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Characterization and Detection of Toric Loops in n-Dimensional Discrete Toric Spaces

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Abstract Since a toric space is not simply connected, it is possible to find in such spaces some loops which are not homotopic to a point: we call them *toric loops*. Some applications, such as the study of the relationship between the geometrical characteristics of a material and its physical properties, rely on three-dimensional discrete toric spaces and require detecting objects having a toric loop.

In this work, we study objects embedded in discrete toric spaces, and propose a new definition of loops and equivalence of loops. Moreover, we introduce a characteristic of loops that we call *wrapping vector*: relying on this notion, we propose a linear time algorithm which detects whether an object has a toric loop or not.

1 Introduction

Topology is used in various domains of image processing in order to perform geometric analysis of objects. In porous material analysis, different topological transformations, such as skeletonisation, are used to study the relationships between the geometrical characteristics of a material and its physical properties.

When simulating a fluid flow through a porous material, the whole material can be approximated by the tessellation of the space made up by copies of one of its samples, under the condition that the volume of the sample exceeds the so-called Representative Elementary Volume (REV) of the material [1]. When the whole Euclidean space is tiled this way, one can remark that the result of the fluid flow simulation is itself the tessellation of the local flow obtained inside any copy of the sample (see Fig. 1-**a**). When considering the flow obtained inside the sample, it appears that the flow leaving the sample by one side comes back by the opposite side (see Fig. 1-**b**). Thus, it is possible to perform the fluid flow simulation only on the sample, under the condition that its opposite sides are joined: with this construction, the sample is embedded inside a toric space [5] [11]. In order to perform geometric analysis of fluid flow through porous materials, we therefore need topological tools adapted to toric spaces.

Considering the sample inside a toric space leads to new difficulties. In a real fluid flow, grains of a material (pieces of the material which are not connected with the borders of the sample) do not have any effect on the final results, as these grains eventually either evacuate the object with the flow or get blocked and connect with the rest of the material. Thus, before performing a fluid flow simulation on a sample, it is necessary to remove its grains (typically, in a finite subset S of $\mathbb{Z}^n$, a grain is a connected component which does not 'touch' the borders of S). However, characterizing a grain inside a toric space, which does not have any border, is more difficult than in $\mathbb{Z}^n$. On the contrary of the discrete space $\mathbb{Z}^n$, n-dimensional discrete toric spaces are not simply connected spaces [11]: some loops, called *toric loops*, are not homotopic to a point (this can be easily seen when considering a 2D torus). In a toric space, a connected component may be considered as a grain if it contains no toric loop. Indeed, when considering a sample embedded inside a toric space, and a tessellation of the Euclidean space made up by copies of this sample, one can remark that the connected components of the sample which do not contain toric loops produce grains in the tessellation, while the connected components con-

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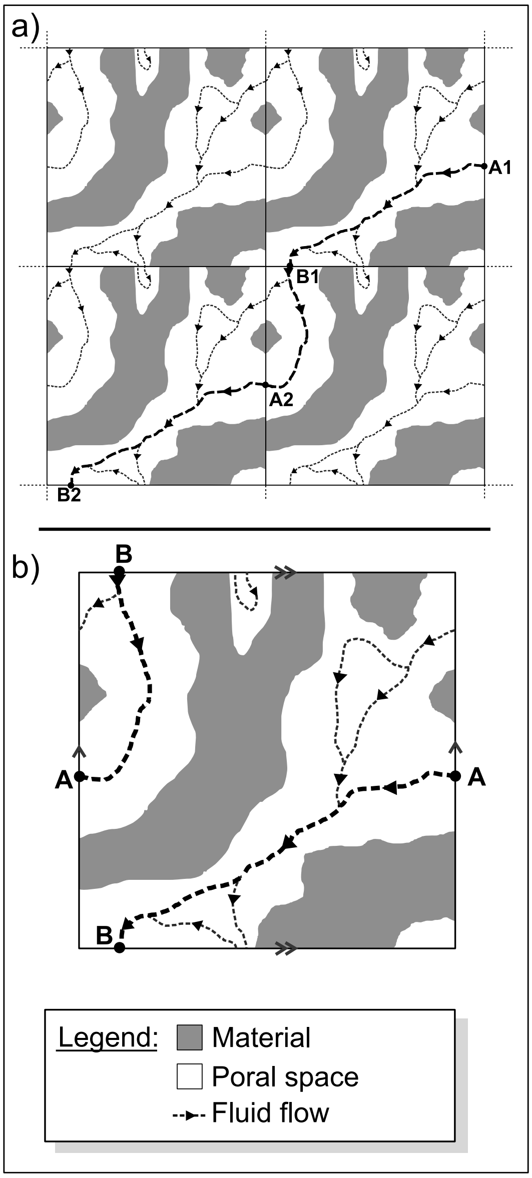

Fig. 1 Simulating a fluid flow - When simulating a fluid flow, a porous material (in gray) can be approximated by the tessellation of one of its samples (see **a**). When the results of the simulation are obtained (the dotted lines), one can see that the fluid flow through the mosaic is the tessellation of the fluid flow simulation results obtained in one sample. For example, one can look at the bold dotted line in **a)**: the flow going from $A1$ to $B1$ is the same than the flow going from $A2$ to $B2$. It is therefore possible to perform the fluid flow simulation through only one sample and, in order to obtain the same results than in **a)**, connect the opposite sides of the sample (see **b**): the sample is embedded inside a toric space.

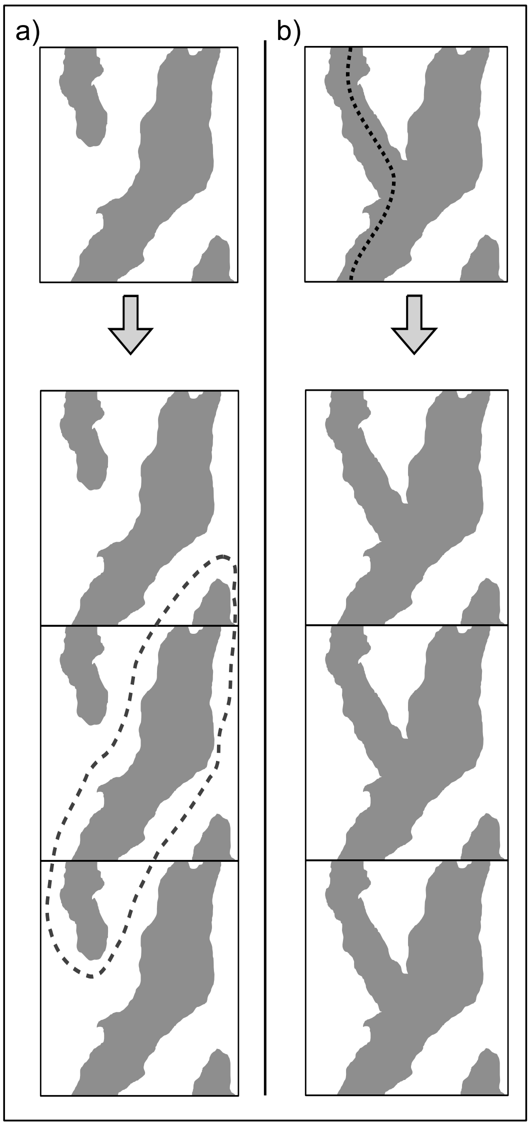

Fig. 2 Grains in toric spaces - The image in **a)** contains no grain based on the 'border criterion'; when the Euclidean space is tessellated with copies of the image, grains appear (the circled connected component is an example of grain). In **b)**, the connected component has toric loops (e.g. the dotted line) and when the Euclidean space is tessellated with copies of the image, no grain appear.

taining toric loops cannot be considered as grains in the tiling (see Fig. 2).

In this work, we give a new definition of loops and homotopy class, adapted to n-dimensional discrete toric spaces. Relying on these notions, we introduce *wrapping vectors*, a new characteristic of loops in toric spaces which is the same for all homotopic loops. Thanks to wrapping vectors, we give a linear time algorithm which allows us to decide whether an n-dimensional object contains a toric loop or not.

This paper is an extension of a paper submitted for a conference [3]. In addition, it contains an algorithm which not only detects when an object contains a toric loop (as the algorithm proposed in [3]) but also builds a basis characterizing all toric loops contained in an object. Furthermore, it contains a comparison between loop homotopy defined in this article and loop equivalence defined in [6].

2 Basic Notions

2.1 Discrete Toric Spaces

A n-dimensional torus is classically defined as the direct product of n circles (see [5]). In the following, we give a discrete definition of toric space, based on modular arithmetic (see [4]).

Given d a positive integer. We set $\mathbb{Z}_d = \{0, ..., d-1\}$, and we denote by $\oplus_d$ the operation such that for all $a, b \in \mathbb{Z}$, $(a \oplus_d b)$ is the element of $\mathbb{Z}_d$ congruent to $(a + b)$ modulo d. We point out that $(\mathbb{Z}_d, \oplus_d)$ is the cyclic group of order d.

Let n be a positive integer, $\mathbf{d} = (d_1, ..., d_n) \in \mathbb{N}^n$, and $\mathbb{T}^n = \mathbb{Z}_{d_1} \times ... \times \mathbb{Z}_{d_n}$, we denote by $\oplus$ the operation such that for all $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{Z}^n$ and $\mathbf{b} = (b_1, ..., b_n) \in \mathbb{Z}^n$, $\mathbf{a} \oplus \mathbf{b} = (a_1 \oplus_{d_1} b_1, ..., a_n \oplus_{d_n} b_n)$. The group $(\mathbb{T}^n, \oplus)$ is the direct product of the n groups $(\mathbb{Z}_{d_i}, \oplus_{d_i})_{(1 \leq i \leq n)}$, and is an *n-dimensional discrete toric space* [5].

The scalar d_i is *the size of the i-th dimension* of $\mathbb{T}^n$, and $\mathbf{d}$ is the *size (vector)* of $\mathbb{T}^n$. For simplicity, the operation $\oplus_{\mathbf{d}}$ will be also denoted by $\oplus$.

2.2 Neighbourhoods in Toric Spaces

As in $\mathbb{Z}^n$, various adjacency relations may be defined in a toric space.

Definition 1 An *m-step* $(0 < m \leq n)$ is a vector $\mathbf{s} = (s_1, ..., s_n)$ of $\mathbb{Z}^n$ such that, for all $i \in [1; n]$, $s_i \in \{-1, 0, 1\}$ and $|s_1| + ... + |s_n| \leq m$.

Two points $\mathbf{a}, \mathbf{b} \in \mathbb{T}^n$ are *m-adjacent* if there exists an m-step $\mathbf{s}$ such that $\mathbf{a} \oplus \mathbf{s} = \mathbf{b}$.

Note that the steps must not be considered as elements of $\mathbb{T}^n$, but rather as elements of $\mathbb{Z}^n$.

