Generic and Mod p Kazhdan-Lusztig Theory for GL_2

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Abstract

Let F be a non-archimedean local field with residue field $\mathbb{F}_q$ and let $\mathbf{G} = GL_{2/F}$. Let $\mathbf{q}$ be an indeterminate and let $\mathcal{H}^{(1)}(\mathbf{q})$ be the generic pro-p Iwahori-Hecke algebra of the p-adic group $\mathbf{G}(F)$. Let $V_{\widehat{\mathbf{G}}}$ be the Vinberg monoid of the dual group $\widehat{\mathbf{G}}$. We establish a generic version for $\mathcal{H}^{(1)}(\mathbf{q})$ of the Kazhdan-Lusztig-Ginzburg antispherical representation, the Bernstein map and the Satake isomorphism. We define the flag variety for the monoid $V_{\widehat{\mathbf{G}}}$ and establish the characteristic map in its equivariant K-theory. These generic constructions recover the classical ones after the specialization $\mathbf{q} = q \in \mathbb{C}$. At $\mathbf{q} = q = 0 \in \overline{\mathbb{F}}_q$, the antispherical map provides a dual parametrization of all the irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}(0)$-modules. When $F = \mathbb{Q}_p$ with $p \geq 5$, we relate our space of mod p Satake parameters to Emerton-Gee's space of semisimple mod p two-dimensional representations of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, thereby arriving at a version in families of Breuil's semisimple mod p Langlands correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$.

Contents

1 Introduction

Let F be a non-archimedean local field with ring of integers o_F and residue field $\mathbb{F}_q$. Let $\mathbf{G}$ be a connected split reductive group over F. Let $\mathcal{H}_k = (k[I \setminus \mathbf{G}(F)/I], \star)$ be the Iwahori-Hecke algebra, i.e. the convolution algebra associated to an Iwahori subgroup $I \subset \mathbf{G}(F)$, with coefficients in an algebraically closed field k. On the other hand, let $\widehat{\mathbf{G}}$ be the Langlands dual group of $\mathbf{G}$ over k, with maximal torus and Borel subgroup $\widehat{\mathbf{T}} \subset \widehat{\mathbf{B}}$ respectively. Let W_0 be the finite Weyl group.

When $k = \mathbb{C}$, the irreducible $\mathcal{H}_\mathbb{C}$-modules appear as subquotients of the Grothendieck group $K^{\widehat{\mathbf{G}}}(\widehat{\mathbf{G}}/\widehat{\mathbf{B}})_\mathbb{C}$ of $\widehat{\mathbf{G}}$-equivariant coherent sheaves on the dual flag variety $\widehat{\mathbf{G}}/\widehat{\mathbf{B}}$. As such they can be parametrized by the isomorphism classes of irreducible tame $\widehat{\mathbf{G}}(\mathbb{C})$-representations of the Weil group $\mathcal{W}_F$ of F, thereby realizing the tame local Langlands correspondence (in this setting also called the Deligne-Lusztig conjecture for Hecke modules): Kazhdan-Lusztig [KL87], Ginzburg [CG97]. Their approach to the Deligne-Lusztig conjecture is based on two steps: the first step develops the theory of the so-called *antispherical representation* leading to a certain dual parametrization of Hecke modules. The second step links these dual data to representations of the group $\mathcal{W}_F$.

The antispherical representation is a distinguished faithful action of the Hecke algebra $\mathcal{H}_\mathbb{C}$ on its maximal commutative subring $\mathcal{A}_\mathbb{C} \subset \mathcal{H}_\mathbb{C}$ via $\mathcal{A}_\mathbb{C}^{W_0}$-linear operators: elements of the subring $\mathcal{A}_\mathbb{C}$ act by multiplication, whereas the standard Hecke operators $T_s \in \mathcal{H}_\mathbb{C}$, supported on double cosets indexed by simple reflections $s \in W_0$, act via the classical Demazure operators [D73, D74]. The link with the geometry of the dual group comes then in two steps. First, the classical Bernstein map $\tilde{\theta}$ identifies the ring of functions $\mathbb{C}[\widehat{\mathbf{T}}]$ with $\mathcal{A}_\mathbb{C}$, such that the invariants $\mathbb{C}[\widehat{\mathbf{T}}]^{W_0}$ become the center $Z(\mathcal{H}_\mathbb{C}) = \mathcal{A}_\mathbb{C}^{W_0}$. Second, the characteristic homomorphism $c_{\widehat{\mathbf{G}}}$ of equivariant K-theory identifies the rings $\mathbb{C}[\widehat{\mathbf{T}}]$ and $K^{\widehat{\mathbf{G}}}(\widehat{\mathbf{G}}/\widehat{\mathbf{B}})_\mathbb{C}$ as algebras over the representation ring $\mathbb{C}[\widehat{\mathbf{T}}]^{W_0} = R(\widehat{\mathbf{G}})_\mathbb{C}$.

When $k = \overline{\mathbb{F}}_q$ any irreducible $\widehat{\mathbf{G}}(\overline{\mathbb{F}}_q)$-representation of $\mathcal{W}_F$ is tame and the Iwahori-Hecke algebra needs to be replaced by the bigger pro-p-Iwahori-Hecke algebra

$$\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q} = (\overline{\mathbb{F}}_q[I^{(1)} \setminus \mathbf{G}(F)/I^{(1)}], \star).$$

Here, $I^{(1)} \subset I$ is the unique pro-p Sylow subgroup of I. The algebra $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ was introduced by Vignéras and its structure theory developed in a series of papers [V04, V05, V06, V14, V15, V16, V17]. More generally, Vignéras introduces and studies a generic version $\mathcal{H}^{(1)}(\mathbf{q}_*)$ of this algebra which is defined over a polynomial ring $\mathbb{Z}[\mathbf{q}_*]$ in finitely many indeterminates $\mathbf{q}_s$. The mod p ring

$\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ is obtained by specialization $\mathbf{q}_s = q$ followed by extension of scalars from $\mathbb{Z}$ to $\overline{\mathbb{F}}_q$, in short $\mathbf{q}_s = q = 0$.

From now on, let $\mathbf{G} = \mathbf{GL_2}$ be the general linear group of rank 2 (in particular, then $\mathbf{q}_s$ is independent of s). Our aim in this article is to show that there is a Kazhdan-Lusztig theory for the generic pro-p Iwahori-Hecke algebra $\mathcal{H}^{(1)}(\mathbf{q})$. On the one hand, it gives back (and actually improves!) the classical theory after passing to the direct summand $\mathcal{H}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ and then specializing $\mathbf{q} = q \in \mathbb{C}$. On the other hand, it gives a genuine mod p theory after specializing to $\mathbf{q} = q = 0 \in \overline{\mathbb{F}}_q$. In the generic situation, the role of the Langlands dual group is taken by its Vinberg monoid $V_{\widehat{\mathbf{G}}}$ and its flag variety. The monoid comes with a fibration $\mathbf{q} : V_{\widehat{\mathbf{G}}} \to \mathbb{A}^1$ and the dual parametrization of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules is achieved by working over the 0-fiber $V_{\widehat{\mathbf{G}},0}$. When $F = \mathbb{Q}_p$ (with $p \geq 5$), we can push further the dual parametrization and arrive at a Langlands parametrization by semisimple two-dimensional $\overline{\mathbb{F}}_p$-representations of the Weil group $\mathcal{W}_{\mathbb{Q}_p}$ or, equivalently, of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Let $k = \overline{\mathbb{F}}_q$ and $\mathbf{q}$ be an indeterminate. We let $\mathbf{T} \subset \mathbf{G}$ be the torus of diagonal matrices. Although our primary motivation is the extreme case $\mathbf{q} = q = 0$, we will prove all our results in the far more stronger generic situation. It also allows us to find the correct normalizations in the extreme case and to recover and improve the classical theory over $\mathbb{C}$ (typically, the formulas become cleaner, e.g. in the Bernstein and in the Satake isomorphism). Let $\mathcal{A}^{(1)}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ be the maximal commutative subring and $\mathcal{A}^{(1)}(\mathbf{q})^{W_0} = Z(\mathcal{H}^{(1)}(\mathbf{q}))$ be its ring of invariants. We let $\tilde{\mathbb{Z}} := \mathbb{Z}[\frac{1}{q-1}, \mu_{q-1}]$ and denote by $\tilde{\bullet}$ the base change from $\mathbb{Z}$ to $\tilde{\mathbb{Z}}$. The algebra $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$ splits as a direct product of subalgebras $\tilde{\mathcal{H}}^{\gamma}(\mathbf{q})$ indexed by the orbits γ of W_0 in the set of characters of the finite torus $\mathbb{T} := \mathbf{T}(\mathbb{F}_q)$. There are regular resp. non-regular components corresponding to $|\gamma| = 2$ resp. $|\gamma| = 1$ and the algebra structure of $\tilde{\mathcal{H}}^{\gamma}(\mathbf{q})$ in these two cases is fundamentally different. We define an analogue of the Demazure operator for the regular components and call it the *Vignéras operator*. Passing to the product over all γ, this allows us to single out a distinguished $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$-linear operator on $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$. Our first main result is the existence of the *generic pro-p antispherical representation*:

Theorem A. (cf. 3.3.1, 4.3.1) *There is a (essentially unique) faithful representation*

$$\tilde{\mathscr{A}}^{(1)}(\mathbf{q}) : \tilde{\mathcal{H}}^{(1)}(\mathbf{q}) \longrightarrow \mathrm{End}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$$

such that

(i) $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})|_{\tilde{\mathcal{A}}^{(1)}(\mathbf{q})} =$ *the natural inclusion* $\tilde{\mathcal{A}}^{(1)}(\mathbf{q}) \subset \mathrm{End}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$

(ii) $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})(T_s) =$ *the Demazure-Vignéras operator on* $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$.

Restricting the representation $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})$ *to the Iwahori component, its base change* $\mathbb{Z}[\mathbf{q}] \to \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$ *coincides with the classical antispherical representation of Kazhdan-Lusztig and Ginzburg.*

We call the left $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-module defined by $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})$ the *generic antispherical module* $\tilde{\mathcal{M}}^{(1)}$.

Let $\mathrm{Mat}_{2\times 2}$ be the $\mathbb{Z}$-monoid scheme of 2×2-matrices. The Vinberg monoid $V_{\widehat{\mathbf{G}}}$, as introduced in [V95], in the particular case of $\mathbf{GL_2}$ is the $\mathbb{Z}$-monoid scheme

$$V_{\mathbf{GL_2}} := \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m.$$

It implies the striking interpretation of the formal indeterminate $\mathbf{q}$ as a regular function. Indeed, denote by z_2 the canonical coordinate on $\mathbb{G}_m$. Let $\mathbf{q}$ be the homomorphism from $V_{\mathbf{GL_2}}$ to the multiplicative monoid $(\mathbb{A}^1, \cdot)$ defined by $(f, z_2) \mapsto \det(f) z_2^{-1}$:

$$\begin{array}{c} V_{\mathbf{GL_2}} \\ \Big\downarrow{\scriptstyle \mathbf{q}} \\ \mathbb{A}^1. \end{array}$$

The fibration $\mathbf{q}$ is trivial over $\mathbb{A}^1\setminus\{0\}$ with fibre $\mathbf{GL_2}$. The special fiber at $\mathbf{q}=0$ is the $\mathbb{Z}$-semigroup scheme

$$V_{\mathbf{GL_2},0} := \mathbf{q}^{-1}(0) = \mathrm{Sing}_{2\times 2}\times\mathbb{G}_m,$$

where $\mathrm{Sing}_{2\times 2}$ represents the singular 2×2-matrices. Let $\mathrm{Diag}_{2\times 2}\subset \mathrm{Mat}_{2\times 2}$ be the submonoid scheme of diagonal 2×2-matrices, and set

$$V_{\widehat{\mathbf{T}}} := \mathrm{Diag}_{2\times 2}\times\mathbb{G}_m \subset V_{\mathbf{GL_2}} = \mathrm{Mat}_{2\times 2}\times\mathbb{G}_m.$$

This is a diagonalizable $\mathbb{Z}$-monoid scheme. Restricting the above $\mathbb{A}^1$-fibration to $V_{\widehat{\mathbf{T}}}$ we obtain a fibration, trivial over $\mathbb{A}^1\setminus\{0\}$ with fibre $\widehat{\mathbf{T}}$. Its special fibre at $\mathbf{q}=0$ is the $\mathbb{Z}$-semigroup scheme

$$V_{\widehat{\mathbf{T}},0} := \mathbf{q}|^{-1}_{V_{\widehat{\mathbf{T}}}}(0) = \mathrm{SingDiag}_{2\times 2}\times\mathbb{G}_m,$$

where $\mathrm{SingDiag}_{2\times 2}$ represents the singular diagonal 2×2-matrices. To ease notion, we denote the base change to $\overline{\mathbb{F}}_q$ of these $\mathbb{Z}$-schemes by the same symbols. Let $\mathbb{T}^\vee$ be the finite abelian dual group of $\mathbb{T}$. We let $R(V^{(1)}_{\widehat{\mathbf{T}}})$ be the representation ring of the extended monoid

$$V^{(1)}_{\widehat{\mathbf{T}}} := \mathbb{T}^\vee\times V_{\widehat{\mathbf{T}}}.$$

Our second main result is the existence of the *generic pro-p Bernstein isomorphism.*

Theorem B. (cf. 6.1.3) *There exists a ring isomomorphism*

$$\mathscr{B}^{(1)}(\mathbf{q}) : \mathcal{A}^{(1)}(\mathbf{q}) \xrightarrow{\ \sim\ } R(V^{(1)}_{\widehat{\mathbf{T}}})$$

with the property: Restricting the isomorphism $\mathscr{B}^{(1)}(\mathbf{q})$ to the Iwahori component, its base change $\mathbb{Z}[\mathbf{q}]\to\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$ recovers[1] *the classical Bernstein isomorphism $\tilde{\theta}$.*

The extended monoid $V^{(1)}_{\widehat{\mathbf{T}}}$ has a natural W_0-action and the isomorphism $\mathscr{B}^{(1)}(\mathbf{q})$ is equivariant. We call the resulting ring isomorphism

$$\mathscr{S}^{(1)}(\mathbf{q}) := \mathscr{B}^{(1)}(\mathbf{q})^{W_0} : \mathcal{A}^{(1)}(\mathbf{q})^{W_0} \xrightarrow{\ \sim\ } R(V^{(1)}_{\widehat{\mathbf{T}}})^{W_0}$$

the *generic pro-p-Iwahori Satake isomorphism.* Our terminology is justified by the following. Let $K=\mathbf{G}(o_F)$. Recall that the spherical Hecke algebra of $\mathbf{G}(F)$ with coefficients in any commutative ring R is defined to be the convolution algebra

$$\mathcal{H}^{\mathrm{sph}}_R := (R[K\backslash\mathbf{G}(F)/K],\star)$$

generated by the K-double cosets in $\mathbf{G}(F)$. We define a *generic spherical Hecke algebra* $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ over the ring $\mathbb{Z}[\mathbf{q}]$. Its base change $\mathbb{Z}[\mathbf{q}]\to R$, $\mathbf{q}\mapsto q$ coincides with $\mathcal{H}^{\mathrm{sph}}_R$. Our third main result is the existence of the *generic Satake isomorphism.*

Theorem C. (cf. 6.2.4) *There exists a ring isomorphism*

$$\mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\ \sim\ } R(V_{\widehat{\mathbf{T}}})^{W_0}$$

with the propery: Base change $\mathbb{Z}[\mathbf{q}]\to\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$ and specialization $\mathbf{q}\mapsto q\in\mathbb{C}$ recovers[1] *the classical Satake isomorphism between $\mathcal{H}^{\mathrm{sph}}_{\mathbb{C}}$ and $R(\widehat{\mathbf{T}})^{W_0}_{\mathbb{C}}$.*

We emphasize that the possibility of having a generic Satake isomorphism is conceptually new and of independent interest. Its definition relies on the deep Kazhdan-Lusztig theory for the intersection cohomology on the affine flag manifold. Its proof follows from the classical case by specialization (to an infinite number of points q).

The special fibre $\mathscr{S}(0)$ recovers Herzig's mod p Satake isomorphism [H11], by choosing certain 'Steinberg coordinates' on $V_{\widehat{\mathbf{T}},0}$.

[1] By 'recovers' we mean 'coincides up to a renormalization'.

As a corollary we obtain the *generic central elements morphism* as the unique ring homomorphism

$$\mathscr{Z}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \longrightarrow \mathcal{A}(\mathbf{q}) \subset \mathcal{H}(\mathbf{q})$$

making the diagram

$$\begin{array}{ccc} \mathcal{A}(\mathbf{q}) & \xrightarrow[\sim]{\mathscr{B}^{(1)}(\mathbf{q})|_{\mathcal{A}(\mathbf{q})}} & R(V_{\widehat{\mathbf{T}}}) \\ {\scriptstyle \mathscr{Z}(\mathbf{q})}\big\uparrow & & \big\uparrow \\ \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) & \xrightarrow[\sim]{\mathscr{S}(\mathbf{q})} & R(V_{\widehat{\mathbf{T}}})^{W_0} \end{array}$$

commutative. The morphism $\mathscr{Z}(\mathbf{q})$ is injective and has image $Z(\mathcal{H}(\mathbf{q}))$. Base change $\mathbb{Z}[\mathbf{q}] \to \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$ and specialization $\mathbf{q} \mapsto q \in \mathbb{C}$ recovers[1] Bernstein's classical central elements morphism. Its specialization $\mathbf{q} \mapsto q = 0 \in \overline{\mathbb{F}}_q$ coincides with Ollivier's construction from [O14].

Our fourth main result is the *characteristic homomorphism* in the equivariant K-theory over the Vinberg monoid $V_{\widehat{\mathbf{G}}}$. The monoid $V_{\widehat{\mathbf{G}}}$ carries an action by multiplication on the right from the $\mathbb{Z}$-submonoid scheme

$$V_{\widehat{\mathbf{B}}} := \mathrm{UpTriang}_{2\times 2} \times \mathbb{G}_m \subset \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m = V_{\widehat{\mathbf{G}}}$$

where $\mathrm{UpTriang}_{2\times 2}$ represents the upper triangular 2×2-matrices. We explain in an appendix how to construct (virtual) quotients in the context of semigroups and how to construct categories of equivariant vector bundles and their K-theory on such quotients. Although maybe well-known, we could not find this material in the literature. The usual induction functor for vector bundles gives a characteristic homomorphism, which is an isomorphism in the case of monoids. Applying this general formalism, the *flag variety* $V_{\widehat{\mathbf{G}}}/V_{\widehat{\mathbf{B}}}$ resp. its extended version $V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}}$ is defined as a $\mathbb{Z}$-monoidoid (instead of a groupoid).

Theorem D. (cf. 5.2.4) *Induction of equivariant vector bundles defines a characteristic isomorphism*

$$c_{V^{(1)}_{\widehat{\mathbf{G}}}} : R(V^{(1)}_{\widehat{\mathbf{T}}}) \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}}}}(V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}}).$$

The ring isomorphism is $R(V^{(1)}_{\widehat{\mathbf{T}}})^{W_0} = R(V^{(1)}_{\widehat{\mathbf{G}}})$-linear and compatible with passage to $\mathbf{q}$-fibres. Over the open complement $\mathbf{q} \neq 0$, its Iwahori-component coincides with the classical characteristic homomorphism $c_{\widehat{\mathbf{G}}}$ between $R(\widehat{\mathbf{T}})$ and $K^{\widehat{\mathbf{G}}}(\widehat{\mathbf{G}}/\widehat{\mathbf{B}})$.

