Erratum — Derived Satake morphisms for p-small weights in characteristic p

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Abstract

We provide a corrected statement and proof of Lemma 2.5.4 of [KP25], as well as a corrected proof of Lemma 2.5.7.

We maintain the notation of [KP25], so that F is a finite, unramified extension of $\mathbb{Q}_p$ of degree $f \geq 1$, and $\mathbf{G}$ is a split connected reductive group over $\mathcal{O}_F$ with fixed split maximal torus $\mathbf{T}$. We also let W denote the Weyl group of $\underline{\mathbf{G}}_{k_F} (= \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}(\mathbf{G})_{k_F})$ relative to $\underline{\mathbf{T}}_{k_F} (= \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p}(\mathbf{T})_{k_F})$, and recall that it decomposes as a direct product of $f = [k_F : \mathbb{F}_p]$ copies of $W(\mathbf{G}, \mathbf{T})$ indexed by the field isomorphisms $\varsigma : k_F \longrightarrow k_F$ over $\mathbb{F}_p$. In what follows, we denote by $\varsigma_0 : k_F \longrightarrow k_F$ the identity map, and define $\varsigma_i(x) := \varsigma_0(x)^{p^i}$, with the indices considered modulo f, so that $\mathrm{Gal}(k_F/\mathbb{F}_p) = \{\varsigma_0, \varsigma_1, \ldots, \varsigma_{f-1}\}$. We then write elements of W in terms of this labeling, namely, as f-tuples $(w_0, w_1, \ldots, w_{f-1})$ of elements of $W(\mathbf{G}, \mathbf{T})$ indexed by $\mathbb{Z}/f\mathbb{Z}$. Similar statements hold for other groups (e.g., $X^*(\underline{\mathbf{T}}), \underline{\Phi}$, etc.). We also use $\mathsf{F} : \underline{\mathbf{G}}_{\mathbb{F}_p} \longrightarrow \underline{\mathbf{G}}_{\mathbb{F}_p}$ to denote the relative Frobenius endomorphism, which stabilizes $\underline{\mathbf{T}}_{\mathbb{F}_p}$.

The omission in [KP25, Lem. 2.5.4] stems from a misuse of [DL76, Thm. 5.13]. Indeed, the fourth condition of [KP25, Lem. 2.5.4] ("for every $\alpha \in \Phi^+$, there exists $i \in \mathbb{Z}/f\mathbb{Z}$ such that $\langle \lambda + \rho, \alpha_i^\vee \rangle \neq p - 1$") amounts to saying that the finite-order character $(\lambda + \rho)|_{T_0}$ is nonsingular ([DL76, Def. 5.15(i)]). By [DL76, Prop. 5.16], this means that $(\lambda + \rho)|_{T_0}$ is in general position, meaning it is not fixed by any non-trivial element of W^{F} ([DL76, Def. 5.15(ii)]). Thus, the characters $(w \cdot \lambda)|_{T_0}$ are all distinct for $w \in W^{\mathsf{F}}$, though not necessarily for $w \in W$. (Incidentally, the notation $(\lambda+\rho)|_{T_0}$ is also a bit abusive, since ρ is not necessarily an element of $X^*(\underline{\mathbf{T}})$. We address this point in the proof below.) The proof of [KP25, Lem. 2.5.7] uses similar ideas, and we give a corrected proof below.

We thank Ben Savoie for bringing the following example to our attention, which helped us identify the gaps.

0.0.1. Example. Suppose F is the quadratic unramified extension of $\mathbb{Q}_p$ with $p > 3$ (so that $f = 2$), and let $\mathbf{G} = \mathrm{GL}_{2/F}$. We define elements $\lambda, \rho' \in X_*(\underline{\mathbf{T}}) \cong (\mathbb{Z}^{\oplus 2})^f$ by

$$\lambda := ((p-1, 0), (0, 0)), \qquad \rho' := ((1, 0), (1, 0)),$$

so that $w \cdot \lambda = w(\lambda + \rho') - \rho'$ for $w \in W$. Thus, all of the conditions of [KP25, Lem. 2.5.4] are satisfied. Note that the characters $(\lambda + \rho')|_{T_0}$ and $(s, s)(\lambda + \rho')|_{T_0}$ are indeed distinct (where s denotes the nontrivial element of the Weyl group): they are given by

$$(\lambda + \rho')\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \overline{x}^{p+1\cdot p} = \overline{x}^{2p}, \qquad (s, s)(\lambda + \rho')\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \overline{y}^{p+1\cdot p} = \overline{y}^{2p}$$