In 2D, the 1- and 2-adjacency relations respectively correspond to the 4- and 8-neighbourhood [7] adapted to two-dimensional toric spaces. In 3D, the 1-, 2- and 3-adjacency relations can be respectively seen as the 6-, 18- and 26-neighbourhood [7] adapted to three- dimensional toric spaces.

If the coordinates of the size vector of $\mathbb{T}^n$ are all greater than 2, then the m-neighbourhood of any element of $\mathbb{T}^n$ is isomorphic to the m-neighbourhood of any element of $\mathbb{Z}^n$.

Based on the *m-adjacency relation* previously defined, we introduce the notion of *m-connectedness*.

Definition 2 A set of points X of $\mathbb{T}^n$ is *m-connected* if, for all $\mathbf{a}, \mathbf{b} \in X$, there exists a sequence $(\mathbf{x_1}, ..., \mathbf{x_k})$ of elements of X such that $\mathbf{x_1} = \mathbf{a}$, $\mathbf{x_k} = \mathbf{b}$ and for all $i \in [1; k-1]$, $\mathbf{x_i}$ and $\mathbf{x_{i+1}}$ are m-adjacent.

2.3 Loops in Toric Spaces

Classically, in $\mathbb{Z}^n$, an m-loop is defined as a sequence of m-adjacent points such that the first point and the last point of the sequence are equal [6]. In this paper, we define a loop as a sequence of m-steps, which describes the direction followed by the loop in the toric space. This new definition will allow us to give simple intermediate properties and proofs leading to our main theorem(see Th. 1).

Definition 3 Given $\mathbf{p} \in \mathbb{T}^n$, an *m-loop (of base point* $\mathbf{p}$*)* is a pair $\mathcal{L} = (\mathbf{p}, V)$, where $V = (\mathbf{v_1}, ..., \mathbf{v_k})$ is a sequence of m-steps such that $(\mathbf{p} \oplus \mathbf{v_1} \oplus ... \oplus \mathbf{v_k}) = \mathbf{p}$.

The number k is the *length of* $\mathcal{L}$. We call *i-th point of* $\mathcal{L}$, with $1 \leq i \leq k+1$, the point $(\mathbf{p} \oplus \mathbf{v_1} \oplus ... \oplus \mathbf{v_{i-1}})$. The loop $(\mathbf{p}, ())$ is called the *trivial loop of base point* $\mathbf{p}$.

Remark 1 In this definition, the $(k+1)$-th point of $\mathcal{L}$ is $\mathbf{p}$, and has been defined in order to make some propositions and proofs more simple.

Remark 2 This definition of loops in toric space allows to remove an ambiguity which can exist in small toric spaces. Indeed, when considering loops as a sequence of m-adjacent points, an ambiguity arises in toric spaces where one dimension has a size equal to 1 or 2. For example, let us consider the two-dimensional toric space $\mathbb{T}^2 = \mathbb{Z}_3 \times \mathbb{Z}_2$, and the 2-adjacency relation on $\mathbb{T}^2$. Let us also consider $\mathbf{x_1} = (1, 0)$ and $\mathbf{x_2} = (1, 1)$ in $\mathbb{T}^2$, and let us consider the sequence of points $\mathcal{L} = (\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_1})$.

The sequence $\mathcal{L}$ could either be the loop passing by $\mathbf{x_1}$ and $\mathbf{x_2}$ and doing a 'u-turn' to come back to $\mathbf{x_1}$, or either be the loop passing by $\mathbf{x_1}$ and $\mathbf{x_2}$, and 'wrapping around' the toric space in order to reach $\mathbf{x_1}$ without making any 'u-turn', as shown on Fig. 3. In toric spaces of small size, defining a loop as a sequence of m-adjacent points may lead to such ambiguity.

However, considering a loop as a sequence of m-steps removes the ambiguity: let $\mathbf{v}$ be the vector $(0, 1)$, the loop passing by $\mathbf{x_1}$ and $\mathbf{x_2}$ and making a u-turn is $(\mathbf{x_1}, (\mathbf{v}, -\mathbf{v}))$ (see Fig. 3-**b**), while the loop wrapping around the toric space is $(\mathbf{x_1}, (\mathbf{v}, \mathbf{v}))$ (see Fig. 3-**c**). Since m-steps are elements of $\mathbb{Z}$, we have $\mathbf{v} \neq -\mathbf{v}$.

3 Loop Homotopy in Toric Spaces

3.1 Homotopic Loops

In this section, we define an equivalence relation between loops, corresponding to an homotopy, inside a discrete toric space. An equivalence relation between

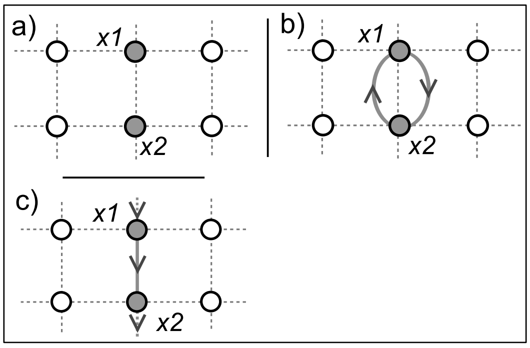

Fig. 3 Loops in toric spaces - In the toric space $\mathbb{Z}_3 \times \mathbb{Z}_2$ (see **a**), the sequence of points $(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_1})$ can be interpreted in two different ways: **b)** and **c)**.

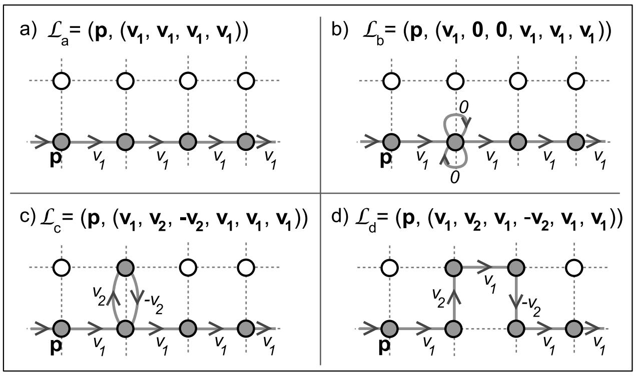

Fig. 4 Homotopic Loops - The 1-loops $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$ and $\mathcal{L}_d$ are homotopic.

loops inside $\mathbb{Z}^2$ and $\mathbb{Z}^3$ has been defined in [6], however, it cannot be adapted to discrete toric spaces (see Sec. 7). Observe that the following definition does not constrain the loops to lie in a subset of the space, on the contrary of the definition given in [6].

Definition 4 Let $\mathcal{K} = (\mathbf{p}, U)$ and $\mathcal{R} = (\mathbf{p}, V)$ be two m-loops of base point $\mathbf{p} \in \mathbb{T}^n$, with $U = (\mathbf{u_1}, ..., \mathbf{u_k})$ and $V = (\mathbf{v_1}, ..., \mathbf{v_r})$. The two m-loops $\mathcal{K}$ *and* $\mathcal{R}$ *are directly homotopic* if one of the three following conditions is satisfied:

1. There exists $j \in [1; r]$ such that $\mathbf{v_j} = 0$ and $U = (\mathbf{v_1}, ..., \mathbf{v_{j-1}}, \mathbf{v_{j+1}}, ..., \mathbf{v_r})$.
2. There exists $j \in [1; k]$ such that $\mathbf{u_j} = 0$ and $V = (\mathbf{u_1}, ..., \mathbf{u_{j-1}}, \mathbf{u_{j+1}}, ..., \mathbf{u_k})$.
3. There exists $j \in [1; k-1]$ such that
   . $V = (\mathbf{u_1}, ..., \mathbf{u_{j-1}}, \mathbf{v_j}, \mathbf{v_{j+1}}, \mathbf{u_{j+2}}, ..., \mathbf{u_k})$, and
   . $\mathbf{u_j} + \mathbf{u_{j+1}} = \mathbf{v_j} + \mathbf{v_{j+1}}$, and
   . $(\mathbf{u_j} - \mathbf{v_j})$ is an n-step.

Remark 3 The last condition ($(\mathbf{u_j} - \mathbf{v_j})$ is an n-step) is not necessary for proving the results presented in this paper. However, it is needed when comparing the loop homotopy and the loop equivalence (see [6]), as done in Sec. 7.

Moreover, this last condition is equivalent to saying that $(\mathbf{u_{j+1}} - \mathbf{v_{j+1}})$ is an n-step.

Remark 4 In the case 1 (resp. 2 and 3), we have $k = r - 1$ (resp. $(r = k - 1)$ and $(r = k)$).

Remark 5 It may be observed that in the above definition, the parameter m is used to specify that we consider m-loops, but it is not taken into account in order to decide if two m-loops are directly homotopic.

Definition 5 Two m-loops $\mathcal{K}$ and $\mathcal{R}$ of base point $\mathbf{p} \in \mathbb{T}^n$ *are homotopic* if there exists a sequence of m-loops $(\mathcal{C}_1, ..., \mathcal{C}_q)$ such that $\mathcal{C}_1 = \mathcal{K}$, $\mathcal{C}_q = \mathcal{R}$ and for all $j \in [1; q-1], \mathcal{C}_j$ and $\mathcal{C}_{j+1}$ are directly homotopic.

Example 1 In the toric space $\mathbb{Z}_4 \times \mathbb{Z}_2$, let us consider the point $\mathbf{p} = (0, 0)$, the 1-steps $\mathbf{v_1} = (1, 0)$ and $\mathbf{v_2} = (0, 1)$, and the 1-loops $\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$ and $\mathcal{L}_d$ (see Fig. 4). The loops $\mathcal{L}_a$ and $\mathcal{L}_b$ are homotopic, the loops $\mathcal{L}_b$ and $\mathcal{L}_c$ are directly homotopic, and the loops $\mathcal{L}_c$ and $\mathcal{L}_d$ are also directly homotopic.

On the other hand, it may be seen that the 1-loops depicted on Fig. 3-**b** and on Fig. 3-**c** are not directly homotopic.

We propose an adaptation of our definition of loop homotopy to $\mathbb{Z}^2$ and $\mathbb{Z}^3$ in Sec. 7, and we show that the resulting definition is equivalent to the definition of loop equivalence given in [6].

3.2 Fundamental Group

Initially defined in the continuous space by Henri Poincaré in 1895 [10], the fundamental group is an essential concept of topology, based on the homotopy relation, which has been transferred into different discrete frameworks (see e.g. [6], [8], [2]).

Given two m-loops $\mathcal{K} = (\mathbf{p}, (\mathbf{u_1}, ..., \mathbf{u_k}))$ and $\mathcal{R} = (\mathbf{p}, (\mathbf{v_1}, ..., \mathbf{v_r}))$ of same base point $\mathbf{p} \in \mathbb{T}^n$, the *product of* $\mathcal{K}$ *and* $\mathcal{R}$ is the m-loop $\mathcal{K}.\mathcal{R} = (\mathbf{p}, (\mathbf{u_1}, ..., \mathbf{u_k}, \mathbf{v_1}, ..., \mathbf{v_r}))$. The identity element of this product operation is the trivial loop $(\mathbf{p}, ())$, and for each m-loop $\mathcal{K} = (\mathbf{p}, (\mathbf{u_1}, ..., \mathbf{u_k}))$, we define the inverse of $\mathcal{K}$ as the m-loop $\mathcal{K}^{-1} = (p, (-\mathbf{u_k}, ..., -\mathbf{u_1}))$.

