated into English two years later [433] but was not widely known either. This explains why several papers appeared, some of which with weaker results, that we describe first. Park, Lee, Sung and Kim [926] improved upon Yang-Guo's bound. Schneider [1024] proved that the number of t-resilient n-variable Boolean functions is less than:

$$\prod_{i=1}^{n-m}\binom{2^i}{2^{i-1}}^{\binom{n-i-1}{m-1}}.$$

but this result was known, see [520]. A general upper bound on the number of Boolean functions whose distances to affine functions are all divisible by 2^t has been obtained in [301]. It implies an upper bound on the number of t-resilient functions which improves upon previous bounds for about half the values of (n, m) (it is better for t large). This bound divides the naive bound by approximately $2^{\sum_{i=0}^{n-m-1}\binom{m-1}{i}-1}$ if $m \geq \frac{n}{2}$ and by approximately $2^{2^{2m+1}-1}$ if $m < \frac{n}{2}$.

An upper bound on t-resilient functions ($m \geq \frac{n}{2}-1$) partially improving upon this latter bound thanks to a refinement of its method was obtained for $\frac{n}{2}-1 \leq m < n-2$ in [285]: the number of n-variable t-resilient functions is lower than:

$$2^{\sum_{i=0}^{n-m-2}\binom{n}{i}} + \frac{\binom{n}{n-m-1}}{2^{\binom{m+1}{n-m-1}+1}}\prod_{i=1}^{n-m}\binom{2^i}{2^{i-1}}^{\binom{n-i-1}{m-1}}.$$

The expressions of these bounds seem difficult to compare mathematically. Tables have been computed in [285].

The problem of counting resilient functions is related to counting integer solutions of a system of linear equations, see [850].

The main result given by Denisov in [433] is an asymptotic formula for the number of t-th order correlation immune functions, where t is negligible compared to n. This formula was later believed incorrect by the author and a correction was given by him in [434], but it has been shown later in [182] that the correct expression was the original one, at least under the condition $1 \leq t \leq (\frac{\ln 2}{6}-\epsilon)\frac{n}{\ln n}$

where $\epsilon > 0$: this number is then equivalent to

$$2^{2^n - t + \sum_{j=0}^{t} j\binom{n}{j}} (2^{n-1}\pi)^{-\frac{\sum_{j=1}^{t}\binom{n}{j}}{2}},$$

and the number of t-resilient functions is equivalent to

$$2^{2^n + \sum_{j=0}^{t} j\binom{n}{j}} (2^{n-1}\pi)^{-\frac{\sum_{j=0}^{t}\binom{n}{j}}{2}}.$$

For large resiliency orders, Tarannikov and Kirienko showed in [1082] that, for every positive integer t, there exists a number $p(m)$ such that for $n > p(m)$, any $(n-m)$-resilient function $f(x_1, \ldots, x_n)$ is equivalent, up to permutation of its input coordinates, to a function of the form $g(x_1, \ldots, x_{p(m)}) \oplus x_{p(m)+1} \oplus \cdots \oplus x_n$. It is then a simple matter to deduce that the number of $(n-m)$-resilient functions equals $\sum_{i=0}^{p(m)} A(m,i)\binom{n}{i}$, where $A(m,i)$ is the number of i-variable $(i-m)$-resilient functions that depend on all inputs $x_1, x_2, \ldots, x_i$ nonlinearly. Hence, it is equivalent to $\frac{A(m,p(m))}{p(m)!} n^{p(m)}$ for t constant when n tends to infinity, and it is at most $A_m\ n^{p(m)}$, where A_m depends on t only. It is proved in [1083] that $3 \cdot 2^{t-2} \leq p(m) \leq (m-1)2^{t-2}$ and in [1082] that $p(4) = 10$; hence the number of $(n-4)$-resilient functions equals $(1/2)n^{10} + \mathcal{O}(n^9)$. It is also shown in [1082] that for $n \geq 10$, there does not exist an unbalanced nonconstant $(n-4)$-th order correlation immune function and that for $n \geq 11$, there does not exist an $(n-4)$-resilient function depending nonlinearly on all its variables.
The classification of first-order correlation immune functions and of 1-resilient functions has been studied in [758], with an exact enumeration for $n = 7$ and a precise estimation for $n = 8$.

7.2 Resilient vectorial Boolean functions

For the convenience of the reader, we recall what we have seen in Section 3.3.1, page 151: an (n,m)-*function* $F(x)$ is t-th order *correlation immune* if its output distribution does not change when at most t coordinates x_i of x are kept constant. S-boxes being better balanced, F is called t-*resilient* if it is balanced and t-th order correlation immune. If such an (n,m,t)-*function* F exists, then we have the bounds $t \leq \left\lfloor \frac{2^{m-1}n}{2^m-1} \right\rfloor$, $t \leq 2\left\lfloor \frac{2^{m-2}(n+1)}{2^m-1} \right\rfloor - 1$, $m \leq n-t$ in general, $m \leq n - \log_2\left[\sum_{i=0}^{t/2}\binom{n}{i}\right]$ if t is even and $m \leq n - \log_2\left[\binom{n-1}{(t-1)/2} + \sum_{i=0}^{(t-1)/2}\binom{n}{i}\right]$ if t is odd, and more complex bounds based on linear programming [78, 520].
Composing a t-resilient (n,m)-function by a permutation of $\mathbb{F}_2^m$ does not change its resiliency order. Function F is t-resilient if and only if one of the following conditions is satisfied (see Proposition 41, page 152):
(i) $\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x} = 0$, for every $u \in \mathbb{F}_2^n$ such that $w_H(u) \leq t$ and every $v \in \mathbb{F}_2^m \setminus \{0_m\}$,
(ii) $\sum_{x\in\mathbb{F}_2^n}(-1)^{g(F(x))\oplus u\cdot x} = 0$, for every $u \in \mathbb{F}_2^n$ such that $w_H(u) \leq t$ and every balanced t-variable Boolean function g.

Finally, F is t-resilient if and only if:
(iii) for every vector $b \in \mathbb{F}_2^m$, the Boolean function $\varphi_b = \delta_{\{b\}} \circ F$ is t-th order correlation immune and has Hamming weight 2^{n-m}.

7.2.1 Constructions of resilient vectorial Boolean functions

Linear or affine resilient functions

The construction of t-resilient linear functions is easy: Bennett et al. [58] and Chor et al. [370] give the connection between linear resilient functions and linear codes (correlation immune functions being related to orthogonal arrays, see [181, 180], this relationship is in fact due to Delsarte [422]). There exists a linear (n, m, t)-function if and only if there exists a binary linear $[n, m, t+1]$ code.

Proposition 127 *[58] Let G be a generating matrix for an $[n, m, d]$ binary linear code. We define $L : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$ by the rule $L(x) = x \times G^T$, where G^T is the transpose of G. Then L is an $(n, m, d-1)$-function.*

This is a direct consequence of Corollary 22, page 320, and of Proposition 41. It can also be seen directly: for every nonzero $v \in \mathbb{F}_2^m$, the vector $v \cdot L(x) = v \cdot (x \times G^t)$ has the form $u \cdot x$ where $u = v \times G$ is a nonzero codeword. Hence, u has Hamming weight at least d and the linear function $v \cdot L$ is $(d-1)$-resilient, since it has at least d independent terms of degree 1 in its ANF.
The converse of Proposition 127 is clearly also true.
Proposition 127 is still straightforwardly true if L is affine instead of linear, that is $L(x) = x \times G^t + a$, where a is a vector of $\mathbb{F}_2^k$.
Stinson [1049] considered the equivalence between resilient functions and what he called large sets of orthogonal arrays. According to Proposition 41, an (n, m)-function is t-resilient if and only if there exists a set of 2^m disjoint binary arrays of dimensions $2^{n-m} \times n$, such that, in any t columns of each array, each of the 2^t elements of $\mathbb{F}_2^t$ occurs in exactly 2^{n-m-t} rows and no two rows are identical.
The construction of (n, m, t)-functions by Proposition 127 can be generalized by considering nonlinear codes of length n (that is subsets of $\mathbb{F}_2^n$) and of size 2^{n-m} whose *dual distance* (see Definition 4, page 32) is at least $t+1$ (see [1050]). In the case of Proposition 127, C is the dual of the code of generating matrix G. The nonlinear code needs also to be *systematic* (that is, there must exist a subset I of $\{1, \ldots, n\}$ called an *information set* of C, necessarily of size $n-m$ since the code has size 2^{n-m}, such that every possible tuple occurs in exactly one codeword within the specified coordinates x_i; $i \in I$; we have seen this notion at page 185) to allow the construction of an $(n, m, d^\perp - 1)$-function: the image of a vector $x \in \mathbb{F}_2^n$ is the unique vector y of $\mathbb{F}_2^n$ such that $y_i = 0$ for every $i \in I$ and such that $x \in y + C$ (in other words, to calculate y, we first determine the unique codeword c of C which matches with x on the information set and we have $y = x + c$). It is deduced in [1050] that, for every $r \geq 3$, a $(2^{r+1}, 2^{r+1} - 2r - 2, 5)$-resilient function exists (the construction is based on the Kerdock code), and that no affine resilient function with such good parameters exists.

Maiorana-McFarland resilient functions

The idea of designing resilient vectorial functions by generalizing the Maiorana-MacFarland construction is natural. One can find a first reference of such construction in a paper by Nyberg [906], but for generating perfect nonlinear functions. This technique has been used by Kurosawa et al. [723], Johansson and Pasalic [648], Pasalic and Maitra [933] and Gupta and Sarkar [580] to produce functions having high resiliency and high nonlinearity[6].

Definition 70 *The class of Maiorana-McFarland (n,m)-functions is the set of those functions F which can be written in the form:*

$$F(x,y) = x \times \begin{pmatrix} \varphi_{11}(y) & \dots & \varphi_{1m}(y) \\ \vdots & \ddots & \vdots \\ \varphi_{r1}(y) & \dots & \varphi_{rm}(y) \end{pmatrix} + H(y), \;\; (x,y) \in \mathbb{F}_2^r \times \mathbb{F}_2^s \qquad (7.17)$$

where r and s are two integers satisfying $r+s=n$, H is any (s,m)-function and, for every $i \le r$ and every $j \le m$, φ_{ij} is a Boolean function on $\mathbb{F}_2^s$.

The concatenation of t-resilient functions being still t-resilient, if the transpose matrix of the matrix involved in Equation (7.17) is the generator matrix of a linear $[r,m,d]$-code for every vector y ranging over $\mathbb{F}_2^s$, then the (n,m)-function F is $(d-1)$-resilient.

After denoting, for every $i \le m$, by ϕ_i the (s,r)-function which admits for coordinate functions the Boolean functions φ_{1i}, ..., φ_{ri} (in i-th column of the matrix above), we can rewrite Relation (7.17) as:

$$F(x,y) = \left(x \cdot \phi_1(y) \oplus h_1(y), \dots, x \cdot \phi_m(y) \oplus h_m(y)\right). \qquad (7.18)$$

Resiliency

Equivalently to what is written above in terms of codes, we have:

Proposition 128 *Let n, m, r and s be integers such that $n = r+s$. Let F be a Maiorana-McFarland (n,m)-function defined as in (7.18) and such that, for every $y \in \mathbb{F}_2^s$, the family $(\phi_i(y))_{i\le m}$ is a basis of an m-dimensional subspace of $\mathbb{F}_2^r$ having $t+1$ for minimum Hamming weight, then F is at least t-resilient.*

Nonlinearity

According to Proposition 53, page 189, the nonlinearity $nl(F)$ of any Maiorana-McFarland's (n,m)-function defined as in Relation (7.18) satisfies:

$$nl(F) = 2^{n-1} - 2^{r-1} \max_{(u,u')\in\mathbb{F}_2^r\times\mathbb{F}_2^s, v\in\mathbb{F}_2^m\setminus\{0_m\}} \left| \sum_{y\in E_{u,v}} (-1)^{v\cdot H(y)\oplus u'\cdot y} \right|, \qquad (7.19)$$

[6] But, as seen in Subsection 3.3.2, this notion of nonlinearity is not relevant to S-boxes for stream ciphers. The generalized nonlinearity, which is the correct notion, needs to be further studied for resilient functions and for *Maiorana-McFarland* (MM) functions.

where $E_{u,v}$ denotes the set $\{y \in \mathbb{F}_2^s;\ \sum_{i=1}^m v_i\phi_i(y) = u\}$.
The bounds given by Relations (7.6) and (7.7), page 322, imply:

$$2^{n-1} - 2^{r-1} \max_{u \in \mathbb{F}_2^r, v \in \mathbb{F}_2^m \setminus \{0_m\}} |E_{u,v}| \leq nl(F) \leq 2^{n-1} - 2^{r-1} \left\lceil \sqrt{\max_{u \in \mathbb{F}_2^r, v \in \mathbb{F}_2^m \setminus \{0_m\}} |E_{u,v}|} \right\rceil .$$

If, for every element y, the vector space spanned by the vectors $\phi_1(y), \ldots, \phi_m(y)$ admits m for dimension and has a minimum Hamming weight strictly larger than k (so that F is t-resilient with $t \geq k$), then we have

$$nl(F) \leq 2^{n-1} - 2^{r-1} \left\lceil \frac{2^{s/2}}{\sqrt{\sum_{i=k+1}^r \binom{r}{i}}} \right\rceil . \tag{7.20}$$

The nonlinearity can be exactly calculated in two situations (at least): if, for every vector $v \in \mathbb{F}_2^m \setminus \{0_m\}$, the (s,r)-function $y \mapsto \sum_{i \leq m} v_i\phi_i(y)$ is injective (resp. takes exactly two times each value of its range), then F admits $2^{n-1} - 2^{r-1}$ (resp. $2^{n-1} - 2^r$) for nonlinearity.
Johansson and Pasalic described in [648] a way to specify the vectorial functions ϕ_1, ..., ϕ_m so that this kind of condition is satisfied. Their result can be generalized in the following form:

Lemma 11 *Let C be a binary linear $[r, m, t+1]$ code. Let $\beta_1, \ldots, \beta_m$ be a basis of the $\mathbb{F}_2$-vector space $\mathbb{F}_{2^m}$, and L_0 a linear isomorphism between $\mathbb{F}_{2^m}$ and C. Then the functions $L_i(z) = L_0(\beta_i z)$, $i = 1, \ldots, m$, are such that, for every $v \in \mathbb{F}_2^m \setminus \{0_m\}$, the function $z \in \mathbb{F}_{2^m} \mapsto \sum_{i=1}^m v_i L_i(z)$ is a bijection from $\mathbb{F}_{2^m}$ into C.*

Proof. For every vector v in $\mathbb{F}_2^m$ and every element z of $\mathbb{F}_{2^m}$, we have $\sum_{i=1}^m v_i L_i(z) = L_0\left(\left(\sum_{i=1}^m v_i\beta_i\right) z\right)$. If the vector v is nonzero, then the element $\sum_{i=1}^m v_i\beta_i$ is nonzero. Hence, the function $z \in \mathbb{F}_{2^m} \mapsto \sum_{i=1}^m v_i L_i(z)$ is a bijection. □

Since the functions $L_1, L_2, \ldots, L_m$ vanish at zero input, they do not satisfy the hypothesis of Proposition 128. A solution to derive a family of vectorial functions also satisfying the hypothesis of Proposition 128 is then to right-compose the functions L_i with a same injective (or 2-to-1) function π from $\mathbb{F}_2^s$ into $\mathbb{F}_{2^m}^*$. Then, for every nonzero vector $v \in \mathbb{F}_2^m \setminus \{0_m\}$, the function $y \in \mathbb{F}_2^s \mapsto \sum_{i=1}^m v_i L_i[\pi(y)]$ is injective (or 2-to-1) from $\mathbb{F}_2^s$ into C^*. This gives the following construction[7]:
Given integers $m < r$, let C be an $[r, m, t+1]$-code such that t is as large as possible (Grassl gives in [570] a precise overview of the best known parameters of codes). Then, define m linear functions L_1, ..., L_m from $\mathbb{F}_{2^m}$ into C as in Lemma 11. Choose an integer s strictly lower than m (resp. lower than or equal to m) and define an injective (resp. 2-to-1) function π from $\mathbb{F}_2^s$ into $\mathbb{F}_{2^m}^*$. Choose any

[7] Another construction based on Lemma 11 involves a family of *nonintersecting codes* (*i.e.* of codes with trivial pairwise intersection) having the same length, dimension and minimum distance; however, this construction is often worse for large resiliency orders, as shown in [319].

(s,m)-function $H = (h_1, \ldots, h_m)$ and denote $r+s$ by n. Then the (n,m)-function F whose coordinate functions are defined by $f_i(x,y) = x \cdot [L_i \circ \pi](y) \oplus h_i(y)$ is t-resilient and admits $2^{n-1} - 2^{r-1}$ (resp. $2^{n-1} - 2^r$) for nonlinearity.

All the primary constructions presented in [648, 723, 933, 907] are based on this principle. The construction of (n,m,t)-functions defined in [580] is also a particular application of this construction, as shown in [319].

Other constructions

Constructions of highly nonlinear resilient vectorial functions, based on elliptic curves theory and on the trace of some power functions $x \mapsto x^d$ on finite fields, have been designed respectively by Cheon [367] and by Khoo and Gong [696]. However, it is still an open problem to design highly nonlinear functions with high algebraic degrees and high resiliency orders with Cheon's method. Besides, the number of functions which can be designed by these methods is very small. In [1159, 1163, 1157] are designed resilient functions whose nonlinearity exceeds the bent concatenation bound.

Zhang and Zheng proposed in [1168, 1170] a secondary construction consisting in the composition $F = G \circ L$ of a linear resilient (n,m,t)-function L with a highly nonlinear (m,k)-function. The resulting function F is obviously t-resilient, admits $2^{n-m} nl(G)$ for nonlinearity where $nl(G)$ denotes the nonlinearity of G and its degree is the same as that of G. Taking for function G the *inverse function* $x \mapsto x^{-1}$ on the finite field $\mathbb{F}_{2^m}$, Zhang and Zheng obtained t-resilient functions having a nonlinearity larger than or equal to $2^{n-1} - 2^{n-m/2}$ and having $m-1$ for algebraic degree. But the linear (n,m)-functions involved in the construction of Zhang and Zheng introduce a weakness: their unrestricted nonlinearity (see Definition 38, page 153) being null, this kind of functions cannot be used as a multi-output combination function in stream ciphers. Nevertheless, this drawback can be avoided by concatenating such functions (recall that the concatenation of t-resilient functions gives t-resilient functions, and a good nonlinearity can be obtained by concatenating functions with disjoint Walsh supports). We obtain this way a modified Maiorana-McFarland's construction, which could be investigated further.

More *secondary constructions* of resilient vectorial functions can be derived from the secondary constructions of resilient Boolean functions (see *e.g.* [180, 225]).

8 Functions satisfying SAC, PC, EPC, or having good GAC

The research on Boolean functions achieving the propagation criterion $PC(l)$ of order $1 \leq l < n$ was active in the 90's. The class of $PC(l)$ functions is a super-class of that of bent functions (bent functions achieve $PC(n)$). For $l \leq n-3$ when n is even and for $l \leq n-1$ when n is odd, its elements can be balanced and highly nonlinear. Strict avalanche property (corresponding to $l = 1$) and propagation properties give more features to Boolean functions in the framework of stream ciphers (see an example with [60]), even if it is more related to block ciphers (and to the differential attack). In the framework of stream ciphers, the invention of algebraic attacks and the difficulty of designing then Boolean functions satisfying all the mandatory criteria have more or less refocused research on Boolean functions meeting mandatory criteria only (including algebraic immunity and fast algebraic immunity). In the framework of block ciphers, studying individually the coordinate or component functions of S-boxes is not the most relevant approach. Nevertheless, to be complete[1], we devote a short chapter to such avalanche criteria.

8.1 $PC(l)$ criterion

For the convenience of the reader, we summarize Definition 24, page 118:

Definition 71 *For $1 \leq l \leq n$, an n-variable Boolean function f satisfies the propagation criterion of order l (in brief, $PC(l)$) if $\mathcal{F}(D_e f) = 0$ for every $e \in \mathbb{F}_2^n$ such that $1 \leq w_H(e) \leq l$. Strict avalanche criterion (SAC) corresponds to $PC(1)$ [516, 581].*

It is shown in [605, 218, 219] that, if n is even, then $PC(n-2)$ implies $PC(n)$; so for n even we can find balanced n-variable $PC(l)$ functions only if $l \leq n-3$. For odd $n \geq 3$, it is also known that the functions which satisfy $PC(n-1)$ are those functions of the form $g(x_1 \oplus x_n, \ldots, x_{n-1} \oplus x_n) \oplus \ell(x)$, where g is bent and ℓ is affine, and that the $PC(n-2)$ functions are those functions of a similar form, but where, for at most one index i, the term $x_i \oplus x_n$ may be replaced by x_i or by x_n (other equivalent characterizations exist [219]).
The *algebraic degree* of $PC(l)$ functions is bounded above by $n-1$. A lower bound

[1] Note that the avalanche and propagation criteria play also a role with hash functions.

on their nonlinearity is easily shown [1169]: if there exists an l-dimensional subspace F such that, for every nonzero $e \in F$, the derivative $D_e f$ is balanced, then $nl(f) \geq 2^{n-1} - 2^{n-\frac{l}{2}-1}$. Indeed, Relation (2.56), page 81, applied with $b = 0_n$ and $E = F^\perp$, shows that every value $W_f^2(u)$ is then bounded above by 2^{2n-l}; this implies, taking $F = \{e \in \mathbb{F}_2^n;\, e \preceq u\}$ for $w_H(u) = l$, that $PC(l)$ functions have nonlinearities bounded below by $2^{n-1} - 2^{n-\frac{l}{2}-1}$. Equality can occur only if $l = n-1$ (n odd) and $l = n$ (n even).
The maximum correlation of Boolean functions satisfying $PC(l)$ (and in particular, of bent functions) with respect to subsets of indices can be deduced from Relations (3.14), page 123, and (2.56), see [187].

There exist characterizations of the propagation criterion. A first obvious one is that, according to Relation (2.54), page 80, f satisfies $PC(l)$ if and only if $\sum_{u\in\mathbb{F}_2^n}(-1)^{a\cdot u}\, W_f^2(u) = 0$ for every nonzero vector a of Hamming weight at most l. A second one (direct consequence of Relation (2.56), page 81) is:

Proposition 129 *[219] Any n-variable Boolean function f satisfies $PC(l)$ if and only if, for every vector u of Hamming weight at least $n-l$, and every vector v: $\sum_{w\preceq u} W_f^2(w+v) = 2^{n+w_H(u)}$.*

Maiorana-McFarland's construction can be used to produce functions satisfying the propagation criterion: the derivative $D_{(a,b)}(x,y)$ of a function of the form (5.1), page 188, being equal to $x \cdot D_b\,\phi(y) \oplus a \cdot \phi(y+b) \oplus D_b g(y)$, the function satisfies $PC(l)$ under the sufficient condition that:
1. for every nonzero $b \in \mathbb{F}_2^s$ of Hamming weight smaller than or equal to l, and every vector $y \in \mathbb{F}_2^s$, the vector $D_b\,\phi(y)$ is nonzero (or equivalently every set $\phi^{-1}(u)$, $u \in \mathbb{F}_2^r$, either is empty or is a singleton or has minimum distance strictly larger than l);
2. every linear combination of at least one and at most l coordinate functions of ϕ is balanced (this condition corresponds to the case $b = 0_s$).
Constructions of such functions have been given in [218, 219, 722].

According to Proposition 129 above, Dobbertin's construction cannot produce functions satisfying $PC(l)$ with $l \geq \frac{n}{2}$. Indeed, if u is for instance the vector with $\frac{n}{2}$ first coordinates equal to 0, and with $\frac{n}{2}$ last coordinates equal to 1, we have, according to Relation (7.9), page 325: $W_h^2(w) = 0$ for every $w \preceq u$.

8.2 $PC(l)$ of order k and $EPC(l)$ of order k criteria

Definition 72 *An n-variable Boolean function satisfies the* propagation criterion *$PC(l)$ of order k (resp. the* extended propagation criterion *$EPC(l)$ of order k) if it satisfies $PC(l)$ when k coordinates of the input x are kept constant (resp. if every derivative $D_e f$, with $e \neq 0_n$ of weight at most l, is k-resilient).*

According to the characterization of resilient functions and its proof, we have:

Proposition 130 *[968] A function f satisfies $EPC(l)$ (resp. $PC(l)$) of order k if and only if, for any vector e of Hamming weight smaller than or equal to l and any vector c of Hamming weight smaller than or equal to k, if $(e, c) \neq (0_n, 0_n)$ (resp. if $(e, c) \neq (0_n, 0_n)$ and if e and c have disjoint supports) then:*

$$W_{D_e f}(c) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) \oplus f(x+e) \oplus c \cdot x} = 0.$$

A characterization by the Walsh transform of f has been deduced in [987].
It has been shown in [970] that $SAC(k)$ (*i.e.* $PC(1)$ of order k) functions have algebraic degrees at most $n - k - 1$. In [797], the criterion $SAC(n-3)$ was characterized through the ANF of the function, and its properties were further studied. A construction of $PC(l)$ of order k functions based on Maiorana-McFarland's method is given in [722] (the mapping ϕ being linear and constructed from linear codes) and generalized in [218, 219] (the mapping ϕ being not linear and constructed from nonlinear codes). A construction of n-variable balanced functions satisfying $SAC(k)$ and having algebraic degree $n - k - 1$ is given, for $n - k - 1$ odd, in [722] and, for $n - k - 1$ even, in [1011] (where balancedness and nonlinearity are also considered).
It is shown in [219] that, for every positive even $l \leq n - 4$ (with $n \geq 6$) and every odd l such that $5 \leq l \leq n - 5$ (with $n \geq 10$), the functions which satisfy $PC(l)$ of order $n - l - 2$ are the functions of the form: $\bigoplus_{1 \leq i < j \leq n} x_i x_j \oplus h(x_1, \ldots, x_n)$, where h is affine.

8.3 Absolute indicator

In [1166] is stated the conjecture that any balanced function on an odd number n of variables satisfies $\Delta_f \geq 2^{(n+1)/2}$. In [820, 527], for $n \in \{15, 21\}$, are given balanced functions with $\Delta_f < 2^{(n+1)/2}$ (an error on the 21-variable functions, found by computer, has been corrected in [681]). In [684], the first construction giving $\Delta_f < 2^{n/2}$ for even n (a balanced 10-variable function with $\Delta_f = 24$) is found. In [1075] is given a construction of n-variable balanced functions with $\Delta_f < 2^{n/2}$, where $n > 44$ and $n \equiv 2$ [mod 4], with specific examples for $n = 18, 22, 26$. In [683], results for $n = 12, 14, \ldots, 26$ are obtained (the journal version to appear also provides n-variable balanced functions with $\Delta_f < 2^{n/2}$, where $n > 50$ and $n \equiv 0$ [mod 4]).
Bounds between the absolute indicator and the nonlinearity are given in [1178].

Remark. The block sensitivity $bs(f)$ of an n-variable Boolean function f equals the maximum number of vectors $a^{(1)}, \ldots, a^{(k)} \in \mathbb{F}_2^n$ with disjoint supports and such that $D_{a^{(i)}} f(x) = 1, \forall i = 1, \ldots, k$. Its (basic) sensitivity $s(f)$ is defined similarly with all vectors $a^{(i)}$ of Hamming weight 1. The 30 year old *sensitivity conjecture* states that there exists a constant C independent of n such that $bs(f) \leq (s(f))^C$ for every f [905]. This conjecture has been proved in [631]. □

9 Algebraic immune functions

The invention of *algebraic attack*s and of *fast algebraic attack*s has deeply modified the research on Boolean functions for stream ciphers. Before 2003, functions had about ten variables (to be fastly computable) and were mainly supposed to be balanced, have large algebraic degree and nonlinearity and in the case of the combiner generator, ensure good trade-off between algebraic degree, nonlinearity and resiliency order. Since 2003, the designer needs also to ensure resistance to the algebraic attack (which needs in practice optimal or almost optimal algebraic immunity) and good resistance to fast algebraic attacks and to the Rønjom-Helleseth attack and its improvements. This implies a larger number of variables (say, between 16 and 20; it can be more if the function is particularly fastly computable) and an algebraic degree close to n (this is a necessary but not sufficient condition for the resistance against fast algebraic attacks). For this reason, the combiner generator seems less adapted nowadays; it needs to be made more complex, for instance with memory. Even the filter generator has posed problem: during five years, no function usable in it could be found (the known functions with optimal algebraic immunity had bad nonlinearity and bad resistance to fast algebraic attacks, see [27]). In 2008 an infinite class of functions possessing all mandatory features was found in [273]. The functions in this class are rather fastly computable, but since stream ciphers need to be faster than block ciphers (which can be used as *pseudorandom generator*s), there is still a need of functions satisfying all mandatory criteria and being very fast to compute, like the hidden weight bit function (HWBF), which has been more recently investigated, see below. To be complete in this introduction, we need to mention that a new way of using Boolean functions came recently with the so-called filter permutator, like in the FLIP cryptosystem [839], which posed new problems on Boolean functions (see [306]); see more in Section 12.2.

9.1 Algebraic immune Boolean functions

For the convenience of the reader, we summarize the definitions seen in Section 3.1 on algebraic immune functions.

Definition 73 *Let f be any n-variable Boolean function. The minimum* algebraic degree *of nonzero annihilators of f or of $f \oplus 1$ (*i.e.* of nonzero multiples of*

$f \oplus 1$ or of f), is called the algebraic immunity *of f and is denoted by $AI(f)$. The* fast algebraic immunity *of f is the integer:*

$$FAI(f) = \min\left(2AI(f), \min\left\{d_{alg}(g) + d_{alg}(fg);\ 1 \leq d_{alg}(g) < AI(f)\right\}\right).$$

The fast algebraic complexity *of f is the integer:*

$$FAC(f) := \min\{\max\left[d_{alg}(g) + d_{alg}(fg), 3d_{alg}(g)\right]; 1 \leq d_{alg}(g) < AI(f)\}.$$

All three parameters are stable under complementation $f \mapsto f \oplus 1$; see more in [324]. We have $AI(f) \leq \min(d_{alg}(f), \lceil \frac{n}{2} \rceil)$ and $FAI(f) \leq FAC(f) \leq n$ for any n-variable function f.

A standard algebraic attack on a stream cipher using some Boolean function f in the combiner model or the filter model is all the more efficient as $AI(f)$ is smaller and many linearly independent lowest degree annihilators of f or $f \oplus 1$ exist. Parameter $FAC(f)$ and its simplified version $FAI(f)$ play a similar role with respect to fast algebraic attacks. In [793] are called *perfect algebraic immune* (PAI) the n-variable Boolean functions f such that, for any pair of strictly positive[1] integers (e, d) such that $e + d < n$ and $e < \frac{n}{2}$, there is no non-zero function g of algebraic degree at most e such that fg has algebraic degree at most d (while we have seen at page 114 that for every n-variable function f and every (e, d) such that $e + d \geq n$, such function g exists). Such functions have perfect immunity against the standard and fast algebraic attacks (indeed, as shown in [793, 789], a PAI function and an almost PAI function with even number of variables have optimal algebraic immunity, where almost PAI is defined similarly with $e+d < n-1$ and $e < \frac{n-1}{2}$ instead of $e+d < n$ and $e < \frac{n}{2}$). It is shown in [485, 793] that perfect algebraic immune functions, when balanced, can exist only if n equals 1 plus a power of 2, and when unbalanced, can exist only if n is a power of 2. Indeed, it is easily seen that, for any perfect algebraic immune function f, we have $d_{alg}(f) \geq n - 1$ and it is proved in [793] that if $d_{alg}(f) = n - 1$ (resp. $d_{alg}(f) = n$), then for $e < \frac{n}{2}$ such that $\binom{n-1}{e} \equiv 1$ [mod 2] (resp. $\binom{n-1}{e} \equiv 0$ [mod 2]), there exists a nonzero function g such that $d_{alg}(g) \leq e$ and $d_{alg}(fg) \leq n-e-1$, and such e exists unless $n = 2^s+1$ (resp. 2^s). It is shown in [791] that no *symmetric Boolean function* can be perfect algebraic immune for $n \geq 5$.

In [929] is introduced a slightly different notion of optimal resistance to FAA: an n-variable Boolean function f is said to satisfy the high degree product property (HDP) of order n if, for every n-variable Boolean function g of algebraic degree e such that $1 \leq e < \lceil \frac{n}{2} \rceil$ and which is not an annihilator of f, we have $d_{alg}(fg) \geq n - e$. Then [929] proves that $f \oplus 1$ has the same property and $AI(f) = \lceil \frac{n}{2} \rceil$; such function is then called algebraic attack resistant (AAR).

As we already saw at page 112, since a Boolean function g is an annihilator of f if and only if $g(x) = 0$ for every element x in the support of f, to

[1] Assuming that e can be null would oblige f to have algebraic degree n and it could then not be balanced.

determine whether f (resp. $f \oplus 1$) admits nonzero annihilators of algebraic degree at most d, we consider a general Boolean function $g(x)$ by its ANF $g(x) = \bigoplus_{\substack{I \subseteq \{1,\dots,n\} \\ |I| \leq d}} a_I \left(\prod_{i \in I} x_i \right)$ and consider the system (see page 112) of the $w_H(f)$ (resp. $2^n - w_H(f)$) equations in the $\sum_{i=0}^{d} \binom{n}{i}$ unknowns $a_I \in \mathbb{F}_2$ expressing that $g(x) = 0$ for $x \in supp(f)$ (resp. $x \notin supp(f)$). The matrix $M_{f,d}$ (resp. $M_{f\oplus 1,d}$) of this system has term $\prod_{i \in I} x_i$ at row indexed by x and column indexed by I, where $x \in supp(f)$ (resp. $x \notin supp(f)$). Calculating the algebraic immunity of a function f by applying the definition consists then in determining the minimum value of d such that the ranks $rk(M_{f,d})$ and $rk(M_{f\oplus 1,d})$ of the matrices of these two systems do not both equal $\sum_{i=0}^{d} \binom{n}{i}$, and the dimension $\dim(An_d(f))$ of the vector space of annihilators of algebraic degree at most d of f equals $\sum_{i=0}^{d} \binom{n}{i} - rk(M_{f,d})$.

The dimension of $An_d(f)$ has been determined for all d in [228] for some classes of functions: minimum weight elements f of the Reed-Muller codes (*i.e.* indicators of affine subspaces of $\mathbb{F}_2^n$), their complements $f \oplus 1$, their sums with affine functions when these are balanced, and complements of threshold functions (see more on these latter functions in Subsection 10.1.7).

Remark. Given an n-variable Boolean function f, denoting by $LDA_n(f)$ the $\mathbb{F}_2$-vector space made of the annihilators of f of algebraic degree $AI(f)$ (assuming that some exist; otherwise we change f into $f \oplus 1$) and the zero function, we have, as observed in [261]:

1. $\dim LDA_n(f) \leq \binom{n}{AI(f)}$, since two distinct annihilators of algebraic degree $AI(f)$ cannot have the same degree $AI(f)$ part in their algebraic normal forms (otherwise, their sum would be a nonzero annihilator of algebraic degree strictly smaller than $AI(f)$);
2. If f is balanced and $AI(f) = \frac{n}{2}$, n even, then $\dim LDA_n(f) \geq \frac{1}{2}\binom{n}{\frac{n}{2}}$, since matrix $M_{f,\frac{n}{2}}$ has 2^{n-1} rows and $\sum_{i=0}^{\frac{n}{2}} \binom{n}{i} = 2^{n-1} + \frac{1}{2}\binom{n}{\frac{n}{2}}$ columns;
3. If f is such that $AI(f) = \frac{n+1}{2}$, n odd, then $\dim LDA_n(f) = \binom{n}{\frac{n+1}{2}}$, since we know that $w_H(f) = 2^{n-1}$, and $M_{f,\frac{n-1}{2}}$ is then a $2^{n-1} \times 2^{n-1}$ square matrix whose rank equals 2^{n-1}; matrix $M_{f,\frac{n+1}{2}}$ has then rank 2^{n-1}. □

9.1.1 General properties of the algebraic immunity and its relationship with some other criteria

We have seen that the algebraic immunity of any n-variable Boolean function is an affine invariant and is bounded above by $\lceil \frac{n}{2} \rceil$. The functions used in stream ciphers must have an algebraic immunity close to this maximum. In the next paragraphs, we give properties and characterizations, some of which are new.

Algebraic immunity of monomial functions

It has been shown in [899, 900] that if the number $r(d)$ of runs of 1's in the binary expansion of the exponent d of a power function $tr_n(ax^d)$ (that is, the number of full subsequences of consecutive 1's) is smaller than $\sqrt{n}/2$, then the algebraic immunity is bounded above by

$$r(d)\lfloor\sqrt{n}\rfloor + \left\lceil \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rceil - 1. \tag{9.1}$$

This comes from the fact that there exists g of algebraic degree $\left\lceil \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rceil$ such that fg has algebraic degree at most (9.1). This property also allows to prove that $FAI(f) \leq r(d)\lfloor\sqrt{n}\rfloor + 2\left\lceil \frac{n}{\lfloor\sqrt{n}\rfloor} \right\rceil - 1$, as observed in [870].
Note that (9.1) is better than the general bound $\lceil \frac{n}{2} \rceil$ for only a negligible part of power mappings, but it addresses all those whose exponents have a constant *2-weight* or a constant number of runs - the *power functions* studied as potential S-boxes in block ciphers enter in this framework. Moreover, the bound is further improved when n is odd and the function is almost bent: the algebraic immunity of such functions is bad since bounded above by $2\lfloor\sqrt{n}\rfloor$. The exact value of the algebraic immunity of the multiplicative *inverse function* $tr_n(ax^{2^n-2})$, $a \neq 0$, has been given in [498]; it equals $\lceil 2\sqrt{n}\rceil - 2$, which is not good either.

Algebraic immunity of a restriction

If the restriction of a function f to an affine space, for instance obtained by fixing x_i to a_i for any $i \in I \subseteq \{1, \ldots, n\}$, has a nonzero annihilator g of some algebraic degree d, then f has for nonzero annihilator the function $g(x)1_A(x)$, equal to $g(x)\left(1 \oplus \prod_{i\in I}(x_i \oplus a_i \oplus 1)\right)$ in the latter example, in which case the algebraic degree equals $d + n - dim(A) = d + |I|$. By applying this to f and to $f \oplus 1$, whose restrictions cannot be both null, the algebraic immunity of the restriction is at least $AI(f) - n + dim(A) = AI(f) - |I|$, as observed in [407]. Moreover, the annihilators of the restriction of f are the restrictions of the annihilators of f.
To have a chance of having large algebraic immunity, a function needs then not only to have large enough algebraic degree but also that each restriction to an affine space of large dimension, for instance the restriction obtained by fixing a few input coordinates, has large enough algebraic degree. This implies that Maiorana-McFarland functions defined by Relation (5.1), page 188, with r large have bad algebraic immunity. It is observed in [279] that a Maiorana-McFarland function $x \cdot \phi(y) \oplus g(y)$, where $x \in \mathbb{F}_2^r, y \in \mathbb{F}_2^{n-r}$, can have algebraic immunity $n - r + 1$ (which is its maximal possible algebraic degree) only if, for every affine subspace A of $\mathbb{F}_2^{n-r}$, we have $\sum_{y\in A}\phi(y) \neq 0$. Indeed, the products of f and $f \oplus 1$ by the indicator function of $\mathbb{F}_2^r \times A$ are annihilators of $f \oplus 1$ and f and are not both null; they equal the products of the restrictions of f and $f \oplus 1$ to $\mathbb{F}_2^r \times A$ (which have algebraic degree $1 + \dim A$ if this non-nullity condition is satisfied and at most $\dim A$ otherwise) and of the function of y equal to the indicator function of A (which has algebraic degree $n - r - \dim A$).

Characterization of annihilators by the Walsh transform

For every $x \in \mathbb{F}_2^n$, we have $(fg)(x) = (\frac{1}{2} - \frac{(-1)^{f(x)}}{2})(\frac{1}{2} - \frac{(-1)^{g(x)}}{2}) = \frac{1}{4}(1 - (-1)^{f(x)} - (-1)^{g(x)} + (-1)^{f(x)\oplus g(x)})$. Recall that the Fourier transform is its own inverse up to a multiplicative factor and that this implies that any integer-valued function φ over $\mathbb{F}_2^n$ is:

1. equal to the zero function if and only if its Fourier transform $\widehat{\varphi}$ is null,
2. constant if and only if $\widehat{\varphi}(a)$ is null at any input $a \neq 0_n$.

We deduce a characterization first observed in [128], that we slightly complete:

Proposition 131 *Let n be any positive integer and f, g any n-variable Boolean functions. Then*

$$g \in An(f) \iff \forall a \in \mathbb{F}_2^n, \quad W_{f\oplus g}(a) + 2^n \delta_0(a) = W_f(a) + W_g(a),$$

where δ_0 is the Dirac (or Kronecker) symbol. Moreover, if f is different from the constant function 1, we have:

$$g \in An(f) \iff \forall a \neq 0, \quad W_{f\oplus g}(a) = W_f(a) + W_g(a).$$

Indeed, the first equivalence is a consequence of observation 1 above applied to the two members of the equality above or to $\varphi = f + g - f \oplus g = 2fg$, using the linearity of the Fourier transform and Relation (2.32), page 74. The second equivalence is then a straightforward consequence of observation 2, since fg constant means $fg = 0$ because $fg = 1$ is impossible, f being not constant function 1.
Note that the bound $AI(f) \leq \lceil \frac{n}{2} \rceil$ shows then that, for every non-constant n-variable Boolean function f, there exists a nonzero n-variable Boolean function g of algebraic degree at most $\lceil \frac{n}{2} \rceil$, such that either $W_{f\oplus g}(a) = W_f(a) + W_g(a)$ for all $a \neq 0_n$ (g being an annihilator of $f \neq 1$) or $W_{f\oplus g}(a) = W_f(a) - W_g(a)$ for all $a \neq 0_n$ (g being an annihilator of $f \oplus 1 \neq 1$).

Moreover, since $(-1)^{(f\oplus g)(x)} = (-1)^{f(x)}(-1)^{g(x)}$, we have by applying Relation (2.45), page 79:

$$2^n W_{f\oplus g} = W_f \otimes W_g,$$

where $W_f \otimes W_g(a) = \sum_{u\in\mathbb{F}_2^n} W_f(a+u) W_g(u)$. We deduce:

Corollary 25 *Let n be any positive integer and f any n-variable Boolean function. We have $g \in An(f)$ if and only if:*

$$\forall a \in \mathbb{F}_2^n, \quad W_f \otimes W_g(a) - 2^n W_g(a) = 2^n W_f(a) - 2^{2n} \delta_0(a) \tag{9.2}$$

and if f is not constant function 1, this condition with $a \neq 0_n$ suffices.

The Walsh transforms of the annihilators of f are then the solutions of the system of the 2^n linear equations (9.2) indexed by a, in the 2^n unknowns $W_g(a)$, $a \in \mathbb{F}_2^n$, whose matrix equals $M - 2^n I$, where I is the identity matrix and M is the matrix whose coefficient at row indexed by a and column indexed by u

equals $W_f(a+u)$. Note that we have $M \times M^t = M \times M = 2^{2n} I$, where M^t is the transpose of M, since, for every $a, b \in \mathbb{F}_2^n$, we have $\sum_{u\in\mathbb{F}_2^n} W_f(a+u)W_f(b+u) = \sum_{u\in\mathbb{F}_2^n} W_f(u)W_f(a+b+u) = 2^{2n}\delta_0(a+b)$, according to the Parseval and Titsworth relations (2.48) and (2.51), page 80.

It is interesting to see that the annihilators of any Boolean function f combine two linear algebraic properties over different fields:

- the set of annihilators of f is an $\mathbb{F}_2$-vector space;
- the set of their Walsh transforms is the intersection between an $\mathbb{R}$-vector space (the set of solutions of the system given above) and the set of integer-valued functions $W : \mathbb{F}_2^n \mapsto \mathbb{Z}$ satisfying the equation $\sum_{u\in\mathbb{F}_2^n} W(a+u)W(u) = 2^{2n}\delta_0(a)$ for every $a \in \mathbb{F}_2^n$ (we know indeed that these 2^n quadratic equations are characteristic of the Walsh transforms of Boolean functions).

Characterization of annihilators by the *NNF*

The determination of annihilators can be handled in a simple way (with two equations over $\mathbb{Z}$, one quadratic and one linear, instead of $w_H(f)$ linear ones over $\mathbb{F}_2$ as we saw with the ANF) through the NNF representation (see Subsection 2.2.4, page 65): let $\sum_{I\subseteq\{1,\dots,n\}} \lambda_I\, x^I$, $\lambda_I \in \mathbb{Z}$, be the NNF of a Boolean function $f(x)$; we know from (2.28), page 69, that an integer-valued function $g(x) = \sum_{I\subseteq\{1,\dots,n\}} \mu_I\, x^I$, $\mu_I \in \mathbb{Z}$, is Boolean if and only if the single quadratic equation:

$$\sum_{I\subseteq\{1,\dots,n\}} 2^{n-|I|} \sum_{J,J'\subseteq\{1,\dots,n\};\ I=J\cup J'} \mu_J\, \mu_{J'} = \sum_{I\subseteq\{1,\dots,n\}} 2^{n-|I|}\mu_I \tag{9.3}$$

is satisfied. We have that g is an annihilator of f if and only if[2] $\sum_{x\in\mathbb{F}_2^n} f(x)g(x) = 0$. Hence, since $\sum_{x\in\mathbb{F}_2^n} x^I = |\{x \in \mathbb{F}_2^n;\ I \subseteq supp(x)\}| = 2^{n-|I|}$:

Proposition 132 *Let f be any n-variable Boolean function and let its NNF equal $\sum_{I\subseteq\{1,\dots,n\}} \lambda_I\, x^I$, $\lambda_I \in \mathbb{Z}$. Then the annihilators of f are the functions $g(x) = \sum_{I\subseteq\{1,\dots,n\}} \mu_I\, x^I$, $\mu_I \in \mathbb{Z}$, which satisfy (9.3) and:*

$$\sum_{I\subseteq\{1,\dots,n\}} 2^{n-|I|} \sum_{J,J'\subseteq\{1,\dots,n\};\ I=J\cup J'} \lambda_J\, \mu_{J'} = 0.$$

Algebraic immunity and codes

It is observed in [600, Theorem 1 and Corollary 1] that the problem of estimating the algebraic immunity of Boolean functions over $\mathbb{F}_{2^n}$ is connected to cyclic codes. We modify the statement of this result (and give a slightly different proof) so as to complete [600] by taking into account the facts that if $f(0) = 1$ then the annihilators g of f must satisfy $g(0) = 0$ and that annihilators of algebraic degree n may exist.

Proposition 133 *Let $f(x)$ be an n-variable Boolean function in univariate form. Then the annihilators of $f(x)$ in univariate representation are those multiples*

[2] Recall that $\sum$ denotes sums in $\mathbb{Z}$.

$g(x)$ of $\gcd(f(x)+1, x^{2^n}+x)$ *in* $\mathbb{F}_{2^n}[x]/(x^{2^n}+x)$ *which satisfy* $(g(x))^2 = g(x)$. *If* $f(0) = 0$, *then the annihilators of algebraic degree at most* $n-1$ *are the codewords of the* cyclic code *of length* $2^n - 1$ *over* $\mathbb{F}_{2^n}$ *and of generator polynomial* $\gcd(f(x)+1, x^{2^n-1}+1)$ *which satisfy* $(g(x))^2 = g(x)$.

Proof. We know that the annihilators of $f \in \mathcal{BF}_n$ are those Boolean functions which are multiples of $f \oplus 1$ in $\mathcal{BF}_n$. These annihilators in univariate representation are then the multiples of $f(x)+1$ in $\mathbb{F}_{2^n}[x]/(x^{2^n}+x)$ which satisfy $(g(x))^2 = g(x)$, and since being such multiple is equivalent to being a multiple of $\gcd(f(x)+1, x^{2^n}+x)$ [mod $x^{2^n}+x$], this proves the first part. The rest is straightforward since, if $f(0) = 0$, then $\gcd(f(x)+1, x^{2^n}+x) = \gcd(f(x)+1, x^{2^n-1}+1)$, and since reducing mod $x^{2^n}+x$ or mod $x^{2^n-1}+1$ a polynomial of algebraic degree at most $n-1$, that is, of degree at most 2^n-2, is the same. □

Corollary 26 *Let $f(x)$ be an n-variable Boolean function in univariate form. Then $AI(f)$ equals the minimum, among all those nonzero elements $g(x)$ of $\mathbb{F}_{2^n}[x]/(x^{2^n}+x)$ which satisfy $(g(x))^2 = g(x)$ and are multiples either of* $\gcd(f(x)+1, x^{2^n}+x)$ *or of* $\gcd(f(x), x^{2^n}+x)$, *of the maximum* 2-weight *of the exponents in the terms of these polynomials.*

It is also shown in [600] that the *spectral immunity* (defined at page 116) of a Boolean function $f(x)$ (in univariate form) is equal to the minimal weight of the nonzero codewords of the cyclic codes over $\mathbb{F}_{2^n}$ of generator polynomials $\gcd(f(x)+1, x^{2^n-1}+1)$ and $\gcd(f(x), x^{2^n-1}+1)$.

In [869] is shown that, given an n-variable Boolean function f, if the minimum distance of the linear code $\{(a_0, \dots, a_{2^n-1}) \in \mathbb{F}_{2^n}^{2^n};\ \sum_{i=0}^{2^n-1} a_i x^i = 0, \forall x \in supp(f)\}$ (*i.e.* the vector space of univariate representations of $\mathbb{F}_{2^n}$-valued annihilators of f), that we shall denote by C_f, is strictly larger than $\sum_{i=0}^{d}\binom{n}{i}$ for a given d, then the minimum algebraic degree of nonzero annihilators of f is strictly larger than d. Indeed, if a nonzero annihilator of f has algebraic degree at most d, then its Hamming weight as a codeword of C_f is at most $\sum_{i=0}^{d}\binom{n}{i}$, a contradiction.
There is however an issue with this result when $f(0) = 0$, since C_f has then minimum distance 2, because it includes the codeword $(1, 0, \dots, 0, 1)$; the result gives then no information in that case. This difficulty can be easily addressed since the codeword $(1, 0, \dots, 0, 1)$ corresponds to an annihilator of algebraic degree n (the indicator of $\{0\}$, *i.e.* function δ_0) and presents then no interest from the view point of algebraic immunity. We can slightly modify the result of [869] by considering, instead of C_f, the code $C'_f = \{(a_0, \dots, a_{2^n-2}) \in \mathbb{F}_{2^n}^{2^n-1};\ \sum_{i=0}^{2^n-2} a_i x^i = 0, \forall x \in supp(f)\}$ (*i.e.* the vector space of univariate representations of $\mathbb{F}_{2^n}$-valued annihilators of f of algebraic degree at most $n-1$); the result also works for C'_f, unless all nonzero annihilators of f have algebraic degree n, that is, unless $f = 1 \oplus \delta_a$ for some $a \in \mathbb{F}_{2^n}$.
We consider now the cyclic code (also introduced in [869]) $\overline{C}_f = \{(a_1, \dots, a_{2^n-1}) \in$

$\mathbb{F}_{2^n}^{2^n-1}; \sum_{i=1}^{2^n-1} a_i x^i = 0, \forall x \in supp(f)\}$ (*i.e.* the subcode of C_f whose elements are the univariate representations of $\mathbb{F}_{2^n}$-valued annihilators of f null at position 0), punctured at 0. The minimum distance (*i.e.* nonzero weight) of C'_f equals at least the minimum distance of $\overline{C}_f$. Indeed, if $f(0) = 1$, then the minimum distances of C_f and $\overline{C}_f$ are equal to each other, and if $f(0) = 0$, then $C_f = \{(0, 0, \ldots, 0, 0), (1, 0, \ldots, 0, 1)\} + \{0\} \times \overline{C}_f$ and the minimum distance of C'_f is then larger than or equal to that of $\overline{C}_f$.
The interest of this observation is that $\overline{C}_f$ is cyclic and we can apply the BCH bound to this cyclic code. This gives a direct lower bound on the minimum algebraic degree of nonzero annihilators of f.

Relationship between normality and algebraic immunity

Normality of order larger than $\frac{n}{2}$ represents a weakness with respect to algebraic immunity:

Proposition 134 *For any positive n and $k \leq n$, if an n-variable function f is k-normal then its algebraic immunity is at most $n - k$.*

Indeed, the fact that $f(x) = \epsilon \in \mathbb{F}_2$ for every $x \in A$, where A is a k-dimensional flat, implies that the indicator of A is an annihilator of $f + \epsilon$. This bound is tight since, being a *symmetric Boolean function*, the majority function (see page 366) is $\lfloor \frac{n}{2} \rfloor$-normal for every n and has algebraic immunity $\lceil \frac{n}{2} \rceil$. Obviously, $AI(f) \leq \ell$ does not imply conversely that f is $(n-\ell)$-normal, since when n tends to infinity, for every $a > 1$, n-variable Boolean functions are almost surely non-$(a \log_2 n)$-normal [222, 224] and the algebraic immunity is always bounded above by $\frac{n}{2}$.

Functions in odd numbers of variables with optimal algebraic immunity

In [188], A. Canteaut has observed the following property:

Proposition 135 *If an n-variable balanced function f, with n odd, admits no nonzero annihilator of algebraic degree at most $\frac{n-1}{2}$, then it has optimal algebraic immunity $\frac{n+1}{2}$.*

This result is a direct consequence of Proposition 136 below, which has been proved later. It means that we do not need to check also that $f \oplus 1$ has no nonzero annihilator of algebraic degree at most $\frac{n-1}{2}$ for showing that f has optimal algebraic immunity[3].
The original proof (simplified in the end) of Proposition 135 is as follows: consider the *Reed-Muller code* of length 2^n and of order $\frac{n-1}{2}$. This code is self-dual (*i.e.* is its own dual), according to Theorem 9, page 177. Let G be a generator matrix of this code. Each column of G is labeled by the vector of $\mathbb{F}_2^n$ obtained by keeping its coordinates of indices $2, \ldots, n + 1$ (assuming that the first row of G is the all-1 vector, corresponding to constant function 1, and that the next n rows

[3] The same has been shown for n even but for (less interesting) unbalanced functions.

correspond to the coordinate functions). Saying that f has no nonzero annihilator of algebraic degree at most $\frac{n-1}{2}$ is equivalent to saying that the matrix obtained by selecting those columns of G corresponding to the elements of the support of f has full rank $\sum_{i=0}^{\frac{n-1}{2}} \binom{n}{i} = 2^{n-1}$. By hypothesis, f has Hamming weight 2^{n-1}. In terms of coding theory, the support of the function is an *information set*. Then the complement of the support of f being an information set of the dual (recall that if $G = [I_k : M]$ is a systematic generator matrix of a linear code, then $[-M^t : I_{n-k}]$ is a parity check matrix of the code) and the code being self-dual, this complement is also an information set of the code (*i.e.* the code is CIS, see page 468).

More relationship between the existence of low degree annihilators of f and of $f \oplus 1$

We have, from [800] (we slightly modify the proof):

Proposition 136 *If, for some $k < \lceil \frac{n}{2} \rceil$, we have $rk(M_{f,k}) = w_H(f)$ (*i.e.* all the rows of $M_{f,k}$ are $\mathbb{F}_2$-linearly independent), then $rk(M_{f\oplus 1,k}) = \sum_{i=0}^{k} \binom{n}{i}$ (*i.e.* $f \oplus 1$ has no nonzero annihilator of algebraic degree at most k).*

Proof. Suppose there exists a nonzero annihilator g of algebraic degree at most k of $f \oplus 1$. We have then $supp(g) \subseteq supp(f)$. Since all the rows of $M_{f,k}$ are $\mathbb{F}_2$-linearly independent, all those of $M_{g,k}$ are $\mathbb{F}_2$-linearly independent, and for every choice of $(b_x)_{x\in supp(g)} \in \mathbb{F}_2^{w_H(g)}$, the system of linear equations whose matrix is $M_{g,k}$ and whose constants are these b_x has a solution. In particular, for every $x \in supp(g)$, there exists g' of algebraic degree at most k such that $gg' = \delta_x$ (the Dirac symbol at x, *i.e.* the indicator function of the singleton $\{x\}$), a contradiction with $d_{alg}(gg') \leq d_{alg}(g) + d_{alg}(g') < n$. □

Minimum Hamming distance to functions of large algebraic immunity bounded below by means of the dimensions of vector spaces of functions

Lobanov has made in two papers [800, 801] the following observations (that we gather in a single proposition):

Proposition 137 *For any n-variable Boolean functions f, h and any integers $0 \leq k, l \leq n$, we have:*

$$d_H(f,h) \geq \dim(An_k(h)) - \dim(An_k(f)) + \dim(An_l(h \oplus 1)) - \dim(An_l(f \oplus 1)).$$

Moreover, if $d \leq AI(f)$, then we have:

$$d_H(f,h) \geq \dim(An_{d-1}(h)) + \dim(An_{d-1}(h \oplus 1)). \tag{9.4}$$

Proof. Among the $rk(M_{f,k})$ linearly $\mathbb{F}_2$-independent rows of $M_{f,k}$ which can be selected, there exist at least $rk(M_{f,k}) - rk(M_{h,k}) = \dim(An_k(h)) - \dim(An_k(f))$ ones which are not rows of $M_{h,k}$, and there are then at least the same number of distinct elements of $\mathbb{F}_2^n$ in the support of f which are not in the support of h. We can apply this to $f \oplus 1$ and $h \oplus 1$ as well, with l in the place of k. This gives the first inequality. Moreover, if $d \leq AI(f)$, then $\dim(An_{d-1}(f)) = \dim(An_{d-1}(f \oplus 1)) = 0$. This completes the proof. $\square$

Lobanov notes that, if $k \geq l$, then the mapping $(g_1, g_2) \mapsto g_1 \oplus g_2$ is an $\mathbb{F}_2$-linear isomorphism between the vector spaces $An_k(h) \times An_l(h \oplus 1)$ and

$$B_{k,l}(h) = \{g \in \mathcal{BF}_n;\ d_{alg}(g) \leq k \text{ and } d_{alg}(hg) \leq l\}.$$

Indeed, the image set of this mapping is included in $B_{k,l}(h)$, since we have $(g_1 \oplus g_2)h = g_2$, and composing it with the mapping $g \in B_{k,l}(h) \mapsto (g \oplus hg, hg) \in An_k(h) \times An_l(h \oplus 1)$ gives identity. Hence, we have:

- $\dim(An_k(h)) + \dim(An_l(h \oplus 1)) = \dim B_{k,l}(h)$,
- $\dim(An_k(f)) + \dim(An_l(f \oplus 1)) = \dim B_{k,l}(f)$,
- $\dim(An_{d-1}(h)) + \dim(An_{d-1}(h \oplus 1)) = \dim B_{d-1,d-1}(h)$.

In [800] is shown[4] that, for every $d \leq \lceil \frac{n}{2} \rceil$ and every function h such that $\dim(An_{d-1}(h)) + \dim(An_{d-1}(h \oplus 1)) > 0$, there exists f for which Bound (9.4) is an equality and such that $AI(f) \geq d$. Let us give a proof of this astonishingly general result. Let C_1 (resp. C_0) be a maximal subset of $supp(h)$ (resp. $supp(h \oplus 1)$) such that the corresponding rows of $M_{h,d-1}$ (resp. $M_{h\oplus 1,d-1}$) are $\mathbb{F}_2$-linearly independent. We have $|C_1| = \sum_{i=0}^{d-1} \binom{n}{i} - \dim(An_{d-1}(h))$ and $|C_0| = \sum_{i=0}^{d-1} \binom{n}{i} - \dim(An_{d-1}(h \oplus 1))$. According to Proposition 136 applied to the indicator function 1_{C_1} (resp. 1_{C_0}) and with $k = d-1$, the ranks of $M_{1_{C_1}\oplus 1,d-1}$ and $M_{1_{C_0}\oplus 1,d-1}$ both equal $\sum_{i=0}^{d-1} \binom{n}{i}$. Since $C_0 \subseteq supp(1_{C_1} \oplus 1)$ (resp. $C_1 \subseteq supp(1_{C_0} \oplus 1)$), there exists outside $C_1 \cup C_0$, a subset C_0' of size $\sum_{i=0}^{d-1} \binom{n}{i} - |C_0| = \dim(An_{d-1}(h \oplus 1))$ (resp. C_1' of size $\sum_{i=0}^{d-1} \binom{n}{i} - |C_1| = \dim(An_{d-1}(h))$) such that the rows of $M_{1_{C_1}\oplus 1,d-1}$ (resp. $M_{1_{C_0}\oplus 1,d-1}$) corresponding to the elements of $C_0 \cup C_0'$ (resp. $C_1 \cup C_1'$) are $\mathbb{F}_2$-linearly independent. Since C_0 and C_1 were taken maximal, we have $C_1' \subseteq supp(h \oplus 1)$ and $C_0' \subseteq supp(h)$. The function $f = h \oplus 1_{C_0'} \oplus 1_{C_1'}$ satisfies $d_H(f, h) = \dim(An_{d-1}(h)) + \dim(An_{d-1}(h \oplus 1))$. And we have $AI(f) \geq d$, since $rk(M_{f,d-1}) \geq |C_1| + |C_1'| = \sum_{i=0}^{d-1} \binom{n}{i}$ and therefore $rk(M_{f,d-1}) = \sum_{i=0}^{d-1} \binom{n}{i}$ and similarly $rk(M_{f\oplus 1,d-1}) = \sum_{i=0}^{d-1} \binom{n}{i}$.

Relationship between algebraic immunity, Hamming weight, algebraic degree, nonlinearity and higher-order nonlinearity

We have seen that nonlinearity and algebraic degree are rather uncorrelated: there are Boolean functions with high nonlinearity and low algebraic degree (since there exist quadratic bent functions), with low nonlinearity and low algebraic degree, with high nonlinearity and high algebraic degree[5], and with low nonlinearity and high algebraic degree. Interestingly, if we replace the algebraic

[4] Originally was assumed the condition that the algebraic degree of h is at most $\lceil \frac{n}{2} \rceil$, but after clarifying the proof with M. Lobanov, we could see that this is not necessary.

[5] But not with maximal nonlinearity and high algebraic degree because of the Rothaus bound.

degree by the algebraic immunity, the latter case can not happen. We need preliminary results which have their own interest.

Proposition 138 *[261] For every n-variable Boolean function, we have:*

$$\sum_{i=0}^{AI(f)-1} \binom{n}{i} \leq w_H(f) \leq \sum_{i=0}^{n-AI(f)} \binom{n}{i}. \tag{9.5}$$

Indeed, if the left-hand side inequality is not satisfied, then $M_{f,AI(f)-1}$ has rank at most $w_H(f) < \sum_{i=0}^{AI(f)-1} \binom{n}{i}$, a contradiction. The right-hand side inequality is obtained from the other one by replacing f by $f \oplus 1$.
This implies again that $AI(f) \leq \lceil \frac{n}{2} \rceil$ (since applied with $AI(f) \geq \lceil \frac{n}{2} \rceil + 1$, it leads to a contradiction, because the lower bound is then strictly larger than the upper bound) and it also implies that a function f such that $AI(f) = \frac{n+1}{2}$ (n odd) must be balanced.
In [261, Lemma 1] has been stated:

Proposition 139 *For any two n-variable Boolean functions f and h, we have:*

$$AI(f) - d_{alg}(h) \leq AI(f \oplus h) \leq AI(f) + d_{alg}(h). \tag{9.6}$$

The proof was incomplete: let $g \neq 0$ be such that $fg = 0$ (resp. $(f \oplus 1)g = 0$) and have algebraic degree $AI(f)$, then we have $(f \oplus h)((h \oplus 1)g) = 0$ (resp. $(f \oplus 1 \oplus h)((h \oplus 1)g) = 0$); it was written that this proves the inequality on the right since $d_{alg}((h \oplus 1)g) \leq AI(f) + d_{alg}(h)$, but this conclusion is correct only if $(h \oplus 1)g \neq 0$. Let us address the case $(h \oplus 1)g = 0$: we have then $(f \oplus h \oplus 1)g = 0$ (resp. $(f \oplus h)g = 0$) and g being a nonzero annihilator of $f \oplus h \oplus 1$ (resp. $f \oplus h$), we have $AI(f \oplus h) \leq AI(f) \leq AI(f) + d_{alg}(h)$. This completes the proof of the inequality on the right. Applying it to $f \oplus h$ instead of f gives then the inequality on the left.

Note that these relations are valid if f and h are defined on different (maybe intersecting) sets of variables and n is the global number of variables (indeed, algebraic immunity does not change if we consider a function with more variables, the additional variables being fictitious). Moreover, if these sets of variables are disjoint, then we have $AI(f) \leq AI(f \oplus h) \leq AI(f) + d_{alg}(h)$, since it is then possible to obtain a nonzero annihilator of algebraic degree $AI(f \oplus h)$ of f or of $f \oplus 1$ as the restriction of a nonzero annihilator of $f \oplus h$ or of $f \oplus h \oplus 1$.

It is deduced in [261] that low nonlinearity implies low algebraic immunity (but high algebraic immunity does not imply high nonlinearity, as well as high nonlinearity does not imply high algebraic immunity): Relation (9.5) applied to $f \oplus h$ with h affine and Relation (9.6) show that:

$$nl(f) \geq \sum_{i=0}^{AI(f)-2} \binom{n}{i}$$

and more generally (by applying Relation (9.5) to $f \oplus h$ with $d_{alg}(h) \leq r$):

$$nl_r(f) \geq \sum_{i=0}^{AI(f)-r-1} \binom{n}{i}. \tag{9.7}$$

These lower bounds, which play a role with respect to probabilistic algebraic attacks, see [792, 793], have been improved in all cases for the first order nonlinearity into

$$nl(f) \geq 2 \sum_{i=0}^{AI(f)-2} \binom{n-1}{i}$$

by Lobanov [798, 799] and in most cases for the r-th order nonlinearity into

$$nl_r(f) \geq 2 \sum_{i=0}^{AI(f)-r-1} \binom{n-r}{i} \tag{9.8}$$

in [228] (in fact, the improvement was slightly stronger than this, but more complex). Another improvement:

$$nl_r(f) \geq \sum_{i=0}^{AI(f)-r-1} \binom{n}{i} + \sum_{i=AI(f)-2r}^{AI(f)-r-1} \binom{n-r}{i} \tag{9.9}$$

(which always improves upon (9.7) and improves upon (9.8) for low values of r) has been subsequently obtained by Mesnager in [849] and slightly later by Lobanov in [800], who gives a general proof for all these bounds, that we recall below. Precisions on the bounds, involving the maximum between the minimal algebraic degree of the nonzero annihilators of f and the minimal algebraic degree of the nonzero annihilators of $f \oplus 1$, have been also given in [998].

Here is Lobanov's general proof: Bound (9.4), page 359, and the observations which follow it imply that, for every n-variable Boolean function f and every positive integer $r \leq n$, we have[6]:

$$nl_r(f) \geq \min_{h \in \mathcal{BF}_n, d_{alg}(h) \leq r} \dim(B_{AI(f)-1, AI(f)-1}(h)). \tag{9.10}$$

Then, if $d_{alg}(h) = r$:

- $\dim(B_{k,k}(h)) \geq \sum_{i=0}^{k-r} \binom{n}{i}$, because all n-variable functions of algebraic degree at most $k-r$ belong to $B_{k,k}(h)$; then (9.10) implies (9.7),
- $\dim(B_{k,k}(h)) \geq 2\sum_{i=0}^{k-r} \binom{n-r}{i}$, because, if $\prod_{i \in I} x_i$ is a monomial of degree r in the ANF of h, then all n-variable functions of the form $hg_1 \oplus (h \oplus 1)g_2$ where g_1, g_2 have algebraic degree at most $k-r$ and depend only on variables $x_i, i \notin I$, belong to $B_{k,k}(h)$ and are distinct since the linear mapping $(g_1, g_2) \mapsto hg_1 \oplus (h \oplus 1)g_2$ has trivial kernel, because $hg_1 \oplus (h \oplus 1)g_2 = 0$ if and only if $hg_1 = (h \oplus 1)g_2 = 0$; then, (9.10) implies (9.8),

[6] A slightly more complex bound is deduced in [801] from the first bound in Proposition 137, which allows to improve upon lower bounds (9.8) and (9.9) in some sub-cases.

- $\dim(B_{k,k}(h)) \geq \sum_{i=0}^{k-r}\binom{n}{i} + \sum_{i=k-2r+1}^{k-r}\binom{n-r}{i}$, because, if $\prod_{i\in I} x_i$ is a monomial of degree r in the ANF of h, then all n-variable functions of the form $g_1 \oplus hg_2$ where g_1, g_2 have algebraic degree at most $k-r$ and g_2 depends only on variables $x_i, i \notin I$, and has only monomials of degree at least $k-2r+1$, belong to $B_{k,k}(h)$ and are distinct since the linear mapping $(g_1, g_2) \mapsto g_1 \oplus hg_2$ has trivial kernel; then (9.10) implies (9.9).

An obvious upper bound on the higher-order nonlinearity exists which also involves the algebraic immunity, as observed in [227]: if $AI(f) \leq r$ and if f is balanced, then we have $nl_r(f) \leq 2^{n-1} - 2^{n-r}$, since by hypothesis, there exists a nonzero function g of algebraic degree at most r such that $g \preceq f$ or $g \preceq f \oplus 1$, and g being nonzero and belonging to the Reed-Muller code of order r, it has Hamming weight at least the minimum distance of this code, that is 2^{n-r}. If $g \preceq f$ for instance, then $d_H(f,g) = w_H(f \oplus g) = w_H(f) - w_H(g) \leq 2^{n-1} - 2^{n-r}$ and $nl_r(f) \leq 2^{n-1} - 2^{n-r}$.

A bound between $nl_r(f)$ and $FAI(f)$ has been also given in [1106]: $nl_r(f) \geq \sum_{i=0}^{\lfloor \frac{FAI(f)-r}{2} \rfloor}\binom{n}{i}$, but the proof has several shortcomings and the result seems false. For instance, with the help of a computer, we can check that the unbalanced function in [1068] with $n = 6$ has FAI 6 and the result above would imply that there exists a function with second-order nonlinearity at least 22, but it is known that the covering radius of RM(2,6) is 18. In fact, $FAI(f) - r$ should be $FAI(f) - r - 1$ in this bound, but even if such correction is made, the proof does not address all issues. Such a result is important to show that some functions cannot have good behavior against fast algebraic attacks, like functions obtained by modifying bent functions (*e.g.* those of [1091]). We give in the next theorem a corrected result (the first bound in Theorem 22 is more or less the only interesting one; we include also the second to give a correct alternative to a bound given in Ref. [1106] and to show what were the difficulties missed by its proof).

Theorem 22 *For any positive integer n and any non-negative integer $r \leq n$, let f be any n-variable function and $k = \min\{d_{alg}(g) + d_{alg}(fg);\ g \neq 0\}$. We have then:*

$$nl_r(f) \geq \sum_{i=0}^{\lfloor \frac{k-r-1}{2} \rfloor}\binom{n}{i}.$$

Moreover, if $nl_r(f) \neq 0$ and if $AI(f) > AI(f \oplus h)$ for at least one function h of algebraic degree at most r such that $d_H(f,h) = nl_r(f)$, then:

$$nl_r(f) \geq \sum_{i=0}^{\lfloor \frac{FAI(f)-r-1}{2} \rfloor}\binom{n}{i}.$$

Proof. Suppose first that $nl_r(f) < \sum_{i=0}^{\lfloor \frac{k-r-1}{2} \rfloor}\binom{n}{i}$. Let h be a Boolean function of algebraic degree at most r whose Hamming distance $w_H(f \oplus h)$ to f equals $nl_r(f)$. Since $f \oplus h$ has Hamming weight strictly smaller than $\sum_{i=0}^{\lfloor \frac{k-r-1}{2} \rfloor}\binom{n}{i}$, the rank of matrix $M_{f\oplus h,\lfloor\frac{k-r-1}{2}\rfloor}$ is also strictly smaller and there exists a nonzero annihilator g of $f\oplus h$ whose algebraic degree is at most $\lfloor\frac{k-r-1}{2}\rfloor$. We have then $fg=hg$ with $g\neq 0$ and $d_{alg}(g)+d_{alg}(fg)=d_{alg}(g)+d_{alg}(hg)\leq 2\lfloor\frac{k-r-1}{2}\rfloor+r<k$, a contradiction.

We address now the second bound. Suppose that $nl_r(f)<\sum_{i=0}^{\lfloor\frac{FAI(f)-r-1}{2}\rfloor}\binom{n}{i}$ and let us fix h of algebraic degree at most r such that $AI(f)>AI(f\oplus h)$. For such h, similarly to above, there exist annihilators $g\neq 0$ of $f\oplus h$ such that $d_{alg}(g)\leq\left\lfloor\frac{FAI(f)-r-1}{2}\right\rfloor$ and one of these annihilators at least has algebraic degree $AI(f\oplus h)<AI(f)$. Then:

- if one of these annihilators equals constant function 1, then $f=h$ and therefore $nl_r(f)=0$, a contradiction;

- in the other case, we arrive to a contradiction as above. □

For instance, if k is near from n and $r=n/2$, we have $nl_{n/2}(f)\geq\sum_{i=0}^{\lambda n}\binom{n}{i}\geq\frac{2^{n\,H_2(\lambda)}}{\sqrt{8n\lambda(1-\lambda)}}$, where $\lambda\approx\frac{1}{4}$ (cf. [809, page 310]), and where $H_2(x)=-x\log_2(x)-(1-x)\log_2(1-x)$ is the entropy function, whose value at $\frac{1}{4}$ equals $\frac{1}{2}+\frac{3}{4}(2-\log_2(3))=2-\frac{3}{4}\log_2(3)\approx 0.8$. Note that $2^{\frac{n}{2}-1}$ is then negligible with respect to $\frac{2^{n\,H_2(\lambda)}}{\sqrt{8n\lambda(1-\lambda)}}$ (this will play a role at page 370).

Remark. If $nl_r(f)\neq 0$, the condition $g\neq 0$ in the definition of k can be replaced by $d_{alg}(g)\geq 1$. Indeed, if there is no other nonzero annihilator of $f\oplus h$ of algebraic degree at most $\lfloor\frac{k-r-1}{2}\rfloor$ than $g=1$, this means that $k\leq r+1$. If $k\leq r$ then $\lfloor\frac{k-r-1}{2}\rfloor<0$ and the result holds, and if $k=r+1$, then the only case where the bound would not hold is if $nl_r(f)=0$, which is excluded. □

9.1.2 The problem of finding functions achieving high algebraic immunity and high nonlinearity

Recall that, in the framework of stream ciphers, we do not have security proofs but we need functions allowing resistance to all known attacks and having enough randomness for hoping they will not be too weak against new attacks. These functions must be as fastly computable as possible.

No known primary construction viewed in Chapters 5, 6 and 7 allows obtaining classes of functions satisfying all important criteria and no secondary construction is known for designing new functions satisfying all the criteria from already defined functions satisfying them. We know however that functions achieving optimal or suboptimal algebraic immunity and at the same time high algebraic degree and high nonlinearity must exist thanks to the results of [437, 1000]. But knowing that almost all functions have high algebraic immunity does not mean that constructing such functions is easy.

Lobanov's bound seen at page 362 does not ensure high enough nonlinearity:

- For n even and $AI(f)=\frac{n}{2}$, it gives $nl(f)\geq 2^{n-1}-2\binom{n-1}{\frac{n}{2}-1}=2^{n-1}-\binom{n}{\frac{n}{2}}$

which is much smaller than the best possible nonlinearity $2^{n-1} - 2^{\frac{n}{2}-1}$ and, more problematically, much smaller than the asymptotic almost sure nonlinearity of Boolean functions, which is, when n tends to ∞, located in the neighborhood of $2^{n-1} - 2^{\frac{n}{2}-1}\sqrt{2n \ln 2}$ as we saw. Until 2008, the best nonlinearity reached by the known functions with optimal AI was that of the majority function and of the iterative construction (see more details below on these functions): $2^{n-1} - \binom{n-1}{\frac{n}{2}} = 2^{n-1} - \frac{1}{2}\binom{n}{\frac{n}{2}}$ [409]. This was a little better than what gives Lobanov's bound, but insufficient.

• For n odd and $AI(f) = \frac{n+1}{2}$, Lobanov's bound gives $nl(f) \geq 2^{n-1} - \binom{n-1}{(n-1)/2} \simeq 2^{n-1} - \frac{1}{2}\binom{n}{(n-1)/2}$ which is a little better than in the n even case, but still far from the average nonlinearity of Boolean functions. Until 2008, the best known nonlinearity was that of the majority function and matched this bound.

Efficient algorithms have been given in [27, 438, 439] for computing the algebraic immunity, with respective complexities $\mathcal{O}\left(2^n\binom{n}{AI(f)}\right)$ and $\mathcal{O}\left(n2^n\binom{n}{AI(f)}\right)$ (the latter being slightly worse but on the other hand the amount of memory needed being smaller). Algorithms for evaluating the immunity to fast algebraic attacks are also given in these references with complexity $\mathcal{O}\left(e\binom{d+1}{e}\binom{n}{e}\sqrt{\binom{n}{d}} + \binom{n}{e}^3\right)$, where e is significantly smaller than $AI(f)$ and d is comparable to $AI(f)$, and $\mathcal{O}\left(n2^n\binom{n}{k}\right)$ where k is the degree of the algebraic system to be solved in the last step of the attacks. They showed the poor resistance of the majority function to FAA. In [997, 999] were introduced three matrices to evaluate the behavior of Boolean functions against fast algebraic attacks using univariate polynomial representation. Later was shown in [794] that one matrix is enough.

9.1.3 The functions with high algebraic immunity found so far and their parameters

Sporadic functions

In [407] are exhibited 7-variable rotation symmetric (RS) functions with nonlinearity 56, resiliency order 2 and algebraic immunity 4, and a large number of 8-variable RS functions with nonlinearity 116, resiliency order 1 and algebraic immunity 4. These authors claimed there exist such functions having good resistance against fast algebraic attacsks, but Siegenthaler's bound shows that this resistance is limited and 8 variables is small; rotation symmetry presents also a risk that the attacker can use such strong structure in specific attacks.

Balanced highly nonlinear functions in up to 20 variables (derived from power functions) with high algebraic immunities have been exhibited in [279] and [27]. Some other interesting ideas of constructions have been proposed, either using simulated annealing [836] (but the number of variables is limited, the gain in terms of nonlinearity is not large, and of course, this cannot produce infinite classes) or by using genetic hill climbing algorithm, starting from the function of Theorem 23 that we shall see at page 368 and applying a few swaps on its truth table [694] (this can increase a little its nonlinearity but it could not lead

to infinite classes either).
In [1031] are calculated the algebraic immunity and nonlinearity of the 20-variable function used as nonlinear filter in the lightweight stream cipher Hitag2. The algebraic immunity is no more than 6.
Note that the construction of Proposition 85, page 262, allows increasing the complexity of Boolean functions while keeping their high nonlinearities and may allow increasing their algebraic immunity as well.

Primary constructions of infinite classes of functions, with insufficient nonlinearity

- The *majority function*, considered by Key, McDonough and Mavron [691] in the context (equivalent to that of algebraic immune functions) of the erasure channel, and rediscovered in the context of algebraic immunity [409, 127]), defined as $f(x) = 1$ if $w_H(x) \geq \frac{n}{2}$ and $f(x) = 0$ otherwise[7], has optimal algebraic immunity. Note that, for n odd, Proposition 135 materializes, in the case of this function, in a rather simple way, since $(f \oplus 1)(x) = f(x + 1_n)$ and f and $f \oplus 1$ are then affine equivalent. The proof of its optimal algebraic immunity is easy. We give it for n odd (the case n even is slightly more technical): an annihilator of $f \oplus 1$ being equal to 0 at every input of Hamming weight at most $\frac{n-1}{2}$, Relation (2.4), page 50, makes that its *ANF* has no term of degree at most $\frac{n-1}{2}$; a nonzero annihilator must then have algebraic degree at least $\frac{n+1}{2}$. The majority function is balanced when n is odd. It is a *symmetric Boolean function* (which can represent a weakness but also allows using it with more variables while ensuring the same or even a better speed), and when n is odd it is the only one with optimal AI, up to the addition of function 1, see more at page 389. It has two main weaknesses: its nonlinearity is weak[8] since, according to [409], it equals $2^{n-1} - \binom{n-1}{\lfloor n/2 \rfloor}$, which for n odd matches Lobanov's bound $2\sum_{i=0}^{\lceil n/2 \rceil -2}\binom{n-1}{i} = 2^{n-1} - \binom{n-1}{(n-1)/2}$, and for n even is slightly larger than Lobanov's bound, equal then to $2^{n-1} - 2\binom{n-1}{n/2-1}$, but is not large enough, and its resistance to *fast algebraic attack*s is bad too (as shown in [27], there exist Boolean functions $g \neq 0$ and h such that $fg = h$, where $d_{alg}(h) = \lfloor n/2 \rfloor + 1$ and $d_{alg}(g) = d_{alg}(h) - 2^j$, where j is maximum so that this number is strictly positive). The nonlinearity has been determined in [409]; the proof using Krawtchouk polynomials is very technical and cannot be included here. A simpler proof has been given by Cusick (and not published). There is a way of showing that the nonlinearity of the majority function (which is adaptable to many other functions similar to it) cannot be good: let us take for instance n odd and apply the second-order Poisson formula (2.57), page 81, with $E = \{x \in \mathbb{F}_2^n;\ x \preceq b\}$ where b is a vector of

[7] Changing $w_H(x) \geq \frac{n}{2}$ into $w_H(x) > \frac{n}{2}$ or $w_H(x) \leq \frac{n}{2}$ or $w_H(x) < \frac{n}{2}$ changes the function into an *affinely equivalent* one, up to addition of the constant 1, and therefore does not change the AI.