We define the *category of Bernstein resp. Satake parameters* $\mathrm{BP}_{\widehat{\mathbf{G}}}$ resp. $\mathrm{SP}_{\widehat{\mathbf{G}}}$ to be the category of quasi-coherent modules on the $\tilde{\mathbb{Z}}$-scheme $V^{(1)}_{\widehat{\mathbf{T}}}$ resp. $V^{(1)}_{\widehat{\mathbf{T}}}/W_0$. By Theorem B, restriction of scalars to the subring $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$ or $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$ defines a functor B resp. P from the category of $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-modules to the categories $\mathrm{BP}_{\widehat{\mathbf{G}}}$ resp. $\mathrm{SP}_{\widehat{\mathbf{G}}}$. For example, the Bernstein resp. Satake parameter of the antispherical module $\tilde{\mathcal{M}}^{(1)}$ equals the structure sheaf $\mathcal{O}_{V^{(1)}_{\widehat{\mathbf{T}}}}$ resp. the quasi-coherent sheaf corresponding to the $R(V^{(1)}_{\widehat{\mathbf{T}}})^{W_0}$-module $K^{V^{(1)}_{\widehat{\mathbf{G}}}}(V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}})$. We call

$$\begin{array}{c} \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \\ \big\downarrow {\scriptstyle P} \\ \mathrm{SP}_{\widehat{\mathbf{G}}} \end{array}$$

the *generic parametrization functor*.

In the other direction, we define the *generic antispherical functor*

$$\begin{array}{c} \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \\ \big\uparrow {\scriptstyle \mathrm{ASph}} \\ \mathrm{SP}_{\widehat{\mathbf{G}}} \end{array}$$

to be the functor $\mathrm{ASph} := (\tilde{\mathcal{M}}^{(1)} \otimes_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))} \bullet) \circ S^{-1}$ where S is the Satake equivalence between $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$-modules and $\mathrm{SP}_{\widehat{\mathbf{G}}}$. Let $\pi : V_{\widehat{\mathbf{T}}}^{(1)} \to V_{\widehat{\mathbf{T}}}^{(1)}/W_0$ be the projection. The relation between all these functors is expressed by the commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) & & \\ & \overset{\mathrm{ASph}}{\nearrow} & \big\downarrow {\scriptstyle B} & \overset{P}{\searrow} & \\ \mathrm{SP}_{\widehat{\mathbf{G}}} & \xrightarrow[\pi^*]{} & \mathrm{BP}_{\widehat{\mathbf{G}}} & \xrightarrow[\pi_*]{} & \mathrm{SP}_{\widehat{\mathbf{G}}} . \end{array}$$

This ends our discussion of the theory in the generic setting.

Then we pass to the special fibre, i.e. we perform the base change $\mathbb{Z}[\mathbf{q}] \to k = \overline{\mathbb{F}}_q$, $\mathbf{q} \mapsto q = 0$. Identifying the k-points of the k-scheme $V_{\widehat{\mathbf{T}},0}^{(1)}/W_0$ with the skyscraper sheaves on it, the antispherical functor ASph induces a map

$$\mathrm{ASph} : \big(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0\big)(k) \longrightarrow \{\text{left } \mathcal{H}_{\overline{\mathbb{F}}_q}^{(1)}\text{-modules}\}.$$

Considering the decomposition of $V_{\widehat{\mathbf{T}},0}^{(1)}/W_0$ into its connected components $V_{\widehat{\mathbf{T}},0}^{\gamma}/W_0$ indexed by $\gamma \in \mathbb{T}^\vee/W_0$, the antispherical map decomposes as a disjoint union of maps

$$\mathrm{ASph}^\gamma : \big(V_{\widehat{\mathbf{T}},0}^{\gamma}/W_0\big)(k) \longrightarrow \{\text{left } \mathcal{H}_{\overline{\mathbb{F}}_q}^{\gamma}\text{-modules}\}.$$

We come to our fifth main result, the mod p dual parametrization of *all* irreducible $\mathcal{H}_{\overline{\mathbb{F}}_q}^{(1)}$-modules via the antispherical map.

Theorem E. (cf. 7.4.9, 7.4.15)

(i) Let $\gamma \in \mathbb{T}^\vee/W_0$ regular. The antispherical map induces a bijection

$$\mathrm{ASph}^\gamma : \big(V_{\widehat{\mathbf{T}},0}^{\gamma}/W_0\big)(k) \xrightarrow{\ \sim\ } \{\textit{simple finite dimensional left } \mathcal{H}_{\overline{\mathbb{F}}_q}^{\gamma}\textit{-modules}\}/\sim .$$

The singular locus of the parametrizing k-scheme

$$V_{\widehat{\mathbf{T}},0}^{\gamma}/W_0 \simeq V_{\widehat{\mathbf{T}},0} = \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m$$

is given by $(0,0) \times \mathbb{G}_m \subset V_{\widehat{\mathbf{T}},0}$ in the standard coordinates, and its k-points correspond to the supersingular Hecke modules through the correspondence ASph^γ.

(ii) Let $\gamma \in \mathbb{T}^\vee/W_0$ be non-regular. Consider the decomposition

$$V_{\widehat{\mathbf{T}},0}^{\gamma}/W_0 = V_{\widehat{\mathbf{T}},0}/W_0 \simeq \mathbb{A}^1 \times \mathbb{G}_m = D(2)_\gamma \cup D(1)_\gamma$$

where $D(1)_\gamma$ is the closed subscheme defined by the parabola $z_2 = z_1^2$ in the Steinberg coordinates z_1, z_2 and $D(2)_\gamma$ is the open complement. The antispherical map induces bijections

$$\mathrm{ASph}^\gamma(2) : D(2)_\gamma(k) \xrightarrow{\ \sim\ } \{\textit{simple 2-dimensional left } \mathcal{H}_{\overline{\mathbb{F}}_q}^{\gamma}\textit{-modules}\}/\sim$$

$$\mathrm{ASph}^{\gamma}(1) : D(1)_{\gamma}(k) \xrightarrow{\sim} \{\textit{antispherical pairs of characters of } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\}/\sim .$$

The branch locus of the covering

$$V_{\widehat{\mathbf{T}},0} \longrightarrow V_{\widehat{\mathbf{T}},0}/W_0 \simeq V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0$$

is contained in $D(2)_{\gamma}$, with equation $z_1 = 0$ in Steinberg coordinates, and its k-points correspond to the supersingular Hecke modules through the correspondence $\mathrm{ASph}^{\gamma}(2)$.

Ultimately, for $F = \mathbb{Q}_p$, we can complete the theory by relating the space $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ of mod p Satake parameters, to the space X of mod p Langlands parameters, defined by Emerton-Gee, cf. [Em19], by means of a *Langlands morphism* $L : V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \to X$. Then, pushing-forward the Satake parameter of the antispherical $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}$ along L, we obtain a quasi-coherent module $L_*S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q})$ on the scheme X, which arranges in a family Breuil's semisimple mod p Langlands correspondence for $\mathbf{GL_2}(\mathbb{Q}_p)$.

To state the result precisely, let $\zeta : Z(G) \to \overline{\mathbb{F}}_q^{\times}$ be a central character of G. There is a natural fibration $\theta : V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \to Z(G)^{\vee}$ where $Z(G)^{\vee}$ is the group scheme of characters of $Z(G)$, and we put

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta} := \theta^{-1}(\zeta).$$

For $F = \mathbb{Q}_p$ with $p \geq 5$, we may then consider the Emerton-Gee moduli curve X_{ζ} parametrizing (isomorphism classes of) two-dimensional semisimple continuous Galois representations over $\overline{\mathbb{F}}_p$ with determinant $\omega\zeta$:

$$X_{\zeta}(\overline{\mathbb{F}}_p) \cong \big\{\text{semisimple continuous } \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(\overline{\mathbb{F}}_p) \ \text{ with } \det\rho = \omega\zeta\big\}/\sim .$$

Here ω is the mod p cyclotomic character. The curve X_{ζ} is expected to be the underlying scheme of a ringed moduli space for the stack of étale (φ, Γ)-modules $\mathcal{X}_2^{\det=\omega\zeta}$ appearing in [EG19] (see also [CEGS19]). For now it is unclear how to define a replacement for X_{ζ} when $F/\mathbb{Q}_p$ is a non trivial finite extension, and this is the reason why we restrict to the case $F = \mathbb{Q}_p$ (and $p \geq 5$).

Theorem F. (cf. 8.3.9) *Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Let $\zeta : Z(G) \to \overline{\mathbb{F}}_p^{\times}$ be a mod p central character of G, and denote by $\mathrm{Mod}^{\mathrm{ladm}}_{\zeta}(\overline{\mathbb{F}}_p[G])$ the category of locally admissible smooth G-representations over $\overline{\mathbb{F}}_p$ with central character ζ.*

There exists a morphism of $\overline{\mathbb{F}}_p$-schemes

$$L_{\zeta} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta} \longrightarrow X_{\zeta}$$

such that the quasi-coherent $\mathcal{O}_{X_{\zeta}}$-module

$$L_{\zeta *}S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})|_{(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}},$$

equal to the push-forward along L_{ζ} of the restriction to $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta} \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ of the Satake parameter $S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})$, interpolates the $I^{(1)}$-invariants of the semisimple mod p Langlands correspondence

$$\begin{array}{ccccc} X_{\zeta}(\overline{\mathbb{F}}_p) & \longrightarrow & \mathrm{Mod}^{\mathrm{ladm}}_{\zeta}(\overline{\mathbb{F}}_p[G]) & \longrightarrow & \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \\ x & \longmapsto & \pi(\rho_x) & \longmapsto & \pi(\rho_x)^{I^{(1)}}, \end{array}$$

in the sense: for all $x \in X_{\zeta}(\overline{\mathbb{F}}_p)$, one has an isomorphism of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules

$$\Big(\big(L_{\zeta *}S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})|_{(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}}\big) \otimes_{\mathcal{O}_{X_{\zeta}}} k(x)\Big)^{\mathrm{ss}} = \Big(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})} (\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p})^{-1}(\mathcal{O}_{L_{\zeta}^{-1}(x)})\Big)^{\mathrm{ss}} \cong \pi(\rho_x)^{I^{(1)}}.$$

Thus, in combination with our computation of the Satake parameter $S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})$ in Theorem D, we see that the semisimple mod p Langlands correspondence is realized in the equivariant K-theory of the dual Vinberg monoid at $\mathbf{q} = 0$, as a natural specialization at $\mathbf{q} = 0$ of Kazhdan-Lusztig's resolution of the Deligne-Langlands conjecture for $\mathbb{C}$-coefficients.

For a more detailed description of the methods used in this article, we refer to the main body of the text. Once the Vinberg monoid is introduced, the generic Satake isomorphism is formulated and the generic antispherical module is constructed, everything else follows from Vignéras' structure theory of the generic pro-p-Iwahori Hecke algebra and her classification of the irreducible representations, and from Paškūnas' parametrization of the blocks of the category $\mathrm{Mod}^{\mathrm{ladm}}_{\zeta}(\overline{\mathbb{F}}_p[G])$.

Notation: In general, the letter F denotes a locally compact complete non-archimedean field with ring of integers o_F. Let $\mathbb{F}_q$ be its residue field, of characteristic p and cardinality q. We denote by $\mathbf{G}$ the algebraic group $\mathbf{GL_2}$ over F and by $G := \mathbf{G}(F)$ its group of F-rational points. Let $\mathbf{T} \subset \mathbf{G}$ be the torus of diagonal matrices. Finally, $I \subset G$ denotes the upper triangular standard Iwahori subgroup and $I^{(1)} \subset I$ denotes the unique pro-p Sylow subgroup of I.

2 The pro-p-Iwahori-Hecke algebra

2.1 The generic pro-p-Iwahori Hecke algebra

2.1.1. We let $W_0 = \{1, s\}$ and $\Lambda = \mathbb{Z} \times \mathbb{Z}$ be the *finite Weyl group* of $\mathbf{G}$ and the *lattice of cocharacters* of $\mathbf{T}$ respectively. If $\mathbb{T} = k^\times \times k^\times$ denote the *finite torus* $\mathbf{T}(\mathbb{F}_q)$, then W_0 acts naturally on $\mathbb{T} \times \Lambda$. The *extended Weyl group* of $\mathbf{G}$ is

$$W^{(1)} = \mathbb{T} \times \Lambda \rtimes W_0.$$

It contains the *affine Weyl group* and the *Iwahori-Weyl group*

$$W_{\mathrm{aff}} = \mathbb{Z}(1, -1) \rtimes W_0 \subseteq W = \Lambda \rtimes W_0.$$

The affine Weyl group W_{aff} is a Coxeter group with set of simple reflexions $S_{\mathrm{aff}} = \{s_0, s\}$, where $s_0 = (1, -1)s$. Moreover, setting $u = (1, 0)s \in W$ and $\Omega = u^{\mathbb{Z}}$, we have $W = W_{\mathrm{aff}} \rtimes \Omega$. The length function ℓ on W_{aff} can then be inflated to W and $W^{(1)}$.

2.1.2. Definition. *Let* $\mathbf{q}$ *be an indeterminate. The* generic pro-p Iwahori Hecke algebra *is the* $\mathbb{Z}[\mathbf{q}]$*-algebra* $\mathcal{H}^{(1)}(\mathbf{q})$ *defined by generators*

$$\mathcal{H}^{(1)}(\mathbf{q}) := \bigoplus_{w \in W^{(1)}} \mathbb{Z}[\mathbf{q}] T_w$$

and relations:

- *braid relations:* $T_w T_{w'} = T_{ww'}$ *for* $w, w' \in W^{(1)}$ *if* $\ell(w) + \ell(w') = \ell(ww')$
- *quadratic relations:* $T_s^2 = \mathbf{q} + c_s T_s$ *if* $s \in S_{\mathrm{aff}}$*, where* $c_s := \sum_{t \in (1,-1)(k^\times)} T_t$.

2.1.3. The identity element is $1 = T_1$. Moreover we set

$$S := T_s, \quad U := T_u \quad \text{and} \quad S_0 := T_{s_0} = USU^{-1}.$$

2.1.4. Definition. *Let* R *be any commutative ring. The* pro-p Iwahori Hecke algebra of G with coefficients in R *is defined to be the convolution algebra*

$$\mathcal{H}^{(1)}_R := (R[I^{(1)} \backslash G / I^{(1)}], \star)$$

generated by the $I^{(1)}$*-double cosets in* G.

2.1.5. Theorem. (Vignéras, [V16, Thm. 2.2]) *Let $\mathbb{Z}[\mathbf{q}] \to R$ be the ring homomorphism mapping $\mathbf{q}$ to q. Then the R-linear map*

$$\mathcal{H}^{(1)}(\mathbf{q}) \otimes_{\mathbb{Z}[\mathbf{q}]} R \longrightarrow \mathcal{H}^{(1)}_R$$

sending T_w, $w \in W^{(1)}$, to the characteristic function of the double coset $I^{(1)} \backslash w / I^{(1)}$, is an isomorphism of R-algebras.

2.2 Idempotents and component algebras

2.2.1. Recall the finite torus $\mathbb{T} = \mathbf{T}(\mathbb{F}_q)$. Let us consider its group algebra $\tilde{\mathbb{Z}}[\mathbb{T}]$ over the ring

$$\tilde{\mathbb{Z}} := \mathbb{Z}[\frac{1}{q-1}, \mu_{q-1}].$$

As $q-1$ is invertible in $\tilde{\mathbb{Z}}$, so is $|\mathbb{T}| = (q-1)^2$. We denote by $\mathbb{T}^\vee$ the set of characters $\lambda : \mathbb{T} \to \mu_{q-1} \subset \tilde{\mathbb{Z}}$, with its natural W_0-action given by ${}^s\lambda(t_1, t_2) = \lambda(t_2, t_1)$ for $(t_1, t_2) \in \mathbb{T}$. The set of W_0-orbits in $\mathbb{T}^\vee / W_0$ has cardinality $\frac{q^2-q}{2}$. Also $W^{(1)}$ acts on $\mathbb{T}^\vee$ through the canonical quotient map $W^{(1)} \to W_0$. Because of the braid relations in $\mathcal{H}^{(1)}(\mathbf{q})$, the rule $t \mapsto T_t$ induces an embedding of $\tilde{\mathbb{Z}}$-*algebras*

$$\tilde{\mathbb{Z}}[\mathbb{T}] \subset \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) := \mathcal{H}^{(1)}(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}.$$

2.2.2. Definition. *For all $\lambda \in \mathbb{T}^\vee$ and for $\gamma \in \mathbb{T}^\vee / W_0$, we define*

$$\varepsilon_\lambda := |\mathbb{T}|^{-1} \sum_{t \in \mathbb{T}} \lambda^{-1}(t) T_t \quad \textit{and} \quad \varepsilon_\gamma := \sum_{\lambda \in \gamma} \varepsilon_\lambda.$$

2.2.3. Lemma. *The elements ε_λ, $\lambda \in \mathbb{T}^\vee$, are idempotent, pairwise orthogonal and their sum is equal to 1. The elements ε_γ, $\gamma \in \mathbb{T}^\vee / W_0$, are idempotent, pairwise orthogonal, their sum is equal to 1 and they are central in $\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$. The $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebra $\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ is the direct product of its sub-$\tilde{\mathbb{Z}}[\mathbf{q}]$-algebras $\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) := \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) \varepsilon_\gamma$:*

$$\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \prod_{\gamma \in \mathbb{T}^\vee / W_0} \mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}).$$

In particular, the category of $\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$-modules decomposes into a finite product of the module categories for the individual component rings $\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) \varepsilon_\gamma$.

Proof. The elements ε_γ are central because of the relations $T_s T_t = T_{s(t)} T_s$, $T_{s_0} T_t = T_{s_0(t)} T_{s_0}$ and $T_u T_t = T_{s(t)} T_u$ for all $t \in (1, -1)k^\times$. □

2.2.4. Following the terminology of [V04], we call $|\gamma| = 2$ a *regular* case and $|\gamma| = 1$ a *non-regular* (or *Iwahori*) case.

2.3 The Bernstein presentation

The inverse image in $W^{(1)}$ of any subset of W along the canonical projection $W^{(1)} \to W$ will be denoted with a superscript $^{(1)}$.