($x, y \in \mathcal{O}_F^\times$). However, the same is not true of the characters $(s, 1)(\lambda + \rho')|_{T_0}$ and $(1, s)(\lambda + \rho')|_{T_0}$: these are given by

$$(s, 1)(\lambda + \rho')\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \overline{x}^{1\cdot p}\overline{y}^p = (\overline{xy})^p, \qquad (1, s)(\lambda + \rho')\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \overline{x}^p\overline{y}^{1\cdot p} = (\overline{xy})^p.$$

The issue here is that although the character $(\lambda + \rho')|_{T_0}$ satisfies the third condition of [KP25, Lem. 2.5.4], the character $(s, 1)(\lambda + \rho')|_{T_0}$ does not.

The corrected statement is as follows. Note that we impose a slightly stronger genericity condition on λ than p-smallness.

0.0.2. Lemma. *Suppose that:*

◇ *[KP25, Assumption 2.1.1] holds;*

◇ *the center of* $\mathbf{G}$ *is connected;*

◇ $\mathbf{M} = \mathbf{T}$ *(so that* ${}^J W = W$*);*

◇ $\lambda \in X^*(\underline{\mathbf{T}})_+$ *satisfies* $\langle \lambda + \rho, \alpha_i^\vee \rangle \leq p - 1$ *for all* $\alpha \in \Phi^+$ *and all* $0 \leq i \leq f-1$*;*

◇ *for every* $\alpha \in \Phi^+$*, the* f*-tuple* $(\langle \lambda + \rho, \alpha_i^\vee \rangle)_{0 \leq i \leq f-1}$ *is not equal to* $(p-1, p-1, \ldots, p-1)$*.*

Then [KP25, Assumption 2.5.2] holds. In particular, we have a splitting

$$\mathrm{R}H^0(U_0, L(\lambda)) \cong \bigoplus_{n=0}^{\dim(U_0)} H^n(U_0, L(\lambda))[-n]$$

in $\mathrm{D}(T_0)$*. Moreover, each* $H^n(U_0, L(\lambda))$ *splits as*

$$H^n(U_0, L(\lambda)) \cong \bigoplus_{\substack{w \in W \\ \ell(w) = n}} k(w \cdot \lambda)$$

in $\mathrm{D}(T_0)^\heartsuit$*, and* $\mathrm{R}H^0(U_0, L(\lambda))$ *is multiplicity-free.*

Proof. We will prove the stronger assertion that if $v, w \in W$ and T_0 acts by the same character on $L_\emptyset(v \cdot \lambda) = k(v \cdot \lambda)$ and $L_\emptyset(w \cdot \lambda) = k(w \cdot \lambda)$, then $v = w$. This will imply that the T_0-modules $k(w \cdot \lambda)$ for $w \in W$ are pairwise distinct, and imply the stronger claimed splitting statement. We proceed in several steps.

Step 1. We first reduce to the case where $\mathbf{G}$ has simply connected derived subgroup. Given $\mathbf{G}$ as in the statement of the lemma, choose a z-extension

$$1 \longrightarrow \widetilde{\mathbf{Z}} \longrightarrow \widetilde{\mathbf{G}} \xrightarrow{q} \mathbf{G} \longrightarrow 1,$$

where $\widetilde{\mathbf{G}}$ is a split connected reductive group over $\mathcal{O}_F$ with simply connected derived subgroup and $\widetilde{\mathbf{Z}}$ is a split central torus. Since $\mathbf{G}$ has connected center, the same is true of $\widetilde{\mathbf{G}}$. The above short exact sequence gives a short exact sequence of maximal tori