[8] This crippling drawback is shared by all the classes of rotation symmetric functions (see definition in Section 10.2, page 392) with optimal AI presented in numerous papers, which are then not mentioned in this book.

Hamming weight $\frac{n-1}{2}$, and $E' = E^\perp = \{x \in \mathbb{F}_2^n;\ x \preceq b + 1_n\}$. For every $a \in E'$, when x ranges over E, the Hamming weight of $a+x$ equals $w_H(a)+w_H(x)$ (since the two vectors have disjoint supports) and is larger than or equal to $\frac{n+1}{2}$ if and only if $w_H(x) \geq \frac{n+1}{2} - w_H(a)$. Hence, we have $\mathcal{F}(h_a) = \sum_{i=0}^{\frac{n-1}{2}-w_H(a)} \binom{\frac{n-1}{2}}{i} - \sum_{i=\frac{n+1}{2}-w_H(a)}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}}{i} = \sum_{i=0}^{\frac{n-1}{2}-w_H(a)} \binom{\frac{n-1}{2}}{i} - \sum_{i=0}^{w_H(a)-1} \binom{\frac{n-1}{2}}{i} =$

$$\begin{cases} \sum_{i=w_H(a)}^{\frac{n-1}{2}-w_H(a)} \binom{\frac{n-1}{2}}{i} & \text{if } w_H(a) \leq \frac{n-1}{4} \\ -\sum_{i=\frac{n+1}{2}-w_H(a)}^{w_H(a)-1} \binom{\frac{n-1}{2}}{i} & \text{if } w_H(a) \geq \frac{n+3}{4} \end{cases}.$$

The absolute value of $\mathcal{F}(h_a)$ is then larger than or equal to $\binom{\frac{n-1}{2}}{\lfloor \frac{n-1}{4} \rfloor}$, unless it is null, that is, unless $\frac{n-1}{2} - w_H(a) = w_H(a) - 1$, which happens only if $n \equiv 3$ [mod 4] and $w_H(a) = \frac{n+1}{4}$. Since $b + 1_n$ has Hamming weight $\frac{n+1}{2}$, the size of E' equals $2^{\frac{n+1}{2}}$. According to (2.57), the arithmetic mean of $W_f^2(u)$ when $u \in E^\perp$ is then at least $(2^{\frac{n+1}{2}} - 1)\left(\binom{\frac{n-1}{2}}{\lfloor \frac{n-1}{4} \rfloor}\right)^2 \approx 2^{\frac{3n-1}{2}}$ (recall that the Stirling formula implies that $\binom{n}{n/2} \sim 2^n\sqrt{\frac{2}{\pi n}}$) and therefore $nl(f)$ is smaller than or equal to approximately $2^{n-1} - 2^{\frac{3n-3}{4}}$. The algebraic degree of the majority function is determined in [409] and equals $2^{\lfloor \log_2(n) \rfloor}$.

Some variants of the majority function have also optimal algebraic immunity and are balanced for n even, but they have more or less the same drawbacks.

It is proved in [127] that, for n even, changing the value at 1_n of the majority function preserves its optimal algebraic immunity, as well as, for $n \geq 8$, changing its values at the inputs of Hamming weights $\frac{n}{2} \pm 4$ and, for $n \geq 10$, making these two changes simultaneously. All such functions happen to be weak against FAA as shown in [27].

- An iterative construction of an infinite class of functions with optimal algebraic immunity has been given in [408] and further studied in [261]; however, the functions it produces are neither balanced (which can be fixed) nor highly nonlinear (which cannot, unless many variables are added) and it is weak against fast algebraic attacks, as also shown in [27].

- More numerous functions with optimal algebraic immunity were given in [230]. Among them are functions with better nonlinearities, but the method did not allow to reach high nonlinearities (see [328]) and some functions constructed in [766, 767] seem still worse from this viewpoint. In [929] was introduced an iterative concatenation method for constructing maximum AI functions with suboptimal FAI, but the nonlinearity of the resulting functions was insufficient. Hence, the question of designing infinite classes of functions achieving all the necessary criteria remained open after these papers.

A first primary construction of an infinite class of functions satisfying all criteria

A function with optimal algebraic immunity, good immunity to fast algebraic attacks, provably much better nonlinearity than the functions mentioned above, and in fact, according to computer investigations, quite sufficient nonlinearity, has been exhibited in 2008 (five years after the invention of algebraic attacks) in [273]. This *primary construction* is defined over the field $\mathbb{F}_{2^n}$. It has been originally defined as the Boolean function whose support equals $\{0, 1, \alpha, \dots, \alpha^{2^{n-1}-2}\}$, where α is any *primitive element* of $\mathbb{F}_{2^n}$. This original function is the Boolean (single-output) case of a class of vectorial functions studied in Ref. [500], where the optimal algebraic immunity was proved. The contribution of [273] (which made that the authors of the subsequent papers gave to these functions the name of *Carlet-Feng functions*) was to observe that all the cryptographic parameters of this function were good (not only the algebraic immunity) and to provide a simpler proof of the optimal algebraic immunity, which gave a better view on why it happens. The proof has been later slightly simplified further in [242]. The authors of the papers which soon after 2008 have modified this function in order to find more functions [1091, 1148] preferred using the function of support $\{\alpha^s, \cdots, \alpha^{2^{n-1}+s-1}\}$ where s is some integer. The two definitions coinciding for $s = 2^{n-1} - 1$ up to addition of constant function 1, and two different values of s giving *linearly equivalent* functions, the two definitions deal with essentially the same function. In the next theorem, we take the modified definition[9].

Theorem 23 *[500, 273] For every positive integer n, every integer s and every primitive element α of $\mathbb{F}_{2^n}$, the balanced Boolean function over $\mathbb{F}_{2^n}$ whose support is $\{\alpha^s, \cdots, \alpha^{2^{n-1}+s-1}\}$ has optimal* algebraic degree $n-1$ *and optimal algebraic immunity $\lceil \frac{n}{2} \rceil$.*

Proof. It is shown in [273] that the univariate representation of the original function equals: $1 + \sum_{i=1}^{2^n-2} \frac{\alpha^i}{(1+\alpha^i)^{1/2}}\, x^i$, where $u^{1/2} = u^{2^{n-1}}$, which shows that the algebraic degree of f equals $n - 1$ (optimal for a balanced function). This proves the first property. Up to linear equivalence, s can be taken equal to 0. Let g be any Boolean function of algebraic degree strictly less than n and $g(x) = \sum_{i=0}^{2^n-2} g_i x^i$, $g_i \in \mathbb{F}_{2^n}$, its univariate representation in the field $\mathbb{F}_{2^n}$ (since g has algebraic degree less than n, we have $g_{2^n-1} = 0$). Then:

$$\begin{pmatrix} g(1) \\ g(\alpha) \\ g(\alpha^2) \\ \vdots \\ g(\alpha^{2^n-2}) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{2^n-2} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(2^n-2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha^{2^n-2} & \alpha^{2(2^n-2)} & \cdots & \alpha^{(2^n-2)(2^n-2)} \end{pmatrix} \times \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{2^n-2} \end{pmatrix}.$$

[9] This same function has been later re-discovered with another presentation by Q. Wang, J. Peng, H. Kan and X. Xue in IEEE Transactions on Inf. Th., as shown in [238] (and in another paper published later by H. Chen, T. Tian and W. Qi in DCC).

If g is an annihilator of f, then $g(1) = g(\alpha) = \cdots = g(\alpha^{2^{n-1}-1}) = 0$ and the coefficients $g_0, \dots, g_{2^n-2}$ satisfy then:

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{2^n-2} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(2^n-2)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \alpha^{2^{n-1}-1} & \alpha^{2(2^{n-1}-1)} & \cdots & \alpha^{(2^{n-1}-1)(2^n-2)} \end{pmatrix} \times \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{2^n-2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If at most 2^{n-1} of the g_i's are nonzero, then erasing $2^{n-1} - 1$ null coefficients (and the corresponding matrix columns) from the system above leads to a homogeneous system of linear equations whose matrix is a $2^{n-1} \times 2^{n-1}$ Vandermonde matrix and is then nonsingular. We have then proved that the vector $(g_0, \cdots, g_{2^n-2})$ is either null or has Hamming weight at least $2^{n-1} + 1$ (in the framework of coding theory, this result is called the *BCH bound*). This implies that any nonzero annihilator of f has algebraic degree at least $\lceil \frac{n}{2} \rceil$ (since otherwise, the number of its nonzero coefficients would be at most 2^{n-1}, because $\sum_{i=0}^{\lceil n/2 \rceil - 1} \binom{n}{i} \leq 2^{n-1}$).
If g is an annihilator of $f \oplus 1$, then we have $g(\alpha^i) = 0$ for every $i = 2^{n-1}, \dots, 2^n - 2$, and for the same reasons as above, the vector $(g_0, \dots, g_{2^n-2})$ has Hamming weight at least 2^{n-1}. Moreover, suppose that function g has algebraic degree at most $\frac{n-1}{2}$ and that the vector $(g_0, \dots, g_{2^n-2})$ has Hamming weight 2^{n-1} exactly. Then n is odd and all the coefficients $g_i; 0 \leq i \leq 2^n - 2, w_2(i) \leq (n-1)/2$, are nonzero[10], but $g_0 \neq 0$ contradicts then $g(0) = 0$. This completes the proof. □

The nonlinearity of the function is also good, at least for values of n for which the function can be used in stream ciphers. In fact, this nonlinearity had been previously studied in [129] (but the algebraic immunity was not considered there) and a lower bound on the nonlinearity was shown, similar to the one later given in [273]:

$$nl(f) \geq 2^{n-1} - n \cdot \ln 2 \cdot 2^{\frac{n}{2}} - 1. \tag{9.11}$$

Bound (9.11) is not sufficient for showing that f has good nonlinearity. It has been improved several times, but the improvements are marginal and insufficient for asserting that the function allows resisting the fast correlation attack. The actual values of the nonlinearity have been computed up to $n = 26$ and happen to be very good and quite sufficient for such a resistance. Note that the nonlinearity depends on the choice of the primitive element α and the bounds mentioned above are in fact bounds on the minimum Hamming distance between f and all functions of the form $tr(ax^j + b)$ where j is co-prime with $2^n - 1$, that we can call the *hyper-nonlinearity* (in relation to the notion of *hyper-bent function* seen in Definition 57, page 270). It is an open question to determine whether a

[10] There is a small inaccuracy in what is written in the proofs provided in [236, 242, 273] since the g_i's are not necessarily in $\mathbb{F}_2$.

significantly better lower bound on the hyper-nonlinearity of f can be proved (some ideas are given in [248, Subsection 4.2]) or if the gap between the bound and the actual hyper-nonlinearity reduces when n takes values larger than 26.

The good resistance to *fast algebraic attack*s has first been checked by computer for $n \leq 12$, using an algorithm from [27], and later shown mathematically in [793] for all n:

Proposition 140 *Let e be a positive integer less than $\frac{n}{2}$ and f be the function of Theorem 23. Then, if $\binom{n-1}{e}$ is even, there exists no non-zero function g with algebraic degree at most e such that fg has algebraic degree at most $n-e-1$, and if $\binom{n-1}{e}$ is odd, there exists no non-zero function g with degree at most e such that fg has degree at most $n-e-2$.*

In particular, f is PAI (see page 352) when n is a power of 2, plus 1 (this was known before only for $n = 3, 5, 9$).

The computation of the function of Theorem 23 is reasonably fast, at least for some values of $n \leq 20$. This may seem surprising, because the complexity of its computation is clearly the same as that of the discrete logarithm, which is known to be asymptotically high (this has led to a whole branch of public key cryptography), but for small values of n (like $n \leq 20$), the function is fast to be computed, all the more if $2^n - 1$ is the product of small factors (this is the case of 18 and 20 for instance), because this allows using the Pohlig-Hellman algorithm; in the case of these two values of n, computing one output bit per cycle is possible with 40,000 transistors, as observed in [238]. This allows avoiding needing using a look-up table (of about one mega-bits, which is too heavy for some devices) for computing the output of the function.

Hence, the functions of this class gather all the properties needed for allowing the stream ciphers using them as filtering functions to resist all the main attacks (the Berlekamp-Massey (BM) algorithm, fast correlation attacks, standard and fast algebraic attacks, and Rønjom-Helleseth attacks).

Modifications of the functions of Theorem 23

- Classes of functions have been proposed, obtained by replacing a part of the support of the function by another part of the same size. In [997] is proposed a matrix approach (instead of the BCH bound) for proving optimal AI, and the (balanced) function of support $\{1, \alpha, \ldots, \alpha^{\sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} \binom{n}{i}}\} \cup U$, where $U \subset \{\alpha^{\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \binom{n}{i}}, \ldots, \alpha^{1+\sum_{i=0}^{\lceil \frac{n}{2} \rceil} \binom{n}{i}}\}$ has size $2^{n-1} - \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \binom{n}{i}$, is proved to have optimal AI. In [1148] are proposed three classes based on the same method; for some values of n, better nonlinearity than with the function of Theorem 23 could be reached and for other values of n, the nonlinearity is worse. The good resistance to FAA has been checked by computer for small values of n.
- Another kind of modification of this same function has been proposed in [1091]. It is based on the $\mathcal{PS}_{ap}$ construction. The so-called *Tu-Deng function* is the $2n$-

variable function defined over $\mathbb{F}_{2^n}^2$ mapping (x, y) to $f(xy^{2^n-2})$, where f is the function of Theorem 23. Note that xy^{2^n-2} equals $\frac{x}{y}$ when $y \neq 0$ and is null when $y = 0$. Since f is balanced, the Tu-Deng function is bent (and therefore has optimal nonlinearity $2^{2n-1} - 2^{n-1}$) as we saw at page 238. Moreover, its AI has optimal value n, up to a combinatorial conjecture which was still an open problem (studied in [513] and other papers but not solved yet) when this book was written, but which has been checked up to $n = 29$; this is quite enough in cryptographic context, since $n = 29$ makes 58 variables. We know that bent functions are not balanced, but it is shown in [1091] that modifying 2^{n-1} output values of the Tu-Deng function can give a balanced function with optimal AI and very large nonlinearity.
Unfortunately, the resulting balanced function lies then at Hamming distance at most 2^{n-1} from the Reed-Muller code of order n and length 2^{2n} (the set of Boolean functions in $2n$ variables of algebraic degree at most n), since because of Theorem 13, page 224, any $2n$-variable bent function has algebraic degree at most n. According to Theorem 22, page 363, and to the observation which follows it, applied with $2n$ instead of n and with $r = n$, the balanced function is weak against fast algebraic attacks (see more precise calculations in [1106, Lemmas 1-2]), as are the 1-resilient functions obtained from it in some papers by modifying a few terms.
The Tu-Deng construction has been generalized to vectorial functions in [501].
- The Tu-Deng function has been modified in [1068] into a class of $2n$-variable functions having the same nice properties as the function of Theorem 23. As recalled in the survey [242]:

Proposition 141 *Let $n = 2^r m \geq 2$, where $r \geq 0$ and $m > 0$ is odd, and let f be the function of Theorem 23. We consider the functions:*

$$f_1(x, y) = f(xy); x, y \in \mathbb{F}_{2^n}, \tag{9.12}$$

$$f_2(x, y) = \begin{cases} f_1(x, y), & x \neq 0 \\ u(y), & x = 0 \end{cases} \tag{9.13}$$

where u is a balanced Boolean function on $\mathbb{F}_{2^n}$ satisfying $u(0) = 0$, $deg(u) = n-1$, and $\max_{a \in \mathbb{F}_{2^n}} |W_u(a)| \leq 2^{\frac{m+1}{2}}$ if $r = 0$ and $\max_{a \in \mathbb{F}_{2^n}} |W_u(a)| \leq \sum_{i=1}^{r} 2^{\frac{n}{2^i}} + 2^{\frac{m+1}{2}}$ if $r \geq 1$. Then f_2 is balanced, f_1 and f_2 have optimal AI (equal to n), f_1 has algebraic degree $2n-2$, f_2 has algebraic degree $2n-1$, $nl(f_1) > 2^{2n-1} - \big(\frac{\ln 2}{\pi} n + 0.42\big)2^n - 1$.

There is a little more complex lower bound on $nl(f_2)$, first given in [1068] and later improved in [1108]; it is slightly smaller than for $nl(f_1)$. Function u does exist, see [1076, 1149].
The proof of optimal AI is obtained up to a conjecture similar to that of Tu-Deng, but slightly different, which has been finally proved in [374]. The same gap between the bound on the nonlinearity of f_2 and its actual values is observed

when computing them up to $n = 19$, see [1068]. The nonlinearities of f_2 and f are similar when they are taken with the same numbers of variables; in some cases $nl(f)$ is better and in some cases $nl(f_2)$ is better. The good behavior of f_2 with respect to FAA has been shown mathematically in [794].

In [789, 790] is introduced a larger class of functions achieving optimal algebraic immunity and almost perfect immunity to fast algebraic attacks. The exact nonlinearity of some functions of this larger class is good (slightly smaller than that of Carlet-Feng function), and some functions of this family have a slightly larger nonlinearity than those of [1068] with the same numbers of variables. The class of [789, 790] also contains a class presented in [644] whose resistance to fast algebraic attacks is also studied in [794, 789, 790] without that a positive answer be clearly obtained. The class of [1068] is modified in [1069] to ensure first-order resiliency.

Other constructions

Constructions, that we shall not detail, are given in [761, 777, 997] and other papers, as well as constructions in [587, 1172, 1124, 796] based on the decomposition of the multiplicative group of $\mathbb{F}_{2^n}^*$ corresponding to what we called *polar representation* at page 191 or more general multiplicative decompositions.

In [776] is proposed a new method, based on deriving new properties of minimal codewords of the punctured *Reed-Muller code* $RM^*(\lfloor \frac{n-1}{2} \rfloor, n)$. Recall that we say that a vector $(a_0, \dots, a_{N-1}) \in \mathbb{F}_2^N$ is *covered* by a vector $(c_0, \dots, c_{N-1}) \in \mathbb{F}_2^N$ if for every $i = 0, \dots, N-1$, we have $c_i = 0 \Rightarrow a_i = 0$, and that the codewords of *cyclic codes* are represented by polynomials (see page 27).

Proposition 142 *[776] Let n be an integer, $\alpha \in \mathbb{F}_{2^n}$ be a primitive element, and let f be the n-variable Boolean function with supp$(f) = \{\alpha^{m_0}, \dots, \alpha^{m_s}\}$, where $m_0 = 0$ and $m_0 < \cdots < m_s < 2^n - 1$. Then $f \oplus 1$ has no annihilator with algebraic degree less than $\lceil \frac{n}{2} \rceil$ if and only if there is no nonzero even weight codeword of the cyclic code $RM^*(\lfloor \frac{n-1}{2} \rfloor, n)$ covered by $c(x) = 1 + x^{m_1} + \cdots + x^{m_s}$.*

This result allows generalizing the function of Theorem 23 for any n, and leads for n odd, thanks to Proposition 135, to large classes of new functions with optimal algebraic immunity and good behavior against fast algebraic attacks, and high nonlinearity.

9.1.4 Secondary constructions of algebraic immune functions

Algebraic immunity and *direct sum*

For any positive integers n, m, any n-variable function f and any m-variable function g depending on disjoint sets of variables, denoting $r = \max(d_{alg}(f), d_{alg}(g))$, we have:

$$\max(AI(f), AI(g)) \leq AI(f \oplus g) \leq \min(AI(f) + AI(g), r). \tag{9.14}$$

Indeed, for some $\epsilon, \eta \in \mathbb{F}_2$, let h be a nonzero annihilator of algebraic degree $AI(f)$ of $f \oplus \epsilon$ and k a nonzero annihilator of algebraic degree $AI(g)$ of $g \oplus \eta$, then the product of h and k is a nonzero[11] annihilator of algebraic degree at most $AI(f) + AI(g)$ of $f \oplus g \oplus \epsilon \oplus \eta$, and we know also that $AI(f \oplus g) \leq d_{alg}(f \oplus g)$. This proves the right-hand side inequality. And if h is a nonzero annihilator of the $(n+m)$-variable function $f \oplus g$, then at least one of its restrictions obtained by fixing x (resp. y) in $f(x) \oplus g(y)$ is nonzero; this proves the left-hand side inequality.

Remark When the sum is not direct, the inequality $AI(f \oplus g) \leq AI(f) + AI(g)$ can be false [227]: let h be an n-variable Boolean function and let l be an n-variable nonzero linear function, then the functions $f = hl$ and $g = h(l \oplus 1)$ have algebraic immunities at most 1, since $f(l \oplus 1) = gl = 0$, and their sum equals h. If $AI(h) > 2$, we obtain a counter-example. □

Of course, the double inequality of (9.14) generalizes to the direct sum of more than two functions. We have also $FAI(f \oplus g) \geq \max(FAI(f), FAI(g))$ and $FAC(f \oplus g) \geq \max(FAC(f), FAC(g))$. These inequalities are not valid if the sum of f and g is not direct.
The algebraic immunity of direct sums of monomials is studied in Section 10.3, see Relation (10.6) at page 395. The upper bound in (9.14) is tight. It is shown in [305] that the upper bound is achieved with equality when the function with the lower algebraic immunity (in a broad sense) is non-constant and the other function f and its complement $f \oplus 1$ have different nonzero annihilator minimum degrees (this is applied in particular to determine the algebraic immunity of the direct sum of a threshold function, see page 390, and affine functions). Another example where the upper bound is achieved with equality is with the direct sum g of an n-variable function f and of a monomial m of degree $AI(f)+1$; as shown in [279], this gives indeed a function of algebraic immunity $AI(f)+1$ because the restriction h_1 to $\mathbb{F}_2^n \times \{0_{AI(f)+1}\}$ of a nonzero degree at most $AI(f)$ annihilator h of g is an annihilator of f, which then either has algebraic degree $AI(f)$ or is null, and in the former case, $gh = 0$ is impossible because mh_1 has degree $2AI(f)+1$ while $m(h_1 \oplus h)$ has degree at most $2AI(f)$ (since each monomial of $h + h_1$ has at least one coordinate in common with m), and in the latter case, fh cannot contain multiples of m and then cannot equal mh since $d_{alg}(h) \leq AI(f) < d_{alg}(m)$. We shall see at page 395 with triangular functions an example of application.
Note that the upper bound in (9.14) shows that the direct sum of two functions can have optimal algebraic immunity only if each has optimal algebraic immunity, except when both are in odd dimension (an example of a direct sum with maximal algebraic immunity of two functions not both having optimal algebraic immunity is function $x_1 \oplus x_2x_3 \oplus x_4x_5x_6$, of algebraic immunity 3 in 6 variables, which is the direct sum of $x_1 \oplus x_2x_3$, of algebraic immunity 2 in 3 variables, and

[11] Thanks to the fact that h and k depend on disjoint sets of variables.

of $x_4x_5x_6$, of algebraic immunity 1 in 3 variables).
The lower bound of (9.14) is also tight. An example where the lower bound is an equality is with two functions $f(x)$ and $g(y)$ whose algebraic immunities equal their algebraic degrees, since we have then $\max(d_{alg}(f), d_{alg}(g)) = \max(AI(f), AI(g)) \leq AI(f \oplus g) \leq d_{alg}(f \oplus g) = \max(d_{alg}(f), d_{alg}(g))$. In [305] is observed that if $d_{alg}(g) > 0$ and if (say) $\max(AI(f), AI(g)) = AI(f)$ and if $f \oplus 1$ has no nonzero annihilator of algebraic degree $AI(f)$, then the lower bound cannot be an equality.

Algebraic immunity and Siegenthaler's construction

Proposition 143 *[261] Let f, g be two n-variable Boolean functions with $AI(f) = d_1$ and $AI(g) = d_2$. Let $h = (1 \oplus x_{n+1})f \oplus x_{n+1}g \in \mathcal{BF}_{n+1}$. Then:*

1. *If $d_1 \neq d_2$ then $AI(h) = \min\{d_1, d_2\} + 1$.*
2. *If $d_1 = d_2 = d$, then $d \leq AI(h) \leq d + 1$, and $AI(h) = d$ if and only if there exists $f_1, g_1 \in \mathcal{BF}_n$ of algebraic degree d such that $\{ff_1 = 0, gg_1 = 0\}$ or $\{(1 \oplus f)f_1 = 0, (1 \oplus g)g_1 = 0\}$ and $d_{alg}(f_1 \oplus g_1) \leq d - 1$.*

Proof. 1. If f has an algebraic degree d_1 nonzero annihilator f_1, and g has an algebraic degree d_2 nonzero annihilator g_1, then we have $(1 \oplus x_{n+1})f_1h = 0$ and $x_{n+1}g_1h = 0$, which proves, after addressing similarly the cases where f_1 is an annihilator of $f \oplus 1$ and/or g_1 is an annihilator of $g \oplus 1$, that $AI(h) \leq \min\{AI(f), AI(g)\} + 1$.
Let $p = (1 \oplus x_{n+1})p_1 \oplus x_{n+1}p_2$ be a lowest algebraic degree nonzero annihilator of h. We have $hp = (1 \oplus x_{n+1})fp_1 \oplus x_{n+1}gp_2 = 0$. So $fp_1 = 0$ and $gp_2 = 0$. Similarly, if p is an annihilator of $h \oplus 1$, then $(1 \oplus f)p_1 = 0$ and $(1 \oplus g)p_2 = 0$.
Now there can be three cases in both scenarios:
(i) p_1 is zero and p_2 is non zero, then $d_{alg}(p_2) \geq d_2$ which gives $d_{alg}(p) \geq d_2 + 1$.
(ii) p_2 is zero and p_1 is non zero, then $d_{alg}(p_1) \geq d_1$ which gives $d_{alg}(p) \geq d_1 + 1$.
(iii) both p_1, p_2 are non zero, then $d_{alg}(p_1) \geq d_1$ and $d_{alg}(p_2) \geq d_2$, which gives $d_{alg}(p) \geq \max\{d_1, d_2\} + 1$, when $d_1 \neq d_2$ and $d_{alg}(p) \geq d$, when $d_1 = d_2 = d$.
So for $d_1 \neq d_2$ we get $AI(h) \geq \min\{d_1, d_2\} + 1$.
2. According to the observations above, we have $d \leq AI(h) \leq d + 1$. And $AI(h)$ equals d if and only if we are in case (iii) and the degree d terms of p_1 and p_2 are the same. □
Corollaries are given in [261] and more complex constructions are studied in [407].

9.1.5 Another direction of research of Boolean functions suitable for stream ciphers

All the functions described in Subsection 9.1.3 are of optimal algebraic immunity and the best ones have good other parameters, given their number of variables. They should be taken with a number of variables large enough for ensuring sufficient resistance to all attacks but also small enough for ensuring good speed.

An alternative method is to find functions with good but not optimal parameters, which would be fastly enough computable for being used with larger numbers of variables, so as to ensure same (and possibly better) resistance to attacks and also same and possibly better speed. The main example of this kind is with the Boolean function (mentioned by Knuth in Vol. 4 of "The Art of Computer Programming") called *hidden weight bit function* (*HWBF*). The principle of this function is as follows: we compute from the input $x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$ a value, say $\phi(x)$ belonging to $\{1, \dots, n\}$, and the output of the function is the value of the coordinate of index $\phi(x)$:

$$f(x_1, \dots, x_n) = x_{\phi(x)}, \text{ where } \phi : \mathbb{F}_2^n \mapsto \{1, \dots, n\}.$$

If the computation of $\phi(x)$ is fast then that of the Boolean function is fast. In the case of HWBF, $\phi(x)$ equals $w_H(x)$ if $x \neq 0_n$ and $\phi(0_n)$ equals any integer between 1 and n (the value of $f(0_n)$ being 0 for any choice). It is proved in [1105] that the function is then balanced and has algebraic degree $n-1$ (optimal) for $n \geq 3$ and that its algebraic immunity is at least $\lfloor \frac{n}{3} \rfloor + 1$, which is quite good since the function can be taken in many more variables than the function of Theorem 23 for instance. But the nonlinearity equals $2^{n-1} - 2\binom{n-2}{\lceil \frac{n-2}{2} \rceil}$ (which is the same as that of the majority function for n odd and worse for n even); this gives a too large bias of the nonlinearity with respect to 2^{n-1}, that is, $\epsilon = \frac{2^{n-1} - nl(f)}{2^n}$; the complexity of the fast correlation attack is then too small, see page 98. The too large value of ϵ is here not compensated by the number of variables, but as shown in [1105], it can be reduced by making a direct sum with a function with large nonlinearity (however, the direct sum represents some risk of attacks). Nevertheless, more functions of this kind need to be investigated. Some attempts have been made but with no significant gain.

9.1.6 An additional condition modifying the study of Boolean functions for stream ciphers

As recalled in [324], a stronger condition than balancedness is necessary in the filter model, if we wish to avoid additionally those attacks which are able, for some choice of the tapping sequence (*i.e.* of the positions inside the LFSR where are taken the inputs to the filter function), to distinguish the *keystream* $(s_i)_{i \in \mathbb{N}}$ output by the pseudorandom generator from a random sequence, by the observation of the distribution of a vectorial sequence of the form $(s_{i+j_1}, \cdots, s_{i+j_n})$, see page 109. We have seen that, for avoiding such attacks, the filter function must have one of the two equivalent forms $x_1 \oplus f(x_2, \cdots, x_n)$ and $f(x_1, \cdots, x_{n-1}) \oplus x_n$ [189, 545, 1044]. Studying if a function of the desired form $f(x_1, \cdots, x_{n-1}) \oplus x_n$ (say) satisfies the criteria listed above is not equivalent to the same study for f (taking a function in $n-1$ variables providing the best trade-off between all criteria and adding the extra variable x_n in order to obtain the desired form gives an algebraic immunity which can be either equal to that of the original function or larger by 1, and it results in functions which no longer ensure the best possible algebraic degree). The constructions in Subsection 9.1.3 have been modified in [324], in order to achieve inside this desired form the best possible values.

Constructions of 1-resilient algebraic immune functions have also been found, but only in even dimension[12], see *e.g.* [261, 1055, 1071, 1076, 1092, 1110, 1115], but many have not good nonlinearity and/or bad resistance to FAA (some because their $\frac{n}{2}$-th order nonlinearity is low) and the behavior of the others may not be optimal.

9.2 Algebraic immune vectorial functions

We have seen at page 148 that algebraic attacks concern also vectorial functions used in stream ciphers and in block ciphers. As far as we know, only standard algebraic attacks have been considered in the literature for stream ciphers using vectorial functions (whose PRG output several bits at each clock-cycle) and fast algebraic attacks do not have reality for block ciphers. Different related notions of *algebraic immunity* exist for vectorial Boolean functions, according to whether these functions are used as multi-output filters in stream ciphers or as S-boxes in block ciphers. They have been studied in [29, 32, 235]. We first give the definition of the algebraic immunity of a set:

Definition 74 *We call annihilator of a subset E of $\mathbb{F}_2^n$ any n-variable Boolean function vanishing on E. We call algebraic immunity of E, and we denote by $AI(E)$, the minimum algebraic degree of all non-zero annihilators of E.*

The algebraic immunity of an n-variable Boolean function f is then equal to $\min(AI(f^{-1}(0)), AI(f^{-1}(1)))$, according to Definition 23, page 111.

The first generalization of algebraic immunity to S-boxes, introduced in [29], is its direct extension:

Definition 75 *The basic algebraic immunity $AI(F)$ of any (n, m)-*function *F is the minimum algebraic immunity of all the pre-images $F^{-1}(z)$ of the elements z of $\mathbb{F}_2^m$ by F.*

The basic algebraic immunity is invariant under affine equivalence. Note that $AI(F)$ also equals the minimum algebraic immunity of the *indicator*s of these pre-images $F^{-1}(z)$ since, the algebraic immunity being a non-decreasing function over sets, we have for every $z \in \mathbb{F}_2^m$:

$$AI(\mathbb{F}_2^n \setminus F^{-1}(z)) \geq AI(F^{-1}(z')), \forall z \neq z'.$$

$AI(F)$ quantifies the resistance to standard algebraic attacks of the stream ciphers using F as a combiner or as a filter function. Indeed, the attacker can

[12] One class of functions in odd dimension has first-order correlation immunity: the concatenation of the majority function f in n even variables and of $f(x + 1_n)$.

combine the output bits of the generator in any way; in other words, he can try a standard algebraic attacks on any stream cipher using Boolean function $h \circ F$ as filter or combiner, where h is any non-constant m-variable Boolean function, and such attack is the most efficient when h has Hamming weight 1 (again because the algebraic immunity is a non-decreasing function over sets).
A second notion of algebraic immunity of vectorial functions [392, 368, 29, 235], more relevant for S-boxes in block ciphers, has been called the graph algebraic immunity.

Definition 76 *The graph algebraic immunity $AI_{gr}(F)$ of any (n, m)-function F is the algebraic immunity of the graph $\{(x, F(x));\ x \in \mathbb{F}_2^n\}$ of the S-box.*

By definition, the graph algebraic immunity is invariant under *CCZ equivalence*.

A third notion, introduced in [235] and called the component algebraic immunity, seems also natural:

Definition 77 *The component algebraic immunity $AI_{comp}(F)$ of any (n, m)-function F is the minimal algebraic immunity of the* component functions $v \cdot F$ *($v \neq 0_m$ in $\mathbb{F}_2^m$) of the S-box.*

The interest of AI_{comp} is that it has a sense for both cases of stream ciphers and block ciphers, and it helps studying the two other notions.

9.2.1 Known bounds on algebraic immunities

Note that we have $AI(F) \leq AI_{comp}(F)$, since $AI_{comp}(F)$ equals $AI(F^{-1}(H))$ for some affine hyperplane H of $\mathbb{F}_2^m$, and since AI is non-decreasing; we also have $AI_{gr}(F) \leq AI_{comp}(F) + 1$, since if g is a nonzero annihilator of $v \cdot F$, $v \neq 0_m$, then the product $h(x, y) = g(x)\,(v \cdot y)$ is a nonzero annihilator of the graph of F and if g is a nonzero annihilator of $v \cdot F \oplus 1$ then $h(x, y) = g(x)\,(v \cdot y) \oplus g(x)$ is a nonzero annihilator of the graph of F. A few observations are deduced in [235].

It has been observed in [29] that, for any (n, m)-function F, we have:

$$AI(F) \leq AI_{gr}(F) \leq AI(F) + m.$$

Indeed, given any minimum degree nonzero annihilator $g(x, y)$ of the graph of F, there exists y such that the function $x \mapsto g(x, y)$ is not the zero function, and this function is a nonzero annihilator of $F^{-1}(y)$, which proves the left-hand side inequality. And given a minimum degree nonzero annihilator g of $F^{-1}(z)$ where z is such that $AI(F^{-1}(z)) = AI(F)$, the function $g(x) \prod_{j=1}^m (y_j \oplus z_j \oplus 1)$ is an annihilator of algebraic degree $AI(F) + m$ of the graph of F; this proves the right-hand side inequality.

Denoting by $d_{n,m}$ the smallest integer such that $\sum_{i=0}^{d_{n,m}} \binom{n}{i} > 2^{n-m}$, we have

$$AI(F) \leq d_{n,m} \leq d_{n,m-1} \leq \cdots \leq d_{n,1} = \left\lceil \frac{n}{2} \right\rceil.$$

Indeed, there is at least one z such that $|F^{-1}(z)| \leq 2^{n-m}$ and according to Relation (9.5), page 361, with $f = 1_{|F^{-1}(z)|}$, we have $\sum_{i=0}^{AI(f)-1} \binom{n}{i} \leq 2^{n-m}$ and therefore $AI(f)-1 < d_{n,m}$. Since $AI(F) \leq AI(f)$, this proves the first inequality, originally observed in [29], and proved tight in [500] thanks to the function that we shall introduce at page 381; the other inequalities are straightforward.
We give in Table 9.1 below taken from [235] the values of $d_{n,m}$, for n ranging from 5 to 20 and for m ranging from 1 to 17.

$n \setminus m$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
5	3	2	1	1	1	0											
6	3	2	2	1	1	1	0										
7	4	3	2	2	1	1	1	0									
8	4	3	2	2	1	1	1	1	0								
9	5	3	3	2	2	1	1	1	1	0							
10	5	4	3	3	2	2	1	1	1	1	0						
11	6	4	4	3	2	2	2	1	1	1	1	0					
12	6	5	4	3	3	2	2	2	1	1	1	1	0				
13	7	5	4	4	3	3	2	2	2	1	1	1	1	0			
14	7	6	5	4	4	3	3	2	2	2	1	1	1	1	0		
15	8	6	5	5	4	3	3	3	2	2	2	1	1	1	1	0	
16	8	7	6	5	4	4	3	3	2	2	2	1	1	1	1	1	0
17	9	7	6	5	5	4	4	3	3	2	2	2	1	1	1	1	1
18	9	8	7	6	5	5	4	4	3	3	2	2	2	1	1	1	1
19	10	8	7	6	5	5	4	4	3	3	3	2	2	2	1	1	1
20	10	8	7	7	6	5	5	4	4	3	3	3	2	2	2	1	1

Table 9.1 THE VALUES OF $d_{n,m}$

Similarly, as also proved in [29], denoting by $D_{n,m}$ the smallest integer such that $\sum_{i=0}^{D_{n,m}} \binom{n+m}{i} > 2^n$, we have

$$AI_{gr}(F) \leq D_{n,m} \leq D_{n,m-1} \leq \cdots \leq D_{n,1} = \left\lceil \frac{n+1}{2} \right\rceil.$$

Note that we have $D_{n,m} = d_{n+m,m}$. In [235] is given the table of the values of $D_{n,m}$, for n ranging from 5 to 20 and for m ranging from 1 to 17.

9.2.2 Bounds on the numbers $d_{n,m}$ and $D_{n,m}$

We have $d_{n,m} \leq n-m$ and $D_{n,m} \leq n$, since $\sum_{i=0}^{n-m} \binom{n}{i} > \sum_{i=0}^{n-m} \binom{n-m}{i} = 2^{n-m}$. The bound $d_{n,m} \leq \left\lceil \frac{n}{2} \right\rceil$ is stronger than $d_{n,m} \leq n-m$ if and only if $m < \frac{n-1}{2}$, and the bound $D_{n,m} \leq \left\lceil \frac{n+1}{2} \right\rceil$ is stronger than $D_{n,m} \leq n$ if and only if $n \geq 3$. The inequality $D_{n,m} \leq \left\lceil \frac{n+1}{2} \right\rceil$ gives $d_{n+m,m} \leq \left\lceil \frac{n+1}{2} \right\rceil$ and therefore, for $n > m$: $d_{n,m} \leq \left\lceil \frac{n-m+1}{2} \right\rceil$, which is stronger than $d_{n,m} \leq \left\lceil \frac{n}{2} \right\rceil$ and than $d_{n,m} \leq n-m$. We know from [809, page 310] that, for any positive number $\lambda \leq 1/2$ and every positive integer n, we have: $\sum_{i=0}^{\lceil \lambda n \rceil} \binom{n}{i} \geq \frac{2^{nH_2(\lambda)}}{\sqrt{8\lambda n(1-\lambda)}}$. This bound implies, for every m:

$$d_{n,m} \leq \min \left\{ \lceil \lambda n \rceil \;/\; n\, H_2(\lambda) - \frac{1}{2}\,(3 + \log_2 n + \log_2 \lambda + \log_2 (1-\lambda)) > n - m \right\}$$

(note that the term in $\frac{1}{2}\,(3 + \log_2 n + \log_2 \lambda + \log_2(1-\lambda))$ is asymptotically negligeable with respect to n). Hence:

Proposition 144 *[235] Let $\lambda \leq 1/2$ be a positive real number. For every positive integers n and m such that:*

$$m > n\,(1 - H_2(\lambda)) + \frac{1}{2}\,(3 + \log_2 n + \log_2 \lambda + \log_2(1-\lambda))\,,$$

where $H_2(x) = -x \log_2(x) - (1-x)\log_2(1-x)$, we have: $d_{n,m} \leq \lceil \lambda n \rceil$.
For any two positive integers n and m such that:

$$m\, H_2(\lambda) > n\,(1 - H_2(\lambda)) + \frac{1}{2}\,(3 + \log_2(n+m) + \log_2 \lambda + \log_2(1-\lambda))\,,$$

we have $D_{n,m} \leq \lceil \lambda(n+m) \rceil$.

We give in Table 9.2 the values of $1 - H_2(\lambda)$ for λ ranging in $\{.1, .2, .3, .4\}$.

λ	.1	.2	.3	.4
$1 - H_2(\lambda)$	.53	.28	.19	.03

Table 9.2 THE VALUES OF $1 - H_2(\lambda)$

These general bounds can be improved for specific values of m.

9.2.3 Consequences on the number of output bits and on the tightness of the bounds

$AI(F)$ can be larger than a number k, only if $m \leq n\,(1 - H_2(k/n)) + \frac{1}{2}\,(3 + \log_2(k(1-k/n)))$, according to Proposition 144. Hence, vectorial (n,m)-functions can be used as combiners or filters only if m is small enough compared to n.

The bound $AI(F) \leq AI_{gr}(F)$ is tight. Indeed:

Proposition 145 *[235] Let F be an (n,m)-function such that, for every $b \in \mathbb{F}_2^m$, there exists $a \in \mathbb{F}_2^n$ such that the ordered pair (a,b) is a linear structure of F (*i.e. *$D_a F$ equals constant function b). Then $AI(F) = AI_{gr}(F)$.*

Proof. Let $(e_1, \ldots, e_m)$ be the canonical basis of $\mathbb{F}_2^m$ and for every $i \leq m$, let (α_i, e_i) be a linear structure of F. Let z be such that $AI(F) = AI(F^{-1}(z))$. We can assume, without loss of generality up to translation, that $z = 0_m$. Let $g(x)$ be a nonzero annihilator of algebraic degree $AI(F)$ of $F^{-1}(0_m)$. Then, let: $h(x,y) = \sum_{b \in \mathbb{F}_2^m} \left(\prod_{i=1}^m (y_i \oplus b_i \oplus 1)\right)\; g\left(x + \sum_{i=1}^m b_i \alpha_i\right)$. Note that $\prod_{i=1}^m (y_i \oplus b_i \oplus 1)$ equals 1 if and only if $y = b$; hence, for every $x \in \mathbb{F}_2^n$, denoting by I

the support of the vector $F(x)$, we have $h(x, F(x)) = g(x + \sum_{i\in I} \alpha_i)$. Since $F(x + \sum_{i\in I} \alpha_i) = F(x) + \sum_{i\in I} e_i = 0_m$, we have $x + \sum_{i\in I} \alpha_i \in F^{-1}(0_m)$ and therefore $h(x, F(x)) = 0$ and h is an annihilator of the graph of F.
Moreover, expanding $h(x, y)$ in the form $\sum_{J\subseteq\{1,\cdots,m\}} \left(\prod_{i\in J} y_i\right) \phi_J(x)$, for every vector $b \in \mathbb{F}_2^m$, denoting by I the support of b, we have: $\prod_{i=1}^{m}(y_i \oplus b_i \oplus 1) = \sum_{\substack{J\subseteq\{1,\cdots,m\}\ /\\ I\subseteq J}} \left(\prod_{i\in J} y_i\right)$, and then, $\phi_J(x) = \sum_{b\in\mathbb{F}_2^m\ /\ supp(b)\subseteq J} g(x+\sum_{i=1}^{m} b_i\alpha_i)$ is a derivative of g of order $|J|$ and has an algebraic degree which is at most $d^\circ g - |J|$. Hence, we have $d^\circ h \leq d^\circ g$ (and in fact $d^\circ h = d^\circ g$, since the part $\phi_\emptyset$ independent of y in $h(x, y)$ equals g(x)). This implies $AI_{gr}(F) \leq AI(F)$, and since we know that $AI(F) \leq AI_{gr}(F)$, then $AI(F) = AI_{gr}(F)$. □

As seen above, we have two upper bounds on the graph algebraic immunity: $AI_{gr}(F) \leq AI(F) + m$ and $AI_{gr}(F) \leq D_{n,m}$. It is shown in [235] that the latter implies that the former cannot be tight when $AI(F) > 0$, $m \geq n/2$ and $n \geq 3$, nor when $AI(F) > 0$, $m \geq n/3$ and $n \geq 25$, and it is deduced that for $n \geq 2$ and $m \geq n/3$ we have $d_{n,m} \leq m$ and for $n \geq 20$ and $m \geq n/4$, we have also $d_{n,m} \leq m$.

The vectorial functions studied in [500, 273] achieve the bound $AI(F) \leq d_{n,m}$ with equality, which shows that this bound is tight for every n, m such that $1 \leq m < n$. It is not known whether the bound $AI_{gr}(F) \leq D_{n,m}$ is tight too. It is shown in [29] that it is tight for $n \leq 14$.

9.2.4 Nonlinearity and higher-order nonlinearity

Lower bounds on the nonlinearity

As proved in [233], the lower bound $nl(f) \geq 2\sum_{i=0}^{AI(f)-2} \binom{n-1}{i}$ due to Lobanov on the nonlinearity of Boolean functions generalizes to (n,m)-*functions* as follows:

$$nl(F) \geq 2^m \sum_{i=0}^{AI(F)-2} \binom{n-1}{i},$$

where $AI(F)$ is the basic algebraic immunity of F. But we have seen that for large m, $AI(F) - 2$ is negative. So a bound involving $AI_{gr}(F)$ is also needed. Applying Lobanov's bound to the component functions of F, we obtain

$$nl(F) \geq 2 \sum_{i=0}^{AI_{comp}(F)-2} \binom{n-1}{i}.$$

The inequality $AI_{comp}(F) \geq AI_{gr}(F) - 1$ implies then

$$nl(F) \geq 2 \sum_{i=0}^{AI_{gr}(F)-3} \binom{n-1}{i}.$$

Lower bounds on the higher-order nonlinearities

For every positive integer r, the r-th order nonlinearity of a vectorial function F is the minimum r-th order nonlinearity of its component functions (recall that the r-th order nonlinearity of a Boolean function equals its minimum Hamming distance to functions of algebraic degree at most r). As proved in [233], the bounds known for Boolean functions generalize to (n, m)-functions as follows:

$$nl_r(F) \geq 2^m \sum_{i=0}^{AI(F)-r-1} \binom{n-r}{i}$$

and

$$nl_r(F) \geq 2^{m-1} \sum_{i=0}^{AI(F)-r-1} \binom{n}{i} + 2^{m-1} \sum_{i=AI(F)-2r}^{AI(F)-r-1} \binom{n-r}{i}$$

(the first of these two bounds can be slightly improved as for Boolean functions).

Applying the bounds valid for Boolean functions to the component functions of F, we have also:

$$nl_r(F) \geq 2 \sum_{i=0}^{AI_{comp}(F)-r-1} \binom{n-r}{i}$$

and

$$nl_r(F) \geq \sum_{i=0}^{AI_{comp}(F)-r-1} \binom{n}{i} + \sum_{i=AI_{comp}(F)-2r}^{AI_{comp}(F)-r-1} \binom{n-r}{i}.$$

The inequality $AI_{comp}(F) \geq AI_{gr}(F) - 1$ implies then

$$nl_r(F) \geq 2 \sum_{i=0}^{AI_{gr}(F)-r-2} \binom{n-r}{i}$$

and

$$nl_r(F) \geq \sum_{i=0}^{AI_{gr}(F)-r-2} \binom{n}{i} + \sum_{i=AI_{gr}(F)-2r-1}^{AI_{gr}(F)-r-2} \binom{n-r}{i}.$$

9.2.5 Constructions of algebraic immune vectorial functions

Feng et al.'s class

In [274] is studied further the class introduced in [500]. We assume that $n \geq 2$ and $1 \leqslant m \leqslant n$. For any fixed integer s, $0 \leqslant s \leqslant 2^n - 2$, $\mathbb{F}_{2^n}$ is a disjoint union of the following 2^m subsets:

$$\begin{aligned} S_0 &= \{\alpha^l \mid s \leqslant l \leqslant s + 2^{n-m} - 2\} \cup \{0\} \\ S_j &= \{\alpha^l \mid s + 2^{n-m}j - 1 \leqslant l \leqslant s + 2^{n-m}(j+1) - 2\};\ 1 \leqslant j \leqslant 2^m - 1 \end{aligned} \tag{9.15}$$

where α is a primitive element. Each integer j, $0 \leqslant j \leqslant 2^m - 1$, has a 2-adic expansion

$$j = j_0 + j_1 2 + \cdots + j_{m-1} 2^{m-1} \qquad (j_0, \cdots, j_{m-1} \in \{0, 1\})$$

and corresponds to the vector $\bar{j} = (j_0, \cdots, j_{m-1}) \in \mathbb{F}_2^m$. For each integer i, $0 \leqslant i \leqslant m - 1$, we define the Boolean function $f_i : \mathbb{F}_{2^n} \to \mathbb{F}_2$ by

$$f_i(x) = \begin{cases} 1, & \text{if } x \in \bigcup_{\substack{0 \leqslant j \leqslant 2^m - 1 \\ j_i = 1}} S_j \\ 0, & \text{otherwise.} \end{cases} \tag{9.16}$$

Then for the (n, m)-function

$$F = (f_0, \cdots, f_{m-1}) : \mathbb{F}_{2^n} \to \mathbb{F}_2^m,$$

we have, for each $\bar{j} = (j_0, \cdots, j_{m-1}) \in \mathbb{F}_2^m$ and $j = \sum_{i=0}^{m-1} j_i 2^i$,

$$\begin{aligned} x \in F^{-1}(\bar{j}) \quad &\Leftrightarrow \quad f_i(x) = j_i \qquad (0 \leqslant i \leqslant m - 1) \\ &\Leftrightarrow \quad x \in \bigcap \{S_k \mid 0 \leqslant k \leqslant 2^m - 1, k_i = j_i\} \qquad (0 \leqslant i \leqslant m - 1) \\ &\Leftrightarrow \quad x \in S_j. \end{aligned}$$

Therefore the (n, m)-function F can be characterized by

$$F^{-1}(\bar{j}) = S_j \qquad (\text{for each } j, 0 \leqslant j \leqslant 2^m - 1) \tag{9.17}$$

It is proved in [274] by observation and calculation of the coefficients of $x^{2^n - 1}$ and $x^{2^n - 2}$ in the univariate representations of the coordinate functions that:

Proposition 146 *(1) For every $1 \leq m \leq n$, function F is balanced.*
(2) We have $d_{\mathrm{alg}}(F) = n - 1$.
(3) We have $d_{min}(F) = n - 1$ if and only if $\frac{\alpha}{1+\alpha}, (\frac{\alpha}{1+\alpha})^2, \ldots, (\frac{\alpha}{1+\alpha})^{2^{m-1}}$ are linearly independent over $\mathbb{F}_2$.

It is also proved in this same paper that the basic algebraic immunity of F is optimal:

Proposition 147 *For every n, m such that $1 \leq m \leq n$, we have $AI(F) = d_{n,m}$.*

The proof is very similar to the Boolean case, by application of the BCH bound.

A lower bound on the (hyper-)nonlinearity of F is also proved in [274] by the use of Gauss sums which allow transforming the expression of the Walsh transform and by bounding from above some trigonometric sums with integrals:

Proposition 148 $nl(F) \geq 2^{n-1} - \frac{2^{\frac{n}{2}+m}}{\pi} \ln(\frac{4(2^n-1)}{\pi}) - 1 \sim 2^{n-1} - \frac{\ln 2}{\pi} 2^{\frac{n}{2}+m} \cdot n.$

A class obtained through group decomposition

In [804] is constructed a class of balanced (n, m)-functions over $\mathbb{F}_{2^n}$ (n even), with $m \leq n/2$, and with high basic algebraic immunity and optimal algebraic degree, based on the decomposition of the multiplicative group of $\mathbb{F}_{2^n}^*$ corresponding to what we called *polar representation* at page 191.

10 Particular classes of Boolean functions

10.1 Symmetric functions

A function is called a *symmetric Boolean function* if it is invariant under the action of the symmetric group (*i*.e. if its output is invariant under permutation of its input bits). Its output depends then only on the Hamming weight of the input (and can be implemented with a number of gates linear in the number of input variables [1117], with a reduced amount of memory required for storing the function). So a Boolean function f is symmetric if and only if there exists a function f from $\{0, 1, \dots, n\}$ to $\mathbb{F}_2$ such that:

$$f(x) = \mathrm{f}(w_H(x)).$$

The vector $(\mathrm{f}(0), \dots, \mathrm{f}(n))$ is sometimes called the *simplified value vector* of f. Such functions are of some interest for cryptography, as they allow to implement in an efficient way nonlinear functions on large numbers of variables. Let us consider for example an LFSR filtered by a 63-variable symmetric function f, whose input is the content of an interval of 63 consecutive flip-flops of the LFSR. This device may be implemented with a cost similar to that of a 6-variable Boolean function, thanks to a 6 bit counter calculating the Hamming weight of the input to f (this counter is incremented if a 1 is shifted in the interval and decremented if a 1 is shifted out). However, the pseudorandom sequence obtained this way has a correlation with transitions (sums of consecutive bits), and a symmetric function should not take all its inputs in a full interval. In fact, it is not yet completely clarified whether the advantage of allowing many more variables and the cryptographic weaknesses these symmetric functions may introduce result in an advantage for the designer or for the attacker.

10.1.1 Representation

Let $r = 0, \dots, n$ and let $1_{E_{n,r}}$ be the Boolean function whose support is the set $E_{n,r}$ of all vectors of Hamming weight r in $\mathbb{F}_2^n$. Then, according to Relation (2.23), page 67, relating the values of the coefficients of the NNF to the values of the function, the coefficient of x^I in the *NNF* of $1_{E_{n,r}}$ equals

$(-1)^{|I|} \sum_{\substack{x\in\mathbb{F}_2^n;\ w_H(x)=r\\ supp(x)\subseteq I}} (-1)^{w_H(x)} = (-1)^{|I|-r}\binom{|I|}{r}$, and we have then:

$$1_{E_{n,r}}(x) = \sum_{I\subseteq\{1,\ldots,n\}} (-1)^{|I|-r}\binom{|I|}{r} x^I. \tag{10.1}$$

Any symmetric function f being equal to $\bigoplus_{r=0}^{n} \mathrm{f}(r)\, 1_{E_{n,r}}$, it equals $\sum_{r=0}^{n} \mathrm{f}(r)\, 1_{E_{n,r}}$, since the functions $1_{E_{n,r}}$ have disjoint supports. The coefficient of x^I in its NNF equals then $\sum_{r=0}^{n} \mathrm{f}(r)(-1)^{|I|-r}\left(\begin{array}{c}|I|\\ r\end{array}\right)$ and depends only on the size of I. Denoting

$$S_i(x) = \sum_{\substack{I\subseteq\{1,\ldots,n\}\\ |I|=i}} x^I = \binom{w_H(x)}{i} = \begin{cases} \frac{w_H(x)\,(w_H(x)-1)\ldots(w_H(x)-i+1)}{i!} & \text{if } w_H(x)\geq i\\ 0 \text{ otherwise,} \end{cases}$$

the NNF of f equals then

$$f(x) = \sum_{i=0}^{n} c_i\, S_i(x), \text{ where } c_i = \sum_{r=0}^{n} \mathrm{f}(r)(-1)^{i-r}\left(\begin{array}{c}i\\ r\end{array}\right). \tag{10.2}$$

According to Relation (10.2), we see by definition of f that this function coincides on $\{0,\ldots,n\}$ with the polynomial:

$$\mathrm{f}(z) = \sum_{i=0}^{n} c_i \binom{z}{i} = \sum_{i=0}^{n} c_i \frac{z\,(z-1)\ldots(z-i+1)}{i!},$$

of degree $\max\{i;\, c_i\neq 0\}$ (which is also the degree of the NNF of f). Note that since this degree is at most n, and the values taken by this polynomial at $n+1$ points are determined by the values of f, this polynomial representation is unique and can be obtained by the Lagrange interpolation formula.
Function $\sigma_i(x) = S_i(x)$ [mod 2] is the i-th elementary symmetric function:

$$\sigma_i(x) = \bigoplus_{1\leq j_1<\cdots<j_i\leq n} \prod_{k=1}^{i} x_{j_k}.$$

According to Lucas' theorem (see page 528 or [809, page 404]), $\sigma_i(x)$ equals 1 if and only if the binary expansion $\sum_{l=1}^{\lfloor\log_2 n\rfloor} i_l\, 2^{l-1}$ of i is covered by that of $w_H(x)$ (*i.e.* writing $w_H(x) = \sum_{l=1}^{\lfloor\log_2 n\rfloor} j_l\, 2^{l-1}$, we have $i_l\leq j_l$, $\forall l=1,\ldots,\lfloor\log_2 n\rfloor$; we write $i\preceq w_H(x)$). Note that this implies that $\sigma_i = \prod_{\substack{l\in\{1,\ldots,\lfloor\log_2 n\rfloor\}\\ i_l=1}} \sigma_{2^l}$. Reducing Relation (10.2) modulo 2, we deduce from Lucas' theorem again that the ANF of f equals:

$$[ANF]\ f(x) = \bigoplus_{i=0}^{n} \epsilon_i\, \sigma_i(x), \text{ where } \epsilon_i = c_i \text{ [mod 2]} = \sum_{r\preceq i} \mathrm{f}(r) \text{ [mod 2]}. \tag{10.3}$$

The algebraic degree of f equals $\max\{i;\, \epsilon_i = 1\}$ (in particular, in the case of $f = 1_{E_{n,r}}$, we have that ϵ_i equals 1 if and only if $r \preceq i$ and the algebraic degree equals $\max\{i \in \{r,\ldots,n\};\, r \preceq i\}$).
Using that the *binary Möbius transform* is involutive, or using that $\sigma_i(x) = 1$ if and only if $\binom{w_H(x)}{i}$ is odd and Lucas' theorem again, we deduce from (10.3) that $\mathrm{f}(j) = \bigoplus_{i \preceq j} \epsilon_i$. The vector $(\epsilon_0,\ldots,\epsilon_n)$ is sometimes called the *simplified ANF vector*, Relation (10.3) gives the expression of the simplified ANF vector by means of the simplified value vector and this relation gives the reverse expression.

According to the observations above, nonzero symmetric Boolean functions are, up to the addition of constant function 1, the component functions of the (n,n)-function $\Sigma(x)$ whose i-th coordinate function is the elementary symmetric function $\sigma_i(x)$. Note that, for every $x, y \in \mathbb{F}_2^n$, we have $w_H(x) = w_H(y)$ if and only if $\Sigma(x) = \Sigma(y)$, since the σ_i's generate by linear combinations all those symmetric Boolean functions null at input 0_n, and two vectors x, y have the same nonzero Hamming weight if and only if every symmetric Boolean function null at 0_n takes the same value at inputs x and y. This translation of an equality between the Hamming weights of two vectors x and y into the equality between the images of x and y by a vectorial function is nicely simple. We have then $w_H(x) = k$ for some non-negative k if and only if, for every $i = 1,\ldots,n$, we have $\sigma_i(x) \equiv \binom{k}{i}$ [mod 2]. We have also $w_H(x) \leq k$ if and only if $\sigma_i(x) = 0$ for all $i > k$ (this necessary condition is sufficient because $\sigma_{w_H(x)}(x) = 1$).

Note that a symmetric Boolean function f has algebraic degree 1 if and only if it equals $\bigoplus_{i=1}^n x_i$ or $\bigoplus_{i=1}^n x_i \oplus 1$, that is, if the binary function $\mathrm{f}(r)$ equals r [mod 2] or $r+1$ [mod 2], and that it is quadratic if and only if it equals $\bigoplus_{1\leq i<j\leq n} x_i x_j$ plus a symmetric function of algebraic degree at most 1, that is, if the function $\mathrm{f}(r)$ equals $\binom{r}{2}$ [mod 2] or $\binom{r}{2} + r$ [mod 2] or $\binom{r}{2} + 1$ [mod 2] or $\binom{r}{2} + r + 1$ [mod 2]. Hence, f has algebraic degree 1 if and only if f satisfies $\mathrm{f}(r+1) = \mathrm{f}(r) \oplus 1$ and it has degree 2 if and only if f satisfies $\mathrm{f}(r+2) = \mathrm{f}(r) \oplus 1$.
As observed in [205], the algebraic degree of a symmetric function f is at most $2^t - 1$, for some positive integer t such that $2^t < n$, if and only if the sequence $(\mathrm{f}(r))_{r\geq 0}$ is periodic with period 2^t (sufficiency is a direct consequence of (10.3) and necessity of the reverse relation). Here again, it is not clear whether this is more an advantage for the designer of a cryptosystem using such symmetric function f (since, to compute the image of a vector x by f, it is enough to compute the number of nonzero coordinates $x_1,\ldots,x_t$) or for the attacker.

10.1.2 Hamming weight

In [173] is given a closed formula for the correlation between any two symmetric Boolean functions (and in particular the weight of a symmetric function). In [532], von zur Gathen and Roche determined all balanced symmetric Boolean functions up to 128 variables. More recently has been proved in [530] that balanced symmetric Boolean functions of fixed algebraic degree $d > 1$ and sufficiently large number of variables are *trivial*. This term means that n is odd and the simplified value vector f is anti-symmetric with respect to the middle of $[0, \ldots, n]$, that is, $\mathrm{f}(n-i) = \mathrm{f}(i) \oplus 1, \forall i$. This same paper also shows (proving a conjecture by Cusick) the nonexistence of trivial balanced elementary symmetric Boolean functions except for $n = 2^{t+1}l - 1$ and $d = 2^t$, where t and l are any nonnegative integers.

10.1.3 Fourier-Hadamard and Walsh transforms

For every $a \in \mathbb{F}_2^n$ and $r \in \{0, \ldots, n\}$, denoting by ℓ the Hamming weight of a, we have $\widehat{1_{E_{n,r}}}(a) = \sum_{x \in \mathbb{F}_2^n;\, w_H(x)=r} (-1)^{a \cdot x} = \sum_{j=0}^{n} (-1)^j \binom{\ell}{j}\binom{n-\ell}{r-j}$, denoting by j the size of $supp(a) \cap supp(x)$. The polynomials $K_{n,r}(X) = \sum_{j=0}^{n} (-1)^j \binom{X}{j}\binom{n-X}{r-j}$ are called *Krawtchouk polynomials*. They are characterized by their generating series:

$$\sum_{r=0}^{n} K_{n,r}(\ell) z^r = (1-z)^{\ell}(1+z)^{n-\ell}$$

and have nice resulting properties (see *e.g.* [809, 328]). By $\mathbb{R}$-linearity, we deduce that the value at a of the Fourier-Hadamard transform of any symmetric function $\sum_{r=0}^{n} \mathrm{f}(r)\, 1_{E_{n,r}}$ equals $\sum_{r=0}^{n} \mathrm{f}(r)\, K_{n,r}(w_H(a))$.
From the Fourier-Hadamard transform, we can deduce the Walsh transform thanks to Relation (2.32), page 74.
In [334] are studied further the exponential sums of symmetric Boolean functions and their asymptotic behavior.

10.1.4 Nonlinearity

If n is even, then the restriction of every symmetric function f on $\mathbb{F}_2^n$ to the $\frac{n}{2}$-dimensional flat $A = \{(x_1, \ldots, x_n) \in \mathbb{F}_2^n\ ;\ x_{i+\frac{n}{2}} = x_i \oplus 1, \forall i \leq \frac{n}{2}\}$ is constant, since all the elements of A have the same Hamming weight $\frac{n}{2}$. Thus, f is $\frac{n}{2}$-normal (see Definition 28, page 126). But Relation (3.15), page 128 does not improve upon the covering radius bound (3.2), page 99. The symmetric functions which achieve this bound, *i.e.* which are bent, have been first characterized by Savicky in [1019]: the bent symmetric functions are the four symmetric functions of algebraic degree 2 already described above: $f_1(x) = \bigoplus_{1 \leq i < j \leq n} x_i x_j$, $f_2(x) = f_1(x) \oplus 1$, $f_3(x) = f_1(x) \oplus x_1 \oplus \cdots \oplus x_n$ and $f_4(x) = f_3(x) \oplus 1$. A stronger result can be proved in a very simple way:

Proposition 149 *[566] For every positive even n, the $PC(2)$ n-variable symmetric functions are the functions f_1, f_2, f_3 and f_4 above.*

Proof. Let f be any $PC(2)$ n-variable symmetric function and let $1 \leq i < j \leq n$. Let us denote by x' the vector: $x' = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$.

Since $f(x)$ is symmetric, it has the form $x_i\, x_j g(x') \oplus (x_i \oplus x_j)\, h(x') \oplus k(x')$. Let us denote by $e_{i,j}$ the vector of Hamming weight 2 whose nonzero coordinates stand at positions i and j. The derivative $D_{e_{i,j}} f$ equals $(x_i \oplus x_j \oplus 1) g(x')$ and is balanced, by hypothesis. Then g must be equal to the constant function 1 (indeed if $g(x') = 1$ for some x', then $(x_i \oplus x_j \oplus 1) g(x')$ equals 1 for half of the inputs (x_i, x_j) and otherwise it equals 1 for none). Hence, the degree at least 2 part of the ANF of f equals $\bigoplus_{1 \le i < j \le n} x_i x_j$. □

Results on the propagation criterion for symmetric functions are in [205].

If n is odd, then the restriction of any symmetric function f to the $\frac{n+1}{2}$-dimensional flat $A = \{(x_1, \ldots, x_n) \in \mathbb{F}_2^n \; ; \; x_{i+\frac{n-1}{2}} = x_i \oplus 1, \forall i \le \frac{n}{2}\}$ is affine, since the Hamming weight function w_H is constant on the hyperplane of A of equation $x_n = 0$ and on its complement. Thus, f is $\frac{n+1}{2}$-weakly-normal. According to Relation (3.15), page 128, this implies that its nonlinearity is upper bounded by $2^{n-1} - 2^{\frac{n-1}{2}}$. It also allows showing that the only symmetric functions achieving this bound with equality are the same as the 4 functions f_1, f_2, f_3 and f_4 above, but with n odd (this has been first proved by Maitra and Sarkar [818], in a more complex way). Indeed:

Proposition 150 *[224] Let n be any positive integer and let f be any symmetric function on $\mathbb{F}_2^n$. Let l be any integer satisfying $0 < l \le \frac{n}{2}$. Denote by h_l the symmetric Boolean function on $n - 2l$ variables defined by $h_l(y_1, \ldots, y_{n-2l}) = f(x_1, \ldots, x_l, x_1 \oplus 1, \ldots, x_l \oplus 1, y_1, \ldots, y_{n-2l})$, where the values of $x_1, \ldots, x_l$ are arbitrary (equivalently, h_l can be defined by $\mathrm{h}_l(r) = \mathrm{f}(r + l)$, for every $0 \le r \le n - 2l$). Then $nl(f) \le 2^{n-1} - 2^{n-l-1} + 2^l nl(h_l)$.*

Proof: Let $A = \{(x_1, \ldots, x_n) \in \mathbb{F}_2^n \mid x_{i+l} = x_i \oplus 1, \forall i \le l\}$. For every element x of A, we have $f(x) = h_l(x_{2l+1}, \ldots, x_n)$. Let us consider the restriction g of f to A as a Boolean function on $\mathbb{F}_2^{n-l}$, say $g(x_1, \ldots, x_l, x_{2l+1}, \ldots, x_n)$. Then, since $g(x_1, \ldots, x_l, x_{2l+1}, \ldots, x_n) = h_l(x_{2l+1}, \ldots, x_n)$, g has nonlinearity $2^l\, nl(h_l)$. According to Relation (3.15) applied with $h_a = g$ and $k = n - l$, we have $nl(f) \le 2^{n-1} - 2^{n-l-1} + 2^l nl(h_l)$. □

The characterizations recalled above of those symmetric functions achieving best possible nonlinearity can be straightforwardly deduced. Moreover, if for some $0 \le l < \lfloor \frac{n-1}{2} \rfloor$, the nonlinearity of an n-variable symmetric function f is strictly larger than $2^{n-1} - 2^{n-l-1} + 2^l \left(2^{n-2l-1} - 2^{\lfloor \frac{n-2l-1}{2} \rfloor} - 1\right) = 2^{n-1} - 2^{\lfloor \frac{n-1}{2} \rfloor} - 2^l$, then, thanks to these characterizations and to Proposition 150, the function h_l must be quadratic, and f satisfies $\mathrm{f}(r+2) = \mathrm{f}(r) \oplus 1$, for all $l \le r \le n - 2 - l$ (this property has been observed in [205, Theorem 6] a little after that [224] was published, and proved slightly differently).

Further properties of the nonlinearities of symmetric functions can be found in [205, 224].

10.1.5 Correlation immunity and resiliency

The correlation immunity of symmetric functions has been studied in [138, 887, 1135] and their resiliency in [370, 557].
There exists a conjecture on symmetric Boolean functions and, equivalently, on functions defined over $\{0,1,\ldots,n\}$ and valued in $\mathbb{F}_2$: if f is a non-constant symmetric Boolean function, then the *numerical degree* of f (hence, the degree of the univariate polynomial representation of f) is larger than or equal to number $n-3$. It is easily shown that this numerical degree is more than $\frac{n}{2}$ (otherwise, the polynomial $\mathrm{f}^2-\mathrm{f}$ would have degree at most n, and being null at $n+1$ points, it would equal the null polynomial, a contradiction with the fact that f is assumed not to be constant). But the gap between $\lfloor \frac{n}{2} \rfloor + 1$ and $n-3$ is open. According to Proposition 118, page 314, the conjecture is equivalent to saying that there does not exist any non-affine symmetric 3-*resilient* function. And proving this conjecture is also a problem on binomial coefficients since the numerical degree of f is bounded above by d if and only if, for every k such that $d < k \leq n$:

$$\sum_{r=0}^{k}(-1)^r\binom{k}{r}\mathrm{f}(r) = 0. \tag{10.4}$$

The conjecture is equivalent to saying that this Relation (10.4), with $d = n-4$, has no binary solution $\mathrm{f}(0),\ldots,\mathrm{f}(n)$. Von zur Gathen and Roche [532] have observed that all symmetric n-variable Boolean functions have numerical degrees larger than or equal to $n-3$, for any $n \leq 128$ (they exhibited Boolean functions with numerical degree $n-3$; see also [557]).
The same authors also observed that, if the number $m = n+1$ is a prime, then all non-constant n-variable symmetric Boolean functions have numerical degree n (and therefore, considering the function $g(x) = f(x) \oplus x_1 \oplus \cdots \oplus x_n$ and applying Proposition 118, all non-affine n-variable symmetric Boolean functions are unbalanced): indeed, m being a prime, the binomial coefficient $\binom{n}{r}$ is congruent with $\frac{(-1)(-2)\ldots(-r)}{1\cdot 2\ldots r} = (-1)^r$, modulo m, and the sum $\sum_{r=0}^{n}(-1)^r\binom{n}{r}\mathrm{f}(r)$ is then congruent with $\sum_{r=0}^{n}\mathrm{f}(r)$, modulo m, and Relation (10.4) with $k = n$ implies then that f must be constant (since it takes its values in $\{0,1\}$ and the sum of them is divisible by $n+1$).
Notice that, applying Relation (10.4) with $k = p-1$, where p is the largest prime less than or equal to $n+1$, shows that the numerical degree of any symmetric non-constant Boolean function is larger than or equal to $p-1$ (or equivalently that no symmetric non-affine Boolean function is $(n-p+1)$-resilient): otherwise, reducing (10.4) modulo p, we would have that the string $\mathrm{f}(0),\ldots,\mathrm{f}(k)$ is constant, and f having univariate degree less than or equal to k, the function f, and thus f itself, would be constant.
More results on the balancedness and resiliency/correlation immunity of symmetric functions can be found in [78, 205, 887, 1135, 1014]. The resiliency order of a symmetric function of algebraic degree d cannot exceed $2^{\lfloor \log_2 d \rfloor + 1} - 2$ [205].

10.1.6 Algebraic immunity and fast algebraic immunity

We have seen in Section 3.1 that, for every n-variable Boolean function f, there exist $g \neq 0$ and h, both of algebraic degree at most $\lceil \frac{n}{2} \rceil$ and such that $f\,g = h$ (equivalently, there exist nonzero annihilators of f or of $f \oplus 1$ of algebraic degree at most $\lceil \frac{n}{2} \rceil$). The same property can be proven when dealing with symmetric functions only: the elementary symmetric functions of degrees at most $\lceil \frac{n}{2} \rceil$ and their products with f give a family of $2\,(\lceil \frac{n}{2} \rceil + 1) > n+1$ symmetric functions, which must be linearly dependent since they live in a vector space of dimension $n+1$. There exist then $g \neq 0$ and h of degree at most $\lceil \frac{n}{2} \rceil$ such that $f\,g = h$ and the conclusion follows (using also the proof of Proposition 25 page 111). However, given an n-variable symmetric function f, there do not necessarily exist symmetric functions $g \neq 0$ and h of algebraic degree as small as $AI(f)$ such that $f\,g = h$.
We have seen that the majority function, which is symmetric, has optimal algebraic immunity. In the case n is odd, it is the only symmetric function having such property, up to the addition of a constant (see [979] which completed a partial result of [765]). In the case n is even, other symmetric functions exist (up to the addition of a constant and to the transformation $x \to \overline{x} = (x_1 \oplus 1, \ldots, x_n \oplus 1)$) with this property and all are known; more precisions and more results on the algebraic immunity of symmetric functions can be found in [127, 364, 774, 785, 941, 978, 979, 976, 1102] and the references therein. In particular, all symmetric functions of optimal algebraic immunity in numbers of variables which are powers of 2 are determined in [785] and it is shown in [127] that for $n = 2^j, 2^j - 1$ and $2^j - 2$, the elementary symmetric function $\sigma_{2^{j-1}}$ has optimal algebraic immunity, and that these are the only cases where an elementary symmetric function can have optimal AI. In [941] is shown thanks to a result of [786] that the corpus of potential annihilators of f or $f \oplus 1$ which needs to be investigated to prove the optimal algebraic immunity of a given function can be reduced in the case it is symmetric (and some necessary conditions on the simplified value vector for symmetric functions to achieve high AI are given), and this allows a description of optimal AI symmetric functions (and also of the sub-optimal ones), whose other parameters are also studied (none is balanced and the nonlinearity is bad).
We have seen at page 352 that, as shown in [791], no symmetric Boolean function can be perfect algebraic immune. Large classes of symmetric functions are very vulnerable to fast algebraic attacks despite their proven resistance against standard algebraic attacks: for $2^m \leq n \leq 2^m + 2^{m-1} - 1$, for every symmetric n-variable function f of algebraic immunity at least 2^{m-1}, there exists g such that $1 \leq d_{alg}(f) \leq n - 2^m + 1$ and $d_{alg}(fg) \leq n - 2^{m-1} + 1$. Even the other cases pose often a problem, since if $d_{alg}(f) > 2^k$ where 2^k does not divide $d_{alg}(f)$, then there exists g such that $d_{alg}(g) \leq e = d_{alg}(f)$ [mod 2^k] and $d_{alg}(fg) \leq d_{alg}(f) - e - 1$, and the FAI of a symmetric function f whose algebraic degree $d_{alg}(f)$ is not a power of 2 is smaller than $d_{alg}(f)$.

10.1.7 The subclass of threshold functions

For every $d \leq n$, we call[1] *threshold function*[2] of index d and we denote by $t_{n,d}$ the n-variable Boolean function whose support equals the set of vectors of Hamming weights at least d. The majority functions are examples. The reservations we made about symmetric functions are of course valid for threshold functions. Moreover, we shall see that threshold functions (as many symmetric functions and as all monotone functions, see page 395) have bad nonlinearity. They may then be improper for use in most cryptographic frameworks. But their output is very fast to compute. They can then be used in many more variables than more complex functions (this is the case for all symmetric functions, but still more for threshold functions). They deserve then some attention since they may present interest in some settings like the FLIP cryptosystem (see page 491).
The class of threshold functions has the interest of being preserved by the action of fixing the values of some variables (fixing one variable to 0 in $t_{n,d}$ gives the function $t_{n-1,d}$, and fixing one variable to 1 gives $t_{n-1,d-1}$). The results on them allow then not only to study their contributions to the resistance against classical attacks, but also against guess and determine attacks (see page 117). This is also true more generally with symmetric functions but less is known on this wider class.
Note that, for each value of d, functions $t_{n,d}$ and $t_{n,n-d+1}$ are *EA equivalent*:

$$\forall x \in \mathbb{F}_2^n, \quad t_{n,n-d+1}(x) = 1 \oplus t_{n,d}(x + 1_n).$$

The majority function for n odd is balanced but all other threshold functions are unbalanced.
We have $t_{n,d}(x) = \sum_{I \subseteq \{1,\dots,n\}} \lambda_I \, x^I$, where, for $d > 0$, $\lambda_\emptyset = t_{n,d}(0) = 0$ and according to Relation (2.23), page 67, for $I \neq \emptyset$:

$$\begin{aligned}\lambda_I &= (-1)^{|I|} \sum_{x \in \mathbb{F}_2^n;\ supp(x) \subseteq I} (-1)^{w_H(x)} t_{n,d}(x) = (-1)^{|I|} \sum_{\substack{x \in \mathbb{F}_2^n;\ supp(x) \subseteq I \\ w_H(x) \geq d}} (-1)^{w_H(x)} \\ &= (-1)^{|I|} \sum_{k=d}^{|I|} (-1)^k \binom{|I|}{k} = (-1)^{|I|-1} \sum_{k=0}^{d-1} (-1)^k \binom{|I|}{k} = (-1)^{|I|-d} \binom{|I|-1}{d-1}\end{aligned}$$

(using $\sum_{k=0}^{|I|} (-1)^k \binom{|I|}{k} = 0$, and the last equality being easily checked by induction on d). According to Lucas' theorem (see page 528 or [809, page 404]), the coefficient of x^I in the *ANF* of $t_{n,d}$ equals 1 (*i.e.* λ_I is odd) if and only if the binary expansion of $d-1$ is covered by (*i.e.* has support included in) that of $|I|-1$, and the algebraic degree of $t_{n,d}$ equals then $k+1$ where k is the largest number smaller than n whose binary expansion covers that of $d-1$, that is, where $k-d+1$ is the largest number smaller than $n-d+1$ whose binary expansion

[1] Our use of the term of threshold function is a little more restrictive than in [914]; more investigation is then needed.
[2] Not to be confused with the threshold implementation of vectorial functions, that we shall address in Subsection 12.1.4, page 472.

is disjoint from that of $d-1$.
Moreover, according to Relation (2.61), page 85, if $u \neq 0_n$, then $W_{t_{n,d}}(u)$ equals $2(-1)^{w_H(u)+1} \sum_{\substack{I\subseteq\{1,\dots,n\}\\ supp(u)\subseteq I}} 2^{n-|I|}\lambda_I$, that is:

$$W_{t_{n,d}}(u) = 2(-1)^{w_H(u)+1} \sum_{\substack{I\subseteq\{1,\dots,n\}\\ supp(u)\subseteq I}} 2^{n-|I|}(-1)^{|I|-d}\binom{|I|-1}{d-1}.$$

Recall from Relation (10.1), page 384, that the NNF of the indicator of the set $E_{n,r}$ of vectors of Hamming weight r has $(-1)^{|I|-r}\binom{|I|}{r}$ for coefficient of x^I. We deduce that $W_{1_{E_{n,r}}}(u) = 2(-1)^{w_H(u)+1} \sum_{\substack{I\subseteq\{1,\dots,n\}\\ supp(u)\subseteq I}} 2^{n-|I|}(-1)^{|I|-r}\binom{|I|}{r}$.
Therefore, for every u, the Walsh transform of function $1_{E_{n,d}}$ at $u \in \mathbb{F}_2^n$ equals the opposite of the Walsh transform of function $t_{n+1,d+1}$ at $(u,1)$ (where "," symbolizes concatenation). And since these two functions are symmetric, this implies that the maximum absolute value of the Walsh transform of $1_{E_{n,d}}$ equals the maximum absolute value of the Walsh transform of $t_{n+1,d+1}$ at nonzero inputs. But the nonlinearities of the two functions are different because the nonlinearity of $1_{E_{n,r}}$ equals its Hamming weight (since this weight is small), and hence, $|W_{1_{E_{n,r}}}|$ takes its maximum at the zero entry. It is easily deduced that

$$nl(t_{n,d}) = \begin{cases} 2^{n-1} - \binom{n-1}{(n-1)/2} & \text{if } d = \frac{n+1}{2}, \\ \sum_{k=d}^{n} \binom{n}{k} = w_H(t_{n,d}) & \text{if } d > \frac{n+1}{2}, \\ \sum_{k=0}^{d-1} \binom{n}{k} = 2^n - w_H(t_{n,d}) & \text{if } d < \frac{n+1}{2}, \end{cases}$$

since this is known from [409] in the case $d = \frac{n+1}{2}$, and for $d > \frac{n+1}{2}$, we have:

$$|W_{1_{E_{n-1,d-1}}}(u)| = 2\,|\sum_{x\in E_{n-1,d-1}} (-1)^{u\cdot x}| \leq 2\, w_H(1_{E_{n-1,d-1}}) = 2\binom{n-1}{d-1},$$

for every $u \neq 0_n$, and since $[|W_{t_{n,d}}(0_n)| = 2^n - 2\sum_{i=d}^{n}\binom{n}{i} = \sum_{i=n-d+1}^{d-1}\binom{n}{i}$, and using Pascal's identity $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$, we deduce that $|W_{t_{n,d}}|$ takes its maximum at the 0_n input, and this completes the proof in this case, and also in the last case according to the identity $t_{n,n-d+1}(x) = 1 \oplus t_{n,d}(x+1_n)$.