2.3.1. Theorem. (Vignéras [V16, Th. 2.10, Cor 5.47]) *The $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}^{(1)}(\mathbf{q})$ admits the following* Bernstein presentation*:*

$$\mathcal{H}^{(1)}(\mathbf{q}) = \bigoplus_{w \in W^{(1)}} \mathbb{Z}[\mathbf{q}] E(w)$$

satisfying

- *braid relations:* $E(w)E(w') = E(ww')$ *for* $w, w' \in W_0^{(1)}$ *if* $\ell(w) + \ell(w') = \ell(ww')$
- *quadratic relations:* $E(s)^2 = \mathbf{q}E(s^2) + c_s E(s)$ *if* $s \in S_0^{(1)}$, *where* $c_{ts} := T_{s(t)}c_s$ *for* $t \in \mathbb{T}, s \in S_0$
- *product formula:* $E(\lambda)E(w) = \mathbf{q}^{\frac{\ell(\lambda)+\ell(w)-\ell(\lambda w)}{2}} E(\lambda w)$ *for* $\lambda \in \Lambda^{(1)}$ *and* $w \in W^{(1)}$
- *Bernstein relations for* $s \in s_\beta^{(1)} \subset S_0^{(1)}$ *and* $\lambda \in \Lambda^{(1)}$ *: set* $V := \mathbb{R}\Phi^\vee$ *and let*

$$\nu : \Lambda^{(1)} \to V$$

be the homomorphism such that $\lambda \in \Lambda^{(1)}$ *acts on* V *by translation by* $\nu(\lambda)$ *; then the Bernstein element*

$$B(\lambda, s) := E(s\lambda s^{-1})E(s) - E(s)E(\lambda)$$

$$\begin{array}{lll} = & 0 & \text{if } \lambda \in (\Lambda^s)^{(1)} \\ = & \operatorname{sign}(\beta \circ \nu(\lambda)) \sum_{k=0}^{|\beta\circ\nu(\lambda)|-1} \mathbf{q}(k,\lambda)c(k,\lambda)E(\mu(k,\lambda)) & \text{if } \lambda \in \Lambda^{(1)} \setminus (\Lambda^s)^{(1)} \end{array}$$

where $\mathbf{q}(k,\lambda)c(k,\lambda) \in \mathbb{Z}[\mathbf{q}][\mathbb{T}]$ *and* $\mu(k,\lambda) \in \Lambda^{(1)}$ *are explicit, cf. [V16, Th. 5.46] and references therein.*

2.3.2. Let

$$\mathcal{A}(\mathbf{q}) := \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[\mathbf{q}]E(\lambda) \subset \mathcal{A}^{(1)}(\mathbf{q}) := \bigoplus_{\lambda \in \Lambda^{(1)}} \mathbb{Z}[\mathbf{q}]E(\lambda) \subset \mathcal{H}^{(1)}(\mathbf{q}).$$

It follows from the product formula that these are *commutative sub-*$\mathbb{Z}[\mathbf{q}]$*-algebras of* $\mathcal{H}^{(1)}(\mathbf{q})$. Moreover, by definition [V16, 5.22-5.25], we have $E(t) = T_t$ for all $t \in \mathbb{T}$, so that $\mathbb{Z}[\mathbb{T}] \subset \mathcal{A}^{(1)}(\mathbf{q})$. Then, again by the product formula, the commutative algebra $\mathcal{A}^{(1)}(\mathbf{q})$ decomposes as the tensor product of the subalgebras

$$\mathcal{A}^{(1)}(\mathbf{q}) = \mathbb{Z}[\mathbb{T}] \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q}).$$

Also, after base extension $\mathbb{Z} \to \tilde{\mathbb{Z}}$, we can set $\mathcal{A}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}) := \mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})\varepsilon_\gamma$, and obtain the decomposition

$$\mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \prod_{\gamma \in \mathbb{T}^\vee/W_0} \mathcal{A}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}) \subset \prod_{\gamma \in \mathbb{T}^\vee/W_0} \mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}).$$

2.3.3. Lemma. *Let* X, Y, z_2 *be indeterminates. There exists a unique ring homomorphism*

$$\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X,Y]/(XY - \mathbf{q}z_2) \longrightarrow \mathcal{A}(\mathbf{q})$$

such that

$$X \longmapsto E(1,0), \quad Y \longmapsto E(0,1) \quad \textit{and} \quad z_2 \longmapsto E(1,1).$$

It is an isomorphism. Moreover, for all $\gamma \in \mathbb{T}^\vee/W_0$,

$$\mathcal{A}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \begin{cases} (\tilde{\mathbb{Z}}\varepsilon_\lambda \times \tilde{\mathbb{Z}}\varepsilon_\mu) \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q}) & \textit{if } \gamma = \{\lambda, \mu\} \textit{ is regular} \\ \tilde{\mathbb{Z}}\varepsilon_\lambda \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q}) & \textit{if } \gamma = \{\lambda\} \textit{ is non-regular.} \end{cases}$$

Proof. For any $(n_1, n_2) \in \mathbb{Z}^2 = \Lambda$, we have $\ell(n_1, n_2) = |n_1 - n_2|$. Hence it follows from product formula that z_2 is invertible and $XY = \mathbf{q}z_2$, so that we get a $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism

$$\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X,Y]/(XY - \mathbf{q}z_2) \longrightarrow \mathcal{A}(\mathbf{q}).$$

Moreover it maps the $\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}]$-basis $\{X^n\}_{n>1} \coprod \{1\} \coprod \{Y^n\}_{n>1}$ to the $\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}]$-basis

$$\{E(n,0)\}_{n>1} \coprod \{1\} \coprod \{E(0,n)\}_{n>1},$$

and hence is an isomorphism. The rest of the lemma is clear since $\mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \tilde{\mathbb{Z}}[\mathbb{T}] \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q})$ and $\tilde{\mathbb{Z}}[\mathbb{T}] = \prod_{\lambda \in \mathbb{T}^\vee} \tilde{\mathbb{Z}}\varepsilon_\lambda$. □

In the following, we will sometimes view the isomorphism of the lemma as an identification and write $X = E(1,0), Y = E(0,1)$ and $z_2 = E(1,1)$.

2.3.4. The rule $E(\lambda) \mapsto E(w(\lambda))$ defines an action of the finite Weyl group $W_0 = \{1, s\}$ on $\mathcal{A}^{(1)}(\mathbf{q})$ by $\mathbb{Z}[\mathbf{q}]$-algebra homomorphisms. By [V05, Th. 4] (see also [V14, Th. 1.3]), the subring of W_0-invariants is equal to the center of $\mathcal{H}^{(1)}(\mathbf{q})$, and the same is true after the scalar extension $\mathbb{Z} \to \tilde{\mathbb{Z}}$. Now the action on $\mathcal{A}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ stabilizes each component $\mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ and then the resulting subring of W_0-invariants is the center of $\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})$. In terms of the description of $\mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ given in Lemma 2.3.3, this translates into :

2.3.5. Lemma. *Let* $\gamma \in \mathbb{T}^{\vee}/W_0$.

- *If* $\gamma = \{\lambda, \mu\}$ *is regular, then the map*

$$\begin{array}{ccc} \mathcal{A}_{\tilde{\mathbb{Z}}}(\mathbf{q}) & \longrightarrow & \mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})^{W_0} = Z(\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})) \\ a & \longmapsto & a\varepsilon_\lambda + s(a)\varepsilon_\mu \end{array}$$

  is an isomorphism of $\tilde{\mathbb{Z}}[\mathbf{q}]$*-algebras. It depends on the choice of order* (λ, μ) *on the set* γ.

- *If* $\gamma = \{\lambda\}$ *is non-regular, then*

$$Z(\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})) = \mathcal{A}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})^{W_0} = \tilde{\mathbb{Z}}[\mathbf{q}][z_2^{\pm 1}, z_1]\varepsilon_\lambda$$

  with $z_1 := X + Y$.

2.3.6. One can express $X, Y, z_2 \in \mathcal{A}^{(1)}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ in terms of the distinguished elements 2.1.3. This is an application of [V16, Ex. 5.30]. We find:

$$(1,0) = s_0 u = us \in \Lambda \Rightarrow X := E(1,0) = (S_0 - c_{s_0})U = U(S - c_s),$$

$$(0,1) = su \in \Lambda \Rightarrow Y := E(0,1) = SU,$$

$$(1,1) = u^2 \in \Lambda \Rightarrow z_2 := E(1,1) = U^2.$$

Also

$$z_1 := X + Y = U(S - c_s) + SU.$$

3 The generic regular antispherical representation

3.1 The generic regular Iwahori-Hecke algebras

Let $\gamma = \{\lambda, \mu\} \in \mathbb{T}^{\vee}/W_0$ be a regular orbit. We define a model $\mathcal{H}_2(\mathbf{q})$ over $\mathbb{Z}$ for the component algebra $\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) \subset \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$. The algebra $\mathcal{H}_2(\mathbf{q})$ itself will not depend on γ.

3.1.1. By construction, the $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebra $\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q})$ admits the following presentation:

$$\mathcal{H}^{\gamma}_{\tilde{\mathbb{Z}}}(\mathbf{q}) = (\tilde{\mathbb{Z}}\varepsilon_\lambda \times \tilde{\mathbb{Z}}\varepsilon_\mu) \otimes'_{\mathbb{Z}} \bigoplus_{w \in W} \mathbb{Z}[\mathbf{q}]T_w,$$

where $\otimes'_{\mathbb{Z}}$ is the tensor product *twisted* by the W-action on $\{\lambda, \mu\}$ through the quotient map $W \to W_0$, together with the

- braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W$ if $\ell(w) + \ell(w') = \ell(ww')$
- quadratic relations: $T_s^2 = \mathbf{q}$ if $s \in S_{\text{aff}}$.

3.1.2. Definition. *Let* $\mathbf{q}$ *be an indeterminate. The* generic second Iwahori-Hecke algebra *is the* $\mathbb{Z}[\mathbf{q}]$*-algebra* $\mathcal{H}_2(\mathbf{q})$ *defined by generators*

$$\mathcal{H}_2(\mathbf{q}) := (\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes'_{\mathbb{Z}} \bigoplus_{w \in W} \mathbb{Z}[\mathbf{q}]T_w,$$

where $\otimes'$ *is the tensor product twisted by the* W*-action on* $\{1, 2\}$ *through the quotient map* $W \to W_0 = \mathfrak{S}_2$*, and relations:*

- *braid relations:* $T_w T_{w'} = T_{ww'}$ *for* $w, w' \in W$ *if* $\ell(w) + \ell(w') = \ell(ww')$
- *quadratic relations:* $T_s^2 = \mathbf{q}$ *if* $s \in S_{\text{aff}}$.

3.1.3. The identity element of $\mathcal{H}_2(\mathbf{q})$ is $1 = T_1$. Moreover we set in $\mathcal{H}_2(\mathbf{q})$

$$S := T_s, \quad U := T_u \quad \text{and} \quad S_0 := T_{s_0} = USU^{-1}.$$

Then one checks that

$$\mathcal{H}_2(\mathbf{q}) = (\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes'_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}], \quad S^2 = \mathbf{q}, \quad U^2 S = SU^2$$

is a presentation of $\mathcal{H}_2(\mathbf{q})$. Note that the element U^2 is invertible in $\mathcal{H}_2(\mathbf{q})$.

3.1.4. Choosing the ordering (λ, μ) on the set $\gamma = \{\lambda, \mu\}$ and mapping $\varepsilon_1 \mapsto \varepsilon_\lambda, \varepsilon_2 \mapsto \varepsilon_\mu$ defines an isomorphism of $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebras

$$\mathcal{H}_2(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}} \xrightarrow{\sim} \mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}),$$

such that $S \otimes 1 \mapsto S\varepsilon_\gamma$, $U \otimes 1 \mapsto U\varepsilon_\gamma$ and $S_0 \otimes 1 \mapsto S_0\varepsilon_\gamma$.

3.1.5. We identify two important commutative subrings of $\mathcal{H}_2(\mathbf{q})$. We define $\mathcal{A}_2(\mathbf{q}) \subset \mathcal{H}_2(\mathbf{q})$ to be the $\mathbb{Z}[\mathbf{q}]$-subalgebra generated by the elements ε_1, ε_2, US, SU and $U^{\pm 2}$. Let X, Y and z_2 be indeterminates. Then there is a unique $(\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}]$-algebra homomorphism

$$(\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2) \longrightarrow \mathcal{A}_2(\mathbf{q})$$

such that $X \mapsto US, Y \mapsto SU, z_2 \mapsto U^2$, and it is an isomorphism. In particular, $\mathcal{A}_2(\mathbf{q})$ is a *commutative* subalgebra of $\mathcal{H}_2(\mathbf{q})$. The isomorphism 3.1.4 identifies $\mathcal{A}_2(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$ with $\mathcal{A}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q})$. Moreover, permuting ε_1 and ε_2, and X and Y, extends to an action of $W_0 = \mathfrak{S}_2$ on $\mathcal{A}_2(\mathbf{q})$ by homomorphisms of $\mathbb{Z}[\mathbf{q}]$-algebras, whose invariants is the center $Z(\mathcal{H}_2(\mathbf{q}))$ of $\mathcal{H}_2(\mathbf{q})$, and the map

$$\begin{array}{rcl} \mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2) & \longrightarrow & \mathcal{A}_2(\mathbf{q})^{W_0} = Z(\mathcal{H}_2(\mathbf{q})) \\ a & \longmapsto & a\varepsilon_1 + s(a)\varepsilon_2 \end{array}$$

is an isomorphism of $\mathbb{Z}[\mathbf{q}]$-algebras. This is a consequence of 3.1.4, 2.3.6, 2.3.3 and 2.3.5. In the following, we will sometimes view the above isomorphisms as identifications. In particular, we will write $X = US, Y = SU$ and $z_2 = U^2$.

3.2 The Vignéras operator

In this subsection and the following, we will investigate the structure of the $Z(\mathcal{H}_2(\mathbf{q}))$-algebra $\operatorname{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$ of $Z(\mathcal{H}_2(\mathbf{q}))$-linear endomorphisms of $\mathcal{A}_2(\mathbf{q})$. Recall from the preceding subsection that $Z(\mathcal{H}_2(\mathbf{q})) = \mathcal{A}_2(\mathbf{q})^s$ is the subring of invariants of the commutative ring $\mathcal{A}_2(\mathbf{q})$.

3.2.1. Lemma. *We have*

$$\mathcal{A}_2(\mathbf{q}) = \mathcal{A}_2(\mathbf{q})^s\varepsilon_1 \oplus \mathcal{A}_2(\mathbf{q})^s\varepsilon_2$$

as $\mathcal{A}_2(\mathbf{q})^s$*-modules.*

Proof. This is immediate from the two isomorphisms in 3.1.5. □

According to the lemma, we may use the $\mathcal{A}_2(\mathbf{q})^s$-basis $\varepsilon_1, \varepsilon_2$ to identify $\operatorname{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$ with the algebra of 2×2-matrices over $\mathcal{A}_2(\mathbf{q})^s = \mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2)$.

3.2.2. Definition. *The endomorphism of* $\mathcal{A}_2(\mathbf{q})$ *corresponding to the matrix*

$$V_s(\mathbf{q}) := \begin{pmatrix} 0 & Y\varepsilon_1 + X\varepsilon_2 \\ z_2^{-1}(X\varepsilon_1 + Y\varepsilon_2) & 0 \end{pmatrix}$$

will be called the Vignéras operator on $\mathcal{A}_2(\mathbf{q})$.

3.2.3. Lemma. *We have* $V_s(\mathbf{q})^2 = \mathbf{q}$.

Proof. This is a short calculation. □

3.3 The generic regular antispherical representation

In the following theorem we define the generic regular antispherical representation of the algebra $\mathcal{H}_2(\mathbf{q})$ on the $Z(\mathcal{H}_2(\mathbf{q}))$-module $\mathcal{A}_2(\mathbf{q})$. Note that the commutative ring $\mathcal{A}_2(\mathbf{q})$ is naturally a subring

$$\mathcal{A}_2(\mathbf{q}) \subset \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q})),$$

an element $a \in \mathcal{A}_2(\mathbf{q})$ acting by multiplication $b \mapsto ab$ on $\mathcal{A}_2(\mathbf{q})$.

3.3.1. Theorem. *There exists a unique* $\mathbb{Z}[\mathbf{q}]$*-algebra homomorphism*

$$\mathscr{A}_2(\mathbf{q}) : \mathcal{H}_2(\mathbf{q}) \longrightarrow \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$$

such that

(i) $\mathscr{A}_2(\mathbf{q})|_{\mathcal{A}_2(\mathbf{q})} =$ *the natural inclusion* $\mathcal{A}_2(\mathbf{q}) \subset \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$

(ii) $\mathscr{A}_2(\mathbf{q})(S) = V_s(\mathbf{q})$.

Proof. Recall that $\mathcal{H}_2(\mathbf{q}) = (\mathbb{Z}\varepsilon_1 \times \mathbb{Z}\varepsilon_2) \otimes'_{\mathbb{Z}} \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$ with the relations $S^2 = \mathbf{q}$ and $U^2 S = SU^2$. In particular $\mathscr{A}_2(\mathbf{q})(S) := V_s(\mathbf{q})$ is well-defined thanks to 3.2.3. Now let us consider the question of finding the restriction of $\mathscr{A}_2(\mathbf{q})$ to the subalgebra $\mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$. As the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{A}_2(\mathbf{q}) \cap \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$ is generated by

$$z_2 = U^2, \quad X = US \quad \text{and} \quad Y = SU,$$

such a $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism exists if and only if there exists

$$\mathscr{A}_2(\mathbf{q})(U) \in \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$$

satisfying

1. $\mathscr{A}_2(\mathbf{q})(U)$ is invertible ;
2. $\mathscr{A}_2(\mathbf{q})(U)^2 = \mathscr{A}_2(\mathbf{q})(U^2) = \mathscr{A}_2(\mathbf{q})(z_2) = z_2 \,\mathrm{Id}$;
3. $\mathscr{A}_2(\mathbf{q})(U) V_s(\mathbf{q}) =$ multiplication by X
4. $V_s(\mathbf{q}) \mathscr{A}_2(\mathbf{q})(U) =$ multiplication by Y.

As before we use the $Z(\mathcal{H}_2(\mathbf{q}))$-basis $\varepsilon_1, \varepsilon_2$ of $\mathcal{A}_2(\mathbf{q})$ to identify $\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$ with the algebra of 2×2-matrices over the ring $Z(\mathcal{H}_2(\mathbf{q})) = \mathcal{A}_2(\mathbf{q})^s$. Then, by definition,

$$V_s(\mathbf{q}) = \begin{pmatrix} 0 & Y\varepsilon_1 + X\varepsilon_2 \\ z_2^{-1}(X\varepsilon_1 + Y\varepsilon_2) & 0 \end{pmatrix}.$$

Moreover, the multiplications by X and by Y on $\mathcal{A}_2(\mathbf{q})$ correspond then to the matrices

$$\begin{pmatrix} X\varepsilon_1 + Y\varepsilon_2 & 0 \\ 0 & Y\varepsilon_1 + X\varepsilon_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Y\varepsilon_1 + X\varepsilon_2 & 0 \\ 0 & X\varepsilon_1 + Y\varepsilon_2 \end{pmatrix}.$$

Now, writing

$$\mathscr{A}_2(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

we have:

$$\mathscr{A}_2(\mathbf{q})(U)^2 = z_2 \,\mathrm{Id} \Longleftrightarrow \begin{pmatrix} a^2 + bc & c(a+d) \\ b(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} z_2 & 0 \\ 0 & z_2 \end{pmatrix},$$

$$\mathscr{A}_2(\mathbf{q})(U) V_s(\mathbf{q}) = \text{ multiplication by } X$$

$$\Longleftrightarrow \begin{pmatrix} cz_2^{-1}(X\varepsilon_1 + Y\varepsilon_2) & a(Y\varepsilon_1 + X\varepsilon_2) \\ dz_2^{-1}(X\varepsilon_1 + Y\varepsilon_2) & b(Y\varepsilon_1 + X\varepsilon_2) \end{pmatrix} = \begin{pmatrix} X\varepsilon_1 + Y\varepsilon_2 & 0 \\ 0 & Y\varepsilon_1 + X\varepsilon_2 \end{pmatrix}$$

and

$$V_s(\mathbf{q})\mathscr{A}_2(\mathbf{q})(U) = \text{ multiplication by } Y$$

$$\Longleftrightarrow \begin{pmatrix} b(Y\varepsilon_1 + X\varepsilon_2) & d(Y\varepsilon_1 + X\varepsilon_2) \\ az_2^{-1}(X\varepsilon_1 + Y\varepsilon_2) & cz_2^{-1}(X\varepsilon_1 + Y\varepsilon_2) \end{pmatrix} = \begin{pmatrix} Y\varepsilon_1 + X\varepsilon_2 & 0 \\ 0 & X\varepsilon_1 + Y\varepsilon_2 \end{pmatrix}.$$

Each of the two last systems admits a unique solution, namely

$$\mathscr{A}_2(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & z_2 \\ 1 & 0 \end{pmatrix},$$

which is also a solution of the first one. Moreover, the determinant

$$ad - bc = -z_2$$

is invertible.