$$1 \longrightarrow \widetilde{\mathbf{Z}} \longrightarrow \widetilde{\mathbf{T}} \longrightarrow \mathbf{T} \longrightarrow 1,$$

where $\widetilde{\mathbf{T}}$ is defined as the preimage of $\mathbf{T}$. Furthermore, by taking special fibers and restricting scalars from k_F to $\mathbb{F}_p$, we obtain short exact sequences of character and cocharacter groups:

$$0 \longrightarrow X^*(\underline{\mathbf{T}}) \xrightarrow{\underline{q}^*} X^*(\underline{\widetilde{\mathbf{T}}}) \longrightarrow X^*(\underline{\widetilde{\mathbf{Z}}}) \longrightarrow 0$$

and

$$0 \longrightarrow X_*(\underline{\widetilde{\mathbf{Z}}}) \longrightarrow X_*(\underline{\widetilde{\mathbf{T}}}) \longrightarrow X_*(\underline{\mathbf{T}}) \longrightarrow 0.$$

We note that the above short exact sequences identify roots (resp., coroots, resp., Weyl groups) of $(\underline{\mathbf{G}}, \underline{\mathbf{T}})$ and $(\underline{\widetilde{\mathbf{G}}}, \underline{\widetilde{\mathbf{T}}})$.

Given $\lambda \in X^*(\underline{\mathbf{T}})_+$ as in the statement of the lemma, the character $\underline{q}^*\lambda \in X^*(\underline{\widetilde{\mathbf{T}}})_+$ satisfies the same hypotheses (i.e., dominance, bounds, and disallowed cases). Suppose then that $(v \cdot \lambda)|_{T_0} = (w \cdot \lambda)|_{T_0}$, so that $v \cdot \lambda - w \cdot \lambda \in (\mathsf{F} - 1)X^*(\underline{\mathbf{T}})$ by [DL76, Eqn. (5.2.2)*]. Pulling back via $\underline{q}$ gives $\underline{q}^*(v \cdot \lambda) - \underline{q}^*(w \cdot \lambda) \in (\mathsf{F} - 1)\underline{q}^* X^*(\underline{\mathbf{T}}) \subset (\mathsf{F} - 1)X^*(\underline{\widetilde{\mathbf{T}}})$. Furthermore, since $v \cdot \lambda \in v(\lambda) + \mathbb{Z}\underline{\Phi}$ (and likewise for w), we deduce that $\underline{q}^*(v \cdot \lambda) - \underline{q}^*(w \cdot \lambda) = v \cdot (\underline{q}^*\lambda) - w \cdot (\underline{q}^*\lambda)$. Thus, we have $(v \cdot \underline{q}^*\lambda)|_{\widetilde{T}_0} = (w \cdot \underline{q}^*\lambda)|_{\widetilde{T}_0}$. Assuming the desired claim for $\widetilde{\mathbf{G}}$, we obtain $v = w$. We may therefore assume that $\mathbf{G}$ has simply connected derived subgroup.

Step 2. Since we may assume $\mathbf{G}$ has simply connected derived subgroup, for each $\alpha \in \underline{\Delta}$, there exists $\lambda_\alpha \in X^*(\underline{\mathbf{T}})$ which satisfies $\langle \lambda_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for $\beta \in \underline{\Delta}$. The element $\rho' := \sum_{\alpha \in \underline{\Delta}} \lambda_\alpha$ is then an element of $X^*(\underline{\mathbf{T}})$, and we have $w \cdot \lambda = w(\lambda + \rho) - \rho = w(\lambda + \rho') - \rho'$. Hence, the condition $(v \cdot \lambda)|_{T_0} = (w \cdot \lambda)|_{T_0}$ is equivalent to $v(\lambda + \rho')|_{T_0} = w(\lambda + \rho')|_{T_0}$. Thus, we may assume $v(\lambda + \rho')|_{T_0} = w(\lambda + \rho')|_{T_0}$, and prove this condition implies $v = w$.

Step 3. The equality of the elements $v, w \in W$ may be checked one irreducible component at a time. Therefore, we may assume that Φ is irreducible.