The symbol $\prod$ will be used for the iteration of the product operation on loops. Given a positive integer w, and an m-loop $\mathcal{K}$ of base point $\mathbf{p}$, we set $\mathcal{K}^w = \prod_{i=1}^{w} \mathcal{K}$ and $\mathcal{K}^{-w} = \prod_{i=1}^{w} \mathcal{K}^{-1}$. We also define $\mathcal{K}^0 = (\mathbf{p}, ())$.

The homotopy of m-loops is a reflexive, symmetric and transitive relation: it is therefore an equivalence relation and the equivalence class, called *homotopy class*, of an m-loop $\mathcal{R}$ is denoted by $[\mathcal{R}]$. The product operation can be extended to the homotopy classes of m-loops of same base point: the product of $[\mathcal{K}]$ and $[\mathcal{R}]$ is $[\mathcal{K}].[\mathcal{R}] = [\mathcal{K}.\mathcal{R}]$. It may be easily seen that this binary operation is well defined since, if $\mathcal{K}' \in [\mathcal{K}]$ and $\mathcal{R}' \in [\mathcal{R}]$, then $\mathcal{K}'.\mathcal{R}' \in [\mathcal{K}.\mathcal{R}]$.

We now define the fundamental group of $\mathbb{T}^n$.

Definition 6 Given an m-adjacency relation on $\mathbb{T}^n$ and a point $\mathbf{p} \in \mathbb{T}^n$, the *m-fundamental group of* $\mathbb{T}^n$ *with base point* $\mathbf{p}$ is the group formed by the homotopy classes of all m-loops of base point $\mathbf{p} \in \mathbb{T}^n$ under the product operation.

The identity element of this group is the homotopy class of the trivial loop, and for each m-loop $\mathcal{K}$ of base point $\mathbf{p}$, the inverse of $[\mathcal{K}]$ is $[\mathcal{K}^{-1}]$, since $[\mathcal{K}.\mathcal{K}^{-1}] = [(\mathbf{p}, ())]$.

The choice of the base point leads to different fundamental groups which are isomorphic to each other ([9], Th. 3.2.16). Thus, in the following, we sometimes talk about the m-fundamental group of $\mathbb{T}^n$, without specifying the base point.

4 Toric Loops in Subsets of $\mathbb{T}^n$

The toric loops, informally evoked in the introduction, can now be formalised using the definitions given in the previous sections.

Definition 7 In $\mathbb{T}^n$, we say that an m-loop is a *toric m-loop* if it does not belong to the homotopy class of a trivial loop.

A connected subset of $\mathbb{T}^n$ is *wrapped in* $\mathbb{T}^n$ if it contains a toric m-loop.

Remark 6 The notion of grain introduced informally in Sec. 1 may now be defined: a connected component of $\mathbb{T}^n$ is *a grain* if it is not wrapped in $\mathbb{T}^n$.

4.1 Algorithm for Detecting Wrapped Subsets of $\mathbb{T}^n$

In order to know whether a connected subset of $\mathbb{T}^n$ is wrapped or not, it is not necessary to build all the m-loops which can be found in the subset: the Wrapped Subset Descriptor (WSD) algorithm (see Alg. 1) answers this question in linear time (more precisely, in $O(N.M)$, where N is the number of points of $\mathbb{T}^n$, and M is the number of distinct m-steps), as stated by the following proposition.

Proposition 1 *Let* $\mathbb{T}^n$ *be an n-dimensional toric space of size vector* $\mathbf{d}$. *A non-empty m-connected subset* X *of* $\mathbb{T}^n$ *is wrapped in* $\mathbb{T}^n$ *if and only if WSD(n,m,* $\mathbb{T}^n$*,*$\mathbf{d}$*,X) is non-empty.*

Algorithm 1: WSD($\mathbf{n}$,$\mathbf{m}$,$\mathbb{T}^n$,$\mathbf{d}$,X)

Data: An n-dimensional toric space $\mathbb{T}^n$ of dimension vector d and a non-empty m-connected subset X of $\mathbb{T}^n$.
Result: A set B of elements of $\mathbb{Z}^n$

```
1  Let p ∈ X; Coord(p) = 0^n; S = {p }; B = ∅;
2  foreach x ∈ X do HasCoord(x) = false;
3  HasCoord(p) = true;
4  while there exists x ∈ S do
5      S = S \{x};
6      foreach non-null n-dimensional m-step v do
7          y = x ⊕_d v;
8          if y ∈ X and HasCoord(y) = true then
9              if Coord(y) ≠ Coord(x) + v then
10                 B = B ∪ ((Coord(x) + v - Coord(y))/ d);
11         else if y ∈ X and HasCoord(y) = false then
12             Coord(y) = Coord(x) + v;
13             S = S ∪{y};
14             HasCoord(y) = true;
15 return B
```

Remark 7 In Alg. 1, the division operation performed on line 10 is a ‘coordinate by coordinate’ division between elements of $\mathbb{Z}^n$.

To summarize, Alg. 1 ‘tries to embed’ the subset X of $\mathbb{T}^n$ in $\mathbb{Z}^n$: if some incompatible coordinates are detected by the test achieved on l. 9 of Alg. 1, then the object has a feature (a toric loop) which is incompatible with $\mathbb{Z}^n$. A toric 2-loop lying in X is depicted in Fig. 5-**f**.

Before proving Prop.1 (see Sec. 5.4), new definitions and theorems must be given: in particular, Th. 1 establishes an important result on homotopic loops in toric spaces. Before, let us study an example of execution of Alg. 1.

Example 2 Let us consider a subset X of points of $\mathbb{Z}_4 \times \mathbb{Z}_4$ (see Fig. 5-**a**) and the 2-adjacency relation. In Fig. 5-**a**, one element of X is chosen as $\mathbf{p}$ and is given the coordinates of the origin (see l. 1 of Alg. 1); then we set $\mathbf{x} = \mathbf{p}$. In Fig. 5-**b**, every neighbour $\mathbf{y}$ of $\mathbf{x}$ (l. 6,7) which is in X (l. 11) is given coordinates depending on its position relative to $\mathbf{x}$ (l. 12) and is added to the set S (l. 13).

Then, in Fig. 5-**c**, one element of S is chosen as $\mathbf{x}$ (l. 4). Every neighbour $\mathbf{y}$ of $\mathbf{x}$ is scanned (l. 6,7). If $\mathbf{y}$ is in X and has already been given some coordinates (l. 8), it is compared with $\mathbf{x}$: as the coordinates of $\mathbf{x}$ and $\mathbf{y}$ are compatible in $\mathbb{Z}^2$ (the test achieved l. 9 returns false), the set B remains empty. Else, if $\mathbf{y}$ is in X and has not previously been given coordinates (l. 11) (see Fig. 5-**d**), then it is given coordinates depending on its position relative to $\mathbf{x}$ (l. 12) and added to the set S.

Finally, in Fig. 5-**e**, another element of S is chosen as $\mathbf{x}$. The algorithm tests one of the neighbours

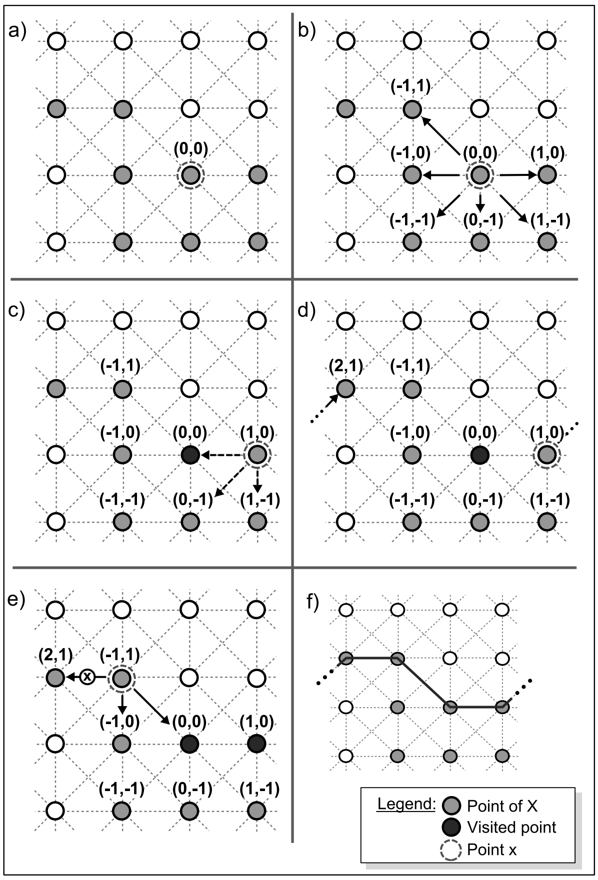

Fig. 5 Example of execution of WSD - see Ex. 2 for a detailed description.

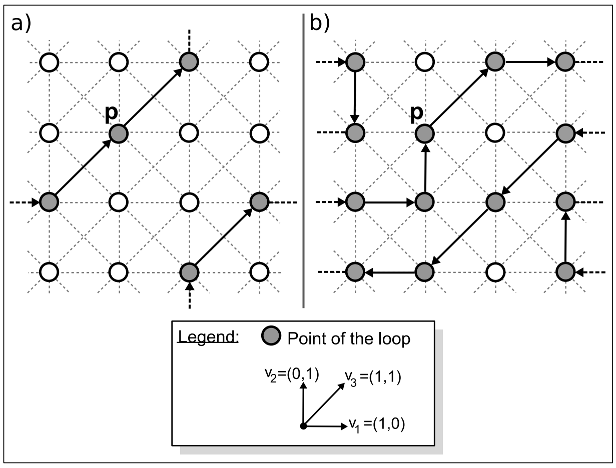

Fig. 6 Wrapping vector - In $\mathbb{T}^2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, the 2-loop in **a)** has a wrapping vector equal to $(4, 4)$, and the 2-loop in **b)** has a wrapping vector equal to $(0, 0)$.

$\mathbf{y}$ of $\mathbf{x}$ (the left neighbour) which is in X and has already some coordinates (l. 8). As the coordinates of $\mathbf{y}$ and $\mathbf{x}$ are incompatible in $\mathbb{Z}^2$ (the points $(-1, 1)$ and $(2, 1)$ are not neighbours in $\mathbb{Z}^2$), the algorithm adds $\frac{(-1,1)+(-1,0)-(2,1)}{4} = (-1, 0)$ to B (l. 10): according to Prop. 1, the subset X is wrapped in $\mathbb{T}^n$.