Finally, $\mathcal{A}_2(\mathbf{q})$ is generated by $\mathcal{A}_2(\mathbf{q}) \cap \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$ together with ε_1 and ε_2. The latter are assigned to map to the projectors

$$\text{multiplication by } \varepsilon_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \text{multiplication by } \varepsilon_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus it only remains to check that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathscr{A}_2(\mathbf{q})(S) = \mathscr{A}_2(\mathbf{q})(S) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathscr{A}_2(\mathbf{q})(S) = \mathscr{A}_2(\mathbf{q})(S) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and similarly with $\mathscr{A}_2(\mathbf{q})(U)$ in place of $\mathscr{A}_2(\mathbf{q})(U)$, which is straightforward. □

3.3.2. Remark. The map $\mathscr{A}_2(\mathbf{q})$, together with the fact that it is an isomorphism (see below), is a rewriting of a theorem of Vignéras, namely [V04, Cor. 2.3]. In loc. cit., the algebra $\mathcal{H}_2(\mathbf{q})$ is identified with the algebra of 2×2-matrices over the ring $\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2)$. In our approach, we have replaced the *abstract rank* 2 *module* underlying the standard representation of this matrix algebra, by the *subring* $\mathcal{A}_2(\mathbf{q})$ *of* $\mathcal{H}_2(\mathbf{q})$ with $\{\varepsilon_1, \varepsilon_2\}$ for the canonical basis. In this way, we are able to formulate the property that the restriction of $\mathscr{A}_2(\mathbf{q})$ to the subring $\mathcal{A}_2(\mathbf{q}) \subset \mathcal{H}_2(\mathbf{q})$ is the action by multiplication. This observation will be crucial to find the analogue of the representation $\mathscr{A}_2(\mathbf{q})$ in the *non*-regular case.

3.3.3. Proposition. *The homomorphism* $\mathscr{A}_2(\mathbf{q})$ *is an isomorphism.*

Proof. It follows from 3.1.3 and 3.1.5 that the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}_2(\mathbf{q})$ is generated by the elements

$$\varepsilon_1,\ \varepsilon_2,\ S,\ U,\ SU$$

as a module over its center $Z(\mathcal{H}_2(\mathbf{q}))$. Moreover, as $SU^2 = U^2 S =: z_2 S$ and $SU =: Y$, we have

$$S = z_2^{-1} YU = z_2^{-1} Y(\varepsilon_1 U + \varepsilon_2 U) = z_2^{-1}(Y\varepsilon_1 + X\varepsilon_2)\varepsilon_1 U + z_2^{-1}(X\varepsilon_1 + Y\varepsilon_2)\varepsilon_2 U,$$

$$U = \varepsilon_1 U + \varepsilon_2 U \quad \text{and} \quad SU = (Y\varepsilon_1 + X\varepsilon_2)\varepsilon_1 + (X\varepsilon_1 + Y\varepsilon_2)\varepsilon_2.$$

Consequently $\mathcal{H}_2(\mathbf{q})$ is generated as a $Z(\mathcal{H}_2(\mathbf{q}))$-module by the elements

$$\varepsilon_1,\ \varepsilon_2,\ z_2^{-1}\varepsilon_1 U,\ \varepsilon_2 U.$$

Since

$$\mathscr{A}_2(\mathbf{q})(U) := \begin{pmatrix} 0 & z_2 \\ 1 & 0 \end{pmatrix},$$

these four elements are mapped by $\mathscr{A}_2(\mathbf{q})$ to

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

As $\mathscr{A}_2(\mathbf{q})$ indentifies $Z(\mathcal{H}_2(\mathbf{q})) \subset \mathcal{H}_2(\mathbf{q})$ with the center of the matrix algebra

$$\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q})) = \mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(Z(\mathcal{H}_2(\mathbf{q}))\varepsilon_1 \oplus Z(\mathcal{H}_2(\mathbf{q}))\varepsilon_2),$$

it follows that the elements ε_1, ε_2, $z_2^{-1}\varepsilon_1 U$, $\varepsilon_2 U$ are linearly independent over $Z(\mathcal{H}_2(\mathbf{q}))$ and that $\mathscr{A}_2(\mathbf{q})$ is an isomorphism. □

We record the following corollary of the proof.

3.3.4. Corollary. *The ring $\mathcal{H}_2(\mathbf{q})$ is a free $Z(\mathcal{H}_2(\mathbf{q}))$-module on the basis $\varepsilon_1, \varepsilon_2, z_2^{-1}\varepsilon_1 U, \varepsilon_2 U$.*

3.3.5. We end this section by noting an equivariance property of $\mathscr{A}_2(\mathbf{q})$. As already noticed, the finite Weyl group W_0 acts on $\mathcal{A}_2(\mathbf{q})$ by $\mathbb{Z}[\mathbf{q}]$-algebra automorphisms, and the action is clearly faithful. Moreover $\mathcal{A}_2(\mathbf{q})^{W_0} = Z(\mathcal{H}_2(q))$. Hence W_0 can be viewed as a subgroup of $\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$, and we can let it act on $\mathrm{End}_{Z(\mathcal{H}_2(\mathbf{q}))}(\mathcal{A}_2(\mathbf{q}))$ by conjugation.

3.3.6. Lemma. *The embedding $\mathscr{A}_2(\mathbf{q})|_{\mathcal{A}_2(\mathbf{q})}$ is W_0-equivariant.*

Proof. Indeed, for all $a, b \in \mathcal{A}_2(\mathbf{q})$ and $w \in W_0$, we have

$$\mathscr{A}_2(\mathbf{q})(w(a))(b) = w(a)b = w(aw^{-1}(b)) = (waw^{-1})(b) = (w\mathscr{A}_2(\mathbf{q})(a)w^{-1})(b).$$

□

4 The generic non-regular antispherical representation

4.1 The generic non-regular Iwahori-Hecke algebras

Let $\gamma = \{\lambda\} \in \mathbb{T}^\vee/W_0$ be a non-regular orbit. As in the regular case, we define a model $\mathcal{H}_1(\mathbf{q})$ over $\mathbb{Z}$ for the component algebra $\mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}) \subset \mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$. The algebra $\mathcal{H}_1(\mathbf{q})$ will not depend on γ.

4.1.1. By construction, the $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebra $\mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q})$ admits the following presentation:

$$\mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}) = \bigoplus_{w \in W} \mathbb{Z}[\mathbf{q}] T_w \varepsilon_\lambda,$$

with

- braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W$ if $\ell(w) + \ell(w') = \ell(ww')$
- quadratic relations: $T_s^2 = \mathbf{q} + (q-1)T_s$ if $s \in S_{\mathrm{aff}}$.

4.1.2. Definition. *Let $\mathbf{q}$ be an indeterminate. The* generic Iwahori-Hecke algebra *is the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}_1(\mathbf{q})$ defined by generators*

$$\mathcal{H}_1(\mathbf{q}) := \bigoplus_{w \in W} \mathbb{Z}[\mathbf{q}] T_w$$

and relations:

- *braid relations: $T_w T_{w'} = T_{ww'}$ for $w, w' \in W$ if $\ell(w) + \ell(w') = \ell(ww')$*
- *quadratic relations: $T_s^2 = \mathbf{q} + (\mathbf{q}-1)T_s$ if $s \in S_{\mathrm{aff}}$.*

4.1.3. The identity element of $\mathcal{H}_1(\mathbf{q})$ is $1 = T_1$. Moreover we set in $\mathcal{H}_1(\mathbf{q})$

$$S := T_s, \quad U := T_u \quad \text{and} \quad S_0 := T_{s_0} = USU^{-1}.$$

Then one checks that

$$\mathcal{H}_1(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}], \quad S^2 = \mathbf{q} + (\mathbf{q}-1)S, \quad U^2 S = SU^2$$

is a presentation of $\mathcal{H}_1(\mathbf{q})$. Note that the element U^2 is invertible in $\mathcal{H}_1(\mathbf{q})$.

4.1.4. Sending 1 to ε_γ defines an isomorphism of $\tilde{\mathbb{Z}}[\mathbf{q}]$-algebras

$$\mathcal{H}_1(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}} \xrightarrow{\sim} \mathcal{H}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q}),$$

such that $S \otimes 1 \mapsto S\varepsilon_\gamma$, $U \otimes 1 \mapsto U\varepsilon_\gamma$ and $S_0 \otimes 1 \mapsto S_0\varepsilon_\gamma$.

4.1.5. We define $\mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q})$ to be the $\mathbb{Z}[\mathbf{q}]$-subalgebra generated by the elements $(S_0 - (\mathbf{q} - 1))U$, SU and $U^{\pm 2}$. Let X, Y and z_2 be indeterminates. Then there is a unique $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism

$$\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][X, Y]/(XY - \mathbf{q}z_2) \longrightarrow \mathcal{A}_1(\mathbf{q})$$

such that $X \mapsto (S_0 - (\mathbf{q}-1))U$, $Y \mapsto SU$, $z_2 \mapsto U^2$, and it is an isomorphism. In particular, $\mathcal{A}_1(\mathbf{q})$ is a *commutative* subalgebra of $\mathcal{H}_1(\mathbf{q})$. The isomorphism 4.1.4 identifies $\mathcal{A}_1(\mathbf{q}) \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$ with $\mathcal{A}^\gamma_{\tilde{\mathbb{Z}}}(\mathbf{q})$. Moreover, permuting X and Y extends to an action of $W_0 = \mathfrak{S}_2$ on $\mathcal{A}_1(\mathbf{q})$ by homomorphisms of $\mathbb{Z}[\mathbf{q}]$-algebras, whose invariants is the center $Z(\mathcal{H}_1(\mathbf{q}))$ of $\mathcal{H}_1(\mathbf{q})$ and

$$\mathbb{Z}[\mathbf{q}][z_2^{\pm 1}][z_1] \xrightarrow{\sim} \mathcal{A}_1(\mathbf{q})^{W_0} = Z(\mathcal{H}_1(\mathbf{q}))$$

with $z_1 := X + Y$. This is a consequence of 4.1.4, 2.3.6, 2.3.3 and 2.3.5. In the following, we will sometimes view the above isomorphisms as identifications. In particular, we will write

$$X = (S_0 - (\mathbf{q}-1))U = U(S - (\mathbf{q}-1)), \quad Y = SU \quad \text{and} \quad z_2 = U^2 \quad \text{in} \quad \mathcal{H}_1(\mathbf{q}).$$

4.1.6. It is well-known that the generic Iwahori-Hecke algebra $\mathcal{H}_1(\mathbf{q})$ is a $\mathbf{q}$-deformation of the group ring $\mathbb{Z}[W]$ of the Iwahori-Weyl group $W = \Lambda \rtimes W_0$. More precisely, specializing the chain of inclusions $\mathcal{A}_1(\mathbf{q})^{W_0} \subset \mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q})$ at $\mathbf{q} = 1$, yields the chain of inclusions $\mathbb{Z}[\Lambda]^{W_0} \subset \mathbb{Z}[\Lambda] \subset \mathbb{Z}[W]$.

4.2 The Kazhdan-Lusztig-Ginzburg operator

As in the regular case, we will study the $Z(\mathcal{H}_1(\mathbf{q}))$-algebra $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$ of $Z(\mathcal{H}_1(\mathbf{q}))$-linear endomorphisms of $\mathcal{A}_1(\mathbf{q})$. Recall that $Z(\mathcal{H}_1(\mathbf{q})) = \mathcal{A}_1(\mathbf{q})^s$ is the subring of invariants of the commutative ring $\mathcal{A}_1(\mathbf{q})$.

4.2.1. Lemma. *We have*

$$\mathcal{A}_1(\mathbf{q}) = \mathcal{A}_1(\mathbf{q})^s X \oplus \mathcal{A}_1(\mathbf{q})^s = \mathcal{A}_1(\mathbf{q})^s \oplus \mathcal{A}_1(\mathbf{q})^s Y$$

as $\mathcal{A}_1(\mathbf{q})^s$*-modules.*

Proof. Applying s, the two decompositions are equivalent; so it suffices to check that $\mathbb{Z}[z_2^{\pm 1}][X, Y]$ is free of rank 2 with basis $1, Y$ over the subring of symmetric polynomials $\mathbb{Z}[z_2^{\pm 1}][X+Y, XY]$. First if $P = QY$ with P and Q symmetric, then applying s we get $P = QX$ and hence $Q(X - Y) = 0$ which implies $P = Q = 0$. It remains to check that any monomial $X^i Y^j$, $i, j \in \mathbb{N}$, belongs to

$$\mathbb{Z}[z_2^{\pm 1}][X+Y, XY] + \mathbb{Z}[z_2^{\pm 1}][X+Y, XY]Y.$$

As $X = (X+Y) - Y$ and $Y^2 = -XY + (X+Y)Y$, the later is stable under multiplication by X and Y; as it contains 1, the result follows. □

4.2.2. Remark. The basis $\{1, Y\}$ specializes at $\mathbf{q} = 1$ to the so-called *Pittie-Steinberg basis* [St75] of $\mathbb{Z}[\Lambda]$ over $\mathbb{Z}[\Lambda]^{W_0}$.

4.2.3. Definition. *We let*

$$D_s := \text{ projector on } \mathcal{A}_1(\mathbf{q})^s Y \text{ along } \mathcal{A}_1(\mathbf{q})^s$$

$$D'_s := \text{ projector on } \mathcal{A}_1(\mathbf{q})^s \text{ along } \mathcal{A}_1(\mathbf{q})^s X$$

$$D_s(\mathbf{q}) := D_s - \mathbf{q}D'_s.$$

4.2.4. Remark. The operators D_s and D'_s specialize at $\mathbf{q} = 1$ to the *Demazure operators* on $\mathbb{Z}[\Lambda]$, as introduced in [D73, D74].

4.2.5. Lemma. *We have*

$$D_s(\mathbf{q})^2 = (1-\mathbf{q})D_s(\mathbf{q}) + \mathbf{q}.$$

Proof. Noting that $Y = z_1 - X$, we have

$$D_s(\mathbf{q})^2(1) = D_s(\mathbf{q})(-\mathbf{q}) = \mathbf{q}^2 = (1-\mathbf{q})(-\mathbf{q}) + \mathbf{q} = ((1-\mathbf{q})D_s(\mathbf{q}) + \mathbf{q})(1)$$

and

$$\begin{aligned} D_s(\mathbf{q})^2(Y) &= D_s(\mathbf{q})(Y - \mathbf{q}z_1) \\ &= Y - \mathbf{q}z_1 - \mathbf{q}z_1(-\mathbf{q}) \\ &= (1-\mathbf{q})(Y - \mathbf{q}z_1) + \mathbf{q}Y \\ &= ((1-\mathbf{q})D_s(\mathbf{q}) + \mathbf{q})(Y). \end{aligned}$$

□

4.3 The generic non-regular antispherical representation

We define the generic non-regular antispherical representation of the algebra $\mathcal{H}_1(\mathbf{q})$ on the $Z(\mathcal{H}_1(\mathbf{q}))$-module $\mathcal{A}_1(\mathbf{q})$. The commutative ring $\mathcal{A}_1(\mathbf{q})$ is naturally a subring

$$\mathcal{A}_1(\mathbf{q}) \subset \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q})),$$

an element $a \in \mathcal{A}_1(\mathbf{q})$ acting by multiplication $b \mapsto ab$ on $\mathcal{A}_1(\mathbf{q})$.

4.3.1. Theorem. *There exists a unique* $\mathbb{Z}[\mathbf{q}]$*-algebra homomorphism*

$$\mathscr{A}_1(\mathbf{q}) : \mathcal{H}_1(\mathbf{q}) \longrightarrow \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$$

such that

(i) $\mathscr{A}_1(\mathbf{q})|_{\mathcal{A}_1(\mathbf{q})} =$ *the natural inclusion* $\mathcal{A}_1(\mathbf{q}) \subset \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$

(ii) $\mathscr{A}_1(\mathbf{q})(S) = -D_s(\mathbf{q})$.

Proof. Recall that $\mathcal{H}_1(\mathbf{q}) = \mathbb{Z}[\mathbf{q}][S, U^{\pm 1}]$ with the relations $S^2 = (\mathbf{q}-1)S + \mathbf{q}$ and $U^2 S = SU^2$. In particular $\mathscr{A}_1(\mathbf{q})(S) := -D_s(\mathbf{q})$ is well-defined thanks to 4.2.5. On the other hand, the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{A}_1(\mathbf{q})$ is generated by

$$z_2 = U^2, \quad X = US + (1-\mathbf{q})U \quad \text{and} \quad Y = SU.$$

Consequently, there exists a $\mathbb{Z}[\mathbf{q}]$-algebra homomorphism $\mathscr{A}_1(\mathbf{q})$ as in the statement of the theorem if and only if there exists

$$\mathscr{A}_1(\mathbf{q})(U) \in \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$$

satisfying

1. $\mathscr{A}_1(\mathbf{q})(U)$ is invertible ;
2. $\mathscr{A}_1(\mathbf{q})(U)^2 = \mathscr{A}_1(\mathbf{q})(U^2) = \mathscr{A}_1(\mathbf{q})(z_2) = z_2 \,\mathrm{Id}$;
3. $\mathscr{A}_1(\mathbf{q})(U)(-D_s(\mathbf{q})) + (1-\mathbf{q})\mathscr{A}_1(\mathbf{q})(U) =$ multiplication by X

4. $-D_s(\mathbf{q})\mathscr{A}_1(\mathbf{q})(U) =$ multiplication by Y.