It is also known from [305] that $AI(t_{n,d}) = \min(d, n-d+1)$, and the vector space of minimum algebraic degree annihilators can be determined. Indeed, applying the transformation $x \mapsto x + 1_n$ changes $t_{n,d}$ into the indicator of the set of vectors of Hamming weight at most $n-d$; the linear combinations over $\mathbb{F}_2$ of the monomials of degrees at least $n-d+1$ vanish over the words of Hamming weight at most $n-d$ and are then annihilators of this indicator; the dimension $\sum_{i=n-d+1}^{n}\binom{n}{i}$ of this vector space of annihilators being equal to the dimension of the vector space of all annihilators, that is, $2^n - w_H(t_{n,d})$, these linear combinations are all the annihilators of the indicator; the annihilators of $t_{n,d}$ are obtained from these linear combinations by the transformation $x \mapsto x + 1_n$. They can have every algebraic degree at least $n - d + 1$. And the annihilators of $1 \oplus t_{n,d}$ are the linear combinations over $\mathbb{F}_2$ of the monomials of degrees at least d. They can have every algebraic degree at least d. Hence $AI(t_{n,d}) = \min(d, n - d + 1)$.

10.2 Rotation symmetric, idempotent and other similar functions

We have already encountered *rotation symmetric* (RS) and *idempotent functions* in Chapters 6 and 7 (see Definitions 59 and 60, page 275). We have seen how, through the choice of a normal basis, the latter are related to the former (see Proposition 89, page 275). RS functions constitute a super-class of symmetric functions, which has been investigated from the viewpoints of bentness and correlation immunity (see *e.g.* [503, 1048]). These functions, which represent an interesting (reasonably small) corpus for computer investigation, have also played a role in the study of nonlinearity. It could be shown in [684, 686], thanks to such computer investigation, that the best nonlinearity of Boolean functions in odd number n of variables is strictly larger than the quadratic bound if and only if $n > 7$. Indeed, a 9-variable function of nonlinearity 241 could be found (while the quadratic bound gives 240, and the covering radius bound 244), and using direct sum with quadratic functions, it gave then 11-variable functions of nonlinearity 994 (while the quadratic bound gives 992 and the covering radius bound 1000), and 13-variable functions of nonlinearity 4036 (while the quadratic bound gives 4032 and the covering radius bound 4050). Later was checked that 241 is the best nonlinearity of 9-variable rotation symmetric functions, but that 9-variable functions whose *truth-tables* (or equivalently ANFs) are invariant under cyclic shifts by 3 steps and under inversion of the order of the input bits can reach nonlinearity 242, which led to 11-variables functions of nonlinearity 996 and 13-variable functions of nonlinearity 4040. Balanced functions in 13 variables beating the quadratic bound could also be found. The construction with RS functions does not beat the nonlinearity of the Patterson-Wiedemann functions for 15 variables.

Hence rotation symmetry is an interesting notion for investigating the parameters of Boolean functions. Cryptographically speaking, the strong structure it provides may represent a risk with respect to attacks, while rotation symmetric functions are more difficult to use with large numbers of variables than symmetric functions (because they are slower to compute in general).

For $n = 2m$ even, we can consider the *bivariate representation* alongside the univariate representation of idempotent functions. We can see how obtaining the univariate (resp. multivariate) form from the bivariate form and vice-versa, and exploit this correspondence to construct more functions; this has been done in

[281] and we follow below this reference. For m odd, the situation is simplified and we place then ourselves in such case: choosing $w \in \mathbb{F}_4 \setminus \mathbb{F}_2$, we have $w^2 = w + 1$, $w^4 = w$, and since $\frac{w^2}{w} \notin \mathbb{F}_{2^m}$, we can take (w, w^2) for a basis of $\mathbb{F}_{2^n}$ over $\mathbb{F}_{2^m}$. Any element of $\mathbb{F}_{2^n}$ is then written in the form $xw + yw^2$, where $x, y \in \mathbb{F}_{2^m}$. Given a normal basis $(\alpha, \alpha^2, \ldots, \alpha^{2^{m-1}})$ of $\mathbb{F}_{2^m}$, a natural normal basis of $\mathbb{F}_{2^n}$ is

$$\left(\alpha w, \alpha^2 w^2, \alpha^4 w, \ldots, \alpha^{2^{m-2}} w^2, \alpha^{2^{m-1}} w, \alpha w^2, \ldots, \alpha^{2^{m-2}} w, \alpha^{2^{m-1}} w^2\right). \quad (10.5)$$

Since $(xw+yw^2)^2 = y^2w+x^2w^2$, the mapping $z \in \mathbb{F}_{2^n} \mapsto z^2 \in \mathbb{F}_{2^n}$ corresponds to the mapping $(x, y) \in \mathbb{F}_{2^m}^2 \mapsto (y^2, x^2) \in \mathbb{F}_{2^m}^2$. Given a function $f(x, y)$ in bivariate form, the related Boolean function over $\mathbb{F}_2^n$ obtained by decomposing the input $xw+yw^2$ over the normal basis (10.5) is then RS if and only if $f(x, y) = f(y^2, x^2)$. Note that applying this identity m times gives $f(x, y) = f(y, x)$ and applying it $m + 1$ times gives $f(x, y) = f(x^2, y^2)$; the double condition "$f(x, y) = f(y, x)$ and $f(x, y) = f(x^2, y^2)$" is necessary and sufficient for f being idempotent.

Definition 78 *A polynomial $f(z)$ over $\mathbb{F}_{2^n}$, $n = 2m \equiv 2 \pmod 4$, is called a weak idempotent if its associate bivariate expression $f(x, y) = f(xw + yw^2)$, $w \in \mathbb{F}_4 \setminus \mathbb{F}_2$, $x, y \in \mathbb{F}_{2^m}$, satisfies $f(x, y) = f(x^2, y^2)$.*

Proposition 151 *For $n \equiv 2 \pmod 4$, idempotents are those polynomials $f(z)$ over $\mathbb{F}_{2^n}$ whose associate bivariate expression $f(x, y) = f(xw+yw^2)$, $w \in \mathbb{F}_4\setminus\mathbb{F}_2$, satisfies $f(x, y) = f(y^2, x^2)$. Their set is included in that of weak idempotents. An idempotent is a weak idempotent invariant under the swap $x \leftrightarrow y$.*

See more in [245, Subsection 5.3]. The corresponding definition at the bit level is obtained by decomposing the univariate representation over the basis (10.5) and the *bivariate representation* over the basis $(\alpha, \alpha^2, \ldots, \alpha^{2^{m-1}})$:

Definition 79 *Let $n = 2m \equiv 2 \pmod 4$. A Boolean function*

$$f(x_0, y_1, x_2, y_3, \ldots, x_{n-2}, y_{n-1})$$

(where each index is reduced modulo m) over $\mathbb{F}_2^n$ is weak RS if it is invariant under the transformation $(x_j, y_j) \mapsto (x_{j+1}, y_{j+1})$.

Note the particular disposition of the indices in $f(x_0, y_1, x_2, y_3, \ldots, x_{n-2}, y_{n-1})$: the index 0 for y does not come at the second position (where we have y_1) but at the m-th position. Since m is odd, the invariance of f under the transformation $(x_j, y_j) \mapsto (x_{j+1}, y_{j+1})$ over (x, y) is equivalent to its invariance under $(x_j, y_j) \mapsto (x_{j+2}, y_{j+2})$. Hence:

Proposition 152 *The Boolean function $f(x_0, y_1, x_2, y_3, \ldots, x_{n-2}, y_{n-1})$ is weak RS if and only if it is invariant under the square of the shift ρ_n.*

Such weak RS function (that some authors call 2-RS function, see *e.g.* [685]) is RS if and only if it is invariant under the swap of x and y. A simple example of a weak RS function is the *direct sum* $f(x) \oplus g(y)$ which is RS when $f = g$, where f and g are RS functions with m variables. More generally, the indirect

sum is studied in [281] (see also [245]), with explicit examples of resulting bent idempotents. There exist also examples of bent and semi-bent weak idempotents [311, 312, 699, 871].
The *secondary constructions* recalled above have led to the construction of RS functions and idempotent bent functions from near-bent RS functions seen at page 278.
The k-variate representation can be studied similarly to the bivariate representation, see [245].

The weights of rotation symmetric functions are studied in [399]. RS functions with optimal algebraic immunity have been constructed (see *e.g.* [1015]), but these functions never reached good nonlinearity.

In [748] is introduced the class of Matriochka symmetric functions, which are the sums of symmetric functions whose sets of variables are different and nested.

The notion of rotation symmetry has been generalized to vectorial functions in [994]. An (n,n)-function is RS if it commutes with the cyclic shift: $F \circ s = s \circ F$. This is equivalent to saying that each coordinate function equals (cyclically) the previous one composed by the cyclic shift. Identifying $\mathbb{F}_2^n$ with $\mathbb{F}_{2^n}$ thanks to a normal basis, this is equivalent to $(F(x))^2 = F(x^2)$ and therefore to the fact that the univariate representation of F has all its coefficients in $\mathbb{F}_2$ (using the uniqueness of such representation). Kavut [680] enumerated all bijective rotation symmetric $(6,6)$-functions with maximum nonlinearity 24, showing that, up to affine equivalence, there are only 4 functions with differential uniformity 4 and algebraic degree 5.

10.3 Direct sums of monomials

Functions $f(x) = \bigoplus_{I\subseteq\{1,\dots,n\}} a_I\, x^I$ where $(a_I = a_J = 1$ and $I \neq J) \Rightarrow (I \cap J = \emptyset)$ are well adapted to situations where Boolean functions must be particularly simple, for instance when they are used with large numbers of variables and when addition and/or multiplication are costly, like in the FLIP cryptosystem (see page 491). As for threshold functions, the class of direct sums of monomials is preserved by the action of fixing the values of some variables and their study addresses then also their behavior against guess and determine attacks resulting in fixing some input values to the functions.
It is convenient to identify a direct sum of monomials whose value at 0_n is 0 by its *direct sum vector* $[m_1, m_2, \dots, m_k]$, of length $k = d_{alg}(f)$, in which each m_i is the number of monomials of degree i (this allows to determine uniquely the function up to permutation of variables). We shall assume that all variables are effective, *i.e.* that the number of variables equals $\sum_{i=1}^k i\, m_i$. The property seen in Relation (6.28), page 258, that the Walsh transform of a direct sum equals the product of the Walsh transforms of the ingredient functions, the *Golomb-Xiao-Massey characterization* of resiliency by the Walsh transform (Theorem 5, page 107), and Relation (3.1), page 99, imply that the resiliency order of f

equals $m_1 - 1$ (with the convention that an unbalanced function has resiliency order -1) and that its nonlinearity equals $2^{n-1} - 2^{m_1-1}\prod_{i=2}^{k}\left(2^i-2\right)^{m_i}$. The algebraic immunity is more complex to determine but it is shown in [306] that if $f(x_1, x_2, x_3, \ldots, x_n)$ is a Boolean function in n variables such that:

$$\forall x \in \mathbb{F}_2^{n-2}\ f(x,0,0) = f(x,0,1) = f(x,1,0),$$

then the Boolean function $f'(x_1, \ldots, x_{n-1})$ defined by:

$$\forall x \in \mathbb{F}_2^{n-2}\ f'(x,1) = f(x,1,1) \text{ and } f'(x,0) = f(x,0,0)$$

satisfies that $AI(f') \leq AI(f)$. Using this property and the algebraic immunity of triangular functions (see below), the algebraic immunity of sums of monomials has been determined in [305]:

$$AI(f) = \min_{0\leq d\leq k}\left(d + \sum_{i=d+1}^{k} m_i\right). \tag{10.6}$$

It is also shown in this same reference that, in some cases, the fast algebraic immunity of such functions can be close to their algebraic immunity.

10.3.1 Triangular functions

Direct sums of monomials are called *triangular functions* when their direct sum vector is the all-1 vector (that is, when they have one monomial of each degree). We assume here also that all variables are effective. The k-th triangular function equals $\bigoplus_{i=1}^{k}\prod_{j=1}^{i} x_{j+i(i-1)/2}$. Its nonlinearity equals $2^{n-1} - \prod_{i=2}^{k}\left(2^i-2\right)$, according to what we have seen with direct sums of monomials, and its algebraic immunity equals k, as first observed in [279] (and used in [839]). This property is easily shown by induction on k since we have seen at page 373 that making the direct sum of a function f and of a monomial of degree $AI(f) + 1$ gives a function of algebraic immunity $AI(f) + 1$.

10.4 Monotone functions

An n-variable Boolean function f is (increasing) *monotone* if, for every $x, y \in \mathbb{F}_2^n$ such that $x \preceq y$ (*i.e.* such that $supp(x) \subseteq supp(y)$, see page 49), we have $f(x) \leq f(y)$. Any *monomial Boolean (multivariate) function* $\prod_{i\in I} x_i$ is monotone. Other examples are threshold functions, see above.

As mentioned in [298, 249], monotone Boolean functions play a role in voting theory (a voting scheme should be monotone), reliability theory (a system currently working should not fail when we replace a defective component by an operative one), hypergraphs (the stability function of a hypergraph, which takes value 1 at x when supp(x) contains at least one edge, is monotone Boolean), learning (monotone Boolean functions are easier to learn). The question addressed here is: can they play also a role with stream ciphers (as filter functions) and our conclusion at the end of this section will be essentially negative.

The balancedness and the algebraic immunity of monotone Boolean functions are addressed in [298], which also recalls what is their ANF and how they can be constructed. This reference studies their Walsh spectrum and their nonlinearity, showing that no monotone bent n-variable function exists for $n \geq 4$, and that every monotone n-variable function f has nonlinearity at most $2^{n-1} - 2^{\frac{n-1}{2}}$ for $n \geq 5$ odd. Let us show how these results are obtained. For every $y \in \mathbb{F}_2^n$ such that $f(y) = 0$, we have, according to the *Poisson summation formula* (2.41), page 78, applied with $a = b = 0_n$ and $E^\perp = \{x \in \mathbb{F}_2^n;\, x \preceq y\}$, $E = \{u \in \mathbb{F}_2^n;\, u \preceq y + 1_n\}$:

$$\sum_{u \in \mathbb{F}_2^n;\, u \preceq y+1_n} W_f(u) = 2^n,$$

and this implies that $\max_{u \in \mathbb{F}_2^n;\, u \preceq y+1_n} |W_f(u)| \geq 2^{w_H(y)}$, since the maximum of a sequence cannot be smaller than its arithmetic mean. And when $f(y) = 1$:

$$\sum_{u \in \mathbb{F}_2^n;\, u \preceq y} (-1)^{1_n \cdot u} W_f(u) = -2^n,$$

and this implies that $\max_{u \in \mathbb{F}_2^n;\, u \preceq y} |W_f(u)| \geq 2^{n - w_H(y)}$. Then:

Proposition 153 *[298] For every odd $n \geq 5$ and every monotone n-variable function f, we have $nl(f) \leq 2^{n-1} - 2^{(n-1)/2}$.*

Indeed, the observations above and Relation (3.1), page 99, imply this bound when there exists y of Hamming weight at least $\frac{n+1}{2}$ such that $f(y) = 0$, or of Hamming weight at most $\frac{n-1}{2}$ such that $f(y) = 1$, and the only case left is when f is the majority function, which has nonlinearity $2^{n-1} - \binom{n-1}{(n-1)/2}$.

But no general upper bound for n even could be shown. Indeed, only the case where $f(x)$ differs from the majority function for at least one input x of Hamming weight different from $n/2$ can be easily handled similarly. The case where $f(x)$ coincides with the majority function for every input x of Hamming weight different from $n/2$ must be handled by other means. Then [298] only conjectured the upper bound $nl(f) \leq 2^{n-1} - 2^{\frac{n}{2}}$ for n even large enough.

This conjecture was proved in [249]. We give its proof (and this will also prove the nonexistence of monotone bent functions). According to the observations above, we can restrict ourselves to the case where n is even and f equals the majority function at every input x of Hamming weight different from $n/2$. We can assume f different from the strict and large majority functions, since the nonlinearity of these two functions, equal to $2^{n-1} - \binom{n-1}{n/2}$, is larger than $2^{n-1} - 2^{n/2}$ for n large enough. What makes the proof work is the *second-order Poisson summation formula* (see Relation (2.57), page 81):

$$\sum_{u \in E^\perp} W_f^2(u) = |E^\perp| \sum_{a \in E'} \left(\sum_{x \in E} (-1)^{f(a+x)} \right)^2, \tag{10.7}$$

valid for any Boolean function f and supplementary subspaces E and E' of $\mathbb{F}_2^n$. For a given y of Hamming weight $n/2$ and such that $f(y) = 0$, let us take $E = \{x \in \mathbb{F}_2^n;\, x \preceq y\}$. Then $E^\perp = \{u \in \mathbb{F}_2^n;\, u \preceq y + 1_n\}$ is supplementary of E and we can then take $E' = E^\perp$; we obtain, f being null on E since it is monotone:

$$\sum_{u\in\mathbb{F}_2^n;\, u\preceq y+1_n} W_f^2(u) = 2^{n/2} \sum_{a\in\mathbb{F}_2^n; a\preceq y+1_n} \left(\sum_{x\in\mathbb{F}_2^n;\, x\preceq y} (-1)^{f(a+x)}\right)^2$$
$$= 2^{3n/2} + 2^{n/2} \sum_{a\in\mathbb{F}_2^n; a\preceq y+1_n; a\neq 0_n} \left(\sum_{x\in\mathbb{F}_2^n;\, x\preceq y} (-1)^{f(a+x)}\right)^2 .$$

Using again that the maximum is bounded below by the mean, we deduce the inequality $\max_{u\in\mathbb{F}_2^n;\, u\preceq y+1_n} W_f^2(u) \geq 2^n + \sum_{a\preceq y+1_n; a\neq 0_n} \left(\sum_{x\in\mathbb{F}_2^n;\, x\preceq y}(-1)^{f(a+x)}\right)^2$. For $a \preceq y + 1_n$, denoting $w_H(a)$ by j, if $x \preceq y$ has Hamming weight strictly less than $n/2 - j$ then $a + x$ has Hamming weight strictly less than $n/2$ and $f(a + x)$ equals 0, and if $x \preceq y$ has Hamming weight strictly larger than $n/2 - j$ then $a + x$ has Hamming weight strictly larger than $n/2$ and $f(a + x)$ equals 1. If $x \preceq y$ has Hamming weight $n/2 - j$, then $a + x$ has weight $n/2$ and the value of $f(a + x)$ is unknown. The value of $\sum_{x\in\mathbb{F}_2^n;\, x\preceq y}(-1)^{f(\mathbf{a}+x)}$ lies then between $\sum_{i=0}^{n/2-1-j}\binom{n/2}{i} - \sum_{i=n/2+1-j}^{n/2}\binom{n/2}{i} - \binom{n/2}{n/2-j}$ and $\sum_{i=0}^{n/2-1-j}\binom{n/2}{i} - \sum_{i=n/2+1-j}^{n/2}\binom{n/2}{i} + \binom{n/2}{n/2-j}$. Replacing $\binom{n/2}{i}$ by $\binom{n/2}{n/2-i}$ in the sum $\sum_{i=n/2+1-j}^{n/2}\binom{n/2}{i}$, we obtain $\sum_{i=0}^{j-1}\binom{n/2}{i}$. Then for $j < n/4$, we have $n/2 - 1 - j \geq j$ and $\left(\sum_{x\in\mathbb{F}_2^n;\, x\preceq y}(-1)^{f(a+x)}\right)^2 \geq \left(\sum_{i=j}^{n/2-1-j}\binom{n/2}{i} - \binom{n/2}{n/2-j}\right)^2 = \left(\sum_{i=j+1}^{n/2-1-j}\binom{n/2}{i}\right)^2$, and for $j > n/4$, we have $j-1 \geq n/2-j$ and $\left(\sum_{x\in\mathbb{F}_2^n;\, x\preceq y}(-1)^{f(a+x)}\right)^2 \geq \left(\sum_{i=n/2-j}^{j-1}\binom{n/2}{i} - \binom{n/2}{n/2-j}\right)^2 = \left(\sum_{i=n/2-j+1}^{j-1}\binom{n/2}{i}\right)^2$. We then deduce that: $\max_{u\in\mathbb{F}_2^n;\, u\preceq y+1_n} W_f^2(u) \geq 2^n +$

$$\sum_{1\leq j<n/4}\binom{n/2}{j}\left(\sum_{i=j+1}^{n/2-1-j}\binom{n/2}{i}\right)^2 + \sum_{n/4<j\leq n/2}\binom{n/2}{j}\left(\sum_{i=n/2-j+1}^{j-1}\binom{n/2}{i}\right)^2 =$$

$$2^n + 2\sum_{1\leq j<n/4}\binom{n/2}{j}\left(\sum_{i=j+1}^{n/2-1-j}\binom{n/2}{i}\right)^2 + \left(\sum_{i=1}^{n/2-1}\binom{n/2}{i}\right)^2 =$$

$$2^n + 2\sum_{1\leq j<n/4}\binom{n/2}{j}\left(2^{n/2} - 2\sum_{i=0}^{j}\binom{n/2}{i}\right)^2 + \left(2^{n/2}-2\right)^2 .$$

And we have $2\sum_{1\leq j<n/4}\binom{n/2}{j}\left(2^{n/2} - 2\sum_{i=0}^{j}\binom{n/2}{i}\right)^2 + \left(2^{n/2}-2\right)^2 \geq 3\cdot 2^n$ for

every $n \geq 10$, since the expression of n equal to

$$2^{-n}\left[2\sum_{1\leq j<n/4}\binom{n/2}{j}\left(2^{n/2}-2\sum_{i=0}^{j}\binom{n/2}{i}\right)^2+\left(2^{n/2}-2\right)^2\right]$$

is non-decreasing and is larger than 3 for $n = 10$. We deduce then:

Proposition 154 *[249] For every even $n \geq 10$ and every monotone n-variable function f, we have $nl(f) \leq 2^{n-1} - 2^{n/2}$.*

Since $2^{n-1} - 2^{(n-1)/2}$ (n odd) and $2^{n-1} - 2^{n/2}$ (n even) are good nonlinearities for Boolean functions in n variables, the bounds above do not tell us if monotone Boolean functions can have good nonlinearity. But a stronger bound, valid for every n, can be proved as also shown in [249]. Indeed, the inequalities $\max_{u\in\mathbb{F}_2^n}|W_f(u)| \geq 2^{w_H(y)}$ for $f(y) = 0$ and $\max_{u\in\mathbb{F}_2^n}|W_f(u)| \geq 2^{n-w_H(y)}$ for $f(y) = 1$ can be refined by using the second-order Poisson summation formula (10.7) again.
- If there exist vectors of Hamming weight strictly larger than $n/2$ whose image by f is 0, let then y have maximal Hamming weight (say, w) among all vectors satisfying $f(y) = 0$. We have with the same arguments as above:

$$\max_{u\in\mathbb{F}_2^n;\, u\preceq y+1_n} W_f^2(u) \geq 2^{2w} + \sum_{a\in\mathbb{F}_2^n;\, a\preceq y+1_n; a\neq 0_n}\left(\sum_{x\in\mathbb{F}_2^n;\, x\preceq y}(-1)^{f(a+x)}\right)^2. \quad (10.8)$$

For every $a \preceq y+1_n$ (of Hamming weight $j \leq n-w$), we have $f(a+x) = 1$ for every $x \preceq y$ such that $a+x$ has Hamming weight at least $w+1$ (that is, for every $x \preceq y$ of Hamming weight at least $w-j+1$), and we deduce $\sum_{x\in\mathbb{F}_2^n;\, x\preceq y}(-1)^{f(a+x)} \leq 2^w - 2\sum_{i=w-j+1}^{w}\binom{w}{i}$. Note that we have $2^w - 2\sum_{i=w-j+1}^{w}\binom{w}{i} \leq 0$ if and only if $w-j+1 \leq \frac{w}{2}$, that is, $j \geq \frac{w}{2}+1$. We have: $\sum_{\substack{a\in\mathbb{F}_2^n; a\neq 0_n\\ a\preceq y+1_n}}\left(\sum_{x\in\mathbb{F}_2^n;\, x\preceq y}(-1)^{f(a+x)}\right)^2 \geq \sum_{j=\lceil\frac{w}{2}\rceil+1}^{n-w}\binom{n-w}{j}\left(2\sum_{i=w-j+1}^{w}\binom{w}{i}-2^w\right)^2$. We deduce then from (10.8) that:

$$\max_{u\in\mathbb{F}_2^n;\, u\preceq y+1_n} W_f^2(u) \geq 2^{2w} + \sum_{j=\lceil\frac{w}{2}\rceil+1}^{n-w}\binom{n-w}{j}\left(2^w-2\sum_{i=0}^{w-j}\binom{w}{i}\right)^2.$$

Denoting $2w = n+k$ (where $k > 0$ has the same partity as n), we have then:

$$\max_{u\in\mathbb{F}_2^n;\, u\preceq y+1_n} W_f^2(u) \geq 2^{n+k} + \sum_{j=\lceil\frac{n+k}{4}\rceil+1}^{\frac{n-k}{2}}\binom{\frac{n-k}{2}}{j}\left(2^{\frac{n+k}{2}}-2\sum_{i=0}^{\frac{n+k}{2}-j}\binom{\frac{n+k}{2}}{i}\right)^2.$$

Hence, we have:

$$nl(f) \leq 2^{n-1} - \frac{1}{2}\sqrt{2^{n+k} + \sum_{j=\lceil\frac{n+k}{4}\rceil+1}^{\frac{n-k}{2}}\binom{\frac{n-k}{2}}{j}\left(2^{\frac{n+k}{2}}-2\sum_{i=0}^{\frac{n+k}{2}-j}\binom{\frac{n+k}{2}}{i}\right)^2}.$$

- If there exist vectors of Hamming weight smaller than $n/2$ and whose image by f equals 1, let y have minimal Hamming weight w such that $f(y) = 1$ ($w < n/2$). Applying the upper bound above to the monotone function $f(x+1_n)\oplus 1$, whose nonlinearity equals that of f, and denoting $w' = n - w = \frac{n+k'}{2}$, where $k' > 0$ has the same partity as n, we have:

$$nl(f) \leq 2^{n-1} - \frac{1}{2}\sqrt{2^{n+k'} + \sum_{j=\lceil \frac{n+k'}{4}\rceil+1}^{\frac{n-k'}{2}} \binom{\frac{n-k'}{2}}{j}\left(2^{\frac{n+k'}{2}} - 2\sum_{i=0}^{\frac{n+k'}{2}-j}\binom{\frac{n+k'}{2}}{i}\right)^2}.$$

- If none of the two cases above happens, then f coincides with the majority function at every input x of Hamming weight different from $n/2$ and either (i) f is a majority function and $nl(f)$ equals then $2^{n-1} - \binom{n-1}{n/2}$ if n is even and $2^{n-1} - \binom{n-1}{(n-1)/2}$ if n is odd, or (ii) n is even and $nl(f) \leq 2^{n-1} - \frac{1}{2}\sqrt{A}$ where A equals $2^n + 2\sum_{1\leq j<n/4}\binom{n/2}{j}\left(2^{n/2} - 2\sum_{i=0}^{j}\binom{n/2}{i}\right)^2 + \left(2^{n/2}-2\right)^2$. We deduce:

Theorem 24 *[249] For every n and every monotone n-variable function f, we have $nl(f) \leq 2^{n-1} - \frac{1}{2}\sqrt{M}$, where $M = \min(A,B,C)$ if n is even and $M = \min(B,C)$ if n is odd, with*

$$A = 2^n + 2\sum_{1\leq j<n/4}\binom{n/2}{j}\left(2^{n/2} - 2\sum_{i=0}^{j}\binom{n/2}{i}\right)^2 + \left(2^{n/2}-2\right)^2,$$

$$B = \min_{\substack{1\leq k\leq n/2\\ n+k\ even}}\left(2^{n+k} + \sum_{j=\lceil\frac{n+k}{4}\rceil+1}^{\frac{n-k}{2}}\binom{\frac{n-k}{2}}{j}\left(2^{\frac{n+k}{2}} - 2\sum_{i=0}^{\frac{n+k}{2}-j}\binom{\frac{n+k}{2}}{i}\right)^2\right),$$

and $C = \left[2\binom{n-1}{\lfloor\frac{n}{2}\rfloor}\right]^2$.

The behavior of A, B and C when n tends to infinity is studied in [249] and shows that $\min(A,B,C)$ is asymptotically equivalent to an expression of n at least equal to $2^{\frac{3n\lambda_n}{2}}$ for some λ_n tending to 1. Tables are given, indicating for each value of n between 4 and 31 the value given by the upper bound of Theorem 24. These tables confirm that the nonlinearity of monotone Boolean functions is bad (much worse than what was suggested by the upper bounds obtained, resp. conjectured, in [298]). This shows that the rather large class of monotone Boolean functions contains no element which could be used as a nonlinear function in a cryptosystem.

11 Highly nonlinear vectorial functions with low differential uniformity

A large nonlinearity is one of the most important criteria for vectorial functions, valid for all uses in stream and block ciphers. Nonlinearity is not the only parameter quantifying the difference in behavior between a vectorial function and *affine functions*, but it is the most important. According to Dib's results [436], the average nonlinearity of vectorial functions is not bad.
Differential uniformity has the same importance as nonlinearity but is specific to *S-boxes* in block ciphers. According to Voloch's results [1098], the average differential uniformity of (n, n)-functions is bad, and this is probably also the case for (n, m)*-functions*. The relationship between nonlinearity and differential uniformity is not completely clarified. For instance, as seen at page 159, there exist vectorial functions with good nonlinearity and bad differential uniformity and *vice versa*, but most known functions with optimal differential uniformity have good nonlinearity. Further work is needed to understand better this relationship. But the work done in general on the study of S-boxes (see a survey in [94]) is significant and has had important practical applications. The design of the AES has taken advantage of the studies (in particular by K. Nyberg) on the notions of nonlinearity and differential uniformity. This has made possible in the AES to use S-boxes working on bytes (at the time, it would not have been possible to find a good 8-bit-to-8-bit S-box by a computer search as this had been done for the 6-bit-to-4-bit S-boxes of the DES). We recommend the book [141].

We briefly recall the main informations given in Subsection 3.2.3, page 137. The *nonlinearity* $nl(F)$ of an (n, m)-function F is the minimum Hamming distance between all *component functions* of F and all affine functions in n variables:

$$nl(F) = 2^{n-1} - \frac{1}{2} \max_{v \in \mathbb{F}_2^m \setminus \{0_m\};\ u \in \mathbb{F}_2^n} |W_F(u, v)|.$$

Nonlinearity quantifies the contribution of functions to the resistance against linear attacks, when they are used as S-boxes in block ciphers, and partly against fast correlation attacks, when they are used as filters or combiners in stream ciphers.
We have seen that the nonlinearity is a CCZ invariant. In particular, if $n = m$ and if F is a permutation, then F and its inverse F^{-1} have the same nonlinearity. We have also seen at page 183 the relationship between the maximal possible nonlinearity of (n, m)-functions and the possible parameters of the linear super-

codes of the Reed-Muller code of order 1. Existence and non-existence results[1] on highly nonlinear vectorial functions are deduced in [1099] and upper bounds on the nonlinearity of (n,m)-functions are derived in [341, 267].

11.1 The covering radius bound; bent/perfect nonlinear functions

As seen at page 139, the covering radius bound is valid for every (n,m)-function:

$$nl(F) \leq 2^{n-1} - 2^{\frac{n}{2}-1}, \tag{11.1}$$

and an (n,m) function is called *bent* if it achieves the covering radius bound *(11.1)* with equality.
The notion of bent vectorial function is invariant under *CCZ equivalence*[2] (since the nonlinearity is) but we have seen at Subsection 6.4, page 297, that CCZ equivalence coincides with EA equivalence for bent vectorial functions. We have also seen that an (n,m)-function is bent if and only if all the component functions $v \cdot F$, $v \neq 0_m$ of F are bent and that bent (n,m)-functions exist if and only if n is even and $m \leq \frac{n}{2}$. Recall also that an (n,m)-function is bent if and only if all its derivatives $D_aF(x) = F(x) + F(x+a)$, $a \in \mathbb{F}_2^n \setminus \{0_n\}$, are balanced, that is, "bent" and "perfect nonlinear (PN)" are equivalent. Bent vectorial functions contribute then also to an optimal resistance to the differential attack of those cryptosystems in which they are involved (but they are not balanced). They can be used to design *authentication schemes* (or codes), see [346].
Thanks to the observations made in Subsection 2.3.7 (where we saw that the evaluation of the multidimensional Walsh transform corresponds in fact to the evaluation of the Walsh transform), it is a simple matter to characterize bent functions as those functions whose squared expression of the multidimensional Walsh transform at L is the same for every L.

Note that if a bent (n,m)-function F is normal in the sense that it is null on (say) an $\frac{n}{2}$-dimensional vector space E, then F is balanced on any translate of E. Indeed, for every $v \neq 0_m$ in $\mathbb{F}_2^m$ and every $u \in \mathbb{F}_2^n \setminus E$, the function $v \cdot F$ is balanced on $u + E$.

We have recalled at Subsections 6.1.15 and 6.1.16 what are the known primary and secondary constructions of bent functions.

11.2 The Sidelnikov-Chabaud-Vaudenay bound

We have seen with Theorem 6, page 140, that a better upper bound than the covering radius bound exists for (n,n)-functions:

$$nl(F) \leq 2^{n-1} - 2^{\frac{n-1}{2}},$$

[1] Using the linear programming bound due to Delsarte.
[2] But the number of bent components of general (n,m)-functions is not.

and that the functions which achieve it with equality (for n necessarily odd) are called *almost bent* (*AB*). There exists a bound on the algebraic degree of AB functions, similar to the bound for bent functions:

Proposition 155 *[257] Let F be any (n,n)-function ($n \geq 3$, odd). If F is AB, then the algebraic degree of F is less than or equal to $(n+1)/2$.*

This is a direct consequence of the fact that the Walsh transform of any function $v \cdot F$ is divisible by $2^{\frac{n+1}{2}}$ and of Theorem 2, page 82. The bound is tight; it is achieved with equality for instance by the inverse of x^3.
Note that the divisibility plays also a role with respect to the algebraic degree of the composition of two vectorial functions: in [204] has been proved (as we recalled in a remark at page 82) that, if the Walsh transform values of a vectorial function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ are divisible by 2^k then, for every vectorial function $G : \mathbb{F}_2^n \to \mathbb{F}_2^n$, the algebraic degree of *composite function* $G \circ F$ is at most equal to the algebraic degree of G plus $n - k$. This means that using AB functions as S-boxes in block ciphers may not be a good idea (suboptimal functions as the multiplicative inverse function, see Chapter 11, may be better, as often in cryptography).

Remark. There is a big gap between the best possible nonlinearity $2^{n-1} - 2^{\frac{n-1}{2}}$ of (n,n)-functions for n odd, achieved by AB functions (see examples below), and the best known nonlinearity $2^{n-1} - 2^{n/2}$ of (n,n)-functions for n even, which is achieved (see below) by the Gold APN functions, the Kasami APN functions, and the multiplicative inverse function x^{2^n-2} (n odd). The gap could seem not so important, but it is, since what matters for the complexity of attacks by linear approximation is not the value of $nl(F)$ but the value of $\frac{2^{n-1}-nl(F)}{2^{n-1}}$. Finding functions with better nonlinearity (and still more relevantly to cryptography, with better nonlinearity and good differential uniformity) or proving that such function does not exist is an open question. $\square$

We recall now the definition of the *differential uniformity* of an (n,m)-function F (see Definition 40, page 157):

$$\delta_F = \max_{\substack{a \in \mathbb{F}_2^n, b \in \mathbb{F}_2^m \\ a \neq 0_n}} |\{x \in \mathbb{F}_2^n;\ D_aF(x) = b\}|$$

is the maximum number of ordered pairs of distinct elements of the graph $\mathcal{G}_F = \{(x,y) \in \mathbb{F}_2^n \times \mathbb{F}_2^m;\ y = F(x)\}$ of F whose sum equals some value $(a,b) \in (\mathbb{F}_2^n \setminus \{0_n\}) \times \mathbb{F}_2^m$. The smaller δ_F, the better the contribution of F to the resistance to differential cryptanalysis. For every (n,m)-function F, we have $\delta_F \geq 2^{n-m}$ (as observed by Nyberg) with equality if and only if F is perfect nonlinear (which can exist if and only if n is even and $m \leq n/2$), and when $m \geq n$, the smallest possible value of δ_F is 2, since δ_F is always even.

We have seen that the differential uniformity is a CCZ invariant (and here also, if $n = m$ and if F is a permutation, then F and its inverse F^{-1} have the same differential uniformity).

11.3 Almost perfect nonlinear and almost bent functions

We have seen in Definition 41, page 159, that differentially 2-uniform (n, n)-functions are called *almost perfect nonlinear* (in brief, *APN*) and contribute to a maximal resistance to differential cryptanalysis.
AB functions contribute to a maximal resistance to both linear and differential cryptanalyses; indeed, according to the proof of the SCV bound and as observed by Chabaud and Vaudenay:

Proposition 156 *For every n odd, AB (n, n)-functions are APN.*

The converse of Proposition 156 is false in general; it is true for quadratic functions in odd dimension [257] and in more general cases that we shall see at page 414. The implication of Proposition 156 can be more precisely changed into a characterization of AB functions:

Proposition 157 *Any vectorial function $F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is AB if and only if F is APN and* plateaued with single amplitude *(see Definition 67, page 302).*

This comes directly from Relations (3.22) and (3.25), page 141. *We shall see in Proposition 163, page 414, that if n is odd, the condition "with the same amplitude" is in fact not necessary.*

AB functions exist for every odd $n \geq 3$. APN functions exist for every $n \geq 2$. Function $F(x) = x^3$, $x \in \mathbb{F}_{2^n}$, is an example; others will be given below.

According to Relations (3.24) and (3.25), and to the two lines following them, APN (n, n)-functions F are characterized[3] by the fact that the power sum of degree 4 of the values of their Walsh transform is minimal:

$$\sum_{v \in \mathbb{F}_2^n, u \in \mathbb{F}_2^n} W_F^4(u, v) = 3 \cdot 2^{4n} - 2 \cdot 2^{3n} \tag{11.2}$$

or equivalently, replacing $\sum_{u \in \mathbb{F}_2^n} W_F^4(u, 0_n)$ by its value 2^{4n}:

Theorem 25 *[341] Any (n, n)-function F is APN if and only if*

$$\sum_{v \in \mathbb{F}_2^n \setminus \{0_n\}, u \in \mathbb{F}_2^n} W_F^4(u, v) = 2^{3n+1}(2^n - 1), \tag{11.3}$$

which is the minimal possible value of this sum for all (n, n)-functions.

[3] This characterization is equivalent to a characterization due to Helleseth [592] in the framework of sequences.

We have seen at page 132 that this implies that the Walsh support of APN (n,n)-functions has size at least $1+(2^n-1)\,2^{n-1}$.

Using Relation (3.10), page 119, F is then APN if and only if $\sum_{v\in\mathbb{F}_2^n\setminus\{0_n\}}\mathcal{V}(v\cdot F)=2^{2n+1}(2^n-1)$. In fact, as observed in [910], F is APN if and only if, for every $a\in\mathbb{F}_2^n\setminus\{0_n\}$, $\sum_{v\in\mathbb{F}_2^n}\mathcal{F}^2(D_a(v\cdot F))=2^n|\{(x,y)\in(\mathbb{F}_2^n)^2;\,D_aF(x)=D_aF(y)\}|$ equals 2^{2n+1} (*i.e.* is minimal), and Theorem 25 can also be referred to [910]. Using Parseval's relation (3.23) and Relation (11.3), any (n,n)-function F is APN if and only if:

$$\sum_{\substack{v\in\mathbb{F}_2^n\setminus\{0_n\}\\ u\in\mathbb{F}_2^n}} W_F^2(u,v)\left(W_F^2(u,v)-2^{n+1}\right)=0. \tag{11.4}$$

This characterization will have nice consequences in the sequel.
It is easily shown as in the proof of the SCV bound, that for every (n,n)-function, the power sum of degree 3: $\sum_{v\in\mathbb{F}_2^n,u\in\mathbb{F}_2^n}\left(\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}\right)^3$ equals

$$2^{2n}\left|\left\{(x,y)\in\mathbb{F}_2^{2n};\,F(x)+F(y)+F(x+y)=0_n\right\}\right|.$$

Applying (with $z=0_n$) the property that, for every APN function F, the relation $F(x)+F(y)+F(z)+F(x+y+z)=0_n$ can be achieved only when $x=y$ or $x=z$ or $y=z$, we have then, for every APN function such that $F(0_n)=0_n$:

$$\sum_{v\in\mathbb{F}_2^n,u\in\mathbb{F}_2^n} W_F^3(u,v)=3\cdot 2^{3n}-2\cdot 2^{2n}. \tag{11.5}$$

But this property is not characteristic (except for plateaued functions, see below) of APN functions among those (n,n)-functions such that $F(0_n)=0_n$, since it is only characteristic of the fact that $\sum_{x\in E}F(x)\neq 0_n$ for every 2-dimensional vector subspace E of $\mathbb{F}_2^n$ (which is more restrictive than for every 2-dimensional flat).
As already seen at page 132, the *spectral complexity* of an APN function satisfies $|\{(u,v)\in\mathbb{F}_2^n\times\mathbb{F}_2^m;\,W_F(u,v)\neq 0\}|\geq\frac{2^{4n}}{3\cdot 2^{2n}-2^{n+1}}\approx\frac{2^{2n}}{3}$.

Note that for every APN function F, we have

$$\left|\{(a,b)\in(\mathbb{F}_2^n)^2\,,a\neq b\,;\,F(a)=F(b)\}\right|\leq 2\cdot(2^n-1)$$

since $F(a)=F(b)$ is equivalent to $D_{a+b}F(a)=0_n$.
Hence, we have $\left|\{(a,b)\in(\mathbb{F}_2^n)^2\,;\,F(a)=F(b)\}\right|=\sum_{z\in\mathbb{F}_2^n}|F^{-1}(z)|^2\leq 3\cdot 2^n-2$ and therefore $|F^{-1}(z)|\leq\left\lfloor\sqrt{3\cdot 2^n-2}\right\rfloor\leq 2^{n/2+1}$, for every $z\in\mathbb{F}_2^n$.

We have seen at page 160 the different ways of expressing that a function is APN. It is observed in [71, Theorem 3] (recalled in [94] and slightly modified in [353]) that, given any linear hyperplane H in $\mathbb{F}_2^n$ and any (n,n)-function F, the necessary property (for F to be APN) that D_aF is 2-to-1 when a is nonzero and belongs to H is also sufficient. Let us give a simple proof: suppose that F is not APN, then there exists an affine plane P in $\mathbb{F}_2^n$, say $P=u+E$ where E is a linear plane, on which F is affine (see page 160). The direction E of P contains at least one nonzero element a of H, because $\dim E + \dim H > n$; then D_aF is not 2-to-1, a contradiction.

We have seen at page 306 that a subclass of APN functions (and superclass of AB quadratic permutations), called crooked functions, has been considered in [57], further studied in [172, 410, 726] and generalized in [727, 729, 80]. There are only two known cases of crooked functions corresponding to the original definition: Gold power AB functions and the class of quadratic AB binomials constructed in [151, 158]. All known crooked functions in the larger sense are quadratic APN and we have several constructions of them. Among the known 487 quadratic AB functions over $\mathbb{F}_{2^7}$ only Gold functions are CCZ equivalent to permutations (among AB functions, permutations are rare). It can be proved [728] that every power crooked function is a Gold function (see definition below).

The maximal algebraic degree of APN functions is unknown: for n odd, it is probably $n-1$ (achieved by x^{2^n-2}), but it is unproven that it is not n, and for n even, it is still more undetermined. All known APN functions (see pages 428 and 434) have algebraic degree at most $n-1$. It has been proved in [156], thanks to characterizations by means of derivatives and power moments of the Walsh transform, that APN functions of algebraic degree n do not exist for $n \geq 3$ within the classes of power functions modified at input 0 (and the nonexistence for power functions modified in one point was checked by computer for $n \leq 13$) and of plateaued functions modified in one point. See more in [153, 654], and in [167] where the notion of APNness is weakened (differently from [24]).

A lower bound given in [297], involving the differential uniformity and the size of the image set of vectorial functions, is equivalent to a bound on the size of the image set of differentially uniform functions (see page 447). In the case of APN functions, this latter bound writes $|Im(F)| \geq \left\lceil \frac{2^{2n}}{3\cdot 2^n-2} \right\rceil$.

11.3.1 Other Characterizations of AB and APN functions

We have seen above the main characterizations but others exist:

Other characterizations by the Walsh transform

We shall see when characterizing general differentially uniform functions in Section 11.6 that other characterizations of APN functions exist by the Walsh transform, as shown in [250].

Characterization by the degrees of univariate polynomials

An (n, n)-function F, given in univariate form, is APN if and only if, for every $a \in \mathbb{F}_{2^n}^*$ and every $b \in \mathbb{F}_{2^n}$, the polynomial $\gcd(x^{2^n} + x, F(x) + F(x+a) + b)$ has degree at most 2 (that is, has degree 0 or 2). Indeed, $x^{2^n} + x$ splits completely over $\mathbb{F}_{2^n}$ and its roots, all simple, are all the elements of $\mathbb{F}_{2^n}$. The polynomial $P(x) = F(x) + F(x+a) + b$ has then a number of zeros in $\mathbb{F}_{2^n}$ equal to the degree of $Q(x) = \gcd(P(x), x^{2^n} + x)$. The degree of $Q(x)$ is 2, that is, the equation $F(x) + F(x+a) = b$ has solutions, if and only if $\gamma_F(a,b) = 1$, where γ_F has been defined at page 254 and will be studied more in detail in Proposition 158 below.

Remark. If F is a quadratic (n,n)-function, the equation $F(x) + F(x+a) = b$ is a linear equation. It admits then at most 2 solutions for every nonzero a and every b if and only if the related homogeneous equation $F(x) + F(x+a) + F(0_n) + F(a) = 0_n$ admits at most 2 solutions for every nonzero a. We shall see that this generalizes to plateaued functions. In the case of a quadratic function, F is APN if and only if the associated bilinear symmetric $(2n,n)$-function $\beta_F(x,y) = F(0_n) + F(x) + F(y) + F(x+y)$ never vanishes when x and y are $\mathbb{F}_2$-linearly independent vectors of $\mathbb{F}_2^n$. For functions of higher degrees, the fact that $\beta_F(x,y)$ (which is no longer bilinear) never vanishes when x and y are linearly independent is only necessary for APNness (sufficient for plateaued functions). □

Characterization by the ANF

By definition, an (n,n)-function is APN if and only if, for every nonzero $a \in \mathbb{F}_2^n$,

$$\delta_0\Big(F(x) + F(x+a) + F(y) + F(y+a)\Big) \oplus \delta_0\Big(x+y\Big) \oplus \delta_0\Big(x+y+a\Big) \equiv 0$$

(where $\equiv 0$ means "equals the zero function"), where $\delta_0(z) = \prod_{i=1}^n (z_i \oplus 1)$ is the Dirac (or Kronecker) symbol. Indeed, this equation expresses that $F(x) + F(x+a) = F(y) + F(y+a)$ if and only if $x = y$ or $x = y + a$. Equivalently, denoting by H_a any linear hyperplane excluding a, function D_aF is injective on H_a, that is:

$$1_{H_a}(x)\, 1_{H_a}(y)\ [\delta_0(F(x) + F(x+a) + F(y) + F(y+a)) \oplus \delta_0(x+y)] \equiv 0.$$

These identities, when considered as multivariate polynomial equalities, need to be viewed in $\mathbb{F}_2[x,y]/(x_i^2 + x_i, y_i^2 + y_i; i = 1, \dots, n)$.
They can also be considered as univariate identities over $\mathbb{F}_{2^n}$, where $\delta_0(z) = 1 + z^{2^n-1}$, and they need then to be reduced modulo $x^{2^n} + x$ and modulo $y^{2^n} + y$ before being checked as identically zero.

Characterization by the ANFs of affine equivalent functions

A necessary condition dealing with *quadratic terms in the ANF of any APN function* has been observed in [71]. Given any APN function F (quadratic or not), every quadratic term x_ix_j $(1 \le i < j \le n)$ must appear with a non-null coefficient in the *algebraic normal form* of F. Indeed, we know that the coefficient of any monomial $\prod_{i\in I} x^i$ in the ANF of F equals $a_I = \sum_{x\in\mathbb{F}_2^n;\ supp(x)\subseteq I} F(x)$ (this sum being calculated in $\mathbb{F}_2^n$). Applied for instance to $I = \{n-1, n\}$, this gives $a_I = F(0,\dots,0,0,0) + F(0,\dots,0,0,1) + F(0,\dots,0,1,0) + F(0,\dots,0,1,1)$, and F being APN, this vector cannot be null. Note that, since the notion of almost perfect nonlinearity is affine invariant (see below), this condition must be satisfied by all of the functions $L' \circ F \circ L$, where L' and L are affine automorphisms of $\mathbb{F}_2^n$. Extended this way (*i.e.* writing that all degree 2 terms have non-null coefficients in the ANF of every affinely equivalent function), the condition becomes necessary and sufficient (indeed, for every distinct x, y, z in $\mathbb{F}_2^n$, there exists an affine automorphism L of $\mathbb{F}_2^n$ such that $L(0, \ldots, 0, 0, 0) = x$, $L(0, \ldots, 0, 1, 0) = y$ and $L(0, \ldots, 0, 0, 1) = z$; so the condition tells that $\sum_{x \in P} F(x)$ is nonzero for every 2-dimensional affine space P).

Characterizations by the Hamming weight and the bentness of associated Boolean functions

The properties of APNness and ABness can be translated in terms of Boolean functions, as observed in [257] and already encountered at page 254:

Proposition 158 *Let F be any (n,n)-function. For every $a, b \in \mathbb{F}_2^n$, let $\gamma_F(a,b)$ equal 1 if the equation $F(x) + F(x+a) = b$ admits solutions, with $a \neq 0_n$. Otherwise, let $\gamma_F(a,b)$ be null. Then:*
1) F is APN if and only if γ_F has Hamming weight $2^{2n-1} - 2^{n-1}$, and we have then, for every $u, v \in \mathbb{F}_2^n$: $W_{\gamma_F}(u,v) = \begin{cases} 2^n \text{ if } (u,v) = (0_n, 0_n) \\ 2^n - W_F^2(u,v) \text{ otherwise;} \end{cases}$
2) F is AB if and only if γ_F is bent. The dual of γ_F is then the indicator *of the Walsh support of F, deprived of $(0_n, 0_n)$.*

Proof.
1) If F is APN, then for every $a \neq 0_n$, the mapping $x \mapsto F(x) + F(x+a)$ is 2-to-1 (that is, the size of the pre-image of any vector equals 0 or 2). Hence, γ_F has Hamming weight $2^{2n-1} - 2^{n-1}$. The converse is also straightforward.
We assume now that F is APN. We have $W_{\gamma_F}(0_n, 0_n) = 2^{2n} - 2w_H(\gamma_F) = 2^n$. For $(u,v) \neq (0_n, 0_n)$, we have:

$$\begin{aligned} W_{\gamma_F}(u,v) &= -2\,\widehat{\gamma_F}(u,v) = -\sum_{a \neq 0_n, x \in \mathbb{F}_2^n} (-1)^{u \cdot a \,\oplus\, v \cdot (F(x) + F(x+a))} \\ &= 2^n - \sum_{a, x \in \mathbb{F}_2^n} (-1)^{u \cdot a \,\oplus\, v \cdot (F(x) + F(x+a))} \\ &= 2^n - \sum_{x, y \in \mathbb{F}_2^n} (-1)^{u \cdot (x+y) \,\oplus\, v \cdot (F(x) + F(y))} = 2^n - W_F^2(u,v). \end{aligned}$$

2) We deduce that F is AB if and only if $W_{\gamma_F}(u,v) = \pm 2^n$ for every $(u,v) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$, *i.e.*, γ_F is bent. Then for every $(u,v) \neq (0_n, 0_n)$, we have $\widetilde{\gamma_F}(u,v) = 0$, that is, $W_{\gamma_F}(u,v) = 2^n$ if and only if $W_F(u,v) = 0$. Hence, the dual of γ_F is the indicator of the Walsh support of F, deprived of $(0_n, 0_n)$. □

Denoting by $L = (L_1, L_2)$ an affine automorphism mapping the graph of F to the graph of G, we have $\gamma_F = \gamma_G \circ \mathcal{L}$, where $\mathcal{L}$ is the linear automorphism such

that $L = \mathcal{L} + cst$. Indeed, we have $G = F_2 \circ F_1^{-1}$, where $F_1(x) = L_1(x, F(x))$ and $F_2(x) = L_2(x, F(x))$; the value $\gamma_G(a, b)$ equals 1 if and only if $a \neq 0_n$ and there exists (x, y) in $\mathbb{F}_2^n \times \mathbb{F}_2^n$ such that $F_1(x) + F_1(y) = a$ and $F_2(x) + F_2(y) = b$, that is, $\mathcal{L}(x, F(x)) + \mathcal{L}(y, F(y)) = \mathcal{L}(x + y, F(x) + F(y)) = (a, b)$. Hence, $\gamma_G \circ \mathcal{L}(a, b) = 1$ if and only if $\gamma_F(a, b) = 1$. Note that different functions may have the same γ_F; see in [561] a study when the function γ_F is the one associated to Gold functions. The linear equivalence between functions γ_F could potentially lead to an equivalence notion strictly more general than CCZ equivalence; this needs to be studied. It is observed in this same reference that if two functions F, F' are such that $\gamma_F = \gamma_{F'}$, then for any function G taken EA equivalent to F, there exists G' which is EA equivalent to F' and such that $\gamma_G = \gamma_{G'}$. In [109] is observed that if two functions F, F' have the same difference distribution table (DDT, see page 158), then for any function G taken CCZ equivalent to F, there exists G' which is CCZ equivalent to F' and such that G and G' have the same DDT (and the same is true with EA instead of CCZ); it is also shown that, for any APN permutation F and any pair $\{a, a'\}$ of distinct nonzero elements, the functions $\gamma_F(a, x)$ and $\gamma_F(a', x)$ are different. It is conjectured in this same reference that two permutations F and G having such property and such that $\gamma_F = \gamma_G$ (*i.e.* with the same DDT) are such that $G(x) = F(x + a) + b$. A guess-and-determine algorithm for reconstructing an S-box from its DDT is given, which is outperformed by an algorithm from [489].
The γ_F functions associated to some AB functions are addressed at page 254 and those associated to some of the known APN functions are determined in [152, 257] (for some other cases it is an open problem).

Remark. Let F be APN. According to Relation (3.3), page 102, we have $nl(F) = \min_{v \in \mathbb{F}_2^n, v \neq 0_n} nl(v \cdot F) \geq 2^{n-2} - \frac{1}{4} \max_{v \in \mathbb{F}_2^n, v \neq 0_n} \min_{e \in \mathbb{F}_2^n, e \neq 0_n} |\mathcal{F}(v \cdot D_e F)| = 2^{n-2} - \frac{1}{2} \max_{v \neq 0_n} \min_{e \neq 0_n} |\sum_{b \in \mathbb{F}_2^n} \gamma_F(e, b)(-1)^{v \cdot b}|$. We obtain then $nl(F) \geq 2^{n-2} - \frac{1}{2} \max_{v \neq 0_n} \min_{e \neq 0_n} |\widehat{\gamma_{F,e}}(v)| \geq 2^{n-2} - \frac{1}{2} \min_{e \neq 0_n} \max_{v \neq 0_n} |\widehat{\gamma_{F,e}}(v)| = \max_{e \neq 0_n} nl(\gamma_{F,e}) - 2^{n-2}$, where $\gamma_{F,e}(b) = \gamma_F(e, b)$. These lower bounds are not efficient for highly nonlinear functions like AB functions, since they are below 2^{n-2} which is much smaller than $2^{n-1} - 2^{\frac{n-1}{2}}$, but since little is known on the nonlinearity of APN non-AB functions, they are worth mentioning. □

Characterizations by the numbers of solutions of systems of equations

There exists a characterization of AB functions by van Dam and Fon-Der-Flaass in [410] similar to the characterization of APN functions by the fact that, for every $(a, b) \neq (0_n, 0_n)$, the system $\begin{cases} x + y & = & a \\ F(x) + F(y) & = & b \end{cases}$ admits 0 or 2 solutions:

Proposition 159 *Any (n,n)-function F is AB if and only if the system:*

$$\left\{ \begin{array}{lcl} x+y+z & = & a \\ F(x)+F(y)+F(z) & = & b \end{array} \right. \tag{11.6}$$

admits $3\cdot 2^n-2$ solutions if $b=F(a)$ and 2^n-2 solutions otherwise.

Indeed, F is AB if and only if, for every $v\in\mathbb{F}_2^n\setminus\{0_n\}$ and every $u\in\mathbb{F}_2^n$, we have $\left(\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}\right)^3=2^{n+1}\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}$, and we know that two *pseudo-Boolean* functions are equal to each other if and only if their Fourier-Hadamard transforms are equal. The value at (a,b) of the Fourier-Hadamard transform of the function of (u,v) equal to $\left(\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}\right)^3$ if $v\neq 0_n$, and to 0 otherwise equals

$$\sum_{\substack{u\in\mathbb{F}_2^n\\ v\in\mathbb{F}_2^n}}\left(\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}\right)^3(-1)^{a\cdot u\oplus b\cdot v}-2^{3n}=$$

$$2^{2n}\left|\left\{(x,y,z)\in\mathbb{F}_2^{3n}\,;\left\{\begin{array}{l} x+y+z=a\\ F(x)+F(y)+F(z)=b\end{array}\right.\right\}\right|-2^{3n},$$

and the value of the Fourier-Hadamard transform of the function which is equal to $2^{n+1}\sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}$ if $v\neq 0_n$, and to 0 otherwise equals

$$2^{3n+1}\left|\left\{x\in\mathbb{F}_2^n\,;\left\{\begin{array}{l} x=a\\ F(x)=b\end{array}\right.\right\}\right|-2^{2n+1}.$$

This proves the result. Note that $3\cdot 2^n-2$ is the number of triples (x,x,a), (x,a,x) and (a,x,x) where x ranges over $\mathbb{F}_2^n$. Hence the condition when $F(a)=b$ means that these particular triples are the only solutions of the system (11.6). This is equivalent to saying that F is APN and we can replace the first condition of van Dam and Fon-Der-Flaass by "F is APN". Denoting $c=F(a)+b$, we have then:

Corollary 27 *Let n be any positive integer and F any APN (n,n)-function. Then F is AB if and only if, for every $c\neq 0_n$ and every a in $\mathbb{F}_2^n$, the equation $F(x)+F(y)+F(a)+F(x+y+a)=c$ has 2^n-2 solutions.*

Let us denote by $\mathcal{A}_2$ the set of 2-dimensional flats of $\mathbb{F}_2^n$ and by Φ_F the mapping $A\in\mathcal{A}_2\to\sum_{x\in A}F(x)\in\mathbb{F}_2^n$. Corollary 27 is equivalent to saying that an APN function is AB if and only if, for every $a\in\mathbb{F}_2^n$, the restriction of Φ_F to those flats which contain a is a $\frac{2^{n-1}-1}{3}$-to-1 function (indeed, there are 6 different ways of ordering the three elements other than a in such flat). Note that the number of 2-dimensional flats of $\mathbb{F}_2^n$ containing a equals $\frac{(2^n-1)(2^n-2)}{(2^2-1)(2^2-2)}=(2^n-1)\frac{2^{n-1}-1}{3}$ and the size of $\mathbb{F}_2^n\setminus\{0_n\}$ equals 2^n-1. We have then:

Corollary 28 *Any (n,n)-function F is APN if and only if Φ_F is valued in $\mathbb{F}_2^n\setminus\{0_n\}$, and F is AB if and only if, additionally, for every $a\in\mathbb{F}_2^n$, the* *restriction of $\Phi_F : \mathcal{A}_2 \to \mathbb{F}_2^n \setminus \{0_n\}$ to those flats which contain a is balanced (that is, has uniform output).*

Note that, for every APN function F and any two distinct vectors a and a', the restriction of Φ_F to those flats which contain a and a' is injective, since for two such distinct flats $A = \{a, a', x, x + a + a'\}$ and $A' = \{a, a', x', x' + a + a'\}$, we have $\Phi_F(A) + \Phi_F(A') = F(x) + F(x + a + a') + F(x') + F(x' + a + a') = \Phi_F(\{x, x + a + a', x', x' + a + a'\}) \neq 0_n$. But this restriction of Φ_F cannot be surjective since the number of flats containing a and a' equals $2^{n-1} - 1$, which is less than $2^n - 1$.

Remark. Other characterizations can be derived with the same method as in Proposition 159's proof. For instance, F is AB if and only if, for every $v \in \mathbb{F}_2^n \setminus \{0_n\}$, $u \in \mathbb{F}_2^n$, we have $\left(\sum_{x\in\mathbb{F}_2^n} (-1)^{v\cdot F(x)\oplus u\cdot x}\right)^4 = 2^{n+1}\left(\sum_{x\in\mathbb{F}_2^n} (-1)^{v\cdot F(x)\oplus u\cdot x}\right)^2$. By applying again the Fourier-Hadamard transform and dividing by 2^{2n}, we deduce that F is AB if and only if, for every (a, b) in $(\mathbb{F}_2^n)^2$, we have:

$$\left|\left\{(x,y,z,t)\in\mathbb{F}_2^{4n}; \left\{\begin{array}{l} x+y+z+t=a \\ F(x)+F(y)+F(z)+F(t)=b \end{array}\right.\right\}\right| - 2^{2n} =$$

$$2^{n+1}\left|\left\{(x,y)\in\mathbb{F}_2^{2n}; \left\{\begin{array}{l} x+y=a \\ F(x)+F(y)=b \end{array}\right.\right\}\right| - 2^{n+1}.$$

Hence, F is AB if and only if the system $\left\{\begin{array}{lcl} x+y+z+t & = & a \\ F(x)+F(y)+F(z)+F(t) & = & b \end{array}\right.$ admits $3 \cdot 2^{2n} - 2^{n+1}$ solutions if $a = b = 0_n$ (this is equivalent to saying that F is APN), $2^{2n} - 2^{n+1}$ solutions if $a = 0_n$ and $b \neq 0_n$ (note that this condition corresponds to adding all the conditions of Corollary 27 with c fixed to b and with a ranging over $\mathbb{F}_2^n$), and $2^{2n} + 2^{n+2}\gamma_F(a, b) - 2^{n+1}$ solutions if $a \neq 0_n$ (indeed, F is APN; note that this gives a new property of AB functions). $\square$

Characterization of APN functions by the minimum distance of related codes, and of AB functions by the weight distribution of these codes

A relationship has been observed in [641] (not exactly in terms of APNness since this notion was not known by the authors) and developed further in [257]) (see also [642, 1099]) between the properties, for an (n, n)-function, of being APN or AB and properties of related codes. This makes that APN functions are generalizations of the cube (n, n)-function x^3, whose related code is the 2-error correcting BCH code (see page 25):

Proposition 160 *[257] Let F be any function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^n}$ such that $F(0) =$*

0. *Let H be the matrix* $\left[\begin{array}{ccccc} 1 & \alpha & \alpha^2 & \dots & \alpha^{2^n-2} \\ F(1) & F(\alpha) & F(\alpha^2) & \dots & F(\alpha^{2^n-2}) \end{array}\right]$*, where α is a* primitive element *of $\mathbb{F}_{2^n}$, where each symbol stands for the column of its coordinates with respect to a basis of the $\mathbb{F}_2$-vector space $\mathbb{F}_{2^n}$, and where only linearly independent rows are kept. Let C_F be the linear code admitting H for parity check matrix. Then, F is APN if and only if C_F has minimum distance 5, and F is AB if and only if $C_F^\perp$ (admitting H for generator matrix) has Hamming weights $0, 2^{n-1}-2^{\frac{n-1}{2}}, 2^{n-1}$ and $2^{n-1}+2^{\frac{n-1}{2}}$ (equivalently, has nonzero Hamming weights between $2^{n-1}-2^{\frac{n-1}{2}}$ and $2^{n-1}+2^{\frac{n-1}{2}}$).*

Proof. Since H contains no zero column, C_F has no codeword of Hamming weight 1 and since all columns of H are distinct vectors, C_F has no codeword of Hamming weight 2. Hence[4], C_F has minimum distance at least 3. This minimum distance is also at most 5, since otherwise, a $[2^n-2, k, d \geq 5]$ code with $k \geq 2^n - 1 - 2n$ would exist by puncturing, and we know from [482] that this is impossible. The fact that C_F has no codeword of weight 3 or 4 is by definition equivalent to the APNness of F, since a vector $(c_0, c_1, \dots, c_{2^n-2}) \in \mathbb{F}_2^{2^n-1}$ is a codeword if and only if $\left\{\begin{array}{l} \sum_{i=0}^{2^n-2} c_i\alpha^i = 0 \\ \sum_{i=0}^{2^n-2} c_iF(\alpha^i) = 0 \end{array}\right.$. The nonexistence of codewords of Hamming weight 3 is then equivalent to the fact that $\sum_{x\in E} F(x) \neq 0$ for every 2-dimensional vector subspace E of $\mathbb{F}_{2^n}$ and the nonexistence of codewords of Hamming weight 4 is equivalent to the fact that $\sum_{x\in A} F(x) \neq 0$ for every 2-dimensional flat A not containing 0. The characterization of ABness through the weights of $C_F^\perp$ comes directly from the characterization of AB functions by their Walsh transform values, respectively by their nonlinearity, and from the fact that the Hamming weight of the Boolean function $v \cdot F(x) \oplus u \cdot x$ equals $2^{n-1} - \frac{1}{2}W_F(u, v)$. □

Remark.
1. If F is APN and $n > 2$, then C_F has dimension $2^n - 1 - 2n$ exactly (*i.e.* all the rows in the matrix $H = \left[\begin{array}{ccccc} 1 & \alpha & \alpha^2 & \dots & \alpha^{2^n-2} \\ F(1) & F(\alpha) & F(\alpha^2) & \dots & F(\alpha^{2^n-2}) \end{array}\right]$ are linearly independent), since according to [482] again, $[2^n - 1, 2^n - 2n, 5]$ codes do not exist. A direct proof of the fact that $C_F^\perp$ has indeed dimension $2n$ is given by Dillon in [447]. This property of $C_F^\perp$ is equivalent to the fact that F has nonzero nonlinearity. Dillon uses Relation (11.2), page 403, and observes that if $v_0 \cdot F$ is affine for some $v_0 \neq 0$ then $\sum_{\substack{u,v\in\mathbb{F}_{2^n} \\ v\notin\{0,v_0\}}} W_F^4(u, v) = (2^n - 2) \cdot 2^{3n}$, which means that all component functions of F except $v_0 \cdot F$ are bent. This allows building a bent $(n, n-1)$-function, a contradiction with Nyberg's result (Proposition 104, page 296). A slightly different proof (also using Nyberg's result) was known earlier, see Proposition 161 below.
2. Any subcode of dimension $2^n - 1 - 2n$ of the $[2^n - 1, n, 3]$ Hamming code is

[4] We can also say that C_F is a subcode of the *Hamming code* (see page 23).

a code C_F for some function F.
3. Proposition 160 assumes that $F(0) = 0$. If we want to express the APN-ness of any (n,n)-function, another matrix can be considered as in [135]: the $(2n+1) \times (2^n - 1)$ matrix $\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & \alpha & \alpha^2 & \dots & \alpha^{2^n-2} \\ F(0) & F(1) & F(\alpha) & F(\alpha^2) & \dots & F(\alpha^{2^n-2}) \end{bmatrix}$.
Then F is APN if and only if the code $\widetilde{C_F}$ admitting this parity check matrix has parameters $[2^n, 2^n - 1 - 2n, 6]$. To prove this, note first that this code does not change if we add a constant to F (contrary to C_F). Hence, by adding the constant $F(0)$, we can assume that $F(0) = 0$. Then, the code $\widetilde{C_F}$ is the *extended code* of C_F (obtained by adding to each codeword of C_F a first coordinate equal to the sum modulo 2 of its coordinates). Since $F(0) = 0$, we can apply Proposition 160 and it is clear that C_F is a $[2^n - 1, 2^n - 1 - 2n, 5]$ code if and only if $\widetilde{C_F}$ is a $[2^n, 2^n - 1 - 2n, 6]$ code, since we know from [482] that C_F cannot have minimum distance larger than 5.
Note that Proposition 156, page 403, means that if $\widetilde{C_F}^{\perp}$ has the highest possible minimum distance $2^{n-1} - 2^{\frac{n-1}{2}}$, then $\widetilde{C_F}$ has minimum distance at least 6.
4. As observed in [135], given two (n,n)-functions F and G such that $F(0) = G(0) = 0$, there exists a linear automorphism[5] which maps $\mathcal{G}_F$ to $\mathcal{G}_G$ if and only if the codes C_F and C_G are equivalent (that is, are equal up to some permutation of the coordinates of their codewords). Indeed, the graph $\mathcal{G}_F$ of F equals the (unordered) set of columns in the parity check matrix of the code C_F, plus an additional point equal to the all-zero vector. Hence, the existence of a linear automorphism which maps $\mathcal{G}_F$ onto $\mathcal{G}_G$ is equivalent to the fact that the parity check matrices[6] of the codes C_F and C_G are equal up to multiplication (on the left) by an invertible matrix and to permutation of the columns. Since two codes with given parity check matrices are equal if and only if these matrices are equal up to multiplication on the left by an invertible matrix, this completes the proof. Similarly, two functions F and G taking any values at 0 are *CCZ equivalent* if and only if the codes $\widetilde{C_F}$ and $\widetilde{C_G}$ are equivalent.
5. For every (n,n)-function F such that $F(0) = 0$, the two first power moments of W_F are known: we have $\sum_{u,v \in \mathbb{F}_{2^n}} W_F(u,v) = 2^n \sum_{v \in \mathbb{F}_{2^n}} (-1)^{v \cdot F(0)} = 2^{2n}$, and $\sum_{u,v \in \mathbb{F}_{2^n}} W_F^2(u,v) = 2^{3n}$ (the former equality is given by the *inverse Walsh transform formula* (2.43), page 78, and the latter is given by the Parseval relation (2.48), page 79). If F is APN then we have also the two next power moments: Relations (11.2) and (11.5), page 404. In the case F is AB, this makes possible to determine the value distribution of W_F and therefore the weight distribution of $C_F^{\perp}$ uniquely[7]. Indeed, there are only 3 nonzero weights, which are known, and

[5] Note that this is a sub-case of CCZ equivalence - in fact, a strict sub-case as shown in [135].
[6] This is true also for the generator matrices of the codes.
[7] The determination of such weight distribution is not known so often (when the code does not contain the all-one vector) since determining the Walsh value distribution of the

we need then only to determine the 3 numbers of codewords of each weight; the four equations obtained, which are linear in these numbers, make this possible. There are 1 codeword of null Hamming weight, $(2^n-1)(2^{n-2}+2^{\frac{n-3}{2}})$ codewords of Hamming weight $2^{n-1}-2^{\frac{n-1}{2}}$, $(2^n-1)(2^{n-2}-2^{\frac{n-3}{2}})$ codewords of Hamming weight $2^{n-1}+2^{\frac{n-1}{2}}$, and $(2^n-1)(2^{n-1}+1)$ codewords of Hamming weight 2^{n-1}. See more details in [257] (the calculations are made there equivalently with the Pless power moment equalities of [958]). We shall see that function x^3 over $\mathbb{F}_{2^n}$ is an AB function (for n odd). The code $C_F^\perp$ corresponding to this function is an important code: the dual of the 2-error-correcting BCH code of length 2^n-1. □

We have seen that if F is APN on $\mathbb{F}_{2^n}$, $n>2$, and $F(0)=0$, the code $C_F^\perp$ has dimension $2n$. Equivalently, the code whose generator matrix equals $\begin{bmatrix} F(1) & F(\alpha) & F(\alpha^2) & \dots & F(\alpha^{2^n-2}) \end{bmatrix}$, and which can therefore be seen as the code $\{tr_n(vF(x);\, v\in\mathbb{F}_{2^n}\}$, has dimension n and intersects the *simplex code* $\{tr_n(ux);\, u\in\mathbb{F}_{2^n}\}$ of generator matrix $\begin{bmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{2^n-2}\end{bmatrix}$ only in the null vector. This can be proved directly:

Proposition 161 *[237] Let F be APN over $\mathbb{F}_2^n$ with $n>2$. Then $nl(F)$ cannot be null and, assuming that $F(0_n)=0_n$, the code $C_F^\perp$ has dimension $2n$.*

Proof. Suppose there exists $v\neq 0_n$ such that $v\cdot F$ is affine. Without loss of generality (by composing F with an appropriate linear automorphism and adding an affine function to F), we can assume that $v=(0,\dots,0,1)$ and that $v\cdot F$ is null. Then, every derivative of F is 2-to-1 and has null last coordinate. Hence, for every $a\neq 0_n$ and every b, the equation $D_aF(x)=b$ has no solution if $b_n=1$ and it has 2 solutions if $b_n=0$. The $(n,n-1)$ function obtained by erasing the last coordinate of $F(x)$ has therefore balanced derivatives; hence it is a bent $(n,n-1)$-function, a contradiction with Nyberg's result (Proposition 104, page 296), since $n-1>\frac{n}{2}$. The last sentence in the statement is straightforward. □
For $n=2$, the nonlinearity can be null; example: function $(x_1,x_2)\to(x_1x_2,0)$.

Remark. As observed at page 159, the nonlinearity and the differential uniformity of general functions do not seem correlated. However, Proposition 161 shows that, for APN functions, a null nonlinearity is impossible. Moreover, all known APN functions have a rather good nonlinearity (probably at least $2^{n-1}-2^{\frac{3n}{5}-1}-2^{\frac{2n}{5}-1}$, but this has to be confirmed since the nonlinearity of the Dobbertin function is unknown except for small values of n). The question of knowing whether it is because we know too few APN functions or because there is some correlation in the case of such optimal functions seems wide open. □

J. Dillon (private communication) observed that the property of Proposition 161 implies that, for every nonzero $c\in\mathbb{F}_{2^n}$, the equation $F(x)+F(y)+F(z)+F(x+y+z)=c$ must have a solution (that is, the function Φ_F introduced

function is much more difficult in general than determining the absolute value distribution, which for an AB function is easily deduced from the single Parseval's relation.

after Corollary 27 is onto $\mathbb{F}_2^n \setminus \{0_n\}$; we have seen this for AB functions since we saw that this function is balanced, but it is new for APN functions). Indeed, otherwise, for every Boolean function $g(x)$, the function $F(x) + c\, g(x)$ would be APN. But this is contradictory with Proposition 161 if we take $g(x) = v_0 \cdot F(x)$ (that is, $g(x) = tr_n(v_0 F(x))$ if we have identified $\mathbb{F}_2^n$ with the field $\mathbb{F}_{2^n}$) with $v_0 \notin c^\perp$, since we have then $v_0 \cdot [F(x) + c\, g(x)] = v_0 \cdot F(x) \oplus g(x)\,(v_0 \cdot c) = 0$.

Characterization of AB functions by *uniformly packed codes*

Proposition 162 *[257] Let F be any (n, n)-function, n odd. Then F is AB if and only if C_F is a uniformly packed code (see Definition 2, page 25) of length $N = 2^n - 1$ with minimum distance 5 and covering radius 3.*

It is deduced in [257, Corollary 3] that an APN function is AB if and only if $\{W_F(u, v);\, u, v \in \mathbb{F}_2^n, v \neq 0_n\}$ has three values.

Characterization of AB functions, among APN functions, by the divisibility of their Walsh transform values (n odd); consequence for *plateaued* functions

We have seen that all AB functions are APN. The converse is false, in general. But if n is odd and if F is APN, then, as shown in [195, 186], there exists a nice necessary and sufficient condition, for F being AB: the weights of $C_F^\perp$ are all divisible by $2^{\frac{n-1}{2}}$ (see also [196], where the divisibilities for several types of such codes are calculated, where tables of exact divisibilities are computed and where proofs are given that a great deal of power functions are not AB). In other words and slightly more generally:

Proposition 163 *Let F be an APN (n, n)-function. Then F is AB if and only if all the values $W_F(u, v)$ of the Walsh spectrum of F are divisible by $2^{\lceil \frac{n+1}{2} \rceil}$.*

Proof. The condition is clearly necessary (with n necessarily odd). Conversely, assume that F is APN and that all the values $W_F(u, v) = \sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot F(x) \oplus u \cdot x}$ are divisible by $2^{\lceil \frac{n+1}{2} \rceil}$. Writing $W_F^2(u, v) = 2^{n+1} \lambda_{u,v}$, where all $\lambda_{u,v}$'s are integers, Relation (11.4), page 404, implies then:

$$\sum_{v \in \mathbb{F}_2^{n*}, u \in \mathbb{F}_2^n} (\lambda_{u,v}^2 - \lambda_{u,v}) = 0, \tag{11.7}$$

and since all the integers $\lambda_{u,v}^2 - \lambda_{u,v}$ are non-negative ($\lambda_{u,v}$ being an integer), we deduce that $\lambda_{u,v}^2 = \lambda_{u,v}$ for every $v \in \mathbb{F}_2^{n*}, u \in \mathbb{F}_2^n$, *i.e.* $\lambda_{u,v} \in \{0, 1\}$. □

Proposition 163 shows that if n is odd and an APN (n, n)-function F is plateaued, or more generally if $F = F_1 \circ F_2^{-1}$ where F_2 is a permutation and the linear combinations of the coordinate functions of F_1 and F_2 are plateaued, then F is AB, since $\sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot F(x) \oplus u \cdot x} = \sum_{x \in \mathbb{F}_2^n} (-1)^{v \cdot F_1(x) \oplus u \cdot F_2(x)}$ is divisible by $2^{\frac{n+1}{2}}$. This makes it possible to deduce easily the AB property of Gold and Kasami functions (see their definitions below) from their APN property, since the Gold

AB functions are quadratic and the Kasami AB functions are equal, when n is odd, to $F_1 \circ F_2^{-1}$ where $F_1(x) = x^{2^{3i}+1}$ and $F_2(x) = x^{2^i+1}$ are quadratic[8].