5 Wrapping Vector and Homotopy Classes in $\mathbb{T}^n$

Deciding if two loops $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to the same homotopy class can be difficult if one attempts to do this by building a sequence of directly homotopic loops which 'link' $\mathcal{L}_1$ and $\mathcal{L}_2$. However, this problem may be solved using the *wrapping vector*, a characteristic which can be easily computed on each loop.

5.1 Wrapping Vector of a Loop

The *wrapping vector* of a loop is the sum of all the elements of the m-step sequence associated to the loop.

Definition 8 Let $\mathcal{L} = (\mathbf{p}, V)$ be an m-loop, with $V = (\mathbf{v_1}, ..., \mathbf{v_k})$. Then the *wrapping vector of* $\mathcal{L}$ is $\mathbf{w}_{\mathcal{L}} = \sum_{i=1}^{k} \mathbf{v_i}$.

Remark 8 In Def. 8, the symbol $\sum$ stands for the iteration of the classical addition operation on $\mathbb{Z}^n$, not of the operation $\oplus$ defined in Sec. 2.1.

Example 3 In $\mathbb{T}^2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, depicted on Fig. 6, the loop $\mathcal{K} = (\mathbf{p}, (\mathbf{v_3}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_1}, \mathbf{v_3}))$ (see Fig. 6-**a**) has a wrapping vector equal to $(4, 4)$, while the loop $\mathcal{L} = (\mathbf{p}, (\mathbf{v_3}, \mathbf{v_1}, \mathbf{v_1}, -\mathbf{v_2}, -\mathbf{v_1}, -\mathbf{v_3}, -\mathbf{v_3}, -\mathbf{v_1}, -\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_1}, \mathbf{v_1}, \mathbf{v_2}))$ has a wrapping vector equal to $(0, 0)$ (see Fig. 6-**b**).

We now define the notion of 'basic loops', which will be used for the proof of Prop. 2 and for building, in Def. 11, a canonical loop for a given wrapping vector.

Definition 9 Let $\mathbb{T}^n$ be an n-dimensional toric space of size vector $\mathbf{d} = (d_1, ..., d_n)$. We denote, for each $i \in [1; n]$, by $\mathbf{b_i}$ the 1-step whose i-th coordinate is equal to 1, and by B_i the 1-step sequence composed of exactly d_i 1-steps $\mathbf{b_i}$.

Given $\mathbf{p} \in \mathbb{T}^n$, for all $i \in [1; n]$, we define *the i-th basic loop of base point* $\mathbf{p}$ as the 1-loop $(\mathbf{p}, B_i)$.

Remark 9 For all $i \in [1; n]$, the wrapping vector of the i-th basic loop of base point $\mathbf{p}$ is equal to $(d_i.\mathbf{b_i})$.

The next property establishes that the wrapping vector of any m-loop can only take specific values in $\mathbb{Z}^n$. The proof may be found in Sec. 8.

Proposition 2 *Let $\mathbb{T}^n$ be an n-dimensional toric space of size vector $\mathbf{d} = (d_1, ..., d_n)$. A vector $\mathbf{w} = (w_1, ..., w_n)$ of $\mathbb{Z}^n$ is the wrapping vector of an m-loop of $\mathbb{T}^n$ if and only if, for all $i \in [1; n]$, w_i is a multiple of d_i.*

Thanks to Prop. 2, we can now define the *normalized wrapping vector* of an m-loop.

Definition 10 Given $\mathbb{T}^n$ of size vector $\mathbf{d} = (d_1, ..., d_n)$, let $\mathcal{L}$ be an m-loop of wrapping vector $\mathbf{w} = (w_1, ..., w_n)$. The *normalized wrapping vector* of $\mathcal{L}$ is $\mathbf{w^*} = (w_1/d_1, ..., w_n/d_n)$.

Remark 10 It may be pointed out that, in Alg. 1, the set B contains the reduced wrapping vectors of loops contained in a set X.

Example 4 The wrapping vector and the normalized wrapping vector give information on how a loop 'wraps around' each dimension of a toric space before 'coming back to its starting point'. For example, let $\mathbb{T}^3 = \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_7$ (hence, the size vector of $\mathbb{T}^3$ is $(2, 5, 7)$). A loop with wrapping vector (4,5,0) has a normalized wrapping vector equal to (2,1,0): it wraps two times in the first dimension, one time in the second, and does not wrap in the third dimension.

On Fig. 6, the normalized wrapping vector of loop $\mathcal{K}$ (see Ex. 3), depicted on Fig. 6-**a**, is equal to $(1, 1)$, while the normalized wrapping vector of $\mathcal{L}$ (see Ex. 3), depicted on Fig. 6-**b**, is equal to $(0, 0)$.

It may easily be seen that, in $\mathbb{T}^n$, for each $i \in [1; n]$, the normalized wrapping vector of the i-th basic loop of any base point is equal to $\mathbf{b_i}$ (see Def. 9).

5.2 Equivalence Between Homotopy Classes and Wrapping Vector

It can be seen that two directly homotopic m-loops have the same wrapping vector, as their associated m-step sequences have the same sum. Therefore, we have the following property.

Proposition 3 *Two homotopic m-loops of $\mathbb{T}^n$ have the same wrapping vector.*

The following definition is necessary in order to understand Prop. 4 and its demonstration, leading to the main theorem of this article.

Definition 11 Let $\mathbf{p}$ be an element of $\mathbb{T}^n$, and $\mathbf{w^*} = (w_1^*, ..., w_n^*) \in \mathbb{Z}^n$.

The *canonical loop of base point* $\mathbf{p}$ *and normalized wrapping vector* $\mathbf{w^*}$ is the 1-loop $\prod_{i=1}^{n} (\mathbf{p}, B_i)^{w_i^*}$, where $(\mathbf{p}, B_i)$ is the i-th basic loop of base point $\mathbf{p}$.

Example 5 Consider $\mathbb{T}^4 = \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_1 \times \mathbb{Z}_2$, $\mathbf{w^*} = (1, 0, 1, -2)$ and $\mathbf{p} = (0, 0, 0, 0)$. The canonical loop of base point $\mathbf{p}$ and normalized wrapping vector $\mathbf{w^*}$ is the 1-loop $(\mathbf{p}, V)$ with:

$$\text{V}=(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-1\end{pmatrix})$$

Proposition 4 *Any m-loop of base point $\mathbf{p} \in \mathbb{T}^n$ and of normalized wrapping vector $\mathbf{w^*} \in \mathbb{Z}^n$ is homotopic to the canonical loop of base point $\mathbf{p}$ and of normalized wrapping vector $\mathbf{w^*}$.*

The proof of the previous proposition may be found in Sec. 8.

The previous proposition shows that the canonical loop of base point $\mathbf{p}$ and of normalized wrapping vector $\mathbf{w^*}$ can be seen as a canonical form for all loops of base point $\mathbf{p}$ and normalized wrapping vector $\mathbf{w^*}$.

From this, we deduce that two m-loops of same base point $\mathbf{p}$ and same normalized wrapping vector $\mathbf{w^*}$ are homotopic, as they both belong to the homotopy class of the canonical loop of base point $\mathbf{p}$ and of normalized wrapping vector $\mathbf{w^*}$.

Example 6 In $\mathbb{T}^2 = \mathbb{Z}^4 \times \mathbb{Z}^4$, let $\mathcal{L}$ be the 2-loop of base point $\mathbf{p}$ represented on Fig. 7-**a**. It can be seen that the normalized wrapping vector of $\mathcal{L}$ is equal to $(1, -1)$: this means that the loop wraps 1 time around the first dimension, and one time around the second dimension. The canonical loop of base point $\mathbf{p}$ and of normalized wrapping vector $(1, -1)$, represented on Fig. 7-**b**, belongs to the same homotopy class as $\mathcal{L}$ (Prop. 4).

We can now state the main theorem of this article, which is a direct consequence of Prop. 3 and Prop. 4.

Theorem 1 *Two m-loops of $\mathbb{T}^n$ of same base point are homotopic if and only if their wrapping vectors are equal.*

Remark 11 According to Th. 1, the homotopy class of the trivial loop $(\mathbf{p}, ())$ is the set of all m-loops of base point $\mathbf{p}$ that have a null wrapping vector.

Thus, the loop depicted on Fig. 6-**b** belongs to the homotopy class of the trivial loop.

5.3 Wrapping Vector and Fundamental Group

Given a point $\mathbf{p} \in \mathbb{T}^n$, we set $\Omega = \{\mathbf{w^*} \in \mathbb{Z}^n/$ *there exists an m-loop in $\mathbb{T}^n$ of base point* $\mathbf{p}$ *and of normalized*

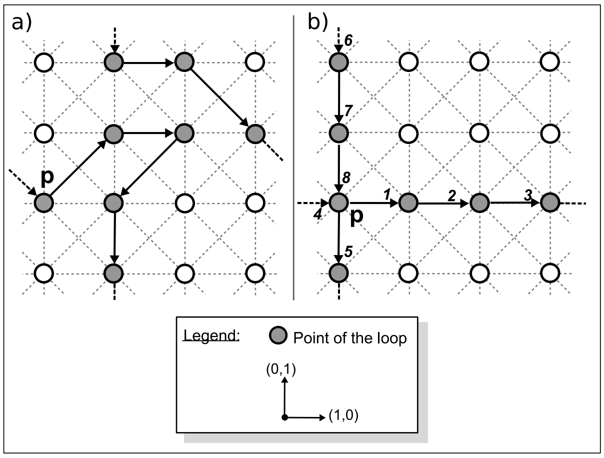

Fig. 7 In $\mathbb{T}^2 = \mathbb{Z}_4 \times \mathbb{Z}_4$, the 2-loop in **a)** has a normalized wrapping vector equal to $(1, -1)$. The 1-loop in **b)** is the canonical loop of base point $\mathbf{p}$ and normalized wrapping vector $(1, -1)$. On **b)**, the numbers represent the positions of the 1-steps in the 1-step sequence associated to the loop.

wrapping vector $\mathbf{w^*}$}. From Prop. 2, it is plain that $\Omega = \mathbb{Z}^n$. Therefore, $(\Omega, +)$ is precisely $(\mathbb{Z}^n, +)$

Theorem 1 states that there exists a bijection between the set of the homotopy classes of all m-loops of base point $\mathbf{p}$ and Ω. The product (see Sec. 3.2) of two m-loops $\mathcal{K}$ and $\mathcal{L}$ of same base point $\mathbf{p}$ and of respective wrapping vectors $\mathbf{w_k}$ and $\mathbf{w_l}$ is the loop $(\mathcal{K}.\mathcal{L})$ of base point $\mathbf{p}$. The wrapping vector of $(\mathcal{K}.\mathcal{L})$ is $(\mathbf{w_k} + \mathbf{w_l})$, therefore we can state that there exists an isomorphism between the fundamental group of $\mathbb{T}^n$ and $(\Omega, +)$.