Let us use the $Z(\mathcal{H}_1(\mathbf{q}))$-basis $1, Y$ of $\mathcal{A}_1(\mathbf{q})$ to identify $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$ with the algebra of 2×2-matrices over the ring $Z(\mathcal{H}_1(\mathbf{q})) = \mathcal{A}_1(\mathbf{q})^s$. Then, by definition,

$$-D_s(\mathbf{q}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \mathbf{q}\begin{pmatrix} 1 & z_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{q} & \mathbf{q}z_1 \\ 0 & -1 \end{pmatrix}.$$

Moreover, as $X = z_1 - Y$, $XY = \mathbf{q}z_2$ and $Y^2 = -XY + (X+Y)Y = -\mathbf{q}z_2 + z_1 Y$, the multiplications by X and by Y on $\mathcal{A}_1(\mathbf{q})$ get identified with the matrices

$$\begin{pmatrix} z_1 & \mathbf{q}z_2 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\mathbf{q}z_2 \\ 1 & z_1 \end{pmatrix}.$$

Now, writing

$$\mathscr{A}_1(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

we have:

$$\mathscr{A}_1(\mathbf{q})(U)^2 = z_2\,\mathrm{Id} \Longleftrightarrow \begin{pmatrix} a^2+bc & c(a+d) \\ b(a+d) & d^2+bc \end{pmatrix} = \begin{pmatrix} z_2 & 0 \\ 0 & z_2 \end{pmatrix},$$

$$\mathscr{A}_1(\mathbf{q})(U)(-D_s(\mathbf{q})) + (1-\mathbf{q})\mathscr{A}_1(\mathbf{q})(U) = \text{ multiplication by } X$$

$$\Longleftrightarrow \begin{pmatrix} a & \mathbf{q}(az_1 - c) \\ b & \mathbf{q}(bz_1 - d) \end{pmatrix} = \begin{pmatrix} z_1 & \mathbf{q}z_2 \\ -1 & 0 \end{pmatrix}$$

and

$$-D_s(\mathbf{q})\mathscr{A}_1(\mathbf{q})(U) = \text{ multiplication by } Y$$

$$\Longleftrightarrow \begin{pmatrix} \mathbf{q}(a+z_1 b) & \mathbf{q}(c+z_1 d) \\ -b & -d \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{q}z_2 \\ 1 & z_1 \end{pmatrix}.$$

Each of the two last systems admits a unique solution, namely

$$\mathscr{A}_1(\mathbf{q})(U) = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} z_1 & z_1^2 - z_2 \\ -1 & -z_1 \end{pmatrix},$$

which is also a solution of the first one. Moreover, the determinant

$$ad - bc = -z_1^2 + (z_1^2 - z_2) = -z_2$$

is invertible. □

4.3.2. The relation between our generic non-regular representation $\mathscr{A}_1(\mathbf{q})$ and the theory of Kazhdan-Lusztig [KL87], and Ginzburg [CG97], is the following. Introducing a square root $\mathbf{q}^{\frac{1}{2}}$ of $\mathbf{q}$ and extending scalars along $\mathbb{Z}[\mathbf{q}] \subset \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$, we obtain the Hecke algebra $\mathcal{H}_1(\mathbf{q}^{\pm\frac{1}{2}})$ together with its commutative subalgebra $\mathcal{A}_1(\mathbf{q}^{\pm\frac{1}{2}})$. The latter contains the elements $\tilde{\theta}_\lambda$, $\lambda \in \Lambda$, introduced by Bernstein and Lusztig, which are defined as follows: writing $\lambda = \lambda_1 - \lambda_2$ with λ_1, λ_2 antidominant, one has

$$\tilde{\theta}_\lambda := \tilde{T}_{e^{\lambda_1}} \tilde{T}_{e^{\lambda_2}}^{-1} := \mathbf{q}^{-\frac{\ell(\lambda_1)}{2}} \mathbf{q}^{\frac{\ell(\lambda_2)}{2}} T_{e^{\lambda_1}} T_{e^{\lambda_2}}^{-1}.$$

They are related to the Bernstein basis $\{E(w),\ w \in W\}$ of $\mathcal{H}_1(\mathbf{q})$ introduced by Vignéras (which is analogous to the Bernstein basis of $\mathcal{H}^{(1)}(\mathbf{q})$ which we have recalled in 2.3.1) by the formula:

$$\forall \lambda \in \Lambda,\ \forall w \in W_0, \quad E(e^\lambda w) = \mathbf{q}^{\frac{\ell(e^\lambda w) - \ell(w)}{2}} \tilde{\theta}_\lambda T_w \quad \in \mathcal{H}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q}^{\pm\frac{1}{2}}).$$

In particular $E(e^\lambda) = \mathbf{q}^{\frac{\ell(e^\lambda)}{2}} \tilde{\theta}_\lambda$, and by the product formula (analogous to the product formula for $\mathcal{H}^{(1)}(\mathbf{q})$, cf. 2.3.1), the $\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$-linear isomorphism

$$\begin{array}{rccc} \tilde{\theta} : & \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda] & \xrightarrow{\sim} & \mathcal{A}_1(\mathbf{q}^{\pm\frac{1}{2}}) \\ & e^\lambda & \longmapsto & \tilde{\theta}_\lambda \end{array}$$

is in fact multiplicative, i.e. it is an isomorphism of $\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$-algebras.

Consequently, if we base change our action map $\mathscr{A}_1(\mathbf{q})$ to $\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}]$, we get a representation

$$\mathscr{A}_1(\mathbf{q}^{\pm\frac{1}{2}}) : \mathcal{H}_1(\mathbf{q}^{\pm\frac{1}{2}}) \longrightarrow \mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}^{\pm\frac{1}{2}}))}(\mathcal{A}_1(\mathbf{q}^{\pm\frac{1}{2}})) \simeq \mathrm{End}_{\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda]^{W_0}}(\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda]),$$

which coincides with the natural inclusion $\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda] \subset \mathrm{End}_{\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda]^{W_0}}(\mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda])$ when restricted to $\mathcal{A}_1(\mathbf{q}^{\pm\frac{1}{2}}) \simeq \mathbb{Z}[\mathbf{q}^{\pm\frac{1}{2}}][\Lambda]$, and which sends S to the opposite $-D_s(\mathbf{q})$ of the $\mathbf{q}$-deformed Demazure operator. Hence, this is the antispherical representation defined by Kazhdan-Lusztig and Ginzburg.

In particular, $\mathscr{A}_1(1)$ is the usual action of the Iwahori-Weyl group $W = \Lambda \rtimes W_0$ on Λ, and $\mathscr{A}_1(0)$ can be thought of as a degeneration of the latter.

4.3.3. Proposition. *The homomorphism $\mathscr{A}_1(\mathbf{q})$ is injective.*

Proof. It follows from 4.1.3 and 4.1.5 that the ring $\mathcal{H}_1(\mathbf{q})$ is generated by the elements

$$1,\ S,\ U,\ SU$$

as a module over its center $Z(\mathcal{H}_1(\mathbf{q})) = \mathbb{Z}[\mathbf{q}][z_1, z_2^{\pm 1}]$. As the latter is mapped isomorphically to the center of the matrix algebra $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$ by $\mathscr{A}_1(\mathbf{q})$, it suffices to check that the images

$$1,\ \mathscr{A}_1(\mathbf{q})(S),\ \mathscr{A}_1(\mathbf{q})(U),\ \mathscr{A}_1(\mathbf{q})(SU)$$

of $1, S, U, SU$ by $\mathscr{A}_1(\mathbf{q})$ are free over $Z(\mathcal{H}_1(\mathbf{q}))$. So let $\alpha, \beta, \gamma, \delta \in Z(\mathcal{H}_1(\mathbf{q}))$ (which is an integral domain) be such that

$$\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} \mathbf{q} & \mathbf{q}z_1 \\ 0 & -1 \end{pmatrix} + \gamma \begin{pmatrix} z_1 & z_1^2 - z_2 \\ -1 & -z_1 \end{pmatrix} + \delta \begin{pmatrix} 0 & -\mathbf{q}z_2 \\ 1 & z_1 \end{pmatrix} = 0.$$

Then

$$\begin{cases} \alpha + \beta\mathbf{q} + \gamma z_1 & = & 0 \\ -\gamma + \delta & = & 0 \\ \beta\mathbf{q}z_1 + \gamma(z_1^2 - z_2) - \delta\mathbf{q}z_2 & = & 0 \\ \alpha - \beta + (\delta - \gamma)z_1 & = & 0. \end{cases}$$

We obtain $\delta = \gamma$, $\alpha = \beta$ and

$$\begin{cases} \alpha(1 + \mathbf{q}) + \gamma z_1 & = & 0 \\ \alpha\mathbf{q}z_1 + \gamma(z_1^2 - z_2 - \mathbf{q}z_2) & = & 0. \end{cases}$$

The latter system has determinant

$$(1 + \mathbf{q})(z_1^2 - z_2 - \mathbf{q}z_2) - \mathbf{q}z_1^2 = z_1^2 - z_2 - 2\mathbf{q}z_2 - \mathbf{q}^2 z_2$$

which is nonzero (its specialisation at $\mathbf{q} = 0$ is equal to $z_1^2 - z_2 \neq 0$), whence $\alpha = \gamma = 0 = \beta = \delta$. □

We record the following two corollaries of the proof.

4.3.4. Corollary. *The ring $\mathcal{H}_1(\mathbf{q})$ is a free $Z(\mathcal{H}_1(\mathbf{q}))$-module on the basis $1, S, U, SU$.*

4.3.5. Corollary. *The homomorphism $\mathscr{A}_1(0)$ is injective.*

4.3.6. We end this section by noting an equivariance property of $\mathscr{A}_1(\mathbf{q})$. As already noticed, the finite Weyl group W_0 acts on $\mathcal{A}_1(\mathbf{q})$ by $\mathbb{Z}[\mathbf{q}]$-algebra automorphisms, and the action is clearly faithful. Moreover $\mathcal{A}_1(\mathbf{q})^{W_0} = Z(\mathcal{H}_1(q))$. Hence W_0 can be viewed as a subgroup of $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$, and we can let it act on $\mathrm{End}_{Z(\mathcal{H}_1(\mathbf{q}))}(\mathcal{A}_1(\mathbf{q}))$ by conjugation.

4.3.7. Lemma. *The embedding $\mathscr{A}_1(\mathbf{q})|_{\mathcal{A}_1(\mathbf{q})}$ is W_0-equivariant.*

Proof. Indeed, for all $a, b \in \mathcal{A}_1(\mathbf{q})$ and $w \in W_0$, we have

$$\mathscr{A}_1(\mathbf{q})(w(a))(b) = w(a)b = w(aw^{-1}(b)) = (waw^{-1})(b) = (w\mathscr{A}_1(\mathbf{q})(a)w^{-1})(b).$$

□

5 Geometric representation theory

5.1 The Vinberg monoid of the dual group $\widehat{\mathrm{G}} = \mathbf{GL_2}$

5.1.1. The Langlands dual group over $k := \overline{\mathbb{F}}_q$ of the connected reductive algebraic group GL_2 over F is $\widehat{\mathbf{G}} = \mathbf{GL_2}$. We recall the k-monoid scheme introduced by Vinberg in [V95], in the particular case of $\mathbf{GL_2}$. It is in fact defined over $\mathbb{Z}$, as the group $\mathbf{GL_2}$. In the following, all the fiber products are taken over the base ring $\mathbb{Z}$.

5.1.2. Definition. *Let* $\mathrm{Mat}_{2\times 2}$ *be the* $\mathbb{Z}$*-monoid scheme of* 2×2*-matrices (with usual matrix multiplication as operation). The* Vinberg monoid for $\mathbf{GL_2}$ *is the* $\mathbb{Z}$*-monoid scheme*

$$V_{\mathbf{GL_2}} := \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m.$$

5.1.3. The group $\mathbf{GL_2} \times \mathbb{G}_m$ is recovered from the monoid $V_{\mathbf{GL_2}}$ as its group of units. The group $\mathbf{GL_2}$ itself is recovered as follows. Denote by z_2 the canonical coordinate on $\mathbb{G}_m$. Then *let* $\mathbf{q}$ *be the homomorphism from* $V_{\mathbf{GL_2}}$ *to the multiplicative monoid* $(\mathbb{A}^1, \cdot)$ *defined by* $(f, z_2) \mapsto \det(f) z_2^{-1}$:

$$\begin{array}{c} V_{\mathbf{GL_2}} \\ \Big\downarrow{\scriptstyle \mathbf{q}} \\ \mathbb{A}^1. \end{array}$$

Then $\mathbf{GL_2}$ is recovered as the fiber at $\mathbf{q} = 1$, canonically:

$$\mathbf{q}^{-1}(1) = \{(f, z_2) : \det(f) = z_2\} \xrightarrow{\sim} \mathbf{GL_2}, \quad (f, z_2) \mapsto f.$$

The fiber at $\mathbf{q} = 0$ is the $\mathbb{Z}$-semigroup scheme

$$V_{\mathbf{GL_2},0} := \mathbf{q}^{-1}(0) = \mathrm{Sing}_{2\times 2} \times \mathbb{G}_m,$$

where $\mathrm{Sing}_{2\times 2}$ represents the singular 2×2-matrices. Note that it has no identity element, i.e. it is a semigroup which is not a monoid.

5.1.4. Let $\mathrm{Diag}_{2\times 2} \subset \mathrm{Mat}_{2\times 2}$ be the submonoid scheme of diagonal 2×2-matrices, and set

$$V_{\widehat{\mathbf{T}}} := \mathrm{Diag}_{2\times 2} \times \mathbb{G}_m \subset V_{\mathbf{GL_2}} = \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m.$$

This is a diagonalizable $\mathbb{Z}$-monoid scheme with character monoid

$$\mathbb{X}^\bullet(V_{\widehat{\mathbf{T}}}) = \mathbb{N}(1,0) \oplus \mathbb{N}(0,1) \oplus \mathbb{Z}(1,1) \subset \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1) = \Lambda = \mathbb{X}^\bullet(\widehat{\mathbf{T}}).$$

In particular, setting $X := e^{(1,0)}$ and $Y := e^{(0,1)}$ in the group ring $\mathbb{Z}[\Lambda]$, we have

$$\widehat{\mathbf{T}} = \mathrm{Spec}(\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]) \subset \mathrm{Spec}(\mathbb{Z}[z_2^{\pm 1}][X, Y]) = V_{\widehat{\mathbf{T}}}.$$

Again, this closed subgroup is recovered as the fiber at $\mathbf{q} = 1$ of the fibration $\mathbf{q}|_{V_{\widehat{\mathbf{T}}}} : V_{\widehat{\mathbf{T}}} \to \mathbb{A}^1$, and the fiber at $\mathbf{q} = 0$ is the $\mathbb{Z}$-semigroup scheme $\mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m$ where $\mathrm{SingDiag}_{2\times 2}$ represents the singular diagonal 2×2-matrices:

$$\begin{array}{ccccc} \widehat{\mathbf{T}} & \hookrightarrow & V_{\widehat{\mathbf{T}}} & \hookleftarrow & \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{q}} & & \Big\downarrow \\ \mathrm{Spec}(\mathbb{Z}) & \xhookrightarrow{1} & \mathbb{A}^1 & \xhookleftarrow{0} & \mathrm{Spec}(\mathbb{Z}). \end{array}$$

In terms of equations, the $\mathbb{A}^1$-family

$$\mathbf{q} : V_{\widehat{\mathbf{T}}} = \mathrm{Diag}_{2\times 2} \times \mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[z_2^{\pm 1}][X, Y]) \longrightarrow \mathbb{A}^1$$

is given by the formula $\mathbf{q}(\mathrm{diag}(x, y), z_2) = \det(\mathrm{diag}(x, y)) z_2^{-1} = xy z_2^{-1}$. Hence, after fixing $z_2 \in \mathbb{G}_m$, the fiber over a point $\mathbf{q} \in \mathbb{A}^1$ is the hyperbola $xy = \mathbf{q} z_2$, which is non-degenerate if $\mathbf{q} \neq 0$, and is the union of the two coordinate axis if $\mathbf{q} = 0$.

5.2 The associated flag variety and its equivariant K-theory

5.2.1. Let $\widehat{\mathbf{B}} \subset \mathbf{GL_2}$ be the Borel subgroup of upper triangular matrices, let $\mathrm{UpTriang}_{2\times 2}$ be the $\mathbb{Z}$-monoid scheme representing the upper triangular 2×2-matrices, and set

$$V_{\widehat{\mathbf{B}}} := \mathrm{UpTriang}_{2\times 2} \times \mathbb{G}_m \subset \mathrm{Mat}_{2\times 2} \times \mathbb{G}_m =: V_{\mathbf{GL_2}}.$$

Then we can apply to this inclusion of $\mathbb{Z}$-monoid schemes the general formalism developed in the Appendix 9. In particular, the *flag variety* $V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}$ is defined as a $\mathbb{Z}$-monoidoid. Moreover, after base changing along $\mathbb{Z} \to k$, we have defined a ring $K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}})$ of $V_{\mathbf{GL_2}}$-equivariant K-theory on the flag variety, together with an induction isomorphism

$$\mathcal{I}nd_{V_{\widehat{\mathbf{B}}}}^{V_{\mathbf{GL_2}}} : R(V_{\widehat{\mathbf{B}}}) \xrightarrow{\sim} K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}})$$

from the ring $R(V_{\widehat{\mathbf{B}}})$ of right representations of the k-monoid scheme $V_{\widehat{\mathbf{B}}}$ on finite dimensional k-vector spaces.

5.2.2. Now, we have the inclusion of monoids $V_{\widehat{\mathbf{T}}} = \mathrm{Diag}_{2\times 2} \times \mathbb{G}_m \subset V_{\widehat{\mathbf{B}}} = \mathrm{UpTriang}_{2\times 2} \times \mathbb{G}_m$, which admits the retraction

$$\begin{array}{ccc} V_{\widehat{\mathbf{B}}} & \longrightarrow & V_{\widehat{\mathbf{T}}} \\ \left(\begin{pmatrix} x & c \\ 0 & y \end{pmatrix}, z_2\right) & \longmapsto & \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, z_2\right). \end{array}$$

Let $\mathrm{Rep}(V_{\widehat{\mathbf{T}}})$ be the category of representations of the commutative k-monoid scheme $V_{\widehat{\mathbf{T}}}$ on finite dimensional k-vector spaces. The above preceding inclusion and retraction define a *restriction functor* and an *inflation functor*

$$\mathrm{Res}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} : \mathrm{Rep}(V_{\widehat{\mathbf{B}}}) \rightleftarrows \mathrm{Rep}(V_{\widehat{\mathbf{T}}}) : \mathrm{Infl}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}}.$$

These functors are exact and compatible with the tensors products and units.

5.2.3. Lemma. *The ring homomorphisms*

$$\mathrm{Res}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} : R(V_{\widehat{\mathbf{B}}}) \rightleftarrows R(V_{\widehat{\mathbf{T}}}) : \mathrm{Infl}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}}$$

are isomorphisms, which are inverse one to the other.

Proof. We have $\mathrm{Res}_{V_{\widehat{\mathbf{T}}}} \circ \mathrm{Infl}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} = \mathrm{Id}$ by construction. Conversely, let M be an object of $\mathrm{Rep}(V_{\widehat{\mathbf{B}}})$. The solvable subgroup $\widehat{\mathbf{B}} \times \mathbb{G}_m \subset V_{\widehat{\mathbf{B}}}$ stabilizes a line $L \subseteq M$. As $\widehat{\mathbf{B}} \times \mathbb{G}_m$ is dense in $V_{\widehat{\mathbf{B}}}$, the line L is automatically $V_{\widehat{\mathbf{B}}}$-stable. Moreover the unipotent radical $\widehat{\mathbf{U}} \subset \widehat{\mathbf{B}}$ acts trivially on L, so that $\widehat{\mathbf{B}} \times \mathbb{G}_m$ acts on L through the quotient $\widehat{\mathbf{T}} \times \mathbb{G}_m$. Hence, by density again, $V_{\widehat{\mathbf{B}}}$ acts on L through the retraction $V_{\widehat{\mathbf{B}}} \to V_{\widehat{\mathbf{T}}}$. This shows that any irreducible M is a character inflated from a character of $V_{\widehat{\mathbf{T}}}$. In particular, the map $R(V_{\widehat{\mathbf{T}}}) \to R(V_{\widehat{\mathbf{B}}})$ is surjective and hence bijective. □

5.2.4. Corollary. *We have a ring isomorphism*

$$c_{V_{\mathbf{GL_2}}} := \mathcal{I}nd_{V_{\widehat{\mathbf{B}}}}^{V_{\mathbf{GL_2}}} \circ \mathrm{Infl}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} : \mathbb{Z}[X, Y, z_2^{\pm 1}] \cong R(V_{\widehat{\mathbf{T}}}) \xrightarrow{\sim} K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}),$$

that we call the characteristic isomorphism in the equivariant K-theory of the flag variety $V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}$.

5.2.5. We have a commutative diagram *specialization at* $\mathbf{q} = 1$

$$\begin{array}{ccc} \mathbb{Z}[X, Y, z_2^{\pm 1}] & \xrightarrow[\sim]{c_{V_{\mathbf{GL_2}}}} & K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}) \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ \mathbb{Z}[X^{\pm 1}, Y^{\pm 1}] & \xrightarrow[\sim]{c_{\mathbf{GL_2}}} & K^{\mathbf{GL_2}}(\mathbf{GL_2}/\widehat{\mathbf{B}}). \end{array}$$

The vertical map on the left-hand side is given by specialization $\mathbf{q}=1$, i.e. by the surjection

$$\mathbb{Z}[X,Y,z_2^{\pm 1}]=\mathbb{Z}[\mathbf{q}][X,Y,z_2^{\pm 1}]/(XY-\mathbf{q}z_2)\longrightarrow \mathbb{Z}[X,Y,z_2^{\pm 1}]/(XY-z_2)=\mathbb{Z}[X^{\pm 1},Y^{\pm 1}].$$

The vertical map on the right-hand side is given by restricting equivariant vector bundles to the 1-fiber of $\mathbf{q}:V_{\mathbf{GL_2}}\to\mathbb{A}^1$, thereby recovering the classical theory.