Proposition 163 also allows to characterize AB functions among APN power functions, thanks to Proposition 21, page 92. Sufficient conditions for power functions not to be AB are given in [194].

Complementary observation on APN functions for n even

If F is APN, then there must exist $v \in \mathbb{F}_2^n \setminus \{0_n\}, u \in \mathbb{F}_2^n$ such that $W_F(u, v)$ is not divisible by $2^{\frac{n}{2}+1}$. Indeed, suppose that all the Walsh values of F have such divisibility, then writing again $W_F^2(u, v) = 2^{n+1}\lambda_{u,v}$, we have Relation (11.7), in which each nonzero $\lambda_{u,v}$ being now even satisfies $\lambda_{u,v}^2 - \lambda_{u,v} > 0$. All the values $\lambda_{u,v}^2 - \lambda_{u,v}$ are then non-negative integers and (for each $v \neq 0_n$) at least one value is strictly positive, a contradiction.

If all the Walsh values of F are divisible by $2^{\frac{n}{2}}$ (*e.g.* if F is plateaued), then we deduce that there must exist $v \in \mathbb{F}_2^n \setminus \{0_n\}, u \in \mathbb{F}_2^n$ such that $W_F(u, v) \equiv 2^{\frac{n}{2}}$ [mod $2^{\frac{n}{2}+1}$]. It is also shown in [24] that for every APN, or more generally *weakly APN*, permutation F (whose derivatives at nonzero directions take strictly more than 2^{n-2} distinct values), at most $\frac{2^n-1}{3}$ component functions of F can be partially-bent (and, in particular, F cannot then be strongly plateaued); indeed, each partially-bent component function of a permutation has a *linear kernel* of dimension at least 2 (bent functions being not balanced), and has then at least 3 constant derivatives at nonzero directions, and if there was $t > \frac{2^n-1}{3}$ partially-bent component functions of F, since $3t > |\mathbb{F}_2^n \setminus \{0_n\}|$, there would exist $a \neq 0_n$ and two distinct nonzero elements v_1, v_2 of $\mathbb{F}_2^n$ such that $v_1 \cdot D_aF$ and $v_2 \cdot D_aF$ are constant, a contradiction since $\{D_aF(x); x \in \mathbb{F}_2^n\}$ would then have at most 2^{n-2} elements.

More can be said in the case of APN plateaued functions, see page 424.

APN functions and finite geometry

We refer the reader to [430] and the references therein for the relations between APN functions and dimensional dual hyperovals or bilinear dimensional dual hyperovals. Other relations with finite geometry are shown in [895].

11.3.2 The particular case of power functions

Identifying $\mathbb{F}_2^n$ with the field $\mathbb{F}_{2^n}$ (in which we can take $x \cdot y = tr_n(xy)$ for inner product), allows considering those (n, n)-functions of the form $F(x) = x^d$, $d \in \mathbb{Z}/(2^n-1)\mathbb{Z}$, called power (n, n)-functions (and sometimes, *monomial vectorial functions*). If such F is APN, then d is called an *APN exponent* over $\mathbb{F}_{2^n}$.

Note that if d is an APN exponent over $\mathbb{F}_{2^n}$ and r divides n, then d [mod (2^r-1)]

[8] The component functions of Kasami APN functions are plateaued for every n even too. This has been proved in [448, Theorem 11] when n is not divisible by 6 and for every n even in [1142].

is an APN exponent over $\mathbb{F}_{2^r}$ (in particular, it cannot be a power of 2 if $r \geq 2$); more generally, if r divides n and $F(x)$ is an APN polynomial function over $\mathbb{F}_{2^n}$ with coefficients in $\mathbb{F}_{2^r}$, then F is APN over $\mathbb{F}_{2^r}$).

Relation between AB power functions and sequences

There is a close relationship between the nonlinearity of power functions and *sequences* used for radars and for spread-spectrum communications. Recall that a binary sequence which can be generated by an LFSR, or equivalently which satisfies a linear recurrence relation $s_i = a_1 s_{i-1} \oplus \cdots \oplus a_n s_{i-n}$, is called an *m-sequence* or a *maximum length sequence* if its period equals $2^n - 1$, which is a maximum. Such a sequence has the form $tr_n(\lambda\alpha^i)$, where $\lambda \in \mathbb{F}_{2^n}$ and α is some primitive element of $\mathbb{F}_{2^n}$. Consequently, its autocorrelation values $\sum_{i=0}^{2^n-2}(-1)^{s_i \oplus s_{i+t}}$ $(1 \leq t \leq 2^n - 2)$ are all equal to -1, that is, are optimal. This is useful for radars and for code division multiple access (CDMA) in telecommunications, since it allows sending a signal easily distinguished from any time-shifted version of itself. Finding a highly nonlinear power function (in particular, an AB power function) x^d on the field $\mathbb{F}_{2^n}$ makes possible to have a d-decimation[9] $s'_i = tr_n(\lambda\alpha^{di})$ of the sequence, whose cross-correlation values $\sum_{i=0}^{2^n-2}(-1)^{s_i \oplus s'_{i+t}} = \sum_{x \in \mathbb{F}_{2^n}^*}(-1)^{tr_n(\lambda(x^d + \alpha^{-t}x))}$ $(0 \leq t \leq 2^n - 2)$ with the sequence s_i have small (minimum) overall magnitude[10] [551, 552, 598]. In the case of an AB function, we speak of a *preferred cross-correlation*, see *e.g.* [174, 598]. The exponents of AB power functions have then been investigated as the decimations with preferred cross-correlation by the researchers on sequences (those whose names have been given to special classes of sequences and will be used for naming the corresponding classes of AB functions, and also S. Golomb [550] who has been one of the main initiators of the theory of sequences, see [555]). They proved the preferred cross-correlation in some cases and made conjectures for others. Hence, when the notion of AB function was invented by Chabaud and Vaudenay, some work had been already done for searching such functions. A survey on cross-correlation distributions is given in [593]. See also [1127].

Simplification of the checking of APNness

When F is a power function, it is enough to check the APN property for $a = 1 \in \mathbb{F}_{2^n}$, since for $a \neq 0$, changing x into ax in the equation $F(x)+F(x+a) = b$ gives $F(x)+F(x+1) = \frac{b}{F(a)}$. Hence, according to what we saw on the characterization by the ANF at page 406, $F(x) = x^d$ is APN if and only if

$$\delta_0\Big(x^d+(x+1)^d+y^d+(y+1)^d\Big)+\delta_0\Big(x+y\Big)+\delta_0\Big(x+y+1\Big)\ \left[\bmod\ x^{2^n}+x, y^{2^n}+y\right]$$

equals the zero function, where $\delta_0(z) = 1 + z^{2^n-1}$, or equivalently

$$1_H(x)1_H(y)\left(\left(x^d+(x+1)^d+y^d+(y+1)^d\right)^{2^n-1}+\left(x+y\right)^{2^n-1}\right),$$

[9] Another m-sequence if d is co-prime with $2^n - 1$.

[10] This makes possible, in code division multiple access, to give different signals to different users.

similarly reduced, equals the zero function, where H is a linear hyperplane excluding 1. Moreover, checking the AB property $\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(vF(x)+ux)} \in \{0,\pm 2^{\frac{n+1}{2}}\}$, for every $u,v\in\mathbb{F}_{2^n}$, $v\neq 0$, is enough for $u=0$ and $u=1$ (and every $v\neq 0$), since changing x into $\frac{x}{u}$ (if $u\neq 0$) in this sum gives $\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(v'F(x)+x)}$, for some $v'\neq 0$. If F is a permutation, then checking the AB property is also enough for $v=1$ and every u, since changing x into $\frac{x}{F^{-1}(v)}$ in this sum gives $\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n\left(F(x)+\frac{u}{F^{-1}(v)}x\right)}$. And in the characterization of Proposition 159, page 408, if $a\neq 0$, then it can be reduced similarly to $a=1$, and if $a=0$, we can assume that $z\neq 0$ and replace x by xz and y by yz, we get $(x^d+y^d+1)z^d=b$ which has the same number of solutions for every nonzero b since F is a permutation; the characterization of ABness reduces then to: the equation $x^d+y^d+(x+y+1)^d=b$ has 2^n-2 solutions for every $b\neq 1$. Then a power permutation F in odd dimension is AB if and only if the function $(x,y)\to x^d+y^d+(x+y+1)^d$ is $(2^n-2)-to-1$ from $\{(x,y)\in(\mathbb{F}_{2^n}\setminus\{1\})^2; x\neq y\}$ to $\mathbb{F}_{2^n}\setminus\{1\}$ (the fact that this never takes value 1 is equivalent to F APN).

Additional information on bijectivity

It was proved in [257] that, when n is even, no APN function exists in a class of permutations including power permutations, that we describe now. Let $k=\frac{2^n-1}{3}$ (which is an integer, since n is even) and let α be a primitive element of the field $\mathbb{F}_{2^n}$. Then $\beta=\alpha^k$ is a primitive element of $\mathbb{F}_4$. Hence, $\beta^2+\beta+1=0$. For every j, the element $(\beta+1)^j+\beta^j=\beta^{2j}+\beta^j$ equals 1 if j is co-prime with 3 (since β^j is then also a primitive element of $\mathbb{F}_4$), and is null otherwise. Let $F(x)=\sum_{j=0}^{2^n-1}\delta_j x^j$, ($\delta_j\in\mathbb{F}_{2^n}$) be an (n,n)-function. According to the observations above, β and $\beta+1$ are the solutions of the equation $F(x)+F(x+1)=\sum_{\gcd(j,3)=1}\delta_j$. Also, the equation $F(x)+F(x+1)=\sum_{j=1}^{2^n-1}\delta_j$ admits 0 and 1 for solutions. Thus:

Proposition 164 *Let n be even and let $F(x)=\sum_{j=0}^{2^n-1}\delta_j x^j$ be any APN (n,n)-function, then $\sum_{j=1}^{\frac{2^n-1}{3}}\delta_{3j}\neq 0$. If F is a power function, $F(x)=x^d$, then 3 divides d and F cannot be a permutation.*

H. Dobbertin gives in [469] a result valid only for power functions but slightly more precise, and he completes it in the case that n is odd:

Proposition 165 *If a power function $F(x)=x^d$ over $\mathbb{F}_{2^n}$ is APN, then for every $x\in\mathbb{F}_{2^n}$, we have $x^d=1$ if and only if $x^3=1$, that is, $F^{-1}(1)=\mathbb{F}_4\cap\mathbb{F}_{2^n}^*$. If n is odd, then $\gcd(d,2^n-1)$ equals 1 and, if n is even, then $\gcd(d,2^n-1)$ equals 3. Consequently, APN power functions are permutations if n is odd, and are three-to-one over $\mathbb{F}_{2^n}^*$ if n is even.*

Proof. Let $x\neq 1$ be such that $x^d=1$. There is a (unique) y in $\mathbb{F}_{2^n}$, $y\neq 0,1$, such that $x=(y+1)/y$. The equality $x^d=1$ implies $(y+1)^d+y^d=0=(y^2+1)^d+(y^2)^d$. By the APN property and since $y^2\neq y$ because $x\neq 1$, we conclude $y^2+y+1=0$. Thus, y, and therefore x, are in $\mathbb{F}_4$ and $x^3=1$. Conversely, if $x \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$ is such that $x^3 = 1$, then 3 divides $2^n - 1$ and n must be even; moreover, d must also be divisible by 3 (indeed, otherwise, the restriction of x^d to $\mathbb{F}_4$ which coincides with the function $x^{\gcd(d,3)}$ would be linear, a contradiction) and then $x^d = 1$. We have then proved that $F^{-1}(1) = \mathbb{F}_4 \cap \mathbb{F}_{2^n}^*$. The rest is straightforward. $\square$.

In [62] is similarly observed that, if all the coefficients in the univariate representation of an APN function $F(x)$ belong to a subfield $\mathbb{F}_{2^r}$ of $\mathbb{F}_{2^n}$, then the equality $D_aF(x) = b$ for some $a, b \in \mathbb{F}_{2^r}$ and $x \in \mathbb{F}_{2^n} \setminus \mathbb{F}_{2^r}$ implies $x^{2^r} = x + a$.

In [406] is shown that, for any n, if an (n,n)-function F fixes 0_n and is such that, for every nonzero $u \in \mathbb{F}_2^n$, the pre-image $F^{-1}(u)$ either is empty or equals a set of three distinct nonzero elements of the form $\{a, b, a+b\}$ (*i.e.* is a 2-dimensional $\mathbb{F}_2$-linear space less 0_n)[11], then F is APN if and only if:

$$\begin{cases} F(x) \neq F(y) \\ F(z) \notin \{F(x), F(y), F(x+y)\} \end{cases} \Longrightarrow F(x)+F(y)+F(z)+F(x+y+z) \neq 0_n.$$

Indeed, this condition is necessary since $\begin{cases} F(x) \neq F(y) \\ F(z) \notin \{F(x), F(y), F(x+y)\} \end{cases}$ implies that $\{x, y, z, x+y+z\}$ is a 2-dimensional flat and the restriction of an APN function to any 2-dimensional flat must not sum up to 0_n; this condition is also sufficient since, if four distinct elements of $\mathbb{F}_2^n$ have null sum as well as their images, then the condition being assumed satisfied, either these four elements come by pairs with the same image in each pair, and this is impossible since if for instance, $F(x) = F(y)$ and $F(z) = F(x+y+z)$, then because of the assumption on the pre-images, we have $F(z) = F(x+y) = F(x) = F(y) = F(x+y+z)$ which is impossible since $F^{-1}(F(z))$ has only three elements, or they are such that $F(z) = F(x+y)$ and the same happens since we have then $F(z) = F(x+y) = F(x+y+z) = F(x) = F(y)$. Note that a sufficient condition for F to be APN is that $F(\mathbb{F}_2^n)$ is a *Sidon set* (see Definition 80, page 420), but it is shown in [406] that such sets of size $\frac{2^n-1}{3} + 1$ do not exist for $n \geq 6$ even.

Nonlinearity

An upper bound valid not only for APN functions but restricted to power functions is proved in [189]:

Proposition 166 *For every n even, if a power function $F(x) = x^d$ on $\mathbb{F}_{2^n}$ is not a permutation (*i.e. *if $\gcd(d, 2^n - 1) > 1$), then the nonlinearity of F is bounded above by $2^{n-1} - 2^{\frac{n}{2}}$. Equality can be achieved only for $\gcd(d, 2^n - 1) = 3$.*

Proof. Let $d_0 = \gcd(d, 2^n - 1)$; for every $v \in \mathbb{F}_{2^n}$, the sum $\sum_{x \in \mathbb{F}_{2^n}} (-1)^{tr_n(vx^d)}$ equals $\sum_{x \in \mathbb{F}_{2^n}} (-1)^{tr_n(vx^{d_0})}$; hence, $\sum_{v \in \mathbb{F}_{2^n}} \left(\sum_{x \in \mathbb{F}_{2^n}} (-1)^{tr_n(vx^d)} \right)^2$ is equal to $2^n \left| \{(x,y),\ x, y \in \mathbb{F}_{2^n},\ x^{d_0} = y^{d_0}\} \right|$. The number of elements in the image of $\mathbb{F}_{2^n}^*$

[11] This needs that n be even, and it happens then with any APN power function, and also with other functions like $x^3 + tr_n(x^9)$.

by the mapping $x \to x^{d_0}$ is $(2^n-1)/d_0$ and every element of this image has d_0 pre-images. Hence, $\sum_{v\in\mathbb{F}_{2^n}^*}\left(\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(vx^d)}\right)^2$ equals $2^n[(2^n-1)d_0+1]-2^{2n} = 2^n(2^n-1)(d_0-1)$ and $\max_{v\in\mathbb{F}_{2^n}^*}\left(\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(vx^d)}\right)^2 \geq 2^n(d_0-1)$. The proof is completed by using that the values of $\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(vx^3)}$ and $\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(vx^5)}$ are known. □

The possible values of the sum $\sum_{x\in\mathbb{F}_{2^n}}(-1)^{tr_n(vx^d)}$ are determined in [62] for APN power functions for n even.

It happens that all known APN power functions have rather good nonlinearity. To clarify the situation for general power APN functions, we need to show lower bounds on their nonlinearity and/or to find such functions with lower nonlinearity. The next bound is shown in [250] (the proof below is from this reference):

Proposition 167 *Let F be any APN power function. Then, if n is odd, we have $nl(F) \geq 2^{n-1} - 2^{\frac{3n-3}{4}}$ and if n is even, we have $nl(F) \geq 2^{n-1} - 2^{\frac{3n-2}{4}}$.*

Proof. If n is odd then, for every $v \neq 0$, the sum $\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,v)$ is independent[12] of the choice of v and, according to the characterization of APN functions by the fourth moment of Walsh transform, equals then 2^{3n+1}. Hence, we have $W_F^4(u,v) \leq 2^{3n+1}$ for every u and the result follows from (3.21), page 139. If n is even, then, since according to Proposition 165, the value of $\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,v)$ does not change when v is multiplied by a nonzero cube, it takes, when v ranges over $\mathbb{F}_{2^n}^*$, $\frac{2^n-1}{3}$ times the value $\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,1)$, $\frac{2^n-1}{3}$ times the value $\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,\alpha)$ and $\frac{2^n-1}{3}$ times the value $\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,\alpha^2)$ (α primitive in $\mathbb{F}_{2^n}$). Hence we have $\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,1)+\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,\alpha)+\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,\alpha^2) = 3\cdot 2^{3n+1}$. We have, by the Cauchy-Schwarz inequality, that $\sum_{u\in\mathbb{F}_2^n} W_F^4(u,v) \geq \frac{\left(\sum_{u\in\mathbb{F}_2^n} W_F^2(u,v)\right)^2}{2^n} = 2^{3n}$ for $v \neq 0$. Hence, we have by complementation that each of the sums $\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,1), \sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,\alpha)$ and $\sum_{u\in\mathbb{F}_{2^n}} W_F^4(u,\alpha^2)$ is bounded above by $3\cdot 2^{3n+1} - 2\cdot 2^{3n} = 2^{3n+2}$. We have then $W_F^4(u,v) \leq 2^{3n+2}$ for every u, v such that $v \neq 0$ and Relation (3.21) completes the proof. □

The bound of Proposition 167 (which for n odd tells again what we saw at page 423) has been extended in [356] to differentially uniform power functions but only for permutations. For explaining the good nonlinearity of known APN functions, there remains to tackle that of quadratic functions for n even (for n odd, APN quadratic functions are AB). Less is known for them; see observations in [250].

Relation with cyclic codes

If F is a power function, then the linear codes C_F and $C_F^\perp$ viewed in Proposition 160, page 410 are *cyclic codes* (see [257] where several results are given in this

[12] Such property will be called CAPNness at page 423 and implies the former inequality.

framework). Indeed, $(c_0,\dots,c_{2^n-2})$ belongs to C_F if and only if $c_0+c_1\alpha+\dots+c_{2^n-2}\alpha^{2^n-2}=0$ and $c_0+c_1\alpha^d+\dots+c_{2^n-2}\alpha^{(2^n-2)d}=0$; this implies (by multiplying these equations by α and α^d, respectively) $c_{2^n-2}+c_0\alpha+\dots+c_{2^n-3}\alpha^{2^n-2}=0$ and $c_{2^n-2}+c_0\alpha^d+\dots+c_{2^n-3}\alpha^{(2^n-2)d}=0$. The *BCH bound* (see page 28) shows in the case $F(x)=x^3$ that C_F has minimum distance (at least) 5 (*i.e.* that F is APN) and (in an original but rather complex way) that the function $x^{2^{\frac{n-1}{2}}+1}$, n odd, is AB: by definition, the *defining set* I of C_F (see page 27) equals the union of the cyclotomic classes of 1 and $2^{\frac{n-1}{2}}+1$, that is, $I=\{1,2,\dots,2^{n-1}\}\cup\{2^{\frac{n-1}{2}}+1,2^{\frac{n+1}{2}}+2,\dots,2^{n-1}+2^{\frac{n-1}{2}},2^{\frac{n+1}{2}}+1,2^{\frac{n+3}{2}}+2,\dots,2^{n-1}+2^{\frac{n-3}{2}}\}$. Since there is no element equal to $2^{n-1}+2^{\frac{n-1}{2}}+1,\dots,2^n-1$ in I, the defining set $\mathbb{Z}/(2^n-1)\mathbb{Z}\setminus\{-i;\, i\notin I\}$ of $C_F^\perp$ contains a string of length $2^{n-1}-2^{\frac{n-1}{2}}-1$. Hence the nonzero codewords of this code have Hamming weights larger than or equal to $2^{n-1}-2^{\frac{n-1}{2}}$. This is not sufficient for concluding that the function is AB, but we can apply the previous reasoning to the *cyclic code* $C_F^\perp\cup(1_{2^n-1}+C_F^\perp)$: the defining set of the dual of this code being equal to that of C_F, plus 0, the defining set of the code equals that of $C_F^\perp$ less 0, which gives a string of length $2^{n-1}-2^{\frac{n-1}{2}}-2$ instead of $2^{n-1}-2^{\frac{n-1}{2}}-1$. Hence the complements of the codewords of $C_F^\perp$ have Hamming weights at least $2^{n-1}-2^{\frac{n-1}{2}}-1$ and the codewords of $C_F^\perp$ have then Hamming weights at most $2^{n-1}+2^{\frac{n-1}{2}}$.
The powerful McEliece Theorem (see *e.g.* [809]) that we recalled at Section 4.1 (page 174) gives the exact divisibility of the codewords of cyclic codes. Translated in terms of vectorial functions, it says that if d is relatively prime to 2^n-1, the exponent e_d of the greatest power of 2 dividing all the Walsh coefficients of the power function x^d is given by $e_d=\min\{w_2(t_0)+w_2(t_1),\ 1\le t_0,t_1<2^n-1;\ t_0+t_1d\equiv 0\ [\text{mod } 2^n-1]\}$. It can be used in relationship with Proposition 163. This has led in [195] to the proof, by Canteaut, Charpin and Dobbertin, of a several decade old conjecture due to Welch.

Note finally that, if F is a power function, then Boolean function γ_F seen in Proposition 158 is within the framework of Dobbertin's *triple construction* [466].

Relation with the notions of Sidon sets and sum-free sets

In [316] is observed that APN exponents have a property involving two well-known notions in additive combinatorics. We refer the reader to this paper and to the references therein for complements.

Definition 80 *A subset of $\mathbb{F}_2^n$ is a* Sidon set *if it does not contain four distinct elements whose sum is null.*

This notion due to Sidon is preserved by affine equivalence and by decreasing inclusion. Denoting by P_S the set of pairs in S, it is equivalent to saying that $\{x,y\}\in P_S\mapsto x+y$ is one-to-one. The size $|S|$ is then such that $\binom{|S|}{2}\le 2^n-1$. Note that an (n,n)-function F is APN if and only if its *graph* $\mathcal{G}_F=\{(x,F(x));x\in\mathbb{F}_2^n\}$ is a Sidon set in $((\mathbb{F}_2^n)^2,+)$, since saying that, given four distinct elements x,y,z,t of $\mathbb{F}_2^n$, if $x+y+z+t=0_n$ then $F(x)+F(y)+F(z)+F(t)\neq 0_n$, is equivalent to saying that, given four distinct elements x, y, z, t, we have $x+y+z+t \neq 0_n$ or $F(x)+F(y)+F(z)+F(t) \neq 0_n$.

Definition 81 *A subset S of $\mathbb{F}_2^n$ is called a* sum-free set *if it does not contain elements x, y, z such that $x+y=z$ (*i.e.*, if $S \cap (S+S) = \emptyset$).*

This notion due to Erdös is preserved by linear equivalence and by decreasing inclusion. The size $|S|$ is then smaller than or equal to 2^{n-1}, because $|S+S| \geq |S|$ and if $|S| > 2^{n-1}$ then the two sets $S+S$ and S have intersection. Note that S cannot contain 0_n. A basic example of a sum-free set (with minimum size) is the complement of a linear hyperplane. The size $|S|$ of a sum-free Sidon set satisfies $\frac{|S|\,(|S|+1)}{2} \leq 2^n - 1$, since $S \cup \{0_n\}$ is then a Sidon set.

Proposition 168 *[316] For every positive integers n and d and for every $j \in \mathbb{Z}/n\mathbb{Z}$, let $e_j = gcd(d-2^j, 2^n-1) \in \mathbb{Z}/(2^n-1)\mathbb{Z}$, and let G_{e_j} be the multiplicative subgroup $\{x \in \mathbb{F}_{2^n}^*; x^{d-2^j} = 1\} = \{x \in \mathbb{F}_{2^n}^*; x^{e_j} = 1\}$ of order e_j. If d is an APN exponent over $\mathbb{F}_{2^n}$, then, for every $j \in \mathbb{Z}/n\mathbb{Z}$, G_{e_j} is a Sidon sum-free set in $\mathbb{F}_{2^n}$.*

Proof. For every $x \in G_{e_j} \setminus \{1\}$, let $s = \frac{x}{x+1}$. Then $x = \frac{s}{s+1}$, and $x^{d-2^j} = 1$ implies $s^{d-2^j} + (s+1)^{d-2^j} = 0$, which implies after multiplication by $s^{2^j}+1 = (s+1)^{2^j}$ that $s^d + (s+1)^d = s^{d-2^j} = \frac{1}{(x+1)^{d-2^j}}$. Note that if $s = \frac{x}{x+1}$ and $s' = \frac{x'}{x'+1}$, with $x \neq 1$ and $x' \neq 1$, then we have $s = s'$ if and only if $x = x'$ and $s = s'+1$ if and only if $x' = x^{-1}$.
Suppose that G_{e_j} is not a Sidon set, then let x, y, z, t be distinct elements of G_{e_j} such that $x+y = z+t$. Making the changes of variables $x \to xt, y \to yt, z \to zt$ and dividing the equality by t, we obtain distinct elements x, y, z of $G_{e_j} \setminus \{1\}$ such that $x+y+z = 1$. Making now the change of variable $y \to zy$, we obtain elements x, y, z in $G_{e_j} \setminus \{1\}$ such that $x+1 = z(y+1)$, $x \neq y$ and $x \neq y^{-1}$. We have then $\frac{1}{(x+1)^{d-2^j}} = \frac{1}{(y+1)^{d-2^j}}$ and $\frac{x}{x+1} \neq \frac{y}{y+1}$, $\frac{x}{x+1} \neq \frac{y}{y+1}+1$, a contradiction with the APNness of F.
Suppose that G_{e_j} is not sum-free, then $G_{e_j} \cap (G_{e_j}+1) \neq \emptyset$. Let $x \in G_{e_j} \cap (G_{e_j}+1)$ and $s = \frac{x}{x+1}$, we have then $\frac{1}{(x+1)^{d-2^j}} = 1$ and $s^d + (s+1)^d = 1$ and the equation $z^d + (z+1)^d = 1$ has four solutions $0, 1, s$, and $s+1$ in $\mathbb{F}_{2^n}$, a contradiction.□

Remark. Denoting $e = gcd(d, 2^n-1)$, we have that G_e itself is a Sidon set since, as recalled above, we have $e = 1$ if n is odd and $e = 3$ if n is even, and $G_1 = \{1\}$, $G_3 = \mathbb{F}_4^*$ are Sidon sets (since they do not contain 4 distinct elements). But G_e is a sum-free set only for n odd, since $\mathbb{F}_4^*$ is not sum-free. □

A geometric characterization of the fact that some integer d co-prime with 2^n-1 is an APN exponent over $\mathbb{F}_{2^n}$ for n odd by means of the *Singer set* $S_d = \{x \in \mathbb{F}_{2^n}; tr_n(x^d) = 1\}$ is also given in [252].

Alternative characterization of APN exponents, relation with the Dickson polynomials

When x ranges over $\mathbb{F}_{2^n}\setminus\{1\}$, $s=\frac{x}{x+1}$ ranges over $\mathbb{F}_{2^n}\setminus\{1\}$ and $s^d+(s+1)^d=\frac{x^d+1}{(x+1)^d}$. Then, considering separately the equation $s^d+(s+1)^d=1$ and the equations $s^d+(s+1)^d=b\neq 1$, we have directly:

Proposition 169 *[316] Let n be any positive integer then a power function $F(x)=x^d$ over $\mathbb{F}_{2^n}$ is APN if and only if the function $x\mapsto\frac{x^d+1}{(x+1)^d}$ is 2-to-1 from $\mathbb{F}_{2^n}\setminus\mathbb{F}_2$ to $\mathbb{F}_{2^n}\setminus\{1\}$.*

By definition, we have $x^d+(x+1)^d=\phi_d(x^2+x)$ where ϕ_d is the *reversed Dickson polynomial*, that is, $\phi_d(X)=D_d(1,X)$, where D_d is classically defined by $D_d(X+Y,XY)=X^d+Y^d$, see [628] and [890, page 227]. Then $F(x)=x^d$ is APN if and only if function ϕ_d is injective over the hyperplane $H=\{x^2+x;x\in\mathbb{F}_{2^n}\}=\{y\in\mathbb{F}_{2^n};tr_n(y)=0\}$. Moreover, if x^d is APN over $\mathbb{F}_{2^n}$ for n even, then ϕ_d is a permutation polynomial of $\mathbb{F}_{2^{n/2}}$, which in turn implies that x^d is APN over $\mathbb{F}_{2^{n/2}}$, see [890, Theorem 8.1.97, page 227].
We also have $\frac{x^d+1}{(x+1)^d}=\psi_d(x+x^{-1})$, where $(\psi_d(X))^2=\frac{D_d(X,1)}{X^d}$, where $D_d(X,1)$ is the *Dickson polynomial* [890] since $\left(\frac{x^d+1}{(x+1)^d}\right)^2=\frac{x^d+x^{-d}}{(x+x^{-1})^d}$. According to Proposition 169, function F is then APN if and only if ψ_d is injective from $\{x+x^{-1};x\in\mathbb{F}_{2^n}\setminus\mathbb{F}_2\}=\{y\in\mathbb{F}_{2^n}^*;tr_n(y^{-1})=0\}$ to $\mathbb{F}_{2^n}\setminus\{1\}$. Note that $\frac{D_d(y^{-1},1)}{(y^{-1})^d}=y^dD_d(y^{-1},1)$ is the value at y of the reciprocal polynomial of $D_d(X,1)$. Hence:

Proposition 170 *[316] For every positive integers n and d, function $F(x)=x^d$ is APN if and only if the reciprocal polynomial $\widetilde{D_d(X,1)}=X^dD_d(X^{-1},1)$ of the Dickson polynomial $D_d(X,1)$ is injective and does not take value 1 over $H^*=\{y\in\mathbb{F}_{2^n}^*;tr_n(y)=0\}$.*

And it has been proved in [316] that for every positive integer d, the reversed Dickson polynomial of index $2d$ and the reciprocal of Dickson polynomial of index d are equal. In fact, as observed with X.-D. Hou, for any characteristic, we have $X^dD_d(\frac{1}{X}-2,1)=D_{2d}(1,X)$.

Search for APN exponents

Dobbertin and Canteaut have independently determined all APN exponents for $n\leq 26$, and Leander-Langevin did the same up to $n=33$ for AB exponents in [753]; all belong to the classical classes of APN exponents that we shall list in Subsection 11.4, page 427. Edel checked all APN exponents for $n\leq 34$ and $n=36,38,40,42$. The main idea for his computer investigation was to consider all the elements in $\mathbb{Z}/(2^n-1)\mathbb{Z}$, discard all those which are not co-prime with 2^n-1 for n odd and do not have gcd equal to 3 with 2^n-1 for n even, and all the remaining exponents whose reduction mod 2^r-1 is not an APN exponent in $\mathbb{F}_{2^r}$ for some divisor r of n. Then, checking APNness was made for one member of each remaining cyclotomic class of 2 modulo 2^n-1 only since x^d and x^{2d} are linearly equivalent. No unclassified APN exponent could be found. A new search has been made in [316], in which were also discarded all those exponents d which were known not satisfying Proposition 168, thanks to a work on the Sidon and sum-free multiplicative subgroups of $\mathbb{F}_{2^n}^*$ made in [315], which shows in particular that $G_e = \{x \in \mathbb{F}_{2^n}^* \mid x^e = 1\}$ is a Sidon set (resp. a sum-free set) if and only if, for every $u \in \mathbb{F}_{2^n}^*$ (resp. for $u = 1$), the polynomial $\gcd(X^e + 1, (X+1)^e + u)$ has at most two zeros in $\mathbb{F}_{2^n}$ (resp. has no zero[13]). The condition for sum-free case is equivalent to saying that $\gcd(X^e + 1, (X+1)^e + 1, X^{2^n} + X)$, that is, $\gcd(X^e + 1, (X+1)^e + 1)$ since $X^e + 1$ divides $X^{2^n - 1} + 1$, equals 1 and this can be handled without computing in the field $\mathbb{F}_{2^n}$ (which needs huge computational power for large values of n) since all the coefficients playing a role in the Euclidean algorithm belong to $\mathbb{F}_2$. Unfortunately, this did not discard enough additional APN candidates for allowing to find new APN exponents.

11.3.3 Componentwise APNness (CAPNness)

Chabaud-Vaudenay's characterization of APN functions by the fourth moment of the Walsh transform (see Relation (11.2), page 403) leads to a notion called componentwise APNness (CAPNness) in [251], stronger than APNness, in which the value on the left hand side of (11.2) is the same for every component function:

Definition 82 *Let n be any positive integer and F any (n,n)-function. We call F* componentwise APN *(CAPN) if, given any nonzero v in $\mathbb{F}_2^n$, its Walsh transform satisfies the equality:*

$$\sum_{u \in \mathbb{F}_2^n} W_F^4(u, v) = 2^{3n+1}. \tag{11.8}$$

Using Relation 3.10, page 119, F is CAPN if and only if $\mathcal{V}(v \cdot F) = 2^{2n+1}$ for every $v \neq 0_n$. This EA invariant notion had been first studied by Berger et al. in [62] without that a specific name be introduced by these authors. They had observed that AB functions and power APN permutations have this property (for straightforward reasons); in particular, all known APN functions in odd dimension are CAPN. They had stated an open question on the existence or nonexistence of such functions for n even. The nonexistence has been proved in [251], by showing that F is CAPN if and only if, for every $w \neq 0_n$, the set $\{(x, y, z) \in \mathbb{F}_2^n;\ F(x) + F(y) + F(z) + F(x + y + z) = w\}$ has size $2^{2n} - 2^{n+1}$ and observing that this size is always divisible by 3, which implies that n is odd.

Relation (11.8) implies $2^{\frac{2n+1}{4}} \leq \max_{u,v \in \mathbb{F}_2^n; v \neq 0_n} |W_F(u, v)| \leq 2^{\frac{3n+1}{4}}$ and therefore $2^{n-1} - 2^{\frac{3(n-1)}{4}} \leq nl(F) \leq 2^{n-1} - 2^{\frac{2n-3}{4}}$.

[13] This condition can be compared to the condition that $X^d + (X+1)^d + 1$ has no zero in $\mathbb{F}_{2^n} \setminus \mathbb{F}_2$, which expresses (as it can be easily checked) that the cyclic code C_F (see Proposition 160, page 410, and see page 419) has no codeword of Hamming weight 3.

11.3.4 Plateaued APN functions

In the case n odd, we have seen in Proposition 163, page 414, that *plateaued* APN n-variable functions are almost bent.

In the case n even, we have seen at page 415 that for any plateaued APN n-variable function F, there must exist $v \in \mathbb{F}_2^{n\,*}$ such that the Boolean function $v \cdot F$ is bent. *Note that this implies that F cannot be a permutation*, according to Proposition 35, page 134, and since a bent Boolean function is never balanced. This was first observed in [62].

When F is plateaued and APN, the numbers $\lambda_{u,v}$ involved in Relation (11.7), page 414, can be divided into two categories (since we know that the amplitude of a plateaued Boolean function equals 2^j with $j \geq \frac{n}{2}$): those such that the function $v \cdot F$ is bent (for each such v, we have $\lambda_{u,v} = 1/2$ for every u and therefore $\sum_{u\in\mathbb{F}_2^n}(\lambda_{u,v}^2 - \lambda_{u,v}) = -2^{n-2}$); and those such that $v \cdot F$ is not bent (then $\lambda_{u,v} \in \{0, 2^i\}$ for some $i \geq 1$ depending on v, and therefore $\lambda_{u,v}^2 = 2^i\lambda_{u,v}$ and we have, thanks to Parseval's relation applied to the Boolean function $v\cdot F$: $\sum_{u\in\mathbb{F}_2^n}(\lambda_{u,v}^2 - \lambda_{u,v}) = (2^i-1)\sum_{u\in\mathbb{F}_2^n}\lambda_{u,v} = (2^i-1)\frac{2^{2n}}{2^{n+1}} = (2^i-1)2^{n-1} \geq 2^{n-1}$). Equation (11.7) implies then that the number B of those v such that $v \cdot F$ is bent satisfies $-B\,2^{n-2} + (2^n-1-B)\,2^{n-1} \leq 0$, which implies that *the number of bent functions among the functions $v \cdot F$ is at least $\frac{2}{3}(2^n-1)$* (this has been first observed in [910] for APN functions with partially-bent components, Nyberg generalizing a result given without a complete proof in [1028] for quadratic functions, and in [62] for plateaued APN functions).

This bound is achieved with equality by the Gold APN functions $F(x) = x^{2^i+1}$, $\gcd(i, n) = 1$ (see page 432). Indeed, we saw at page 230 that the function $tr_n(vF(x))$ is bent if and only if v is not the third power of an element of $\mathbb{F}_{2^n}$.

Note that, given an APN plateaued function F, saying that the number of bent functions among the functions $tr_n(vF(x))$ equals $\frac{2}{3}(2^n-1)$ is equivalent to saying, according to the observations above, that there is no v such that $\lambda_{u,v} = \pm 2^i$ with $i > 1$, that is, F has nonlinearity $2^{n-1} - 2^{\frac{n}{2}}$ and it is also equivalent to saying that F has the same extended Walsh spectrum as the Gold functions.

The fact that an APN function F has same extended Walsh spectrum as the Gold functions can be characterized by using a similar method as for proving Corollary 27, page 409: this situation happens if and only if, for every $v \in \mathbb{F}_2^n \setminus \{0_n\}$ and every $u \in \mathbb{F}_2^n$, we have $W_F(u,v) \in \{0, \pm 2^{\frac{n}{2}}, \pm 2^{\frac{n+2}{2}}\}$ (where $W_F(u,v) = \sum_{x\in\mathbb{F}_2^n}(-1)^{v\cdot F(x)\oplus u\cdot x}$), that is

$$W_F(u,v)\left(W_F^2(u,v) - 2^{n+2}\right)\left(W_F^2(u,v) - 2^n\right) = 0,$$

or equivalently $W_F^5(u,v) - 5\cdot 2^n\,W_F^3(u,v) + 2^{2n+2}W_F(u,v) = 0$. Applying the Fourier-Hadamard transform and dividing by 2^{2n}, this is equivalent to the fact that

$$\left|\left\{(x_1,\dots,x_5)\in\mathbb{F}_2^{5n};\left\{\begin{array}{c}\sum_{i=0}^5 x_i = a\\ \sum_{i=0}^5 F(x_i) = b\end{array}\right.\right\}\right| - 2^{3n} -$$

$$5\cdot 2^n\left(\left|\left\{(x_1,\dots,x_3)\in\mathbb{F}_2^{3n};\left\{\begin{array}{l}\sum_{i=0}^3 x_i=a\\ \sum_{i=0}^3 F(x_i)=b\end{array}\right\}\right|-2^n\right)+$$

$$2^{2n+2}\left(\left|\left\{x\in\mathbb{F}_2^n;\left\{\begin{array}{l}x=a\\ F(x)=b\end{array}\right\}\right|-2^{-n}\right)=0$$

for every $a,b\in\mathbb{F}_2^n$. A necessary condition is (taking $b=F(a)$ and using that F is APN) that, for every $a,b\in\mathbb{F}_2^n$, we have

$$\left|\left\{(x_1,\dots,x_5)\in\mathbb{F}_2^{5n};\left\{\begin{array}{l}\sum_{i=0}^5 x_i=a\\ \sum_{i=0}^5 F(x_i)=b\end{array}\right\}\right|=$$

$$2^{3n}+5\cdot 2^n(3\cdot 2^n-2-2^n)-2^{2n+2}(1-2^{-n})=2^{3n}+3\cdot 2^{2n+1}-3\cdot 2^{n+1}.$$

There exist APN quadratic functions whose Walsh spectra are different from the Gold functions. K. Browning *et al.* [135] exhibit such function in 6 variables: $F(x)=x^3+\alpha^{11}x^5+\alpha^{13}x^9+x^{17}+\alpha^{11}x^{33}+x^{48}$ (α primitive), for which 46 functions $tr_6(vF(x))$ are bent, 16 are plateaued with amplitude 16 and one is plateaued with amplitude 32. For $n=8$, among the 8179 quadratic APN functions identified in [1146], there are 487 functions with the spectrum $\{*-64^6,-32^{2240},-16^{20880},0^{15600},16^{23664},32^{2880},64^{10}*\}$ and 12 functions with the spectrum $\{*-64^{12},-32^{2100},-16^{21360},0^{14880},16^{24208},32^{2700},64^{20}*\}$ and the rest have Gold-like Walsh spectrum [655]. $\square$

For all n, characterizations of APN functions among plateaued vectorial functions

Thanks to the characterizations of plateaued functions recalled in Subsection 6.5, page 302, we shall see that all the main results known for quadratic APN functions generalize to plateaued APN functions, simplifying the study of the APNness of (n,n)-functions when they are known to be plateaued[14].

In particular, it is much used in papers on APN functions that, if a function F is quadratic, then given $a\neq 0_n$, the property that all equations $F(x)+F(x+a)=v$ (which are then linear equations) have at most 2 solutions is equivalent (as we saw already) to the fact that the single homogeneous equation $F(x)+F(x+a)=F(0_n)+F(a)$ has exactly 2 solutions. Proving APNness results then in proving that, for every $a\neq 0_n$, this equation has 0_n and a for only solutions. This is probably the main reason why many results on APN functions [80, 147, 151, 158, 157, 160, 239, 283, 1118, 1146] were found for quadratic functions. The property above generalizes to all plateaued functions:

Proposition 171 *[247] Any plateaued (n,n)-function F is APN if and only if, for every $a\neq 0_n$ in $\mathbb{F}_2^n$, the equation $F(x)+F(x+a)=F(0_n)+F(a)$ has the 2 solutions 0_n and a only.*

[14] In [247] are also given characterizations of plateaued functions among APN functions.

Indeed, for every $v \in \mathbb{F}_2^n$, the size $|\{(a,b) \in (\mathbb{F}_2^n)^2\,;\, F(x) + F(x+a) + F(x+b) + F(x+a+b) = v\}|$ does not depend on $x \in \mathbb{F}_2^n$, according to Theorem 18, page 304, and we can reduce ourselves in this characterization to a and b linearly independent. Function F is APN if and only if, for such a, b, this size is null for $v = 0_n$. This completes the proof by taking $x = 0_n$, and fixing $a \neq 0_n$.
Another particularity of plateaued functions, extending that of *quadratic functions*, is the sufficiency for APNness of the necessary condition (11.5), page 404:

Proposition 172 *[247] Let F be any* plateaued *(n,n)-function. Assume that $F(0_n) = 0_n$. Then F is APN if and only if the set $\{(x,a) \in (\mathbb{F}_2^n)^2 \,|\, F(x) + F(x+a) + F(a) = 0_n\}$ has size $3 \cdot 2^n - 2$. Equivalently:*

$$\sum_{u,v \in \mathbb{F}_2^n, v \neq 0_n} W_F^3(u,v) = 2^{2n+1}(2^n - 1).$$

Indeed, each equation $F(x) + F(x+a) = F(a)$, $a \neq 0_n$ has at least a and 0_n for solutions; Proposition 171 shows then the first assertion; and we have $\sum_{(u,v) \in (\mathbb{F}_2^n)^2} W_F^3(u,v) = 2^{2n} \left|\{(x,a) \in (\mathbb{F}_2^n)^2 \,|\, F(x) + F(x+a) + F(a) = 0_n\}\right|$. See in [247] several inequalities by means of power moments of the Walsh transform, valid for all vectorial functions, and achieved with equality by APN functions only.

The case of unbalanced component functions: Theorem 19, page 308, implies:

Proposition 173 *[247] Let F be any plateaued (n,n)-function having all its component functions unbalanced, then*

$$\left|\{(a,b) \in (\mathbb{F}_2^n)^2\,, a \neq b\,;\, F(a) = F(b)\}\right| \geq 2 \cdot (2^n - 1), \tag{11.9}$$

with equality if and only if F is APN.

Hence, the APNness of plateaued (n,n)-functions with unbalanced component functions depends only on their value distribution. For instance, any plateaued (n,n)-function, n even, having similar value distribution as APN power functions (that is, mapping 0 to 0 and being 3-to-1 for the rest of the inputs), is APN and, since $\sum_{a,b \in \mathbb{F}_2^n} (-1)^{v \cdot D_a D_b F(x)} = W_F^2(0_n, v)$, has the same extended Walsh spectrum as the APN Gold functions (particular cases of this result are given in [283]). The case of power functions simplifies further [247].
We have $\left|\{(a,b) \in (\mathbb{F}_2^n)^2\,, a \neq b\,;\, F(a) = F(b)\}\right| = \sum_{a \in \mathbb{F}_2^n; a \neq 0_n} \left|(D_a F)^{-1}(0_n)\right|$ and this is the parameter Nb_F of page 135. Each set $(D_a F)^{-1}(0_n)$ has then size exactly 2. Any function F having this latter property is called *zero-difference 2-balanced*[15], see [451, 462]. The zero-difference 2-balancedness of some classes of quadratic APN functions seen in [283] is a corollary of Proposition 173 since the functions in these classes have unbalanced components. Note, as observed in [283], that for every δ, all quadratic zero-difference δ-balanced functions are differentially δ-uniform.

[15] Such ZDB functions have however more applications when they are over cyclic groups.

11.4 The known infinite classes of AB functions

We begin with AB functions, because when dealing subsequently with APN functions, we shall just complete the list, and also for historical reasons, since AB functions were considered first (under different names in the domain of sequences, as seen at page 416). All the functions in this subsection and the next one are viewed within the structure of the finite field $\mathbb{F}_{2^n}$, n odd; that of semifield has been used in [77, 896]; we refer the reader to these papers for more details.

11.4.1 Power AB functions

The first known examples of AB functions have been power functions $x \mapsto x^d$ on the field $\mathbb{F}_{2^n}$ (n odd) for reasons also explained at page 416. The exponents d of these power functions are (1) those given below (and summarized in Table 11.1), whose largest classes are the two first and (2) the inverses modulo $2^n - 1$ of these values. These inverses have been studied in [908, 731].

• $d = 2^i + 1$ with $\gcd(i, n) = 1$ and $1 \leq i \leq \frac{n-1}{2}$ (proved by Gold, see [540, 908]). The condition $1 \leq i \leq \frac{n-1}{2}$ (here and below) is not necessary but we mention it because the other values of i give *EA equivalent* functions. These power functions are called *Gold AB functions*.

• $d = 2^{2i} - 2^i + 1 = \frac{2^{3i}+1}{2^i+1}$ with $\gcd(i, n) = 1$ and $2 \leq i \leq \frac{n-1}{2}$ (we exclude $i = 1$ since then the function is the cube function, that is a Gold function). The AB property of these functions is equivalent to a result historically due to Welch, but never published by him, and is a particular case of a result of Kasami [669]; see other proofs in [470] and [443]. These power functions are called *Kasami AB functions* (some authors call them *Kasami-Welch functions*). Note that, denoting by $G_i(x)$ the Gold AB function x^{2^i+1} over $\mathbb{F}_{2^n}$, and by $L(x)$ the linear function $x^{2^{2i}} + x$, Kasami function $K_i(x)$ not only equals $G_{3i} \circ G_i^{-1}$ but also equals $G_i \circ L \circ G_i^{-1}(x) + x^{2^{2i}} + x^{2^i} + x$ (and is therefore EA equivalent to $G_i \circ L \circ G_i^{-1}$, that is 2-to-1 while K_i is 1-to-1); more generally, for every nonzero $\mu \in \mathbb{F}_{2^n}$, denoting $L_\mu(x) = x^{2^{2i}} + \mu\, x$, function $\mu\, K_i(x)$ equals $G_i \circ L_\mu \circ G_i^{-1}(x) + x^{2^{2i}} + \mu^{2^i} x^{2^i} + \mu^{2^i+1} x$; indeed, $G_i \circ L_\mu \circ G_i^{-1}(x) = \left(x^{\frac{2^{2i}}{2^i+1}} + \mu\, x^{\frac{1}{2^i+1}}\right)^{2^i+1} = x^{2^{2i}} + \mu\, x^{\frac{2^{3i}+1}{2^i+1}} + \mu^{2^i} x^{\frac{2^{2i}+2^i}{2^i+1}} + \mu^{2^i+1} x = x^{2^{2i}} + \mu\, K_i(x) + \mu^{2^i} x^{2^i} + \mu^{2^i+1} x$. More is observed in [144].

• $d = 2^{(n-1)/2} + 3$ (conjectured by Welch and proved by Canteaut, Charpin and Dobbertin, see [471, 195, 196]). These functions are called *Welch functions*.

• $d = 2^{(n-1)/2} + 2^{(n-1)/4} - 1$ if $n \equiv 1 \pmod 4$, $d = 2^{(n-1)/2} + 2^{(3n-1)/4} - 1$ if $n \equiv 3 \pmod 4$ (conjectured by Niho, proved by Hollmann and Xiang, after the work by Dobbertin, see [472, 608]). These functions are called *Niho functions*.

The almost bentness can be proved in two steps: (1) prove the almost perfect nonlinearity; the non-easy cases (Kasami, Welch and Niho) can be treated

Table 11.1 Known AB exponents on $\mathbb{F}_{2^n}$ (n odd) up to equivalence and to inversion.

Functions	Exponents d	Conditions
Gold	$2^i + 1$	$\gcd(i, n) = 1$
Kasami	$2^{2i} - 2^i + 1$	$\gcd(i, n) = 1$
Welch	$2^t + 3$	$n = 2t + 1$
Niho	$2^t + 2^{\frac{t}{2}} - 1$, t even $2^t + 2^{\frac{3t+1}{2}} - 1$, t odd	$n = 2t + 1$

by Dobbertin's general method introduced in [472] and further developed in [474], called the multivariate method (see the end of Appendix for an example of this method); (2) prove then ABness by using Proposition 163, page 414, and McEliece's Theorem, see page 179, in the cases of the Welch and Niho functions. The global proofs of ABness are not easy except in the case of Gold functions, and too long for being included here.

The direct proof that the Gold function above is AB is easy by using the properties of quadratic functions. Since it is a power permutation, we can restrict the study of the Walsh transform to the component function $tr_n(x^{2^i+1})$. The linear kernel of this component function $\{x \in \mathbb{F}_{2^n};\, tr_n(x^{2^i}y + xy^{2^i}) = tr_n((x^{2^i} + x^{2^{n-i}})\, y) = 0, \forall y \in \mathbb{F}_{2^n}\}$ has equation $x^{2^{2i}} + x = 0$, and equals then $\mathbb{F}_2$, since $\gcd(2^{2i} - 1, 2^n - 1) = 1$. Function $tr_n(x^{2^i+1} + ax)$ is constant on $\mathbb{F}_2$ if and only if $tr_n(a) = 1$. The value at a of the Walsh transform equals then $\pm 2^{\frac{n+1}{2}}$ if $tr_n(a) = 1$ and is null otherwise. This proves ABness. The support of $W_F(u, v)$ has equation $tr_n\left(\frac{u}{v^{\frac{1}{2^i+1}}}\right) = 1$. The Walsh transform sign is studied in [734].

The inverse of x^{2^i+1} is x^d, where $d = \sum_{k=0}^{\frac{n-1}{2}} 2^{2ik}$, and x^d has therefore the algebraic degree $\frac{n+1}{2}$ [908] (hence, the bound of Proposition 155, page 402, is tight). It has been proved in [443, Theorem 7] and [448, Theorem 15] that, if $3i$ is congruent with 1 mod n, then the Walsh support of the Kasami Boolean function $tr_n(x^{2^{2i}-2^i+1})$ equals[16] the support of the Gold Boolean function $tr_n(x^{2^i+1})$ (*i.e.* the set $\{x \in \mathbb{F}_{2^n};\, tr_n(x^{2^i+1}) = 1\}$). The Walsh support of the Kasami functions is also determined in [742] when $5i \equiv 1$ [mod n] (it is more complex). The knowledge of the Walsh support gives the absolute value (but not the sign) of the Walsh transform of the Kasami function, this function being a permutation. It has been shown in [734, 548] that, for every AB power function x^d over $\mathbb{F}_{2^n}$ whose restriction to any subfield of $\mathbb{F}_{2^n}$ is also AB, the value $\sum_{x \in \mathbb{F}_{2^n}} (-1)^{tr_n(x^d+x)}$ equals $2^{\frac{n+1}{2}}$ if $n \equiv \pm 1$ [mod 8] and $-2^{\frac{n+1}{2}}$ if $n \equiv \pm 3$ [mod 8].

Note that the knowledge of the support of the Walsh transform gives also an information on autocorrelation: according to the Wiener-Khintchine formula, the Fourier-Hadamard transform of function $a \to \mathcal{F}(D_a f) = \sum_{x \in \mathbb{F}_2^n} (-1)^{D_a f(x)}$ equals the square of the Walsh transform of f. In the case that $3i$ is congruent

[16] For n even, it equals the set $\{x \in \mathbb{F}_{2^n};\, tr_2^n(x^{2^i+1}) = 0\}$, where tr_2^n is the trace function from $\mathbb{F}_{2^n}$ to the field $\mathbb{F}_{2^2}$: $tr_2^n(x) = x + x^4 + x^{4^2} + \cdots + x^{4^{\frac{n}{2}-1}}$.

with 1 mod n for instance, since the value at b of the square of the Walsh transform of f equals $2^{n+1}tr_n(x^{2^i+1})$, then by applying the inverse Fourier-Hadamard transform (that is, by applying the Fourier-Hadamard transform again and dividing by 2^n), $\mathcal{F}(D_af)$ equals twice the Fourier-Hadamard transform of the function $tr_n(x^{2^i+1})$. We deduce that, except at the zero vector, $\mathcal{F}(D_af)$ equals the opposite of the Walsh transform of the function $tr_n(x^{2^i+1})$.
It is proved in [429] (see also [159, 1140]) that power functions are CCZ equivalent if and only if their exponents or their inverses are in the same cyclotomic coset. The algebraic degrees of functions in Table 11.1 show their pairwise CCZ inequivalence in general.
It was conjectured by Hans Dobbertin that the list of power AB functions is complete. No counter-example to this conjecture has been found (see page 422).

11.4.2 Non-power AB functions

It had been conjectured in [257] that all AB functions are equivalent to power functions (and then to permutations). This conjecture has been disproved, in a first step by exhibiting in [163] AB functions which are EA inequivalent to any power function and to any permutation, but which are by construction CCZ equivalent to the Gold function $x \to x^3$, and in a second step by finding AB functions which are CCZ inequivalent to power functions (at least for some values of n) [158]. Note that an easy case where a function is provably EA inequivalent to power functions is when a component function $tr_n(vF)$ has algebraic degree larger than 1 and different from the algebraic degree of F [163].

AB functions CCZ equivalent to power functions

To construct APN (n,n)-functions, and AB functions from known ones by using CCZ equivalence, is needed, given such a function F, to find an affine permutation $\mathcal{L}$ of $\mathbb{F}_2^n \times \mathbb{F}_2^n$ such that, denoting $\mathcal{L}(x,y) = (L_1(x,y), L_2(x,y))$, where $L_1(x,y), L_2(x,y) \in \mathbb{F}_2^n$, the function $F_1(x) = L_1(x, F(x))$ is a permutation. This is a necessary and sufficient condition for the image of the graph of F by $\mathcal{L}$ to be the graph of a function. Two cases of such $\mathcal{L}$ were found in [162, 163] for the function $F(x) = x^{2^i+1}$ where $(i,n) = 1$, giving new classes of AB functions:
• The function $F(x) = x^{2^i+1} + (x^{2^i} + x)\, tr_n(x^{2^i+1} + x)$, where $n > 3$ is odd and $\gcd(n,i) = 1$, is AB. It is provably EA inequivalent to any power function [162, 163] and it is EA inequivalent to any permutation [163, 771], which disproved the conjecture above.
• For n odd, $m \,|\, n$, $m \neq n$ and $\gcd(n,i) = 1$, the (n,n)-function:

$$x^{2^i+1} + tr_m^n(x^{2^i+1}) + x^{2^i}\, tr_m^n(x) + x\; tr_m^n(x)^{2^i} +$$
$$[tr_m^n(x)^{2^i+1} + tr_m^n(x^{2^i+1}) + tr_m^n(x)]^{\frac{1}{2^i+1}}(x^{2^i} + tr_m^n(x)^{2^i} + 1) +$$
$$[tr_m^n(x)^{2^i+1} + tr_m^n(x^{2^i+1}) + tr_m^n(x)]^{\frac{2^i}{2^i+1}}(x + tr_m^n(x))$$

where tr_m^n denotes the trace function $tr_m^n(x) = \sum_{i=0}^{n/m-1} x^{2^{mi}}$ from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^m}$, is an AB function of algebraic degree $m+2$ which is provably EA inequivalent to any power function; the question of knowing whether it is EA inequivalent to any permutation is open.

It would be good to find similarly classes of AB functions by using CCZ equivalence with Kasami (resp. Welch, Niho) functions. For n odd, the Kasami $x^{4^k-2^k+1}$ function equals $F_2 \circ F_1^{-1}(x)$, where $F_1(x)$ and $F_2(x)$ are respectively the Gold functions x^{2^k+1} and $x^{2^{3k}+1}$. Hence, the first step would be to investigate permutations of the form $L_1(x^{2^k+1})+L_2(x^{2^{3k}+1})$, that is, to find L_1^1 and L_1^2 linear such that for every $u \neq 0$ and every x, we have $L_1^1(x^{2^k}u+xu^{2^k}+u^{2^k+1})+L_1^2(x^{2^{3k}}u+xu^{2^{3k}}+u^{2^{3k}+1}) \neq 0$. However, it is conjectured in [145] that for a non-Gold power APN (or AB) function, CCZ equivalence coincides with EA equivalence together with inverse transformation, and it is proven (with the help of a check by computer) that this conjecture is true for $n \leq 8$.

• The AB functions constructed in [162, 163] cannot be obtained from power functions by applying EA equivalence and inverse transformation, but Budaghyan shows in [140] that some AB functions EA inequivalent to power functions can be constructed by only applying EA equivalence and inverse transformation to power AB functions, for instance the function $\left(x^{\frac{1}{2^i+1}} + tr_3^n(x+x^{2^{2i}})\right)^{-1}$.

AB functions CCZ inequivalent to power functions

The problems of (in)existence of AB functions CCZ inequivalent to power functions and of quadratic APN functions EA inequivalent to Gold functions remained open after finding the two classes above. A paper by Edel, Kyureghyan and Pott [493] introduced two quadratic APN functions from $\mathbb{F}_{2^{10}}$ (resp. $\mathbb{F}_{2^{12}}$) to itself. The first one, $x^3+\alpha x^{36}$, was proved CCZ inequivalent to power functions. These two (quadratic) APN functions were sporadic and this left open the question of knowing whether a whole infinite class of APN functions being not CCZ equivalent to power functions could be exhibited. Moreover, the question of existence of such AB functions was still open.

• The new following class of *binomial AB functions* for n divisible by 3 was found in [151, 158] by Budaghyan, the author, Felke and Leander:

Proposition 174 *Let s and k be positive integers with* $\gcd(s,3k)=1$ *and* $t \in \{1,2\}$, $i = 3-t$. *Let* $d = 2^{ik}+2^{tk+s}-(2^s+1)$, $g_1 = \gcd(2^{3k}-1, d/(2^k-1))$ *and* $g_2 = \gcd(2^k-1, d/(2^k-1))$. *If* $g_1 \neq g_2$ *then the function*

$$F : \mathbb{F}_{2^{3k}} \to \mathbb{F}_{2^{3k}}$$
$$x \mapsto x^{2^s+1} + \alpha^{2^k-1} x^{2^{ik}+2^{tk+s}}$$

where α is primitive in $\mathbb{F}_{2^{3k}}$ is AB when k is odd and APN when k is even.

It could be proved (mathematically) in [151, 158] that some of these functions are EA inequivalent to power functions and CCZ inequivalent to some AB power functions, deducing that they are CCZ inequivalent to all power functions for some values of n:

Proposition 175 *Let s and $k \geq 4$ be positive integers such that $s \leq 3k-1$, $\gcd(k,3) = \gcd(s,3k) = 1$, and $i = sk$ [mod 3], $t = 2i$ [mod 3], $n = 3k$. If $a \in \mathbb{F}_{2^n}$ has the order $2^{2k}+2^k+1$ then the function $F(x) = x^{2^s+1} + ax^{2^{ik}+2^{tk+s}}$ is an AB permutation on $\mathbb{F}_{2^n}$ when n is odd and is APN when n is even. It is EA inequivalent to power functions and CCZ inequivalent to Gold and Kasami mappings as shown by a computer-free proof.*

This class was the first infinite family of APN and AB functions CCZ inequivalent to power functions, disproving a conjecture from [257] on the nonexistence of quadratic AB functions inequivalent to Gold functions. This class has been generalized in [116, 118] (see page 439), with Walsh spectra determined in [117].

• It has been shown by Budaghyan, the author and Leander in [160] that:

Proposition 176 *For every odd positive integer, the function $x^3 + tr_n(x^9)$ is AB on $\mathbb{F}_{2^n}$ (and that it is APN for n even).*

This function is one of the only examples[17] with x^3 of a function AB for any n odd. It is CCZ inequivalent to any Gold, inverse and Dobbertin functions on $\mathbb{F}_{2^n}$ if $n \geq 7$ and EA inequivalent to power functions [160]. It has been extended in the same reference into the AB function $x^3 + a^{-1}tr_n(a^3x^9)$ (which is CCZ inequivalent to all power functions according to [1140], since it has been proved that it is not equivalent to Gold), and in [161] for n divisible by 3 and odd into $x^3 + a^{-1}tr_n^3(a^3x^9 + a^6x^{18})$ and $x^3 + a^{-1}tr_n^3(a^6x^{18} + a^{12}x^{36})$. Coefficient a for all three functions can be reduced to $a = 1$ up to equivalence[18]. The principle of adding a Boolean function to an APN function has been generalized into the so-called switching method (see page 440).

• The eighth entry of Table 11.4, page 444 (displaying the known classes of quadratic APN polynomials CCZ inequivalent to power functions) is potentially an infinite class, and has been found in [142] by applying to the cube function x^3 the so-called *isotopic shift* $F \mapsto F_L(x) = F(x+L(x)) - F(x) - F(L(x))$ (where L is linear), adapted from an equivalence notion originally defined by Albert in the study of presemifields in odd characteristic[19].

An open question is to find infinite classes of AB functions CCZ inequivalent to power functions and to quadratic functions. Actually the very existence of such functions is an open problem too. A former question on the existence of AB

[17] If we do not take into account that n is present in the definition of tr_n.
[18] The situation is different for n even, with two different functions for $a = 1$ and a primitive.
[19] All quadratic APN (6,6)-functions can be obtained from x^3 by isotopic shift; an extension in [143], where instead of $xL(x)^{2^i} + x^{2^i}L(x)$, given by the isotopic shift of x^{2^i+1}, is taken $xL_1(x)^{2^i} + x^{2^i}L_2(x)$, with L_1 and L_2 linear, leads to 15 new APN (9,9)-functions.

functions CCZ inequivalent to permutations has been solved: for $n = 7$, all 484 quadratic AB functions found in [1146] are CCZ inequivalent to permutations (Yuyin Yu, private communication, 2018). At the moment, the only known AB functions CCZ equivalent to permutations are the power AB functions and the binomials of Proposition 174.

11.5 The known infinite classes of APN functions

We list below the known infinite classes of APN functions (those which are not already seen as AB functions).

11.5.1 Sporadic APN (and AB) functions

In 4 and 5 variables, all APN functions are known (they are classified under EA and CCZ equivalences by Brinkmann and Leander in [134]; for $n = 4$, there are two EA equivalence classes one of which is not EA equivalent to a power function and there is one CCZ equivalence class; for $n = 5$, there are seven EA equivalence classes two of which are not EA equivalent to any power function, and all APN functions are CCZ equivalent to one of three power functions). In 6-8 variables, known APN functions lying outside the known infinite classes are listed by Browning, Dillon, Kibler and McQuistan and by Yu, Wang and Li in [135, 1146] (see a few more, some of which in more variables, in [493, 494, 1118]). We refer the reader to these papers for the tables they contain, which are useful when trying to state conjectures on APN functions and for having a precise knowledge of all known APN functions. For $n = 6$, the classification of quadratic APN functions is complete: 13 quadratic APN functions are given in [135] and, as proven in [492], up to CCZ equivalence, these are the only quadratic APN functions. Only one non-quadratic APN function is known outside the infinite classes up to CCZ equivalence (it is, for $n = 6$, the Brinkmann-Leander-Edel-Pott function [494], see page 440). In [1146], by establishing a correspondence between quadratic APN functions and those $n \times n$ matrices over $\mathbb{F}_{2^n}$ which are symmetric with only zeros on the main diagonal and such that every nonzero linear combination of the rows has rank $n-1$, it is shown that there are at least 490 CCZ inequivalent APN $(7,7)$-functions (487 of which are quadratic) and at least 8180 for $n = 8$ (8179 quadratic). For n odd, all power APN functions and the known APN binomials (see Proposition 174) are permutations. For n even, the only known APN permutation is constructed in [136] for $n = 6$. The existence of APN permutations for even $n \geq 8$ is an open problem.

11.5.2 Power APN functions

As in the case of AB functions, the first known APN functions have been power functions $x \mapsto x^d$ over $\mathbb{F}_{2^n}$; the exponents d of these power functions are those given below (and summarized in Table 11.2) and their inverses (when n is odd); we do not repeat below the exponents of AB functions, but we do in Table 11.2:

• $d = 2^i + 1$ with $\gcd(i, n) = 1$, n even and $1 \leq i \leq \frac{n-2}{2}$ (*Gold APN functions*, see [540, 908]). The proof that these functions are APN (whatever is the parity of n) is easy: the equality $F(x) + F(x + 1) = F(y) + F(y + 1)$ is equivalent to $(x + y)^{2^i} = (x + y)$, and thus implies that $x + y = 0$ or $x + y = 1$, since i and n are co-prime. Hence, any equation $F(x) + F(x + 1) = b$ admits at most two solutions. Gold functions being quadratic are plateaued.

• $d = 2^{2i} - 2^i + 1$ with $\gcd(i, n) = 1$, n even and $2 \leq i \leq \frac{n-2}{2}$ (*Kasami APN functions*, see [641], see also [468]). The proof that such function is APN is difficult. It comes down to showing that the restriction to the hyperplane of equation $tr_n(x) = 0$ of a function ϕ such that $F(x) + F(x + 1) = \phi(x^2 + x)$ (which exists since $F(x) + F(x + 1)$ is invariant by translation by 1) is injective; Dobbertin shows a close connection between ϕ and the polynomial $P(x) = \frac{(tr_i(x))^{2^i+1}}{x^{2^i}}$, called *Müller-Cohen-Matthews polynomial* (*MCM polynomial*) [380] and proves that ϕ is a bijection in [468, 474]. Kasami APN functions are plateaued as proved when 3 does not divide n in [448] and for every even n in [1142].

• $d = 2^n - 2$, n odd. The corresponding so-called *multiplicative inverse permutation* (or simply *inverse function*) $x \mapsto F(x) = x^{2^n-2}$ (which equals $\frac{1}{x}$ if $x \neq 0$, and 0 otherwise) is APN [71, 908]. Indeed, the equation $x^{2^n-2} + (x+1)^{2^n-2} = b$ (that we can take with $b \neq 0$, since the inverse function is a permutation) admits 0 and 1 for solutions if and only if $b = 1$; and it (also) admits (two) solutions different from 0 and 1 if and only if there exists $x \neq 0, 1$ such that $\frac{1}{x} + \frac{1}{x+1} = b$, that is, $x^2 + x = \frac{1}{b}$. It is well-known that such existence is equivalent to the fact that $tr_n\left(\frac{1}{b}\right) = 0$ (since 0 and 1 do not satisfy this latter equation). Hence, F is APN if and only if $tr_n(1) = 1$, that is, if n is odd.
Consequently, the functions $x \mapsto x^{2^n-2^i-1}$, which are *linearly equivalent* to F (through the linear isomorphism $x \mapsto x^{2^i}$) are also APN, if n is odd.
If n is even, then the equation $x^{2^n-2} + (x+1)^{2^n-2} = b$ admits at most 2 solutions if $b \neq 1$ and admits 4 solutions (the elements of $\mathbb{F}_4$) if $b = 1$, which means that F opposes a good (but not optimal) resistance against differential cryptanalysis. The inverse function is not plateaued, since we have seen that the set of values of its Walsh spectrum equals the set of all integers $s \equiv 0$ [mod 4] in the range $[-2^{\frac{n}{2}+1}+1; 2^{\frac{n}{2}+1}+1]$. The values of $\mathcal{V}(v \cdot F)$ are calculated in [351]. A connection between the differential properties of function x^3 and of the multiplicative inverse function (and more generally between functions x^{2^t-1} and $x^{2^{n-t+1}-1}$, using that $x^{2^t-1} + (x+1)^{2^t-1} + 1 = \frac{(x^{2^{t-1}}+x)^2}{x^2+x}$) is shown in [93], with a focus on $t = 3$. In [757] is proved that any function $F(x) = x^{-1} + G(x)$, where G is any non-affine polynomial, is APN on at most a finite number of fields $\mathbb{F}_{2^n}$.

Table 11.2 Known APN exponents up to equivalence (any n) and up to inversion (n odd).

Functions	Exponents d	Conditions
Gold	2^i+1	$\gcd(i,n)=1$
Kasami	$2^{2i}-2^i+1$	$\gcd(i,n)=1$
Welch	2^t+3	$n=2t+1$
Niho	$2^t+2^{\frac{t}{2}}-1$, t even $2^t+2^{\frac{3t+1}{2}}-1$, t odd	$n=2t+1$
Inverse	$2^{2t}-1$	$n=2t+1$
Dobbertin	$2^{4t}+2^{3t}+2^{2t}+2^t-1$	$n=5t$

- $d=2^{\frac{4n}{5}}+2^{\frac{3n}{5}}+2^{\frac{2n}{5}}+2^{\frac{n}{5}}-1$, with n divisible by 5 (*Dobbertin function*, see [473]). It has been shown by Canteaut, Charpin and Dobbertin [196] that this function cannot be AB: they showed that $C_F^\perp$ contains words whose Hamming weights are not divisible by $2^{\frac{n-1}{2}}$ (the Walsh spectrum values of F are divisible by $2^{\frac{n}{5}}$ but not all by $2^{\frac{2n}{5}+1}$). The proof that the Dobbertin function is APN is also difficult and comes down as well to showing that some mapping is a permutation. Neither the nonlinearity nor the Walsh spectrum of the Dobbertin function are known. The Dobbertin function is not plateaued as seen in [473].

Nonlinearity

For n even, the Gold, Kasami and inverse functions have the best known nonlinearity $2^{n-1}-2^{\frac{n}{2}}$ [540, 669] (knowing whether there exist (n,n)-functions with nonlinearity strictly larger than this value when n is even is an open question). This is easily shown in the former case by using the properties of quadratic functions; it has been proved by Kasami in the second case, and it was first shown in [333] in the latter case. Dobbertin functions have worse nonlinearity.
The *inverse function* $x\mapsto x^{2^n-2}=x^{-1}$ has been chosen for the S-boxes of the AES with $n=8$ because of its bijectivity, good nonlinearity, good differential uniformity (which is suboptimal[20], equal to 4), highest possible algebraic degree $n-1$, non-plateauedness, simplicity... The computation of its output can be adapted to the device on which it is done thanks to the fact that 8 is a power of 2 and x^{-1} can then be computed by decomposition over subfields. An example of such decomposition is as follows: we can write $x^{-1}=x^{2^{n/2}}(x^{2^{n/2}+1})^{-1}$; we have then a product between $x^{2^{n/2}}$ which is a linear function over $\mathbb{F}_{2^n}$ and the inverse of $x^{2^{n/2}+1}$ which lives in the subfield $\mathbb{F}_{2^{n/2}}$. This method can be iterated; note that over $\mathbb{F}_{2^2}$, the inverse function equals x^2 and is then linear. This allows minimizing the number of nonlinear multiplications needed for computing x^{-1} which plays a role with respect to countermeasures against side channel attacks (see page 469).
For n odd, Gold and Kasami functions are AB. The nonlinearity of inverse func-

[20] We speak here of the sub-S-boxes; the global AES S-box is the concatenation of 16 differentially 4-uniform functions and has then differential uniformity $4\cdot 2^{15\cdot 8}=2^{122}$.

tion equals the highest even number bounded above by $2^{n-1}-2^{\frac{n}{2}}$, as also shown in [333] (this result has drawn K. Nyberg to focus on the inverse function in [908], which contributed to the invention of AES). Lachaud and Wolfmann proved (as we already mentioned at page 240) in [733] that the set of values of its Walsh spectrum equals the set of all integers $s \equiv 0$ [mod 4] in the range $[-2^{\frac{n}{2}+1}+1; 2^{\frac{n}{2}+1}+1]$, whatever is the parity of n; see more in [601]. See [196] for a list of all known *permutations* with best known nonlinearity. See also [467].

Inequivalence between functions

It is shown in [159] that distinct Gold functions are CCZ inequivalent. We have seen at page 309 that two power functions are CCZ equivalent if and only if they are EA equivalent or one of them is EA equivalent to the inverse of the other. Hence, given the algebraic degrees of Kasami functions, any two distinct functions taken among Gold and Kasami functions[21] are CCZ inequivalent (which was already shown in [159] when one function is Gold and the other is Kasami). And in [1140] is shown the CCZ inequivalence between any n-variable Gold function and any Niho function for $n \geq 9$. It is also shown in [247] that any plateaued function in even dimension which is CCZ equivalent to a Gold or Kasami APN function is necessarily EA equivalent to it. This result is revisited in [1141, Corollary 1] after a study of the more general framework of plateaued APN functions. Inverse and Dobbertin functions are CCZ inequivalent to all other known APN functions and between them because of their peculiar Walsh spectra, as also first observed in [159]. The situation is summarized in:

Proposition 177 *[1140, Proposition 2]*
(i) The Gold functions x^{2^s+1} and x^{2^t+1} on $\mathbb{F}_{2^n}$; $s,t<n/2$, are CCZ- equivalent if and only if $s=t$.
(ii) The Gold function x^{2^s+1} and the Kasami function $x^{4^r-2^r+1}$ on $\mathbb{F}_{2^n}$; $s,r< n/2$, are CCZ- equivalent if and only if either $s=r=1$ or $(n,s,r)=(5,1,2)$.
(iii) On $\mathbb{F}_{2^n}$ with n odd and $n\geq 9$, the Gold function x^{2^s+1} and the Welch function are always CCZ inequivalent.
(iv) On $\mathbb{F}_{2^n}$ with n odd and $n\geq 9$, the Gold function x^{2^s+1} and the Niho function are always CCZ inequivalent.
(v) The Kasami functions $x^{4^r-2^r+1}$ and $x^{4^s-2^s+1}$ on $\mathbb{F}_{2^n}$; $r,s<n/2$, are CCZ equivalent if and only if $r=s$.
(vi) On $\mathbb{F}_{2^n}$ with n odd and $n\geq 9$, the Kasami function $x^{4^r-2^r+1}$ and the Welch function are always CCZ inequivalent.
(vii) On $\mathbb{F}_{2^n}$ with n odd and $n\geq 9$, the Kasami function $x^{4^r-2^r+1}$ and the Niho function are always CCZ inequivalent.
(viii) On $\mathbb{F}_{2^n}$ with n odd and $n\geq 11$, the Welch function and the Niho function are always CCZ inequivalent.

It is proven in [134] that there exists no APN function CCZ inequivalent to

[21] A Gold function with $i=1$ and a Kasami function with $i=1$ as well are not distinct.

power mappings on $\mathbb{F}_{2^n}$ for $n \leq 5$. See also [494, Table 3] where the so-called switching classes related to the *switching* construction described at page 440 are investigated for $n = 5$; there are three classes with representatives x^3, x^5 and x^{-1}; in general, a switching class containing an APN function F is not included in the CCZ equivalence class containing F, but in the case of $\mathbb{F}_{2^5}$, they are the same. This fact is given as a comment in the second paragraph after Remark 2 in this same paper [494], where is also indicated that, in the case $n = 8$, the EA switching class of the Gold function x^3 contains 17 CCZ inequivalent functions.

Remark. Proving the CCZ inequivalence between two functions is mathematically (and also computationally) difficult, unless some CCZ invariant parameters can be proved different for the two functions. Examples of direct proofs of CCZ inequivalence using only the definition can be found in [158, 159, 160]. $\square$

Examples[22] of *CCZ invariant* parameters are the following (see [135] and [494] where they are introduced and used, as well as *group algebra* interpretations):

- The extended Walsh spectrum.
- The equivalence class of the code $\widetilde{C_F}$ defined at page 411 (under the relation of equivalence of codes), and all the invariants related to this code (the weight enumerator of $\widetilde{C_F}$, the weight enumerator of its dual - but it corresponds to the extended Walsh spectrum of the function - the automorphism group etc..., which coincide with some of the invariants below).
- The Γ-rank: let $\mathcal{G} = \mathbb{F}_2[\mathbb{F}_2^n \times \mathbb{F}_2^n]$ be the *group algebra* of $\mathbb{F}_2^n \times \mathbb{F}_2^n$ over $\mathbb{F}_2$, consisting of the formal sums $\sum_{g\in\mathbb{F}_2^n\times\mathbb{F}_2^n} a_g\, g$ where $a_g \in \mathbb{F}_2$. If S is a subset of $\mathbb{F}_2^n \times \mathbb{F}_2^n$, then it can be identified with the element $\sum_{s\in S} s$ of $\mathcal{G}$. The dimension of the ideal of $\mathcal{G}$ generated by the graph $\mathcal{G}_F = \{(x, F(x));\, x \in \mathbb{F}_2^n\}$ of F is called the Γ-*rank* of F. The Γ-rank equals (see [494]) the rank of the matrix $M_{\mathcal{G}_F}$ whose term indexed by $(x, y) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$ and by $(a, b) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$ equals 1 if $(x, y) \in (a, b) + \mathcal{G}_F$ and equals 0 otherwise.
- The Δ-rank, that is, the dimension of the ideal of $\mathcal{G}$ generated by the set $D_F = \{(a, F(x) + F(x + a));\, a,\, x \in \mathbb{F}_2^n;\, a \neq 0\}$ (according to Proposition 158, this set has size $2^{2n-1} - 2^{n-1}$ and is a difference set when F is AB). The Δ-rank equals the rank of the matrix M_{D_F} whose term indexed by (x, y) and by (a, b) equals 1 if $(x, y) \in (a, b) + D_F$ and equals 0 otherwise.
- The order of the automorphism group of the design $dev(\mathcal{G}_F)$, whose points are the elements of $\mathbb{F}_2^n \times \mathbb{F}_2^n$ and whose blocks are the sets $(a, b) + \mathcal{G}_F$ (and whose incidence matrix is $M_{\mathcal{G}_F}$); this is the group of all those permutations on $\mathbb{F}_2^n \times \mathbb{F}_2^n$ which map every such block to a block.
- The order of the automorphism group of the design $dev(D_F)$, whose points are the elements of $\mathbb{F}_2^n \times \mathbb{F}_2^n$ and whose blocks are the sets $(a, b) + D_F$ (and whose incidence matrix is M_{D_F}).

[22] There are other CCZ invariants known; we describe all those efficient for APN functions.

- The order of the automorphism group $\mathcal{M}(\mathcal{G}_F)$ of the so-called multipliers of $\mathcal{G}_F$, that is, the permutations π of $\mathbb{F}_2^n \times \mathbb{F}_2^n$ such that $\pi(\mathcal{G}_F)$ is a translate $(a,b)+\mathcal{G}_F$ of $\mathcal{G}_F$. This order is easier to compute and it makes it possible in some cases to prove CCZ inequivalence easily. As observed in [135], $\mathcal{M}(\mathcal{G}_F)$ is the automorphism group of the code $\widetilde{C_F}$.
- The order of the automorphism group $\mathcal{M}(D_F)$.
- A CCZ invariant lower bound on the minimum distance to other APN functions [153].