Step 4. Next, we obtain preliminary bounds on $v(\lambda + \rho') - w(\lambda + \rho')$. Set $\mu := \lambda + \rho'$, and suppose we have $v, w \in W$ which satisfy $v(\mu)|_{T_0} = w(\mu)|_{T_0}$. By [DL76, Eqn. (5.2.2)*], this means that

$$v(\mu) - w(\mu) = (\mathsf{F} - 1)\nu', \qquad (\diamondsuit)$$

where $\nu' \in X^*(\underline{\mathbf{T}})$. (Explicitly, we have $\mathsf{F}((\nu'_0, \nu'_1, \ldots, \nu'_{f-1})) = (p\nu'_1, \ldots, p\nu'_{f-1}, p\nu'_0)$ for $(\nu'_0, \nu'_1, \ldots, \nu'_{f-1}) \in X^*(\underline{\mathbf{T}})$.) Using the fact that the center of $\mathbf{G}$ is connected, we see that the right-hand side of ($\diamondsuit$) lies in $(\mathsf{F} - 1)X^*(\underline{\mathbf{T}}) \cap \mathbb{Z}\underline{\Phi} = (\mathsf{F} - 1)\mathbb{Z}\underline{\Phi}$ (see [DL76, pf. of Thm. 5.13, point (c)]), so $\nu' \in \mathbb{Z}\underline{\Phi}$. Setting $\nu := \mathsf{F}(v^{-1})(\nu') \in \mathbb{Z}\underline{\Phi}$, we may rewrite ($\diamondsuit$) as

$$\mu - v^{-1}w(\mu) = (\mathsf{F} - v^{-1}\mathsf{F}(v))\nu. \qquad (\diamondsuit\diamondsuit)$$

Let $\alpha \in \Phi$. Pairing equation ($\diamondsuit$) with $\alpha_i^\vee$ gives

$$\langle v(\mu), \alpha_i^\vee \rangle - \langle w(\mu), \alpha_i^\vee \rangle = p\langle \nu', \alpha_{i+1}^\vee \rangle - \langle \nu', \alpha_i^\vee \rangle.$$

Hence, we obtain

$$\begin{aligned} \sum_{j=0}^{f-1} \langle v(\mu) - w(\mu), \alpha_{i+j}^\vee \rangle\, p^j &= \sum_{j=0}^{f-1} \Big(p^{j+1} \langle \nu', \alpha_{i+j+1}^\vee \rangle - p^j \langle \nu', \alpha_{i+j}^\vee \rangle \Big) \\ &= (p^f - 1)\langle \nu', \alpha_i^\vee \rangle. \end{aligned}$$

Since the "p-adic digits" of the left-hand side are between $-2p+2$ and $2p-2$, we conclude that $\langle \nu', \alpha_i^\vee \rangle \in \{\pm 2, \pm 1, 0\}$ for all $\alpha \in \Phi$ and all $0 \le i \le f-1$. Using $\nu = \mathsf{F}(v^{-1})(\nu')$, we also obtain $\langle \nu, \alpha_i^\vee \rangle \in \{\pm 2, \pm 1, 0\}$.

Finally, we see that pairing equation ($\diamondsuit\diamondsuit$) with $\alpha_i^\vee$ gives

$$\langle \mu, \alpha_i^\vee \rangle - \langle v^{-1}w(\mu), \alpha_i^\vee \rangle = p\langle \nu, \alpha_{i+1}^\vee \rangle - \langle v^{-1}\mathsf{F}(v)(\nu), \alpha_i^\vee \rangle = p\langle \nu, \alpha_{i+1}^\vee \rangle - \langle \nu, v_{i-1}^{-1}v_i(\alpha^\vee)_i \rangle, \qquad (\diamondsuit\diamondsuit\diamondsuit)$$

where we have used that $\mathsf{F}(v)_i = v_{i-1}$. The bounds on $\lambda + \rho$ imply the left-hand side is at least $-2p+2$, and at most $2p-2$. We will use equation ($\diamondsuit\diamondsuit\diamondsuit$) to obtain more precise information about the $\langle \nu, \alpha_i^\vee \rangle$.