5.2.6. Let $\mathrm{Rep}(V_{\mathbf{GL_2}})$ be the category of right representations of the k-monoid scheme $V_{\mathbf{GL_2}}$ on finite dimensional k-vector spaces. The inclusion $V_{\widehat{\mathbf{B}}}\subset V_{\mathbf{GL_2}}$ defines a restriction functor

$$\mathrm{Res}^{V_{\mathbf{GL_2}}}_{V_{\widehat{\mathbf{B}}}}:\mathrm{Rep}(V_{\mathbf{GL_2}})\longrightarrow\mathrm{Rep}(V_{\widehat{\mathbf{B}}}),$$

whose composition with $\mathrm{Res}^{V_{\widehat{\mathbf{B}}}}_{V_{\widehat{\mathbf{T}}}}$ is the restriction from $V_{\mathbf{GL_2}}$ to $V_{\widehat{\mathbf{T}}}$:

$$\mathrm{Res}^{V_{\mathbf{GL_2}}}_{V_{\widehat{\mathbf{T}}}}=\mathrm{Res}^{V_{\widehat{\mathbf{B}}}}_{V_{\widehat{\mathbf{T}}}}\circ\mathrm{Res}^{V_{\mathbf{GL_2}}}_{V_{\widehat{\mathbf{B}}}}:\mathrm{Rep}(V_{\mathbf{GL_2}})\longrightarrow\mathrm{Rep}(V_{\widehat{\mathbf{T}}}).$$

These restriction functors are exact and compatible with the tensors products and units.

5.2.7. The action of the Weyl group W_0 on $\Lambda=\mathbb{X}^\bullet(\widehat{\mathbf{T}})$ stabilizes $\mathbb{X}^\bullet(V_{\widehat{\mathbf{T}}})\subset\mathbb{X}^\bullet(\widehat{\mathbf{T}})$, consequently W_0 acts on $V_{\widehat{\mathbf{T}}}$ and the inclusion $\widehat{\mathbf{T}}\subset V_{\widehat{\mathbf{T}}}$ is W_0-equivariant. Explicitly, $W_0=\{1,s\}$ and s acts on $V_{\widehat{\mathbf{T}}}=\mathrm{Diag}_{2\times 2}\times\mathbb{G}_m$ by permuting the two diagonal entries and trivially on the $\mathbb{G}_m$-factor.

5.2.8. Lemma. *The ring homomorphism*

$$\mathrm{Res}^{V_{\mathbf{GL_2}}}_{V_{\widehat{\mathbf{T}}}}:R(V_{\mathbf{GL_2}})\longrightarrow R(V_{\widehat{\mathbf{T}}})$$

is injective, with image the subring $R(V_{\widehat{\mathbf{T}}})^{W_0}\subset R(V_{\widehat{\mathbf{T}}})$ of W_0-invariants. The resulting ring isomorphism

$$\chi_{V_{\mathbf{GL_2}}}:R(V_{\mathbf{GL_2}})\xrightarrow{\ \sim\ }R(V_{\widehat{\mathbf{T}}})^{W_0}$$

is the character isomorphism of $V_{\mathbf{GL_2}}$.

Proof. This is a general result on the representation theory of $V_{\widehat{\mathbf{G}}}$. Note that in the case of $\widehat{\mathbf{G}}=\mathbf{GL_2}$, we have

$$R(V_{\widehat{\mathbf{T}}})^{W_0}=\mathbb{Z}[X+Y,XYz_2^{-1}=:\mathbf{q},z_2^{\pm 1}]\subset\mathbb{Z}[X,Y,z_2^{\pm 1}]=R(V_{\widehat{\mathbf{T}}}).$$

□

6 Dual parametrization of generic Hecke modules

We keep all the notations introduced in the preceding section. In particular, $k=\overline{\mathbb{F}}_q$.

6.1 The generic Bernstein isomorphism

Recall from 2.3.2 the subring $\mathcal{A}(\mathbf{q})\subset\mathcal{H}^{(1)}(\mathbf{q})$ and the remarkable Bernstein basis elements $E(1,0)$, $E(0,1)$ and $E(1,1)$. Also recall from 5.1.4 the representation ring $R(V_{\widehat{\mathbf{T}}})=\mathbb{Z}[X,Y,z_2^{\pm 1}]$ of the diagonalizable k-submonoid scheme $V_{\widehat{\mathbf{T}}}\subset V_{\widehat{\mathbf{G}}}$ of the Vinberg k-monoid scheme of the Langlands dual k-group $\widehat{\mathbf{G}}=\mathbf{GL_2}$ of $GL_{2,F}$.

6.1.1. Theorem. *There exists a unique ring homomorphism*

$$\mathscr{B}(\mathbf{q}):\mathcal{A}(\mathbf{q})\longrightarrow R(V_{\widehat{\mathbf{T}}})$$

such that

$$\mathscr{B}(\mathbf{q})(E(1,0))=X,\quad \mathscr{B}(\mathbf{q})(E(0,1))=Y,\quad \mathscr{B}(\mathbf{q})(E(1,1))=z_2\quad\textit{and}\quad \mathscr{B}(\mathbf{q})(\mathbf{q})=XYz_2^{-1}.$$

It is an isomorphism.

Proof. This is a reformulation of the first part of 2.3.3. □

6.1.2. Then recall from 2.3.2 the subring $\mathcal{A}^{(1)}(\mathbf{q}) = \mathbb{Z}[\mathbb{T}] \otimes_{\mathbb{Z}} \mathcal{A}(\mathbf{q}) \subset \mathcal{H}^{(1)}(\mathbf{q})$ where $\mathbb{T}$ is the finite abelian group $\mathbf{T}(\mathbb{F}_q)$. Let $\mathbb{T}^\vee$ be the finite abelian dual group of $\mathbb{T}$. As $\mathbb{T}^\vee$ has order prime to p, it defines a constant finite diagonalizable k-group scheme, whose group of characters is $\mathbb{T}$, and hence whose representation ring $R(\mathbb{T}^\vee)$ identifies with $\mathbb{Z}[\mathbb{T}]$: $t \in \mathbb{T} \subset \mathbb{Z}[\mathbb{T}]$ corresponds to the character ev_t of $\mathbb{T}^\vee$ given by evaluation at t. Set

$$V_{\widehat{\mathbf{T}}}^{(1)} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}}}.$$

6.1.3. Corollary. *There exists a unique ring homomorphism*

$$\mathscr{B}^{(1)}(\mathbf{q}) : \mathcal{A}^{(1)}(\mathbf{q}) \longrightarrow R(V_{\widehat{\mathbf{T}}}^{(1)})$$

such that

$$\mathscr{B}^{(1)}(\mathbf{q})(E(1,0)) = X, \quad \mathscr{B}^{(1)}(\mathbf{q})(E(0,1)) = Y, \quad \mathscr{B}^{(1)}(\mathbf{q})(E(1,1)) = z_2, \quad \mathscr{B}^{(1)}(\mathbf{q})(\mathbf{q}) = XYz_2^{-1}$$

$$\text{and} \quad \forall t \in \mathbb{T},\ \mathscr{B}^{(1)}(\mathbf{q})(T_t) = \mathrm{ev}_t\,.$$

It is an isomorphism, that we call the generic (pro-p) Bernstein isomorphism.

6.1.4. Also, setting $V_{\widehat{\mathbf{B}}}^{(1)} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{B}}}$, we have from 5.2.3 the ring isomorphism

$$\mathrm{Infl}_{V_{\widehat{\mathbf{T}}}^{(1)}}^{V_{\widehat{\mathbf{B}}}^{(1)}} = \mathrm{Id}_{\mathbb{Z}[\mathbb{T}]} \otimes_{\mathbb{Z}} \mathrm{Res}_{V_{\widehat{\mathbf{T}}}}^{V_{\widehat{\mathbf{B}}}} : R(V_{\widehat{\mathbf{T}}}^{(1)}) = \mathbb{Z}[\mathbb{T}] \otimes_{\mathbb{Z}} R(V_{\widehat{\mathbf{T}}}) \xrightarrow{\ \sim\ } R(V_{\widehat{\mathbf{B}}}^{(1)}) = \mathbb{Z}[\mathbb{T}] \otimes_{\mathbb{Z}} R(V_{\widehat{\mathbf{B}}}),$$

and setting $V_{\widehat{\mathbf{G}}}^{(1)} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{G}}}$, we have from 9.5.2 the ring isomorphism

$$\mathcal{I}nd_{V_{\widehat{\mathbf{B}}}^{(1)}}^{V_{\widehat{\mathbf{G}}}^{(1)}} : R(V_{\widehat{\mathbf{B}}}^{(1)}) \xrightarrow{\ \sim\ } K^{V_{\widehat{\mathbf{G}}}^{(1)}}(V_{\widehat{\mathbf{G}}}^{(1)}/V_{\widehat{\mathbf{B}}}^{(1)});$$

hence by composition we get the *characteristic isomorphism*

$$c_{V_{\widehat{\mathbf{G}}}^{(1)}} : R(V_{\widehat{\mathbf{T}}}^{(1)}) \xrightarrow{\ \sim\ } K^{V_{\widehat{\mathbf{G}}}^{(1)}}(V_{\widehat{\mathbf{G}}}^{(1)}/V_{\widehat{\mathbf{B}}}^{(1)}).$$

Whence a ring isomorphism

$$c_{V_{\widehat{\mathbf{G}}}^{(1)}} \circ \mathscr{B}^{(1)}(\mathbf{q}) : \mathcal{A}^{(1)}(\mathbf{q}) \xrightarrow{\ \sim\ } K^{V_{\widehat{\mathbf{G}}}^{(1)}}(V_{\widehat{\mathbf{G}}}^{(1)}/V_{\widehat{\mathbf{B}}}^{(1)}).$$

6.1.5. The representation ring $R(V_{\widehat{\mathbf{T}}})$ is canonically isomorphic to the ring $\mathbb{Z}[V_{\widehat{\mathbf{T}}}]$ of regular functions of $V_{\widehat{\mathbf{T}}}$ considered now as a diagonalizable monoid scheme over $\mathbb{Z}$. Also recall from 2.2.1 the ring extension $\mathbb{Z} \subset \tilde{\mathbb{Z}}$, and denote by $\tilde{\bullet}$ the base change functor from $\mathbb{Z}$ to $\tilde{\mathbb{Z}}$. For example, we will from now on write $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$ instead of $\mathcal{A}_{\tilde{\mathbb{Z}}}^{(1)}(\mathbf{q})$. We have the constant finite diagonalizable $\tilde{\mathbb{Z}}$-group scheme $\mathbb{T}^\vee$, whose group of characters is $\mathbb{T}$, and whose ring of regular functions is

$$\tilde{\mathbb{Z}}[\mathbb{T}] = \prod_{\lambda \in \mathbb{T}^\vee} \tilde{\mathbb{Z}}\varepsilon_\lambda.$$

Hence applying the functor Spec to $\tilde{\mathscr{B}}^{(1)}(\mathbf{q})$, we obtain the commutative diagram of $\tilde{\mathbb{Z}}$-schemes

$$\begin{array}{ccc} \mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})) & \xleftarrow[\sim]{\ \mathrm{Spec}(\tilde{\mathscr{B}}^{(1)}(\mathbf{q}))\ } & V_{\widehat{\mathbf{T}}}^{(1)} = \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}}} \\ & {\scriptstyle \pi_0 \times \mathbf{q}} \searrow \qquad \swarrow {\scriptstyle \mathrm{Id} \times \mathbf{q}} & \\ & (\mathbb{A}^1)^{(1)} := \mathbb{T}^\vee \times \mathbb{A}^1 & \end{array}$$

where $\pi_0 : \mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})) \to \mathbb{T}^\vee$ is the decomposition of $\mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$ into its connected components. In particular, for each $\lambda \in \mathbb{T}^\vee$, we have the subring $\tilde{\mathcal{A}}^\lambda(\mathbf{q}) = \tilde{\mathcal{A}}^{(1)}(\mathbf{q})\varepsilon_\lambda$ of $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$ and the isomorphism

$$\mathrm{Spec}(\tilde{\mathcal{A}}^\lambda(\mathbf{q})) \xleftarrow[\sim]{\mathrm{Spec}(\tilde{\mathscr{B}}^\lambda(\mathbf{q}))} \{\lambda\} \times V_{\widehat{\mathbf{T}}}$$

of $\tilde{\mathbb{Z}}$-schemes over $\{\lambda\} \times \mathbb{A}^1$. In turn, each of these isomorphisms admits a model over $\mathbb{Z}$, obtained by applying Spec to the ring isomorphism in 4.1.5

$$\mathscr{B}_1(\mathbf{q}) : \mathcal{A}_1(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}}).$$

6.2 The generic Satake isomorphism

Recall part of our notation: $\mathbf{G}$ is the algebraic group $\mathbf{GL_2}$ (which is defined over $\mathbb{Z}$), F is a local field and $G := \mathbf{G}(F)$. We have denoted by o_F the ring of integers of F. Now we set $K := \mathbf{G}(o_F)$.

6.2.1. Definition. *Let R be any commutative ring. The* spherical Hecke algebra of G with coefficients in R *is defined to be the convolution algebra*

$$\mathcal{H}_R^{\mathrm{sph}} := (R[K\backslash G/K], \star)$$

generated by the K-double cosets in G.

6.2.2. By the work of Kazhdan and Lusztig, the R-algebra $\mathcal{H}_R^{\mathrm{sph}}$ depends on F only through the cardinality q of its residue field. Indeed, choose a uniformizer $\varpi \in o_F$. For a dominant cocharacter $\lambda \in \Lambda^+$ of $\mathbf{T}$, let $\mathbb{1}_\lambda$ be the characteristic function of the double coset $K\lambda(\varpi)K$. Then $(\mathbb{1}_\lambda)_{\lambda\in\Lambda^+}$ is an R-basis of $\mathcal{H}_R^{\mathrm{sph}}$. Moreover, for all $\lambda, \mu, \nu \in \Lambda^+$, there exist polynomials

$$N_{\lambda,\mu;\nu}(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}]$$

depending only on the triple (λ, μ, ν), such that

$$\mathbb{1}_\lambda \star \mathbb{1}_\mu = \sum_{\nu\in\Lambda^+} N_{\lambda,\mu;\nu}(q)\mathbb{1}_\nu$$

where $N_{\lambda,\mu;\nu}(q) \in \mathbb{Z} \subset R$ is the value of $N_{\lambda,\mu;\nu}(\mathbf{q})$ at $\mathbf{q} = q$. These polynomials are uniquely determined by this property since when F vary, the corresponding integers q form an infinite set. Their existence can be deduced from the theory of the spherical algebra with coefficients in $\mathbb{C}$, as $\mathcal{H}_R^{\mathrm{sph}} = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}}$ and $\mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}} \subset \mathcal{H}_{\mathbb{C}}^{\mathrm{sph}}$ (e.g. using arguments similar to those in the proof of 6.2.4 below).

6.2.3. Definition. *Let $\mathbf{q}$ be an indeterminate. The* generic spherical Hecke algebra *is the $\mathbb{Z}[\mathbf{q}]$-algebra $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ defined by generators*

$$\mathcal{H}^{\mathrm{sph}}(\mathbf{q}) := \oplus_{\lambda\in\Lambda^+}\mathbb{Z}[\mathbf{q}]T_\lambda$$

and relations:

$$T_\lambda T_\mu = \sum_{\nu\in\Lambda^+} N_{\lambda,\mu;\nu}(\mathbf{q})T_\nu \quad \textit{for all } \lambda, \mu \in \Lambda^+.$$

6.2.4. Theorem. *There exists a unique ring homomorphism*

$$\mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \longrightarrow R(V_{\widehat{\mathbf{T}}})$$

such that

$$\mathscr{S}(\mathbf{q})(T_{(1,0)}) = X + Y, \quad \mathscr{S}(\mathbf{q})(T_{(1,1)}) = z_2 \quad \textit{and} \quad \mathscr{S}(\mathbf{q})(\mathbf{q}) = XYz_2^{-1}.$$

It is an isomorphism onto the subring $R(V_{\widehat{\mathbf{T}}})^{W_0}$ of W_0-invariants

$$\mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}})^{W_0} \subset R(V_{\widehat{\mathbf{T}}}).$$

In particular, the algebra $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ is commutative.

Proof. Let

$$\mathscr{S}_{\mathrm{cl}} : \mathcal{H}^{\mathrm{sph}}_{\mathbb{C}} \xrightarrow{\sim} \mathbb{C}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})]^{W_0}$$

be the 'classical' isomorphism constructed by Satake [Sat63]. We use [Gr98] as a reference.

For $\lambda \in \Lambda^+$, let $\chi_\lambda \in \mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})]^{W_0}$ be the character of the irreducible representation of $\widehat{\mathbf{G}}$ of highest weight λ. Then $(\chi_\lambda)_{\lambda\in\Lambda^+}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})]^{W_0}$. Set $f_\lambda := \mathscr{S}_{\mathrm{cl}}^{-1}(q^{\langle\rho,\lambda\rangle}\chi_\lambda)$, where $2\rho = \alpha := (1,-1)$. Then for each $\lambda,\mu \in \Lambda^+$, there exist polynomials $d_{\lambda,\mu}(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}]$ such that

$$f_\lambda = \mathbb{1}_\lambda + \sum_{\mu<\lambda} d_{\lambda,\mu}(q)\mathbb{1}_\mu \in \mathcal{H}^{\mathrm{sph}}_{\mathbb{C}},$$

where $d_{\lambda,\mu}(q) \in \mathbb{Z}$ is the value of $d_{\lambda,\mu}(\mathbf{q})$ at $\mathbf{q} = q$; the polynomial $d_{\lambda,\mu}(\mathbf{q})$ depends only on the couple (λ,μ), in particular it is uniquely determined by this property. As $(\mathbb{1}_\lambda)_{\lambda\in\Lambda^+}$ is a $\mathbb{Z}$-basis of $\mathcal{H}^{\mathrm{sph}}_{\mathbb{Z}}$, so is $(f_\lambda)_{\lambda\in\Lambda^+}$. Then let us set

$$f_\lambda(\mathbf{q}) := T_\lambda + \sum_{\mu<\lambda} d_{\lambda,\mu}(\mathbf{q})T_\mu \in \mathcal{H}^{\mathrm{sph}}(\mathbf{q}).$$

As $(T_\lambda)_{\lambda\in\Lambda^+}$ is a $\mathbb{Z}[\mathbf{q}]$-basis of $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$, so is $(f_\lambda(\mathbf{q}))_{\lambda\in\Lambda^+}$.

Next consider the following $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$-linear map:

$$\begin{aligned}\mathscr{S}_{\mathrm{cl}}(\mathbf{q}) : \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) &\longrightarrow \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})] = \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}][\mathbb{X}^\bullet(\widehat{\mathbf{T}})]\\ 1\otimes f_\lambda(\mathbf{q}) &\longmapsto \mathbf{q}^{\langle\rho,\lambda\rangle}\chi_\lambda.\end{aligned}$$

We claim that it is a ring homomorphism. Indeed, for $h_1(\mathbf{q}), h_2(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{sph}}(\mathbf{q})$, we need to check the identity

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(h_1(\mathbf{q})h_2(\mathbf{q})) = \mathscr{S}_{\mathrm{cl}}(\mathbf{q})(h_1(\mathbf{q}))\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(h_2(\mathbf{q})) \in \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}][\mathbb{X}^\bullet(\widehat{\mathbf{T}})].$$

Projecting in the $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$-basis $\mathbb{X}^\bullet(\widehat{\mathbf{T}})$, the latter corresponds to (a finite number of) identities in the ring $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$ of polynomials in the variable $\mathbf{q}^{\frac{1}{2}}$. Now, by construction and because $\mathscr{S}_{\mathrm{cl}}$ is a ring homomorphism, the desired identities hold after specialyzing $\mathbf{q}$ to any power of a prime number; hence they hold in $\mathbb{Z}[\mathbf{q}^{\frac{1}{2}}]$. Also note that $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})$ maps $1 = T_{(0,0)}$ to $1 = \chi_{(0,0)}$ by definition.