CCZ equivalence does not preserve crookedness nor the algebraic degree.

Exceptional exponents

The exponents d such that the function x^d is APN on infinitely many extensions $\mathbb{F}_{2^n}$ of $\mathbb{F}_2$ are called *exceptional* (see [642, 444, 135]). We have seen above that a power function x^d is APN if and only if the function $x^d + (x+1)^d + 1$ (we write "+1" so that 0 is a root, which simplifies presentation) is 2-to-1 and that, for every (n,n)-function F over $\mathbb{F}_{2^n}$, there exists a polynomial P such that $F(x) + F(x+1) + F(1) = P(x + x^2)$. In each case of the Gold and Kasami functions, one of these polynomials P is an exceptional polynomial (*i.e.* is a permutation over infinitely many fields $\mathbb{F}_{2^n}$); from there comes the term. In the case of the Gold function x^{2^i+1}, we have $P(x) = x + x^2 + x^{2^2} + \cdots + x^{2^{i-1}}$ which is a linear function over the algebraic closure of $\mathbb{F}_2$ having kernel $\{x \in \mathbb{F}_{2^i}\,;\, tr_i(x) = 0\}$ and is therefore a permutation over $\mathbb{F}_{2^n}$ for every n co-prime with i. In the case of Kasami exponents, the polynomial is related to the MCM polynomials, see page 433. It had been conjectured in [642] that Gold and Kasami exponents are the only exceptional exponents. This conjecture has been shown in [603]. It has been shown in [40] that if the degree of a function given in univariate representation is not divisible by 4, and if this degree is not a Gold or a Kasami exponent, and if the polynomial contains a term of odd degree, then the function can not be APN over infinitely many extensions of $\mathbb{F}_2$. See more on *exceptional* APN functions in [419, 418] and the references therein.

11.5.3 Non-power APN functions

As for AB functions, it had been wrongly conjectured that all APN functions were *EA equivalent* to power functions.

APN functions CCZ equivalent to power functions

Using also the notion of CCZ equivalence, two more infinite classes of APN functions have been introduced by Budaghyan, the author and Pott in [162, 163] which disprove the conjecture above:

- The function $F(x) = x^{2^i+1} + (x^{2^i} + x + 1)\, tr_n(x^{2^i+1})$, where $n \geq 4$ is even and

Table 11.3 Some APN functions CCZ equivalent to Gold functions and EA inequivalent to power functions on $\mathbb{F}_{2^n}$ (constructed in [162, 163]).

Functions	Conditions	d_{alg}
$x^{2^i+1}+(x^{2^i}+x+tr_n(1)+1)tr_n(x^{2^i+1}+x\ tr_n(1))$	$n\geq 4$ $\gcd(i,n)=1$	3
$[x+tr_{n/3}(x^{2(2^i+1)}+x^{4(2^i+1)})+tr_n(x)tr_{n/3}(x^{2^i+1}+x^{2^{2i}(2^i+1)})]^{2^i+1}$	$6\|n$ $\gcd(i,n)=1$	4
$x^{2^i+1}+tr_m^n(x^{2^i+1})+x^{2^i}tr_m^n(x)+x\ tr_m^n(x)^{2^i}$ $+[tr_m^n(x)^{2^i+1}+tr_m^n(x^{2^i+1})+tr_m^n(x)]^{\frac{1}{2^i+1}}(x^{2^i}+tr_m^n(x)^{2^i}+1)$ $+[tr_m^n(x)^{2^i+1}+tr_m^n(x^{2^i+1})+tr_m^n(x)]^{\frac{2^i}{2^i+1}}(x+tr_m^n(x))$	$m\neq n$ n odd $m\|n$ $\gcd(i,n)=1$	$m+2$

$\gcd(n,i)=1$, which is EA inequivalent to any power function.
• For n even and divisible by 3, the function $F(x)$ equal to

$$[x+tr_{n/3}(x^{2(2^i+1)}+x^{4(2^i+1)})+tr_n(x)\,tr_{n/3}(x^{2^i+1}+x^{2^{2i}(2^i+1)})]^{2^i+1},$$

where $\gcd(n,i)=1$, is APN and is EA inequivalent to any known APN function. We display in Table 11.3 the APN functions found this way. Finding classes of APN functions by using CCZ equivalence with Kasami (resp. Welch, Niho, Dobbertin, inverse) functions is an open problem.
In [1167] are made some observations about the fact that, starting from a quadratic APN function, it is possible to obtain a CCZ equivalent function which is EA inequivalent to any quadratic function.

APN functions CCZ inequivalent to power functions

• As recalled at page 430, two quadratic APN functions from $\mathbb{F}_{2^{10}}$ (resp. $\mathbb{F}_{2^{12}}$) to itself have been introduced in [493]. The first one: $F(x)=x^3+ux^{36}$, where $u\in\mathbb{F}_4\setminus\mathbb{F}_2$, was proved to be CCZ inequivalent to any power function by computing its Δ-rank. Proposition 174, page 430, which gives binomial AB functions when n is odd, gives *binomial APN functions* when n is even, which generalize the second function: $F(x)=x^3+\alpha^{15}x^{528}$, where α is a *primitive element* of $\mathbb{F}_{2^{12}}$. Some of them can be proven CCZ inequivalent to Gold and Kasami mappings, as seen in Proposition 175, and therefore, they are CCZ inequivalent to all power mappings due to a result of [1140] (that if a quadratic function is CCZ equivalent to a power function, then it is EA equivalent to a Gold map). A similar class but with n divisible by 4 was later given in [157]. A common framework exists for the class of Proposition 175 and this new class:

Proposition 178 *Let $n=tk$ be a positive integer, with $t\in\{3,4\}$, and s be such that t,s,k are pairwise co-prime and such that t is a divisor of $k+s$. Let α be a primitive element of $\mathbb{F}_{2^n}$ and $w=\alpha^e$, where e is a multiple of 2^k-1, co-prime with 2^t-1. Then the following function is APN:*

$$F(x)=x^{2^s+1}+wx^{2^{k+s}+2^{k(t-1)}}.$$

For $n \geq 12$, these functions are EA inequivalent to power functions and CCZ inequivalent to Gold and Kasami mappings as shown by a computer-free proof [158]. This implies that they are CCZ inequivalent to all power mappings [1140].
• Proposition 175, page 431, has been generalized[23] in [118] for n divisible by 3 by Bracken, Byrne, Markin and McGuire into *quadrinomial APN functions*:

$$F(x) = u^{2^k} x^{2^{2k}+2^{k+s}} + ux^{2^s+1} + vx^{2^{2k}+1} + wu^{2^k+1} x^{2^{k+s}+2^s} \tag{11.10}$$

is APN on $\mathbb{F}_{2^{3k}}$, when $3 \,|\, k+s$, $(s, 3k) = (3, k) = 1$ and u is primitive in $\mathbb{F}_{2^{3k}}$, $v \neq w^{-1} \in \mathbb{F}_{2^k}$. It contains the trinomials introduced in [116]. The Walsh spectrum has been determined in [123]. The general class of Proposition 178 is not generalized for the n divisible by 4 case.
• The same paper [118] obtained *multinomial APN functions* for $n \equiv 2$ [mod 4]:

$$F(x) = bx^{2^s+1} + b^{2^k} x^{2^{k+s}+2^k} + cx^{2^k+1} + \sum_{i=1}^{k-1} r_i x^{2^{i+k}+2^i} \tag{11.11}$$

where k, s are odd and co-prime, $b \in \mathbb{F}_{2^{2k}}$ is not a cube, $c \in \mathbb{F}_{2^{2k}} \setminus \mathbb{F}_{2^k}$, $r_i \in \mathbb{F}_{2^k}$, is APN on $\mathbb{F}_{2^{2k}}$. Recently, in [146], has been proved that these functions (11.11) are EA equivalent to the functions of Proposition 181 below, which is itself generalized by the class in [239], see below.

• The construction of AB functions of Proposition 176, page 431, works for APN functions: *for any positive integer n, function $x^3 + tr_n(x^9)$ is APN on $\mathbb{F}_{2^n}$*. It is shown in [160] that, if F is an APN quadratic (n,n)-function and f is quadratic Boolean such that, for every $a \in \mathbb{F}_{2^n}^*$, there exists a linear Boolean function ℓ_a satisfying $\beta_f(x,a) = f(x+a)+f(x)+f(a)+f(0) = \ell_a(\beta_F(x,a))$, then $F(x)+f(x)$ is APN provided that, if $\beta_F(x,a) = 1$ for some $x \in \mathbb{F}_{2^n}$, then $\ell_a(1) = 0$.
Function $x^3 + tr_n(x^9)$ is CCZ inequivalent to any Gold function on $\mathbb{F}_{2^n}$ if $n \geq 7$, as proved in [160], and therefore it is CCZ inequivalent to any power function [1140]. The extended Walsh spectrum of this function is the same as for the Gold functions as shown in [114].
The approach which has led to function $x^3 + tr_n(x^9)$ has been generalized (as for AB functions) and new functions have been deduced: $x^3 + a^{-1} tr_n(a^3 x^9)$ in [160], and for n divisible by 3, $x^3 + a^{-1} tr_n^3(a^3 x^9 + a^6 x^{18})$ and $x^3 + a^{-1} tr_n^3(a^6 x^{18} + a^{12} x^{36})$ in [161] (for n even, each such function defines two CCZ inequivalent functions, one for $a = 1$ and one for any $a \neq 1$). Their APNness is proved in this latter reference by showing for n even that if L is linear and $F(x) = x + L(x^3)$ is a permutation[24] over $\mathbb{F}_{2^n}$, then $F(x^3)$ is APN, and by showing a more complex result for n odd. The Walsh spectra of the functions above are determined in [114, 166] (they are Gold like Walsh spectra). When F is a Gold function, all possible APN mappings $F(x) + f(x)$, where f is a Boolean function have been computed until dimension 15. The only possibilities different from $x^3 + tr_n(x^9)$,

[23] Note that Proposition 174 covers a larger class of APN functions than Proposition 175.
[24] Sufficient conditions are given in [161] for that.

are for $n = 5$ the function $x^5 + tr_n(x^3)$ (CCZ equivalent to Gold functions) and for $n = 8$ the function $x^9 + tr_n(x^3)$ (CCZ inequivalent to power functions), see more in [1096]. In Table 11.4 are displayed all the known classes of APN functions CCZ inequivalent to power functions. It is shown in [161] that:

Proposition 179 *Let n be any positive integer and K some field extension of $\mathbb{F}_{2^n}$. Let L be an $\mathbb{F}_2$-linear mapping from $\mathbb{F}_{2^n}$ to $\mathbb{F}_{2^n}$ extended to an $\mathbb{F}_2$-linear mapping from K to K. Let E be a coset in K of a vector space containing $L(\mathbb{F}_{2^n})$. Assume that $F(x) = x + L(x^3)$ is injective on E and that the set $\{x^2 + x + 1; x \in E\}$ contains the set of elements y of $\mathbb{F}_{2^n}$ such that $tr_n(y) = 0$. Then $F(x^3)$ is APN over $\mathbb{F}_{2^n}$.*

Note that if the output of $x^3 + tr_n(x^9)$ is decomposed over an $\mathbb{F}_2$-basis of $\mathbb{F}_{2^n}$ which contains element 1, function $x^3 + tr_n(x^9)$ differs from x^3 for only one coordinate function. This is an example of the idea originally due to Dillon (after [160]) and developed in [494] of the *switching* construction: starting with a known APN function F, one of the coordinate functions is changed to give G, or equivalently if we view F and G valued in $\mathbb{F}_{2^n}$: $G(x) = F(x) + f(x)\,u$ where $u \in \mathbb{F}_{2^n}^*$ and f is a Boolean function; this gives a function which is differentially 4-uniform and in some rare cases, APN. In general, each change of a coordinate function in an S-box can at most multiply its differential uniformity δ by 2. Indeed, changing for instance the last coordinate function and denoting by F' the function obtained from F by erasing the last coordinate, the equation $F'(x)+F'(x+a) = b'$ corresponds to $F(x)+F(x+a) = (b', 0)$ or $F(x)+F(x+a) = (b', 1)$, and the differential uniformity of any function obtained by changing the last coordinate function is then at most 2δ (but this change can lower down the nonlinearity to 0). It is observed in [494] that if F is APN and $G(x) = F(x) + f(x)\,u$ as above, then G is APN if and only if we have $f(x)+f(x+a)+f(y)+f(y+a) = 0$ for every $x, y, a \in \mathbb{F}_{2^n}$ such that $F(x) + F(x + a) + F(y) + F(y + a) = u$; the proof is straightforward. This has led in [494] to an APN (6,6)-function CCZ inequivalent to power functions and to quadratic functions (the only known, currently), which had been already found in [134] but missed as a non-quadratic function; we shall call it the *Brinkmann-Leander-Edel-Pott function*; it seems that in the function given in [494], a coefficient was missing; a correct version is:

$$x^3 + \alpha^{17}(x^{17} + x^{18} + x^{20} + x^{24}) + \alpha^{14}[\alpha^{18}x^9 + \alpha^{36}x^{18} + \alpha^9x^{36} + x^{21} + x^{42}] + \alpha^{14}\,Tr_1^6(\alpha^{52}x^3 + \alpha^6x^5 + \alpha^{19}x^7 + \alpha^{28}x^{11} + \alpha^2x^{13}),$$

where α is primitive.

On the basis of a generalized switching construction, Göloglŭ has proposed in [547] the function $x^{2^k+1} + (tr_m^n(x))^{2^k+1}$, where $\gcd(k, n) = 1$ and $n = 2m$, m even, but this function was proved affine equivalent to the Gold function in [166].

• An idea of J. Dillon [445] was that (n, n)-functions (over $\mathbb{F}_{2^n}$) of the form:

$$F(x) = x(Ax^2 + Bx^q + Cx^{2q}) + x^2(Dx^q + Ex^{2q}) + Gx^{3q},$$

where $q = 2^{\frac{n}{2}}$, n even, have good chances to be differentially 4-uniform. Such

F is quadratic. For $a \in \mathbb{F}_{2^n}^*$, we consider then the equation $G_1 := F(x+a) + F(x) + F(a) = a_1x + a_2x^2 + a_3x^q + a_4x^{2q} = 0$, where $a_1, \dots, a_4 \in \mathbb{F}_{2^n}$. We deduce $G_2 := a_2^q G_1 + a_4 G_1^q = b_1x + b_2x^2 + b_3x^q = 0$, $G_3 := b_3^2 G_1 + a_3b_3G_2 + a_4G_2^2 = c_1x + c_2x^2 + c_3x^4 = 0$. If either c_1, c_2 or c_3 is nonzero, then F is differentially 4-uniform and possibly APN. This idea was applied to more general functions by Budaghyan and the author in [147]; this resulted in *trinomial APN functions*:

Proposition 180 *Let n be even and let $\gcd(i, \frac{n}{2}) = 1$. Set $q = 2^{\frac{n}{2}}$ and let $c, b \in \mathbb{F}_{2^n}$ be such that $c^{q+1} = 1$, $c \notin \{\lambda^{(2^i+1)(q-1)}, \lambda \in \mathbb{F}_{2^n}\}$, $cb^q + b \neq 0$. Then:*

$$F(x) = x^{2^{2i}+2^i} + bx^{q+1} + cx^{q(2^{2i}+2^i)}$$

is APN on $\mathbb{F}_{2^n}$. Such vectors b, c do exist if and only if $\gcd(2^i+1, q+1) \neq 1$. For $\frac{n}{2}$ odd, this is equivalent to saying that i is odd.

The extended Walsh spectrum of these functions is the same as that of the Gold functions [1181]. But it has been recently proved in [146] that these functions are EA equivalent to the functions of the next proposition.

• The method also resulted in a class of *hexanomial APN functions*:

Proposition 181 *[147] Let n be even and i be co-prime with $\frac{n}{2}$. Set $q = 2^{\frac{n}{2}}$ and let $c \in \mathbb{F}_{2^n}$ and $a \in \mathbb{F}_{2^n} \setminus \mathbb{F}_q$. If the polynomial $X^{2^i+1} + cX^{2^i} + c^qX + 1$ has no zero $x \in \mathbb{F}_{2^n}$ such that $x^{q+1} = 1$ (in particular if it is irreducible over $\mathbb{F}_{2^n}$), then the following function is APN on $\mathbb{F}_{2^n}$:*

$$F(x) = x(x^{2^i} + x^q + cx^{2^iq}) + x^{2^i}(c^qx^q + ax^{2^iq}) + x^{(2^i+1)q}.$$

The condition was shown achievable by computer investigation, then mathematically in [122] and [97]; finally in [547], all the polynomials satisfying it have been characterized, constructed and counted. This class was generalized (up to CCZ equivalence) in [239], see below (the question whether this generalizing bivariate construction gives new functions up to CCZ equivalence is open). It was checked with a computer that some of the APN functions provided by Proposition 181 are CCZ inequivalent to power functions for $n = 6, 8, 10$. It remains open to prove the same property for every even $n \geq 12$.
Cases where the hypothesis of Proposition 181 is satisfied were exhibited in [97, 122, 980]. The polynomials $X^{2^i+1} + cX^{2^i} + c^qX + 1$ are directly related to the polynomials $X^{2^i+1} + X + a$. In [122] are characterized the coefficients $a \in \mathbb{F}_{2^n}^*$ such that this latter polynomial has no zero in $\mathbb{F}_{2^n}$ when $\gcd(i, n) = 1$ and n is even. In particular for $i = 1$, the polynomial $X^3 + X + a$ has no zero (*i.e.* is irreducible) if and only if $a = u + u^{-1}$ where u is not a cube in $\mathbb{F}_{2^n}$. Note that $X^3 + X$ is the *Dickson polynomial* D_3 (seen at page 422).
As shown in [146], this hexanomial construction is more general than those of Proposition 180 and those of (11.11) (strictly more general, because it is also defined for n even divisible by 4 while these are not).
• A method has been introduced by the author in [239] for constructing APN

functions from bent functions. Let B be a bent $(n, \frac{n}{2})$-function and let G be a function from $\mathbb{F}_2^n$ to $\mathbb{F}_2^{\frac{n}{2}}$. Let $F : x \in \mathbb{F}_2^n \to (B(x), G(x)) \in \mathbb{F}_2^{\frac{n}{2}} \times \mathbb{F}_2^{\frac{n}{2}}$. Then F is APN if and only if, for every nonzero $a \in \mathbb{F}_2^n$, and for every $c, d \in \mathbb{F}_2^{\frac{n}{2}}$, the system of equations $\begin{cases} B(x) + B(x+a) & = & c \\ G(x) + G(x+a) & = & d \end{cases}$ has 0 or 2 solutions. Since B is bent, the number of solutions of the first equation always equals $2^{\frac{n}{2}}$ and such regularity can help. Functions EA equivalent to the Brinkmann-Leander-Edel-Pott function can be obtained in the form $(B(x,y), G(x,y))$ with $B(x,y) = sx^3 + ty^3 + ux^2y + vx\,y^2$.
Taking B equal to the Maiorana-McFarland function $B(x,y) = xy$ on $\mathbb{F}_{2^{\frac{n}{2}}} \times \mathbb{F}_{2^{\frac{n}{2}}}$, where xy is the product of x and y in the field $\mathbb{F}_{2^{\frac{n}{2}}}$, and writing then (a, b) with $a, b \in \mathbb{F}_{2^{\frac{n}{2}}}$ instead of $a \in \mathbb{F}_2^n$, the system of equations above becomes, after changing c into $c+ab$: $\begin{cases} bx + ay & = & c \\ G(x,y) + G(x+a, y+b) & = & d \end{cases}$. Then, by considering separately the cases $a = 0, b \neq 0$ and $a \neq 0$, F is APN if and only if:

1. for every $c \in \mathbb{F}_{2^{\frac{n}{2}}}$, the function $y \in \mathbb{F}_{2^{\frac{n}{2}}} \to G(c, y)$ is APN;
2. for every $b, c \in \mathbb{F}_{2^{\frac{n}{2}}}$, the function $x \in \mathbb{F}_{2^{\frac{n}{2}}} \to G(x, bx + c)$ is APN (this is easily seen by replacing b and c respectively by ab and ac in the latter system).

This leads to the following *bivariate APN functions*:

Proposition 182 *[239] Let n be any even integer; let i, j be integers such that $\gcd(n/2, i-j) = 1$, and let $s \neq 0$, $t \neq 0$, u and v be elements of $\mathbb{F}_{2^{n/2}}$. Set $G(x,y) = sx^{2^i+2^j} + ux^{2^i}y^{2^j} + vx^{2^j}y^{2^i} + ty^{2^i+2^j}$. Then the function*

$$F : (x,y) \in \mathbb{F}_{2^{n/2}} \times \mathbb{F}_{2^{n/2}} \to (x\,y, G(x,y)) \in \mathbb{F}_{2^{n/2}} \times \mathbb{F}_{2^{n/2}}$$

is APN if and only if $G(x,1) = sx^{2^i+2^j} + ux^{2^i} + vx^{2^j} + t$ has no zero in $\mathbb{F}_{2^{n/2}}$.

For $j = 0$ and $s = 1$ (as in Proposition 181), such polynomials are called "projective" by some authors. In [239] are investigated examples where the condition of Proposition 182 is satisfied and is shown that Propositions 180 and 181 and the APN functions of Bracken, Byrne, Markin and McGuire recalled above are cases of application of Proposition 182 (note that the bivariate function $(x,y) \in \mathbb{F}_{2^{n/2}}^2 \mapsto xy$ is EA equivalent to the univariate function $x \in \mathbb{F}_{2^n} \mapsto x^{2^{n/2}+1}$), see more in [243] where the univariate representation of these functions is investigated and generalized in several ways. Note that it was mentioned (but unpublished) by Göloglŭ, Krasnayova and Lisoněk at the conference Fq13, in 2017, that any APN function of the particular form $x^3 + ax^{3\cdot 2^{\frac{n}{2}}} + bx^{2\cdot 2^{\frac{n}{2}}+1} + cx^{2+2^{\frac{n}{2}}}$, $a, b, c \in \mathbb{F}_{2^{\frac{n}{2}}}$, is equivalent to x^3 or to $x^{2^{\frac{n}{2}-1}+1}$ or when $n = 6$ to the so-called Kim function (see the definition of this function at page 443).
The construction of function $(x,y) \mapsto (xy, x^{2^i+1} + ay^{2^j(2^i+1)})$, where $a \neq 0$ is assumed impossible to be written in the form $b^{2^i+1}(c + c^{2^i})^{1-2^j}$, has been pro-

posed by Zhou and Pott in [1182]. It is similar to the one of Proposition 182 but different. It has been generalized in [243]:

Proposition 183 *Let n be any even integer; let i be any integer co-prime with $m = n/2$, and let P, Q, R and S be linear homomorphisms of $\mathbb{F}_{2^m}$. Set $G(x,y) = P(x^{2^i+1}) + Q(x^{2^i}y) + R(xy^{2^i}) + S(y^{2^i+1})$. Then the function*

$$F : (x,y) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m} \to (x\,y, G(x,y)) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^m}$$

is APN if and only if, for every a and b in $\mathbb{F}_{2^m}$ such that $(a,b) \neq (0,0)$, the linear function $T_{a,b}(y) := P(a^{2^i+1}y) + Q(a^{2^i}by) + R(ab^{2^i}y) + S(b^{2^i+1}y)$ satisfies:
- if m is odd then $T_{a,b} : \mathbb{F}_{2^m} \mapsto \mathbb{F}_{2^m}$ is bijective;
- if m is even, then $(\ker T_{a,b}) \cap \{u^{2^i+1}(t^{2^i} + t);\ u \in \mathbb{F}^*_{2^m}, t \in \mathbb{F}_{2^m}\} = \{0\}$.

• A new class has been found by Zhou and Pott in [1182], see Table 11.4.
• Very recently has been found in [1077] the penultimate entry in Table 11.4, which enters in the framework of the proposition above.
• Still more recently has been found in [165] the last entry in the table.

Note that each class of functions can be described in many different ways due to equivalences. The reader must then not be surprised if there are small differences between the representations given in Table 11.4 and in the body of the section. The descriptions in the table are in some cases for subclasses, for which inequivalences between the different entries could be shown. This table as well as other tables with data on APN functions is periodically renewed at https://boolean.h.uib.no/mediawiki/index.php/Tables.

Note that we know then a little more than 20 infinite polynomial families of APN functions, while, as we saw, more than 8000 CCZ inequivalent APN functions are known already over $\mathbb{F}_{2^8}$; hence, even taking into account that some families contain several functions for each n (but some are not defined for $n = 8$), the currently known families cover only a tiny fraction of all APN functions.

An APN permutation and the *big open APN problem*

The APN functions listed above for n even are not permutations. This is problematic since for implementation reasons (see *e.g.* page 434), n even is preferred. Block ciphers using bijective APN $(7,7)$-functions and $(9,9)$-functions as S-boxes exist, such as the MISTY block cipher [830] and its variant KASUMI [687], but have drawbacks. Most block ciphers use then differentially 4-uniform permutations in even dimension. The question (called the big open APN problem by Dillon) of knowing whether there exist APN permutations when n is even (which would allow simplifying the structure) was wide open (as first mentioned in [910] and answered negatively for $n = 4$ in [624] thanks to a computer investigation and in [176] mathematically) until Browning, Dillon, McQuistan and Wolfe exhibit in [136] an APN permutation (of algebraic degree $n-2$ and nonlinearity $2^{n-1} - 2^{\frac{n}{2}}$) in $n = 6$ variables (used later in the cryptosystem Fides [84] which has been subsequently broken due to its weaknesses in the linear component). This permutation is CCZ equivalent to the so-called Kim function $x^3 + x^{10} + \alpha x^{24}$

Table 11.4 Known classes of quadratic APN polynomials CCZ inequivalent to power functions on $\mathbb{F}_{2^n}$.

Functions	Conditions	Proven
$x^{2^s+1}+\alpha^{2^k-1}x^{2^{ik}+2^{tk+s}}$	$n=pk$, $\gcd(k,p)$=$\gcd((s,pk))$=1, $p\in\{3,4\}$, $i=sk \bmod p$, $t=p-i$, $n\geq 12$, α primitive in $\mathbb{F}_{2^n}^*$	Prop. 174, 175 178; [158]
$ax^{2^i(q+1)}+x^{2^i+1}+x^{q(2^i+1)}$ $+x^{q+1}+cx^{2^iq+1}+c^qx^{2^i+q}$	$q=2^m$, $n=2m$, $\gcd(i,m)$=1, $c\in\mathbb{F}_{2^n}$, $a\in\mathbb{F}_{2^n}\setminus\mathbb{F}_q$, $X^{2^i+1}+cX^{2^i}+c^qX+1$ has no zero in $\mathbb{F}_{2^n}$ s.t. $x^{q+1}=1$	Prop. 181 [147]
$x^3+a^{-1}tr_n(a^3x^9)$	$a\neq 0$	[160]
$x^3+a^{-1}tr_3^n(a^3x^9+a^6x^{18})$	$3\|n$, $a\neq 0$	[161]
$x^3+a^{-1}tr_3^n(a^6x^{18}+a^{12}x^{36})$	$3\|n$, $a\neq 0$	[161]
$\alpha x^{2^s+1}+\alpha^{2^k}x^{2^{2k}+2^{k+s}}+$ $vx^{2^{2k}+1}+w\alpha^{2^k+1}x^{2^s+2^{k+s}}$	$n=3k$, $\gcd(k,3)$=$\gcd(s,3k)$=1, $v,w\in\mathbb{F}_{2^k}$, $vw\neq 1$, $3\|(k+s)$ α primitive in $\mathbb{F}_{2^n}^*$	see page 439 [116]
$(x+x^{2^m})^{2^k+1}+$ $\beta(\alpha x+\alpha^{2^m}x^{2^m})^{(2^k+1)2^i}+$ $\alpha(x+x^{2^m})(\alpha x+\alpha^{2^m}x^{2^m})$	$n=2m$, $m\geq 2$ even, $\gcd(k,m)=1$ and $i\geq 2$ even α primitive in $\mathbb{F}_{2^n}^*$, $\beta\in\mathbb{F}_{2^m}$ not cube	[1182]
$a^2x^{2^{2m+1}+1}+b^2x^{2^{m+1}+1}+$ $ax^{2^{2m}+2}+bx^{2^m+2}+$ $(c^2+c)x^3$	$n=3m$, m odd; U subgroup of $\mathbb{F}_{2^n}^*$ of order $2^{2m}+2^m+1$, $L(x)=$ $ax^{2^{2m}}+bx^{2^m}+cx\in\mathbb{F}_{2^n}[x]$, s.t. $\forall v,t\in U$, $L(v)\notin\{0,v\}$ and $v^2L(t)$ $+tL(v)^2\neq 0\Rightarrow\frac{t^2L(v)+vL(t)^2}{v^2L(t)+tL(v)^2}\notin\mathbb{F}_{2^m}$	[142]
$\alpha(\alpha^qx+\alpha x^q)(x^q+x)+$ $(\alpha^qx+\alpha x^q)^{2^{2i}+2^{3i}}+$ $a(\alpha^qx+\alpha x^q)^{2^{2i}}(x^q+x)^{2^i}+$ $b(x^q+x)^{2^i+1}$	$q=2^m, n=2m, \gcd(i,m)=1$ $X^{2^i+1}+aX+b$ has no zero in $\mathbb{F}_{2^m}$	[1077]
$x^3+\beta(x^{2^i+1})^{2^k}+\beta^2x^{3\cdot 2^m}+$ $(x^{2^{i+m}+2^m})^{2^k}$	$n=2m$; m odd; $3\not\|m$ $i=m-2$ or $i=(m-2)^{-1}$ [mod n] β primitive in $\mathbb{F}_4$	[165]

(given in [135]), whose associated code C_F (see Proposition 160) is therefore a double simplex code (see page 25). It is EA equivalent to an involution and is studied further in [944, 199] where the *butterfly construction* is introduced. This construction works with concatenations of bivariate functions $R(x,y)$ over $\mathbb{F}_{2^{n/2}}$ ($n/2$ odd) which are viewed as $R_y(x)$ and are such that R_y is bijective for every y. The resulting butterflies have two CCZ equivalent representations, one of which, called closed butterfly, can be taken quadratic (and may not be bijective) and has the form $(R_y(x),R_x(y))$, while the other, called open butterfly, is the involution of the form $(R_{R_y^{-1}(x)}(y),R_y^{-1}(x))$. The graph of the closed butterfly writes:

$$\mathcal{G}_F=\{(x,y,R_y(x),R_x(y));x,y\in\mathbb{F}_2^n\},$$

and operating on this graph the linear permutation $(x, y, z, t) \mapsto (z, y, t, x)$ gives:

$$\mathcal{G}'_F = \{(R_y(x), y, R_x(y), x); x, y \in \mathbb{F}_2^n\}.$$

This set is invariant when swapping x and y, which corresponds to swapping $(R_y(x), y)$ and $(R_x(y), x)$. By the change of variable $x = R_y^{-1}(x')$, which transforms $R_y(x)$ into x', we have:

$$\mathcal{G}'_F = \{(x', y, R_{R_y^{-1}(x')}(y), R_y^{-1}(x')); x', y \in \mathbb{F}_2^n\},$$

that is the graph of the open butterfly, which is an involution, according to the invariance mentioned above. This construction includes the APN permutation of [136], but unfortunately it is shown in [199, 943] that it does not allow obtaining APN permutations in more than 6 variables. The butterfly construction gives differentially 4-uniform involutions, see page 456.

The question of existence of APN permutations in even dimension n remains open for $n \geq 8$. There exist nonexistence results within the following classes:

- plateaued functions (when APN, they have bent components, see page 424),
- a class of functions including power functions (see page 415),
- functions whose univariate representation coefficients lie in $\mathbb{F}_{2^{\frac{n}{2}}}$, or in $\mathbb{F}_{2^4}$ for n divisible by 4 [624],
- functions whose univariate representation coefficients satisfy $\sum_{i=0}^{\frac{2^n-1}{3}} a_{3i} = 0$ [184],
- functions having at least one partially-bent component; it is indeed proved in [176] (starting from the same idea as in [24], see page 415) that no component function of an APN permutation can be partially-bent (this improves upon several previous results on the components of APN permutations): n being even, the linear kernel of such balanced partially-bent component function $v \cdot F$ would have dimension at least 2, and since for every a, b, we have $D_a(v \cdot F)(x) \oplus D_b(v \cdot F)(x) = D_{a+b}(v \cdot F)(x + a)$, there would exist $a \neq 0$ such that $D_a(v \cdot F) = 0$, and since $Im(D_a F)$ has 2^{n-1} elements because F is APN, $Im(D_a F)$ would include 0, a contradiction with the bijectivity of F.

Finding infinite classes of APN functions CCZ inequivalent to power functions and to quadratic functions is an open problem too.

11.5.4 The extended Walsh spectra of known APN functions

For n odd, the known APN functions have three possible spectra (all satisfying $\mathcal{V}(v \cdot F) = 2^{2n+1}$ for every $v \neq 0$, since they are power permutations and since we have already seen that F is APN if and only if $\sum_{v \in \mathbb{F}_2^n \setminus \{0_n\}} \mathcal{V}(v \cdot F) = 2^{2n+1}(2^n - 1)$), see *e.g.* [62]:

- the spectrum of the AB functions which gives a nonlinearity of $2^{n-1} - 2^{\frac{n-1}{2}}$,

- the spectrum of the inverse function, which takes any value divisible by 4 in $[-2^{\frac{n}{2}+1}+1; 2^{\frac{n}{2}+1}+1]$ and gives a nonlinearity close to $2^{n-1}-2^{\frac{n}{2}}$,
- the spectrum of the *Dobbertin function* which is more complex (it is divisible by $2^{n/5}$ and not divisible by $2^{2n/5+1}$ [196]); its nonlinearity seems to be bounded below by approximately $2^{n-1}-2^{3n/5-1}-2^{2n/5-1}$ - maybe equal - but this has to be proven (or disproven).

For n even, the spectra may be more diverse:

- the Gold functions (and all known infinite classes of quadratic APN functions [117, 123, 1061]) whose component functions are bent for a third of them and have nonlinearity $2^{n-1}-2^{\frac{n}{2}}$ for the rest of them; the Kasami functions which have the same extended spectra,
- the Dobbertin function (same observation as above),
- As soon as $n \geq 6$, we find (quadratic) APN functions with different spectra (*e.g.* $x^3+\alpha^{11}x^5+\alpha^{13}x^9+x^{17}+\alpha^{11}x^{33}+x^{48}$, for $n=6$, with a 7-valued Walsh spectrum found by Dillon).

The nonlinearities seem also bounded below by approximately $2^{n-1}-2^{3n/5-1}-2^{2n/5-1}$ (but this has to be proven ... or disproven too). Note that the question of classifying APN functions is open even when restricting ourselves to quadratic APN functions in more than 6 variables (even classifying their Walsh spectra is open for even numbers of variables). There is only one known example of quadratic APN function (with $n=6$) having non-Gold-like nonlinearity, see [445].

11.5.5 Conclusion on known APN functions

As we can see, very few functions usable as S-boxes have emerged so far. The only known APN permutations are in odd dimension or in dimension 6, which is not convenient for implementation. Besides, Gold functions, all the other found quadratic functions and the Welch functions have too low algebraic degrees for being widely chosen for the design of new S-boxes. The Kasami functions themselves seem too closely related to quadratic functions. The *inverse function* has many very nice properties: large Walsh spectrum and good nonlinearity, differential uniformity of order at most 4, fast implementation. But differential uniformity 2 in a dimension equal to a power of 2 would be better and the inverse function has a potential weakness against algebraic attacks, which did not lead yet to efficient attacks, but may in the future. So further studies on APN permutations seem essential for the future designs of SP networks.

11.6 Differentially uniform functions

In [297] is shown, for every (n,m)-function F, the following lower bound involving the differential uniformity of F and the size of its image set:

$$\delta_F \geq \left\lceil \frac{\frac{2^{2n}}{|Im(F)|} - 2^n}{2^n - 1} \right\rceil.$$

This is equivalent to the lower bound:

$$|Im(F)| \geq \left\lceil \frac{2^{2n}}{(2^n-1)\,\delta_F + 2^n} \right\rceil.$$

11.6.1 Characterizations by the Walsh transform

We have seen that APN functions are nicely characterized by their Walsh transform through Relation (11.2), page 403. It is shown in [250] that other characterizations by the Walsh transform exist for APN functions and that more generally, for each value of δ, several (in fact, an infinity of) characterizations by the Walsh transform of differentially δ-uniform functions exist. We follow here the presentation of [252]. Denoting for every $a, b \in \mathbb{F}_2^n$ and every (n,m)-function F by $N_F(b,a)$ the size of the set $\{x \in \mathbb{F}_2^n; D_aF(x) = D_aF(b)\}$, we have that F is differentially δ-uniform if and only if, for every $a \neq 0_n$ in $\mathbb{F}_2^n$ and every $b \in \mathbb{F}_2^n$, we have $N_F(b,a) \in \{2, 4, \ldots, \delta\}$. For every polynomial $\phi_\delta(X) = \sum_{j\geq 0} A_j X^j \in \mathbb{R}[X]$ such that $\phi_\delta(u) = 0$ for $u = 2, 4, \ldots, \delta$ and $\phi_\delta(u) > 0$ for every even $u \in \{\delta+2, \ldots, 2^n\}$, we have then for every (n,m)-function F that:

$$\sum_{j\geq 0} A_j \sum_{\substack{a,b\in\mathbb{F}_2^n\\ a\neq 0_n}} (N_F(b,a))^j = \sum_{\substack{a,b\in\mathbb{F}_2^n\\ a\neq 0_n}} \sum_{j\geq 0} A_j\,(N_F(b,a))^j \geq 0,$$

and that F is differentially δ-uniform if and only if this inequality is an equality. The sum $\sum_{a,b\in\mathbb{F}_2^n} (N_F(b,a))^j$ is easily expressed by means of the Walsh transform of F:

Lemma 12 *Let F be any (n,m)-function. We have, for $j \geq 1$:*

$$\sum_{a,b\in\mathbb{F}_2^n} (N_F(b,a))^j = \sum_{a,b\in\mathbb{F}_2^n, a\neq 0_n} (N_F(b,a))^j + 2^{n(j+1)} =$$

$$2^{-j(m+n)} \sum_{\substack{u_1,\ldots,u_j\in\mathbb{F}_2^n\\ v_1,\ldots,v_j\in\mathbb{F}_2^m}} W_F^2\left(\sum_{i=1}^j u_i, \sum_{i=1}^j v_i\right) \prod_{i=1}^j W_F^2(u_i, v_i). \tag{11.12}$$

This technical lemma is proved in [250] by raising at the j-th power the equality $N_F(b,a) = 2^{-m} \sum_{x\in\mathbb{F}_2^n, v\in\mathbb{F}_2^m} (-1)^{v\cdot(D_aF(x)+D_aF(b))}$, obtaining $(N_F(b,a))^j = 2^{-jm} \sum_{\substack{x_i\in\mathbb{F}_2^n, v_i\in\mathbb{F}_2^m\\ i=1,\ldots,j}} (-1)^{\bigoplus_{i=1}^j v_i\cdot(F(x_i)+F(x_i+a)+F(b)+F(b+a))}$ and using that $y_i =$

$x_i + a$ if and only if $\sum_{u_i\in\mathbb{F}_2^n}(-1)^{u_i\cdot(x_i+y_i+a)} = 2^n$ (idem for $c = b + a$). We deduce from Lemma 12:

Theorem 26 *[250] Let n, m and δ be positive integers, with δ even, and let F be any (n,m)-function. Let*

$$\phi_\delta(X) = \sum_{j\geq 0} A_j X^j \in \mathbb{R}[X]$$

be any polynomial such that $\phi_\delta(u) = 0$ for $u = 2, 4, \ldots, \delta$ and $\phi_\delta(u) > 0$ for every even $u \in \{\delta+2, \ldots, 2^n\}$. Then we have:

$$2^n(2^n-1)A_0 + \sum_{j\geq 1} 2^{-j(n+m)} A_j \left((W_F^2)^{\otimes(j+1)}(0_n, 0_m) - 2^{(2j+1)n+jm}\right) \geq 0, \tag{11.13}$$

where $(W_F^2)^{\otimes(j+1)}$ is the $(j+1)$-th order convolutional product of W_F:

$$(W_F^2)^{\otimes(j+1)}(0_n, 0_m) = \sum_{\substack{u_1,\ldots,u_j\in\mathbb{F}_2^n \\ v_1,\ldots,v_j\in\mathbb{F}_2^m}} W_F^2\left(\sum_{i=1}^j u_i, \sum_{i=1}^j v_i\right) \prod_{i=1}^j W_F^2(u_i, v_i).$$

Moreover, this inequality is an equality if and only if F is differentially δ-uniform.

Relation (11.2), page 403, can then be deduced from Theorem 26 by choosing $\phi_2(X) = X - 2$. Theorem 26 gives other interesting characterizations. We shall give one for each case $\delta = 2$ and $\delta = 4$, but more can be found in [250]. Taking $\phi_2(X) = (X-2)(X-4)$, we obtain:

Corollary 29 *[250] Every (n,n)-function F is APN if and only if:*

$$\sum_{\substack{u_1,u_2\in\mathbb{F}_2^n; v_1,v_2\in\mathbb{F}_2^n \\ v_1\neq 0_n, v_2\neq 0_n, v_1\neq v_2}} W_F^2(u_1, v_1) W_F^2(u_2, v_2) W_F^2(u_1+u_2, v_1+v_2) = 2^{5n}(2^n-1)(2^n-2). \tag{11.14}$$

Moreover, every (n,n)-function satisfies a version of (11.14) with "$\geq$" in the place of "$=$", but to show this, Theorem 26 must be applied with $\phi_4(X) = (X-2)(X-4)$ and also with $\phi_2(X) = X - 2$ (see [250]).
Applying Theorem 26 with $\phi_4(X) = (X-2)(X-4)$ when $m = n-1$ gives:

Corollary 30 *[250] Every $(n, n-1)$-function F is differentially 4-uniform if and only if:*

$$\sum_{\substack{u_1,u_2\in\mathbb{F}_2^n; v_1,v_2\in\mathbb{F}_2^{n-1} \\ v_1\neq 0_{n-1}, v_2\neq 0_{n-1}, v_1\neq v_2}} W_F^2(u_1, v_1) W_F^2(u_2, v_2) W_F^2(u_1+u_2, v_1+v_2) = 2^{5n}(2^{n-1}-1)(2^{n-1}-2). \tag{11.15}$$

And every $(n, n-1)$-functions satisfies a version of (11.15) with "$\geq$" in the place of "$=$".
Note the similarity between these two corollaries, which shows that the two

optimal notions of APN (n,n)-function and differentially 4-uniform $(n,n-1)$-function are close.

In [252], Theorem 26 is generalized into a characterization of all the criteria on vectorial functions dealing with the numbers of solutions of equations of the form $\sum_{i\in I} F(x+u_{i,a})+L_a(x)+u_a=0_m$, with L_a linear. In particular injective functions are characterized this way. A characterization of o-polynomials originally given in [314] can also be obtained by this generalization. And a generalization to differentially δ-uniform functions of a characterization by Nyberg of APN functions by means of the Walsh transforms of their *derivatives* is also derived.

11.6.2 Componentwise Walsh uniformity (CWU)

We have seen at page 423 that the characterization of APNness by Relation (11.2) leads to a stronger notion called CAPNness, in which the relation is satisfied by each component function. The characterizations of APN (n,n)-functions and differentially 4-uniform $(n,n-1)$-functions by Relations (11.14) and (11.15) in Corollaries 29 and 30 lead similarly to the following EA invariant notion introduced in [251]: we call *componentwise Walsh uniform* (*CWU*) those functions $F:\mathbb{F}_2^n\mapsto\mathbb{F}_2^m$, with $m\in\{n-1,n\}$, which satisfy (11.14), respectively (11.15), for each pair of *component functions*.

Definition 83 *[251] An (n,m)-function F with $m\in\{n-1,n\}$ is called CWU if, for all distinct nonzero $v_1,v_2\in\mathbb{F}_2^m$, we have:*

$$\sum_{u_1,u_2\in\mathbb{F}_2^n} W_F^2(u_1,v_1)W_F^2(u_2,v_2)W_F^2(u_1+u_2,v_1+v_2)=2^{5n}.$$

If $m=n$, then CWUness implies APNness. The converse is not true in general, but we have the following result (we refer the reader to [251] for the proof):

Proposition 184 *Any crooked function (in particular, any quadratic APN (n,n)-function) is CWU.*

An investigation of CWU functions among all known non-quadratic APN power functions was made in [251] for $n\le 11$. Two potential infinite classes of non-quadratic CWU power functions arised:

- all Kasami APN functions (n odd or even)
- the inverse of the Gold APN permutation $x^{2^{\frac{n-1}{2}}+1}$ (n odd).

None of the other known classes of non-quadratic APN power functions is made of CWU functions only. We have:

Proposition 185 *[251] The compositional inverse of the Gold APN permutation $x^{2^{\frac{n-1}{2}}+1}$ is CWU for every odd n.*

The proof is rather long, so we refer to [251] for it. Finding a proof (after confirmation of the investigation results) of the same property for Kasami functions is an open problem (see some observations in [251] and more in [252]).

11.6.3 Cyclic difference sets, cyclic-additive difference sets and the CWU property

We have seen the notion of additive difference set in $\mathbb{F}_2^n$ when dealing with bent functions at page 220: every nonzero element can be written in the same number of ways as the difference $x-y$ (that is, $x+y$) between two elements of the set. This notion exists for every group structure. When this group structure is that of $\mathbb{F}_{2^n}^*$, we speak of cyclic difference set, since $\mathbb{F}_{2^n}^*$ is cyclic (and $x-y$ has to be replaced by $\frac{x}{y}$). A particular case which plays a role with APN functions (see [448]) is the following:

Definition 84 *A subset Δ of size 2^{n-1} of the multiplicative group $\mathbb{F}_{2^n}^*$ is called a* cyclic difference set with Singer parameters *if, for all distinct v_1, v_2 in $\mathbb{F}_{2^n}^*$, we have $|\{(x,y)\in\Delta^2;\ v_1x+v_2y=0\}|=2^{n-2}$.*

Equivalently, the symmetric difference between Δ and $a\Delta$ equals 2^{n-1} for every $a\in\mathbb{F}_{2^n}^*\setminus\{1\}$, that is, $\sum_{x\in\mathbb{F}_{2^n}^*}(-1)^{1_\Delta(x)\oplus 1_\Delta(ax)}=-1$ (*i.e.* the sequence $s_i=1_\Delta(\alpha^i)$, where α is primitive, has *ideal autocorrelation*). Any *Singer set* (already seen at page 421) $S_d=\{x\in\mathbb{F}_{2^n}; tr_n(x^d)=1\}$, $\gcd(d,2^n-1)=1$, has such property since $x\mapsto x^d$ is a (multiplicative) group automorphism and the intersection between two distinct affine hyperplanes has dimension $n-2$. Maschietti [825] (see also [443]) proves that, for every d co-prime with 2^n-1 and such that the mapping $x\mapsto x+x^d$ is 2-to-1 over $\mathbb{F}_{2^n}$, the complement of the image of this mapping is a cyclic difference set with Singer parameters. It is proved in [443, 448] that, for every APN Kasami function $F(x)=x^{4^k-2^k+1}$ over $\mathbb{F}_{2^n}$, $n=3k\pm1$, the set $\{F(x)+F(x+1); x\in\mathbb{F}_{2^n}\}$ (or its complement if it contains 0) is a cyclic difference set with Singer parameters (note that $x\mapsto F(x)+F(x+1)$ is also 2-to-1) and in [448] that the complement of its translation by 1, that is, of $\Delta_F=\{F(x)+F(x+1)+1; x\in\mathbb{F}_{2^n}\}$ is a cyclic difference set with Singer parameters (the proof is deduced from an elegant calculation of the Fourier transform of the indicator of the set $D_F=\{x^{\frac{1}{2^i+1}}; x\in\Delta_F\}$), under the weaker condition that $\gcd(k,n)=1$[25]. Known facts are summarized with their proofs and a few new observations are made in [252].

A relationship is shown in [251] between CWU power permutations and a new notion similar to the cyclic difference set property.

Definition 85 *A set $\Delta\subseteq\mathbb{F}_{2^n}$ is called a* cyclic-additive difference set *if, for every distinct nonzero v_1, v_2 in $\mathbb{F}_{2^n}$, we have:*

$$|\{(x,y,z)\in\Delta^3;\ v_1x+v_2y+(v_1+v_2)z=0\}|=2^{2n-3}.$$

Every power function F is APN if and only if the set $\{F(x)+F(x+1)+1; x\in\mathbb{F}_{2^n}\}$ has size 2^{n-1}.

Proposition 186 *Let F be any power APN permutation. Then, F is CWU if and only if the set $\{F(x)+F(x+1)+1; x\in\mathbb{F}_{2^n}\}$ (or equivalently its complement) is a cyclic-additive difference set.*

[25] For Gold APN functions, we have the same, but the set is the *classical Singer set* S_1.

There are differences between cyclic and cyclic-additive difference sets:

- the notion of cyclic difference set is invariant under raising the elements to a power co-prime with $2^n - 1$, and if two sets Δ and Δ' are such that $W_{1_\Delta}(a) = W_{1_{\Delta'}}(a^k)$ where k is co-prime with $2^n - 1$, then Δ is a cylic difference set if and only if Δ' is one, while these properties are not true for cyclic-additive difference sets,
- the notion of cyclic-additive difference set is invariant under translation $x \mapsto x + a$ while that of cyclic difference set is not (see [448]).

It seems impossible to deduce the cyclic-additive property of Δ_F in the case of Kasami APN permutations from the fact proved by Dillon and Dobbertin in [448] that the Fourier transform of the *indicator* of the set $D_F = \{x^{\frac{1}{2^i+1}}; x \in \Delta_F\}$ takes at any input $a \in \mathbb{F}_{2^n}$ the same value as the Walsh transform of the Boolean function $tr_n(x^3)$ at $a^{\frac{2^i+1}{3}}$.

11.6.4 The known differentially 4-uniform (n, n)-permutations, n even

For computational reasons (explained at page 434 for the inverse function but valid in a more general context), (n, n)-functions are better used as S-boxes when n is even, the best being when n is a power of 2. In practice, we have most often $n = 4$ (for lightweight cryptosystems, to be implemented for instance on cheaper smart cards) and $n = 8$ (for cryptosystems implemented on more powerful devices), since $n = 16$ seems still too large for current computational means. We have seen that only one APN permutation, in 6 variables, is known. It is then important to find as many differentially 4-uniform permutations as possible in even dimension.

Note that if these permutations are involutions, this allows reducing further the complexity of the algorithm, since the same implemented function can then be used for encryption and decryption. Several block ciphers like AES, Khazad, Anubis or PRINCE use involutive functions (up to affine equivalence) in their S-boxes. Note that, as already mentioned at page 444 and shown in [136], the 6-variable permutation exhibited in this reference is EA equivalent to an involution.

The smallest differential uniformity and largest nonlinearity achievable by a $(4, 4)$-permutation are respectively 4 and 4 [756, 183]. Up to affine equivalence, there are 16 classes of such permutations. All have algebraic degree 3 and are also optimal against algebraic attacks. Half have a component function of algebraic degree 2, which should be avoided and half have all their component functions cubic. There are 6 CCZ equivalence classes.

The smallest differential uniformity and largest nonlinearity achievable by an $(8, 8)$-permutation are respectively 4 and 112 (achieved by Gold, Kasami and inverse functions).

We describe now the known infinite classes of differentially 4-uniform (n, n)-

functions. We begin with the functions obtained by *primary constructions*, starting with power functions:

- The *inverse function* x^{2^n-2} (the only known involutive differentially 4-uniform power (n,n)-permutation, see [521]) for n even first proposed in [908] is used (composed by an affine permutation) for the *S-box of the AES*[26] with $n=8$. This class of functions has best known nonlinearity $2^{n-1}-2^{n/2}$ and has maximum algebraic degree $n-1$. It is the worst possible against algebraic attacks (which are not efficient on the AES, but some risk exists that they will be improved as they were for stream ciphers) since if we denote $y=x^{-1}$ then we have the bilinear relation $x^2y=x$.
- The *Gold functions* x^{2^i+1} where $\gcd(i,n)=2$ are differentially 4-uniform and they are bijective (*i.e.* $\gcd(2^i+1,2^n-1)=1$) if and only if $n\equiv 2$ [mod 4] since, 2^i-1 and 2^i+1 being co-prime, we have $\gcd(2^i+1,2^n-1)=\frac{\gcd(2^{2i}-1,2^n-1)}{\gcd(2^i-1,2^n-1)}=\frac{2^{\gcd(2i,n)}-1}{2^{\gcd(i,n)}-1}$ (but n is then not a power of 2 and these functions are quadratic). They have best known nonlinearity. Gold functions are never involutive.
- The *Kasami functions* $x^{2^{2i}-2^i+1}$ such that $n\equiv 2$ [mod 4] and $\gcd(i,n)=2$ are differentially 4-uniform as proved in [669] (see also [604]) and bijective as well, since $\gcd(2^i+1,2^n-1)=1$ and since $2^{2i}-2^i+1=\frac{2^{3i}+1}{2^i+1}$ implies $\gcd(2^{2i}-2^i+1,2^n-1)=\gcd(2^{3i}+1,2^n-1)=\frac{\gcd(2^{6i}-1,2^n-1)}{\gcd(2^{3i}-1,2^n-1)}=\frac{2^{\gcd(6i,n)}-1}{2^{\gcd(3i,n)}-1}=1$ since $2^{3i}-1$ and $2^{3i}+1$ are co-prime. They are not quadratic but they have the same Walsh spectrum as the Gold functions (thus, with best known nonlinearity). They are in fact rather closely related to quadratic functions, since they have the form $F=R_1\circ R_2^{-1}$ where R_1 and R_2 are quadratic permutations, which has some similarity with a function CCZ equivalent to a quadratic function. There is a threat that this could be used in a modified version of the higher-order differential attack (but adapting the attack to such kind of function is an open problem). This class of functions never reaches the maximum algebraic degree $n-1$ (but this is not really a problem). Kasami functions are never involutive.

  Remark. While all APN permutations have, by definition of APNness, the same differential spectrum, this is different for differentially 4-uniform permutations. For instance, the inverse function and the Gold and Kasami functions have different differential spectra (the inverse function has a better differential spectrum, in which value 4 is obtained less often). More on power functions can be found in [92]. □

- The function $x^{2^{n/2+n/4+1}}$ introduced by Dobbertin [467] and shown by Bracken

[26] Often represented by a double-entry look-up table with 16 rows and 16 columns, whose indices belong to $\mathbb{F}_2^4$ (and can be written in *hexadecimal* from 0 to f), which provides the $16^2=256$ entries (which, when represented in hexadecimal, belong to $\{00,\dots,\mathrm{ff}\}$).

and Leander to be differentially 4-uniform [120] has best known nonlinearity $2^{n-1} - 2^{n/2}$ as well. It is bijective (but not involutive) if n is divisible by 4 but not by 8; in this case, n is not a power of 2; the function has algebraic degree 3 which is rather low.

- The APN binomials of Proposition 175 share many properties of Gold power functions. In particular, relaxing conditions on involved parameters leads to differentially 2^t-uniform permutations [121]. Let $n = 3k$ and t be a divisor of k where $3 \nmid k$ and $\frac{k}{t}$ is odd. Let s be an integer such that $\gcd(3k, s) = t$ and $3|(k + s)$. Then the functions $\alpha x^{2^s+1} + \alpha^{2^k} x^{2^{-k}+2^{k+s}}$, where α is a primitive element of $\mathbb{F}_{2^n}$, are differentially 2^t-uniform and bijective. This class of functions has nonlinearity $2^{n-1} - 2^{(n+t-2)/2}$ if $n+t$ is even and $2^{n-1} - 2^{(n+t-3)/2}$ if $n + t$ is odd. A conjecture of [121] that quadratic quadrinomial APN functions (11.10) allow similar generalization to differentially 2^t-uniform permutations is disproved in [983].
  It is shown in [356], among other results, that for n even, any quadratic differentially 4-uniform permutation has all its $\delta_F(a, b)$ values in $\{0, 4\}$ for $a \neq 0$, has nonlinearity $2^{n-1} - 2^{n/2}$, and is plateaued with single amplitude.
- The author proposed in [241], for constructing differentially 4-uniform permutations, to use the structure of the field $\mathbb{F}_{2^{n+1}}$ instead of that of $\mathbb{F}_{2^n}$ (of course, $\mathbb{F}_{2^{n+r}}$ could be also tried with $r \geq 2$). The idea consists in finding an (N, N)-function, where $N = n + 1$, whose restriction to an affine hyperplane of $\mathbb{F}_{2^N}$ has for image set an affine hyperplane. This restriction provides then an (n, n)-permutation since any affine hyperplane of $\mathbb{F}_{2^N}$ is affinely equivalent to $\mathbb{F}_2^n$. Let the affine hyperplane be $A = u + E$ where E is a linear hyperplane, the restriction to A is differentially 4-uniform if the restriction to A of any derivative in a nonzero direction belonging to E is 2-to-1. An example given in [241] is with the Dickson polynomials D_k, seen at page 422. Recall that every element of $\mathbb{F}_{2^N}^*$ can be expressed uniquely in the form $h + \frac{1}{h}$ where $h \in \mathbb{F}_{2^{2N}}^*$ - more precisely, $h \in \mathbb{F}_{2^N}^* \cup U$, where $U = \{x^{2^N-1}; x \in \mathbb{F}_{2^{2N}}^*\}$ is the multiplicative subgroup of $\mathbb{F}_{2^{2N}}^*$ of order $2^N + 1$. Then $D_k(h + \frac{1}{h}, 1) = h^k + \frac{1}{h^k}$ by definition. Moreover, the image of $\mathbb{F}_{2^N}$ by this function equals $\{\frac{1}{x}; x \in \mathbb{F}_{2^N}, tr_N(x) = 0\}$ (with the usual convention $\frac{1}{0} = 0$) and the image of $U \setminus \{1\}$ equals $\{\frac{1}{x}; x \in \mathbb{F}_{2^N}, tr_N(x) = 1\}$. If k is co-prime with $2^N + 1$, then the mapping $h \mapsto h^k$ is a permutation of $U \setminus \{1\}$ and induces then a permutation of $\{\frac{1}{x}; x \in \mathbb{F}_{2^N}, tr_N(x) = 1\}$ whose expression coincides with $D_k(x, 1)$ on this set. The function $\frac{1}{D_k(\frac{1}{x},1)}$ is then a permutation of the hyperplane H of $\mathbb{F}_{2^N}$ of equation $tr_N(x) = 1$. For $k = 3$, N must be even, and we have $D_3(x, 1) = x^3 + x$ and $\frac{1}{\frac{1}{x^3}+\frac{1}{x}} = \frac{x^3}{x^2+1} = x + \frac{1}{x+1} + \frac{1}{x^2+1}$, which is EA equivalent to $\frac{1}{x} + \frac{1}{x^2}$ and is differentially 4-uniform over H. But the argument given in [241] for this last property is not correct: it is written that the function $x \to x + x^2$ is 2-to-1 which is true, and that the inverse function is APN which is false since N is even. The correct argument is that H excluding 0, the equation

$x^{2^n-2}+(x+a)^{2^n-2}=b$ (where $a\neq 0$ and therefore $b\neq 0$ since the inverse function is a permutation) is equivalent to $\frac{1}{x}+\frac{1}{x+a}=b$, that is, $x^2+ax=\frac{a}{b}$ and has then at most 2 solutions.

This differentially 4-uniform permutation is in odd dimension. We complete here the study by addressing the case n even: if k is co-prime with 2^N-1, then the mapping $h\mapsto h^k$ is a permutation of $\mathbb{F}_{2^N}$ and induces then a permutation of $\{\frac{1}{x}; x\in\mathbb{F}_{2^N}, tr_N(x)=0\}$ whose expression coincides with $D_k(x,1)$ on this set. The function $\frac{1}{\frac{1}{x^3}+\frac{1}{x}}=x+\frac{1}{x+1}+\frac{1}{x^2+1}$ restricted now to the hyperplane of equation $tr_N(x)=0$ is bijective. It is differentially 4-uniform since N being odd, the inverse function is APN. And it is shown in [241] that the nonlinearity is at least $2^{n-1}-2^{\frac{n}{2}+1}$ (not optimal) and the algebraic degree equals $n-1$. A similar proposal in even dimension is given in [1107].

In [772], Li and Wang have used again the idea of working with functions F over $\mathbb{F}_{2^N}$ with $N=n+1$, and of taking their restrictions to hyperplanes. They need F to be a quadratic APN permutation over $\mathbb{F}_{2^N}$ and they take N odd so that n is even (then this permutation is AB but this is not used). They consider for every nonzero $u\in\mathbb{F}_{2^N}$ the linear function $L_u(x)=F(x+u)+F(x)+F(u)+F(0)$, whose range H_u is a linear hyperplane (since F is APN) such that $F(u)+F(0)\notin H_u$ (since F is bijective). Then the restriction of $L_u\circ F^{-1}$ to H_u is injective because $L_u\circ F^{-1}(x_1)=L_u\circ F^{-1}(x_2)$ and $x_1\neq x_2$ imply $F^{-1}(x_1)+F^{-1}(x_2)=u$, since F is APN, and therefore, $L_u\circ F^{-1}(x_2)=F(F^{-1}(x_2)+u)+F(F^{-1}(x_2))+F(u)+F(0)=x_1+x_2+F(u)+F(0)$, and the relation $x_1+x_2=F(u)+F(0)+L_u(F^{-1}(x_2))\in(F(u)+F(0)+H_u)\cap H_u=\emptyset$ is impossible. This permutation is differentially 4-uniform by construction since L_u is 2-to-1 and F^{-1} is APN; it is then a differentially 4-uniform permutation of H_u. They obtained with F equal to Gold functions three classes of differentially 4-uniform bijections in even dimension with best known nonlinearity $2^{n-1}-2^{n/2}$ and algebraic degree $\frac{n}{2}+1$.

It seems difficult to prove that any of the functions presented in this paragraph is CCZ inequivalent to power functions and to quadratic functions (but we see no reason why such equivalence could happen since the field structures of $\mathbb{F}_{2^n}$ and $\mathbb{F}_{2^N}$ are independent of each other).

We continue now with permutations obtained by modifications of known differentially 4-uniform bijections.

- Qu, Tan, Tan and Li [982] proposed two classes of differentially 4-uniform bijections in even dimension. These functions were obtained through the switching construction, by adding Boolean functions to the inverse function. The first class has the form $x^{2^n-2}+tr_n(x^2(x+1)^{2^n-2})$. It has optimal algebraic degree $n-1$ and nonlinearity larger than $2^{n-1}-2^{n/2+1}-2$. The second class has the form $x^{2^n-2}+tr_n\big(x^{(2^n-2)d}+(x^{2^n-2}+1)^d\big)$, where

$d = 3(2^t + 1)$, $2 \leq t \leq n/2 - 1$. It has algebraic degree $n - 1$ as well and nonlinearity at least $2^{n-2} - 2^{n/2-1} - 1$. The authors did not study whether their functions are CCZ inequivalent to the inverse function, but this can be checked for even $n = 6, \ldots, 12$ with a computer. A generalized method for constructing differentially 4-uniform permutations in even dimension is presented in the same reference (by determining conditions for their differential uniformity) which includes the former two classes of functions, and produces many CCZ inequivalent differentially 4-uniform bijections in even dimension (the authors could show that the number of CCZ inequivalent differentially 4-uniform permutations over $\mathbb{F}_{2^{2k}}$ grows exponentially when k increases). Zha, Hu, Sun and Shan in [1151], Qu, Tan, Li and Gong in [981] (who made more systematic the approach of [982] and obtained more functions), Peng and Tan in [938], and Chen, Deng, Zhu and Qu in [362] proposed more functions of the form $x^{2^n-2}+g(x)$, where g is a Boolean function (thanks to more precise conditions in the latter reference). In [1145] are also built differentially 4-uniform permutations by swapping two values of the inverse function (it is observed that a function $I_{(u,v)}$ over $\mathbb{F}_{2^{2m}}$, obtained from the inverse power function by swapping its values at two different points $u \neq 0$ and $v \neq 0$, is a differentially 4-uniform permutation if and only if $tr_{2m}(uv^{-1})tr(u^{-1}v) = 1$).

- The author, Tang, Tang and Liao proposed in [325] the following construction, for $n \geq 6$ even, of differentially 4-uniform (n,n)-permutations of algebraic degree $n - 1$:

  $$(x_1, \cdots, x_{n-1}, x_n) \mapsto \begin{cases} (1/x', f(x')), & \text{if } x_n = 0 \\ (c/x', f(x'/c) + 1), & \text{if } x_n = 1, \end{cases}$$

  where $c \in \mathbb{F}_{2^{n-1}} \setminus \mathbb{F}_2$ is such that $tr_{n-1}(c) = tr_{n-1}(1/c) = 1$, $x' \in \mathbb{F}_{2^{n-1}}$ is identified with $(x_1, \cdots, x_{n-1}) \in \mathbb{F}_2^{n-1}$ and f is an arbitrary Boolean function defined on $\mathbb{F}_{2^{n-1}}$.

  It is shown in [363] that the particular functions corresponding to $f(x') = tr_{n-1}\left(\frac{1}{x'+1}\right)$ have high nonlinearity and are CCZ inequivalent to all known differentially 4-uniform power permutations and to quadratic functions. It is also shown that the functions in the general class are CCZ inequivalent to the inverse function, and for $n = 2k$, $k = 4, \ldots, 7$, to the sums of the inverse function and of Boolean functions.

- Zha, Hu and Sun [1150] presented two classes of differentially 4-uniform bijections by applying affine transformations to the inverse function on some subfields. Their functions have maximum algebraic degree $n-1$. The lower bounds on the nonlinearity of these functions would need to be worked further. In [1130] is provided another infinite family of differentially 4-uniform permutations with the same "piecewise" method but starting from Gold functions. Inequivalence to known functions needs to be checked.

  Peng and Tan in [939], Peng, Tan and Wang in [940] and Xu and Qu in [1132] presented similar transformations.

Tang, the author and Tang [1070], for any even $n \geq 6$, introduced a class of subsets U of $\mathbb{F}_{2^n}$ such that the function equal to $(x+1)^{2^n-2}$ if $x \in U$ and to x^{2^n-2} otherwise gives a differentially 4-uniform permutation. For every even n, at least $2^{2^{n-3}-2^{n/2-2}}$ different such sets U are designed. For every even $n \geq 12$, it is proved that if the size of U is such that $0 < |U| < (2^{n-1} - 2^{n/2})/3 - 2$ then the functions are CCZ inequivalent to known differentially 4-uniform power functions and to quadratic functions. A table of comparison with these other functions is given.

- Li, Wang and Yu in [773] modified the inverse function by cyclically shifting the images of the function over some subset $\{\alpha_0, \alpha_1, \ldots, \alpha_m\}$ of $\mathbb{F}_{2^n}$. Fu and Feng [521] proposed new families with such cycles of length 3.
- Perrin, Udovenko and Biryukov have introduced in [944] the interesting *butterfly construction*, already seen at page 444, generalized by Canteaut, Duval, and Perrin in [199]. It is shown in [944, 523] and [199] that the resulting function is differentially 4-uniform with (best known) nonlinearity $2^{n-1} - 2^{\frac{n}{2}}$ when, respectively: $R_y(x) = (x+ay)^3 + y^3$, with $a \in \mathbb{F}^*_{2^{\frac{n}{2}}}$, $R_y(x) = (x+ay)^{2^i+1} + y^{2^i+1}$, with $a \in \mathbb{F}^*_{2^{\frac{n}{2}}}$, $\gcd(i,n) = 1$, and $R_y(x) = (x+ay)^3 + by^3$, with $a, b \in \mathbb{F}^*_{2^{\frac{n}{2}}}$, $b \neq (1+a)^3$. More differentially 4-uniform permutations with nonlinearity $2^{n-1} - 2^{\frac{n}{2}}$ are obtained with the butterfly construction in [523].

Fu and Feng studied in [521] if some functions among those recalled in the present subsection could be involutions. They obtained the following involutive differentially 4-uniform permutations:

- functions of the form $x^{2^n-2} + 1_U(x)$ or $(x+1)^{2^n-2}$ if $x \in U$ and to x^{2^n-2} otherwise, where $U = \mathbb{F}_4$ and in the case $n \equiv 2$ [mod 4], $U = \mathbb{F}_2$ or $U = \mathbb{F}_4 \setminus \mathbb{F}_2$;
- the Peng and Tan function [939]: $F(x) = \begin{cases} b\,(x+1)^{2^n-2} + a \text{ if } x \in \mathbb{F}_{2^t}, \\ x^{2^n-2} \text{ otherwise,} \end{cases}$ where t divides n and $a = b = 1$, or $a = 0, b = 1, t$ even, or $a = 0, b = 1, t = 1, 3, \frac{n}{2}$ odd, or $a = b \in \mathbb{F}_4 \setminus \mathbb{F}_2, t = 2, \frac{n}{2}$ odd, or $a = 1, tr_n(b^{-1}) = 1$ and $\frac{n}{t}$ odd;
- the Peng, Tan and Wang function [940]: $F(x) = \begin{cases} (c\,x)^{2^n-2} \text{ if } x \in U, \\ x^{2^n-2} \text{ otherwise,} \end{cases}$ where $U = \bigcup_{g \in G} g\,\Gamma$, where Γ is the cyclic multiplicative group generated by γ and $\{g^{-1}, g \in G\} = G$ and $tr_n(\gamma) = tr_n(\gamma^{-1})$ and $tr_n\left(\frac{\gamma}{\frac{g}{g'}\gamma^l + \frac{g'}{g}\gamma^{-l}}\right) = 1$ for every $g, g' \in G$ and every l (with l not divisible by $|\Gamma|$ if $g = g'$);
- the particular case given by Fu and Feng [521] of the Li, Wang and Yu function [773], modifying the inverse function by cyclically shifting a triple $\{\alpha_0, \alpha_1, \alpha_2\}$ of $\mathbb{F}^*_{2^n}$, where $(\alpha_0, \alpha_1, \alpha_2) = (0, \gamma, \gamma^{-1})$ with $\gamma \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$, $tr_n(\gamma) = tr_n(\frac{1}{\gamma+1}) = 1$ or $(\alpha_0, \alpha_1, \alpha_2) = (1, \gamma, \gamma^{-1})$ with $\gamma \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$, $tr_n(\gamma) = tr_n(\frac{1}{\gamma}) = tr_n(\frac{1}{\gamma+1}) = 1$, $tr_n(\frac{1}{(\gamma+1)^3}) = 0$;

- the functions $F(x_1,\cdots,x_{n-1},x_n)=\begin{cases}(1/x',f(x')), & \text{if } x_n=0\\ (c/x',f(x'/c)+1) & \text{if } x_n=1\end{cases}$ from [325] where $supp(f)=\emptyset$ or $=\{0\}$.

It is observed in [967] that the indicators of the graphs of all the known differentially 4-uniform (n,n)-permutations (n even) have algebraic immunity 2. In [1039] is studied the differential uniformity of the composition of two functions and are constructed new differentially 4-uniform permutations from known ones.

11.6.5 Other differentially 4-uniform (n,n)-functions

There are differentially 4-uniform functions in odd dimension, that we do not list here since they are less interesting, practically. There are also differentially 4-uniform functions in even dimension which are not permutations. These can be obtained from APN functions by adding a Boolean function, or composing them (on the right or on the left) by 2-to-1 affine functions. Differentially 4-uniform functions which are faster and less costly to compute can be obtained by concatenating the outputs of a bent function and of another function [239]:

1. The function $(x,y)\to(xy,(x^3+w)(y^3+w'))$, where w, w' and $\frac{w}{w'}$ belong to $\mathbb{F}_{2^{n/2}}\setminus\{x^3,\, x\in\mathbb{F}_{2^{n/2}}\}$, with $n/2$ even.
2. The function $(x,y)\to(xy,x^3(y^2+y+1)+y^3)$, with $n/2$ odd.
3. The function $F: X\in\mathbb{F}_{2^n}\to(X^{2^{n/2}+1},(X^{2^{n/2}+1})^3+wX^3+(wX^3)^{2^{n/2}})$.

Such functions are not bijective but, because of their low implementation complexity, have an advantage when we wish to protect the cryptosystem against side channel attacks (see Section 12.1, page 460). For instance, Function 1 above has been used in the cryptosystem PICARO [957]. See more in [239]. Other examples of differentially 4-uniform functions are the function $ax^{2^{2s}+1}+bx^{2^s+1}+cx^{2^{2s}+2^s}$ such that $\gcd(s,n)=1$ as shown in [115], the function $x^{2^{n-1}-1}+ax^5$ (n odd, $a\in\mathbb{F}_{2^n}$) as shown in [120], and several classes given in [243].
Some constructions of differentially 4-uniform functions have been given in [896], in connection with the structure of commutative semifield already seen in the chapter on bent functions. A semifield is a finite algebraic structure $(E,+,\circ)$ such that (1) $(E,+)$ is an Abelian group, (2) the operation $\circ$ is distributive on the left and on the right with respect to $+$, (3) there is no nonzero divisor of 0 in E and (4) E contains an identity element with respect to $\circ$. This structure has been very useful for constructing planar functions in odd characteristic. In characteristic 2, it may lead to new APN functions and differentially 4-uniform permutations by considering for instance the function $(x\circ x)\circ x$ in a classical semifield (there are two classes of them, whose underlying Abelian group is the additive group of $\mathbb{F}_{2^n}$: the Albert semifields, in which the multiplication is $x\circ y=xy+\beta(xy)^\sigma$, where $x\to x^\sigma$ is an automorphism of the field $\mathbb{F}_{2^n}$ which is not a generator and $\beta\notin\{x^{\sigma+1};\, x\in\mathbb{F}_{2^n}\}$, and the Knuth semifield where the multiplication is $x\circ y=xy+(xtr(y)+ytr(x))^2$, where tr is a trace function from $\mathbb{F}_{2^n}$ to a suitable subfield).

11.6.6 Other differentially uniform (n, n)-functions

Some results have been found on differential uniformities 6 and 8 for (n, n)-functions, see [93, 95, 1129] and the references therein. In [1160] are proposed two methods for constructing balanced (n, m)-functions (with $m < n$ unfortunately) with nonlinearity strictly larger than $2^{n-1} - 2^{n/2}$ and with "good" other parameters[27].

11.6.7 On the best differential uniformity of (n, m)-functions

When $m < n$, (n, m)-functions cannot be used in substitution-permutation networks but they can be used in *Feistel ciphers*, like in the DES cipher which has 8 S-boxes each mapping 6 bits to 4 bits. When n is even and $m \leq \frac{n}{2}$, these functions can be bent (*i.e.* PN), which allows them to oppose optimal resistance against differential and linear attacks, but they are then not balanced and the number of their output bits is small. When $\frac{n}{2} < m < n$, little theoretical work has been done on differentially uniform (n, m)-functions. We know that the differential uniformity of such functions is bounded below by $2^{n-m}+2$. We call this bound *Nyberg's bound*. Characterizing the pairs (n, m) for which this bound is tight is an open question.

- *In the case $m = n-1$, Nyberg's bound is tight.* There is indeed a simple way of designing differentially 4-uniform $(n, n-1)$-functions: any function of the form $L \circ F$, where F is an APN (n, n)-function and L is a surjective affine $(n, n-1)$-function is indeed differentially 4-uniform. Using such S-box in a Feistel cipher can be seen as using the APN function itself.
  In [255] is studied an alternate way to construct differentially 4-uniform $(n, n-1)$-functions by defining their lookup table (LUT) as the concatenation of the LUT of two APN $(n-1, n-1)$-functions; the corresponding function $S(x, x_n) = x_n F(x) + (1+x_n) G(x)$ is a differentially 4-uniform $(n, n-1)$-function if and only if, for every $a \in \mathbb{F}_{2^{n-1}}$, the function $F(x) + G(x + a)$ is at most 2-to-1 (*i.e.* each value in the image set has at most two corresponding pre-images).
  The particular case where the two APN functions differ by an affine function provides, when one of these functions is a Gold function, the family of quadratic differentially 4-uniform $(n, n-1)$-functions $(x, x_n) \mapsto x^{2^i+1} + x_n x$ where $x \in \mathbb{F}_{2^{n-1}}$ and $x_n \in \mathbb{F}_2$ with $\gcd(i, n-1) = 1$, whose Walsh transform and nonlinearity are studied, as well as the CCZ inequivalence to all functions of the form $L \circ F$ above.
- In [259], (n, m)-functions achieving Nyberg's bound with equality are studied in the (Maiorana-McFarland) form $F(x, z) = I(x)\phi(z)$, where $I(x)$ is the

[27] An inappropriate comparison is made in this paper with a permutation - the one used as S-box in the AES - and with the S-box used in PICARO (designed to resist side channel attacks, and therefore a little weaker with respect to the other features).

(m,m)-*inverse function*[28] and $\phi(z)$ is an $(n-m,m)$-function. An infinite family of differentially $(2^{m-1}+2)$-uniform $(2m-1,m)$-functions with $m \geq 3$ is designed (which also have high nonlinearity and not too low algebraic degree). Hence, *Nyberg's bound is tight for* $m = \frac{n+1}{2}$, $n \geq 5$ *odd*.
Differentially 4-uniform $(m+1,m)$-functions in this form are also designed and a method is proposed to construct infinite families of $(m+k,m)$-functions with low differential uniformity, leading to an infinite family of $(2m-2,m)$-functions with $\delta \leq 2^{m-1} - 2^{m-6} + 2$ for any $m \geq 8$. But this does not provide functions achieving Nyberg's bound with equality and the existence of such (n,m)-functions for $\frac{n}{2} + 1 \leq m \leq n-2$ is open.
In fact, it is even an open problem to determine whether there exist differentially δ-uniform $(n, n-k)$ functions with $k \geq 2$, k significantly smaller than $\frac{n}{2}$, $\delta < 2^{k+1}$, and $n > 5$ ($\delta = 2^{k+1}$ is easily reached with functions $L \circ F$ where F is an APN (n,n)-function and L is an affine surjective $(n, n-k)$-function). In particular, the existence of differentially 6-uniform $(n, n-2)$-functions for $n > 5$ is an open question (differentially 6-uniform $(5,3)$-functions are known [255]). In [15] are built more differentially 4-uniform $(n, n-1)$-functions and differentially 8-uniform $(n, n-2)$-functions. In [949], several evolutionary algorithms and problem sizes have been explored in order to find such functions. The results of this investigation show that the problem, which is easy in dimensions 4 and 5, is very difficult for larger n.

[28] The only function which returned positive results when we made a computer investigation.

12 Recent uses of Boolean and vectorial functions and related problems

Many mathematical problems in computer science result in questions on Boolean functions (or on vectorial functions). Cryptography has been no exception since the 50's and new roles of Boolean functions still emerge nowadays. In this chapter, we give several examples of recent problematics in cryptography which result in new questions on Boolean functions, vectorial functions and related codes, or which renew the interest of some known notions.