Step 5. We verify the lemma when Φ is of type A_1. Let $\alpha \in \Phi^+$ denote the unique positive root, and note that $w^{-1}v(\alpha_i^\vee) = \pm\alpha_i^\vee$ for all choices of $v, w \in W$ and all i. In particular, the left-hand side of equation ($\diamondsuit\diamondsuit\diamondsuit$) is either 0 or $2\langle \mu, \alpha_i^\vee \rangle$, from which we deduce $\langle \nu, \alpha_{i+1}^\vee \rangle \ge 0$ for all i. Furthermore, since $\nu \in \mathbb{Z}\underline{\Phi} = \bigoplus_{0 \le i \le f-1} \mathbb{Z}\alpha_i$, the quantity $\langle \nu, \alpha_{i+1}^\vee \rangle$ must be even. Hence, by Step 4, we have $\langle \nu, \alpha_{i+1}^\vee \rangle \in \{0, 2\}$ for all i. We consider 3 cases:

- ◇ If $\langle \nu, \alpha_i^\vee \rangle = 0$ for all i, then $\nu = 0$. By equation ($\diamondsuit\diamondsuit$) we obtain $\mu = v^{-1}w(\mu)$, which implies $v^{-1}w = 1 \in W$.
- ◇ If $\langle \nu, \alpha_i^\vee \rangle = 2$ for all i, then equation ($\diamondsuit\diamondsuit\diamondsuit$) and the bounds on $\lambda + \rho$ imply that we must have

  $$\langle \mu - v^{-1}w(\mu), \alpha_i^\vee \rangle = 2p - 2$$

  for all i. Since we have $\langle \mu - v^{-1}w(\mu), \alpha_i^\vee \rangle = 2\langle \mu, \alpha_i^\vee \rangle$ or 0, we conclude that $\langle \mu, \alpha_i^\vee \rangle = p - 1$ for all i, contradicting the assumptions of the lemma.
- ◇ Suppose we are not in one of the two above cases (in particular, $f \ge 2$), so that there exists an embedding i_0 such that $\langle \nu, \alpha_{i_0}^\vee \rangle = 0$ and $\langle \nu, \alpha_{i_0+1}^\vee \rangle = 2$. The right-hand side of ($\diamondsuit\diamondsuit\diamondsuit$) (for the embedding i_0) then becomes $2p$, which forces $\langle \mu, \alpha_{i_0}^\vee \rangle = p$, contradicting the bounds on $\lambda + \rho$.

Thus, in the remainder of the proof we may assume that Φ is not of type A_1.

Step 6. We now obtain stronger bounds on $\langle \nu, \alpha_i^\vee \rangle$ when $\alpha^\vee$ is not the highest coroot. Note that we have the following bounds on the left-hand side of equation ($\diamondsuit\diamondsuit\diamondsuit$):

$$\langle \mu - v^{-1}w(\mu), \alpha_i^\vee \rangle \in \begin{cases} [-p+2, 2p-2] & \text{if } \alpha \in \Phi^+, \\ [-2p+2, p-2] & \text{if } \alpha \in \Phi^-. \end{cases} \qquad (\clubsuit)$$

Define a sequence of roots $\beta^{(1)}, \beta^{(2)}, \ldots$ by $\beta^{(j)} = v_{i-j}^{-1}v_i(\alpha)$. Applying equation ($\diamondsuit\diamondsuit\diamondsuit$) to the root $\beta^{(j)}$ and the embedding $i - j$ gives

$$\langle \mu, \beta_{i-j}^{(j),\vee} \rangle - \langle v^{-1}w(\mu), \beta_{i-j}^{(j),\vee} \rangle = p\langle \nu, \beta_{i-j+1}^{(j),\vee} \rangle - \langle \nu, v_{i-j-1}^{-1}v_{i-j}(\beta^{(j),\vee})_{i-j} \rangle = p\langle \nu, \beta_{i-j+1}^{(j),\vee} \rangle - \langle \nu, \beta_{i-j}^{(j+1),\vee} \rangle.$$

Hence, the bounds on $\langle \mu - v^{-1}w(\mu), \alpha_i^\vee \rangle$ from equation ($\clubsuit$) and on $\langle \nu, \alpha_i^\vee \rangle$ from Step 4 show that $\langle \nu, \beta_{i-j+1}^{(j),\vee} \rangle = 2$ (resp., $\langle \nu, \beta_{i-j+1}^{(j),\vee} \rangle = -2$) implies $\langle \nu, \beta_{i-j}^{(j+1),\vee} \rangle = 2$ and $\beta^{(j)} \in \Phi^+$ (resp., $\langle \nu, \beta_{i-j}^{(j+1),\vee} \rangle = -2$ and $\beta^{(j)} \in \Phi^-$).