It can also be seen that $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})$ is injective using a specialization argument: if $h(\mathbf{q}) \in \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ satisfies $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(h(\mathbf{q})) = 0$, then the coordinates of $h(\mathbf{q})$ (in the basis $(1\otimes f_\lambda(\mathbf{q}))_{\lambda\in\Lambda^+}$ say, one can also use the basis $(1\otimes T_\lambda)_{\lambda\in\Lambda^+}$) are polynomials in the variable $\mathbf{q}^{\frac{1}{2}}$ which must vanish for an infinite number of values of $\mathbf{q}$, and hence they are identically zero.

Let us describe the image of $\mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \subset \mathbb{Z}[\mathbf{q}^{\frac{1}{2}}] \otimes_{\mathbb{Z}[\mathbf{q}]} \mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ under the ring embedding $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})$. By construction, we have

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(\mathcal{H}^{\mathrm{sph}}(\mathbf{q})) = \bigoplus_{\lambda\in\Lambda^+} \mathbb{Z}[\mathbf{q}]\mathbf{q}^{\langle\rho,\lambda\rangle}\chi_\lambda.$$

Explicitly,

$$\Lambda^+ = \mathbb{N}(1,0) \oplus \mathbb{Z}(1,1) \subset \mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1) = \Lambda,$$

so that

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(\mathcal{H}^{\mathrm{sph}}(\mathbf{q})) = \left(\bigoplus_{n\in\mathbb{N}} \mathbb{Z}[\mathbf{q}]\mathbf{q}^{\frac{n}{2}}\chi_{(n,0)}\right) \otimes_{\mathbb{Z}} \mathbb{Z}[\chi_{(1,1)}^{\pm 1}].$$

On the other hand, recall that the ring of symmetric polynomials in the two variables $e^{(1,0)}$ and $e^{(0,1)}$ is a graded ring generated the two characters $\chi_{(1,0)} = e^{(1,0)} + e^{(0,1)}$ and $\chi_{(1,1)} = e^{(1,0)}e^{(0,1)}$:

$$\mathbb{Z}[e^{(1,0)}, e^{(0,1)}]^s = \bigoplus_{n\in\mathbb{N}} \mathbb{Z}[e^{(1,0)}, e^{(0,1)}]^s_n = \mathbb{Z}[\chi_{(1,0)}, \chi_{(1,1)}].$$

As $\chi_{(1,0)}$ is homogeneous of degree 1 and $\chi_{(1,1)}$ is homogeneous of degree 2, this implies that

$$\mathbb{Z}[e^{(1,0)}, e^{(0,1)}]^s_n = \bigoplus_{\substack{(a,b)\in\mathbb{N}^2\\ a+2b=n}} \mathbb{Z}\chi_{(1,0)}^a\chi_{(1,1)}^b.$$

Now if $a+2b=n$, then $\mathbf{q}^{\frac{n}{2}}\chi_{(1,0)}^{a}\chi_{(1,1)}^{b}=(\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)})^{a}(\mathbf{q}\chi_{(1,1)})^{b}$. As the symmetric polynomial $\chi_{(n,0)}$ is homogeneous of degree n, we get the inclusion

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(\mathcal{H}^{\mathrm{sph}}(\mathbf{q}))\subset\mathbb{Z}[\mathbf{q}][\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)},\mathbf{q}\chi_{(1,1)}]\otimes_{\mathbb{Z}}\mathbb{Z}[\chi_{(1,1)}^{\pm1}]=\mathbb{Z}[\mathbf{q}][\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)},\chi_{(1,1)}^{\pm1}].$$

Since by definition of $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})$ we have $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(f_{(1,0)}(\mathbf{q}))=\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}$, $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(f_{(1,1)}(\mathbf{q}))=\chi_{(1,1)}$ and $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})(f_{(-1,-1)}(\mathbf{q}))=\chi_{(-1,-1)}=\chi_{(1,1)}^{-1}$, this inclusion is an equality. We have thus obtained the $\mathbb{Z}[\mathbf{q}]$-algebra isomorphism:

$$\mathscr{S}_{\mathrm{cl}}(\mathbf{q})|_{\mathcal{H}^{\mathrm{sph}}(\mathbf{q})}:\mathcal{H}^{\mathrm{sph}}(\mathbf{q})\xrightarrow{\sim}\mathbb{Z}[\mathbf{q}][\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)},\chi_{(1,1)}^{\pm1}].$$

Also note that $T_{(1,0)}\mapsto\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)}$ and $T_{(1,1)}\mapsto\chi_{(1,1)}$ since $T_{(1,0)}=f_{(1,0)}(\mathbf{q})$ and $T_{(1,1)}=f_{(1,1)}(\mathbf{q})$.

Finally, recall that $V_{\widehat{\mathbf{T}}}$ being the diagonalizable k-monoid scheme $\mathrm{Spec}(k[X,Y,z_2^{\pm1}])$, we have

$$R(V_{\widehat{\mathbf{T}}})^{W_0}=\mathbb{Z}[X,Y,z_2^{\pm1}]^{W_0}=\mathbb{Z}[X+Y,XY,z_2^{\pm1}]=\mathbb{Z}[X+Y,XYz_2^{-1},z_2^{\pm1}].$$

Hence we can define a ring isomorphism

$$\iota:\mathbb{Z}[\mathbf{q}][\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)},\chi_{(1,1)}^{\pm1}]\xrightarrow{\sim}R(V_{\widehat{\mathbf{T}}})^{W_0}$$

by $\iota(\mathbf{q}):=XYz_2^{-1}$, $\iota(\mathbf{q}^{\frac{1}{2}}\chi_{(1,0)})=X+Y$ and $\iota(\chi_{(1,1)})=z_2$. Composing, we get the desired isomorphism

$$\mathscr{S}(\mathbf{q}):=\iota\circ\mathscr{S}_{\mathrm{cl}}(\mathbf{q})|_{\mathcal{H}^{\mathrm{sph}}(\mathbf{q})}:\mathcal{H}^{\mathrm{sph}}(\mathbf{q})\xrightarrow{\sim}R(V_{\widehat{\mathbf{T}}})^{W_0}.$$

Note that $\mathscr{S}(\mathbf{q})(T_{(1,0)})=X+Y$, $\mathscr{S}(\mathbf{q})(T_{(1,1)})=z_2$, $\mathscr{S}(\mathbf{q})(\mathbf{q})=XYz_2^{-1}$, and that $\mathscr{S}(\mathbf{q})$ is uniquely determined by these assignments since the ring $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$ is the polynomial ring in the variables $\mathbf{q}$, $T_{(1,0)}$ and $T_{(1,1)}^{\pm1}$, thanks to the isomorphism $\mathscr{S}_{\mathrm{cl}}(\mathbf{q})|_{\mathcal{H}^{\mathrm{sph}}(\mathbf{q})}$. □

6.2.5. Remark. The choice of the isomorphism ι in the preceding proof may seem *ad hoc*. However, it is natural from the point of view of the Vinberg fibration $\mathbf{q}:V_{\widehat{\mathbf{T}}}\to\mathbb{A}^1$.

First, as pointed out by Herzig in [H11, §1.2], one can make the classical complex Satake transform $\mathscr{S}_{\mathrm{cl}}$ integral, by removing the factor $\delta^{\frac{1}{2}}$ from its definition, where δ is the modulus character of the Borel subgroup. Doing so produces a ring embedding

$$\mathcal{S}':\mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}}\hookrightarrow\mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})].$$

The image of $\mathcal{S}'$ is not contained in the subring $\mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})]^{W_0}$ of W_0-invariants. In fact,

$$\mathcal{S}'(T_{(1,0)})=qe^{(1,0)}+e^{(0,1)}\quad\text{and}\quad\mathcal{S}'(T_{(1,1)})=e^{(1,1)},$$

so that

$$\mathcal{S}':\mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}}\xrightarrow{\sim}\mathbb{Z}[(qe^{(1,0)}+e^{(0,1)}),e^{\pm(1,1)}]\subset\mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})].$$

Now,

$$\mathbb{Z}[\mathbb{X}^\bullet(\widehat{\mathbf{T}})]=\mathbb{Z}[\widehat{\mathbf{T}}]=\mathbb{Z}[V_{\widehat{\mathbf{T}},1}],$$

where $\widehat{\mathbf{T}}\cong V_{\widehat{\mathbf{T}},1}$ is the fiber *at* 1 of the fibration $\mathbf{q}:V_{\widehat{\mathbf{T}}}\to\mathbb{A}^1$ considered over $\mathbb{Z}$. But the algebra $\mathcal{H}_{\mathbb{Z}}^{\mathrm{sph}}$ is the specialisation *at* q of the generic algebra $\mathcal{H}^{\mathrm{sph}}(\mathbf{q})$. From this perspective, the morphism $\mathcal{S}'$ is unnatural, since it mixes a 1-fiber with a q-fiber. To restore the $\mathbf{q}$-compatibility, one must consider the composition of $\mathbb{Q}\otimes_{\mathbb{Z}}\mathcal{S}'$ with the isomorphism

$$\begin{array}{rcl}\mathbb{Q}[V_{\widehat{\mathbf{T}},1}]=\mathbb{Q}[X,Y,z_2^{\pm1}]/(XY-z_2) & \xrightarrow{\sim} & \mathbb{Q}[V_{\widehat{\mathbf{T}},q}]=\mathbb{Q}[X,Y,z_2^{\pm1}]/(XY-qz_2)\\ X & \mapsto & q^{-1}X\\ Y & \mapsto & Y\\ z_2 & \mapsto & z_2.\end{array}$$

But then one obtains the formulas

$$\begin{array}{rcl} \mathcal{H}^{\mathrm{sph}}_{\mathbb{Q}} & \xrightarrow{\sim} & \mathbb{Q}[V_{\widehat{\mathbf{T}},q}] = \mathbb{Q}[X,Y,z_2^{\pm 1}]/(XY - qz_2) \\ T_{(1,0)} & \mapsto & X+Y \\ T_{(1,1)} & \mapsto & z_2 \\ q & \mapsto & XYz_2^{\pm 1}. \end{array}$$

This composed map is defined over $\mathbb{Z}$, its image is the subring $\mathbb{Z}[V_{\widehat{\mathbf{T}},q}]^{W_0}$ of W_0-invariants, and it is precisely the specialisation $\mathbf{q} = q$ of the isomorphism $\mathscr{S}(\mathbf{q})$ from 6.2.4.

6.2.6. Definition. *We call*

$$\mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{T}}})^{W_0}$$

the generic Satake isomorphism*.*

6.2.7. Composing with the inverse of the character isomorphism $\chi_{V_{\widehat{\mathbf{G}}}}^{-1} : R(V_{\widehat{\mathbf{T}}})^{W_0} \xrightarrow{\sim} R(V_{\widehat{\mathbf{G}}})$ from 5.2.8, we arrive at an isomorphism

$$\chi_{V_{\widehat{\mathbf{G}}}}^{-1} \circ \mathscr{S}(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} R(V_{\widehat{\mathbf{G}}}).$$

6.2.8. Next, recall the generic Iwahori-Hecke algebra $\mathcal{H}_1(\mathbf{q})$ 4.1.2, and the commutative subring $\mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q})$ 4.1.5 together with the isomorphism $\mathscr{B}_1(\mathbf{q})$ in 6.1.5.

6.2.9. Definition. *The* generic central elements morphism *is the unique ring homomrphism*

$$\mathscr{Z}_1(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \longrightarrow \mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q})$$

making the diagram

$$\begin{array}{ccc} \mathcal{A}_1(\mathbf{q}) & \xrightarrow[\sim]{\mathscr{B}_1(\mathbf{q})} & R(V_{\widehat{\mathbf{T}}}) \\ {\scriptstyle \mathscr{Z}_1(\mathbf{q})}\uparrow & & \uparrow \\ \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) & \xrightarrow[\sim]{\mathscr{S}(\mathbf{q})} & R(V_{\widehat{\mathbf{T}}})^{W_0} \end{array}$$

commutative.

6.2.10. By construction, the morphism $\mathscr{Z}_1(\mathbf{q})$ is injective, and is uniquely determined by the following equalities in $\mathcal{A}_1(\mathbf{q})$:

$$\mathscr{Z}_1(\mathbf{q})(T_{(1,0)}) = z_1, \quad \mathscr{Z}_1(\mathbf{q})(T_{(1,1)}) = z_2 \quad \text{and} \quad \mathscr{Z}_1(\mathbf{q})(\mathbf{q}) = \mathbf{q}.$$

Moreover the group W_0 acts on the ring $\mathcal{A}_1(\mathbf{q})$ and the invariant subring $\mathcal{A}_1(\mathbf{q})^{W_0}$ is equal to the center $Z(\mathcal{H}_1(\mathbf{q})) \subset \mathcal{H}_1(\mathbf{q})$. As the isomorphism $\mathscr{B}_1(\mathbf{q})$ is W_0-equivariant by construction, we obtain that the image of $\mathscr{Z}_1(\mathbf{q})$ indeed is equal to the *center of the generic Iwahori-Hecke algebra* $\mathcal{H}_1(\mathbf{q})$:

$$\mathscr{Z}_1(\mathbf{q}) : \mathcal{H}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\sim} Z(\mathcal{H}_1(\mathbf{q})) \subset \mathcal{A}_1(\mathbf{q}) \subset \mathcal{H}_1(\mathbf{q}).$$

6.2.11. Under the identification $R(V_{\widehat{\mathbf{T}}}) = \mathbb{Z}[V_{\widehat{\mathbf{T}}}]$ of 6.1.5, the elements $\mathscr{S}(\mathbf{q})(T_{(1,0)}) = X + Y$, $\mathscr{S}(\mathbf{q})(\mathbf{q}) = \mathbf{q}$, $\mathscr{S}(\mathbf{q})(T_{(1,1)}) = z_2$, correspond to the *Steinberg choice of coordinates* z_1, $\mathbf{q}$, z_2 on the affine $\mathbb{Z}$-scheme $V_{\widehat{\mathbf{T}}}/W_0 = \mathrm{Spec}(\mathbb{Z}[V_{\widehat{\mathbf{T}}}]^{W_0})$. On the other hand, the *Trace of representations morphism* $\mathrm{Tr} : R(V_{\widehat{\mathbf{G}}}) \to \mathbb{Z}[V_{\widehat{\mathbf{G}}}]^{\widehat{\mathbf{G}}}$ fits into the commutative diagram

$$\begin{array}{ccc} R(V_{\widehat{\mathbf{T}}})^{W_0} & \xleftarrow[\sim]{\chi_{V_{\widehat{\mathbf{G}}}}} & R(V_{\widehat{\mathbf{G}}}) \\ \Vert & & \downarrow {\scriptstyle \mathrm{Tr}} \\ \mathbb{Z}[V_{\widehat{\mathbf{T}}}]^{W_0} & \xleftarrow[\sim]{\mathrm{Ch}} & \mathbb{Z}[V_{\widehat{\mathbf{G}}}]^{\widehat{\mathbf{G}}} \end{array}$$

where $\chi_{V_{\widehat{\mathbf{G}}}}$ is the character isomorphism of 5.2.8, and Ch is the *Chevalley isomorphism* which is constructed for the Vinberg monoid $V_{\widehat{\mathbf{G}}}$ by Bouthier in [Bo15, Prop. 1.7]. So we have the following commutative diagram of $\mathbb{Z}$-schemes

$$\begin{array}{ccccc}
\operatorname{Spec}(\mathcal{A}_1(\mathbf{q})) & \xleftarrow[\sim]{\operatorname{Spec}(\mathscr{B}_1(\mathbf{q}))} & V_{\widehat{\mathbf{T}}} & \hookrightarrow & V_{\widehat{\mathbf{G}}} \\
\Big\downarrow{\scriptstyle \operatorname{Spec}(\mathscr{Z}_1(\mathbf{q}))} & & \Big\downarrow & & \Big\downarrow \\
\operatorname{Spec}(\mathcal{H}^{\mathrm{sph}}(\mathbf{q})) & \xleftarrow[\sim]{\operatorname{Spec}(\mathscr{S}(\mathbf{q}))} & V_{\widehat{\mathbf{T}}}/W_0 & \xrightarrow[\sim]{\operatorname{Spec}(\mathrm{Ch})} & V_{\widehat{\mathbf{G}}}//\widehat{\mathbf{G}} \\
& {\scriptstyle (T_{(1,0)},\mathbf{q},T_{(1,1)})}\searrow\sim & & \sim\swarrow{\scriptstyle (z_1,\mathbf{q},z_2)} & \\
& & \mathbb{A}^2\times\mathbb{G}_m. & &
\end{array}$$

Note that for $\widehat{\mathbf{G}}=\mathbf{GL_2}$, the composed *Chevalley-Steinberg map* $V_{\widehat{\mathbf{G}}}\to\mathbb{A}^2\times\mathbb{G}_m$ is given explicitly by attaching to a 2×2 matrix its characteristic polynomial (when $z_2=1$).

6.2.12. We have recalled that for the generic pro-p-Iwahori-Hecke algebra $\mathcal{H}^{(1)}(\mathbf{q})$ too, the center can be described in terms of W_0-invariants, namely $Z(\mathcal{H}^{(1)}(\mathbf{q}))=\mathcal{A}^{(1)}(\mathbf{q})^{W_0}$, cf. 2.3.4. As the generic Bernstein isomorphism $\mathscr{B}^{(1)}(\mathbf{q})$ is W_0-equivariant by construction, cf. 6.1.3, we can make the following definition.

6.2.13. Definition. *We call*

$$\mathscr{S}^{(1)}(\mathbf{q}):=\mathscr{B}^{(1)}(\mathbf{q})^{W_0}:\mathcal{A}^{(1)}(\mathbf{q})^{W_0}\xrightarrow{\sim}R(V^{(1)}_{\widehat{\mathbf{T}}})^{W_0}$$

the generic pro-p-Iwahori Satake isomorphism.

6.2.14. Note that with $V^{\gamma}_{\widehat{\mathbf{T}}}:=\coprod_{\lambda\in\gamma}V_{\widehat{\mathbf{T}}}$ we have $V^{(1)}_{\widehat{\mathbf{T}}}=\mathbb{T}^{\vee}\times V_{\widehat{\mathbf{T}}}=\coprod_{\gamma\in\mathbb{T}^{\vee}/W_0}V^{\gamma}_{\widehat{\mathbf{T}}}$ and the W_0-action on this scheme respects these γ-components. We obtain the decomposition into connected components

$$V^{(1)}_{\widehat{\mathbf{T}}}/W_0=\coprod_{\gamma\in\mathbb{T}^{\vee}/W_0}\Big(\coprod_{\lambda\in\gamma}V_{\widehat{\mathbf{T}}}\Big)/W_0=\coprod_{\gamma\in\mathbb{T}^{\vee}/W_0}V^{\gamma}_{\widehat{\mathbf{T}}}/W_0$$

If γ is regular, then $V^{\gamma}_{\widehat{\mathbf{T}}}/W_0\simeq V_{\widehat{\mathbf{T}}}$, the isomorphism depending on a choice of order on the set γ, cf. 2.3.5. Hence, passing to $\tilde{\mathbb{Z}}$ as in 6.1.5, with $\tilde{\mathcal{H}}^{(1)}(\mathbf{q}):=\mathcal{H}^{(1)}_{\tilde{\mathbb{Z}}}(\mathbf{q})$, we obtain the following commutative diagram of $\tilde{\mathbb{Z}}$-schemes.

$$\begin{array}{ccc}
\operatorname{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})) & \xleftarrow[\sim]{\operatorname{Spec}(\tilde{\mathscr{B}}^{(1)}(\mathbf{q}))} & V^{(1)}_{\widehat{\mathbf{T}}} \\
\Big\downarrow & & \Big\downarrow \\
\operatorname{Spec}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) & \xleftarrow[\sim]{\operatorname{Spec}(\tilde{\mathscr{S}}^{(1)}(\mathbf{q}))} & V^{(1)}_{\widehat{\mathbf{T}}}/W_0 \\
\Big\downarrow\wr & & \wr\Big\downarrow{\scriptstyle 2.3.5} \\
(\mathbb{A}^2\times\mathbb{G}_m)^{\mathbb{T}^{\vee}/W_0} & \xleftarrow[\sim]{} & \coprod_{(\mathbb{T}^{\vee}/W_0)_{\mathrm{reg}}}V_{\widehat{\mathbf{T}}}\coprod_{(\mathbb{T}^{\vee}/W_0)_{\text{non-reg}}}V_{\widehat{\mathbf{T}}}/W_0,
\end{array}$$

where the bottom isomorphism of the diagram is given by the standard coordinates (x,y,z_2) on the regular components and by the Steinberg coordinates $(z_1,\mathbf{q},z_2)$ on the non-regular components.