12.1 Physical attacks and related problems on functions and codes

Until the 90's, cryptographers implicitly considered the *black box attacker model* only, in which the cryptanalyst has access to ciphertexts (in the ciphertext-only attacker model) or to plaintext-ciphertext pairs (in the known-plaintext and the chosen-plaintext models), but has no information beyond input/output. This was realistic when the ciphers were run only on computers, all the more if these were protected (by a Faraday cage for instance). But nowadays, cryptographic algorithms are run often on mobile devices, on smart cards (which include a part of hardware and work with software implementations), or on light hardware devices (*e.g.* field-programmable gate arrays FPGA, application-specific integrated circuits ASIC). Side-channel information (through the running-time, power consumption, electromagnetic emanations, etc.) is then accessible.
The *side channel attacks* (*SCA*), see [712, 713, 984], on the implementations of block ciphers[1] in such embedded systems, see [823], take advantage of this additional information obtained through the physical environment. They are able to treat this information for extracting the secret parameters of the algorithm and are in practice extremely powerful. They assume an *attacker model* different from classical attacks: the *grey box attacker model*, in which the adversary has also access to leakage. This additional information is all the more usable on block ciphers, which are iterative: each round involves diffusion layers and substitution layers; both kinds are necessary for the security, and the diffusion needs several rounds before being effective; the SCA can then be very efficient by attacking the first round (while in the black box model, only the global cipher is attackable)

[1] SCA also exist on asymmetric ciphers, but this is out of the scope of this book, and they have not been as developed for stream ciphers as they were for block ciphers.

or the last round (the first round of the reverse cipher), see the survey [320]. The exploited *leakage* is a measurable quantity (in the case of a so-called *monovariate attack* or *univariate attack* on a single leakage[2]) depending on the data manipulated by the algorithm (the key is mixed with the data, and any leakage that depends on this data can be used as an oracle). The important data for SCA are the values of the so-called *sensitive variable*s of the algorithm. These are variables whose values are in general stored in registers and which depend on the (varying) input to the algorithm (assumed known by the attacker), and on the (constant) secret key (or better for the attacker, on a part of the secret key, since this allows for a divide-and-conquer approach, where the key is recovered byte by byte, a customary case in block ciphers being when the cipher computes the sum of a public binary vector and of a sub-sequence of the round-key). The length n of such variable is a number depending on the cipher (4 if the cipher works on nibbles, 8 if it works on bytes, 16 if it works on words, ...).
The attacker records, for instance, the emanations emitted by the register on which the values of the sensitive variable are stored, which can be approximated as a real-valued function $\mathcal{L}$ of the sensitive variable (the register is a micrometric object, whose contents cannot, in general, be measured directly). For instance, in the so-called *Hamming weight leakage model*, $\mathcal{L}(Z)$ equals the Hamming weight of Z; in the *Hamming distance leakage model*, $\mathcal{L}(Z)$ is the Hamming distance between two consecutive values of the register where Z is stored; in more general *linear leakage model*s, $\mathcal{L}(Z)$ equals a linear combination with real coefficients of the bits of Z (we speak then of a static linear leakage model, needed to ensure that the leakages corresponding to different shares are independent), or of the differences between the bits at two consecutive states of the register. In what the attacker records, $\mathcal{L}$ is added with inevitable noise $\mathcal{N}$, generally viewed as a white Gaussian variable, due to the activity in the device around the register (an attacker can only measure an aggregated function of each computing element's leakage, such as the total current drawn by the circuit) and depending on the choice of the leakage model (a good choice minimizes the noise). The part independent of the noise in the leak is called the *deterministic leak*. The attacker tests exhaustively all the possible values of the key bits involved in the sensitive variable, computing for each choice the corresponding *modeled leakage* value, the correct key values being those which maximize statistically (for a series of runs with the same key and different plaintexts) the dependency between the modeled leakage (which depends on the tested key value and the plaintext, and also on the leakage model chosen for the attack) and the measured leakage. This dependency can be evaluated by different statistical methods, leading to different SCA. For instance, in *differential power analysis* (*DPA*) [713] or more general differential analyses, the statistical distinguisher is the difference of means between the two cases (among all plaintexts used) where the leak is larger, resp. smaller, than some fixed value. If the guessed key is correct, then the modeled

[2] Multivariate attacks are more difficult to perform in practice.

difference should be close to a nonzero constant, while if it is wrong, it should be close to zero (the two means measuring then a same random variable). In *correlation power analysis* (*CPA*) or more general correlation analyses, the statistical distinguisher is Pearson's (linear) correlation coefficient (which is more complex to evaluate but more efficient), equal to the co-variance between the two values, divided by the product of their standard deviations.

The attacker starts with a first order attack, in which the leakage is handled as is. It can be proved that this first order attack is successful if the conditional expectation $\mathbb{E}(\mathcal{L}|Z = z)$ depends on z. If it does not, then the attacker can try successively a second order attack, which mixes the observations of two leakages (and if these two leakages are the same, the attacker takes then the square of a single leakage[3] in a so-called zero-offset CPA [1100]; we shall consider only this case in the sequel, to simplify the presentation), a third order attack, ..., increasing the order of the attack until it is successful. The complexity of such *higher-order side channel attack* (*HO-SCA*) [879, 920, 1047, 803, 381] depends then on the smallest value of the order j such that the conditional expectation $\mathbb{E}(\mathcal{L}^j|Z = z)$ depends on z, see [264]. It is shown in this latter reference that the complexity of the attack (in time and in the number of measuring events – called traces, see below – which is needed) is exponential in the order, essentially because the noise associated to $\mathcal{L}^j$ is exponential in j. Relatively to the noise, the leaked information decreases exponentially with the order j; it is proportional to V^{-j} where V is the variance of the noise $\mathcal{N}$. This is where the choice of the leakage model plays a role: a bad choice will increase the variance of the noise.

SCA are mainly statistical attacks, and the measures are made several times, each time providing a so-called *leakage trace*. Usually, traces are assumed independent and identically distributed[4]. The measure quantifies as we saw above the running-time, the power consumption, the electromagnetic radiations of the cryptographic computation or even the photonic emission. Depending on the execution platform, the part of the leakage due to one bit can be modeled according to its activity (the leakage is observed when the bit changes values; this is the case of complementary metal-oxide-semiconductor [CMOS] technology), or its value (the leakage differs according to the bit's state; this can be viewed as a particular case of the former case). If every bit of a sensitive variable leaks an identical amount, irrespective of its neighbors, we are in the so-called Hamming distance (resp. weight) leakage model (see more in [262]). The measure is inevitably imprecise and noisy as we saw above with HO-SCA, but if the cryptosystem is not protected against SCA, the resulting attack can be devastating (an unprotected AES can be attacked in a few seconds with a few traces while its

[3] This case is more frequent in hardware; two distinct leakages are more exploitable in software because it is easier to determine the exact distinct timings of two leakages than to distinguish them when they happen in parallel; note however that the improved capabilities of modern microprocessors more and more allow parallel software computing.

[4] Adaptative adversarial strategies are seldom conferring a significant advantage, compared to non-adaptative strategies.

security against classical attacks is still nowadays of 128 bits, which corresponds to a huge amount of computation time, even for thousands of computers in parallel). In particular, *continuous side-channel attacks* in which the adversary gets information at each invocation of the cryptosystem are especially threatening [713].
SCA are not the only threats on block ciphers, since *fault injection attack*s (*FIA*) can also be performed, which aim at extracting the secret key when the algorithm is running over some device, by injecting some fault in the computation, so as to obtain exploitable differences at the output. For instance, differential fault analysis (DFA) attacks, first proposed by Biham and Shamir [83], use information obtained from an incorrectly functioning implementation of an algorithm to derive the secret information. The AES can be attacked this way (see *e.g.* [91]) as well as stream ciphers [606]. These attacks can be non-invasive and perturb internal data (for example with electromagnetic impulses), without damaging the system, and leaving then no evidence that they have been perpetrated.

Masking. The implementations of cryptosystems need to include countermeasures to *physical attack*s (SCA and FIA). A sound approach against SCA is to use a *secret sharing scheme* (see page 168), often called *masking* in the context of side-channel attacks[5]. This method, which aims at amplifying the impact of the noise in the adversary's observations and at randomizing the secret-dependent internal values of the algorithm from one execution to another, is efficient both for implementations in smart cards and FPGA or ASIC (in the former case, the shares are usually manipulated in serial, while in the latter, they are manipulated in parallel). This approach consists, for a given *masking order* d, in splitting each sensitive variable[6] Z of the implementation into $d+1$ shares $M_0,\dots,M_d$ such that Z can be recovered from these shares but no information can be recovered from less than $d+1$ shares, *i.e.*, Z is a deterministic function of all the M_i, but is independent of $(M_i)_{i\in I}$ if $|I|\leqslant d$. The simplest way (called *Boolean masking*) of achieving this is to draw $M_1,\dots,M_d$ at random from the space in which lives Z (the M_i's are then called *masks* and are redrawn fresh at every encryption) and to take M_0 such that $M_0+\cdots+M_d$ equals Z, where $+$ is a relevant group operation (in practice, the bitwise XOR). The masks change at every computation. This countermeasure allows resisting the SCA of order d. For instance, for $d=1$ and if the leakage is the Hamming weight w_H, then instead of having traces corresponding to $w_H(Z)$, the attacker will have traces corresponding to $w_H(Z+M,M)=w_H(Z+M)+w_H(M)$ (note that the individual leak from any of the two shares is useless since it does not give information, being individually random); we assume here that the attacker cannot separate the two leaks (which is more difficult with hardware than with smart cards, as we explained above); if he can, the designer needs to take d larger. It can be checked that the first order

[5] Other methods exist: threshold implementations and multiparty computation, see below.
[6] We denote random variables by capital letters.

attack is then no longer successful. It has been also proposed (see [974, 562]) to use Shamir's $(\ell, d+1)$ secret sharing scheme (see page 168) rather than Boolean masking (for which the information on the shared data is relatively easy to rebuild from the observed shares, and this simplifies the task of the attacker). The advantages of such masking method are studied in [340], where is shown that it may be more advantageous for the attacker (in terms of attack complexity) to observe strictly more than $d+1$ shares (while it could seem natural that observing strictly more than $d+1$ shares is inappropriate for the attacker since it provides more noise), thanks to the existence of so-called *linear exact repairing codes* (which allow reconstruction from less information than Lagrange's interpolation, thanks to polynomial interpolation formulae that optimize the amount of information which needs to be extracted), and that the choice of the public points (the α_i's at page 169) has an impact on the countermeasure strength.

Security. We see with HO-SCA that, since the complexity of mounting a successful side-channel attack increases exponentially with the order of the attack, then when applied against a masked implementation, it grows exponentially with the masking order. Hence, it is always possible, theoretically, to protect a cryptosystem against SCA by masking, but this needs practically to change in the algorithm every function $x \mapsto F(x)$ (that we shall assume to be an (n,n)-function to simplify; the general case is similar) into a function[7] $(m_0, \dots, m_d) \mapsto (m'_0, \dots, m'_d)$ such that, if $m_0, \dots, m_d$ are shares of x then $m'_0, \dots, m'_d$ are shares of $F(x)$ (we shall say that such function $(m_0, \dots, m_d) \mapsto (m'_0, \dots, m'_d)$ is the *masked version of function* F), and such that the d-th order security property is satisfied. The latter property, which is equivalent to the *probing security model* introduced in [637], states that every tuple of d or less intermediate variables is independent of the secret parameters of the algorithm[8]. When satisfied, it guarantees that no attacker able to learn at most d intermediate results (called *probes*) of a computation can succeed in an attack of order lower than or equal to d. This model, which is a simplified version of the behavior of a device in the real-world (in which physical leakages reveal some information on the whole computation), allows thanks to its simplicity to build efficient compilers transforming (at a cost which is quadratic in d) any circuit into a secure one in the probing model (see a survey in the introduction of [49]). A more realistic and more complex model was proposed in [880], and improved in [973] into the *noisy leakage model*, which was studied further and improved in [487], where the so-called *statistical distance* was introduced, allowing to show that constructions proven secure in the probing security model are also secure in the noisy leakage model, provided that the probing order is a large enough function of the noisy leakage order. A last improvement can be found in [564].

[7] We denote set or space elements by lower case letters.
[8] Note that when the algorithm handles vectors, in $\mathbb{F}_2^n$, there are different ways of interpreting the definition, according to whether it refers to intermediate variables as vectors or as individual bits; we shall then specify *bit-probing security* when needed.

An *a priori* weaker notion of d-th order resistance has been introduced in [897] to characterize the security of parallel implementations, for which higher-order probing security can never be achieved because all shares are treated within one single cycle. It is called the *bounded moment security model* and has been studied in several papers (see *e.g.* [49]). A masking scheme is secure at order d in this model if no moment of degree d in the intermediate variables depends on the secret. It is more appropriate for hardware. Indeed, the appropriate model and, hence, the kind of masking scheme to be applied, depends on the capabilities of the execution platform: embedded software devices like smartcards can execute operations sequentially, but need to rely on smaller memories (which are constrained resources); therefore, functions like S-boxes are preferentially *recomputed* [972, Sec. 2.1], while FPGAs are able to execute several operations in parallel, and can leverage on large memory blocks (called BRAM–Block Random Access Memory); in such context, masked functions can be simply tabulated, *i.e.*, computed in one clock cycle (such strategy is also referred to as *Global Look-Up Table* [972, Sec. 3.2]). Therefore, masked computations in smartcards require end-to-end security, whereas masked computations in FPGAs can resort to large tabulated functions where only data representation (*i.e.*, tables input & output) shall be secured. Note however that if the need in memory appears too important, we can change the algorithm so that, instead of working on $\mathbb{F}_{2^n}$, it works in $\mathbb{F}_{2^{n/k}}$ for some k which provides a time-memory trade-off.
Reductions between the leakage security models seen above are studied in [49] (it is proved in particular that probing security for a serial implementation implies bounded moment security for its parallel counterpart, and that simple refreshing algorithms with linear complexity that are not secure in the continuous probing model are secure in the continuous bounded moment model). When probing and bounded moment security models are considered at the bit level, then they are equivalent [577]. Note that there also exists a parameter quantifying the resistance of S-boxes to DPA, called the (modified) transparency order [343]; we shall not address it here.
Until recently, no method was known for securely composing masked (elementary) functions ensuring d-probing security with a (tight) number $d+1$ of shares. This problem has been solved in [48] thanks to the introduction of the security notions of t*-(strong) non interference (S)NI* (any set of at most t intermediate variables can be perfectly simulated with at most t shares of each input, and in the case of strong NI, at most $t-t_{out}$ shares, where t_{out} is the number of output variables among the t ones), and optimized in [55] (where is shown that some masked S-boxes may be composed without refreshing).
Security at order one against SCA is nowadays considered insufficient in both models for most practical operational environments. Detecting a single fault is also insufficient. Second-order resistance to both side-channel analysis and fault injection resistance (in a “mask then encode” procedure, which is more efficient than “encode then mask” in terms of variable size growth, but care must be

taken on the way the redundancy is applied) may be sufficient[9] (but without a security margin taking into account future improvements of SCA).

Masking functions. If a function F is linear (like diffusion layers - MixColumns and ShiftRows in AES), then we can take $m'_i = F(m_i)$ for designing its masked version. A little more generally, if a function is affine (like the round key addition - AddRoundKey in AES), it can be masked at no extra cost.
If a function F is not affine (which is the case of a substitution layer - SubBytes in AES), then we can design its masked version as follows: assuming that the input to F lives in $\mathbb{F}_{2^n}$ (which is always possible since we assume it lives in a vector space over $\mathbb{F}_2$, and $\mathbb{F}_{2^n}$ is an n-dimensional vector space over $\mathbb{F}_2$), it is a univariate polynomial function (see page 58) and its computation can be decomposed into a sequence of additions and multiplications in the field. The operations of addition, scalar multiplication and squaring being linear functions, they can be masked at no extra cost (see above). For masking multiplication, there is a method called the *ISW algorithm* (Ishai-Sahai-Wagner), which is introduced in [637] for the case of $\mathbb{F}_2$ and generalized to $\mathbb{F}_{2^n}$ in [996]:

Algorithm 3: Higher-Order Masking Scheme ISW for the Multiplication

Input : sharings $(a_0, a_1, \cdots, a_d)$ and $(b_0, b_1, \cdots, b_d)$ of a and b in $\mathbb{F}_{2^n}$
Output: a sharing $(c_0, c_1, \cdots, c_d)$ of $c = a \times b$

Randomly generate $d(d+1)/2$ elements $r_{ij} \in \mathbb{F}_{2^n}$ indexed such that $0 \leqslant i < j \leqslant d$
for $i = 0$ **to** d **do**
    for $j = i + 1$ **to** d **do**
        $r_{j,i} \leftarrow (r_{i,j} + a_i \times b_j) + a_j \times b_i$
    end
end
for $i = 0$ **to** d **do**
    $c_i \leftarrow a_i \times b_i$
    for $j = 0$ **to** d, $j \neq i$ **do**
        $c_i \leftarrow c_i + r_{i,j}$
    end
end
return $(c_0, c_1, \ldots, c_d)$

The time complexity and the amount of random data which needs to be generated for the ISW algorithm are both quadratic in d (see more in [320]).
Other methods[10] exist like in [974] (they are surveyed in [54], see also [563]), that we do not detail since they do not pose, so far, new questions on Boolean and vectorial functions.
We see that the designer of the block cipher implementation has some advantage over the attacker, because increasing d raises exponentially the complexity of the attack and only quadratically the complexity of the countermeasure. However, countermeasures are costly in terms of running time and program executable file

[9] Some palliative countermeasures may then be needed, consisting for instance in desynchronization, random interrupts or dummy operations; such palliative countermeasures are not sufficient on their own.
[10] One (the threshold implementation) will be seen in Subsection 12.1.4, page 472; its complexity is higher while it addresses a more difficult situation than ISW algorithm.

size (in software applications) or of implementation area (in hardware applications). For example, in software with 8-bit AVR architecture, an AES without masking runs in few hundreds of cycles or few thousands, while with masking it needs already about 40000 cycles for first order; moreover, the program executable file size is also increased because of the need of masking S-boxes (see https://github.com/ANSSI-FR/secAES-ATmega8515/). In hardware, the implementation area is roughly tripled. The cost overhead may be too high for real-world products, all the more when the order of probing security is larger than 1. But the implementation (including masking) must be efficient today while the SCA can be performed in the future.
We need then to minimize the implementation and memory complexities of the countermeasures. This is where Boolean and vectorial functions can play a role.

12.1.1 A new role of correlation immunity and of the dual distance of codes related to side channel attack countermeasures

Correlation immune Boolean functions (see Definition 21, page 105), allow reducing in two possible ways the cost overhead due to masking, while keeping the same resistance to d-th order SCA in the bounded moment security model (and possibly the same order of probing security, but this depends on the implementation), when the leak is a linear combination over the reals of the bits of the sensitive variable, added with a Gaussian noise (this assumption on the leak is rather realistic and the assumption on the noise is almost always the case in literature):

- by applying a method called *leakage squeezing*, which allows achieving with one single mask the same protection of registers against higher-order SCA as with d ones, where d is an integer strictly larger than 1 that we shall define. This method, which allows making optimal the representation of the shares and maximizes the resistance order against high-order side-channel attacks, has been introduced in [811] and further studied in [810, 263] (an extremely close counter-measure has been introduced independently in [132]). It uses a bijective vectorial function F which is applied to modify the mask (this is a reason why the method is better adapted to hardware since we need to apply F^{-1} at some point, and in hardware this can be made more easily as we explained, but millions of smart cards built by industry nowadays include leakage squeezing as well). The pair (M_0, M_1) such that $M_0 + M_1 = Z$ is not processed as is in the device, but in the form of $(M_0, F(M_1))$. The condition for achieving resistance to d-th order SCA in the bounded moment security model is proved in [811, 810, 263]: assuming that the leakage model is a pseudo-Boolean function of numerical degree (see Definition 13, page 66) at most d (which is the case of the d-th power of a degree 1 leakage), it is that the graph *indicator* of F, that is, the $2n$-variable Boolean function whose support equals the graph

$\{(x, y);\ y = F(x)\}$ of F, is d-th order correlation immune. Such graph is a *complementary information set* code (*CIS code* for short), in the sense that it admits (at least) two *information sets* (see page 344) which are complementary of each other, see [280]. The condition that the indicator of this CIS code is d-th order correlation immune is equivalent to saying that the *dual distance* of this code is at least $d+1$ (according to Corollary 6, page 108), which is coherent with what was observed by Massey [827] already in 1993. For instance, a rate $\frac{1}{2}$ $[16, 8, 5]$ linear code can be used. But it is shown in [810] that there exists a non-linear code which achieves better: it is the Nordstrom-Robinson code of parameters $(16, 256, 6)$. A comprehensive study of CIS codes has been made in [280] and it is shown in [264] that the mutual information between the sensitive data and the leakage vanishes exponentially with the noise variance, at a rate which is proportional to the dual distance.
The method of leakage squeezing has been later generalized in [262] to several masks. Compared to first-order leakage squeezing, second-order leakage squeezing is more efficient, since it increments by one unit at least the resistance against high-order attacks, with an appropriate (*a priori* different) code. In fact, it improves it more, since better improvements have been realized by relevant choices of squeezing bijections. But the optimal solutions are more difficult to find than in the case with one mask. When the masking is applied on bytes (as in AES), optimal leakage squeezing with one mask resists HO-SCA of orders up to 5 (with the Nordstrom-Robinson code), and with two masks, resistance against HO-SCA of order 7 is provided. The study of the corresponding higher-order CIS codes has been made in [277]. A rate $\frac{1}{3}$ [24,8,8] linear code (maximal minimal distance) with three disjoint information sets fulfills the conditions.

- an alternative way of resisting higher-order SCA with one single mask consists in avoiding processing the mask at all: for every sensitive variable Z which is the input to some box S in the block cipher, Z is replaced by $Z + M$ where M is drawn at random, and $Z + M$ is the input to a "masked" box S_M whose output is a masked version of $S(Z)$ (and the process of masking continues similarly during the whole implementation, only the very last step being eventually unmasked to give the result). This method is called *rotating S-box masking* (RSM) [898]. It needs, for each box S in the cipher, to implement a look-up table for each masked box S_M. This is particularly well adapted to hardware: all S-boxes are then addressed in parallel, for a better throughput; the attacker is not able to know which S-box is addressed for a given value of M; he/she is only able to identify that the S-boxes have been looked up, but the order in which they are queried is indistinguishable from his/her standpoint; he/she is limited to collecting an aggregated function of all S-boxes. This being said, many smart cards implement RSM nowadays as well, still more than leakage squeezing.
To reduce the cost, M is not drawn at random in the whole set of binary

vectors of the same length as Z, but in a smaller set of such vectors, say E. The condition for achieving resistance to d-th order SCA in the bounded moment security model is that the indicator function 1_E is a d-th order correlation immune function, *i.e.* that E viewed as a code has dual distance at least $d+1$ [74, 286]. This is because, for any $j \leq d$, the mean of $w_H^j(Z+M)$ when Z has some fixed value z equals $\frac{1}{|E|}\sum_{m\in E} w_H^j(z+m) = \frac{1}{|E|}\sum_{m\in\mathbb{F}_2^n} 1_E(m)\, w_H^j(z+m) = \frac{1}{|E|}1_E \otimes w_H^j(z) = \frac{1}{2^n|E|}\,\widehat{\widehat{1_E}\times\widehat{w_H^j}}(z)$, according to Relation (2.45), page 79, and this mean is independent of z if and only if $\widehat{1_E}\times\widehat{w_H^j}(a) = 0$ for every $a \neq 0_n$, while we know that w_H^j, which has numerical degree j, satisfies that $\widehat{w_H^j}(a) = 0$ if and only if $w_H(a) > j$. Given d, we wish to choose this d-th order correlation immune function 1_E with lowest possible (nonzero) weight, since the size of the overhead due to the masked look-up tables is proportional to the size of the set[11].

In [289] is shown that the security notion (at bit level, *i.e.*, in $\mathbb{F}_2$) corresponds in these two cases to d-probing and d-th order bounded moment security models.

Leakage squeezing and RSM needing correlation immune functions of low weights (with a particular shape in the case of leakage squeezing since the function must then be the graph indicator of a permutation, see more in [244]), this has posed a new problematic on Boolean functions, that we begun to address at pages 332 and following (further work is needed). Most of the numerous studies made (mostly in the nineties) on correlation immune functions in the framework of stream ciphers (see page 105) dealt with *resilient* (balanced) functions and do not apply to low weight correlation immune functions.

12.1.2 Vectorial functions in univariate form: minimizing the number of nonlinear multiplications for reducing the cost of countermeasures

In [297] are studied properties that an S-box could possess for being more resilient against side-channel attacks, like the (almost) preservation of Hamming weight and a small Hamming distance between input and output; the incidences on the nonlinearity and differential uniformity are determined.

Additional protections, like masking, are in any case unavoidable. We have seen that the complexity of masking additions and linear multiplications (like, for instance, $x \times x$) is negligible compared to that of masking nonlinear multiplications. To efficiently mask an algorithm, we need to minimize the *masking complexity* of each S-box, that is, the minimum number of nonlinear multiplications needed to implement it. This parameter is affine invariant.

[11] Note however that if the cipher is made like the AES, with identical substitution boxes up to affine equivalence, the substitution layer can be slightly modified so as to be masked at no extra cost: the affine equivalent boxes are replaced by masked versions of a same box; namely, the 16 byte masks which can be applied to the 16 boxes are the codewords of the $[8,4,4]$ self-dual code.

When the S-box is a power function $F(x) = x^d$ like in the AES, minimizing the number of nonlinear multiplications results in a variant of the classical problem of minimizing addition chains in a group, see [284]; determining the masking complexity amounts to finding the addition chain for d with the least number of additions which are not doublings. For instance, the *inverse function* $x \to x^{254} = x^{-1}$ in $\mathbb{F}_{2^8}$ can be implemented with 4 nonlinear multiplications, in many ways (we saw one at page 434, note that the well-known square-and-multiply algorithm for computing the inverse needs more than 4 multiplications).
When the S-box is a general polynomial, minimizing the number of nonlinear multiplications is a new paradigm. It is proved in [382] that, for every positive integer n, there exists a polynomial $P(x) \in \mathbb{F}_{2^n}[x]$ with masking complexity:

$$\mathcal{MC}(P) \geq \sqrt{\frac{2^n}{n}} - 2. \tag{12.1}$$

There exist several methods for trying to minimize the multiplicative complexity[12], $\mathcal{MC}(P)$ of polynomials P and allowing their probing secure evaluation at minimized cost. We refer to [284, 320] for more details. The two first methods are provable and the two last are heuristic[13] (and more efficient in practice):

- The *cyclotomic method* consists in rewriting $P(x)$ in the form:

  $$P(x) = u_0 + \sum_{i=1}^{q} L_i(x^{\alpha_i}) + u_{2^n-1}x^{2^n-1} \ ,$$

  where q is a positive integer and $(L_i)_{i\leqslant q}$ is a family of linear functions. Since the transformations $x \in \mathbb{F}_{2^n} \mapsto x^{2^j}$ are $\mathbb{F}_2$-linear, their masking complexity is null. This implies that the masking complexity of $\sum_{i=1}^{q} L_i(x^{\alpha_i})$ is bounded above by the number of non-linear multiplications required to evaluate all the monomials x^{α_i}, that is, by $\sum_{\delta|(2^n-1)} \frac{\varphi(\delta)}{\mu(\delta)} - 1$, where $\mu(m)$ denotes the multiplicative order of 2 modulo m and φ the Euler's totient function.
- The *Knuth-Eve method* is based on a recursive use of the observation that any polynomial $P(x)$ of degree t over $\mathbb{F}_{2^n}[x]$ can be written in the form

  $$P(x) = P_1(x^2) \oplus P_2(x^2)x$$

  where $P_1(x)$ and $P_2(x)$ have degrees bounded above by $\lfloor t/2 \rfloor$. This implies that the masking complexity of $P(x)$ is at most:

  $$\begin{cases} 3 \cdot 2^{(n/2)-1} - 2 & \text{if } n \text{ is even,} \\ 2^{(n+1)/2} - 2 & \text{if } n \text{ is odd.} \end{cases}$$

[12] This term is meant here at the $\mathbb{F}_{2^n}$ field level; it can be also considered at the bit level, in relation with bitsliced implementations, see *e.g.* [565] and the references therein.
[13] In the sense of "not proved".

- The *Coron-Roy-Vivek (CRV) method* [382] starts with a union $\mathcal{C}$ of cyclotomic classes $\mathcal{C}_i$ in $\mathbb{Z}/(2^n-1)\mathbb{Z}$, such that all power functions x^j, $j \in \mathcal{C}$, can be processed with a global small enough number of nonlinear multiplications. This set of monomials x^j spans a subspace $\mathcal{P}$ of $\mathbb{F}_{2^n}[x]$. A polynomial $R \in \mathbb{F}_{2^n}[x_1, \cdots, x_t]$ is searched such that:

$$P(x) = R\left(P_1(x), \cdots, P_t(x)\right),$$

where the P_i's are taken in $\mathcal{P}$. Denoting by μ the number of non-linear multiplications required to build $\mathcal{C}$, the search tries to minimize $\mathcal{MC}(R)+\mu$. A heuristic approach (in order to speed up the process) is proposed:

1. Build the union set $\mathcal{C}$ such that all the powers of P's monomials are in $\mathcal{C}+\mathcal{C}$,
2. Choose and fix a set of r polynomials $P_1(x)$, ..., $P_r(x)$ in $\mathcal{P}$ and search for $r+1$ polynomials $P_{r+1}(x)$, ..., $P_{2r+1}(x)$ in $\mathcal{P}$ such that:

$$P(x) = \sum_{i=1}^{r} P_i(x) \times P_{r+i}(x) + P_{2r+1}(x) \ . \tag{12.2}$$

Thanks to the fact that $P_1(x)$, ..., $P_r(x)$ have been fixed, this results in solving a linear system of $n2^n$ Boolean equations in at most $\min(r, |\mathcal{C}|) \times |\mathcal{C}| + |\mathcal{C}|$ unknowns. The condition $2^n \leqslant |\mathcal{C}| \times (1+\min(r,|\mathcal{C}|))$ ensures then that the method outputs at least one solution. The complexity of the resulting probing secure method is $\mathcal{O}(\sqrt{2^n/n})$, which is asymptotically better than the complexity of Knuth-Eve's method. Moreover, a comparison of Coron's complexity with Inequality (12.1) shows that it is asymptotically optimal.

- The *CPRR method* [321] is more recent and based on another algebraic decomposition heuristic principle. It decomposes $P(x)$ by means of functions of low algebraic degree, and designs efficient probing-secure evaluation methods for such low-degree functions. The decomposition step starts by deriving a family of generators: $\begin{cases} G_1(x) = F_1(x) \\ G_i(x) = F_i\big(G_{i-1}(x)\big) \end{cases}$ where the F_i are random polynomials of algebraic degree s. Then it randomly generates t polynomials $Q_i = \sum_{j=1}^{r} L_j \circ G_j$, where the L_j are linearized polynomials. Eventually, it searches for t polynomials P_i of algebraic degree s and for $r+1$ linearized polynomials L_i such that:

$$P(x) = \sum_{i=1}^{t} P_i\big(Q_i(x)\big) + \sum_{i=1}^{r} L_i\big(G_i(x)\big) + L_0(x) \ .$$

As in the CRV method, the search for polynomials P_i and L_i amounts to solving a system of linear equations over $\mathbb{F}_{2^n}$.

For masking a function F of algebraic degree at most s, the method uses that, for every function from $\mathbb{F}_{2^n}$ to itself of algebraic degree at most s, the mapping

$$\beta_F^{(s)}(a_1, a_2, \ldots, a_s) = \sum_{I \subseteq \{1,\ldots,s\}} F\Big(\sum_{i\in I} a_i\Big)$$

is multilinear (which is easily seen and has been first observed in [209]), which allows proving that, for every $d \geq s$:

$$F\Big(\sum_{i=1}^{d} a_i\Big) =$$

$$\sum_{1\leq i_1<\cdots<i_s\leq d} \beta_F^{(s)}(a_{i_1}, \ldots, a_{i_s}) + \sum_{j=0}^{s-1} \eta_{d,s}(j) \sum_{\substack{I\subseteq\{1,\ldots d\}\\ |I|=j}} F\Big(\sum_{i\in I} a_i\Big),$$

where $\eta_{d,s}(j) = \binom{d-j-1}{s-j-1} \bmod 2$ for every $j \leq s-1$, and deducing that:

$$F\Big(\sum_{i=1}^{d} a_i\Big) = \sum_{j=0}^{s} \mu_{d,s}(j) \sum_{\substack{I\subseteq\{1,\ldots,d\}\\ |I|=j}} F\Big(\sum_{i\in I} a_i\Big),$$

where $\mu_{d,s}(j) = \binom{d-j-1}{s-j} \bmod 2$ for every $j \leq s$. This reduces the complexity of the d-masking of a degree s function to several s-maskings. An alternative (tree-based) method is also proposed. It is shown that the processing of any S-box of dimension $n = 8$ can be split into 11 evaluations of quadratic functions, or into 4 evaluations of cubic functions.

12.1.3 Vectorial functions and algebraic side channel attacks

In [990] is introduced an attack on block ciphers called algebraic side channel attack, which combines the two approaches of algebraic attacks and side channel attacks. In [272] is studied the algebraic phase of this attack. The notion of algebraic immunity is modified to include the information from the leakage on Hamming weight or on Hamming distance, and it is studied how this can allow obtaining enough equations of degree one to be able to solve the algebraic system with Gröbner methods. We refer to these two papers for the technical details.

12.1.4 Vectorial functions and threshold implementation

The countermeasures against SCA presented so far suppose, for having good efficiency, that the leakage has some regularity. Building hardware with such property is expensive in practice. In particular, hardware *glitches*, which are transient faults (coming when the input signals of the combinational logic arrive at different moments in time when they should come simultaneously; signal switches then several times when it should switch once) common in CMOS technology, change the leaking into functions $\mathcal{L}$ having numerical degree larger than one, because of the interactions between bits that they cause, and which moreover vary with time. Glitch-free hardware is very expensive. We present here the main known solutions for avoiding needing it.

1. The problem of building implementations secure against d-th order side channel attacks in the presence of glitches is equivalent to the problem of securing the processing of a function with several semi-honest players (see [974]). A related way of masking S-boxes is the so-called *polynomial masking*, introduced separately in [974] and [562], and which gives a solution to this problem without needing sophisticated hardware. The idea is to make the global circuit glitch-free-like by the implementation itself, splitting the circuit implementing the S-box into several sub-circuits communicating with each other on the basis of a *multiparty computation* protocol (see page 169), like the one in [59]. The masking operation of a sensitive data $z \in \mathbb{F}_{2^n}$ is based on Shamir's secret sharing seen in Subsection 3.6.1, page 168. It consists in constructing a function $f_z(x) = \sum_{i=1}^{m-1} a_i x^i + z$, where $(a_i)_{1\leq i\leq m-1}$ are some random secret coefficients, then as in Boolean masking, z can be represented by m shares $(z_0, \ldots, z_{m-1})$, with $z_i = (\alpha_i, f_z(\alpha_i))$ for $0 \leq i \leq m-1$ for some random inputs $(\alpha_i)_{0\leq i\leq m-1}$. To get z (unmasked), we have to reconstruct f_z by polynomial interpolation[14], and finally calculate $z = f_z(0)$. The advantage of this method is that it is grounded by a well studied theory (multi-party computation) and the security models are clear. Its disadvantage is that it is not very efficient, especially when first-order SCA is considered.

2. Another S-box masking method, also based on ideas of multiparty computation and aiming at solving the problem posed by glitches, is *threshold implementation*[15] (*TI*). Threshold schemes are attractive from an academic viewpoint, because they come with an information-theoretic proof of resistance against first-order DPA while allowing realistic-size circuits[16]. Introduced in [903] and presented more completely in [904], they pose interesting challenges on vectorial functions. In the TI of an (n, m)-*function* F, the shares of the output of F are the outputs of several functions of the shares of the input, each such function being independent of at least one of the shares of the input to F (a different one for each function); these two properties, called correctness and incompleteness, provide first order probing security (if the implementation is done properly). More precisely:

- The masked version with t masks (*i.e.* with $t+1$ shares) of each input variable x_i will be denoted by $\mathbf{x_i} = (x_i^{(1)}, \ldots, x_i^{(t+1)}) \in \mathbb{F}_2^{t+1}$. We shall denote the sum $x_i^{(1)} \oplus \cdots \oplus x_i^{(t+1)}$ of the coordinates of $\mathbf{x_i}$ by $s(\mathbf{x_i})$; we have $s(\mathbf{x_i}) = x_i$ for every i. Extending s to a function over $(\mathbb{F}_2^n)^{t+1}$, we have then $s(\mathbf{x}) = x$, for every $x = (x_1, \ldots, x_n)$.
- A t-mask (*i.e.* $(t+1)$-share) *realization* of an (n, m)-function F is a vector

[14] Better methods have been very recently found, see [340].
[15] Not to be confused with threshold functions, seen in Subsection 10.1.7.
[16] Which can still be attacked by (univariate) mutual information and higher-order analyses.

$\mathbf{F} = (F_1, \dots, F_{t+1})$ of $((t+1)n, m)$-functions, *i.e.* a function from $(\mathbb{F}_2^n)^{t+1}$ to $(\mathbb{F}_2^m)^{t+1}$, such that, for all $\mathbf{x} \in (\mathbb{F}_2^n)^{t+1}$, denoting also $\sum_{j=1}^{t+1} F_j(\mathbf{x})$ by $s(\mathbf{F}(\mathbf{x}))$, we have:

$$\text{if } x = s(\mathbf{x}), \text{ then } F(x) = s(\mathbf{F}(\mathbf{x})).$$

This property is called *correctness*. In practice, the numbers of input and output shares may be different but we take them equal to simplify the presentation. To obtain d-th order security against univariate attacks in a so-called HO-TI, we would need to have td masks and $\binom{td+1}{t}$ shares, and in a so-called consolidated masking scheme, d masks and $(d+1)^t$ output shares later synchronized in a register and compressed back to $d+1$ shares (which is often compared to the ISW multiplication), but with an extra register to protect against glitches; we shall not develop this and refer the reader to [85] and [992].

In terms of function graphs, let $\mathcal{G}_F = \{(x, F(x)); x \in \mathbb{F}_2^n\}$ and $\mathcal{G}_{\mathbf{F}} = \{(\mathbf{x}, \mathbf{F}(\mathbf{x})); \mathbf{x} \in (\mathbb{F}_2^n)^{t+1}\}$ be the graphs of functions F and $\mathbf{F}$; correctness corresponds to the fact that the linear function:

$$(\mathbf{x}, \mathbf{y}) \mapsto (s(\mathbf{x}), s(\mathbf{y}))$$

maps $\mathcal{G}_{\mathbf{F}}$ to $\mathcal{G}_F$.

Correctness can be characterized by the Walsh transform in the following way:

Proposition 187 *Given a $((t+1)n, (t+1)m)$-function:*

$$\mathbf{F} = (F_1, \dots, F_{t+1}) : \mathbf{x} \in (\mathbb{F}_2^n)^{t+1} \mapsto \mathbf{F}(\mathbf{x}) \in (\mathbb{F}_2^m)^{t+1}$$

and an (n, m)-function $F : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$, we have:

$$s(\mathbf{F}(\mathbf{x})) = F(s(\mathbf{x}))$$

if and only if:

$$\forall u^{(1)}, \dots, u^{(t+1)} \in \mathbb{F}_2^n, \forall v \in \mathbb{F}_2^m,\ W_{\mathbf{F}}((u^{(1)}, \dots, u^{(t+1)}), (v, \dots, v)) =$$

$$\begin{cases} 2^{tn}\, W_F(u^{(1)}, v) \text{ if } u^{(1)} = \dots = u^{(t+1)} \\ 0 \text{ otherwise.} \end{cases}$$

Proof. We have $(F_1 + \dots + F_{t+1})(x^{(1)}, \dots, x^{(t+1)}) = F(x^{(1)} + \dots + x^{(t+1)})$ if and only if these two functions have the same Walsh transform, that is, if for every $u^{(1)}, \dots, u^{(t+1)} \in \mathbb{F}_2^n$ and $v \in \mathbb{F}_2^m$, $W_{F_1+\dots+F_{t+1}}((u^{(1)}, \dots, u^{(t+1)}), v)$ equals the value at $((u^{(1)}, \dots, u^{(t+1)}), v)$ of the Walsh transform of function $(x^{(1)}, \dots, x^{(t+1)}) \mapsto F(x^{(1)} + \dots + x^{(t+1)})$, that is,

$$\sum_{(x^{(1)}, \dots, x^{(t+1)}) \in (\mathbb{F}_2^n)^{t+1}} (-1)^{v \cdot F(x^{(1)} + \dots + x^{(t+1)}) + u^{(1)} \cdot x^{(1)} \oplus \dots \oplus u^{(t+1)} \cdot x^{(t+1)}},$$

which, by changing $x^{(1)}$ into $x^{(1)} + \cdots + x^{(t+1)}$, equals:

$$\sum_{(x^{(1)},\dots,x^{(t+1)})\in(\mathbb{F}_2^n)^{t+1}} (-1)^{v\cdot F(x^{(1)})\oplus u^{(1)}\cdot x^{(1)}\oplus(u^{(1)}+u^{(2)})\cdot x^{(2)}\oplus\cdots\oplus(u^{(1)}\oplus u^{(t+1)})\cdot x^{(t+1)}}.$$

The rest is straightforward. □

A necessary and sufficient condition for a function G to be the realization of some function is then as follows:

Corollary 31 *Given $G = (G_1, \dots, G_{t+1}) : (\mathbb{F}_2^n)^{t+1} \mapsto (\mathbb{F}_2^m)^{t+1}$, the function $(G_1+\cdots+G_{t+1})(x^{(1)}, \dots, x^{(t+1)})$ depends only on $x^{(1)}+\cdots+x^{(t+1)}$ if and only if, for every $u^{(1)}, \dots, u^{(t+1)} \in \mathbb{F}_2^n, v \in \mathbb{F}_2^m$, $W_G((u^{(1)}, \dots, u^{(t+1)}), (v, \dots, v))$ equals zero when the equalities $u^{(1)} = \cdots = u^{(t+1)}$ are not all satisfied. Then, we have that $W_G((u^{(1)}, \dots, u^{(t+1)}), (v, \dots, v))$ is divisible by 2^{tn} for every $u^{(1)}, \dots, u^{(t+1)}$.*

Proof. The condition that $W_G((u^{(1)}, \dots, u^{(t+1)}), (v, \dots, v))$ equals zero when the equalities $u^{(1)} = \cdots = u^{(t+1)}$ are not all satisfied is necessary, according to Proposition 187. It is also sufficient since we have then, according to the *inverse Walsh transform formula* 2.43, page 78:

$$2^{(t+1)n}(-1)^{v\cdot(G_1+\cdots+G_{t+1})(x^{(1)},\dots,x^{(t+1)})} =$$
$$\sum_{(u^{(1)},\dots,u^{(t+1)})\in(\mathbb{F}_2^n)^{t+1}} (-1)^{u^{(1)}\cdot x^{(1)}\oplus\cdots\oplus u^{(t+1)}\cdot x^{(t+1)}} W_G((u^{(1)}, \dots, u^{(t+1)}), (v, \dots, v))$$
$$= \sum_{u\in\mathbb{F}_2^n} (-1)^{u\cdot(x^{(1)}+\cdots+x^{(t+1)})} W_G((u, \dots, u), (v, \dots, v)) \tag{12.3}$$

and we have that, for every v, function $v \cdot (G_1 + \cdots + G_{t+1})(x^{(1)}, \dots, x^{(t+1)})$ depends then only on $x^{(1)} + \cdots + x^{(t+1)}$. It is easily seen that $(G_1 + \cdots + G_{t+1})(x^{(1)}, \dots, x^{(t+1)})$ depends then only on $x^{(1)} + \cdots + x^{(t+1)}$.
Using now Relation (12.3) with $x^{(2)} = \cdots = x^{(t+1)} = 0_n$ and with x instead of $x^{(1)}$ and applying the inverse Walsh transform formula to the resulting function of x, we have $2^{tn} \sum_{x\in\mathbb{F}_2^n} (-1)^{v\cdot(G_1+\cdots+G_{t+1})(x,0_n,\dots,0_n)\oplus u\cdot x} = W_G((u, \dots, u), (v, \dots, v))$ and this proves the divisibility property. □

Correctness is a constraint on the realization $\mathbf{F}$, not really on F itself. But in threshold implementation, a second property is required for $\mathbf{F}$, and a third one is desired too; both put also constraints on F:

- In a threshold implementation $\mathbf{F} = (F_1, \dots, F_{t+1})$, every function F_j should be independent of the j-th coordinate of each $\mathbf{x_i}$ (in the sense that this j-th coordinate should not appear at all in the ANF of F_j), i.e. F_j should be independent of the j-th share of $\mathbf{x}$. This property is called *non-completeness*[17] and implies that the output of F_j, individually, is uncorrelated to any input

[17] In (univariate) HO-TI of order d, this property becomes: any combination of up to d *component functions* of $\mathbf{F}$ must be independent of at least one input share.

variable x_i (assuming, as in the subsections above, that any vector of less than $t+1$ shares of x_i is uncorrelated to x_i). Note that if F has algebraic degree at most t, then it is easy to build such $\mathbf{F}$: starting from the ANF of $F(x)$, replacing each x_i by $x_i^{(1)} \oplus \cdots \oplus x_i^{(t+1)}$ and expanding, we obtain a sum of monomials in each of which at least one upper index is not appearing, and then, starting with $j = 1$ and incrementing j at each step, we store in F_j all those monomials involving variables whose upper indices are different from j and which have not yet been stored. This way guarantees both correctness and non-completeness.
For instance, applying this method to the Boolean function $f(x) = x_1x_2$ gives:

$$\begin{array}{l}
f_1((x_1^{(1)}, x_1^{(2)}, x_1^{(3)}), (x_2^{(1)}, x_2^{(2)}, x_2^{(3)})) = x_1^{(2)}x_2^{(2)} \oplus x_1^{(2)}x_2^{(3)} \oplus x_1^{(3)}x_2^{(2)}\\
f_2((x_1^{(1)}, x_1^{(2)}, x_1^{(3)}), (x_2^{(1)}, x_2^{(2)}, x_2^{(3)})) = x_1^{(3)}x_2^{(3)} \oplus x_1^{(1)}x_2^{(3)} \oplus x_1^{(3)}x_2^{(1)}\\
f_3((x_1^{(1)}, x_1^{(2)}, x_1^{(3)}), (x_2^{(1)}, x_2^{(2)}, x_2^{(3)})) = x_1^{(1)}x_2^{(1)} \oplus x_1^{(1)}x_2^{(2)} \oplus x_1^{(2)}x_2^{(1)}.
\end{array}$$

Non-completeness can be characterized by the Walsh transform in the following way:

Proposition 188 *Given a $((t+1)n, (t+1)m)$-function:*

$$\mathbf{F} = (F_1, \dots, F_{t+1}) : \mathbf{x} \in (\mathbb{F}_2^n)^{t+1} \mapsto \mathbf{F}(\mathbf{x}) \in (\mathbb{F}_2^m)^{t+1},$$

each function F_j is independent of the j-th coordinate of each $\mathbf{x_i}$ if and only if:

$$\forall j \in \{1, \dots, t+1\}, \forall u^{(1)}, \dots, u^{(t+1)} \in \mathbb{F}_2^n, \forall v^{(1)}, \dots, v^{(t+1)} \in \mathbb{F}_2^m,$$

$$\left(\begin{array}{c} v^{(k)} = 0_m, \forall k \neq j, \\ \text{and } u^{(j)} \neq 0_n \end{array} \right) \Rightarrow (W_{\mathbf{F}}((u^{(1)}, \dots, u^{(t+1)}), (v^{(1)}, \dots, v^{(t+1)})) = 0). \tag{12.4}$$

Indeed, according to the Walsh and inverse Walsh transform formulae, F_j is independent of $(x_1^{(j)}, \dots, x_n^{(j)})$ if and only if, for every $(u^{(1)}, \dots, u^{(t+1)}) \in (\mathbb{F}_2^n)^{t+1}$ such that $u^{(j)} \neq 0_n$, we have $W_{F_j}(u^{(1)}, \dots, u^{(t+1)}) = 0$.

Definition 86 *We call t-mask (*i.e. *$(t+1)$-share) TI, or t-th order TI, of an (n,m)-function any function $\mathbf{F}$ from $(\mathbb{F}_2^n)^{t+1}$ to $(\mathbb{F}_2^m)^{t+1}$ satisfying correctness and non-completeness.*

We shall call F the TI-masked function.

- The following property, called *uniformity (of TI)*, is desired too but is often not included in the very definition of TI: for every $\mathbf{b} = (b^{(1)}, b^{(2)}, \dots, b^{(t+1)})$ in $(\mathbb{F}_2^m)^{t+1}$, the number of $\mathbf{x}$ in $(\mathbb{F}_2^n)^{t+1}$ for which $\mathbf{F}(\mathbf{x}) = \mathbf{b}$ is equal to $2^{t(n-m)}$ times the number of x in $\mathbb{F}_2^n$ for which $F(x) = s(\mathbf{b})$ (if F is a permutation of $\mathbb{F}_2^n$ then this is equivalent[18] to saying that $\mathbf{F}$ is a permutation

[18] If the output of F was shared in more shares than the input, it would be equivalent to saying that $\mathbf{F}$ is balanced.

of $(\mathbb{F}_2^n)^{t+1}$). This property is needed to make sure that, if the masking of the input to $\mathbf{F}$ is uniform, then the output of $\mathbf{F}$ is also a uniform masking of the output of F. The uniformity property of a TI is then important when the output of the TI is the input to another block (which is always the case in an iterative cryptographic primitive such as a block cipher). If we use a non-uniform TI of a function, we need to add sufficient refreshing (this is how HO-TI can be multivariate secure). Such possibility is used quite often in practice but is expensive.

Uniformity is the hard property among the three described above. The usual method for trying to achieve it is by adding so-called correction terms to the output of TI when they do not ensure uniformity; these are terms which are added in pairs to shares in a way preserving non-completeness. The terms in a pair canceling each other when the sum of output shares is made, correctness is preserved as well. But this method is difficult and has to be applied on each S-box; it is not really doable for infinite classes. How many shares are needed for a uniform TI without extra randomness is also currently a question without formal answer. The answer for $(3,3)$-functions and $(4,4)$-functions was given in [86, 87] by exhaustive search.

In [904, Theorem 1 and Corollary 1], the authors observe that correctness, non completeness and the fact that the sharing at input is uniform suffices to ensure that each of the output shares is statistically independent of the input variables and the output variables and that the same holds for all intermediate results. Hence, if the power consumption of each shared sub-circuit implementing one of the functions F_j is independent of the other sub-circuits, the implementation resists first order SCA, even in the presence of glitches. Uniformity ensures additionally that no more information than a possible bias in the output distribution of the TI-masked function is provided. This uses more random values during the setup and this is a disadvantage already acknowledged in [903], but it does not need fresh randomness during the process. We shall specify "TI with uniformity" when needed (that is, when the TI achieves uniformity without additional fresh randomness).

Uniformity can be characterized by means of the Walsh transforms of F and $\mathbf{F}$ as well: the condition is equivalent to $\sum_{\mathbf{x}\in(\mathbb{F}_2^n)^{t+1},\mathbf{v}\in(\mathbb{F}_2^m)^{t+1}}(-1)^{\mathbf{v}\cdot(\mathbf{F}(\mathbf{x})+\mathbf{b})} = 2^{tn}\sum_{x\in\mathbb{F}_2^n,v\in\mathbb{F}_2^m}(-1)^{v\cdot(F(x)+b)}$, for every $\mathbf{b}$, where $\mathbf{v}=(v^{(1)},\dots,v^{(t+1)})$ and $b=s(\mathbf{b})$, that is: $\forall\,\mathbf{b}\in(\mathbb{F}_2^m)^{t+1}$,

$$\sum_{\mathbf{v}\in(\mathbb{F}_2^m)^{t+1}}(-1)^{\mathbf{v}\cdot\mathbf{b}}\,W_{\mathbf{F}}(0_{n(t+1)},\mathbf{v}) = 2^{tn}\sum_{v\in\mathbb{F}_2^m}(-1)^{v\cdot s(\mathbf{b})}W_F(0_n,v).$$

Algebraic degree of functions admitting a t-mask TI

A drawback of threshold implementation is that functions F of algebraic degree t can have TI with at least t masks only[19] (a necessary and sufficient condition for the existence of a t-mask TI with or without uniformity is then that the algebraic degree be at most t). This has been first observed in [903, Theorem 1] (with incomplete statement and proof).

Proposition 189 *Let F be any (n,m)-function admitting a t-mask (*i.e.* a $(t+1)$-share) TI with or without uniformity. Then $d_{alg}(F) \leq t$.*

Proof. Consider the ANF of F:

$$F(x) = \sum_{I \subseteq \{1,\ldots,n\}} a_I x^I, \quad a_I \in \mathbb{F}_2^n,$$

where $x^I = \prod_{i \in I} x_i$. Let $\mathbf{F}$ be a t-mask TI of F. Because of correctness, the (unique) ANF of the $((t+1)n, m)$-function $(s(\mathbf{F}))(\mathbf{x_1}, \ldots, \mathbf{x_n})$ is obtained by expanding:

$$F(s(\mathbf{x_1}), \ldots, s(\mathbf{x_n})) = \sum_{I \subseteq \{1,\ldots,n\}} a_I \prod_{i \in I} (x_i^{(1)} \oplus \cdots \oplus x_i^{(t+1)}). \tag{12.5}$$

Suppose that $d_{alg}(F) \geq t+1$ and consider a monomial x^I of degree $d_{alg}(F)$. Then the ANF of $F(s(\mathbf{x_1}), \ldots, s(\mathbf{x_n}))$ contains all the monomials of the form $\prod_{i \in I} x_i^{(j_i)}$ where $j_i \in \{1, \ldots, t+1\}$, with nonzero coefficients. Indeed, two distinct monomials x^I and $x^{I'}$ of degree $d_{alg}(F)$ in the ANF of F provide disjoint sets of monomials in the expansion of (12.5), which cannot then cancel each others. Moreover, none of the monomials of the form $\prod_{i=1}^k x_i^{(j_i)}$ where $i \mapsto j_i$ is onto $\{1, \ldots, t+1\}$ can be obtained from $(s(\mathbf{F}))(\mathbf{x_1}, \ldots, \mathbf{x_n})$ because of non-completeness. A contradiction. $\square$

Hence for instance, the inverse function $F(x) = x^{2^n-2}$ used in the AES which has algebraic degree $n-1$ cannot have an $(n-1)$-share (with $(n-2)$-masks) TI. A question is: can it have an n-share (with $(n-1)$-masks) TI with uniformity? Many such questions are open. For instance, recall that, for n odd and $t = \frac{n+1}{2}$, any almost bent function F has algebraic degree at most t. Does any AB function have an $\frac{n+1}{2}$-mask TI with uniformity?

Even for *quadratic functions*, there does not always exist a TI with uniformity of minimum number of masks (that is, with 2 masks): see [87, Corollaries 1 and 2]. In fact, for any $t \geq 2$, we do not know how characterizing the functions for which a t-mask TI exists, despite the theoretical results of [73].

This is a concern since the implementation cost of a function increases then exponentially with its degree: according to Proposition 189, a monomial of degree t results in the sum of $(t+1)^t$ monomials. This drawback is bypassed by expressing (when it is possible) high algebraic degree functions as the compositions of lower algebraic degree functions for which TI can be found, see [86, 87, 724], see

[19] For multivariate 1st order security, or td masks for univariate higher-order security.

also [321] and page 471 for a general method (the CPRR method) addressing this problem. Then uniformity can be ensured by introducing (when necessary) fresh randomness when making the composition of two threshold implementations, that is, by adding to the output of $\mathbf{F}$, a vector $\mathbf{c}$ such that $s(\mathbf{c}) = 0_m$ and such that any sub-vector of d components is random (this is sometimes called re-masking). Note that when creating a masked implementation of a decomposition, the TI of the lower degree components have to be separated by a register stage to stop glitches (and reducing the number of these register stages is needed). Of course, it is preferred to minimize the number of the lower algebraic degree functions which are composed for giving the S-box.

The TI of small S-boxes has been studied, see [113, 86, 87] and the references therein. In particular in [86] is designed the threshold implementation with at most 4 masks for all $(3,3)$-permutations and $(4,4)$-permutations[20]. But $n \leq 4$ is interesting for lightweight ciphers only. In [87] are studied APN $(5,5)$-permutations (affine equivalent to the AB power functions x^3, x^5, x^7, x^{11} and x^{15}) and the sole known APN $(6,6)$-permutation by different methods, in particular by expressing them as the compositions of quadratic functions (which needs re-masking and does not provide a TI, properly speaking). But designing TI with uniformity for these functions is an open question, as well as for the multiplicative inverse differentially 4-uniform $(8,8)$-function used in AES. Note that the *inverse function* plays not only a role in relation with the AES since we know, see [331, 1184], that any (n,n)-permutation can be expressed as the composition of functions $x \mapsto ax + b$ and of the inverse permutation.

An alternative approach for designing TI is, given some *secondary construction* of functions, to deduce the TI of the built function from the TI of the used functions. In [1094] is studied the following construction of an $(n+1, n+1)$-function H from two (n,n)-functions F and G and two n-variable Boolean functions f and g:

$$H : (x, x_{n+1}) \in \mathbb{F}_2^n \times \mathbb{F}_2 \mapsto x_{n+1}(F(x), f(x)) + (x_{n+1} \oplus 1)(G(x), g(x)) \in \mathbb{F}_2^n \times \mathbb{F}_2.$$

Clearly, every $(n+1, n+1)$-function can be obtained this way. Note that H is a permutation if and only if it is injective, that is, if and only if $x \mapsto (F(x), f(x))$ and $x \mapsto (G(x), g(x))$ are injections (which is a necessary and sufficient condition for the fact that two distinct inputs of the same form $(x, 0)$ or of the same form $(x, 1)$ do not give the same output) and have disjoint value sets (which is a necessary and sufficient condition for the fact that an input $(x, 0)$ and an input $(y, 1)$ do not give the same output). If F and G are permutations, then the condition simplifies into $f \circ F^{-1} \oplus \; g \circ G^{-1} = 1$. It is then shown that if F and f have t-mask TI with uniformity $\mathbf{F}$ and $\mathbf{f}$, and if $G(x_1, \ldots, x_n)$ equals either $F(x_1, \ldots, x_n)$ or $F(x_1, \ldots, x_{i-1}, x_i \oplus 1, x_{i+1}, \ldots, x_n)$ for some $i = 1, \ldots, n$, and g is taken such that $f \circ F^{-1} \oplus \; g \circ G^{-1} = 1$, then H has a t-mask TI with uniformity. The idea of the proof in the slightly more complex latter case is

[20] All $(2,2)$-permutations being affine, they do not need to be studied.

to decompose $(F(x), f(x))$ in the form $x_i(F_i(x), f_i(x)) + (x_i \oplus 1)(F_i'(x), f_i'(x))$ where $(F_i(x), f_i(x))$ (resp. $(F_i'(x), f_i'(x))$) is the restriction of $(F(x), f(x))$ to the hyperplane of equation $x_i = 0$ (resp. $x_i = 1$) and to observe that $H(x, x_{n+1})$ equals $(x_{n+1} \oplus x_i \oplus 1)(F_i(x), f_i(x)) + (x_{n+1} \oplus x_i)(F_i'(x), f_i'(x))$. It would be nice if less restrictive cases within this general construction could be addressed.
In [105] are constructed $(8, 8)$-functions based on a Feistel network, a substitution permutation network or the (special case of) MISTY network [830], all using quadratic 4-bit S-boxes, which admit a TI implementation while still maintaining a good level of differential uniformity and nonlinearity.

A recent general alternative technique, called the *changing of the guards*, and which represents a nice step forward, has been presented in [402] and applied to the Keccak S-box. It builds a $(t + 1)$-share threshold implementation with uniformity of any invertible S-box layer of algebraic degree t, after a transformation (the S-box is subdivided into several stages, separated by registers and some shares receive additional components, see the details in [402]); each share at the output of S-box i is made uniform by bitwise adding to it one or two shares from the input of S-box $i - 1$; this solves the problem of threshold implementation with uniformity (but only after such transformation of the S-box). In a next important step forward, a modification of the changing of the guards has been given in [1056] and applied to the AES S-box, at the cost of a significant extension of the AES S-box design, but ensuring in a nice way uniformity with 3-share TI (while Daemen's method in [402], as is, needs $t + 1$ shares, that is, 8 in the case of the AES S-box). The method includes a generic way to construct a uniform sharing for any function, by changing (by extension and reduction) the function to an invertible one while maintaining its essential functionality unchanged[21], which can be, in the case of AES S-box, decomposed into quadratic bijections. The extension of an (n, m)-function F is the $(n+m, n+m)$-permutation $(x, y) \mapsto (x, F(x)+y)$. If $\mathbf{F}$ is a 3-share TI of F, then a 3-share TI with uniformity of the extension is, still denoting $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$, the function $((x_1, y_1), (x_2, y_2), (x_3, y_3)) \mapsto ((x_2, \mathbf{F}_1(\mathbf{x}) + y_3), (x_3, \mathbf{F}_2(\mathbf{x}) + y_1), (x_1, \mathbf{F}_3(\mathbf{x}) + y_2))$. The reduction of an $(n + m, n + m)$-function G is the $(n + m, n + m)$-function $(x, y) \mapsto (0_n, y)$. If $\mathbf{G}$ is a 3-share TI of G, then a 3-share TI with uniformity of the reduction is the function $((x_1, y_1), (x_2, y_2), (x_3, y_3)) \mapsto ((x_2 + x_3, y_2), (x_3, y_3), (x_2, y_1))$. Composing these two 3-share TI provides a 3-share TI with uniformity of $(x, y) \mapsto (0_n, F(x) + y)$. See more in [1056].

Invariance of the existence of a t-mask TI

The existence of a t-mask TI of $F(x)$ is invariant when changing $F(x)$ into $F(x + a)$ (by changing $\mathbf{F}(\mathbf{x})$ into $\mathbf{F}(\mathbf{x} + \mathbf{a})$ for some $\mathbf{a}$ such that $s(\mathbf{a}) = a$,

[21] So the success of the method does not contradict Proposition 189.

which also preserves uniformity), and it is also invariant under linear equivalence $F \sim L \circ F \circ L'$ (which preserves uniformity as well), as observed (and proved rather informally) in [86, Theorem 2]. Indeed, applying L' on each share of $\mathbf{x}$ and L on each share of the output of $\mathbf{F}$ preserves correctness since L and L' are linear and it preserves uniformity since L and L' are bijective, and it also preserves non-completeness, since the applications of L and L' are made separately on each share. Hence the existence of a t-mask TI is an *affine invariant*, as well as the existence of a t-mask TI with uniformity. And the existence of a t-mask TI is also invariant under adding an affine function but it is not clear whether this preserves uniformity (in other words, affine equivalence preserves the existence of a uniform TI but extended affine equivalence does probably not).

Remark. Given a permutation F having a t-mask TI, function F^{-1} does not necessarily have a t-mask TI, since there are quadratic functions having 2-mask TI and whose inverses are not quadratic[22] and therefore do not have a 2-mask TI, according to Proposition 189. In particular, if $\mathbf{F}$ is a t-mask TI of F, function $\mathbf{F}^{-1}$ is not necessarily a t-mask TI of F^{-1}. This is because the condition "for every $j = 1, \ldots, t+1$, the j-th coordinate function of $\mathbf{F}$ is independent of the j-th coordinate of each $\mathbf{x_i}$" is not equivalent to the condition "for every $j = 1, \ldots, t+1$, the j-th coordinate function of $\mathbf{F}^{-1}$ is independent of the j-th coordinate of each $\mathbf{x_i}$". This subtle difference can be more easily seen with the characterization by Condition (12.4), which is not the same when applied to $\mathbf{F}$ and to its inverse: the hypothesis "$v_k = 0_n, \forall k \neq j$ and $u_j \neq 0_n$" of the implication, when it is applied to $\mathbf{F}^{-1}$, becomes "$u_k = 0_n, \forall k \neq j$ and $v_j \neq 0_n$", since we know that changing a function into its inverse corresponds for the Walsh transform to swapping the parameter(s) living in the domain and the one (those) living in the codomain, the same value of the Walsh transform being then kept for this new input. □

3. A more recent way to protect against SCA in the presence of glitches is domain oriented rather than function oriented; it organizes properly the ISW computations (we described above the methods based on decompositions of polynomials and related masking) and implements the concept of share domains (keeping each domain independent from the others). Each share of a variable is associated with one share domain. This method is called *domain-oriented masking*, see [575, 574], and is an alternative to threshold implementation requiring less chip area and less randomness, all the more when raising the protection order. □

12.1.5 Linear complementary dual codes and complementary pairs of codes used for direct sum masking

Direct sum masking has a weaker relationship with Boolean functions than with codes. We however briefly describe it, since a large number of recent papers deal

[22] This should be checked among quadratic functions with known 2-TI, though.

with the related notions of linear complementary dual codes and complementary pairs of codes, and also because the dual distance of codes playing a central role, correlation immune functions are closely related.
The impact of codes on protection against fault injection attacks is well studied; the number of detected faults relates to their minimum distance. The (explicit) use of codes for protecting against SCA is more recent. The *direct sum masking* (*DSM*) countermeasure [131, 288] is a generalization of *Boolean masking* consisting in:

- encoding (see definition at pages 19 and 22) the sensitive data, say x, that we consider here as living in $\mathbb{F}_2^k$, into a codeword of a k-dimensional linear subcode C of $\mathbb{F}_2^n$,
- encoding the mask y drawn at random from $\mathbb{F}_2^{n-k}$ into a codeword of an $(n-k)$-dimensional linear subcode D of $\mathbb{F}_2^n$.

The masked version of x equals then the sum of these two codewords. This is only a first-order masking scheme in terms of probing security, if there is a reuse of the mask.
If G is a generator matrix of C and G' a generator matrix of D, we take then:

$$z = x \times G + y \times G'; \qquad x \in \mathbb{F}_2^k, y \in \mathbb{F}_2^{n-k}. \tag{12.6}$$

For allowing the final demasking at the end of the computation, it must be possible to recover x from z (but to avoid leaks, the algorithm should not include a computation of x, unless it has arrived to its end). This means that C and D must have trivial intersection, that is, be supplementary[23]:

$$\mathbb{F}_2^n = C \oplus D.$$

Every vector $z \in \mathbb{F}_2^n$ can then be written in a unique way as in (12.6). Note that this provides the possibility of removing the mask without the knowledge of it.
As mentioned above, d-th order masking is a particular case of DSM: we have then $n = (d+1)k$, $C = \mathbb{F}_2^k \times \{0_{dk}\}$, where 0_{dk} is the all-zero vector of length dk, $G = [I_{k,k} : 0_{k,dk}]$, where $I_{k,k}$ is the $k \times k$ identity matrix and $0_{k,dk}$ is the $k \times dk$ all-zero matrix, $D = \{(y_0, \ldots, y_d); y_i \in \mathbb{F}_2^k, \sum_{i=0}^d y_i = 0_k\}$ and, for instance:

$$G' = \begin{bmatrix} I_{k,k} & I_{k,k} & 0_{k,k} & \ldots & 0_{k,k} \\ I_{k,k} & 0_{k,k} & I_{k,k} & \ldots & 0_{k,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_{k,k} & 0_{k,k} & \ldots & \ldots & I_{k,k} \end{bmatrix}.$$

But an advantage of DSM is that, when C and D are properly chosen, it can be also a countermeasure against FIA (while classical masking cannot), which helps reduce the cost of the overall countermeasure against SCA and FIA.
A pair (C, D) of supplementary codes is called a *linear complementary pair*

[23] We prefer using this term rather than complementary, which is ambiguous; we use the classical notation $\oplus$ to denote such direct sum, which needs not to be confused with XOR.

(LCP) of codes. It is shown in [131] that if the monovariate leak $\mathcal{L}$ (which is a *pseudo-Boolean* function) has numerical degree 1, the encoding with an LCP of codes (C, D) as described above protects against d-th order HO-SCA if and only if the dual distance of D satisfies $d(D^\perp) > d$. Moreover, as shown in [960, Section 3.2], d-th order bit-probing security is then ensured, because by definition, less than $d(D^\perp)$ of those equations expressing the coordinates of $z = x \times G + y \times G'$ by means of the coordinates of x and y, do not allow to eliminate the coordinates of y since less than $d(D^\perp)$ columns of G' are linearly independent (since as we saw in a remark at page 32, the dual distance of a linear code equals the minimum nonzero number of linearly dependent columns in its generator matrix). The encoding protects against the injection of faults of Hamming weights at most d if and only if the minimum distance of C satisfies $d(C) > d$. Note that when encoding is made over bits, the ensured security is the so-called *bit-probing security*, whose order can be higher than the probing security order when the attacker can probe symbols belonging to a larger alphabet.

According to the observations above, the security parameter against both HO-SCA and FIA is $\min\{d(C), d(D^\perp)\} - 1$. But taking this minimum as the sole security parameter supposes that the orders of needed protection against SCA and FIA are comparable. This is not always the case. In a safety context like autonomous trains and cars, it is crucial to ensure the detection of faults and side-channel leakage is a lesser risk, while in the context of internet of things (IoT) or banking, minimizing side-channel leakage is a premium objective. We must then take the pair $(d(C) - 1, d(D^\perp)\} - 1)$ for security parameter.

In the case of *Boolean masking*, the codes, seen above, $C = \mathbb{F}_2^k \times \{0_{dk}\}$ and $D = \{(y_0, \ldots, y_d); y_i \in \mathbb{F}_2^k, \sum_{i=0}^d y_i = 0_k\}$ satisfy $d(C) = 1$ and $d(D^\perp) = d + 1$. This confirms that Boolean masking protects against SCA but not FIA.

If D equals the dual $C^\perp$ of C in an LCP of codes, then C and D are so-called *linear complementary dual* (LCD) codes. Such codes are well adapted to the cases where the need for protection is the same for SCA and FIA ; the security parameter of an LCD code C when used in so-called *orthogonal direct sum masking* ($ODSM$) [131, 288] is simply $d(C) - 1$.

The notion of LCD code is anterior to DSM. In [1134], Yang and Massey introduced it as an optimal linear coding solution for a rather particular problematic: the two user binary adder channel. They provided a necessary and sufficient condition under which a *cyclic code* is LCD. In [75], Bhasin et al. have shown how also using and implementing LCD codes and LCP of codes to strengthen encoded circuit against hardware Trojan horses, while minimizing the cost.

Note that $D = C^\perp$ if and only if G' is a parity check matrix of C, that is, $G \times {G'}^t = 0_{k,k}$, where ${G'}^t$ is the transposed matrix of matrix G'. We can denote then G' by H and use an orthogonal projection to recover x and y from z: the relation $z = x \times G + y \times H$ implies $z \times H^t = y \times H \times H^t$ and $z \times G^t = x \times G \times G^t$, and this provides x and y since "C is LCD", "the matrix $H \times H^t$ is invertible" and "the matrix $G \times G^t$ is invertible" are equivalent.

Since the introduction of DSM, and the investigation of numerous constructions of LCD codes in [288], many papers have studied constructions of such codes and of LCP of codes, see a description in [295], where a problematic is also described: faults are detected by verifying that masks have not been altered during processing; checking this requires to have access to the masks. A possibility is to mask with $z = (x \times G + y \times G', y)$ instead of $z = x \times G + y \times G'$. The leak is then modified. Some *MDS* codes can keep the same protection ability when changing this way the encoding (the code of generator matrix $[G' : I_{n-k}]$, where I_{n-k} is the identity matrix, can have the same dual distance as the code of generator matrix G', which can be MDS).

Remark. A particular case[24] of DSM[25] is *inner product masking* (*IPM*), whose principle for masking a sensitive data $x \in \mathbb{F}_{2^n}$ is to generate a vector over $\mathbb{F}_{2^n}$ whose inner product with some public vector equals x, see [491, 46, 44, 45] (see also [960, 289, 366]; the latter reference expresses the side-channel resistance of IPM in terms of, classically, the minimum distance of $D^\perp$, and less classically, the first nonzero coefficient in its weight enumerator). With the public vector $(1, l_1, \dots, l_{n-1})$, since we want $x = (x \times G + y \times G') \cdot (1, l_1, \dots, l_{n-1})$; $x \in \mathbb{F}_{2^n}$, $y \in \mathbb{F}_{2^n}^{n-1}$ as explained above, we can take (as shown in [960]): $G = (1, 0, \dots, 0)$ (and C has then minimum distance 1, which does not allow detecting FIA) and $y \times G' = (y \cdot (l_1, \dots, l_{n-1}), y_1, \dots, y_{n-1})$, and code D has then generator matrix:

$$\begin{bmatrix} l_1 & 1 & 0 & \dots & 0 \\ l_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ l_{n-1} & 0 & \dots & 0 & 1 \end{bmatrix}$$

(the masked information is $z = (x + \sum_{i=1}^{n-1} l_i\, y_i, y_1, \dots, y_{n-1})$ and we have $x = z \cdot (1, l_1, \dots, l_{n-1})$). Code $D^\perp$ has generator matrix $(1, l_1, \dots, l_{n-1})$ (and the order of protection against SCA equals the Hamming weight of $(l_1, \dots, l_{n-1})$).
IPM also contains *Boolean masking* as a particular case (take $(l_1, \dots, l_{n-1}) = 1_{n-1}$) as well as the methods of masking using secret sharing [974] (see Relation (3.43), page 169). It has been recently modified in [365], so as to allow fault injection detection as well. □
An important issue is to compute (in particular, multiply) efficiently over encodings. In the case of IPM, solutions exist [46, 44], but for DSM, it remains an open challenge, except in a simplified framework addressed in [260].

[24] With a practical difference, though: IPM works over field elements while DSM often works over bits; hence probing security may not mean the same in both cases. Nevertheless, it is proved in [960] that if the deterministic leaks of the shares are linear functions of their bits, the bounded moment security order of the IPM is equal to the probing security order of the bitwise encoding obtained by decomposing the elements of $\mathbb{F}_{2^n}$ over a basis over $\mathbb{F}_2$.

[25] And also of leakage squeezing, that we saw at page 467, and which is also a particular case of DSM, but only when F is linear.

12.1.6 Robust codes, AMD codes and vectorial functions

In many cases of error detection, the assumption that the most probable errors have low Hamming weight cannot be guaranteed. As shown successively in [666, 667, 664], the classical method of error detection by codes having large enough minimum distance is then often not efficient. In fact, it is in many cases almost impossible to predict the error patterns (*e.g.* in the case of address decoder errors, or with power glitches, or when data is compressed for transmission and decompressed after). For instance, the error characteristics in silicon devices like memories are changing and in many instances can be unpredictable. This is becoming still more true while embedded systems are becoming ubiquitous, and their roles are becoming more mission critical for sensitive applications. Taking again the example of memories, embedded ones are exposed to unpredictable environments (when for instance moved in a plane from sea level, where cosmic rays are weak, to high sky where the error rate can be large) and their reliability is now a matter of critical importance and safety. This situation of unpredictability is similar to fault injection attacks (FIA) on hardware implementations of cryptographic algorithms (as also observed in the references above): classical methods of error detection are ineffective when the error distribution within a device is controlled by an adversary. For instance, when an attacker induces stress (see [82]) resulting in that bits vanish one after the other in the processed data, this results in the injection of an error on each bit which was equal to 1; when increasing the stress, the error distribution turns to almost uniform. Classical codes (and the codes seen in Subsection 12.1.5) may fail then to detect errors, when the adversary succeeds in producing an error changing a correct codeword into a wrong codeword. The *worst error masking probability* is the maximal probability that a given error e transforms a codeword into a codeword. In the case of linear codes, the undetectable errors are the codewords themselves, and the attacker only needs, for his error injection, to know the code which is used in the device and to be able to inject codewords as errors. The worst error masking probability being then equal to 1 (worst possible), linear codes are not adapted to minimizing worst error masking probability.

Codes robust against fault injection with unknown error probability

In [667, 717, 665] has been presented the notion of robust code, aiming at providing uniform protection against all errors, without any assumption on the error distribution, or on the capabilities of an attacker:

Definition 87 *Given a positive integer R, an unrestricted (*i.e. *non-necessarily linear) code $C \subset \mathbb{F}_q^n$ is called R-robust if the size of the intersection of C and any of its translates $e + C$, where $e \in \mathbb{F}_q^n, e \neq 0_n$, is bounded above by R. Given a code C, the smallest possible value of R having this property shall be denoted* *by R_C:*

$$R_C = \max_{0_n \neq e \in \mathbb{F}_q^n} |C \cap (e + C)|. \tag{12.7}$$

A binary R-robust code C of length n with $M = |C|$ is denoted by a triple (n, M, R).

The code can be *systematic* (recall that this means there exists a subset I of positions in codewords, called an *information set* of C, such that every possible tuple in $\mathbb{F}_q^I$ occurs in exactly one codeword within the specified coordinates x_i; $i \in I$; this implies that $M = q^{|I|}$). Systematic codes are more practical for error detection in computer hardware thanks to the separation between information bits and check bits. The code equals then, up to a permutation of the codeword coordinates, the graph of a function: $\{(x, F(x)); x \in \mathbb{F}_q^I\}$, for some (not necessarily linear) function F. But we shall see that the code cannot then be perfect robust.
The probability of missing an algebraic manipulation with a code C equals the so-called probability of error masking, which for each possible error e is denoted by $Q(e)$ and is defined as:

$$Q(e) = \frac{|C \cap (e + C)|}{|C|}. \tag{12.8}$$

The *worst error masking probability* $\max_{e \neq 0_n} Q(e)$ equals then $\frac{R_C}{|C|}$. As observed in [666], we have $\max_{e \neq 0_n} Q(e) \geq \frac{|C|-1}{q^n-1}$ (with equality if and only if the code is uniformly robust, see below), for any code of length n over $\mathbb{F}_q$ (which is easily shown by using that the maximum of a sequence of values is always larger than or equal to the arithmetic mean, and equals it if and only if the sequence is constant, and observing that $\sum_{e \neq 0_n} Q(e) = \frac{2}{|C|}\binom{|C|}{2} = |C| - 1$); a little more can be shown by using that the numerator in (12.8) is an integer. Note that there is a slight error in [667] about this result: it is written that for every code, we have $Q(e) \geq \frac{|C|-1}{q^n-1}$ for every $e \neq 0_n$, which is false (suppose that the minimum distance of the code is larger than 1 and take e smaller than the minimum distance).
A code is called a *robust code* if its worst error masking probability is strictly less than 1, and it is called a *uniformly robust code* (or *perfect robust code*) if $Q(e)$ is constant for $e \neq 0_n$, that is, $Q(e) = \frac{|C|-1}{q^n-1}, \forall e \neq 0_n$ (note that the minimum distance of such code is necessarily 1). This is equivalent to saying that C is a *difference set* in $(\mathbb{F}_q^n, +)$, that is, in the case $q = 2$ and assuming that C is neither equal to $\{0_n\}$ nor to $\mathbb{F}_q^n$, that the *indicator* function of C is bent.

Robustness and worst error masking probability for supports of Boolean functions and graphs of vectorial functions
1. Let C be the support of an n-variable Boolean function. Then if we denote by Δ the symmetric difference between two sets, we have that $|C \,\Delta\, (e+C)| = 2\,|C| - 2\,|C \cap (e+C)|$ for every $e \in \mathbb{F}_2^n$, and therefore $|C \cap (e+C)| = |C| - \frac{1}{2}|C \,\Delta\, (e+C)|$. Let us revisit the case where the function is *bent*: we know then that $|C| =$

$2^{n-1} + (-1)^\epsilon\, 2^{\frac{n}{2}-1}$ for some $\epsilon \in \mathbb{F}_2$, and $|C\,\Delta\,(e+C)| = 2^{n-1}$, for every $e \neq 0_n$; this gives $|C \cap (e+C)| = 2^{n-2} + (-1)^\epsilon\, 2^{\frac{n}{2}-1}$, $Q(e) = \frac{2^{n-2}+(-1)^\epsilon\, 2^{\frac{n}{2}-1}}{2^{n-1}+(-1)^\epsilon\, 2^{\frac{n}{2}-1}}$ (equal to the optimum $\frac{2^{n-1}+(-1)^\epsilon\, 2^{\frac{n}{2}-1}-1}{2^n-1}$), and C is uniformly robust (but cannot be systematic since its size is not a power of 2). In [666] (and the references therein) has been proposed for f the basic Maiorana-McFarland function $f(x,y) = x \cdot y$; $x, y \in \mathbb{F}_2^{\frac{n}{2}}$, n even, but any (binary) bent function would behave the same. In this same reference is also proposed to take $C = \{(x,y) \in (\mathbb{F}_q^{\frac{n}{2}})^2; x \cdot y = u\}$, where q is a power of a prime and n is even. This improves $Q(e)$ in some cases (with a different value according to whether u is zero or not). This same reference also investigates codes which are the unions of the codes $\{(x,y) \in (\mathbb{F}_q^{\frac{n}{2}})^2; x \cdot y = u\}$ for some values of u.