Consequently, we retrieve in our discrete framework a well-known property of the fundamental group of toric spaces [5].

Proposition 5 *The fundamental group of* $\mathbb{T}^n$ *is isomorphic to* $(\mathbb{Z}^n, +)$.

5.4 Proof of Alg. 1

Proof (of Prop. 1) For all $\mathbf{y} \in X$ such that $\mathbf{y} \neq \mathbf{p}$, there exists a point $\mathbf{x}$ such that the test performed on l. 11 of Alg. 1 is true: we call $\mathbf{x}$ *the label predecessor of* $\mathbf{y}$.

• At the end of the execution of Alg. 1, if the set B is empty, then the test performed l. 9 was never true. Let $\mathcal{L} = (\mathbf{p}, V)$ be an m-loop contained in X, with $V = (\mathbf{v_1}, ..., \mathbf{v_k})$, and let us denote by $\mathbf{x_i}$ the i-th point of $\mathcal{L}$. As the test performed l. 9 was always false, we have the following:

$$\begin{cases} \text{for all } i \in [1; k-1], \mathbf{v_i} = Coord(\mathbf{x_{i+1}}) - Coord(\mathbf{x_i}) \\ \mathbf{v_k} = Coord(\mathbf{x_1}) - Coord(\mathbf{x_k}) \end{cases}$$

The wrapping vector of $\mathcal{L}$ is

$$\mathbf{w} = \sum_{i=1}^{k-1}(Coord(\mathbf{x_{i+1}}) - Coord(\mathbf{x_i})) \\ + Coord(\mathbf{x_1}) - Coord(\mathbf{x_k}) = \mathbf{0}$$

Thus, if the algorithm returns false, each m-loop of X has a null wrapping vector and, according to Th. 1, belongs to the homotopy class of a trivial loop: there is no toric m-loop in X which is therefore not wrapped in $\mathbb{T}^n$.

• If B is not empty, then, there exists $\mathbf{x}, \mathbf{y} \in X$ and an m-step $\mathbf{a}$ such that $\mathbf{x} \oplus \mathbf{a} = \mathbf{y}$ and $Coord(\mathbf{y}) - Coord(\mathbf{x}) \neq \mathbf{a}$.

It is therefore possible to find two sequences γ_x and γ_y of m-adjacent points in X, with $\gamma_x = (\mathbf{p} = \mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_q} = \mathbf{x})$ and $\gamma_y = (\mathbf{y} = \mathbf{y_t}, ..., \mathbf{y_2}, \mathbf{y_1} = \mathbf{p})$, such that, for all $i \in [1; q-1], \mathbf{x_i}$ is the label predecessor of $\mathbf{x_{i+1}}$, and for all $i \in [1; t-1], \mathbf{y_i}$ is the label predecessor of $\mathbf{y_{i+1}}$. Therefore, we can set

$$\begin{cases} \text{. for all } i \in [1; q-1], \mathbf{u_i} = Coord(\mathbf{x_{i+1}}) - Coord(\mathbf{x_i}) \\ \quad \text{is an m-step such that } \mathbf{x_i} \oplus \mathbf{u_i} = \mathbf{x_{i+1}} \\ \text{. for all } i \in [1; t-1], \mathbf{v_i} = Coord(\mathbf{y_i}) - Coord(\mathbf{y_{i+1}}) \\ \quad \text{is an m-step such that } \mathbf{y_{i+1}} \oplus \mathbf{v_i} = \mathbf{y_i} \end{cases}$$

Let $\mathcal{N}_{\mathbf{x},\mathbf{y},\mathbf{a}} = (\mathbf{p}, V)$ be the m-loop such that $V = (\mathbf{u_1}, ..., \mathbf{u_{q-1}}, \mathbf{a}, \mathbf{v_{t-1}}, ..., \mathbf{v_1})$. The m-loop $\mathcal{N}_{\mathbf{x},\mathbf{y},\mathbf{a}}$ is lying in X and its wrapping vector $\mathbf{w}$ is equal to:

$$\mathbf{w} = \sum_{i=1}^{q-1}\mathbf{u_i} + \mathbf{a} + \sum_{i=1}^{t-1}\mathbf{v_i} = \mathbf{a} - (Coord(\mathbf{y}) - Coord(\mathbf{x})) \\ \neq \mathbf{0}$$

Thus, when the algorithm returns a non-empty set, it is possible to find, inside X, an m-loop with a non-null wrapping vector: by Th. 1, there is a toric m-loop in X which is therefore wrapped in $\mathbb{T}^n$. □

The algorithm proposed in [3] returns a boolean telling whether the subset X is a wrapped subset of $\mathbb{T}^n$ or not. To obtain this algorithm from the code given in Alg. 1, it is sufficient to replace l. 10 by 'return true' and to replace l. 15 by 'return false'. The advantage of Alg. 1 is that it gives more information on the toric loops lying inside a wrapped subset X, as shown in Sec. 5.5.

5.5 Computing a Basis For Toric Loops in a Subset of $\mathbb{T}^n$

In this section, we show that Alg. 1 builds a basis for all normalized wrapping vectors of all toric m-loops contained in a subset of $\mathbb{T}^n$.

Given $\mathbb{T}^n$ of size vector $\mathbf{d}$ and an m-connected subset X of $\mathbb{T}^n$, we consider having run WSD$(n, m, \mathbb{T}^n, \mathbf{d}, X)$, and we will use *Coord*, the function built on l. 12 of Alg. 1.

Given an m-step $\mathbf{v}$ and two points $\mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{x} \oplus \mathbf{v} = \mathbf{y}$, *the points* $\mathbf{x}$ *and* $\mathbf{y}$ *are conflictive through* $\mathbf{v}$ if $Coord(\mathbf{x}) + \mathbf{v} \neq Coord(\mathbf{y})$. Observe that, for all conflictive pairs of points x, y through v contained in the subset X of $\mathbb{T}^n$, the vector $((Coord(\mathbf{x}) + \mathbf{v} - Coord(\mathbf{y}))/\mathbf{d})$ is added to the set B built on l. 10 of Alg. 1.

The next lemma establishes that, in order to calculate the wrapping vector of an m-loop (and therefore, its homotopy class, as stated by Th. 1), only the conflictive pairs of points in the loop need to be considered:

Lemma 1 *Let* $\mathbf{p} \in X$ *and let* $\mathcal{K} = (\mathbf{p}, V)$ *be an m-loop in* X*, with* $V = (\mathbf{v_1}, ..., \mathbf{v_k})$*. For all* $i \in [1; k+1]$*, we denote by* $\mathbf{x_i}$ *the i-th point of* $\mathcal{K}$*, and we set* $C = \{i \in [1; k] | \mathbf{x_i}$ *and* $\mathbf{x_{i+1}}$ *are conflictive through* $\mathbf{v_i}\}$*. Let* $\mathbf{w}$ *be the wrapping vector of* $\mathcal{K}$*. We have:*

$$\mathbf{w} = \sum_{j \in C} (Coord(\mathbf{x_j}) + \mathbf{v_j} - Coord(\mathbf{x_{j+1}}))$$

Proof The wrapping vector w of $\mathcal{K}$ is by definition:

$$\begin{aligned} \mathbf{w} &= \sum_{j=1}^{k} \mathbf{v_j} = \sum_{j \notin C} \mathbf{v_j} + \sum_{j \in C} \mathbf{v_j} \\ &= \sum_{j \notin C} (Coord(\mathbf{x_{j+1}}) - Coord(\mathbf{x_j})) + \sum_{j \in C} \mathbf{v_j} \\ &= \sum_{j=1}^{k} (Coord(\mathbf{x_{j+1}}) - Coord(\mathbf{x_j})) \\ &- \sum_{j \in C} (Coord(\mathbf{x_{j+1}}) - Coord(\mathbf{x_j})) + \sum_{j \in C} \mathbf{v_j} \end{aligned}$$

As $\sum_{j=1}^{k} (Coord(\mathbf{x_{j+1}}) - Coord(\mathbf{x_j}))) = Coord(\mathbf{x_{k+1}}) - Coord(\mathbf{x_1}) = 0$, we get the lemma proved. □

We now focus on the set B, result of WSD$(n, m, \mathbb{T}^n, \mathbf{d}, X)$. For all $\mathbf{x}, \mathbf{y} \in X$ that are conflictive through an m-step $\mathbf{v}$, the vector $((Coord(\mathbf{x}) + \mathbf{v} - Coord(\mathbf{y}))/\mathbf{d})$ is in B. The next proposition states that B can be seen as a generating set for all (normalized) wrapping vectors of all m-loops of X.

Proposition 6 *Let the set* $B = \{\mathbf{w_1}, ..., \mathbf{w_k}\}$ *be the result of WSD(n,m,* $\mathbb{T}^n$*,*$\mathbf{d}$*, X). A vector* $\mathbf{w^*} \in \mathbb{Z}^n$ *is the normalized wrapping vector of an m-loop of* X *if and only if there exists* k *non-negative integers* $\alpha_1, ..., \alpha_k$ *such that*

$$\mathbf{w^*} = \sum_{i=1}^{k} \alpha_i . \mathbf{w_i} \qquad (1)$$

Remark 12 If $\mathbf{x}$ and $\mathbf{y}$ are conflictive through $\mathbf{v}$, then $\mathbf{y}$ and $\mathbf{x}$ are conflictive through $(-\mathbf{v})$: therefore, if $\mathbf{u}$ belongs to B, then $-\mathbf{u}$ also belongs to B. This is why it is possible, in Prop. 6, to restrain the choice of the coefficients $\alpha_1, ..., \alpha_k$ to the set of non-negative integers.

Proof If $\mathcal{L}$ is an m-loop in X of normalized wrapping vector w^*, then, by Lem. 1 and by construction of B, we deduce that w^* satisfies Equ. 1.

Now, let w^* be a vector which satisfies Equ. 1. For each $\mathbf{b} \in B$, there exists $\mathbf{x}$ and $\mathbf{y}$ in X and an m-step $\mathbf{a}$ such that $\mathbf{x}$ and $\mathbf{y}$ are conflictive through $\mathbf{a}$ and such that $\mathbf{b} = (Coord(\mathbf{x}) + \mathbf{a} - Coord(\mathbf{y}))/(\mathbf{d})$. Consider the m-loop $\mathcal{N}_{\mathbf{x},\mathbf{y},\mathbf{a}}$ (see the second part of proof of Prop. 1), lying inside X, and whose wrapping vector is equal to $(Coord(\mathbf{x}) + \mathbf{a} - Coord(\mathbf{y}))$: the normalized wrapping vector of $\mathcal{N}_{\mathbf{x},\mathbf{y},\mathbf{a}}$ is $\mathbf{b}$.