Now let $\widehat{\alpha} \in \Phi^+$ denote the highest short root, so that $\widehat{\alpha}^\vee \in \Phi^{\vee,+}$ is the highest (long) coroot. Let $\alpha \in \Phi^+ - \{\widehat{\alpha}\}$, and note that the dominance condition on λ implies $1 \le \langle \mu, \alpha_i^\vee \rangle \le p - 2$ for all $0 \le i \le f-1$. In particular, the left-hand side of ($\diamondsuit\diamondsuit\diamondsuit$) lies between $-p+2$ and $2p-3$, from which we conclude $\langle \nu, \alpha_{i+1}^\vee \rangle \in \{\pm 1, 0\}$. Furthermore, we claim that $\langle \nu, v_{i-1}^{-1}v_i(\alpha^\vee)_i \rangle \ne \pm 2$. Indeed, if we had $\langle \nu, v_{i-1}^{-1}v_i(\alpha^\vee)_i \rangle = \pm 2$ then the previous paragraph would imply that for the sequence of roots $\beta^{(1)}, \beta^{(2)}, \ldots$ associated to α, we would have $\langle \nu, \beta_{i-j}^{(j+1),\vee}\rangle = \pm 2$ for all j. In particular, we would get

$$\langle \mu - v^{-1}w(\mu), \alpha_i^\vee\rangle = \langle \mu - v^{-1}w(\mu), \beta_i^{(f),\vee}\rangle = p\langle \nu, \beta_{i+1}^{(f),\vee}\rangle - \langle \nu, \beta_i^{(f+1),\vee}\rangle = \pm(2p-2),$$

contradicting the lower and upper bounds of $-p+2$ and $2p-3$.

Combining these observations with the bounds

$$\langle \mu - v^{-1}w(\mu), \alpha_i^\vee\rangle \in \begin{cases} [-p+2, p-3] & \text{if } w_i^{-1}v_i(\alpha) \in \Phi^+, \\ [2, 2p-3] & \text{if } w_i^{-1}v_i(\alpha) \in \Phi^-, \end{cases}$$

we conclude

$$\langle \mu - v^{-1}w(\mu), \alpha_i^\vee\rangle \in \begin{cases} \{\pm 1, 0\} & \text{if } w_i^{-1}v_i(\alpha) \in \Phi^+, \\ \{p \pm 1, p\} & \text{if } w_i^{-1}v_i(\alpha) \in \Phi^-. \end{cases}$$

Hence, we see that

$$w_i^{-1}v_i(\alpha) \in \Phi^+ \text{ implies } \langle \nu, \alpha_{i+1}^\vee\rangle = 0, \text{ and } w_i^{-1}v_i(\alpha) \in \Phi^- \text{ implies } \langle \nu, \alpha_{i+1}^\vee\rangle = 1. \qquad (\clubsuit\clubsuit)$$

Step 7. We prove that $v(\mu)|_{T_0} = w(\mu)|_{T_0}$ implies $v = w$. Consider the highest coroot $\widehat{\alpha}^\vee = \sum_{\alpha\in\Delta} m_\alpha \alpha^\vee$, and note that $\langle \nu, \widehat{\alpha}_i^\vee\rangle \in \{0, 1, 2\}$ for all $0 \le i \le f-1$ by ($\clubsuit\clubsuit$). We examine three cases:

- ◇ Suppose first that $\langle \nu, \widehat{\alpha}_{i+1}^\vee\rangle = 0$ for some embedding $i+1$. The positivity of the m_α then implies that $\langle \nu, \alpha_{i+1}^\vee\rangle = 0$ for all $\alpha \in \Delta$, which by ($\clubsuit\clubsuit$) implies $w_i^{-1}v_i(\alpha) \in \Phi^+$ for all $\alpha \in \Delta$. Thus, we get $w_i^{-1}v_i = 1$, which by equation ($\diamondsuit\diamondsuit\diamondsuit$) implies

$$p\langle \nu, \alpha_{i+1}^\vee\rangle - \langle \nu, v_{i-1}^{-1}v_i(\alpha_i^\vee)\rangle = \langle \mu - v^{-1}w(\mu), \alpha_i^\vee\rangle = 0$$

  for all $\alpha \in \Phi$. In particular, applying the above equation to $v_i^{-1}v_{i-1}(\alpha)$ for $\alpha \in \Delta$ shows that $\langle \nu, \alpha_i^\vee\rangle = 0$ for all $\alpha \in \Delta$ (and thus $\langle \nu, \widehat{\alpha}_i^\vee\rangle = 0$). Hence, by induction we conclude that $w_i^{-1}v_i = 1$ for all $0 \le i \le f-1$.

- ◇ Suppose next that $\langle \nu, \widehat{\alpha}_{i+1}^\vee\rangle = 1$ for some embedding $i+1$. Hence $\nu_{i+1} \neq 0$, and ($\clubsuit\clubsuit$) implies ν_{i+1} is dominant. By [Bou81, Ch. VI, Exers. du §1, Exer. 24(c)], ν_{i+1} is equal to a fundamental dominant weight ϖ_α for some α which satisfies $m_\alpha = 1$. However, [Bou81, Ch. VI, §2.3, Cor.] implies that $\varpi_\alpha \notin \bigoplus_{\beta\in\Delta} \mathbb{Z}\beta$, and we arrive at a contradiction.

- ◇ By the above two bullet points, we may assume $\langle \nu, \widehat{\alpha}_i^\vee\rangle = 2$ for all $0 \le i \le f-1$. In this case, equations ($\diamondsuit\diamondsuit\diamondsuit$) and ($\clubsuit$) imply

$$\langle \mu - v^{-1}w(\mu), \widehat{\alpha}_i^\vee\rangle = 2p-2,$$

  which gives $\langle \mu, \widehat{\alpha}_i^\vee\rangle = p-1$ for all i. However, this contradicts the assumptions of the lemma.

□

0.0.3. Remark. Suppose $\mathbf{G} = \mathrm{GL}_{2/F}$, where F is an unramified extension of $\mathbb{Q}_p$ of degree f. In this case, we can verify Lemma 0.0.2 assuming our original p-small bounds on $\lambda+\rho$, namely that $\langle \lambda+\rho, \alpha_i^\vee\rangle \le p$ for all $\alpha \in \Phi^+$ and all $0 \le i \le f-1$, subject to the following exclusions: the f-tuple $(\langle \lambda+\rho, \alpha_i^\vee\rangle)_{0\le i\le f-1}$ is not in either of the following forms:

- ◇ $(p-1, p-1, \ldots, p-1)$, or
- ◇ containing a substring of length at least 2 of the form $(p, p-1, \ldots, p-1, 1)$ (with indices considered modulo f).

Indeed, we may proceed exactly as in Steps 1 through 4 of the above proof. In Step 5, we consider the third case, and note that if we are not in the first two cases (so that $f \ge 2$), the f-tuple $(\langle \nu, \alpha_i^\vee\rangle)_{0\le i\le f-1}$ contains a substring of the form $(0, 2, \ldots, 2, 0)$ (with indices considered modulo f, and the entry 2 appearing at least once). Hence, equation ($\diamondsuit\diamondsuit\diamondsuit$) and the p-smallness assumption imply that $(\langle \mu - v^{-1}w(\mu), \alpha_i^\vee\rangle)_{0\le i\le f-1}$ contains a substring of length at least 2 of the form $(2p, 2p-2, \ldots, 2p-2, 2)$. Since we have $\langle \mu - v^{-1}w(\mu), \alpha_i^\vee\rangle = 2\langle \mu, \alpha_i^\vee\rangle$ or 0, we conclude that $(\langle \mu, \alpha_i^\vee\rangle)_{0\le i\le f-1}$ contains a substring of length at least 2 of the form $(p, p-1, \ldots, p-1, 1)$, contradicting the above assumption.