2. Let C be now systematic, *i.e.* the graph of a vectorial function. We have, by slightly completing [717]:

Proposition 190 *Let $C = \{(x, F(x)), x \in \mathbb{F}_2^k\}$ be the graph of a vectorial function F from $\mathbb{F}_2^k$ to $\mathbb{F}_2^r$, with k and r non-negative. The worst error masking probability of C equals the differential uniformity of F divided by 2^k. It is then bounded below by 2^{-r} and equals this optimum if and only if F is perfect nonlinear.*

Indeed, denoting $e = (a,b)$, we have:

$$|C \cap (e+C)| = \left| \left\{ (x,y) \in (\mathbb{F}_2^k)^2;\ \left\{ \begin{array}{l} x = y + a \\ F(x) = F(y) + b \end{array} \right. \right\} \right| = \left| (D_a F)^{-1}(b) \right|.$$

For $a = 0_k$ and $b \neq 0_k$, this size is null and $\max_{e \neq 0_{k+r}} |C \cap (e+C)|$ equals then the differential uniformity of F (see Definition 40, page 157). We know from the bound due to Nyberg that it is then bounded below by 2^{k-r} with equality if and only if F is *perfect nonlinear* (in which case the derivatives are balanced). Since C has size 2^k, this gives the result.
Note that for this code, the value $\max_{e \neq 0_{k+r}} Q(e)$, equal to 2^{-r}, is larger than $\frac{|C|-1}{q^n-1} = \frac{2^k-1}{2^n-1}$ (no systematic code can be perfect robust).
Note also that, according to Nyberg's result[26] (Proposition 104, page 296), the best codes C from Proposition 190 can exist only if k is even and $r \leq \frac{k}{2}$, that is, the length $n = k + r$ and the dimension k satisfy $n \leq \frac{3k}{2}$ (*i.e.* their *transmission rate* is at least $\frac{2}{3}$). If this double condition is not satisfied, we can take F almost perfect nonlinear, and the worst error masking probability of C is then 2^{-r+1}.
It is observed in [717] that, thanks to the fact that the robust codes above are nonlinear, the error detection for these codes depends on the encoded data (while for a linear code, the set of missed errors is the same for all encoded data). This makes for the attacker the set of necessary errors harder to determine, all the more when the data depends on the secret key or when randomization is applied. But this makes also, as observed in [690, 304], that the efficiency of

[26] The situation is different in odd characteristic; then, PN (n,n)-functions exist for every n.

these codes depends on the fact that the data be uniformly distributed, which is not reasonable in many situations, in particular when the information bits of messages are also controllable by an attacker. This limitation can be overcome by algebraic manipulation detection (AMD) codes, that we address in the next paragraph.

Codes and algebraic manipulation

A model for error injection has been introduced in [395] under the name of *algebraic manipulation*. This model assumes that the attacker is able to modify the value of some abstract data storage device, without having read-access to the data. Such a device is denoted by $\sum(G)$ and can hold an element g (corresponding to some secret s), from a public finite Abelian (additive) group G. The attacker is not able to obtain any information about the element g stored in the device $\sum(G)$. However, he can change the stored element g by adding an error $e \in G$ of his choice. After such algebraic manipulation (tampering), the abstract storage device $\sum(G)$ will store the value $g+e$. The attacker can choose the value e only on the basis of what he already knew about g before it was stored in the device (his *a priori* knowledge of g). This models for instance the situation with *linear secret sharing schemes* (see Subsection 3.6.1, page 168), in which the correctness of the secret s reconstructed from the shares of a qualified coalition of players is guaranteed only if all these shares are correct. If the coalition contains dishonest players and if the honest players in it are not able to reconstruct s on their own (*i.e.* if they do not constitute a qualified coalition), then the dishonest players can cause the reconstruction of a modified secret s', and they can control the difference between s and s', thanks to the linearity of the secret sharing. In particular, in a minimal qualified coalition of players, a single corrupted player can cause the reconstruction of an incorrect secret.
Two types of fault injection attacks can be considered. In the weaker ones, the adversary cannot choose the input. So, from the attacker's point of view, the source s is uniformly distributed; he can only inject an error e in the storage device $\sum(G)$, but he cannot change value s at his own discretion. In the stronger version, the adversary knows the value s and moreover can choose it, and change it after he got some information from the device (in a kind of adaptive chosen attack). In both types of fault injection attacks, the value g stored in $\sum(G)$ is hidden from the attacker.

The countermeasure against algebraic manipulation consists in using so-called *algebraic manipulation detection codes*, which were introduced in [395] after that observations were made in [481]. AMD codes encode an original information $s \in S$ as an element of $g \in G$ in such way that any algebraic manipulation is detected with high probability. No secret key is needed, contrary to the case of message authentication codes.

Definition 88 *An* AMD code *is a pair of two functions: a probabilistic encoding function $E : S \to G$ from a set S into a finite Abelian group G, and a determin-*

istic decoding function $D : G \to S \cup \{\perp\}$*, where* $\perp \notin S$ *symbolizes that algebraic manipulation has been detected, satisfying that* $D(E(s)) = s$ *with probability* 1 *for every* $s \in S$.
The AMD code is called ϵ*-secure for* $\epsilon > 0$ *if, for every* $s \in S$ *and for every* $e \in G$*, the probability that* $D(E(s) + e) \notin \{s, \perp\}$ *is at most* ϵ*. It is called weak* ϵ*-secure if, for every* $e \in G$ *and for every* $s \in S$ *sampled from* S *with uniform distribution (independently of* e*, then), the probability that* $D(E(s)+e) \notin \{s, \perp\}$ *is at most* ϵ*.*
A systematic *AMD code is an AMD code in which set* S *is a group and the encoding function* E *has the form*

$$\begin{aligned} E : S &\to G = S \times G_1 \times G_2 \\ s &\to (s, x, F(x, s)), \end{aligned} \tag{12.9}$$

where G_1 *and* G_2 *are groups,* F *is a function and* x *is randomly chosen with uniform probability in* G_1*. The decoding is then* $D(s', x', r') = s'$ *if* $F(x', s') = r'$*, and* $D(s', x', r') = \perp$ *otherwise.*

Given an AMD code, $E(s)$ can safely be stored on $\sum(G)$ (supposed protected from reading) so that the adversary who manipulates the stored value by adding some nonzero e can cause it to decode to some $s' \neq s$ with probability at most ϵ, only. AMD codes also allow the protection of hardware and memories against FIA (seen in Section 12.1), see[27] [1113].
Note that if the AMD code is ϵ-secure, then for every $s \in S$, the size $|D^{-1}(s)|$ of the pre-image of s by D is necessarily at least $\frac{1}{\epsilon}$ (this property will be used below), since denoting by E_s the set of all possible images of s by E, and choosing e so that there exists in $E_s + e$ an element x of $E_{s'}$ with $s' \neq s$ (and so $D(x) = s' \notin \{s, \perp\}$), the size of E_s needs to be at least $\frac{1}{\epsilon}$ for allowing the probability that $D(E(s) + e) \notin \{s, \perp\}$ to be at most ϵ.
Deterministic weak secure AMD codes are a randomization of systematic robust codes (seen above), with $\max_{e \neq 0} Q(e) = \max_{e \neq 0} \text{Prob}\,[D(E(s) + e) \notin \{s, \perp\}]$.
Note that, as already seen, every $(|G|, |S|, \lambda)$-difference set D (see page 220) in $(G, +)$ where $G = S \times G_1 \times G_2$ (assuming that S is an additive group and that such difference set exists) provides then a weak ϵ-secure AMD code with $\epsilon = \frac{\lambda}{|S|}$, by taking for E any bijection between S and D. In fact, it is enough (and necessary) that every nonzero element e in $S \times G_1 \times G_2$ can be written in at most (rather than exactly) λ ways as the difference between two elements of D, that is, $|D \cap (e + D)| \leq \lambda$. The graphs of perfect nonlinear (resp. almost perfect nonlinear) (r, s)-functions have such property with $\lambda = 2^{r-s}$ (resp. $\lambda = 2^{r-s+1}$).
In [395] is proposed the systematic AMD code with $F(x, s) = x^{d+2} + \sum_{i=1}^{d} s_i x^i$, where $s \in \mathbb{F}_q^d$, $x \in \mathbb{F}_q$, which provides a systematic $\frac{d+1}{q}$-secure AMD code, thanks to the fact that, when $e_x \neq 0$, $(x + e_x)^{d+2} - x^{d+2}$ equals a polynomial of degree exactly $d+1$ (which matches any value at most $d+1$ times), and when $e_x = 0$ and

[27] In this reference is required for a systematic AMD code that, for any nonzero (e_x, e_s), D_{e_x,e_s} is non-constant on any section $G_1 \times \{s\}$.

$e_s \neq 0$, $\sum_{i=1}^{d}(s_i+[e_s]_i)x^i - \sum_{i=1}^{d} s_i x^i$ is a nonzero polynomial of degree at most d (which matches any value at most d times). This construction is generalized in [396], where is shown (by extending an idea from [1112] which worked with generalized Reed-Muller codes) how systematic AMD codes can be deduced from classical codes: from any subset S of $G_2^{G_1}$ (*i.e.* any code of length $|G_1|$ over G_2 whose codewords are indexed in G_1), we take for $F(x,s)$ the coordinate of index x in the codeword s; the condition for the ϵ-security of such AMD code is given in [396, 397]. The AMD code from [395] given above corresponds to the case where S equals the subset of a Reed-Solomon code (viewed as in the remark on RS codes at page 62) whose elements correspond to monic polynomials of degree $d+2$ with no term of degree $d+1$. Other examples of AMD codes are given in [397, 668, 1112]. In [535] are proposed modifications of AMD codes, which can have minimum distances larger than 1, and then not only detect injected faults but also correct errors caused by natural reasons.
It is shown in [916] (which dealt with cheating detection in secret sharing) and recalled in [396] that:

- for any ϵ-secure AMD code, we have $|G| \geq \frac{|S|-1}{\epsilon^2}+1$;
  indeed, given $s \in S$, applying the inequality $|D^{-1}(s')| \geq \frac{1}{\epsilon}$ for each $s' \neq s$, we have that the probability that $D(E(s)+e) \notin \{s, \perp\}$ when e is chosen uniformly at random in $G \setminus \{0\}$ is at least $\frac{|S|-1}{\epsilon(|G|-1)}$, and we have by hypothesis that this probability is at most ϵ;
  - as observed in [396], the inequality $|G| \geq \frac{|S|-1}{\epsilon^2}+1$ cannot be an equality for systematic codes, since for such codes, the size of E_s (the set of all possible images of s by E) equals $|G_1|$, which is then at least $\frac{1}{\epsilon}$, and we have also $|G_2| \geq \frac{1}{\epsilon}$ because, for every $s \in S$ and $(e_s, e_x) \in S \times G_1 \setminus \{(0,0)\}$, we have $\max_{e_F \in G_2} \text{Prob}\,[D(E(s)+(e_s,e_x,e_F)) \notin \{s,\perp\}] \geq \frac{1}{|G_2|}$, as this probability equals that of the event $F(x+e_x, s+e_s) - F(x,s) = e_F$, where $F(x+e_x, s+e_s) - F(x,s) \in G_2$; these two inequalities imply $|G| \geq \frac{|S|}{\epsilon^2}$;
  - moreover, it is shown in [396] (which adapted a proof from [1112] dealing with a different notion of AMD codes) that, for any systematic ϵ-secure AMD code with $\epsilon < 1$, we have $|G_1| \geq \frac{\log|S|}{\epsilon \log|G_2|}$, where log is (for instance) the base 2 logarithm; indeed, the code over G_2 of all functions $x \mapsto F(x,s)+e_F$, where (s,e_F) ranges over $S \times G_2$, contains $|S|\,|G_2|$ codewords of length $|G_1|$ and has minimum distance at least $|G_1|(1-\epsilon) > 0$ (since the code is ϵ-secure); the bound is then deduced from the *Singleton bound*[28] (see page 21): $|G_1|(1-\epsilon) \leq |G_1| - \log_{|G_2|}(|S|\,|G_2|)+1$, that is, $|G_1|\epsilon \geq \log_{|G_2|}(|S|)$;
- for weak ϵ-secure AMD codes, we have $|G| \geq \frac{|S|-1}{\epsilon}+1$, indeed, all that we can then say is that $|D^{-1}(s')| \geq 1$.

A stronger definition of AMD codes has been proposed in [1112, 1113], in which the condition becomes that the probability that $D(E(s)+e) \notin \{\perp\}$ is at most ϵ

[28] Recall that this bound is valid for (unrestricted) codes over any alphabet.

(hence, in this definition, every undetected algebraic manipulation is treated as a success of the adversary, while in Definition 88, when the source message is unaltered, it is not). This is preferred for some applications (like non-malleable secret sharing schemes [558]). More precisely, systematic AMD codes detect algebraic manipulation for errors (e_s, e_x, e_F), under the condition that the information part contains an error: $e_s \neq 0$ while this stronger version detects errors with zero information part $(e_s = 0, e_x, e_F)$; for some secure architectures, the integrity of redundant bits of the codes is indeed also important. The AMD codes described above from [395] satisfy this stronger requirement as well as the main construction in [397]. Lower bounds on the values of ϵ such that such codes can be ϵ-secure are studied in [1113] (with other notation) and constructions are given.

The first domains of application of AMD codes have been, as indicated in [395], robust secret sharing schemes (which ensure that, given a coalition S of players able to reconstruct some secret value s, no sub-coalition of (dishonest) players unable on their own to reconstruct s can modify their shares and lead with the other players from S to the reconstruction of some value $s' = s + t$, where $t \neq 0$ could be controlled by the dishonest players; this is achieved by applying a linear secret sharing scheme to an encoding of the secret by an AMD code rather than to the secret itself) and robust fuzzy extractors (enabling to recover a uniformly random key from a noisy and non-uniform secret, such as those obtained by biometrics, in such way that the key can be recovered from any value close to the secret) [395]. Other cryptographic applications are the message authentication codes which remain secure when the adversary can manipulate the key, unconditionally secure *multiparty computation* protocols with a dishonest majority, anonymous message transmission (and quantum communication), and more applications mentioned in [396, 397]. Applications to memory security have been developed in [1114, 534].

12.2 Fully homomorphic encryption and related questions on Boolean functions

We refer to [839, 306, 305] for the present section. We observe nowadays two complementary phenomena: the proliferation of small embedded devices having growing but still limited computing (and data storage) facilities, and the development of cloud services with extensive storage and computing means. The cloud becomes then a more and more unavoidable complement to embedded devices. But the outsourcing of data processing raises new privacy concerns. The users want to prevent the servers from learning about their data, while these servers are needed to help computing values from them. Gentry's *fully homomorphic encryption* (*FHE*) scheme [536, 537] gives a theoretical solution to this problem, by allowing encryption $\mathbf{C}^H$ preserving both operations of addition and multiplication:

$$\mathbf{C}^H(m+m') = \mathbf{C}^H(m) + \mathbf{C}^H(m');\ \mathbf{C}^H(mm') = \mathbf{C}^H(m)\,\mathbf{C}^H(m'). \qquad (12.10)$$

Given a vectorial function F from a finite field to itself (possibly, to a subfield), if Alice wants to compute $F(m)$ and needs the help of the cloud for that, she can send $\mathbf{C}^H(m)$ to Claude[29], who computes $F(\mathbf{C}^H(m))$, which equals $\mathbf{C}^H(F(m))$, thanks to (12.10) and since F has a polynomial representation. After decryption, Alice gets $F(m)$ but the server has not learned anything about m nor about $F(m)$.

But repetitive use of homomorphic encryption requires more computational power and storage capacity than what can offer small devices (see more in [305]). A solution to this problem is that Alice uses a *hybrid symmetric-FHE encryption* protocol, which works according to the following phases:

1. *Initialization.* Alice sends to Claude her homomorphic public key pk^H and the homomorphic ciphertext of her symmetric key $\mathbf{C}^H(\mathsf{sk}^S)$ (which is much easier to compute than $\mathbf{C}^H(m)$ since sk^S is much shorter than m, and which needs to be computed once for all further communication with Claude).
2. *Storage.* Alice encrypts her data m with the symmetric encryption scheme $\mathbf{C}^S$, and sends $\mathbf{C}^S(m)$ to Claude.
3. *Evaluation.* Claude calculates $\mathbf{C}^H(\mathbf{C}^S(m))$ and homomorphically evaluates the decryption of the symmetric scheme on Alice's data and gets $\mathbf{C}^H(m)$.
4. *Computation.* Claude homomorphically executes the treatment of F on Alice's data, and gets $\mathbf{C}^H(F(m))$.
5. *Result.* Claude sends $\mathbf{C}^H(F(m))$ and Alice gets $F(m)$ by deciphering (deciphering being much less costly than enciphering in FHE).

However, the best adapted generations of FHE, that is, 2nd and 3rd generations, are noise-based (being built on the learning with errors [LWE] problem) and need expensive "bootstrapping" when the noise grows too much. It is then mandatory to reduce the error growth during evaluation-computation and this is more or less equivalent to reducing the number of multiplications for the 2nd generation (more precisely, to reduce the multiplication depth), and the number of additions for the 3rd generation (in fact, the correct parameter is much more complex, it also depends of multiplications, see [537], but describing it precisely would be too long). The choice of the symmetric cipher $\mathbf{C}^S$ is then central for reducing the cost.

12.2.1 The FLIP cipher

The multiplicative depth of AES being too large (and its additive depth being still larger), other symmetric encryption schemes have been proposed: block ciphers, like *LowMC* [11], *Rasta* and *Agrasta* [480], and the stream cipher *Kreyvium*

[29] The name used now in cryptography to personify the cloud.

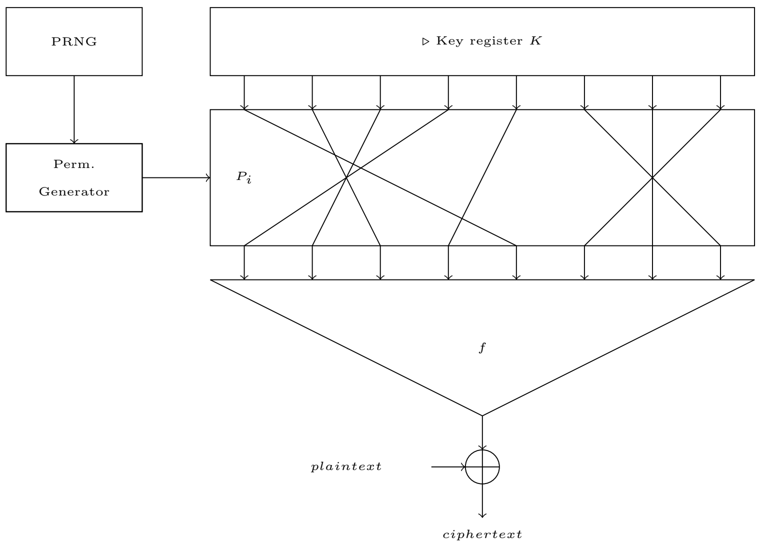


Figure 12.1 Filter permutator construction.

[192]. These solutions have drawbacks: Kreyvium is expensive (all the more if it needs to be started again, which can happen often), and lowMC has low complexity rounds, but their iteration makes it unadapted, as almost any other block cipher (if we look precisely how they can work with HeLib [584] for instance), except for Rasta and Agrasta, which are also well adapted for *multiparty computation*, but whose originality is not in the choice of the S-box, which is why we do not describe them here.

The filter permutator and the FLIP cipher

The *FLIP cipher* is an encryption scheme described in [839], which tries to minimize the parameters mentioned above (in particular the multiplicative depth). It is based on a new stream cipher model, called the *filter permutator* (see Figure 12.1 below), consisting in updating at each clock cycle a key register by a permutation of the coordinates, piloted by a pseudorandom number generator (PRNG), and in filtering the resulting permuted key with a Boolean function f whose input is the whole register[30] and whose output provides the *keystream*. Applying the non-linear filtering function directly on the key bits allows reducing the noise level when used in hybrid symmetric-FHE encryption protocols. In theory, there is no big difference between the filter model seen at page 39 and the filter permutator since the LFSR is simply replaced by a permutator. But in practice, there is much difference since the filter function has hundreds of input bits instead of about 20, and there is another important difference that we shall see in the next subsection.

[30] A future version of FLIP gets rid of this constraint.

In the versions of the cipher proposed in [839], function f has $n = n_1+n_2+n_3 \geq 500$ variables, where n_2 is even and n_3 equals $\frac{k(k+1)}{2}t$ for some k and t. It is defined as:

$$f(x_0,\ldots,x_{n_1-1},y_0,\ldots,y_{n_2-1},z_0,\ldots,z_{n_3-1}) = \bigoplus_{i=0}^{n_1-1} x_i \oplus \bigoplus_{i=0}^{n_2/2-1} y_{2i}\, y_{2i+1} \oplus$$

$$\bigoplus_{j=1}^{t} T_k\left(z_{\frac{(j-1)k(k+1)}{2}}, z_{\frac{(j-1)k(k+1)}{2}+1}, \ldots, z_{\frac{(j-1)k(k+1)}{2}+\frac{k(k+1)}{2}-1}\right),$$

where *triangular function* T_k is defined as:

$$T_k(z_0,\ldots,z_{j-1}) = z_0 \oplus z_1z_2 \oplus z_3z_4z_5 \oplus \cdots \oplus z_{\frac{k(k-1)}{2}} \cdots z_{\frac{k(k+1)}{2}-1}.$$

We have seen in Subsection 6.2.6, page 292, how calculating the nonlinearity of direct sums, in Subsection 9.1.4, page 372, how calculating their algebraic immunity, and we have calculated in Subsection 10.3.1, page 395 the values of the nonlinearities and algebraic immunities of triangular functions.
Four sets of parameters were proposed for the filtering function. The Hamming weight of the input to the function being forced to $\frac{n}{2}$ where n is the size of the register, the four proposed instances, displayed in Table 12.1, ensure that $\binom{n}{n/2} \geq 2^\lambda$, where λ is a security parameter (the number of elementary operations needed for a cryptanalysis by exhaustive search being 2^λ). There exists a guess

Name	n	n_1	n_2	t	k	λ
FLIP-530	530	42	128	8	9	80
FLIP-662	662	46	136	4	15	80
FLIP-1394	1394	82	224	8	16	128
FLIP-1704	1704	86	238	5	23	128

Table 12.1 n: total number of variables, n_1: linear part, n_2: quadratic part, t: number of triangular functions, k: degree of the triangular functions; λ: resulting security parameter.

and determine attack on a preliminary version of FLIP [490]. It is not efficient on the regular versions of FLIP. As checked in [839], FLIP is well suited for reducing the increase of the noise in homomorphic encryption, particularly for the 3rd generation, and even for the 2nd generation.

12.2.2 Boolean functions with restricted inputs

It was asserted in [839] that function f has sufficiently good cryptographic parameters (small balance bias, large algebraic degree, large nonlinearity, large algebraic immunity and fast algebraic immunity), but by definition in the filter permutator, the input to f has constant Hamming weight (equal to the weight of the secret key) while the study of f was made over the whole space $\mathbb{F}_2^n$. An important question has then been to see if the filtering function proposed in [839] maintains good behavior with respect to classical attacks when its domain is restricted. This has been established in a subsequent paper [306]. The work consisted in:

- reconsidering all classical attacks in the framework of Boolean functions restricted to some generic subset E of $\mathbb{F}_2^n$ (resulting from the specifications of the cryptosystem which uses them, and also possibly of the cryptanalysis performed on it, for instance a guess and determine attack), and in particular to a set of vectors of constant Hamming weight,
- studying how a generic function can contribute to the resistance against each attack in such framework,
- revisiting all related criteria, and studying constructions of functions satisfying the new versions of these criteria,
- studying specifically FLIP's function and seeing if it provides a good trade-off.

Set E may change when processing the algorithm or during the cryptanalysis. We may also want the function to be usable in a variety of situations. We are then interested in Boolean functions achieving good trade-off between all important cryptographic criteria, when they are restricted to each set E in some family $\mathcal{E}$. A particular family plays a special role for FLIP, as explained above:

$$\mathcal{E} = \{E_{n,1}, \dots, E_{n,n-1}\}, \text{ where } E_{n,k} = \{x \in \mathbb{F}_2^n;\ w_H(x) = k\}.$$

These sets are called *slices* in some papers (see *e.g.* [504, 505]). Note that symmetric functions (see Section 10.1, page 383), among which are balanced functions, bent functions and functions with optimal algebraic immunity, are constant on each set $E_{n,k}$ and lose then completely their desirable properties. We shall see other examples of similar degradation.
We recall below from [306] the general study of the most important cryptographic criteria in such general framework and how they particularize when E lives in class $\mathcal{E}$ above.
Note that for the FLIP cipher, Siegenthaler's correlation attack (see page 106) does not seem to apply. We do not study then the resilience of restricted Boolean functions, but such study could be useful for other ciphers and for the resistance to guess and determine attacks.

Remark. A probabilistic and asymptotic study has been made in [504, 507, 508] on the restrictions of Boolean functions to sets of inputs of fixed Hamming weight. We refer the reader interested to these papers (which also contain other interesting results); we deal here with fixed (generic) numbers of variables.
The nonlinearity of Boolean functions under non-uniform input distribution (which is another, possibly more general, way of not reducing the study of Boolean functions to the usual framework) has been also studied in [525], but the chosen distribution is binomial and does not fit with the framework of FLIP nor that of guess and determine attacks. □

Balance
We denote by $w_H(f)_k$ the Hamming weight of the restriction of f to $E_{n,k}$:

$$w_H(f)_k = |\{x \in \mathbb{F}_2^n, w_H(x) = k, f(x) = 1\}|.$$

For all $n \geq 2$, there exist balanced Boolean functions which are unbalanced on $E_{n,k}$ for every $k \in [1, n-1]$; these functions can even be $(n-1)$-resilient (and remain then balanced when at most $n-1$ of their variables are arbitrarily fixed): an example is the first elementary symmetric Boolean function $\sigma_1(x) = \bigoplus_{i=1}^n x_i = w_H(x)$ [mod 2]. But there exist, for some values of n, balanced functions which are balanced on each $E_{n,k}$; $k \in [1, n-1]$:

Definition 89 *We call* weightwise perfectly balanced *the functions which are balanced on any $E_{n,k}$ for $k = 1, \ldots, n-1$, that is, such that:*

$$\forall k \in [1, n-1],\ w_H(f)_k = \frac{\binom{n}{k}}{2}, \tag{12.11}$$

and such that $f(0_n) = 0$ and $f(1_n) = 1$.

The double condition "$f(0_n) = 0$ and $f(1_n) = 1$" makes f globally balanced and is not restrictive for balanced functions satisfying (12.11), up to the addition of constant 1. Of course, such functions can exist only if $\binom{n}{k}$ is even for every $k = 1, \ldots, n-1$, *i.e.* n is a power of 2.
Necessary conditions on the Algebraic Normal Form of Boolean functions to be weightwise perfectly balanced are given in [306]. A *secondary construction* based on the "indirect sum" (see Theorem 21, page 329) has been given in this same reference. We recall the proof.

Proposition 191 *[306] Let f, f' and g be weightwise perfectly balanced n-variable functions and let g' be any n-variable Boolean function, then*

$$h(x, y) = f(x) \oplus \prod_{i=1}^n x_i \oplus g(y) \oplus (f(x) \oplus f'(x))g'(y); \quad x, y \in \mathbb{F}_2^n$$

is a weightwise perfectly balanced $2n$-variable function.

Proof.

- If $w_H(x, y) = 0$ then $h(x, y) = 0$.
- If $k \in \{1, \ldots, n-1\}$, then, the set $\{(x, y) \in \mathbb{F}_2^{2n}; w_H(x, y) = k\}$ equals the disjoint union of the following sets:
  - $\{0_n\} \times \{y \in \mathbb{F}_2^n; w_H(y) = k\}$, on which $h(x, y)$ equals $g(y)$ and is then balanced;
  - $\{x \in \mathbb{F}_2^n; w_H(x) = i\} \times \{y\}$, where $1 \leq i \leq k$ and $w_H(y) = k - i$, on each of which $h(x, y)$ equals $f(x) \oplus g(y)$ if $g'(y) = 0$ and $f'(x) \oplus g(y)$ if $g'(y) = 1$; in both cases, it is balanced;
- If $k = n$, then the set $\{(x, y) \in \mathbb{F}_2^{2n}; w_H(x, y) = k\}$ equals the disjoint union of the following sets:

  - $\{(0_n, 1_n)\} \cup \{(1_n, 0_n)\}$, on which $h(x, y)$ equals respectively 1 and 0 and is then globally balanced;
  - $\{x \in \mathbb{F}_2^n; w_H(x) = i\} \times \{y\}$, where $1 \le i \le n-1$ and $w_H(y) = n - i$, on each of which $h(x, y)$ equals $f(x) \oplus g(y)$ if $g'(y) = 0$ and $f'(x) \oplus g(y)$ if $g'(y) = 1$; in both cases, it is balanced;
- If $k \in \{n+1, \ldots, 2n-1\}$, then the set $\{(x, y) \in \mathbb{F}_2^{2n}; w_H(x, y) = k\}$ equals the disjoint union of the following sets:
  - $\{1_n\} \times \{y \in \mathbb{F}_2^n; w_H(y) = k - n\}$, on which $h(x, y)$ equals $g(y)$ and is then balanced;
  - $\{x \in \mathbb{F}_2^n; w_H(x) = i\} \times \{y\}$, where $k - n + 1 \le i \le n - 1$ and $w_H(y) = k - i$, on each of which $h(x, y)$ equals $f(x) \oplus g(y)$ if $g'(y) = 0$ and $f'(x) \oplus g(y)$ if $g'(y) = 1$; in both cases, it is balanced;
- If $k = 2n$, then $w_H(x, y) = k$ is equivalent to $x = y = 1_n$, then $h(x, y) = 1$. $\square$

Noting that $f(x_1, x_2) = x_1$ is weightwise perfectly balanced, we can recursively build weightwise perfectly balanced Boolean functions of 2^ℓ variables, for all ℓ in $\mathbb{N}^*$. For instance, with $f = f'$, we obtain the following class:

$$f(x_1, x_2, \ldots, x_{2^\ell}) = \bigoplus_{a=1}^{\ell} \bigoplus_{i=1}^{2^{\ell-a}} \prod_{j=0}^{2^{a-1}-1} x_{i+j2^{\ell-a+1}}.$$

In [787] is proposed another construction based on the nice idea that if a Boolean function f on $\mathbb{F}_{2^n}$ satisfies $f(0_n) = 0$, $f(1_n) = 1$ and $f(x^2) = f(x) \oplus 1$ for all $x \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$, the function over $\mathbb{F}_2^n$ obtained by decomposing x over a normal basis is weightwise perfectly balanced. Indeed, the transformation $x \mapsto x^2$ results in a cyclic shift. These functions are invariant under a shift by two positions of the input (we already evoked in Section 10.2, page 392, the interest and risk of such rotation symmetry). The restricted nonlinearities of the functions (see definition below) are also studied.
In [1074] is given a large family of Boolean functions which are weightwise perfectly balanced if n is equal to a power of 2 and weightwise almost perfectly balanced (see below) otherwise, and which have optimal algebraic immunity and keep good algebraic immunity when restricted.
It is possible to extend the construction of Proposition 191 to get for all n *weightwise almost perfectly balanced* functions, satisfying by definition that for all $k \in [1, n-1]$, $w_H(f)_k$ equals $\frac{\binom{n}{k}}{2}$ when $\binom{n}{k}$ is even and $\frac{\binom{n}{k} \pm 1}{2}$ when $\binom{n}{k}$ is odd, see the proof and more results in [306].
The transformation $f \mapsto (g \to \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+g(x)})$, where g ranges over the set of all symmetric Boolean functions null at zero input, is introduced in this same reference. This transformation is similar to the Walsh transform, but with symmetric functions playing the role played normally by affine functions. It is shown that, for every n-variable Boolean function f, the quadratic mean of the sequence: $k \to \sum_{w_H(x)=k} (-1)^{f(x)}$ equals $\frac{1}{\sqrt{n+1}}$ times the quadratic mean of the sequence: $g \to \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)+g(x)}$.

Nonlinearity
The Hamming distance between a function f and a linear function $\ell_a(x) = a \cdot x$ on inputs ranging over some set E equals

$$d_E(f, \ell_a) = \frac{|E|}{2} - \frac{1}{2} \sum_{x \in E} (-1)^{f(x) \oplus a \cdot x}$$

(sum performed in $\mathbb{Z}$). The minimal distance $nl_E(f)$ between f and affine functions over E, that we shall call *nonlinearity with inputs in* E, equals then:

$$nl_E(f) = \frac{|E|}{2} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} \Big| \sum_{x \in E} (-1)^{f(x) \oplus a \cdot x} \Big|.$$

Since $\sum_{a \in \mathbb{F}_2^n} \left(\sum_{x \in E} (-1)^{f(x) \oplus a \cdot x} \right)^2 = 2^n\, |E|$, we have then:

$$nl_E(f) \leq \frac{|E|}{2} - \frac{\sqrt{|E|}}{2}. \tag{12.12}$$

For $E \subsetneq \mathbb{F}_2^n$, this bound is in general not achievable with equality (contrary to the unrestricted case for n even). In the case of $E = E_{n,k}$, it is never tight, except maybe for two particular pairs (n, k): $(50, 3)$ and $(50, 47)$, since Erdös showed that the binomial coefficient $\binom{n}{k}$ with $3 \leq k \leq n/2$ is the square of an integer for the single case $\binom{50}{3}$.

Bound (12.12) can be improved:

Proposition 192 *[306] Let E be a subset of $\mathbb{F}_2^n$ and f a Boolean function over E. Then:*

$$nl_E(f) \leq \frac{|E|}{2} - \frac{1}{2} \sqrt{|E| + \lambda},$$

where

$$\lambda = \max_{a \in \mathbb{F}_2^n; a \neq 0_n} \Big| \sum_{\substack{(x,y) \in E^2 \\ x+y=a}} (-1)^{f(x) \oplus f(y)} \Big|.$$

Proof. For every nonzero $a \in \mathbb{F}_2^n$, we have:

$$\begin{aligned} \sum_{b \in \mathbb{F}_2^n;\, a \cdot b = 0} \left(\sum_{x \in E} (-1)^{f(x) \oplus b \cdot x} \right)^2 &= \sum_{(x,y) \in E^2} (-1)^{f(x) \oplus f(y)} \sum_{b \in \mathbb{F}_2^n;\, a \cdot b = 0} (-1)^{b \cdot (x+y)} \\ &= 2^{n-1} \sum_{\substack{(x,y) \in E^2 \\ x+y \in \{0_n, a\}}} (-1)^{f(x) \oplus f(y)}, \end{aligned}$$

which implies:

$$\max_{b \in \mathbb{F}_2^n;\, a \cdot b = 0} \Big| \sum_{x \in E} (-1)^{f(x) \oplus a \cdot x} \Big| \geq \sqrt{|E| + \sum_{\substack{(x,y) \in E^2 \\ x+y=a}} (-1)^{f(x) \oplus f(y)}}.$$

If $\sum_{\substack{(x,y)\in E^2\\ x+y=a}}(-1)^{f(x)\oplus f(y)}$ is negative, then we can apply this inequality to function $f'(x)=f(x)\oplus v\cdot x$ where $v\cdot a=1$; we have $\sum_{\substack{(x,y)\in E^2\\ x+y=a}}(-1)^{f'(x)\oplus f'(y)} = -\sum_{\substack{(x,y)\in E^2\\ x+y=a}}(-1)^{f(x)\oplus f(y)}$. Relation (3.1), page 99, completes the proof. □

Note that this result applied for $E=\mathbb{F}_2^n$ proves again that the derivatives of bent functions are all balanced.

More observations are made for $E=E_{n,k}$ in [306] and the case of direct sums is studied. Proposition 192 is a particular case of a more general and slightly more complex result given in this same reference, which has been generalized in [878], where the consequences are studied in detail.

The maximal value of $nl_E(f)$ is the covering radius of the punctured first order Reed-Muller code obtained by deleting all the coordinates whose indices lie outside E and is then at least $\frac{d}{2}$, where d is the minimum distance of this code. For $E=E_{n,k}$, this minimum distance has been determined by Dumer and Kapralova [488]; we have:

- for $0\le k<n/2$, $d=\binom{n-1}{k-1}$
- for $k=n/2$, $d=\binom{n-2}{k-2}$
- for $n/2<k\le n-1$, $d=\binom{n-1}{k}$
- and for $k=n$, $d=1$.

The maximal value of $nl_{E_{n,k}}(f)$ is then nonzero except for particular values of k.

Nevertheless, fixing the input Hamming weight of some functions may deteriorate their nonlinearity in an extreme way: for every n, there exists f of large nonlinearity such that $nl_k(f)=0, \forall k=0,\dots,n$. For instance, the (bent) elementary symmetric function σ_2 (n even) has this latter property (like any other symmetric function). We leave open the determination of all the bent n-variable Boolean functions such that $nl_k(f)=0, \forall k=0,\dots,n$. Those which are quadratic have been studied in [306], but the proof was incomplete, because the third item of the next technical lemma was viewed as straightforward while it is not.

Lemma 13 *Let n be any positive integer.*
1. The n-variable Boolean functions such that $nl_k(f)=0$ for every $k=1,\dots,n$ are the functions of the form

$$f(x)=\bigoplus_{i=1}^{n} x_i\,\varphi_i(x)\oplus\varphi_0(x), \tag{12.13}$$

where $\varphi_0,\varphi_1,\dots,\varphi_n$ are symmetric Boolean functions.
2. Up to the addition of an affine function, such function equals

$$\bigoplus_{i=1}^{n}\ell_i(x)\sigma_i(x), \tag{12.14}$$

where σ_i is the i-th elementary symmetric Boolean function and where the ℓ_i's are all affine.

3. If $n \geq 6$, then f is quadratic if and only if, up to the addition of an affine function, we have:

$$f(x) = \ell(x)\,\sigma_1(x) \oplus \epsilon\,\sigma_2(x), \tag{12.15}$$

where $\epsilon \in \mathbb{F}_2$, and $\ell(x)$ is a linear function.

Proof. 1. Any function of the form (12.13) coincides with an affine function on every $E_{n,k}$ since each symmetric function is constant on it, and conversely, if a Boolean function f coincides on every $E_{n,k}$ with an affine function, say with $\ell_k(x) = \sum_{i \in I_k} x_i \oplus \epsilon_k$, then defining, for every $x \in E_{n,k}$ and every $i = 1, \ldots, n$, that $\varphi_i(x) = 1$ if $i \in I_k$ and $\varphi_i(x) = 0$ otherwise, and $\varphi_0(x) = \epsilon_k$, we have $f(x) = \bigoplus_{i=1}^n x_i\,\varphi_i(x) \oplus \varphi_0(x)$, where the φ_i's are symmetric functions.
2. Expressing each function $\varphi_0, \ldots, \varphi_n$ by means of the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$, we obtain, up to the addition of an affine function, $f(x) = \bigoplus_{i=1}^n \ell_i(x)\sigma_i(x)$, where the ℓ_i's are all affine.
3. All the terms obtained after expansion of $\bigoplus_{i=3}^n \ell_i(x)\sigma_i(x)$ in (12.14) have degree at least 3 and, using the uniqueness of the ANF of a Boolean function, f is quadratic if and only if all those whose degree is at least 4 cancel and those of degree 3 are cancelled by those from $\ell_2(x)\sigma_2(x)$ (the expression of the function can then be taken equal to the quadratic part of (12.14) expanded). Let us translate this into explicit conditions on (12.14).
For every $i, j = 1, \ldots, n$, we have $x_j\,\sigma_i(x) = (\bigoplus_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = i, j \in I}} x^I) \oplus (\bigoplus_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = i+1, j \in I}} x^I)$. We deduce that, writing $\ell_i(x) = \bigoplus_{j \in J_i} x_j \oplus \epsilon_i$, we have, for $i < n$:

$$\ell_i(x)\,\sigma_i(x) = \left(\bigoplus_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = i, |I \cap J_i|\ [\text{mod } 2] = \epsilon_i \oplus 1}} x^I \right) \oplus \left(\bigoplus_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = i+1, |I \cap J_i|\ \text{odd}}} x^I \right), \tag{12.16}$$

since each x_j, $j \in J_i$, contributes once for each x^I such that $|I| = i$ and $j \in I$ and once for each x^I such that $|I| = i+1$ and $j \in I$. And for $i = n$, $\ell_n(x)\,\sigma_n(x)$ equals $\sigma_n(x)$ if $|J_n|$ [mod 2]$= \epsilon_n \oplus 1$ and is zero otherwise. Hence:

- for $i = n$, we have $\ell_n(x)\,\sigma_n(x) = (|J_n|\,[\text{mod } 2] \oplus \epsilon_n)\,\sigma_n(x)$,
- for $1 \leq i \leq n-1$, specifying the values of the two sub-sums in (12.16), we have:
  - if $0 < |J_i| < n$, then $\ell_i(x)\,\sigma_i(x)$ contains terms of degree i but not all of them (since both parities can be achieved by $|I \cap J_i|$ when $|I| = i$) and terms of degree $i+1$ but, if $i \leq n-2$, not all of them as well, and if $i = n-1$ the part in $\sigma_{i+1} = \sigma_n$ has coefficient $|J_{n-1}|\,[\text{mod } 2]$,
  - if $|J_i| = 0$, then $\ell_i(x)\,\sigma_i(x) = \epsilon_i\,\sigma_i(x)$ (note that, for $i = n-1$, the coefficient of σ_{i+1}, which is then 0, takes the same value $|J_{n-1}|\,[\text{mod } 2]$, obtained above for $0 < |J_i| < n$),
  - if $|J_i| = n$, then $\ell_i(x)\,\sigma_i(x) = (\sigma_1(x) \oplus \epsilon_i)\sigma_i(x) = (i\ [\text{mod } 2] \oplus \epsilon_i)\,\sigma_i(x) \oplus (i+1\ [\text{mod } 2])\sigma_{i+1}(x)$ (and the coefficient $i+1$ [mod 2] of σ_n for $i = n-1$ matches the value $|J_{n-1}|\,[\text{mod } 2]$ above as well).

It cannot then happen, when the function is quadratic, that $0 < |J_i| < n$ for some value of $i \geq 3$ and $|J_i| = 0$ or $|J_i| = n$ for another value of $i \geq 3$.

Then f is quadratic if and only if we have $\epsilon_n = (|J_n| + |J_{n-1}|)$ [mod 2] and, addressing first the two latter cases above and then the first case:

- either, for every $i = 2, \ldots, n-1$, we have $|J_i| = \eta_i\, n$ with $\eta_i \in \{0,1\}$ and for $i \geq 3$, $\epsilon_i = (\eta_i + \eta_{i-1})\, i$ [mod 2],
- or, for every $i = 3, \ldots, n-1$, we have $0 < |J_i| < n$ and the two following sets $\{I \subseteq \{1,\ldots,n\}; |I| = i \text{ and } |I \cap J_i| \text{ [mod 2]} = \epsilon_i \oplus 1\}$ and $\{I \subseteq \{1,\ldots,n\}; |I| = i \text{ and } |I \cap J_{i-1}| \text{ odd}\}$ are equal. Denoting by z_i the vector of $\mathbb{F}_2^n$ of support J_i, by $B_{\geq 3}$ the set of vectors of $\mathbb{F}_2^n$ of Hamming weight at least 3, by E^0 (rather than $E^\perp$) the orthogonal of an $\mathbb{F}_2$-vector space E and by E^1 its complement, the condition writes: $\{0_n, z_i\}^{\epsilon_i \oplus 1} \cap B_{\geq 3} = \{0_n, z_{i-1}\}^1 \cap B_{\geq 3}$, or equivalently: $\{0_n, z_i\}^{\epsilon_i} \cap B_{\geq 3} = \{0_n, z_{i-1}\}^0 \cap B_{\geq 3}$. If $n \geq 6$ then the linear space $\{0_n, z_{i-1}\}^0$ contains elements of Hamming weight at least 5 and every of its elements of weight at most 2 is then the sum of two elements of $\{0_n, z_{i-1}\}^0 \cap B_{\geq 3}$ (one of weight at least 5 and one of weight at least 3); hence the vector space $\langle \{0_n, z_{i-1}\}^0 \cap B_{\geq 3}\rangle$ spanned by $\{0_n, z_{i-1}\}^0 \cap B_{\geq 3}$ equals $\{0_n, z_{i-1}\}^0$; the same is true for $\{0_n, z_i\}^{\epsilon_i} \cap B_{\geq 3}$ if $\epsilon_i = 0$, in which case we have $z_i = z_{i-1}$, that is, $J_i = J_{i-1}$, and if $\epsilon_i = 1$ then $\{0_n, z_{i-1}\}^0$ equals $\langle \{0_n, z_i\}^{\epsilon_i} \cap B_{\geq 3}\rangle$ which contains $\{0_n, z_i\}^0$ for the same reasons as above and can not be reduced to a hyperplane, and is then equal to $\mathbb{F}_2^n$, a contradiction.

Summarizing, we have, up to the addition of an affine function:

- we are in the first case above and $f(x)$ equals the quadratic part of a function of the form $\ell(x)\,\sigma_1(x) \oplus (\eta\,\sigma_1(x) \oplus \epsilon)\,\sigma_2(x)$, where $\epsilon, \eta \in \mathbb{F}_2$, and $\ell(x)$ is a linear function. Since $\eta\,\sigma_1(x)\,\sigma_2(x) = \eta\,\sigma_3(x)$, we can take $\eta = 0$ and we obtain then (12.15).
- or we are in the second case above with $z_2 = z_3 = \cdots = z_{n-1}$ and $\epsilon_3 = \cdots = \epsilon_{n-1} = 0$, and $f(x)$ is the quadratic part of $\ell(x)\sigma_1(x) \oplus \epsilon\,\sigma_2(x) \oplus \ell'(x)(\bigoplus_{i=2}^{n-1} \sigma_i(x)) \oplus \epsilon_n \sigma_n(x)$, where ℓ and ℓ' are linear and $\epsilon_n = \ell'(1)$, that is, $\ell(x)\sigma_1(x) \oplus \epsilon\,\sigma_2(x) \oplus \ell'(x)(\sigma_1(x) \oplus 1 \oplus \delta_0(x)) = \ell(x)\sigma_1(x) \oplus \epsilon\,\sigma_2(x) \oplus \ell'(x)(\sigma_1(x) \oplus 1)$ and this second case happens then to be equivalent, up to the addition of an affine function, to a particular case of the first. □

Remark. The same proof shows that a function (12.14) has algebraic degree at most k if and only if all the terms of degree at least $k+2$ in $\bigoplus_{i=k+1}^{n} \ell_i(x)\sigma_i(x)$ cancel and those of degree $k+1$ are cancelled by those from $\ell_k(x)\sigma_k(x)$. The expression of the function equals then the part of degree at most k in (12.14), which is the part of degree at most k in $\bigoplus_{i=1}^{k-1} \ell_i(x)\sigma_i(x) \oplus (\eta\,\sigma_1(x) \oplus \epsilon)\sigma_k(x)$, where $\ell_i(x)$ is affine for every $i = 1, \ldots k-1$, and $\epsilon, \eta \in \mathbb{F}_2$. Since the degree k part of $\sigma_1(x)\sigma_k(x)$ equals $\sigma_k(x)$ if k is odd and equals 0 otherwise, we obtain $f(x) = \bigoplus_{i=1}^{k-1} \ell_i(x)\sigma_i(x) \oplus \epsilon\,\sigma_k(x)$, where the ℓ_i's are affine and $\epsilon \in \mathbb{F}_2$. □

Proposition 193 *[306] For every even $n \geq 6$, the quadratic bent functions satisfying $nl_k(f) = 0$ for every k are, up to the addition of an affine function, the functions $f(x) = \ell(x)\,\sigma_1(x) \oplus \sigma_2(x)$, where ℓ is linear and $\ell(1_n) = 0$.*

Proof. The symplectic form $(x, y) \to f(x + y) \oplus f(x) \oplus f(y) \oplus f(0_n)$ associated with the function in (12.15) equals:

$$\ell(x)\sigma_1(y) \oplus \ell(y)\sigma_1(x) \oplus \epsilon\,\Big(\bigoplus_{1\leq j\neq i\leq n} x_j y_i\Big).$$

Denoting $\ell(x) = \bigoplus_{i=1}^{n} l_i x_i$, the kernel

$$E = \{x \in F_2^n; \forall y \in F_2^n, f(x + y) \oplus f(x) \oplus f(y) \oplus f(0_n) = 0\}$$

of this symplectic form is the $\mathbb{F}_2$-vector space of the solutions of the equations:

$$(L_i) : \ell(x) \oplus l_i\,\Big(\bigoplus_{j=1}^{n} x_j\Big) \oplus \epsilon\,\Big(\bigoplus_{j\neq i} x_j\Big) = 0, \quad i = 1, \ldots, n.$$

If $\epsilon = 0$, then since the hyperplane of equation $\bigoplus_{j=1}^{n} x_j = 0$ has non-trivial intersection with the kernel of ℓ (because $n \geq 3$), and since every element in this intersection satisfies all equations, f cannot be bent. We assume then that $\epsilon = 1$. For all those $x \in E$ such that $\bigoplus_{j=1}^{n} x_j = 0$, the equation:

$$(L_i + L_{i'}) : (l_i \oplus l_{i'})\left(\bigoplus_{j=1}^{n} x_j\right) \oplus (x_i \oplus x_{i'}) = 0,$$

valid for every $i \neq i'$, results in $x_i \oplus x_{i'} = 0$ and implies that either all x_i's are null (in which case (L_i) is of course satisfied), or all are equal to 1, in which case (L_i) becomes (since n is even) $\ell(1_n) = 1$. Hence, $\epsilon = 1$ and $\ell(1_n) = 0$ is a necessary condition for the function to be bent. It is also sufficient, because for all $x \in E$ such that $\bigoplus_{j=1}^{n} x_j = 1$, according to Equation $(L_i + L_{i'})$ again, all those x_i such that $l_i = 0$ are equal to some value $\eta \in \mathbb{F}_2$ and all those x_i such that $l_i = 1$ are equal to another value, which can be only $\eta \oplus 1$, since $\bigoplus_{j=1}^{n} x_j = 1$, and the number of i such that $l_i = 1$ is odd, that is, $\ell(1_n) = 1$, a contradiction. This completes the proof. □

It is shown in [306] that for the direct sum of any n-variable function f and any m-variable function g, we have:

$$nl_{E_{n+m,k}}(f \oplus g) \geq \sum_{i=0}^{k} \binom{n}{i} nl_{E_{m,k-i}}(g) + \sum_{i=0}^{k} nl_{E_{n,i}}(f)\left(\binom{m}{k-i} - 2nl_{E_{m,k-i}}(g)\right).$$

We refer to this reference for the proof.

Algebraic immunity

The majority function being, as every symmetric function, constant on all inputs of the same Hamming weight, and having optimal algebraic immunity, it is an extreme example of degradation of the algebraic immunity when inputs are restricted to $E_{n,k}$. We call *algebraic immunity with inputs in* E of a given Boolean function f the non-negative integer:

$$\begin{aligned} AI_E(f) = &\quad \min\{\max(d_{alg}(g), d_{alg}((fg)_{|E})); g \not\equiv 0 \text{ on } E\} \\ = &\min\{d_{alg}(g); (fg)_{|E} \equiv 0 \text{ or } ((f \oplus 1)g)_{|E} \equiv 0; g \not\equiv 0 \text{ on } E\}, \end{aligned}$$

where $d_{alg}((fg)_{|E})$ equals the minimum algebraic degree of Boolean functions over $\mathbb{F}_2^n$ which coincide with fg over E, and we call annihilators of f over E the functions g such that $(fg)_{|E} \equiv 0$.
The equality between these two minima is shown easily: if g and $h = fg$ achieving the former minimum coincide on E, we have then $g \oplus h = g(f \oplus 1) = 0$ on E, where g has nonzero restriction to E, and if they do not, then after multiplication of equality $h = fg$ by f, we have $(g \oplus h)f = 0$, where $g \oplus h$ has nonzero restriction to E; this proves that the former minimum is bounded below by the latter. The inequality in the other order is still more obvious since the set over which the latter minimum is taken is a subset of the set over which the former is taken.

Remark. Taking the restriction to E may (often) decrease the algebraic immunity (we shall see examples below) since it weakens the condition on g to be an annihilator of f or of $f \oplus 1$, but it may also increase the algebraic immunity, because it strengthens the condition on g to be nonzero. Take for instance an $(n-1)$-variable function f of algebraic immunity at least 2, and define $f'(x,0) = f(x)$, $f'(x,1) = 0$, for every $x \in \mathbb{F}_2^{n-1}$. Then the indicator of $\mathbb{F}_2^{n-1} \times \{1\}$ being an annihilator, we have $AI(f') = 1$, while $AI_{\mathbb{F}_2^{n-1}\times\{0\}}(f') = AI(f) \geq 2$. Moreover, for the same reason, we have $AI_k(f') = 1$ for every $k \in \{1, \ldots, n-1\}$, while for some functions f, we can have $AI_k(f) \geq 2$ for some k. □

The upper bound of Proposition 26, page 112, has been adapted to the algebraic immunity with inputs in E.

Proposition 194 *[306] Let $E \subseteq \mathbb{F}_2^n$ and let f be defined over E. Let d and e be non-negative integers. Let $M_{d,E}$ be the $(\sum_{i=0}^{d} \binom{n}{i}) \times |E|$ matrix whose term at row indexed by $u \in \mathbb{F}_2^n$ such that $w_H(u) \leq d$, and at column indexed by $x \in E$, equals $\prod_{i=1}^{n} x_i^{u_i}$.*
If $rank(M_{d,E}) + rank(M_{e,E}) > |E|$, then there exist two Boolean functions g and h on E, such that g is not identically null on E and:

$$d_{alg}(g) \leq e,\ d_{alg}(h) \leq d \text{ and } fg = h \text{ on } E.$$

We have then:

$$AI_E(f) \leq \min\left\{e;\ rank(M_{e,E}) > \frac{|E|}{2}\right\}. \tag{12.17}$$

Proof. By definition, $rank(M_{d,E})$ equals the maximum size of a free family $\mathcal{F}_d$ of restrictions to E of monomials x^u of algebraic degree $w_H(u) \leq d$ (such family generates the restrictions to E of the Boolean functions of algebraic degree at most d) and $|E|$ is the dimension of the $\mathbb{F}_2$-vector space of Boolean functions over E. If $rank(M_{d,E})+rank(M_{e,E}) > |E|$, the elements of $\mathcal{F}_d$ and the products between f and the elements of a maximum size free family $\mathcal{F}_e$ are necessarily $\mathbb{F}_2$-linearly dependent. Gathering the part of this linear combination dealing with the elements of $\mathcal{F}_d$ and those dealing with $\mathcal{F}_e f$, this linear dependence gives two functions h and g of degrees at most d and e, respectively, such that $(fg)_{|_E} = h_{|_E}$ and $(g_{|_E}, h_{|_E}) \not\equiv (0,0)$, *i.e.* $g_{|_E} \not\equiv 0$. Inequality (12.17) is then straightforward by taking $d = e$. □

In the case of fixed input weights, a recurring relation on the rank of $M_{d,E_{n,k}}$ has been found in [306] (we refer to this reference for the proof, which is a little too long for being given), where has been deduced that this rank equals:

$$\binom{n}{\min(d,k,n-k)}.$$

For $k \leq n/2$, Relation (12.17) implies then that, for every n-variable Boolean function f:

$$AI_{E_{n,k}}(f) \leq \min\left\{e;\ 2\binom{n}{e} > \binom{n}{k}\right\}.$$

It is deduced in this same reference (by technical observations dealing with binomial coefficients) that the best possible algebraic immunity of a function with constrained input Hamming weight is lower than for unconstrained functions.
A lower bound exists on the algebraic immunity of the direct sum of two Boolean functions. We recall the proof from [306]:

Proposition 195 *Let $(f\oplus g)(x,y) = f(x)\oplus g(y)$, $x \in \mathbb{F}_2^n$, $y \in \mathbb{F}_2^m$, where $n \leq m$. Let k be such that $n \leq k \leq m$. Then the following relation holds:*

$$AI_k(f \oplus g) \geq AI(f) - d_{alg}(g). \tag{12.18}$$

Proof. Let $h(x,y)$ be a nonzero annihilator of $f\oplus g$ over $E_{n+m,k}$. Let $(a,b) \in \mathbb{F}_2^{n+m}$ have Hamming weight k and be such that $h(a,b) = 1$. Since (a,b) has Hamming weight k with $n \leq k \leq m$, we may, up to changing the order of the coordinates of b (and without loss of generality), assume that, for every $j = 1,\dots,n$, we have $b_j = a_j \oplus 1$ and for every $j = n+1,\dots k$, we have $b_j = 1$ (so that for every $j = k+1,\dots m$, we have $b_j = 0$). This is possible since $k \geq n$ and in all cases, the last 1 in (a,b) is at position $2n + (k-n) = n+k \leq n+m$. We define the following affine function over $\mathbb{F}_2^n$:

$$L(x) = (x_1 \oplus 1, x_2 \oplus 1, \dots, x_n \oplus 1, 1, \dots, 1, 0, \dots, 0),$$

where the length of the part "$1,\dots,1$" equals $k-n$. We have $L(a) = b$. The n-variable function $h(x, L(x))$ is then non-zero and is an annihilator of $f(x) \oplus g(L(x))$ over $\mathbb{F}_2^n$. If $g(b) = 0$, then function $h(x,L(x))\,(g(L(x)) \oplus 1)$ is a non-zero annihilator of f and has algebraic degree at most $d_{alg}(h) + d_{alg}(g)$; then we have

$d_{alg}(h) + d_{alg}(g) \geq AI(f)$. If $g(b) = 1$, then by applying the same reasoning to $f \oplus 1$ instead of f and $g \oplus 1$ instead of g, we have $d_{alg}(h) + d_{alg}(g) \geq AI(f)$. If $h(x, y)$ is a non-null annihilator of $f \oplus g \oplus 1$ over $E_{n+m,k}$, we have the same conclusion by replacing f by $f \oplus 1$ or g by $g \oplus 1$. This completes the proof. □

Bound (12.18) may seem loose because of the presence of $-d_{alg}(g)$, but it is not. Let us see with an example (given in [306]) that making the direct sum with some non-constant Boolean functions g may indeed contribute to a decrease of the algebraic immunity over inputs of fixed Hamming weight: take n odd, $f(x) = 1 \oplus maj(x)$ where maj is the majority function over n variables (which has optimal algebraic immunity $\frac{n+1}{2}$) and $g(y) = maj(y)$ over n variables as well. Then the $2n$-variable function $f \oplus g$ is null at fixed input weight n, because if $w_H(x) + w_H(y) = n$, then either $w_H(x) \leq \frac{n-1}{2}$ and $w_H(y) \geq \frac{n+1}{2}$, and we have then $f(x) = g(y) = 1$, or $w_H(x) \geq \frac{n+1}{2}$ and $w_H(y) \leq \frac{n-1}{2}$, and we have then $f(x) = g(y) = 0$. The algebraic immunity with input weight n equals then 0.
Bound (12.18) also shows that, if $k \geq n$, then taking $g = 0$ (*i.e.* adding $m \geq k$ virtual variables to f) gives $AI_k(f \oplus 0) \geq AI(f)$; this latter bound is tight (take for f a function whose algebraic immunity equals its algebraic degree). Another example showing the tightness of Bound (12.18) when $d_{alg}(g) = 1$ is also given in this same reference.

It is shown in [306] that for the direct sum of any n-variable function f and any m-variable function g, we have:

$$AI_k(f \oplus g) \geq \min_{0 \leq \ell \leq k} (\max[AI_\ell(f), AI_{k-\ell}(g)]).$$

We refer to this reference for the proof.

Impact of Boolean functions with restricted input on FLIP

Balancedness

For given k, let $p_k = Pr_{x \in E_{n,k}}[f(x) = 1] = \frac{1}{2} - \epsilon_k$. The amount of data needed for an attacker to detect the bias ϵ_k is equal to $\frac{1}{{\epsilon_k}^2}$. In the case of the FLIP cipher, we have $k = \frac{n}{2}$. It has been checked in [306] that the bias is not exploitable, even in the case of guess and determine attacks.

Nonlinearity

In a fast correlation attack (approximating the keystream equations by linear approximation of the filtering function and using a decoding method), the attacker builds a linear system which can be seen as an instance of the learning parity with noise (LPN) problem [99], where the noise parameter is $\eta_k = \frac{nl_{E_{n,k}}}{\binom{n}{k}}$. The data complexity of the attack is $\mathcal{O}(2^h \eta_k^{-2(r+1)})$ where the parameters h and r depend on the algorithm used and on the number of variables. It could be shown in [306] that $nl_{E_{n,\frac{n}{2}}}(f)$ is large enough for allowing the FLIP cipher to resist the fast correlation attack, combined or not with a guess and determine attack.

Algebraic immunity
Proposition 195 has allowed to bound $nl_{E_{n,\frac{n}{2}}}(f)$ from below, with the help of the next proposition.

Proposition 196 *Let $f(x)$ be an n-variable Boolean function such that:*

$$\forall x \in \mathbb{F}_2^{n-2} \; f(0,0,x) = f(0,1,x) = f(1,0,x).$$

Let $f'(X, x_3, \cdots, x_n)$ be the Boolean function in $n-1$ variables defined by :

$$\forall x \in \mathbb{F}_2^{n-2} \; f'(1,x) = f(1,1,x) \text{ and } f'(0,x) = f(0,0,x).$$

If $AI(f) \leq d$ then $AI(f') \leq d$.

This proposition, whose proof can be found in [306], implies that if f is the direct sum of d monomials and if, for every $i \in [k, d]$, f has a monomial of degree i, where k is the smallest degree of all monomials of f, then $AI(f) = d$. This allowed to show that all instances of the FLIP cipher resist the algebraic attack. Determining whether they resist the algebraic attack combined with the guess and determine attack is open. An interesting point observed in [306] is that the high number of triangular functions used in FLIP to prevent the guess and determine attack combined with fast algebraic attack may reduce the algebraic immunity and there is then a trade-off to be found.

12.3 Local pseudorandom generators and related criteria on Boolean functions

Recall that the principle of *pseudorandom generators* is to allow expanding short random strings (like private keys), called seeds, into pseudorandom strings, whose length is significantly larger (say, polynomial, that is, in $\mathcal{O}(n^s)$ where n is the length of the seed, with $s > 1$). They are called *local* if each output bit depends on a constant number d of input bits. This property, related to the design of cryptographic primitives that can be evaluated in constant time while using polynomially many cores, allows a wide variety of applications. The only known example of a local pseudorandom generator is the so-called Goldreich's PRG, which applies a simple d-variable Boolean function (Goldreich calls it a d-ary predicate) to public random subsets of size d of the seed.

12.3.1 The Goldreich pseudorandom generator

In [541] has been proposed a *one-way function* (OWF), which is an asymptotic construction aiming at being a "simplest possible function that we do not know how to invert efficiently". This OWF is built as a *random local function* (see below) and has been later modified into a *local pseudorandom generator* [543] with nice applications (making possible, with constant computational overhead, a secure two-party computation of any Boolean circuit, and having other applications, see [636, 393]).

Let n and m be two integers, let $(S_1, \ldots, S_m)$ be a list of m subsets of $\{1, \ldots, n\}$ of size d, where d is small compared to n (it can be logarithmic in n or even constant), and let f be a Boolean function in d variables (the so-called predicate). The corresponding *Goldreich's function* $G : \mathbb{F}_2^n \mapsto \mathbb{F}_2^m$ is defined as $G(x) = f(S_1(x)), f(S_2(x)), \ldots, f(S_m(x))$ for every $x \in \mathbb{F}_2^n$, where $S_i(x)$ is a vector made of those bits of x indexed by S_i. Originally, m was of size comparable to n. The one-way property of the function was related to the choice of the predicate and to the fact that $(S_1, \ldots, S_m)$ was an *expander graph*[31] (which corresponds to saying in the particular framework we are in, that for some k, every k subsets cover $k + \Omega(n)$ elements of $\{1, \ldots, n\}$ (which happens to be the case with probability tending to 1 for subsets drawn at random).

Goldreich's pseudorandom generator proposed later takes a larger value of m, polynomial in n. The integer d is called the locality of the PRG and many works have focused on a framework called "polynomial-stretch local" in which d is constant and $m = n^s$ where $s > 1$ (s is called the *stretch*). For these polynomial-stretch local PRG, the security is considered asymptotically, relative to the class of polynomial adversaries as linear distinguishers. They are conjectured secure under some necessary conditions on the predicate and on the subsets S_i (see the survey [22] and, for a faster overview, [393, Section 1.2]). For instance, to avoid an attack by Gaussian elimination, the predicate f must be non-linear in a basic sense. Moreover, the higher is the algebraic degree, the better, since a random local function with a predicate of algebraic degree s cannot be pseudorandom for a stretch as large as s. The predicate must also be such that, when fixing some number r of input bits to f, its algebraic degree remains large. Note that this has a close relation with algebraic immunity (since if the algebraic degree of f falls down to k when r input bits are fixed, we know that $AI(f) \leq r + k$) and algebraic attacks have been actually further investigated, and the algebraic immunity $AI(f)$ (sometimes called rational degree among people working on Goldreich's PRG) happens to play a direct role and should be large enough (larger than s). There is also an attack [889] when the output of the function is correlated with a number of its input bits smaller than or equal to $\frac{s}{2}$, and f should then be *resilient* with a sufficient order (all the more since this attack has been extended to cases where $m \geq \lambda n$ for large λ); in [915] has been shown that f should be (at least) 2-resilient. The very simple 5-variable function $f(x) = x_1 \oplus x_2 \oplus x_3 \oplus x_4 x_5$ (whose structure is similar to that of the FLIP function, but simpler and with considerably smaller parameters) has been proved resisting some attacks based on $\mathbb{F}_2$-linear distinguishers when $m \in \mathcal{O}(n^s)$ with $s < 1.5$ (note that its algebraic degree, resiliency order and algebraic immunity are all three equal to 2), but of course does not resist the attacks evoked above for larger stretches nor resists

[31] A hypergraph whose hyper-edges have size d is said to be (α, β)-expanding if, for any choice of $k \leq \alpha m$ edges, their union has size at least $\beta k d$.

algebraic attacks. A general structure has been proposed in [23] for predicates: the direct sum of the full linear function $\bigoplus_{i=1}^{k} x_i$ and of the *majority function* (see page 366) in $n-k$ variables. No attack is known on such functions when $k \geq 2s$ and $\lceil \frac{n-k}{2} \rceil \geq s$.
Of course, constraints also exist on the choice of the subsets $(S_1, \ldots, S_m)$, more precisely on the hypergraph (see page 89) given by them, which needs to be sufficiently expanding, but in practice, an overwhelming proportion of hypergraphs are sufficiently expanding.

As we can see, Goldreich's PRG gives one more example (and an important one) where the classical notions on Boolean functions for cryptography play central roles in frameworks different from stream or block ciphers. An open question is asked in this context by Applebaum and Lovett in [23]: given two positive integers e and k, what is the smallest number of variables (the reference writes "the smallest arity") for which there exists a Boolean function (a predicate) of algebraic immunity (of rational degree) at least e and of resiliency order at least k? The parameters of the direct sum of the full linear function $\bigoplus_{i=1}^{k} x_i$ and of the majority function in $2e-1$ variables show that this number is at most $k+2e-1$ as observed in [838] (Applebaum and Lovett give the bound $k+2e$ and propose as example the direct sum of function $\bigoplus_{i=1}^{k} x_i$ and of the majority function in $2e$ variables; note that this very function is not k-resilient, at least in the usual sense, because the majority function in $2e$ variables is not balanced, but they probably think of a majority function modified into a balanced function with the same AI). As we can see, the upper bound $k+2e$ is not optimal and even $k+2e-1$ may not be optimal.
This problem is clearly related to another open question: is there, for $k \leq n-2$, an upper bound on the algebraic immunity of k-resilient functions which would be sharper than $\min(n-k-1, \lceil \frac{n}{2} \rceil)$ (implied by the *Siegenthaler bound* and the *Courtois-Meier bound*)? We specify $k \leq n-2$ because for $k = n-1$, Siegenthealer's bound gives 1 and not 0 (and the two $(n-1)$-resilient functions equal to the full linear n-variable function and its complement have algebraic immunity 1). For very large values of k, the reply to the latter open question is probably no (for instance, for $k = n-2$, it is clearly no, and for $k = n-3$, there exist $(n-3)$-resilient functions of algebraic immunity 2, which are easy to obtain with Maiorana-McFarland's construction; an example is the direct sum of the full linear $(n-2)$-variable function and of the 2-variable majority function $x_{n-1}x_n$). Note that many infinite classes of 1-resilient functions of optimal algebraic immunity have been found in even numbers of variables, but as far as we know, none has been found being 2-resilient, nor 1-resilient in odd numbers of variables (recall that, if $f(x)$ is a 1-resilient function with optimal AI in odd number n of variables, then $f(x) \oplus x_{n+1}$ is a 2-resilient function with optimal AI in $n+1$ variables). So already for $k=2$ the question is open.

12.4 The Gowers norm on pseudo-Boolean functions

The Gowers uniformity norm has been introduced in 2001 in the paper [568] which provided a new proof of a result originally shown by van der Waerden, on the existence, for any positive integers k, r, of a positive integer M such that, in any r-partition of $\{1, 2, \ldots, M\}$, there exists at least one class containing an arithmetic progression of length k. We shall not try to summarize the content of this rather dense 129-page long paper (available on the internet), in which the norm was defined for functions over $\mathbb{Z}/N\mathbb{Z}$. The Gowers norm can also be expressed, as in [571], in terms of pseudo-Boolean functions (see below). Since 2001, the Gowers norm has been intensively studied and applied in additive combinatorics and in the probabilistic testing of specific properties of Boolean functions (knowing only a few of their values, see the thesis [361], see also [13] where are addressed the Reed-Muller codes and [542]). When applied to the sign function of a Boolean function f, it deals, as we shall see, with the higher-order derivatives of f (whose definition and notation have been given at page 56). It results in a measure related to the higher-order nonlinearity. We shall see with Corollary 32 below that smaller is the Gowers U_k norm of f, higher is then the contribution of f to the resistance to attacks by approximations by Boolean functions of algebraic degree at most $k-1$ (these attacks are listed after Definition 20, page 102). The definition of the Gowers U_k norm, valid for all pseudo-Boolean functions, is as follows:

Definition 90 *[568, 569, 571] Let k, n be positive integers such that $k < n$. Let $\varphi : \mathbb{F}_2^n \mapsto \mathbb{R}$ be a pseudo-Boolean function. The k-th order Gowers uniformity norm of φ equals:*

$$||\varphi||_{U_k} = \left(\mathbb{E}_{x,x_1,\ldots,x_k\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k\}}\varphi\left(x+\sum_{i\in S}x_i\right)\right]\right)^{\frac{1}{2^k}},$$

*where $\mathbb{E}_{x,x_1,\ldots,x_k\in\mathbb{F}_2^n}$ is the notation for arithmetic mean (*i.e.* for expectation in uniform probability).*

Note that, by considering separately the cases where S does not contain k and those where it does, and using that $(x, x_k) \mapsto (x, x + x_k)$ is a permutation of $(\mathbb{F}_2^n)^2$, the expression $\mathbb{E}_{x,x_1,\ldots,x_k\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k\}}\varphi\left(x+\sum_{i\in S}x_i\right)\right]$ is equal to $\mathbb{E}_{x_1,\ldots,x_{k-1}\in\mathbb{F}_2^n}\left[\left(\mathbb{E}_x\left[\prod_{S\subseteq\{1,\ldots,k-1\}}\varphi\left(x+\sum_{i\in S}x_i\right)\right]\right)^2\right]$. Being then always non-negative, the expression does admit a 2^k-th root.

Equality $||\varphi||_{U_k} =$

$$\left(\mathbb{E}_{x_1,\ldots,x_{k-1}\in\mathbb{F}_2^n}\left[\left(\mathbb{E}_x\left[\prod_{S\subseteq\{1,\ldots,k-1\}}\varphi\left(x+\sum_{i\in S}x_i\right)\right]\right)^2\right]\right)^{\frac{1}{2^k}} \tag{12.19}$$

has its own interest. For $k=1$, it shows that:

$$||\varphi||_{U_1} = \left|\mathbb{E}_{x\in\mathbb{F}_2^n}\left[\varphi(x)\right]\right| = 2^{-n}\left|\sum_{x\in\mathbb{F}_2^n}\varphi(x)\right| \tag{12.20}$$

(which is then not a norm), and for $k=2$ that:

$$||\varphi||_{U_2} = \left(\mathbb{E}_{x_1\in\mathbb{F}_2^n}\left[\left(\mathbb{E}_x\left[\varphi(x)\varphi(x+x_1)\right]\right)^2\right]\right)^{\frac{1}{4}}.$$

Another identity is also useful:

$$||\varphi||_{U_k} = \mathbb{E}_{x_k\in\mathbb{F}_2^n}\left[\mathbb{E}_{x,x_1,\dots,x_{k-1}\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\dots,k-1\}}\psi_{x_k}\left(x+\sum_{i\in S}x_i\right)\right]\right], \tag{12.21}$$

where $\psi_{x_k}(x)=\varphi(x)\varphi(x+x_k)$.

Proposition 197 *For every pseudo-Boolean function φ, the sequence $(||\varphi||_{U_k})_{k\geq 1}$ is non-decreasing:*

$$||\varphi||_{U_1} \leq ||\varphi||_{U_2} \leq \cdots \leq ||\varphi||_{U_k} \leq \dots \tag{12.22}$$

This is due to the inequality:

$$\mathbb{E}_{x_1,\dots,x_{k-1}\in\mathbb{F}_2^n}\left[\mathbb{E}_{x\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\dots,k-1\}}\varphi_S\left(x+\sum_{i\in S}x_i\right)\right]\right] \leq$$

$$\left(\mathbb{E}_{x_1,\dots,x_{k-1}\in\mathbb{F}_2^n}\left[\left(\mathbb{E}_{x\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\dots,k-1\}}\varphi_S\left(x+\sum_{i\in S}x_i\right)\right]\right)^2\right]\right)^{\frac{1}{2}},$$

which is a direct consequence of the Cauchy-Schwarz inequality, since it is equivalent to inequality $\left(\sum_{x_1,\dots,x_{k-1}\in\mathbb{F}_2^n}\left(\mathbb{E}_{x\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\dots,k-1\}}\varphi_S\left(x+\sum_{i\in S}x_i\right)\right]\right)\right)^2 \leq$ $2^{(k-1)n}\sum_{x_1,\dots,x_{k-1}\in\mathbb{F}_2^n}\left(\left(\mathbb{E}_{x\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\dots,k-1\}}\varphi_S\left(x+\sum_{i\in S}x_i\right)\right]\right)^2\right)$.

As shown in [568], for every $k\geq 2$, $||\cdot||_{U_k}$ is a norm. The triangular inequality

$$||\varphi+\psi||_{U_k} \leq ||\varphi||_{U_k} + ||\psi||_{U_k}$$

can be checked as follows: expanding $\prod_{S\subseteq\{1,\dots,k\}}(\varphi+\psi)\left(x+\sum_{i\in S}x_i\right)$ leads to 2^{2^k} terms of the form $\prod_{S\subseteq\{1,\dots,k\}}\varphi_S\left(x+\sum_{i\in S}x_i\right)$, where each function φ_S

is either φ or ψ; for each of these terms, we have, using the Cauchy-Schwarz inequality, and Relation (12.19) (in both ways):

$$\left|\mathbb{E}_{x,x_1,\ldots,x_k\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k\}}\varphi_S\left(x+\sum_{i\in S}x_i\right)\right]\right|=$$

$$\left|\mathbb{E}_{x_1,\ldots,x_{k-1}\in\mathbb{F}_2^n}\left[\left(\mathbb{E}_{x\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k-1\}}\varphi_S\left(x+\sum_{i\in S}x_i\right)\right]\right)\cdot\right.\right.$$

$$\left.\left.\left(\mathbb{E}_{x'\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k-1\}}\varphi_{S\cup\{k\}}\left(x'+\sum_{i\in S}x_i\right)\right]\right)\right]\right|\leq$$

$$\left(\mathbb{E}_{x_1,\ldots,x_{k-1}\in\mathbb{F}_2^n}\left[\left(\mathbb{E}_{x\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k-1\}}\varphi_S\left(x+\sum_{i\in S}x_i\right)\right]\right)^2\right]\right)^{\frac{1}{2}}\cdot$$

$$\left(\mathbb{E}_{x_1,\ldots,x_{k-1}\in\mathbb{F}_2^n}\left[\left(\mathbb{E}_{x\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k-1\}}\varphi_{S\cup\{k\}}\left(x+\sum_{i\in S}x_i\right)\right]\right)^2\right]\right)^{\frac{1}{2}}=$$

$$\left(\mathbb{E}_{x,x_1,\ldots,x_k\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k\}}\varphi_{S\setminus\{k\}}\left(x+\sum_{i\in S}x_i\right)\right]\right)^{\frac{1}{2}}\cdot$$

$$\left(\mathbb{E}_{x,x_1,\ldots,x_k\in\mathbb{F}_2^n}\left[\prod_{S\subseteq\{1,\ldots,k\}}\varphi_{S\cup\{k\}}\left(x+\sum_{i\in S}x_i\right)\right]\right)^{\frac{1}{2}}\leq\cdots$$

$$\leq\prod_{S\subseteq\{1,\ldots,k\}}\left|\mathbb{E}_{x,x_1,\ldots,x_k\in\mathbb{F}_2^n}\left[\prod_{S'\subseteq\{1,\ldots,k\}}\varphi_S\left(x+\sum_{i\in S'}x_i\right)\right]\right|^{\frac{1}{2^k}},$$

resulting in an upper estimate by $||\varphi||_{U_k}^r||\psi||_{U_k}^{2^k-r}$, where r and 2^k-r are the numbers of times that φ_S equals φ and ψ respectively, and this proves that $(||\varphi+\psi||_{U_k})^{2^k}\leq\sum_{r=0}^{2^k}\binom{2^k}{r}||\varphi||_{U_k}^r||\psi||_{U_k}^{2^k-r}$, that is, $||\varphi+\psi||_{U_k}\leq||\varphi||_{U_k}+||\psi||_{U_k}$.

Let now f be an n-variable Boolean function. We can consider the U_k norm of the sign function $f_\chi=(-1)^f$ or that of f itself, viewed as a function from $\mathbb{F}_2^n$ to $\{0,1\}$. In the former case, which is the most studied one, we have from the very definition:

Proposition 198 *Let k, n be positive integers such that $k < n$. Let f be an n-variable Boolean function. Then $||f_\chi||_{U_k}$ equals the 2^k-th root of the average value of $2^{-n}\mathcal{F}(D_{a_1}D_{a_2}\dots D_{a_k}f)$, where $\mathcal{F}(g) = \sum_{x\in\mathbb{F}_2^n}(-1)^{g(x)}$, when $a_1, a_2, \dots, a_k$ range independently over $\mathbb{F}_2^n$.*

For any $k \geq 1$ and any Boolean function f, according to Proposition 198 and to the fact that, for every n-variable Boolean function g, we have $\mathcal{F}(g) \leq 2^n$ with equality if and only if g is the null function, we have that $||f_\chi||_{U_k}$ is bounded above by 1, with equality if and only if all k-th order derivatives of f are null, and we know, according to Proposition 5, page 55, that this is equivalent to saying that f has algebraic degree at most $k-1$.

Relation (12.21) shows that $||f_\chi||_{U_k}$ satisfies the "recurrence" relation:

$$||f_\chi||_{U_k} = \left(\mathbb{E}_{h\in\mathbb{F}_2^n}\left[||(D_hf)_\chi||_{U_{k-1}}^{2^{k-1}}\right]\right)^{\frac{1}{2^k}}. \tag{12.23}$$

This relation can be iterated and shows then again the role of higher-order derivatives.

Note that, for $k = 2$, according to Proposition 198 and to Relation (3.9) page 119, $||f_\chi||_{U_2}$ is related to the second moment $\mathcal{V}(f)$ of the autocorrelation coefficients by:

$$(||f_\chi||_{U_2})^4 = 2^{-3n}\,\mathcal{V}(f), \tag{12.24}$$

and all the observations made at page 119 on $\mathcal{V}(f)$ give then corresponding equalities and bounds on $||f_\chi||_{U_2}$ (hence, studying the U_2 norm has limited interest). For instance, we have $nl(f) \leq 2^{n-1} - 2^{n-1}(||f_\chi||_{U_2})^2 \leq 2^{n-1} - 2^{\frac{3n}{4}-1}||f_\chi||_{U_2}$, with equality on the left-hand side if and only if f is plateaued and overall equality if and only if f is bent. We have also, according to Relation (12.24) and Relation (3.10), page 119, that $||f_\chi||_{U_2}$ equals the normalized quartic mean of the Walsh transform of f:

$$||f_\chi||_{U_2} = 2^{-n}\left(\sum_{b\in\mathbb{F}_2^n} W_f^4(b)\right)^{\frac{1}{4}}. \tag{12.25}$$

In fact, it is easily shown that, for any pseudo-Boolean function φ, we have:

$$||\varphi||_{U_2} = 2^{-n}\left(\sum_{b\in\mathbb{F}_2^n} \widehat{\varphi}^4(b)\right)^{\frac{1}{4}}. \tag{12.26}$$

Relations (12.23) and (12.25) and Relation (6.5), page 222, directly imply that a bent function and its dual have the same U_3 norm (as observed in [528]).
Of course, thanks to Relation (12.23) iterated $k-2$ times, Relation (12.25) results in a similar relation between $||f_\chi||_{U_k}$ and the average quartic mean of the Walsh transforms of the $(k-2)$-th derivatives of f.