Therefore, for each $\mathbf{b} \in B$, there exists an m-loop $\mathcal{L}_{\mathbf{b}}$ inside X, whose normalized wrapping vector is equal to $\mathbf{b}$. Let $\mathcal{L}^* = \prod_{i=1}^{k} (\mathcal{L}_{\mathbf{w_i}})^{\alpha_i}$. By construction, $\mathcal{L}^*$ is contained in X, and its wrapping vector is equal to $\mathbf{w^*}$. □

Thus, algorithm 1 builds a (non-minimal) basis allowing to compute the normalized wrapping vector of any m-loop of X: the normalized wrapping vector of any m-loop lying inside X is the linear combination of elements of B with non-negative coefficients. The set B, result of Alg. 1, allows to get information on how X wraps inside the toric space.

6 Conclusion

In this article, we give a formal definition of loops and homotopy, which suits all dimensions, inside discrete toric spaces in order to define various notions such as the fundamental group and the wrapping vector. Moreover, we show that wrapping vectors completely characterize toric loops (see Th. 1) and lead to a linear time algorithm for the detection of such loops in a subset X of $\mathbb{T}^n$. In addition, this algorithm allows to build, for each subset X of $\mathbb{T}^n$, a basis of vectors which characterizes all toric loops contained in X and describes how X wraps around $\mathbb{T}^n$.

In Sec. 1, we have seen that detecting toric loops is important in order to filter grains from a material's sample and perform a fluid flow simulation on the sample. The WSD algorithm proposed in this article detects which subsets of a sample, embedded inside a toric space, will create grains and should be removed. Future works will include analysis of the relationship between other topological characteristics of materials and their physical properties: for example, studying the skeleton of the pore space of a material could help to find new methods for performing fluid flow analysis.

7 Annex A: More About Loop Homotopy

An homotopy relation between loops in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, called *loop equivalence*, was defined in [6]. In this section, we first recall this definition and we show that, when adapted to toric spaces, it may give unwanted results. Then, we show that when adapted to $\mathbb{Z}^2$ and $\mathbb{Z}^3$, our notion of *loop homotopy* is equivalent to the *loop equivalence* defined in [6].

7.1 Loop Equivalence [6]

From now, we consider the discrete grid $\mathbb{Z}^2$ or $\mathbb{Z}^3$, and a subset X of grid points: the points of X (resp. $\overline{X}$) are called black (resp. white) points. As in [6], the loops we will consider are constrained, as in [6], to lie in X.

A *black m-loop of base point* $\mathbf{p} \in X$ is an m-loop $\mathcal{L} = (\mathbf{p}, U)$ (see Def. 3), with $U = (\mathbf{u_1}, ..., \mathbf{u_k})$, such that, for all $j \in [1; k]$, the j-th point of $\mathcal{L}$ is in X.

Definition 12 Let $\mathcal{K} = (\mathbf{p}, U)$ and $\mathcal{R} = (\mathbf{p}, V)$ be two black m-loops in $\mathbb{Z}^n$ ($n = 2$ or $n = 3$), with $U = (\mathbf{u_1}, ..., \mathbf{u_k})$ and $V = (\mathbf{v_1}, ..., \mathbf{v_r})$. For all $i \in [1; k+1]$, we denote by $\mathbf{x_i}$ the i-th point of $\mathcal{K}$, and for all $i \in [1; r+1]$, we denote by $\mathbf{y_i}$ the i-th point of $\mathcal{R}$.

The m-loops $\mathcal{K}$ and $\mathcal{R}$ *differ in a unit lattice square or unit lattice cube* J of $\mathbb{Z}^n$ if

. $k = r$, and
. $\mathbf{x_i} \in J$ if $\mathbf{y_i} \in J$, and
. $\mathbf{x_i} = \mathbf{y_i}$ if $\mathbf{y_i} \notin J$.

Definition 13 Let $\mathcal{K} = (\mathbf{p}, U)$ and $\mathcal{R} = (\mathbf{p}, V)$ be two black m-loops, with $(n, m) \in \{(2, 1); (2, 2); (3, 1); (3, 3)\}$. Let k and r be respectively the lengths of $\mathcal{K}$ and $\mathcal{R}$.

The two m-loops $\mathcal{K}$ and $\mathcal{R}$ *are directly equivalent* if one of the two following conditions is matched:

. Let $\tilde{U}$ and $\tilde{V}$ be the sequences obtained respectively from U and V by removing all null steps. We have $\tilde{U} = \tilde{V}$.
. There exists a unit lattice cube or unit lattice square J such that $\mathcal{K}$ and $\mathcal{R}$ differ in J, and if $(n, m) = (3, 1)$, the cube J does not contain two diametrically opposite white points.

Definition 14 Two black loops $\mathcal{K}$ and $\mathcal{R}$ *are equivalent* if there exists a sequence $(\mathcal{K} = \mathcal{C}_1, ..., \mathcal{C}_i = \mathcal{R})$ of black loops such that, for all $j \in [1; i-1], \mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are directly equivalent.

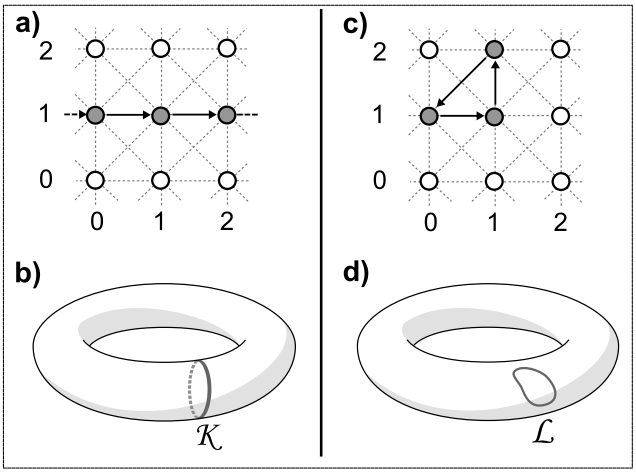

Fig. 8 Equivalent loops - In **a** and **c**: in $\mathbb{T}^2 = \mathbb{Z}_3 \times \mathbb{Z}_3$, the loops in **a)** and **c)** are equivalent (see Def. 13) but not homotopic (see Def. 4). In **b** and **d**: the two loops do not belong to the same homotopy class in $\mathbb{T}^2$, as one wraps around the toric space, and not the other.

7.2 Loop Equivalence in Toric Spaces Gives Unwanted Results

In order to adapt Def. 13 to our discrete toric framework it is necessary to replace all occurrences of '$\mathbb{Z}^n$' by '$\mathbb{T}^n$' and 'unit lattice cube' (resp. 'unit lattice square') by 'toric unit lattice cube' (resp. 'toric unit lattice square'). Moreover, all the points of $\mathbb{T}^n$ are black, therefore all conditions depending on the colours of the points of the space can be ignored.

The following example pinpoints that Def. 13, adapted to our discrete toric framework, can produce unwanted results.

Example 7 Given a two-dimensional toric space $(\mathbb{T}^2, \oplus)$ whose points are all black, with $\mathbb{T}^2 = \mathbb{Z}_3 \times \mathbb{Z}_3$, let us consider the element $\mathbf{p} = (0; 1)$, the 2-steps $\mathbf{v_1} = (1; 0)$, $\mathbf{v_2} = (0; 1)$, $\mathbf{v_3} = (-1; -1)$ and the 2-loops $\mathcal{K} = (\mathbf{p}, (\mathbf{v_1}, \mathbf{v_1}, \mathbf{v_1}))$ and $\mathcal{L} = (\mathbf{p}, (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}))$. It can be seen on Fig. 8 that $\mathcal{K}$ and $\mathcal{L}$ do not belong to the same homotopy class in $\mathbb{T}^2$ ($\mathcal{K}$ wraps around the toric space, but $\mathcal{L}$ does not), however, based on Def. 13 adapted to toric spaces, they are equivalent.

The fundamental group of $\mathbb{T}^2$ obtained from Def. 13 is trivial, which is in contradiction with Prop. 5.

This is why we introduced a new definition of loop homotopy for toric spaces in this article (see Def. 4 and Def. 5).

7.3 Comparing Black Loop Homotopy and Black Loop Equivalence in $\mathbb{Z}^2$ and $\mathbb{Z}^3$

It is possible to adapt all definitions given previously in this article to $\mathbb{Z}^n$, by replacing the operation '$\oplus$' by the usual operation '+'. This way, we define the direct homotopy of black m-loops in $\mathbb{Z}^n$: two black m-loops of same base point $p \in \mathbb{Z}^n$ *are directly homotopic* if they are directly homotopic in the sense of definition 4 adapted to $\mathbb{Z}^n$.

We accordingly define the homotopy of black m-loops in $\mathbb{Z}^n$: two black m-loops $\mathcal{K}$ and $\mathcal{R}$ of same base point $p \in \mathbb{Z}^n$ *are homotopic* if there exists a sequence $(\mathcal{K} = \mathcal{C}_1, ..., \mathcal{C}_i = \mathcal{R})$ of black m-loops such that, for all $j \in [1; i-1], \mathcal{C}_i$ and $\mathcal{C}_{i+1}$ are directly homotopic.

A non self-intersecting loop is an m-loop $(\mathbf{p}, (\mathbf{u_1}, ..., \mathbf{u_k}))$ such that, for all $i \in [1; k[$, and for all $j \in]i; k]$, with $(i, j) \neq (1; k), \sum_{h=i}^{j} \mathbf{u_h} \neq 0$. We now introduce a lemma which will be used in the proof of the forecoming proposition.

Lemma 2 *In $\mathbb{Z}^3$, any non self-intersecting black 1-loop contained in a unit lattice cube which does not contain two diametrically opposite white points, is homotopic to a trivial loop.*

Proof A non self-intersecting 1-loop $(\mathbf{p}, U)$ holding inside a unit lattice cube is such that $|U| \in \{0, 2, 4, 6, 8\}$.

When considering all possible symmetries and rotations in the unit lattice cube of $\mathbb{Z}^3$, only 5 kinds of non self-intersecting (and non trivial) black 1-loops can be built (as shown on Fig. 9). For example, only one kind of 1-loop composed of eight 1-steps can be built (see Fig. 9a): let us call it $(\mathbf{p}, (\mathbf{u_1}, ..., \mathbf{u_8}))$. It is plain that, in order to pass by each of the cube's eight vertices once and only once, we must have $\{\mathbf{u_1}, ..., \mathbf{u_8}\} = \{\mathbf{v_a}, \mathbf{v_a}, -\mathbf{v_a}, -\mathbf{v_a}, \mathbf{v_b}, -\mathbf{v_b}, \mathbf{v_c}, -\mathbf{v_c}\}$, with $\mathbf{v_a}, \mathbf{v_b}, \mathbf{v_c}$ being 1-steps of $\mathbb{Z}^3$ such that $\mathbf{v_a} \neq \pm\mathbf{v_b}$, $\mathbf{v_a} \neq \pm\mathbf{v_c}$ and $\mathbf{v_b} \neq \pm\mathbf{v_c}$. In order to avoid the loop to self-intersect, we must have $\mathbf{u_1} = \mathbf{v_a}$ or $\mathbf{u_2} = \mathbf{v_a}$.