There is an interesting link between the splitting behavior of the complex $\mathrm{R}H^0(U_0, L(\lambda))$ in the GL_2 case, and the geometry of the Emerton–Gee stack. Let us fix a p-small $\lambda \in X^*(\underline{\mathbf{T}})_+$. On the one hand, if λ is not in one of the excluded cases, the above paragraph shows that the complex $\mathrm{R}H^0(U_0, L(\lambda))$ splits in $\mathrm{D}(T_0)$. On the other hand, the $\mathbf{G}(k_F)$-representation $L(\lambda)$ corresponds to an irreducible component $\mathcal{X}(L(\lambda))$ of the reduced Emerton–Gee stack $\mathcal{X}_{2,\mathrm{red}}$, and this component is non-smooth if and only if λ belongs to one of the excluded cases ([KS25, Thm. 1.1]). We do not yet know of a direct relationship between the splitting of $\mathrm{R}H^0(U_0, L(\lambda))$ and the smoothness of $\mathcal{X}(L(\lambda))$, however.

Lemma 2.5.7 of [KP25] is correct as stated, though our proof contains a gap (namely, nontriviality of the character $(w \cdot 0)|_{T_0}$ for nontrivial $w \in W$ does not immediately imply that $(v \cdot 0)|_{T_0} \neq (w \cdot 0)|_{T_0}$ for distinct $v, w \in W$). We provide the corrected proof below.

Proof of Lemma 2.5.7 of [KP25]. We adapt the proof of Lemma 0.0.2 (for $\lambda = 0$), with the following modifications:

- ◇ Steps 1 through 3 hold as before. (In fact, we may skip Step 3: the proof below does not require that Φ be irreducible.)
- ◇ In Step 4, since we have dropped the assumption that the center of $\mathbf{G}$ is connected, we can no longer conclude that $\nu' \in \mathbb{Z}\underline{\Phi}$. On the other hand, the quantity $\langle v(\rho') - w(\rho'), \alpha_i^\vee \rangle = \mathrm{ht}(v^{-1}(\alpha_i^\vee)) - \mathrm{ht}(w^{-1}(\alpha_i^\vee))$ lies between $-2h + 2$ and $2h - 2$. Since $h \leq p - 2$ by [KP25, Assumption 2.1.1], we see that $\langle \nu', \alpha_i^\vee \rangle$ and $\langle \nu, \alpha_i^\vee \rangle$ both lie in $\{\pm 1, 0\}$ for all choices of $\alpha \in \Phi$ and $0 \leq i \leq f - 1$.
- ◇ Suppose there exist $\alpha \in \Delta$ and $0 \leq i \leq f - 1$ which satisfy $w_i^{-1} v_i(\alpha) \in \Phi^-$ (so that $w^{-1}v(\alpha_i) \in \underline{\Phi}^-$). We then have

  $$\langle \rho' - v^{-1}w(\rho'), \alpha_i^\vee \rangle = \mathrm{ht}(\alpha_i^\vee) + \mathrm{ht}(-w^{-1}v(\alpha_i^\vee)) = 1 + \mathrm{ht}(-w^{-1}v(\alpha_i^\vee)) \in [2, h] \subset [2, p - 2].$$

  By equation (◇◇◇), this quantity equals $p\langle \nu, \alpha_{i+1}^\vee \rangle - \langle \nu, v_{i-1}^{-1} v_i(\alpha^\vee)_i \rangle$. Using the bounds on $\langle \nu, \alpha_i^\vee \rangle$ from the previous bullet point, we arrive at a contradiction. Therefore, we conclude that $w_i^{-1} v_i(\alpha) \in \Phi^+$ for all choices of $\alpha \in \Delta$ and $0 \leq i \leq f - 1$, from which we deduce $v = w$.

□

1 Conflict of interest statement

On behalf of all authors, Cédric Pépin states that there is no conflict of interest.

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