An important property of the Gowers uniformity norm is that $||\varphi||_{U_k}$ is an upper bound for the normalized correlations $\left|\mathbb{E}_{x\in\mathbb{F}_2^n}\varphi(x)(-1)^{g(x)}\right|$ between pseudo-Boolean function φ and the sign functions $(-1)^g$ of all Boolean functions g of algebraic degree at most $k-1$ (and in fact, between φ and a larger set of pseudo-Boolean functions, see below). This is a direct corollary of Proposition 197:

Corollary 32 *[568, 572] Let k, n be positive integers such that $k < n$. Let φ be an n-variable pseudo-Boolean function and g an n-variable Boolean function of algebraic degree at most $k-1$. Then:*

$$\left|\mathbb{E}_{x\in\mathbb{F}_2^n}\varphi(x)(-1)^{g(x)}\right| \leq ||\varphi||_{U_k}.$$

Indeed, according to Relation (12.22), we have that $\left|\mathbb{E}_{x\in\mathbb{F}_2^n}\varphi(x)(-1)^{g(x)}\right| = ||\varphi(-1)^g||_{U_1} \leq ||\varphi(-1)^g||_{U_k} = ||\varphi||_{U_k}$.
The result applies more generally when replacing $(-1)^g$ by what is called, in the Gowers norm domain, a "polynomial of degree at most $k-1$", that is, a pseudo-Boolean function ψ such that $\prod_{S\subseteq\{1,\ldots,k\}}\psi\left(x+\sum_{i\in S}x_i\right)$ equals the constant function 1 for every choice of $x_1,\ldots,x_k$.

For $\varphi = f_\chi$, we have $\min(d_H(f,g), d_H(f,g\oplus 1)) = 2^{n-1}(1-\left|\mathbb{E}_{x\in\mathbb{F}_2^n}\varphi(x)(-1)^{g(x)}\right|)$, and taking the minimum for all Boolean functions g of algebraic degree at most $k-1$, Corollary 32 implies:

$$nl_{k-1}(f) \geq 2^{n-1}\big(1-||f_\chi||_{U_k}\big). \tag{12.27}$$

Relation (12.27) means (similarly to what we announced at page 509) that the functions with small U_k norm have large $(k-1)$-th order nonlinearity. Recall that we have seen at page 103 that, asymptotically and for every $\epsilon > 0$, almost all Boolean functions are such that $nl_{k-1}(f) > 2^{n-1}(1-\epsilon)$.

The *Gowers inverse conjecture* (GIC) is that if $||\varphi||_{U_k}$ is positive for a given pseudo-Boolean function of absolute value bounded above by 1, then φ correlates (at a level to be determined for each k) with a polynomial of algebraic degree $k-1$ (as defined above).
This is straightforwardly true for $k=1$, according to Relation (12.20) (taking constant polynomial 1).

The GIC is also easily checked for $k = 2$: Relation (12.26) gives:

$$\begin{aligned}
||\varphi||_{U_2} &= 2^{-n}\left(\sum_{b\in\mathbb{F}_2^n}\widehat{\varphi}^4(b)\right)^{\frac14}\\
&\le 2^{-n}\left(\left(\max_{b\in\mathbb{F}_2^n}\widehat{\varphi}^2(b)\right)\sum_{b\in\mathbb{F}_2^n}\widehat{\varphi}^2(b)\right)^{\frac14}\\
&= 2^{-n}\left(2^n\left(\max_{b\in\mathbb{F}_2^n}\widehat{\varphi}^2(b)\right)\sum_{x\in\mathbb{F}_2^n}\varphi^2(x)\right)^{\frac14}\\
&\le 2^{-\frac{n}{2}}\left(\max_{b\in\mathbb{F}_2^n}\widehat{\varphi}^2(b)\right)^{\frac14},
\end{aligned}$$

and we observe that $|\widehat{\varphi}(b)|$ measures the correlation between φ and $(-1)^{b\cdot x}$.
The GIC is proved for $\varphi = (-1)^f$ and $k = 3$ in [1010, Appendix A]. The proof is a little too long for being included here.
But for generic values of k, the GIC has been independently refuted by Green and Tao [573] and by Lovett, Meshulam and Samorodnitsky [805] (a counter-example φ for $k = 4$ is the sign function of the elementary symmetric function σ_4: its maximum normalized correlation with the sign functions of cubic Boolean functions tends to 0 when n tends to infinity, but its Gowers U_4 norm is bounded below by a strictly positive number). Bergelson, Tao and Ziegler [9] have proposed and proved a modification of the inverse Gowers conjecture valid in low characteristic, but in characteristic 2, their result does not relate to distances and a better adapted modification needs then to be found.

In [528] is studied $||f_\chi||_{U_3}$ for some Maiorana-McFarland bent functions (the value is determined when the permutation involved in the definition of the function, see Relation 6.9, page 233, is APN) and of some cubic monomial functions.

13 Open questions

In this chapter, we list open problems related to the main chapters of this book; some have been already mentioned in References [245, 248]. We avoid stating those which seem elusive like the determination of all bent functions. Some open questions are however quite difficult, while others, more recent, may be easier to address, and some (which have never been proposed until now) may even be easy.

13.1 Questions of general cryptography dealing with functions

1. Generalize the higher-order differential attack to block ciphers using *S-box*es *CCZ equivalent* to *quadratic function*s.
2. Find an expression of the period of general nonlinear-feedback shift register sequences (*NFSR*) by means of the initialization and the feedback function.

13.2 General questions on Boolean functions and vectorial functions

1. Determine, for all values of n, the exact minimum *numerical degree* of n-variable Boolean functions depending on all their variables (this value is near $\log_2 n$, according to Proposition 15, page 86, and to the few lines after its proof); determine the functions having such numerical degree.
2. Determine the set of all possible *coset leader*s of the first order *Reed-Muller code*, *i.e.* of those Boolean functions whose *nonlinearity* equals the *Hamming weight*, that is, such that the null function is a best affine approximation, or equivalently such that $W_f(0_n) = \max_{a\in\mathbb{F}_2^n} |W_f(a)|$ (see page 98).
3. Find simple formulae for the number of *balanced quadratic function*s in n variables and for the weight distribution of the dual $RM(n-3,n)$ of the 2nd order Reed-Muller code.
4. Determine the possible *Hamming weight*s in the 3rd order Reed-Muller code (we know they are diverse, see Section 5.3, page 204); determine the weight distribution of this code (in particular, determine the number of balanced cubic functions).

5. Determine, for all values of n and all $3 \leq k \leq n-2$, those n-variable Boolean functions whose Walsh transform is divisible by 2^k (see the second remark at page 83).
6. Use the numerical normal form (see Definition 12, page 65) to design relevant secondary constructions of Boolean functions (see some ideas of constructions in [248, Subsection 4.1]).
7. Determine the best nonlinearities of Boolean functions in odd dimension $n \geq 9$; in particular, find 9-variable Boolean functions having nonlinearity larger than 242 or show they do not exist.
8. Determine the best nonlinearities of *balanced* Boolean functions in dimension $n \geq 8$; in particular, find an 8-variable (resp. 10-variable) balanced Boolean function with nonlinearity 118 (resp. 494) or show they do not exist. Prove or disprove Dobbertin's conjecture for balanced functions (see page 325).
9. Find a better upper bound on the nonlinearity of (n, m)-*functions* than the known ones when:
   - n is odd and $m < n$;
   - n is even and $\frac{n}{2} < m < n$.
10. Characterize those n-variable Boolean functions f such that $\sum_{a\in\mathbb{F}_2^n} W_f^4(a) = \pm 2^n \sum_{a\in\mathbb{F}_2^n} W_f^3(a)$ (among which, all plateaued functions, according to Corollary 17, page 288), and those (n, m)-functions F such that $\sum_{a\in\mathbb{F}_2^n} W_F^4(a, v) = \pm 2^n \sum_{a\in\mathbb{F}_2^n} W_F^3(a, v)$ for every v (among which all plateaued (n, m)-functions, according to Corollary 20, page 309).
11. Given any Boolean function $f(x) = \bigoplus_{I\subseteq\{1,\ldots,n\}} a_I\, x^I$ and writing the binary expansion of each coefficient of x^I in its NNF as $\lambda_I = \sum_{j\geq 0} \lambda_{I,j} 2^j$, let us denote $f^{(j)}(x) = \bigoplus_{I\subseteq\{1,\ldots,n\}} \lambda_{I,j}\, x^I$. Study the properties of the transformations: $f \mapsto f^{(j)}$ for $j \geq 1$ (for $j = 0$, it is identity) and the cryptographic properties of the functions $f^{(j)}$ related to particular classes of Boolean functions f, such as affine, bent etc.

13.3 Bent functions and plateaued functions

1. Characterize all bent functions of *algebraic degree* 3; extend this characterization to plateaued functions for any amplitude.
2. Determine an efficient lower bound on the number of n-variable bent functions (see page 268); same question for plateaued functions of any amplitude.
3. Any n-variable Boolean function (n even) of *algebraic degree* at most $\frac{n}{2}$ is it the sum of two *bent* functions? (this is called the bent sum decomposition problem, see page 268 as well).
4. Determine an efficient upper bound on the number of n-variable bent functions (see page 269); same question for plateaued functions of any amplitude.
5. What is the minimum, for all n-variable bent Boolean functions, of the maximal dimension of those affine subspaces of $\mathbb{F}_2^n$ on which they are constant? (we

know it is strictly smaller than $n/2$, according to the existence of non-normal bent functions); same question for plateaued functions of any amplitude.

6. Characterize all *self-dual bent functions*, see page 222 (quadratic ones have been determined in [626]).
7. Characterize the *algebraic normal forms* of the elements of class $\mathcal{PS}$ (see page 237) or their trace representations.
8. Investigate the structure of $\mathcal{PS}$, find a constructive definition of $\mathcal{PS}$ functions.
9. Evaluate the size of $\mathcal{PS}$; determine whether the subclass of those $\mathcal{PS}$ functions which are related to full spreads is a large part of it (as observed by Dillon, the $\mathcal{PS}$ functions related to those partial spreads which can be extended to spreads of larger sizes - in particular, those related to full spreads - have necessarily algebraic degree $n/2$).
10. What are the possible algebraic degrees of $\mathcal{PS}^+$ bent functions?
11. Determine whether all Kasami bent functions $tr_n(ax^{2^{2k}-2^k+1})$ are non-weakly-normal for $n \geq 14$ not divisible by 3 and $1 < k < n/2$ co-prime with n and $a \in \mathbb{F}_4 \setminus \mathbb{F}_2$ (see page 279 and [270]).
12. Clarify what can be all the *univariate representations* of those Niho bent functions (see page 246) related to known o-polynomials like Subiaco and Adelaide.
13. Determine the duals of the Niho-bent functions numbers 1 and 3 of pages 247 and foll. (the dual of function number 2 has been determined in [311]).
14. Determine what is the largest possible number of distinct affine *derivatives* of a non-quadratic *bent* n-variable function (n even), which results in determining what is the maximal dimension of this vector space (it is easy to see that it is at least $n-3$: take a quadratic bent function g in $n-6$ variables and a cubic bent function h in 6 variables; then $g(x)$ has 2^{n-6} affine derivatives and it is easy to see that h can have 2^3 affine derivatives (take for instance the Maiorana McFarland function $x_1y_1 \oplus x_2y_2 \oplus (x_1x_2 \oplus x_3)y_3$). Then $g(x) \oplus h(y)$, $x \in \mathbb{F}_2^{n-6}, y \in \mathbb{F}_2^6$, has 2^{n-3} distinct affine derivatives.
15. Find codes with the same parameters as the *Kerdock codes* (see page 280) and which are not equivalent to subcodes of the second order *Reed-Muller code*.
16. Find new simple and general constructions of *perfect nonlinear/bent* (n, m)-functions (see pages 295 and 297).
17. Find *hyper-bent functions* (see page 270) EA inequivalent to $\mathcal{PS}_{ap}$ functions in more than 4 variables (a sporadic example exists in 4 variables [278]).
18. Determine if there exist Boolean functions in more than 3 variables whose second-order *derivatives* D_aD_bf are all *balanced* when a and b are $\mathbb{F}_2$-linearly independent, see page 284.
19. Find nonquadratic bent functions in class C^+, see page 253.
20. Find a geometric interpretation of plateauedness, similar to the one found with generalized partial spreads for bent functions (see page 267).

13.4 Correlation immune and resilient functions

1. Determine an efficient lower bound on the number of n-variable k-resilient functions (see page 341).
2. Determine an efficient upper bound on the number of n-variable k-resilient functions (see page 342).
3. Determine whether there exists any non-affine 3-*resilient symmetric Boolean function* (see page 388).
4. Determine whether the minimum nonzero *Hamming weight* $\omega_{n,t}$ of n-variable t-th order correlation immune functions satisfies $\omega_{n,t} \leq \omega_{n+1,t}$ for every n and t (see page 334).
5. Determine $\omega_{n,t}$ for every n and t.
6. Determine those Boolean functions achieving the Siegenthaler bound (see Proposition 117, page 313) with equality (this is equivalent to determining the Boolean functions whose numerical degree equals the algebraic degree, thanks to Proposition 118, page 314).

13.5 Algebraic immune functions

1. Determine, for n odd and for n even, an efficient lower bound on the number of n-variable functions of maximal algebraic immunity (see page 112).
2. Determine, for n odd and for n even, an efficient upper bound on the number of n-variable functions of maximal algebraic immunity.
3. Determine a lower bound on the *hyper-nonlinearity* of the indicator function of $\{\alpha, \dots, \alpha^{2^{n-1}}\}$ in $\mathbb{F}_{2^n}$ (α *primitive element*), see page 368, which would be not far from the values of the nonlinearity computed for $n \leq 26$.
4. Find a class of Boolean functions which would be as fast to compute as the *hidden weight bit function* (see page 374) and with provably not bad algebraic immunity and fast algebraic immunity, and whose nonlinearity would be good.
5. Determine, for any n, what is the best possible resiliency order of n-variable Boolean functions with optimal algebraic immunity.
6. Determine, for any n, what is the best possible nonlinearity of Boolean functions with optimal algebraic immunity.

13.6 Highly nonlinear vectorial functions with low differential uniformity

1. Determine whether there exist (n, n)-functions with *nonlinearity* strictly larger than $2^{n-1} - 2^{\frac{n}{2}}$ when n is even (see page 402).
2. Determine whether non-quadratic crooked functions exist (see page 306).
3. Find APN functions (see page 159), new up to CCZ equivalence (see page 45), by means of their ANF.

4. Find new APN exponents (see page 422) or prove that all are known.
5. Determine whether infinite classes of *APN* functions with bad nonlinearity exist.
6. Determine whether the APN binomials of Proposition 178, page 438 can be generalized for $t = 4$ to trinomials or quadrinomials.
7. Find simple and general *secondary constructions* of *APN* and *AB* functions (see page 141), and of differentially 4-uniform (n, n)-functions (see page 157) different from the switching construction (see page 440), and if possible more systematic.
8. Find classes of AB functions by using *CCZ equivalence* with Kasami (resp. Welch, Niho) functions (see pages 430 and 444).
9. Find an example of AB function CCZ inequivalent to *power functions* and to *quadratic functions* (we have only one APN function known, with $n = 6$, having such property [494]).
10. Find infinite classes of *APN* and *AB* functions CCZ inequivalent to power functions and to quadratic functions.
11. Determine whether there exist *componentwise APN* (*CAPN*) functions (see page 423) which are neither AB nor power permutations.
12. Determine whether there exist APN functions in odd dimension which are not CAPN.
13. Determine whether the CAPNness of permutations is equivalent to the CAPNness of their compositional inverses, and more generally, whether CAPNness is CCZ-invariant.
14. Determine whether *Kasami APN functions* are componentwise Walsh uniform (CWU, see page 449).
15. Find a systematic way, given an APN function F, to build another (EA inequivalent) function F' such that $\gamma_{F'} = \gamma_F$ (see Proposition 158, page 407).
16. Find an APN permutation in even dimension $n \geq 8$, or better, an infinite class (this is the so-called "big APN problem"; see observations on this problem in [136, 199, 248] and how to work with CCZ equivalence to reach EA inequivalent functions in [145]).
17. Derive new simple and general constructions of *APN*/*AB* functions from *perfect nonlinear* functions (see page 442), and *vice versa*.
18. If possible, *classify APN* functions, or at least their extended Walsh spectra, or at least their nonlinearities.
19. Determine whether differentially 6-uniform $(n, n-2)$-functions exist for $n > 5$.
20. Determine the pairs (n, m) for which *Nyberg's bound* (see page 458) is tight.
21. Construct infinite classes of CWU differentially 4-uniform $(n, n-1)$-functions.
22. Determine all the APN $(2n, 2n)$-functions equal to the concatenation of homogeneous bivariate polynomials over $\mathbb{F}_{2^n}$, that is, of the form:

$$F(x, y) = \Big(\sum_{\substack{0 \leq i,j \leq 2^n - 1 \\ i+j=k}} a_{i,j} x^i y^j, \sum_{\substack{0 \leq i,j \leq 2^n - 1 \\ i+j=k}} a'_{i,j} x^i y^j \Big)\ ;\ x, y, a_{i,j}, a'_{i,j} \in \mathbb{F}_{2^n}.$$

13.7 Recent uses of Boolean and vectorial functions and related problems

1. Characterize for $t \geq 2$ (or at least for $t = 2$) those functions which admit a *threshold implementation* (*TI*) with t masks (*i.e.* a t-th order TI) and with uniformity, see pages 472 and foll.
2. The multiplicative *inverse function* $F(x) = x^{2^n-2}$ can it have an $(n-1)$-th order TI with uniformity, in particular for $n = 8$?
3. Any *AB* function has it an $\frac{n+1}{2}$-th order TI with uniformity?
4. Find generic *primary constructions* of TI with uniformity.
5. Provide cases of secondary constructions of TI with uniformity more general than those exhibited in [1094] (see page 479).
6. Determine all non-quadratic *bent* functions whose restrictions to the set of binary vectors of length n and *Hamming weight* k have null *nonlinearity* (*i.e.* coincide with an affine function), for every $k = 1, \dots, n-1$ (the determination of the quadratic ones is given in Proposition 193, page 502).
7. Determine, for every $1 \leq k \leq n$, the smallest integer e such that $2\binom{n}{e} > \binom{n}{k}$ (providing an upper bound on algebraic immunity with input restricted to Hamming weight k, see page 504), and study its asymptotic behavior relatively to the standard algebraic immunity upper bound $\lceil n/2 \rceil$.
8. Determine whether the four instances of the FLIP cipher (see Subsection 12.2.1) resist algebraic attacks combined with guess and determine attacks.
9. Given two positive integers e and k, what is (see page 508) the smallest number of variables for which there exists a Boolean function of algebraic immunity at least e and resiliency order at least k?
10. Is there, for $k \leq n-2$, an upper bound on the algebraic immunity of k-resilient functions which would be sharper than $\min(n-k-1, \lceil \frac{n}{2} \rceil)$?
11. Find a modification of the inverse Gowers conjecture (see page 513) which would be true in characteristic 2 and would involve Hamming distances.

14 Appendix: finite fields

We briefly recall the basics on finite fields and the main properties used in the body of our monograph. In the limit of this appendix, we are far from complete, and we refer then to the books [775, 890], whose sizes show the extent of the state of the art that we briefly summarize here.

Reminder: A field $(\mathbb{F}, +, *)$ is by definition such that:
- $(\mathbb{F}, +)$ is an Abelian group (we denote its neutral element by 0),
- $(\mathbb{F} \setminus \{0\}, *)$ is an Abelian group (we denote its neutral element by 1),
- and $*$ is distributive with respect to $+$.
Notation: $\mathbb{F} \setminus \{0\}$ can be denoted by $\mathbb{F}^*$.
Important property: $\mathbb{F}$ has no nonzero "zero divisor" (we call this way any element α of $\mathbb{F}$ such that there exists $\beta \neq 0$ such that $\alpha * \beta = 0$).

Exercise: Show (by Euclidean division and factorization) that a polynomial of degree n over a field can have at most n zeros in this field.

14.1 Prime fields and fields with 4, 8 and 9 elements

14.1.1 Characteristic of a finite field

The cardinality of a field is called its *order*. Let $\mathbb{F}$ be a finite field (*i.e.* a field with a finite order, also called a Galois field). The mapping $m \in \mathbb{N} \to m \cdot 1 = 1 + \cdots + 1 \in \mathbb{F}$ cannot be injective. Hence there exist positive integers m, m' such that $m < m'$ and $m \cdot 1 = m' \cdot 1$. Then we have $(m' - m) \cdot 1 = 0$ and $m' - m > 0$. The smallest positive integer p such that $p \cdot 1 = 0$ is called the *characteristic* of $\mathbb{F}$.

Remark. For every $x \in \mathbb{F}$, we have $p \cdot x = (p \cdot 1) * x = 0$, *i.e.* the iterated addition of any element with itself is "mod p". □

Theorem *The characteristic of any finite field is a prime integer.*
Proof: Suppose that $p = kl$ for some integers $1 < k < p$ and $1 < l < p$. We have $(kl) \cdot 1 = 0$, then $k \cdot 1$ and $l \cdot 1$ are nonzero divisors of zero. A contradiction. □

Notation: we will now write p instead of $p \cdot 1$ and the multiplication between field elements will be (classically) represented with a lack of symbol instead of $*$.

14.1.2 Prime fields

For every field $\mathbb{F}$ of characteristic p (prime), we have $\{0, 1, \ldots, p-1\} \subseteq \mathbb{F}$. As observed already, $0, 1, \ldots, p-1$ have here to be considered as integers mod p. Hence they belong to $\mathbb{Z}/p\mathbb{Z}$. Hence, more precisely, we have $\mathbb{Z}/p\mathbb{Z} \subseteq \mathbb{F}$.

Theorem *Let p be any prime integer. Every field of characteristic p admits $\mathbb{Z}/p\mathbb{Z}$ as a subfield, and $\mathbb{Z}/p\mathbb{Z}$ does not have a proper subfield (and is then called a prime field).*
Indeed, $\mathbb{Z}/p\mathbb{Z}$ is itself a field (and it is the smallest field of characteristic p): it is a ring (*i.e.* $(\mathbb{Z}/p\mathbb{Z}, +)$ is an Abelian group and $*$ is associative and distributive with respect to $+$) and every nonzero element a has an inverse, since the mapping $x \in \mathbb{Z}/p\mathbb{Z} \to ax \in \mathbb{Z}/p\mathbb{Z}$ being injective, it is bijective.

If n is not a prime, then $\mathbb{Z}/n\mathbb{Z}$ is not a field, since it has zero divisors.

Remark. According to Bézout's identity, for p prime and $a \in \mathbb{Z}/p\mathbb{Z}^*$, since a and p are co-prime, there exist u and v such that $1 = au + pv$ and (u mod p) is the inverse of a. It can be calculated by the (extended) Euclidean algorithm. □

Example: operations in $\mathbb{Z}/7\mathbb{Z}$:

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Notation: since $\mathbb{Z}/p\mathbb{Z}$ is a field, we shall denote it by $\mathbb{F}_p$.

14.1.3 Possible size of a finite field

Let $\mathbb{F}$ be a finite field of characteristic p. Since $\mathbb{F}_p$ is a subfield, $\mathbb{F}$ *is a vector space over* $\mathbb{F}_p$, and since $\mathbb{F}$ is a finite set, $\mathbb{F}$ must have a finite basis, say of size n. Let $\{b_1, b_2, \ldots, b_n\}$ denote such a basis. Any element of $\mathbb{F}$ can be written uniquely as a linear combination $c_1b_1 + c_2b_2 + \cdots + c_nb_n$ and there are then p^n elements in $\mathbb{F}$.

Theorem *A finite field of characteristic p must have size $q = p^n$ for some natural number n.*
Number n is called the *degree*.

14.1.4 Extending prime fields; fields with 4, 8 and 9 elements

Reminder: $\mathbb{C}$ is an extension field of $\mathbb{R}$ as follows: every element (a_0, a_1) of $\mathbb{R}^2$ is identified to the polynomial $a_0 + a_1x$; $\mathbb{C}$ equals the set of polynomials $a_0 + a_1x$ with the usual addition and with multiplication mod $x^2 + 1$; x is written i.
$\mathbb{C}$ has no nonzero zero divisor because the polynomial $x^2 + 1$ is irreducible over $\mathbb{R}$ (*i.e.* cannot be factored). $\mathbb{C}$ is a field. It is the smallest field containing $\mathbb{R}$ and an additional element i, solution of the equation $x^2 + 1 = 0$.
$\mathbb{C}$ *is an extension field of degree 2 of* $\mathbb{R}$.

The field $\mathbb{F}_4$: It is easily checked that the polynomial $x^2 + x + 1$ is irreducible over $\mathbb{F}_2$. Any element (a_0, a_1) of $\mathbb{F}_2^2$ is identified to the polynomial $a_0 + a_1x$, and $\mathbb{F}_4$ equals $\mathbb{F}_2^2$ with usual addition and multiplication mod $x^2 + x + 1$.

$+$	0	1	x	$1{+}x$
0	0	1	x	$1{+}x$
1	1	0	$1{+}x$	x
x	x	$1{+}x$	0	1
$1{+}x$	$1{+}x$	x	1	0

$*$	0	1	x	$1{+}x$
0	0	0	0	0
1	0	1	x	$1{+}x$
x	0	x	$1{+}x$	1
$1{+}x$	0	$1{+}x$	1	x

$\mathbb{F}_4$ has no nonzero zero divisor because $x^2 + x + 1$ is irreducible. The mapping $x \in \mathbb{F}_4 \to ax \in \mathbb{F}_4$ being injective for $a \neq 0$, it is bijective and $\mathbb{F}_4 = \{0, 1, x, 1{+}x\}$ is a field. We have also $\mathbb{F}_4 = \{0, 1, x, x^2\}$, that is, x is a generator of $\mathbb{F}_4$, which is related to the fact that $x^2 + x + 1$ is a primitive polynomial (see below).
Notation: x is written α. It is a root in $\mathbb{F}_4$ of the polynomial $x^2 + x + 1$. We have:

$+$	0	1	α	$1{+}\alpha$
0	0	1	α	$1{+}\alpha$
1	1	0	$1{+}\alpha$	α
α	α	$1{+}\alpha$	0	1
$1{+}\alpha$	$1{+}\alpha$	α	1	0

$*$	0	1	α	$1{+}\alpha$
0	0	0	0	0
1	0	1	α	$1{+}\alpha$
α	0	α	$1{+}\alpha$	1
$1{+}\alpha$	0	$1{+}\alpha$	1	α

, that is:

$+$	0	1	α	α^2
0	0	1	α	α^2
1	1	0	α^2	α
α	α	α^2	0	1
α^2	α^2	α	1	0

$*$	0	1	α	α^2
0	0	0	0	0
1	0	1	α	α^2
α	0	α	α^2	1
α^2	0	α^2	1	α

The first representation of the elements 0, 1, α, $1{+}\alpha$ of $\mathbb{F}_4$ is called the additive form and the second one 0, 1, α, α^2, the multiplicative form.

Exercise: calculate the tables of $+$ and $*$, in the two representations, for
- $\mathbb{F}_8$, constructed with $\mathbb{F}_2$ and the irreducible (primitive) polynomial $x^3 + x + 1$
- $\mathbb{F}_9$ and $\mathbb{F}_{27}$, constructed with $\mathbb{F}_3$ and the irreducible (primitive) polynomials $x^2 + x + 2$, $x^3 + 2x^2 + 1$.

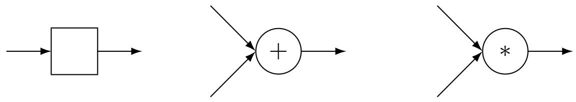
14.2 General finite fields: construction, primitive element

Let p be a prime number, n a positive integer and $f(x)$ an irreducible polynomial of degree n over $\mathbb{F}_p$ (we shall see below that such polynomial exists for any

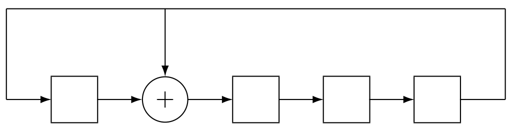
p and any n). Then Kronecker's construction is to identify $\mathbb{F}_p^n$ with the set of polynomials of degrees at most $n-1$ over $\mathbb{F}_p$ and to endow it with the usual addition and with the multiplication mod $f(x)$. In precise mathematical terms, we construct $\mathbb{F}_p[x]/(f(x))$, the quotient of the ring $\mathbb{F}_p[x]$ by the ideal $(f(x))$ equal to the set of multiples of $f(x)$. In more practical words, by denoting x by α, we take this symbol and allow it to add and multiply with elements of $\mathbb{F}_p$ and with itself, with the restriction that $f(\alpha) = 0$. We obtain a ring, denoted by $\mathbb{F}_p(\alpha)$, with no nonzero divisor of zero. Since this ring is finite, it is a field: the mapping $x \in \mathbb{F}_p(\alpha) \to ax \in \mathbb{F}_p(\alpha)$ being injective, it is bijective.

Theorem *Let $f(x)$ be an irreducible polynomial over any finite field $\mathbb{K}$ (in practice, a prime field). The set $\mathbb{K}[x]/(f(x))$ obtained from Kronecker's construction is again a field.*

Electronic circuits for calculating in finite fields are based on flip-flops (for storing bits), adders and multipliers:



For instance in $\mathbb{F}_{2^4}$ with irreducible polynomial $X^4 + X + 1$, the operation of multiplication by α results in:



Notation: $\mathbb{F}_p(\alpha)$ is denoted by $\mathbb{F}_{p^n}$ (the same notation[1] for all choices of $f(x)$, for reasons which will appear later).

Remark. Different time/memory trade-offs exist in the literature for implementing multiplications. For hardware implementations and large dimensions n, several works have been published among which the Omura-Massey method, the Sunar-Koç method, the Karatsuba algorithm. For software implementations in small dimensions (e.g. $n \leq 10$), the number of pertinent possibilities is reduced. See a survey in [320]. □

Remark. In characteristic 2, Newton's formula $(u+v)^k = \sum_{i=0}^{k} \binom{k}{i} u^i v^{k-i}$ has to be applied in conjunction with Lucas' theorem (see page 528) which says that $\binom{k}{i}$ [mod 2] equals 1 if and only if the binary expansion of k covers that of i. □

[1] But some authors write $GF(p^n)$ instead of $\mathbb{F}_{p^n}$.

Calculating the inverse: recall that there is a Euclidean algorithm for polynomials similar to the one for integers. Let $a(x)$, $b(x)$ be a pair of polynomials. The (extended) Euclidean algorithm for polynomials provides their greatest common divisor $d(x)$ and also a pair of polynomials $s(x)$, $t(x)$ such that $s(x)a(x) + t(x)b(x) = d(x)$.
In our situation, one of the polynomials, $b(x) = f(x)$, is irreducible and the other, $a(x)$ representing a nonzero element of the field, is of degree lower than the irreducible one, so that their greatest common divisor is 1. Then we can find polynomials $s(x)$ and $t(x)$ such that $s(x)a(x) + t(x)f(x) = 1$.
Denoting as usual x by α, we have $f(\alpha) = 0$, then we see that we have found that $s(\alpha)$ is an inverse of $a(\alpha)$.

Example [Euclidean algorithm for $x^2 + 1$ in $\mathbb{F}_{27}$ constructed with the irreducible polynomial $x^3 + 2x^2 + 1$ over $\mathbb{F}_3$]. We detail the operations mathematically, as an illustration. A simpler method will be possible when we arrive to the notion of primitive element (thanks to the multiplicative representation). First we divide $f(x) = x^3 + 2x^2 + 1$ by $a(x) = x^2 + 1$. We get $x^3 + 2x^2 + 1 = (x+2)(x^2+1) + 2x + 2$. The remainder $2x + 2$ is not of degree zero, so we must divide x^2+1 by $2x+2$; we get $x^2+1 = (2x+1)(2x+2)+2$. Since the remainder is a constant, the Euclidean algorithm stops; the two polynomials are relatively prime with greatest common divisor equal to 1 (after division by 2). Expressing each remainder by its expression obtained in each division, from bottom to top, we have $2 = x^2 + 1 - (2x + 1)(2x + 2) = x^2 + 1 - (2x + 1)[x^3 + 2x^2 + 1 - (x+2)(x^2+1)] = -(2x+1)(x^3+2x^2+1) + (2x^2+2x)(x^2+1)$ and therefore $(2x^2+2x)(x^2+1) \equiv 2$ [mod x^3+2x^2+1]. If we divide by 2 (*i.e.* multiply by 2), we conclude that the inverse of $x^2 + 1$ is $x^2 + x$.

14.2.1 The fundamental equation over finite fields

Let $\mathbb{F}_q$ be a field of order $q = p^n$, where p is a prime. Consider the multiplicative group $\mathbb{F}_q^*$ of nonzero elements of $\mathbb{F}_q$. It has order $q-1$. Let β denote an arbitrary element of the multiplicative group. By Lagrange's Theorem (saying that, for any finite group G, the order of every subgroup divides the order of G), an element of a group raised to the order of the group equals the identity, that is, $\beta^{q-1} = 1$ (indeed, the set of all powers of β is a subgroup of $\mathbb{F}_q^*$ whose order equals the order of β). This property is called Fermat's little theorem for finite fields.
This equation is valid for any nonzero β. Multiplying through by β yields

$$\beta^q = \beta$$

which is still valid for nonzero elements and now also valid for zero.
In particular, for all $j \in \mathbb{Z}/p\mathbb{Z}$, we have $j^p = j$.

Another proof of equation $\beta^q = \beta$ (without using Lagrange's Theorem) is to say that if $a_1, a_2, \ldots, a_{q-1}$ denote the nonzero elements of $\mathbb{F}_q$ and β is any nonzero element of $\mathbb{F}_q$, the elements $\beta a_1, \beta a_2, \ldots, \beta a_{q-1}$ are all different and are then all nonzero elements of $\mathbb{F}_q$; hence we have $a_1 a_2 \ldots a_{q-1} = \beta a_1 \beta a_2 \ldots \beta a_{q-1} = \beta^{q-1} a_1 a_2 \ldots a_{q-1}$ and therefore $\beta^{q-1} = 1$. The rest is similar.

Exercise: Let $P(x)$ be a polynomial over $\mathbb{F}_q$. Show that $P(x)$ is in fact over $\mathbb{F}_p$ if and only if $(P(x))^p = P(x^p)$.

Note that the polynomial $x^q - x$ having degree q, it cannot have more than q zeros (roots) and we know then all its q distinct zeros in $\mathbb{F}_q$, that is, $x^q - x$ factors completely into linear factors in $\mathbb{F}_q$ (*i.e.* splits): $x^q - x = \prod_{\beta \in \mathbb{F}_q} (x - \beta)$. We say that $\mathbb{F}_q$ is the *splitting field* of $x^q - x$:

$$\mathbb{F}_q = \{x;\ x^q - x = 0\}.$$

Let us now prove that any irreducible polynomial $f(x)$ of degree n over $\mathbb{F}_p$ is a divisor of $x^q - x$. As we saw, this polynomial $f(x)$ does not have any zero in $\mathbb{F}_p$, but Kronecker's construction makes possible to construct an extension field $\mathbb{F}_q$ in which $f(x)$ does have a zero. We know that in $\mathbb{F}_q$ the polynomial $x^q - x$ has all elements of the field as zeros, so it must have a zero in common with $f(x)$, and since $f(x)$ is irreducible, $\gcd(f(x), x^q - x)$, which is not trivial, equals $f(x)$.

Theorem *Let $q = p^n$ be a power of a prime. Then every irreducible polynomial of degree n over the prime field $\mathbb{F}_p$ divides the polynomial $x^q - x$.*

Moreover, since $x^q - x$ splits in $\mathbb{F}_q$, then $f(x)$ also splits in $\mathbb{F}_q$. Since $f(\beta^p) = (f(\beta))^p$ for every $\beta \in \mathbb{F}_q$, because the Newton formula reduces to $(\beta + \beta')^p = \beta^p + \beta'^p$, and because $j^p = j$ for every $j \in \mathbb{Z}/p\mathbb{Z}$, the elements of the form α^{p^j}, $j = 0, \ldots, n-1$ where α denotes a zero of $f(x)$ in $\mathbb{F}_q$, are zeros of $f(x)$ and they are then all the zeros of $f(x)$ since the degree of this polynomial is assumed equal to n and all the elements α^{p^j} are distinct (otherwise, $f(x)$ would be divisible by a polynomial of degree strictly less than n with coefficients in $\mathbb{F}_p$ and would then not be irreducible). Note that if we do not assume anymore that $f(x)$ has degree n, its degree d is necessarily a divisor of n and $x^{p^d} - x$ divides $x^{p^n} - x$.

14.2.2 Existence of finite fields

Let $q = p^n$ be a power of a prime. If we wish to build $\mathbb{F}_q$, the only thing we need is an irreducible polynomial of degree n over $\mathbb{F}_p$.

Lemma *There exist irreducible polynomials of every degree n over every prime field $\mathbb{F}_p$.*

Proof. Let $F_d(x)$ denote the product of all polynomials irreducible over $\mathbb{F}_p$ of degree d. Then since distinct irreducible polynomials are co-prime, we have

$$x^q - x = \prod_{d|n} F_d(x).$$

Let N_d denote the number of irreducible polynomials of degree d over $\mathbb{F}_p$. By equating degrees on both sides of this relation, we see that $p^n = \sum_{d|n} dN_d$.
By the Möbius inversion formula [635], denoting by μ the Möbius function $\mu(d) = \begin{cases} 0 \text{ if } d \text{ is divisible by a square} \\ (-1)^{n_d} \text{ otherwise, where } n_d \text{ is the number of primes dividing } d \end{cases}$, we have then:

$$nN_n = \sum_{d|n} \mu(d)\, p^{n/d} > 0. \qquad \square$$

Hence, for any power of a prime q, there exists a field with q elements.

Reminder: we have $\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases}$, which implies the Möbius inversion formula.

14.2.3 Uniqueness of finite fields

We show now that when building $\mathbb{F}_q$ by Kronecker's construction, any irreducible polynomial of degree n gives the same field up to isomorphism.

Definition *Let $\mathbb{F}$ and $\mathbb{K}$ be two fields, finite or not. An isomorphism between F and $\mathbb{K}$ is a bijective (one-to-one) correspondence ϕ from $\mathbb{F}$ to $\mathbb{K}$ such that for any a and b in $\mathbb{F}$, the following equalities hold:*

$$\phi(a + b) = \phi(a) + \phi(b)$$
$$\phi(ab) = \phi(a)\phi(b).$$

If such an isomorphism exists, the tables of operations $+$ and $*$ are the same in the two *isomorphic* fields, but with different orderings, or names, of the elements. We shall understand this as: the two fields are in fact the same field.

Exercise: Let $q = p^n$ with p prime. Let α be a zero of some irreducible polynomial $f(x)$ over $\mathbb{F}_p$ and let $\mathbb{F}_q = \mathbb{F}_p(\alpha)$. Let $\mathbb{K}$ be a field of the same order q.
1. Show that the image of α by any isomorphism from $\mathbb{F}_q$ to $\mathbb{K}$ must be a zero of $f(x)$.
2. Prove that sending α to a zero of $f(x)$ in $\mathbb{K}$ gives an isomorphism from $\mathbb{F}_q$ to $\mathbb{K}$.

Theorem *All finite fields of the same size are isomorphic and can be obtained with Kronecker's construction.*

Example: Let $\mathbb{F}_{27}$ denote the field obtained by adjoining a zero α of $f(x) = x^3 + 2x^2 + 1$ to $\mathbb{F}_3$ and let $\mathbb{K}_{27}$ be the field obtained by adjoining a zero β of $g(x) = x^3 + 2x + 1$. To find an isomorphism from $\mathbb{F}_{27}$ to $\mathbb{K}_{27}$, we can factor the polynomial $f(x) = x^3 + 2x^2 + 1$ in $\mathbb{K}_{27}$. We know that $f(x)$ must factor completely in $\mathbb{K}_{27}$. So to factor $f(x)$ we need to only look for its three zeros in $\mathbb{K}_{27}$, and there are only 27 elements to try (actually, 24, because no element of $\mathbb{F}_3$ is a zero). We obtain $f(x) = (x + \beta^2 + 2)(x + \beta^2 + \beta)(x + \beta^2 + 2\beta)$, from which we can read that the three zeros are $2\beta^2 + 1, 2\beta^2 + 2\beta$, and $2\beta^2 + \beta$. The isomorphism is now given by sending α into any of the zeros of $f(x)$ in $\mathbb{K}_{27}$.

Exercise: show that $x^{p^n} - x$ divides $x^{p^m} - x$ if and only if n divides m. Deduce that $\mathbb{F}_{p^n}$ is a subfield of $\mathbb{F}_{p^m}$ if and only if n divides m.

14.2.4 Frobenius automorphism

In $\mathbb{F}_q$, the mapping $\phi : x \mapsto x^p$ is an automorphism (that we already encountered above), since:
- $(x + y)^p = x^p + y^p$, $\forall x, y \in \mathbb{F}_q$, since p being a prime $\binom{p}{i} = \frac{p}{i}\binom{p-1}{i-1}$ is divisible by p for every $0 < i < p$;
- $(xy)^p = x^p y^p$, and $x \mapsto x^p$ is bijective since its (additive or multiplicative) kernel is trivial. It is called the *Frobenius automorphism*.

This implies for instance that $(x + y)^{\sum_{i\in I} p^i} = \prod_{i\in I}(x^{p^i} + y^{p^i})$, which for $p = 2$ implies *Lucas' theorem* [809, page 404]: $\binom{n}{j}$ is odd if and only if the binary expansion of j is covered by (*i.e.* has support included in) that of n.

Exercise: We saw that any polynomial $f(X)$ over $\mathbb{F}_p$ satisfies $f(\phi(\beta)) = \phi(f(\beta))$ for every $\beta \in \mathbb{F}_q$. Show that this is characteristic of polynomials over $\mathbb{F}_p$ among polynomials over $\mathbb{F}_q$.

The automorphisms of $\mathbb{F}_q$ are the powers of the Frobenius automorphism; their set, with the composition operation, is a group, called the Galois group of $\mathbb{F}_q$.

14.2.5 Primitive element

When studying $\mathbb{F}_4$, we have seen that the element that we denoted by α is such that $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$. The lemma below implies that, for every power q of a prime, the multiplicative group $\mathbb{F}_q^*$ is also cyclic, that is, there exists $\alpha \in \mathbb{F}_q$ such that $\mathbb{F}_q = \{0, 1, \alpha, \dots, \alpha^{q-2}\}$ (*i.e.* $q - 1$ is the smallest possible integer i such that $\alpha^i = 1$). Such α will be called a *primitive element*.

Exercise: Recall that any irreducible polynomial $P(x) \in \mathbb{F}_p[x]$ of degree $n > 1$ is a divisor of $x^{p^n-1} - 1$. We say that $P(x)$ is primitive if one of its zeros is a primitive element of $\mathbb{F}_{p^n}$ (and then, all its zeros are).
1. Show that if $P(x)$ is a primitive polynomial then $\min\{m;\ P(x) \,|\, x^m - 1\} =$

$p^n - 1$.
2. Show conversely that if $P(x)$ is any irreducible polynomial of degree n such that $\min\{m;\ P(x) \,|\, x^m - 1\} = p^n - 1$ then $P(x)$ is primitive.

Lemma *Let G be a finite multiplicative group with k elements, in which for every $m \le k$, there are at most m solutions for the equation $x^m = 1$. Then G is a cyclic group.*
Proof: Denote by a_m the number of elements of G of order m (that is, satisfying $x^m = 1$ and $x^{m'} \neq 1$ for $m' < m$). Note that, if m does not divide k, then $a_m = 0$, according to Lagrange's theorem. If $a_m \neq 0$, then there exists $g \in G$ of order m. Then, according to the hypothesis that there are at most m solutions for the equation $x^m = 1$, the m powers of g are all solutions of $x^m = 1$. Moreover, g^i has order $\frac{m}{\gcd(i,m)}$. Hence, the group generated by g has $\phi(m)$ generators, where $\phi(m)$ is Euler's totient function (whose value is the number of elements in $\{1, \ldots, m\}$ which are co-prime with m). Thus, if a_m is nonzero, then it is precisely $\phi(m)$. But $\sum_{m|k} \phi(m) = k$ and $\sum_{m|k} a_m = \sum_{m=1}^{k} a_m = k$. Hence $a_m = \phi(m)$ for every m which divides k.
In particular, $a_k = \phi(k) > 0$, so G contains elements of order k, and is cyclic. □

Exercise: We know that every multiplicative subgroup of $\mathbb{F}_q^*$ has for order a divisor of $q-1$. Show that, for each divisor k of $q-1$, there exists a unique multiplicative subgroup of $\mathbb{F}_q^*$ of order k. Show that a generator of this subgroup is $\alpha^{\frac{q-1}{k}}$, where α is a primitive element of $\mathbb{F}_q$.

14.3 Representation (additive and multiplicative) ; trace function

For $q = p^n$, Kronecker's construction with any irreducible polynomial of degree n over $\mathbb{F}_p$ leads to the additive representation of any element $x \in \mathbb{F}_q$:

$$x = x_0 + x_1\alpha + x_2\alpha^2 + \cdots + x_{n-1}\alpha^{n-1}; \quad x_0, \ldots, x_{n-1} \in \mathbb{F}_p.$$

This is called the *additive representation* of x
For α primitive, since the fundamental polynomial $x^q - x$ splits in $\mathbb{F}_q$, this α is the zero of an irreducible (a primitive) polynomial over $\mathbb{F}_p$. In other words, for every n, there exists a primitive polynomial of degree n.
Then any nonzero element $x \in \mathbb{F}_q$ can be written:

$$x = \alpha^i\ ; \quad \text{for } i \in \mathbb{Z}/(q-1)\mathbb{Z} \text{ that is } i \in \{0, \ldots, q-2\}.$$

This is the *multiplicative representation.*

Remark. Denote by $f_{n,\alpha}(i,j)$ the bivariate function over $\mathbb{Z}/(p^n-1)\mathbb{Z}$ such that $\alpha^i + \alpha^j = \alpha^{f_{n,\alpha}(i,j)}$. There is no known expression of $f_{n,\alpha}(i,j)$ (see for instance [890, Subsection 2.1.7.5, page 27], but we have $f_{n,\alpha}(i+1, j+1) = f_{n,\alpha}(i,j) + 1, f_{n,\alpha}(pi, pj) = p f_{n,\alpha}(i,j)$; if k is co-prime with $p^n - 1$, then we have $f_{n,\alpha^k}(i,j) = f_{n,\alpha}(ki,kj)$, and if β is a primitive element of $\mathbb{F}_{p^{rn}}$ and $\alpha = \beta^{(p^{rn}-1)/(p^n-1)}$, then we have $f_{rn,\beta}((p^{rn}-1)i/(p^n-1),(p^{rn}-1)j/(p^n-1)) = (p^{rn}-1)f_{n,\alpha}(i,j)/(p^n-1)$. □

Exercise: Show that, for every integer i, we have

$$\{x^i,\, x \in \mathbb{F}_q\} = \{x^{\gcd(i,q-1)},\, x \in \mathbb{F}_q\}.$$

14.3.1 Absolute trace function

Let $q = p^n$. Recall that the Frobenius automorphism $\Phi : x \to x^p$ satisfies $\Phi^n = Id$ and that $\mathbb{F}_p$ is the set of solutions of equation $\Phi(x) = x$. Two elements are called conjugate if they correspond through Φ^i for some integer i. We have seen that the zeros of any irreducible polynomial over $\mathbb{F}_p$ are conjugate. The function:

$$\begin{aligned} tr_{q/p}(x) &= x + \Phi(x) + \Phi^2(x) + \cdots + \Phi^{n-1}(x) \\ &= x + x^p + x^{p^2} + \cdots + x^{p^{n-1}} \end{aligned}$$

is linear over $\mathbb{F}_q$. In the body of our monograph and below, for $p = 2$ and $q = 2^n$, we simply write tr_n instead of $tr_{q/p}$.

Exercise: Show that

$$tr_{q/p}(\Phi(x)) = \Phi(tr_{q/p}(x)) = tr_{q/p}(x) \text{ for every } x \in \mathbb{F}_q.$$

Hence $tr_{q/p}(x) \in \mathbb{F}_p$ for every element x of $\mathbb{F}_q$ (but not for any element of a super-field of $\mathbb{F}_q$, if we extend the polynomial function $tr_{q/p}$ to this super-field) and $tr_{q/p}$ is an $\mathbb{F}_p$-linear form over the space $\mathbb{F}_q$. It is called the *absolute trace function* over $\mathbb{F}_q$.

Exercise: 1. Show that, for every $a \in \mathbb{F}_q$ the function $tr_{q/p}(ax)$ is identically null on $\mathbb{F}_q$ if and only if $a = 0$.
2. Deduce that the set of all $\mathbb{F}_p$-linear forms over $\mathbb{F}_q$ equals the set of these functions.

Remark. For every nonzero $a \in \mathbb{F}_q$, the set $\{x \in \mathbb{F}_q;\ tr_{q/p}(ax) = 0\}$ is a hyperplane (a vector space of co-dimension 1) in the vector space $\mathbb{F}_q$ over $\mathbb{F}_p$. If $(\alpha_1,\dots,\alpha_n)$ is a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$ and $(\beta_1,\dots,\beta_n)$ an orthonormal basis (such that $tr_{q/p}(\alpha_i\beta_j) = 1$ if $i = j$ and $tr_{q/p}(\alpha_i\beta_j) = 0$ otherwise) and if $a = \sum_{i=1}^n a_i\,\beta_i, x = \sum_{i=1}^n x_i\,\alpha_i$, then the equation of this hyperplane in $\mathbb{F}_p^n$ is $\sum_{i=1}^n a_i x_i = 0$. □

Exercise: Show that every hyperplane of $\mathbb{F}_q$ (vectorspace over $\mathbb{F}_p$) has this form.

Exercise: Show that, for every $i \in \mathbb{F}_p$, we have

$$tr_{q/p}(X) - i = \prod_{u \in \mathbb{F}_q; tr_{q/p}(u)=i} (X - u).$$

Exercise: 1. Recall why a binary linear recurring sequence

$$s_n = a_1 s_{n-1} \oplus \cdots \oplus a_L s_{n-L} \ ; \quad a_1, \ldots, a_L \in \mathbb{F}_2 \tag{14.1}$$

has ultimate period at most $2^L - 1$.
2. Show that if $a_L \neq 0$ then the sequence is fully periodic.
3. We assume that the polynomial $f(x) = x^L + a_1 x^{L-1} + \cdots + a_L$ is primitive.
a. Show that any sequence of the form $s_n = tr_L(a\alpha^n)$ where $a \in \mathbb{F}_{2^n}^*$ and α is a zero of $f(x)$ satisfies Relation (14.1).
b. Deduce that all the nonzero sequences satisfying Relation (14.1) are of the form $s_n = tr_L(a\alpha^n)$ where $a \in \mathbb{F}_{2^n}^*$. Show that they admit $2^L - 1$ as minimal period.
4. Conversely, we assume that the nonzero sequences s_n satisfying Relation (14.1) admit $2^L - 1$ as minimal period. Show that $f(x)$ is primitive.

These sequences are called *m-sequences*.

Exercise: Determine the kernel of the linear mapping $x \in \mathbb{F}_{2^n} \to x + x^2$ and deduce that, for every $u \in \mathbb{F}_{2^n}$, there exists a solution of the equation $x^2 + x = u$ in $\mathbb{F}_{2^n}$ if and only if $tr_n(u) = 0$.

14.3.2 Subfields and other trace functions

Kronecker's construction can be applied the same way as before to any finite field instead of a prime field: let q be a prime power and $f(x)$ be an irreducible polynomial of degree k over $\mathbb{F}_q$. Then $\mathbb{F}_q[x]/(f(x))$ is a field of order q^k. This implies again that, if n divides m, then (up to isomorphism) $\mathbb{F}_{p^n}$ is a subfield of $\mathbb{F}_{p^m}$. Recall that, conversely, if $\mathbb{F}_{p^n}$ is a subfield of $\mathbb{F}_{p^m}$, then n divides m.

The trace function from $\mathbb{F}_{q^k}$ to $\mathbb{F}_q$ is the function:

$$tr_{q^k/q}(x) = x + x^q + x^{q^2} + \cdots + x^{q^{k-1}}.$$

Exercise: Prove that $tr_{q^k/q}$ is a $\mathbb{F}_q$-linear form over $\mathbb{F}_{q^k}$.

Exercise: Check that, if $n \,|\, m \,|\, s$, then $tr_{p^s/p^n} = tr_{p^m/p^n} \circ tr_{p^s/p^m}$.

14.4 Permutations on a finite field

Exercise: Show that, for every prime power q, every function $f(x)$ from $\mathbb{F}_q$ to $\mathbb{F}_q$ is a polynomial function of degree at most $q-1$ over $\mathbb{F}_q$, that is:

$$f(x) = \sum_{i=0}^{q-1} a_i x^i\,; \quad a_i \in \mathbb{F}_q$$

and that this representation is unique.
This polynomial is seen as an element of $\mathbb{F}_q[x]/(x^q - x)$. It is called a permutation polynomial if the function $f(x)$ is bijective.

14.4.1 Examples of permutation polynomials

Affine polynomials: $P(x) = ax + b$, $a \neq 0$
Bijective linearized polynomials (or p-polynomials): $P(x) = \sum_{i=0}^{n-1} a_i x^{p^i}$, $a_i \in \mathbb{F}_q$, such that $ker(P) = \{0\}$. These polynomials being viewed [mod $x^q - x$], the exponents i are viewed in $\mathbb{Z}/n\mathbb{Z}$ where $q = p^n$.

Exercise: Show that the gcd of two linearized polynomials $L(x) = \sum_{i=0}^{n-1} a_i x^{p^i}$, $a_i \in \mathbb{F}_p$, and $L'(x) = \sum_{i=0}^{n-1} b_i x^{p^i}$, $b_i \in \mathbb{F}_p$ (*p-polynomials* over $\mathbb{F}_p$), equals $\sum_{i=0}^{n-1} c_i x^{p^i}$, $c_i \in \mathbb{F}_p$, where $\sum_{i=0}^{n-1} c_i x^i = \gcd(\sum_{i=0}^{n-1} a_i x^i, \sum_{i=0}^{n-1} b_i x^i)$. Deduce that $L(x)$ is a permutation polynomial over $\mathbb{F}_q$ if and only if $\sum_{i=0}^{n-1} a_i x^i$ (its p-associate polynomial) is co-prime with $x^n - 1$.
This result extends to polynomials $\sum_{i=0}^{n-1} a_i x^{q^i}$, $a_i \in \mathbb{F}_q$, over $\mathbb{F}_{q^n}$ for $q = p^r$.

The sums of such polynomials with constants (affine permutations).
Power (monomial) functions: recall that if α is a primitive element of $\mathbb{F}_q$, then for every $i \in \mathbb{Z}/(q-1)\mathbb{Z}$, α^i is primitive if and only if $\gcd(i, q-1) = 1$.

Exercise: Show that, for every $i \in \mathbb{Z}/(q-1)\mathbb{Z}$, the function $x \to x^i$ is a permutation of $\mathbb{F}_q$ if and only if $\gcd(i, q-1) = 1$.

Exercise: Show for $i = 2^j + 1$ and $q = 2^n$ that $\gcd(i, q-1) = \frac{\gcd(2^{2j}-1, q-1)}{\gcd(2^j-1, q-1)} = \frac{2^{\gcd(2j,n)}-1}{2^{\gcd(j,n)-1}}$ and that $x \to x^i$ is a permutation of $\mathbb{F}_q$ if and only if $\frac{n}{\gcd(j,n)}$ is odd.

***Dickson polynomial*s**.

Theorem *For every positive integer k, there exists a polynomial D_k over $\mathbb{Z}$ satisfying the formal equality $D_k\left(x + \frac{1}{x}\right) = x^k + \frac{1}{x^k}$. We have $D_1(x) = x$, $D_2(x) = x^2 - 2$ and $D_k(x) = xD_{k-1}(x) - D_{k-2}(x)$ for $k \geq 3$.*
Proof. By induction on k: if k is odd then $\left(x + x^{-1}\right)^k = \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{i} \left(x^{2i-k} + x^{k-2i}\right)$ and therefore, assuming the property valid until $k-1$, it is proved for k and

$x^k = \sum_{i=0}^{\frac{k-1}{2}} \binom{k}{i} D_{k-2i}(x)$ and

$$D_k(x) = x^k - \sum_{i=1}^{\frac{k-1}{2}} \binom{k}{i} D_{k-2i}(x)$$

and if k is even then $\left(x + x^{-1}\right)^k = \sum_{i=0}^{\frac{k}{2}-1} \binom{k}{i} \left(x^{2i-k} + x^{k-2i}\right) + \binom{k}{\frac{k}{2}}$ implies $x^k = \sum_{i=0}^{\frac{k}{2}-1} \binom{k}{i} D_{k-2i}(x) + \binom{k}{\frac{k}{2}}$ and therefore

$$D_k(x) = x^k - \sum_{i=1}^{\frac{k}{2}-1} \binom{k}{i} D_{k-2i}(x) - \binom{k}{\frac{k}{2}}.$$

The rest of the proof is by expanding $(x + x^{-1})(x^{k-1} + x^{-(k-1)})$. □

Exercise: 1. Let $q = 2^n$. Recall why the equation $x^2 + x = c$ has solutions if and only if $tr_n(c) = 0$. Deduce that, for $a \neq 0$, the equation $x^2 + ax = b$ has solutions if and only if $tr_n\left(\frac{b}{a^2}\right) = 0$.
2. a. Let $q = 2^n$. Show that any element x of $\mathbb{F}_q$ satisfies $tr_{2n}(x) = 0$. Deduce that, for every $x \in \mathbb{F}_q^*$, there exist two elements h of $\mathbb{F}_{q^2}^*$ such that $h + h^{-1} = x$ (by transforming this equation in h into an equation of degree 2 in $\frac{h}{x}$) and that these two elements are inverse of each other.
b. Show that h belongs to $\mathbb{F}_q$ if and only if $tr_n(x^{-1}) = 0$ and that, otherwise, h belongs to the multiplicative subgroup of order $q + 1$ of $\mathbb{F}_{q^2}$.
3. Let $q = p^n$ with p odd.
a. Does any element of $\mathbb{F}_q$ have a square root in $\mathbb{F}_q$?
b. Show that, given a primitive element α of $\mathbb{F}_{q^2}$, every element $x \in \mathbb{F}_q^*$ can be written $\alpha^{i(q+1)}$ and has a square root in $\mathbb{F}_{q^2}$.
c. Show that, for every $x \in \mathbb{F}_q^*$, there exist two elements h of $\mathbb{F}_{q^2}^*$ such that $h + h^{-1} = x$ and that these two elements are inverse of each other.

Exercise:
1. Show that the Dickson polynomial defines a function from $\mathbb{F}_q$ to $\mathbb{F}_q$.
2. Show that, if $\gcd(k, q^2 - 1) = 1$, then D_k is a permutation polynomial over $\mathbb{F}_q$.
3. To prove the converse, assume that $\gcd(k, q^2 - 1) = d > 1$.
a. If d is even, show that $D_k(-x) = D_k(x)$ and $-x \neq x$.
b. If d is odd, then let r be an odd prime dividing d. Show that there exist two distinct elements b, c in $\mathbb{F}_{q^2}^*$ such that $b + \frac{1}{b} \in \mathbb{F}_q$, $c + \frac{1}{c} \in \mathbb{F}_q$, $b \neq \frac{1}{c}$ and $b^r = c^r$ (the two cases where r divides $q - 1$ and r divides $q + 1$ can be distinguished). Check that $D_k(b + \frac{1}{b}) = D_k(c + \frac{1}{c})$ and $b + \frac{1}{b} \neq c + \frac{1}{c}$.
Then D_k is not a permutation polynomial over $\mathbb{F}_q$.

14.4.2 General results on permutation polynomials

Exercise:

1. Show that $\sum_{b\in\mathbb{F}_q} b^t = \begin{cases} 0 \text{ if } t = 0,\dots,q-2 \\ -1 \text{ if } t = q-1 \end{cases}$.

2. Deduce that, for any polynomial $P(x) = \sum_{i=0}^{q-1} p_i x^i$ over $\mathbb{F}_q$, we have $p_{q-1} = -\sum_{b\in\mathbb{F}_q} P(b)$.

Exercise: With the usual convention $0^0 = 1$, show that, given $b \in \mathbb{F}_q$, we have

$$\sum_{t=0}^{q-2} b^t = \begin{cases} 0 \text{ if } b \neq 0,1 \\ 1 \text{ if } b = 0 \\ -1 \text{ if } b = 1 \end{cases}$$

and

$$\sum_{t=0}^{q-1} b^t = \begin{cases} 1 \text{ if } b \neq 1 \\ 0 \text{ if } b = 1 \end{cases}.$$

Lemma *Let $a_0,\dots,a_{q-1}$ be elements of $\mathbb{F}_q$. These elements are all distinct if and only if*

$$\sum_{i=0}^{q-1} a_i^t = \begin{cases} 0 \textit{ if } t = 0,\dots,q-2 \\ -1 \textit{ if } t = q-1 \end{cases}.$$

Proof: If $a_0,\dots,a_{q-1}$ are distinct then $\{a_0,\dots,a_{q-1}\} = \mathbb{F}_q$ and, denoting by α a primitive element of $\mathbb{F}_q$, we have:

$$\sum_{i=0}^{q-1} a_i^t = 0^t + \sum_{j=0}^{q-2} \alpha^{jt} = 0^t + \sum_{j=0}^{q-2} (\alpha^t)^j = \begin{cases} 0 \text{ if } t = 0,\dots,q-2 \\ -1 \text{ if } t = q-1 \end{cases}.$$

Conversely, if $\sum_{i=0}^{q-1} a_i^t = \begin{cases} 0 \text{ if } t = 0,\dots,q-2 \\ -1 \text{ if } t = q-1 \end{cases}$, then

$$P(x) = \sum_{i=0}^{q-1}\sum_{t=0}^{q-1} a_i^t x^{q-1-t} = -1.$$

For every $b \in \mathbb{F}_q^*$, we have $P(b) = \sum_{i=0}^{q-1}\left(\sum_{t=0}^{q-1}\left(\frac{a_i}{b}\right)^t\right) = |\{i = 0,\dots,q-1;\ a_i \neq b\}|$ [mod p], according to the previous exercise. Hence, $|\{i = 0,\dots,q-1;\ a_i = b\}| \neq 0$ and the same happens for $b = 0$. This completes the proof. □

Theorem *(Hermite's criterion) A polynomial $P(x)$ over $\mathbb{F}_q$ is a permutation polynomial if and only if the following two conditions hold:*

1. *$P(x)$ has a single root in $\mathbb{F}_q$;*
2. *for each integer $t = 1,\dots,q-2$ not divisible by p, the polynomial $(P(x))^t$ [mod $x^q - x$] has degree at most $q-2$.*

Proof. Condition 1 is necessary and implies that $\sum_{b\in\mathbb{F}_q}(P(b))^{q-1} = -1$. Condition 2 is equivalent to $\sum_{b\in\mathbb{F}_q}(P(b))^t = 0$ for every $t = 1, \dots, q-2$ not divisible by p. For $t = pt'$, we have $\sum_{b\in\mathbb{F}_q}(P(b))^t = \left(\sum_{b\in\mathbb{F}_q}(P(b))^{t'}\right)^p$. The lemma above completes the proof.

Exercise: Show that $\sum_{v\in\mathbb{F}_q} e^{\frac{2i\pi tr_{q/p}(va)}{p}} = \begin{cases} q \text{ if } a = 0 \\ 0 \text{ otherwise.} \end{cases}$

Theorem *(characterization through* component functions*) A polynomial $P(x)$ over $\mathbb{F}_q$ is a permutation polynomial if and only if, for every $v \in \mathbb{F}_q^*$, the function $tr_{q/p}(vP(x))$ is balanced (that is, takes every value of $\mathbb{F}_p$ the same number of times). Equivalently, for every $v \in \mathbb{F}_q^*$:*

$$\sum_{c\in\mathbb{F}_q} e^{\frac{2i\pi tr_{q/p}(vP(c))}{p}} = 0.$$

Proof. If $P(x)$ is a permutation polynomial over $\mathbb{F}_q$ then, for every $v \in \mathbb{F}_q^*$, the function $tr_{q/p}(vP(x))$ is balanced since the function $tr_{q/p}$ is balanced over $\mathbb{F}_q$. This implies that $\sum_{c\in\mathbb{F}_q} e^{\frac{2i\pi tr_{q/p}(vP(c))}{p}}$ is proportional to $\sum_{j=0}^{p-1}(e^{\frac{2i\pi}{p}})^j = 0$. Conversely, if the condition is satisfied then for every $b \in \mathbb{F}_q$ we have:

$$|\{c \in \mathbb{F}_q;\ P(c) = b\}| = \frac{1}{q}\sum_{c\in\mathbb{F}_q}\sum_{v\in\mathbb{F}_q} e^{\frac{2i\pi tr_{q/p}(vP(c))}{p}} e^{-\frac{2i\pi tr_{q/p}(vb)}{p}} = 1. \qquad \square$$

Remark. As proved by Carlitz in [330], all permutation polynomials over $\mathbb{F}_q$ with $q > 2$ are generated through composition by the multiplicative inverse monomial x^{q-2} and the degree 1 polynomials $ax + b$ with $a \in \mathbb{F}_q^*, b \in \mathbb{F}_q$. □

14.5 Equations over finite fields

Fundamental equation and general equations

We have seen that, for every prime power q, the equation $x^q - x = 0$ admits $\mathbb{F}_q$ as set of solutions. Consequently, given a prime p and two positive integers r and s, the equation $x^{p^s} - x = 0$ has for solutions in $\mathbb{F}_{p^r}$ the elements of $\mathbb{F}_{p^r} \cap \mathbb{F}_{p^s} = \mathbb{F}_{p^{\gcd(r,s)}}$ and in $\mathbb{F}_{p'^r}$ with $p' \neq p$, it has solutions $0, 1$.

Important remark. More generally and for the same reason, finding the solutions in $\mathbb{F}_q$ of an equation $P(x) = 0$ over $\mathbb{F}_q$ is equivalent to finding the solutions of the equation $\gcd(P(x), x^q - x) = 0$. Since the polynomial $x^q - x$ splits over $\mathbb{F}_q$, the number of solutions equals the degree of $\gcd(P(x), x^q - x)$.

Equations of degree 1

$ax + b = 0$, $a \neq 0$, has solution $-b/a$.

Equations of degree 2

$ax^2+bx+c=0$, $a\neq 0$, behaves differently according to whether $p=2$ or not:

- if $p\neq 2$ then the usual resolution works;
- if $p=2$ then, if $a\neq 0$ and $b\neq 0$, $ax^2+bx+c=0$ is equivalent to $\left(\frac{ax}{b}\right)^2+\frac{ax}{b}=\frac{ac}{b^2}$. Hence, solving the equation $ax^2+bx+c=0$ of degree 2 reduces to solving the equation $x^2+x=\beta$ for some β. Let $c\in\mathbb{F}_{2^n}$ and $x=\sum_{j=1}^{n-1}\beta^{2^j}(\sum_{k=0}^{j-1}c^{2^k})$, then $x+x^2=\sum_{j=1}^{n-1}\beta^{2^j}(\sum_{k=0}^{j-1}c^{2^k})+\sum_{j=2}^{n}\beta^{2^j}(\sum_{k=1}^{j-1}c^{2^k})=\sum_{j=1}^{n-1}\beta^{2^j}(\sum_{k=0}^{j-1}c^{2^k})+\sum_{j=2}^{n}\beta^{2^j}(\sum_{k=0}^{j-1}c^{2^k})+\sum_{j=2}^{n}\beta^{2^j}c=\beta^2c+\beta(\sum_{k=1}^{n-1}c^{2^k})+\sum_{j=2}^{n-1}\beta^{2^j}c=$ $c\,tr_n(\beta)+\beta\,tr_n(c)$. This equality and what we have seen already in an exercise imply:

  Theorem *Let n be any positive integer and $\beta\in\mathbb{F}_{2^n}$. A necessary and sufficient condition for the existence of solutions in $\mathbb{F}_{2^n}$ of the equation $x^2+x=\beta$ is that $tr_n(\beta)=0$. Assuming that this condition is satisfied, the solutions of the equation are $x=\sum_{j=1}^{n-1}\beta^{2^j}(\sum_{k=0}^{j-1}c^{2^k})$ and $x=1+\sum_{j=1}^{n-1}\beta^{2^j}(\sum_{k=0}^{j-1}c^{2^k})$, where c is any (fixed) element such that $tr_n(c)=1$.*

  Note that if $n=2m$ and $\beta\in\mathbb{F}_{2^m}$, then the condition $tr_n(\beta)=0$ is satisfied and x simplifies into $x=\beta\,(\sum_{k=0}^{m-1}c^{2^k})+\sum_{j=1}^{m-1}\beta^{2^j}(\sum_{k=j}^{m+j-1}c^{2^k})=\sum_{j=0}^{m-1}(\beta d)^{2^j}$, where $d=\sum_{k=0}^{m-1}c^{2^k}$, and since $tr_n(c)=tr_m^n(d)$, we have that $x=\sum_{j=0}^{m-1}(\beta d)^{2^j}$, where d is any element of $\mathbb{F}_{2^n}$ such that $tr_m^n(d)=1$. Then, for every $u\neq 0$ and v in $\mathbb{F}_{2^m}$, the equation $x^2+ux=v$ has for solutions x and $x+u$ in $\mathbb{F}_{2^n}$, where $x=u\sum_{j=0}^{m-1}\left(\frac{vd}{u^2}\right)^{2^j}$, where $d\in\mathbb{F}_{2^n}, tr_m^n(d)=1$.

Remark. For the more general equation $x+x^{2^k}=b$ where k is odd, $\gcd(k,n)=1$ and $tr_n(b)=0$, let $S_{n,k}(x)=\sum_{i=0}^{n-1}x^{2^{ki}}$ and $U=\{\zeta\in\mathbb{F}_{2^{2n}}\mid\zeta^{2^n+1}=1\}$; let $\zeta\in U\setminus\{1\}$; then, for any $b\in\mathbb{F}_{2^n}^*$, we have $\{x\in\mathbb{F}_{2^n}\mid x+x^{2^k}=b\}=S_{n,k}\left(\frac{b}{\zeta+1}\right)+\mathbb{F}_2$, see [700]. □

Equations of degree 3

Theorem *[64, 1119] Let t_1,t_2 denote the roots of $t^2+bt+a^3=0$ in $\mathbb{F}_{2^{2n}}$, where $a\in\mathbb{F}_{2^n}, b\in\mathbb{F}_{2^n}^*$. Then the factorization of $f(x)=x^3+ax+b$ over $\mathbb{F}_{2^n}$ is characterized as follows:*
- f has three zeros in F_{2^n} if and only if $tr_n\left(\frac{a^3}{b^2}+1\right)=0$ and t_1,t_2 are cubes in $\mathbb{F}_{2^n}$ (n even), $\mathbb{F}_{2^{2n}}$ (n odd).
- f has exactly one zero in $\mathbb{F}_{2^n}$ if and only if $tr_n\left(\frac{a^3}{b^2}+1\right)=1$.
- f has no zero in $\mathbb{F}_{2^n}$ if and only if $tr_n\left(\frac{a^3}{b^2}+1\right)=0$ and t_1,t_2 are not cubes in

$\mathbb{F}_{2^n}$ *(n even)*, $\mathbb{F}_{2^{2n}}$ *(n odd)*.
We refer to [64, 1119] for the proof.

Equations of degree 4 over $\mathbb{F}_{2^n}$

The equation $x^4 + ax^3 + bx^2 + cx + d = 0$ is linear if $a = 0$ and otherwise, substituting x with $x + e$ gives the equation $x^4 + ax^3 + (ae + b)x^2 + (ae^2 + c)x + e^4 + ae^3 + be^2 + ce + d = 0$ and choosing e such that $ae^2 + c = 0$ (which is always possible since $a \neq 0$) gives an equation with no term in x; substituting then x with x^{-1} (assuming that the constant term of the resulting equation was nonzero) and normalizing leads to a linear equation. So we assume that $a = 0$. Then if the homogeneous equation $x^4 + bx^2 + cx = 0$ has a nonzero solution u (i.e. if $x^4 + bx^2 + cx$ is not a permutation) then the equation writes $(x^2 + ux)^2 + (u^2 + b)(x^2 + ux) = d$. Substituting then $x^2 + ux$ with $(u^2 + b)y$ gives the equation $y^2 + y = \frac{d}{u^4+b^2}$ and we are sent back to the resolution of equations of degree 2.

Equation $x^{2^k+1} + x + a = 0$

This equation, first studied in [96, 594, 595], is solved in [700] for $\gcd(k, n) = 1$ (which is a breakthrough).

Power equations

The image set of a power function x^i equals the union of $\{0\}$ and of a multiplicative subgroup of $\mathbb{F}_q^*$ of order $\frac{q-1}{\gcd(i,q-1)}$. The equation $x^i = a$ has one solution if $a = 0$, no solution if a does not belong to this subgroup and $\gcd(i, q - 1)$ solutions if a belongs to it, since there exist integers k (co-prime with $q - 1$) and l (co-prime with i), such that $ik + j(q - 1) = \gcd(i, q - 1)$.

Multivariate method: an example

Hans Dobbertin has developed a method for solving some kinds of equations which play a role when proving that some vectorial functions are APN. The method applies if n is a multiple of a small positive number, say 2 or 3. Assume for instance that n is a multiple of 3, denote $k = n/3$ and assume that we are given some equation in which x appears with exponents which are linear combinations of 1, 2^k and 2^{2k}. The idea of the method is then to introduce the two new variables $y = x^{2^k}$ and $z = y^{2^k}$, to express the equation and its 2^k and 2^{2k} powers by means of the unknowns x, y, z and to eliminate (for instance by using resultants) some of these variables from these three equations. Then even if y and z are eliminated, it happens that the resulting equation is different from the original one. We give an example of this method taken from [158].

Let s and k be positive integers with $\gcd(s, 3k) = 1$, and $n = 3k$. Let

$$d = 2^{2k} + 2^{k+s} - (2^s + 1), g_1 = \gcd(2^{3k} - 1, d/(2^k - 1)), g_2 = \gcd(2^k - 1, d/(2^k - 1)),$$

and let $a \in \mathbb{F}_{2^n}^*$ have the order $2^{2k} + 2^k + 1$ (*i.e.* $a = \alpha^{2^k-1}$ for some primitive element α of $\mathbb{F}_{2^n}^*$). Let

$$\Delta_a(x) = a\left(x^{2^{2k}} + x^{2^{k+s}}\right) + x^{2^s} + x.$$

The equation $\Delta_a(x) = 0$ has $x = 0$ and $x = 1$ for zeros. Let us show that if $g_1 \neq g_2$ then there are no other zero.

We denote $y = x^{2^k}$, $z = y^{2^k}$ and $b = a^{2^k}$, $c = b^{2^k}$ the equation $\Delta_a(x) = 0$ can be rewritten as $a(z + y^{2^s}) + (x^{2^s} + x) = 0$. By definition, a is always a $(2^k - 1)$-th power and thus $abc = 1$. Besides, $a \notin \mathbb{F}_2$. Considering also the conjugated equations we derive the following system of equations

$$\begin{array}{llll} f_1 = & \Delta_a(x) & = a(z + y^{2^s}) + x^{2^s} + x & = 0 \\ f_2 = & f_1^{2^k} & = b(x + z^{2^s}) + y^{2^s} + y & = 0 \\ f_3 = & f_1^{2^{2k}} & = \frac{1}{ab}(y + x^{2^s}) + z^{2^s} + z & = 0. \end{array}$$

Eliminating y and z from these equations gives an equation in x. It happens that this equation is in general different from the original equation and is often simpler: we compute

$$R_1 = b(f_1)^{2^s} + a^{2^s} f_2 = a^{2^s} b y^{2^{2s}} + a^{2^s} y^{2^s} + a^{2^s} y + b x^{2^{2s}} + b x^{2^s} + a^{2^s} b x$$

$$R_2 = \frac{1}{a(b+1)}\left(b f_1 + a f_2 + ab f_3\right) = y^{2^s} + \frac{a+1}{ab+a} y + \frac{1}{a} x^{2^s} + \frac{ab+b}{ab+a} x$$

to eliminate z. To eliminate $y^{2^{2s}}$, we compute:

$$\begin{aligned} R_3 &= R_1 + a^{2^s} b (R_2)^{2^s} \\ &= \frac{a^{2^s}(b+1)^{2^s} + (a+1)^{2^s} b}{(b+1)^{2^s}} y^{2^s} + a^{2^s} y + \frac{a^{2^s} b^{2^s+1} + b}{b^{2^s} + 1} x^{2^s} + a^{2^s} b x. \end{aligned}$$

Using equations R_2 and R_3 we can eliminate y^{2^s} by computing

$$R_4 = R_3 + \frac{a^{2^s}(b+1)^{2^s} + (a+1)^{2^s} b}{(b+1)^{2^s}} R_2 = P(a)(y + (b+1)x^{2^s} + bx),$$

where

$$P(a) = \frac{(ab)^{2^s+1} + (ab)^{2^s} + a^{2^s} b + a^{2^s} + ab + b}{(b+1)^{2^s+1} a}.$$

Computing

$$\begin{aligned} R_5 &= (R_4)^{2^s} + P(a)^{2^s} R_2 = P(a)^{2^s} \\ &\times \left(\frac{a+1}{ab+a} y + (b^{2^s} + 1) x^{2^{2s}} + \frac{ab^{2^s} + 1}{a} x^{2^s} + \frac{ab+b}{ab+a} x\right) \end{aligned}$$

we finally get our desired equation by

$$R_6 = \frac{a+1}{ab+a} P(a)^{2^s - 1} R_4 + R_5 = P(a)^{2^s} (b+1)^{2^s} \left(x^{2^{2s}} + x^{2^s}\right).$$

Obviously if x is a solution of $\Delta_a(x) = 0$ then $R_6(x) = 0$. For $P(a)^{2^s}(b+1) \neq 0$

this is equivalent to $x = 0, 1$. Thus to prove the result, it is sufficient to show that $P(a)$ does not vanish for elements a fulfilling the equation

$$a = \left(\alpha v^{2^k+2^s+1}\right)^{2^k-1} \tag{14.2}$$

Note that, if a satisfies (14.2), then a is not a $(2^k + 2^s + 1)$-th power, since α^{2^k-1} is not: $g_2 = \gcd(2^k - 1, 2^k + 2^s + 1)$ is by hypothesis a strict divisor of $g_1 = \gcd(2^n - 1, 2^k + 2^s + 1)$ and α being a primitive element, it cannot be a (g_1/g_2)-th power.

Consequently, it is sufficient to show, that if $P(a) = 0$ then a is a (2^k+2^s+1)-th power. For $a \notin \mathbb{F}_2$ the equation $P(a) = 0$ is equivalent to

$$a = \left(\frac{a+1}{c+1}\right)^{2^s+1} c^{2^s+1} \left(\frac{b+1}{a+1}\right) a = \left(\frac{a+1}{c+1}c\right)^{2^k+2^s+1},$$

as can be easily seen by dividing this equality by a, simplifying it by $(a+1)$, and then expanding it, using that $c = 1/ab$. Note that the right hand side is always a $(2^k + 2^s + 1)$-th power. This proves the property.

14.6 Factoring polynomials

Factoring polynomials is more demanding than solving polynomial equations. A nicely practical method given in [775, 4.1] is as follows:

Theorem *Let $F(x)$ and $H(x)$ be two monic polynomials over $\mathbb{F}_q$ such that $F(x)$ divides $(H(x))^q - H(x)$. Then $F(x) = \prod_{c\in\mathbb{F}_q} \gcd(F(x), H(x) - c)$.*

The proof is straightforward by observing that $(H(x))^q - H(x) = \prod_{c\in\mathbb{F}_q}(H(x) - c)$, that $F(x)$ divides then $\prod_{c\in\mathbb{F}_q} \gcd(F(x), H(x) - c)$ and that conversely each polynomial $\gcd(F(x), H(x) - c)$ divides $F(x)$ and these greatest common divisors are co-prime.

Then for factoring $F(x)$ into irreducible factors, we can factor separately each polynomial $\gcd(F(x), H(x) - c)$.

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