If we choose $\mathbf{u_2} = \mathbf{v_a}$, then, in order to avoid the loop to self-intersect, we need $\mathbf{u_6} = \mathbf{v_a}$ and $\mathbf{u_4} = \mathbf{u_8} = -\mathbf{v_a}$. Then, we set $\mathbf{u_1} = \mathbf{v_b}$, and consequently, $\mathbf{u_3} = \mathbf{v_c}$, $\mathbf{u_5} = -\mathbf{v_b}$, and $\mathbf{u_7} = -\mathbf{v_c}$. Choosing $\mathbf{u_1} = \mathbf{v_a}$ leads to a symmetrical loop.

A similar reasoning, in the case $|U| = 6$, shows that the two loops $(\mathbf{p}, (\mathbf{v_a}, \mathbf{v_b}, \mathbf{v_c}, -\mathbf{v_a}, -\mathbf{v_b}, -\mathbf{v_c}))$ (Fig. 9b) and $(\mathbf{p}, (\mathbf{v_a}, \mathbf{v_b}, -\mathbf{v_a}, \mathbf{v_c}, -\mathbf{v_b}, -\mathbf{v_c}))$ (see Fig. 9c) are the only configurations of non self-intersecting loops that can be built in the unit lattice cube of $\mathbb{Z}^3$. The cases $|U| = 4$ and $|U| = 2$ are even simpler, each with one possible coniguration of non-self intersecting loop:

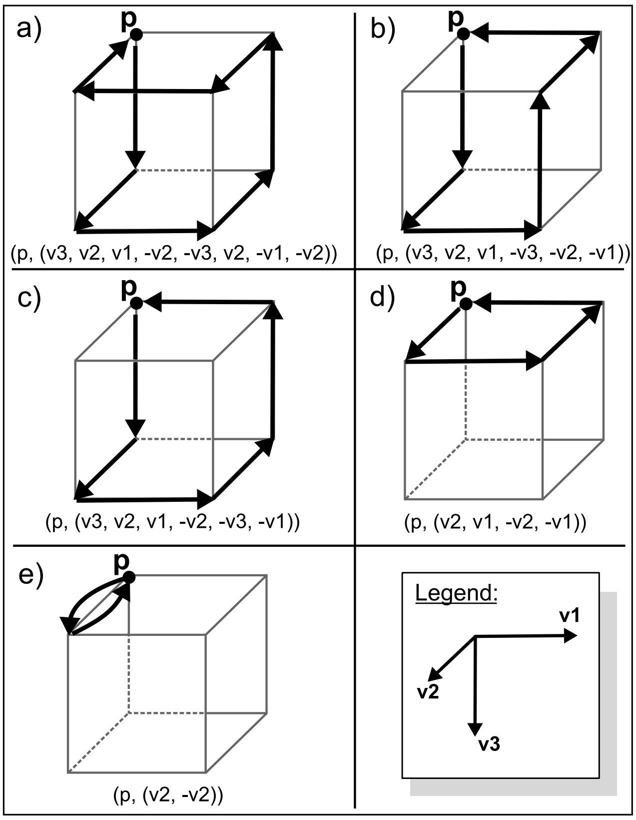

Fig. 9 In a unit lattice cube, when considering all possible symmetries, only 5 different non self-intersecting black 1-loops can be built. If the cube does not contain two diametrically opposite white points, the black 1-loops are all equivalent to a trivial loop.

$(\mathbf{p}, (\mathbf{v_a}, \mathbf{v_b}, -\mathbf{v_a}, -\mathbf{v_b}))$ (see Fig. 9d) and $(\mathbf{p}, (\mathbf{v_a}, -\mathbf{v_a}))$ (see Fig. 9e).

The five kinds of non self-intersecting black 1-loop which can exist in a unit lattic cube are represented on Fig.9. It is plain that each of these loops can be reduced to a trivial loop if the cube does not contain two diametrically opposite white points. □

The next proposition establishes that, in $\mathbb{Z}^n$, black m-loop homotopy and black m-loop equivalence defined in [6] (see Def. 13) are equivalent.

Proposition 7 *Two black m-loops $\mathcal{K} = (\mathbf{p}, U)$ and $\mathcal{R} = (\mathbf{p}, V)$ in $\mathbb{Z}^n$ (with $(n, m) \in \{(2, 1); (2, 2); (3, 1); (3, 3)\}$) are equivalent if and only if they are homotopic.*

Proof In the following proof, we set $U = (\mathbf{u_1}, ..., \mathbf{u_k})$, $V = (\mathbf{v_1}, ..., \mathbf{v_r})$, and, for all $i \in [1; k+1]$, we denote by $\mathbf{x_i}$ the i-th point of $\mathcal{K}$, and for all $i \in [1; r+1]$, we denote by $\mathbf{y_i}$ the i-th point of $\mathcal{R}$. We have, for all $i \in [1; k]$, $\mathbf{u_i} = (\mathbf{x_{i+1}} - \mathbf{x_i})$ and, for all $i \in [1; r]$, $\mathbf{v_i} = (\mathbf{y_{i+1}} - \mathbf{y_i})$.

If $k \neq r$, then it can be easily seen that $\mathcal{K}$ and $\mathcal{R}$ are directly equivalent if and only if they are homotopic.

Now, let us consider the case where $k = r$. We define

$D_K = \{\mathbf{x} \in \mathcal{K}|$ there exists $i \in [2;k]$ such that $\mathbf{x_i} \neq \mathbf{y_i}\}$

$D_R = \{\mathbf{y} \in \mathcal{R}|$ there exists $i \in [2;k]$ such that $\mathbf{y_i} \neq \mathbf{x_i}\}$.

Note that, for all $i \in [2;k]$, $x_i \in D_K$ if and only if $y_i \in D_R$. Thus, $(D_K \cup D_R)$ is the set of all points of $\mathcal{K}$ which differ from the corresponding point of $\mathcal{R}$, and vice versa.

i) Suppose that $\mathcal{K}$ and $\mathcal{R}$ are directly homotopic (see Def. 4, case 3), then there exists $j \in [1;k-1]$ such that $V = (\mathbf{u_1}, ..., \mathbf{u_{j-1}}, \mathbf{v_j}, \mathbf{v_{j+1}}, \mathbf{u_{j+2}}, ..., \mathbf{u_k})$, where $(u_j - v_j)$ is an n-step and $(u_j + u_{j+1} = v_j + v_{j+1})$. Obviously, we have $(D_K \cup D_R) = \{\mathbf{x_{j+1}}, \mathbf{y_{j+1}}\}$.

As $(\mathbf{x_{j+1}} - \mathbf{y_{j+1}}) = (\mathbf{u_j} - \mathbf{v_j})$, the points $\mathbf{x_{j+1}}$ and $\mathbf{y_{j+1}}$ lie in a same unit lattice square or cube. Furthermore, if $n = 3$ and $m = 1$, $(\mathbf{u_j} - \mathbf{v_j})$ is a 2-step, proving that $\mathbf{x_{j+1}}$ and $\mathbf{y_{j+1}}$ lie in a same unit lattice square (no diametrically opposite white points to matter): the m-loops $\mathcal{K}$ and $\mathcal{R}$ are directly equivalent.

ii) Reciprocally, suppose that $\mathcal{K}$ and $\mathcal{R}$ are directly equivalent.

• In the case where $(n, m) \in \{(2, 2), (3, 3)\}$, we set, for all $h \in [1;k]$, $S_h = (\mathbf{v_1}, ..., \mathbf{v_{h-1}}, \mathbf{x_{h+1}} - \mathbf{y_h}, \mathbf{u_{h+1}}, ..., \mathbf{u_k})$ and $\mathcal{C}_h = (p, S_h)$.

First, we prove that for all $h \in [1;k]$, $\mathcal{C}_h$ is an m-loop of base point p, by proving that $(\mathbf{x_{h+1}} - \mathbf{y_h})$ is an m-step. As $\mathcal{K}$ and $\mathcal{R}$ are directly equivalent, we either have $\mathbf{x_h} = \mathbf{y_h}$ or $\mathbf{x_{h+1}} = \mathbf{y_{h+1}}$ (the result is then directly obtained), or we have $\mathbf{x_h}, \mathbf{y_h}, \mathbf{x_{h+1}}$ and $\mathbf{y_{h+1}}$ lying in a same unit lattice cube or square: $(\mathbf{x_{h+1}} - \mathbf{y_h})$ is therefore an n-step, and also an m-step since $n = m$.

We are going to prove that for all $h \in [1;k-1], \mathcal{C}_h$ and $\mathcal{C}_{h+1}$ are directly homotopic by proving that they match the case 3 of Def. 4. We set $S_h = (\mathbf{a_1}, ..., \mathbf{a_k})$, and $S_{h+1} = (\mathbf{b_1}, ..., \mathbf{b_k})$:

- $S_{h+1} = (\mathbf{a_1}, ..., \mathbf{a_{h-1}}, \mathbf{b_h}, \mathbf{b_{h+1}}, \mathbf{a_{h+2}}, ..., \mathbf{a_k})$,
- $(\mathbf{a_h} + \mathbf{a_{h+1}}) = \mathbf{x_{h+1}} - \mathbf{y_h} + \mathbf{u_{h+1}} = \mathbf{x_{h+2}} - \mathbf{y_h}$, and $(\mathbf{b_h} + \mathbf{b_{h+1}}) = \mathbf{v_h} + \mathbf{x_{h+2}} - \mathbf{y_{h+1}} = \mathbf{x_{h+2}} - \mathbf{y_h}$,
- $(\mathbf{a_h} - \mathbf{b_h}) = \mathbf{x_{h+1}} - \mathbf{y_h} - \mathbf{v_h} = \mathbf{x_{h+1}} - \mathbf{y_{h+1}}$ is an n-step, as either $\mathbf{x_{h+1}} = \mathbf{y_{h+1}}$ or $\mathbf{x_{h+1}}$ and $\mathbf{y_{h+1}}$ belong to a same unit lattice cube or square, and also an m-step since $n = m$.

Finally, by pointing out that $\mathcal{C}_1$ is equal to $\mathcal{K}$ and that $\mathcal{C}_k$ is equal to $\mathcal{R}$, we conclude that $\mathcal{K}$ and $\mathcal{R}$ are homotopic.

• In the case where $(n, m) = (3, 1)$, let us assume that the set D_K (resp. D_R) contains only consecutive points of the loop $\mathcal{K}$ (resp. $\mathcal{R}$): if it was not the case, the following reasoning could still be performed on each consecutive elements of D_K and D_R in order to obtain the same result.

Thus, there exists $i \in [2;k]$ and $j \in [i;k]$ such that $(D_K \cup D_R) = \{\mathbf{x_i}, ..., \mathbf{x_j}, \mathbf{y_i}, ..., \mathbf{y_j}\}$ is included in a unit lattice square or a unit lattice cube which does not contain two diametrically opposite white points. Therefore, we have $V = (\mathbf{u_1}, ..., \mathbf{u_{i-2}}, \mathbf{v_{i-1}}, ..., \mathbf{v_j}, \mathbf{u_{j+1}}, ..., \mathbf{u_k})$. It is possible to simplify the problem in two ways:

- As $m = 1$, $\mathbf{y_i} - \mathbf{x_{i-1}}$ and $\mathbf{x_i} - \mathbf{x_{i-1}}$ are 1-steps. Therefore, $\mathbf{x_{i-1}}, \mathbf{x_i}$ and $\mathbf{y_i}$ are in a same unit lattice square and, as $\mathbf{x_i} \neq \mathbf{y_i}$, we find that $\mathbf{x_{i-1}}$ lie in the same unit lattice cube or square than the elements of $(D_K \cup D_R)$. The same way, we prove that $\mathbf{x_{j+1}}$ lie in the same unit lattice cube or square than the elements of $(D_K \cup D_R)$.
  It may be seen that $\mathcal{K}$ is homotopic to the black 1-loop $\mathcal{K}' = (\mathbf{p}, (\mathbf{u_1}, ..., \mathbf{u_j}, -\mathbf{v_j}, ..., -\mathbf{v_{i-1}}, \mathbf{v_{i-1}}, ..., \mathbf{v_j}, \mathbf{u_{j+1}}, ..., \mathbf{u_k}))$.
  Hence, proving that $\mathcal{K}'$ and $\mathcal{R}$ are homotopic can be achieved by proving that the black 1-loop $(\mathbf{x_{i-1}}, (\mathbf{u_{i-1}}, ..., \mathbf{u_j}, -\mathbf{v_j}, ..., -\mathbf{v_{i-1}}))$, whose points are contained inside the same unit lattice cube or square than $(D_K \cup D_R)$, is homotopic to the trivial loop $(\mathbf{x_{i-1}}, ())$.
- Let $\mathcal{C} = (\mathbf{p}, (\mathbf{w_1}, ..., \mathbf{w_i}, ..., \mathbf{w_j}, ..., \mathbf{w_k}))$ be a self intersecting black 1-loop such that $\mathbf{p} + \mathbf{w_1} + ... + \mathbf{w_i} = \mathbf{p} + \mathbf{w_1} + ... + \mathbf{w_j}$. The problem of showing that $\mathcal{C}$ is homotopic to $(\mathbf{p}, ())$ can be decomposed into two smaller problems: to prove that $\mathcal{C}' = (\mathbf{p} + \mathbf{w_1} + ... + \mathbf{w_i}, (\mathbf{w_{i+1}}, ..., \mathbf{w_j}))$ is homotopic to $(\mathbf{p} + \mathbf{w_1} + ... + \mathbf{w_i}, ())$, and then to prove that $\mathcal{C}'' = (\mathbf{p}, (\mathbf{w_1}, ..., \mathbf{w_i}, \mathbf{w_{j+1}}, ..., \mathbf{w_k}))$ is homotopic to $(\mathbf{p}, ())$. Therefore, in order to prove that a black 1-loop is homotopic to a trivial loop, we can consider only, without loss of generality, non self-intersecting black 1-loops.

Therefore, in order to prove that the two black 1-loops $\mathcal{K}$ and $\mathcal{L}$ are homotopic, it is sufficient to prove that any non self-intersecting black 1-loop, contained in a unit lattice cube which does not contain two diametrically opposite white points, is homotopic to a trivial loop: this is established by Lem. 2.

As the case $(n, m) = (2, 1)$ is included in the case $(n, m) = (3, 1)$, it can be concluded that $\mathcal{K}$ and $\mathcal{R}$ are homotopic. □

8 Annex B: Lemmas and proofs

Proof of Prop. 2 - First, let $\mathcal{L} = (\mathbf{p}, V)$ be an m-loop of wrapping vector $\mathbf{w} = (w_1, ..., w_n)$, with $\mathbf{p} = (p_1, ..., p_n)$. As $\mathcal{L}$ is a loop, for all $i \in [1;n]$, $p_i \oplus_{d_i} w_i = p_i$. Hence, for all $i \in [1;n]$, $w_i \equiv 0(mod\ d_i)$, proving that w_i is a multiple of d_i for all $i \in [1;n]$.

Let $\mathbf{w} = (w_1, ..., w_n)$ be a vector of $\mathbb{Z}^n$ such that for all $i \in [1;n]$, w_i is a multiple of d_i. If we denote by $(\mathbf{p}, B_i)$ the i-th basic loop of base point $\mathbf{p}$, we see that

$(\prod_{i=1}^{n}(\mathbf{p}, B_i)^{w_i/d_i})$ is an m-loop whose wrapping vector is equal to $\mathbf{w}$. □

Lemma 3 *Any m-loop $\mathcal{L} = (\mathbf{p}, V)$ is homotopic to a 1-loop.*

Proof Let us write $V = (\mathbf{v_1}, ..., \mathbf{v_k})$ and let $j \in [1; n]$ be such that $\mathbf{v_j}$ is not a 1-step. The m-loop $\mathcal{L}$ is directly homotopic to $\mathcal{L}_1 = (\mathbf{p}, V_1)$, with $V_1 = (\mathbf{v_1}, ..., \mathbf{v_{j-1}}, \mathbf{v_j}, \mathbf{0}, \mathbf{v_{j+1}}, ..., \mathbf{v_k})$. As $\mathbf{v_j}$ is not a 1-step, there exists an (m-1)-step $\mathbf{v'_j}$ and a 1-step $\mathbf{v_{j1}}$ such that $\mathbf{v_j} = (\mathbf{v_{j1}} + \mathbf{v'_j})$. The m-loop $\mathcal{L}_1$ is directly homotopic to $\mathcal{L}_2 = (\mathbf{p}, V_2)$, with $V_2 = (\mathbf{v_1}, ..., \mathbf{v_{j-1}}, \mathbf{v_{j1}}, \mathbf{v'_j}, \mathbf{v_{j+1}}, ..., \mathbf{v_k})$. By iteration, it is shown that $\mathcal{L}$ is homotopic to a 1-loop. □

Lemma 4 *Let $\mathcal{L}_A = (\mathbf{p}, V_A)$ and $\mathcal{L}_B = (\mathbf{p}, V_B)$ be two m-loops such that $V_A = (\mathbf{v_1}, ..., \mathbf{v_{j-1}}, \mathbf{v_{j1}}, \mathbf{v_{j2}}, \mathbf{v_{j+1}}, ..., \mathbf{v_k})$ and $V_B = (\mathbf{v_1}, ..., \mathbf{v_{j-1}}, \mathbf{v_{j2}}, \mathbf{v_{j1}}, \mathbf{v_{j+1}}, ..., \mathbf{v_k})$ where $\mathbf{v_{j1}}$ and $\mathbf{v_{j2}}$ are 1-steps. Then, $\mathcal{L}_A$ and $\mathcal{L}_B$ are homotopic.*

Proof As $\mathbf{v_{j1}}$ and $\mathbf{v_{j2}}$ are 1-steps, they have at most one non-null coordinate. If $(\mathbf{v_{j1}} - \mathbf{v_{j2}})$ is an n-step, the two loops are directly homotopic. If $(\mathbf{v_{j1}} - \mathbf{v_{j2}})$ is not an n-step, then necessarily $\mathbf{v_{j1}} = (-\mathbf{v_{j2}})$. Therefore, $\mathcal{L}_A$ is directly homotopic to $\mathcal{L}_C = (\mathbf{p}, V_C)$, with $V_C = (\mathbf{v_1}, ..., \mathbf{v_{j-1}}, \mathbf{0}, \mathbf{0}, \mathbf{v_{j+1}}, ..., \mathbf{v_k})$. Furthermore, $\mathcal{L}_C$ is also directly homotopic to $\mathcal{L}_B$. □

Proof of Prop. 4 - Let $\mathbf{a}$ and $\mathbf{b}$ be two non-null 1-steps. Let i (resp. j) be the index of the non-null coordinate of $\mathbf{a}$ (resp $\mathbf{b}$). We say that $\mathbf{a}$ *is index-smaller than* $\mathbf{b}$ if $i < j$.

Let $\mathcal{L} = (\mathbf{p}, V)$ be an m-loop of normalized wrapping vector $\mathbf{w^*} \in \mathbb{Z}^n$.

. **1 -** The m-loop $\mathcal{L}$ is homotopic to a 1-loop $\mathcal{L}_1 = (\mathbf{p}, V_1)$ (see Lem. 3).
. **2 -** By Def. 4 and 5, the 1-loop $\mathcal{L}_1$ is homotopic to a 1-loop $\mathcal{L}_2 = (\mathbf{p}, V_2)$, where V_2 contains no null vector.
. **3 -** Let $\mathcal{L}_3 = (\mathbf{p}, V_3)$ be such that V_3 is obtained by iteratively permuting all pairs of consecutive 1-steps $(\mathbf{v_j}, \mathbf{v_{j+1}})$ in V_2 such that $\mathbf{v_{j+1}}$ is index-smaller than $\mathbf{v_j}$. Thanks to Lem. 4, $\mathcal{L}_3$ is homotopic to $\mathcal{L}_2$.
. **4 -** Consider $\mathcal{L}_4 = (\mathbf{p}, V_4)$, where V_4 is obtained by iteratively replacing all pairs of consecutive 1-steps $(\mathbf{v_j}, \mathbf{v_{j+1}})$ in V_3 such that $\mathbf{v_{j+1}} = (-\mathbf{v_j})$ by two null vectors, and then removing these two null vectors. The loop $\mathcal{L}_4$ is homotopic to $\mathcal{L}_3$.

The 1-loop $\mathcal{L}_4$ is homotopic to $\mathcal{L}$, it has therefore the same normalized wrapping vector $\mathbf{w^*} = (w_1^*, ..., w_n^*)$ (see Prop. 3). By construction, each pair of consecutive 1-steps $(\mathbf{v_j}, \mathbf{v_{j+1}})$ of V_4 is such that $\mathbf{v_j}$ and $\mathbf{v_{j+1}}$ are non-null and either $\mathbf{v_j} = \mathbf{v_{j+1}}$ or $\mathbf{v_j}$ is index-smaller than $\mathbf{v_{j+1}}$.

Let $\mathbf{d} = (d_1, ..., d_n)$ be the size vector of $\mathbb{T}^n$. As the normalized wrapping vector of $\mathcal{L}_4$ is equal to $\mathbf{w^*}$, we deduce that the $(d_1.|w_1^*|)$ first elements of V_4 are equal to $(\frac{w_1^*}{|w_1^*|}.\mathbf{b_1})$ (see Def. 9). Moreover, the $(d_2.|w_2^*|)$ next elements are equal to $(\frac{w_2^*}{|w_2^*|}.\mathbf{b_2})$, etc. Therefore, we have $\mathcal{L}_4 = (\prod_{i=1}^{n}(\mathbf{p}, B_i)^{w_i^*})$. □

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