6.3 The generic parametrization

We keep the notation $\mathbb{Z}\subset\tilde{\mathbb{Z}}$ for the ring extension of 2.2.1. Then we have defined the $\tilde{\mathbb{Z}}$-scheme $V^{(1)}_{\widehat{\mathbf{T}}}$ in 6.1.5, and we have considered in 6.2.14 its quotient by the natural W_0-action. Also recall that $\widehat{\mathbf{G}}=\mathbf{GL_2}$ is the Langlands dual k-group of $GL_{2,F}$.

6.3.1. Definition. *The category of quasi-coherent modules on the $\tilde{\mathbb{Z}}$-scheme $V_{\widehat{\mathbf{T}}}^{(1)}/W_0$ will be called the* category of Satake parameters*, and denoted by* $\mathrm{SP}_{\widehat{\mathbf{G}}}$*:*

$$\mathrm{SP}_{\widehat{\mathbf{G}}} := \mathrm{QCoh}\big(V_{\widehat{\mathbf{T}}}^{(1)}/W_0\big).$$

For $\gamma \in \mathbb{T}^\vee/W_0$, we also define $\mathrm{SP}_{\widehat{\mathbf{G}}}^\gamma := \mathrm{QCoh}\big(V_{\widehat{\mathbf{T}}}^\gamma/W_0\big)$, where as above $V_{\widehat{\mathbf{T}}}^\gamma = \coprod_{\lambda\in\gamma} V_{\widehat{\mathbf{T}}}$.

6.3.2. Now, over $\tilde{\mathbb{Z}}$, we have the isomorphism

$$i_{\tilde{\mathscr{S}}^{(1)}(\mathbf{q})} := \mathrm{Spec}(\tilde{\mathscr{S}}^{(1)}(\mathbf{q})) : V_{\widehat{\mathbf{T}}}^{(1)}/W_0 \xrightarrow{\ \sim\ } \mathrm{Spec}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})))$$

from the scheme $V_{\widehat{\mathbf{T}}}^{(1)}/W_0$ to the spectrum of the center $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$ of the generic pro-p-Iwahori Hecke algebra $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$, cf. 6.2.14.

6.3.3. Corollary. *The category of modules over $Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$ is equivalent to the category of Satake parameters:*

$$S := (i_{\tilde{\mathscr{S}}^{(1)}(\mathbf{q})})^* : \mathrm{Mod}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) \underset{\longleftarrow}{\xrightarrow{\ \sim\ }} \mathrm{SP}_{\widehat{\mathbf{G}}} : (i_{\tilde{\mathscr{S}}^{(1)}(\mathbf{q})})_*.$$

The equivalence S will be referred to as the functor of Satake parameters*. The quasi-inverse $(i_{\tilde{\mathscr{S}}^{(1)}(\mathbf{q})})_*$ will be denoted by S^{-1}.*

6.3.4. Still from 6.2.14, these categories decompose as products over $\mathbb{T}^\vee/W_0$ (considered as a finite set), compatibly with the equivalences: for all $\gamma \in \mathbb{T}^\vee/W_0$,

$$S^\gamma := (i_{\tilde{\mathscr{S}}^\gamma(\mathbf{q})})^* : \mathrm{Mod}(Z(\tilde{\mathcal{H}}^\gamma(\mathbf{q}))) \underset{\longleftarrow}{\xrightarrow{\ \sim\ }} \mathrm{SP}_{\widehat{\mathbf{G}}}^\gamma : (i_{\tilde{\mathscr{S}}^\gamma(\mathbf{q})})_*,$$

where

$$\mathrm{SP}_{\widehat{\mathbf{G}}}^\gamma \simeq \begin{cases} \mathrm{QCoh}(V_{\widehat{\mathbf{T}}}) & \text{if } \gamma \text{ is regular} \\ \mathrm{QCoh}(V_{\widehat{\mathbf{T}}}/W_0) & \text{if } \gamma \text{ is non-regular.} \end{cases}$$

In the regular case, the latter isomorphism depends on a choice of order on the set γ.

6.3.5. In particular, we have the trivial orbit $\gamma := \{1\}$. The corresponding component $\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})$ of $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$ is canonically isomorphic to the $\tilde{\mathbb{Z}}$-base change of the generic non-regular Iwahori-Hecke algebra $\mathcal{H}_1(\mathbf{q})$. Hence from 6.2.10 we have an isomorphism

$$\tilde{\mathscr{Z}}_1(\mathbf{q}) : \tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q}) \xrightarrow{\ \sim\ } Z(\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})) \subset \tilde{\mathcal{A}}^{\{1\}}(\mathbf{q}) \subset \tilde{\mathcal{H}}^{\{1\}}(\mathbf{q}) \subset \tilde{\mathcal{H}}^{(1)}(\mathbf{q}).$$

Using these identifications, the equivalence S^γ for $\gamma := \{1\}$ can be rewritten as

$$S^{\{1\}} : \mathrm{Mod}(\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})) \xrightarrow{\ \sim\ } \mathrm{SP}_{\widehat{\mathbf{G}}}^{\{1\}}.$$

6.3.6. Definition. *The category of quasi-coherent modules on the $\tilde{\mathbb{Z}}$-scheme $V_{\widehat{\mathbf{T}}}^{(1)}$ will be called the* category of Bernstein parameters*, and denoted by* $\mathrm{BP}_{\widehat{\mathbf{G}}}$*:*

$$\mathrm{BP}_{\widehat{\mathbf{G}}} := \mathrm{QCoh}\big(V_{\widehat{\mathbf{T}}}^{(1)}\big).$$

6.3.7. Over $\tilde{\mathbb{Z}}$, we have the isomorphism

$$i_{\tilde{\mathscr{B}}^{(1)}(\mathbf{q})} := \mathrm{Spec}(\tilde{\mathscr{B}}^{(1)}(\mathbf{q})) : V_{\widehat{\mathbf{T}}}^{(1)} \xrightarrow{\ \sim\ } \mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$$

from the scheme $V_{\widehat{\mathbf{T}}}^{(1)}$ to the spectrum of the commutative subring $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$ of the generic pro-p-Iwahori Hecke algebra $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$, cf. 6.1.5. Also we have the *restriction functor*

$$\mathrm{Res}^{\tilde{\mathcal{H}}^{(1)}(\mathbf{q})}_{\tilde{\mathcal{A}}^{(1)}(\mathbf{q})} : \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \longrightarrow \mathrm{Mod}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})) \cong \mathrm{QCoh}(\mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})))$$

from the category of left $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-modules to the one of $\tilde{\mathcal{A}}^{(1)}(\mathbf{q})$-modules, equivalently of quasi-coherent modules on $\mathrm{Spec}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q}))$.

6.3.8. Definition. *The* functor of Bernstein parameters *is the composed functor*

$$B := (i_{\tilde{\mathscr{B}}^{(1)}(\mathbf{q})})^* \circ \mathrm{Res}^{\tilde{\mathcal{H}}^{(1)}(\mathbf{q})}_{\tilde{\mathcal{A}}^{(1)}(\mathbf{q})} : \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \longrightarrow \mathrm{BP}_{\widehat{\mathbf{G}}}\,.$$

6.3.9. Still from 6.1.5, the category $\mathrm{BP}_{\widehat{\mathbf{G}}}$ decomposes as a product over the finite group $\mathbb{T}^\vee$:

$$\mathrm{BP}_{\widehat{\mathbf{G}}} \cong \prod_{\lambda\in\mathbb{T}^\vee} \mathrm{BP}^\lambda_{\widehat{\mathbf{G}}}, \quad \text{where} \quad \forall\lambda\in\mathbb{T}^\vee,\ \mathrm{BP}^\lambda_{\widehat{\mathbf{G}}} \simeq \mathrm{QCoh}(V_{\widehat{\mathbf{T}}}).$$

6.3.10. Denoting by $\pi : V^{(1)}_{\widehat{\mathbf{T}}} \to V^{(1)}_{\widehat{\mathbf{T}}}/W_0$ the canonical projection, the compatibilty between the functors S and B of Satake and Bernstein parameters is expressed by the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) & \xrightarrow{\;B\;} & \mathrm{BP}_{\widehat{\mathbf{G}}} \\ {\scriptstyle \mathrm{Res}^{\tilde{\mathcal{H}}^{(1)}(\mathbf{q})}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}}\downarrow & & \downarrow{\scriptstyle \pi_*} \\ \mathrm{Mod}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) & \xrightarrow[\sim]{\;S\;} & \mathrm{SP}_{\widehat{\mathbf{G}}}\,. \end{array}$$

6.3.11. Definition. *The* generic parametrization functor *is the functor*

$$P := S \circ \mathrm{Res}^{\tilde{\mathcal{H}}^{(1)}(\mathbf{q})}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))} = \pi_* \circ B :$$

$$\begin{array}{c} \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \\ \downarrow \\ \mathrm{SP}_{\widehat{\mathbf{G}}}\,. \end{array}$$

6.3.12. It follows from the definitions that for all $\gamma \in \mathbb{T}^\vee/W_0$, the fiber of P over the direct factor $\mathrm{SP}^\gamma_{\widehat{\mathbf{G}}} \subset \mathrm{SP}_{\widehat{\mathbf{G}}}$ is the direct factor $\mathrm{Mod}(\tilde{\mathcal{H}}^\gamma(\mathbf{q})) \subset \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))$:

$$P^{-1}(\mathrm{SP}^\gamma_{\widehat{\mathbf{G}}}) = \mathrm{Mod}(\tilde{\mathcal{H}}^\gamma(\mathbf{q})) \subset \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})).$$

Accordingly the parametrization functor P decomposes as the product over the finite set $\mathbb{T}^\vee/W_0$ of functors

$$P^\gamma : \mathrm{Mod}(\tilde{\mathcal{H}}^\gamma(\mathbf{q})) \longrightarrow \mathrm{SP}^\gamma_{\widehat{\mathbf{G}}}\,.$$

6.3.13. In the case of the trivial orbit $\gamma := \{1\}$, it follows from 6.3.5 that $P^{\{1\}}$ factors as

$$\begin{array}{ccc} \mathrm{Mod}(\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})) & & \\ {\scriptstyle \mathrm{Res}^{\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})}_{\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})}}\downarrow & \searrow^{P^{\{1\}}} & \\ \mathrm{Mod}(\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})) & \xrightarrow[\sim]{\;S^{\{1\}}\;} & \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}}}\,. \end{array}$$

6.4 The generic antispherical module

Recall the generic regular and non-regular antispherical representations $\mathscr{A}_2(\mathbf{q})$ 3.3.1 and $\mathscr{A}_1(\mathbf{q})$ 4.3.1 of $\mathcal{H}_2(\mathbf{q})$ and $\mathcal{H}_1(\mathbf{q})$. Thanks to 3.1.4 and 4.1.4, they are models over $\mathbb{Z}$ of representations $\tilde{\mathscr{A}}^\gamma(\mathbf{q})$ of the regular and non-regular components $\tilde{\mathscr{A}}^\gamma(\mathbf{q})$, $\gamma \in \mathbb{T}^\vee/W_0$, of the generic pro-$p$-Iwahori Hecke algebra $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$ over $\tilde{\mathbb{Z}}$, cf. 2.2.3 and 2.3.2. Taking the product over $\mathbb{T}^\vee/W_0$ of these representations, we obtain a representation

$$\tilde{\mathscr{A}}^{(1)}(\mathbf{q}) : \tilde{\mathcal{H}}^{(1)}(\mathbf{q}) \longrightarrow \mathrm{End}_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}(\tilde{\mathcal{A}}^{(1)}(\mathbf{q})).$$

By construction, the representation $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})$ depends on a choice of order on each regular orbit γ.

6.4.1. Definition. *We call* $\tilde{\mathscr{A}}^{(1)}(\mathbf{q})$ *the* generic antispherical representation*, and the corresponding left* $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$*-module* $\tilde{\mathcal{M}}^{(1)}$ *the* generic antispherical module*.*

6.4.2. Proposition.

1. *The generic antispherical representation is faithful.*
2. *The Bernstein parameter of the antispherical module is the structural sheaf:*
$$B(\mathcal{M}^{(1)}) = \mathcal{O}_{V^{(1)}_{\widehat{\mathbf{T}}}}.$$
3. *The Satake parameter of the antispherical module is the* $\tilde{R}(V^{(1)}_{\widehat{\mathbf{G}}})$*-module of* $V^{(1)}_{\widehat{\mathbf{G}}}$*-equivariant* K*-theory of the flag variety of* $V^{(1)}_{\widehat{\mathbf{G}}}$*:*
$$\tilde{c}_{V^{(1)}_{\widehat{\mathbf{G}}}} : S(\mathcal{M}^{(1)}) \xrightarrow{\sim} \tilde{K}^{V^{(1)}_{\widehat{\mathbf{G}}}}(V^{(1)}_{\widehat{\mathbf{G}}}/V^{(1)}_{\widehat{\mathbf{B}}}).$$

Proof. Part 1. follows from 3.3.3 and 4.3.3, part 2. from the property *(i)* in 3.3.1 and 4.3.1, and part 3. from the characteristic isomorphism in 6.1.4. □

6.4.3. Now, being a left $\tilde{\mathcal{H}}^{(1)}(\mathbf{q})$-module, the antispherical module $\tilde{\mathcal{M}}^{(1)}$ defines a functor
$$\tilde{\mathcal{M}}^{(1)} \otimes_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))} \bullet : \mathrm{Mod}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) \longrightarrow \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})).$$
On the other hand, recall the canonical projection $\pi : V^{(1)}_{\widehat{\mathbf{T}}} \to V^{(1)}_{\widehat{\mathbf{T}}}/W_0$ from 6.3.10. Then point 2. of 6.4.2 has the following consequence.

6.4.4. Corollary. *The diagram*
$$\begin{array}{ccc} \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) & \xrightarrow{B} & \mathrm{BP}_{\widehat{\mathbf{G}}} \\ {\scriptstyle \tilde{\mathcal{M}}^{(1)}\otimes_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))}\bullet}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ \mathrm{Mod}(Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))) & \xrightarrow[\sim]{S} & \mathrm{SP}_{\widehat{\mathbf{G}}} \end{array}$$
is commutative.

6.4.5. Definition. *The* generic antispherical functor *is the functor*
$$\mathrm{ASph} := (\tilde{\mathcal{M}}^{(1)} \otimes_{Z(\tilde{\mathcal{H}}^{(1)}(\mathbf{q}))} \bullet) \circ S^{-1} :$$
$$\mathrm{SP}_{\widehat{\mathbf{G}}} \longrightarrow \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})).$$

6.4.6. Corollary. *The diagram*
$$\begin{array}{ccccc} & & & & \mathrm{Mod}(\tilde{\mathcal{H}}^{(1)}(\mathbf{q})) \\ & \nearrow^{\mathrm{ASph}} & & & \downarrow{\scriptstyle P} \\ \mathrm{SP}_{\widehat{\mathbf{G}}} & \xrightarrow[\pi^*]{} & \mathrm{BP}_{\widehat{\mathbf{G}}} & \xrightarrow[\pi_*]{} & \mathrm{SP}_{\widehat{\mathbf{G}}} \end{array}$$
is commutative.

Proof. One has $P \circ \mathrm{ASph} = \pi_* \circ (B \circ \mathrm{ASph}) = \pi_* \circ \pi^*$ by the preceding corollary. □

6.4.7. By construction, the antispherical functor ASph decomposes as a product of functors ASph^{γ} for $\gamma \in \mathbb{T}^{\vee}/W_0$, and accordingly the previous diagram decomposes over $\mathbb{T}^{\vee}/W_0$.

6.4.8. In particular for $\gamma = \{1\}$ we have the commutative diagram
$$\begin{array}{ccccccc} & & & & \mathrm{Mod}(\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})) & & \\ & \nearrow^{\mathrm{ASph}^{\{1\}}} & & & \downarrow{\scriptstyle P^{\{1\}}} & \searrow^{\mathrm{Res}^{\tilde{\mathcal{H}}^{\{1\}}(\mathbf{q})}_{\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})}} & \\ \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}}} & \xrightarrow[\pi^*]{} & \mathrm{BP}^{\{1\}}_{\widehat{\mathbf{G}}} & \xrightarrow[\pi_*]{} & \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}}} & \xleftarrow[S^{\{1\}}]{\sim} & \mathrm{Mod}(\tilde{\mathcal{H}}^{\mathrm{sph}}(\mathbf{q})). \end{array}$$

7 The theory at $\mathbf{q} = q = 0$

We keep all the notations introduced in the preceding section. In particular, $k = \overline{\mathbb{F}}_q$.

7.1 Geometric representation theory at $\mathbf{q} = 0$

7.1.1. Recall from 5.1 the k-semigroup scheme

$$V_{\mathbf{GL_2},0} = \mathrm{Sing}_{2\times 2} \times \mathbb{G}_m,$$

which can even be defined over $\mathbb{Z}$, and which is obtained as the 0-fiber of

$$\begin{array}{c} V_{\mathbf{GL_2}} \\ \Big\downarrow \mathbf{q} \\ \mathbb{A}^1. \end{array}$$

7.1.2. It admits

$$V_{\widehat{\mathbf{T}},0} = \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m$$

as a commutative subsemigroup scheme. The latter has the following structure: it is the pinching of the monoids

$$\mathbb{A}^1_X \times \mathbb{G}_m := \mathrm{Spec}(k[X, z_2^{\pm 1}]) \quad \text{and} \quad \mathbb{A}^1_Y \times \mathbb{G}_m := \mathrm{Spec}(k[Y, z_2^{\pm 1}])$$

along the sections $X = 0$ and $Y = 0$. These monoids are semisimple, with representation rings

$$R(\mathbb{A}^1_X \times \mathbb{G}_m) = \mathbb{Z}[X, z_2^{\pm 1}] \quad \text{and} \quad R(\mathbb{A}^1_Y \times \mathbb{G}_m) = \mathbb{Z}[Y, z_2^{\pm 1}].$$

There are three remarkable elements in $V_{\widehat{\mathbf{T}},0}$, namely

$$\varepsilon_X := (\mathrm{diag}(1,0), 1), \quad \varepsilon_Y := (\mathrm{diag}(0,1), 1) \quad \text{and} \quad \varepsilon_0 := (\mathrm{diag}(0,0), 1).$$

They are idempotents. Now let M be a finite dimensional k-representation of $V_{\widehat{\mathbf{T}},0}$. The idempotents act on M as projectors, and as the semigroup $V_{\widehat{\mathbf{T}},0}$ is commutative, the k-vector space M decomposes as a direct sum

$$M = \bigoplus_{(\lambda_X, \lambda_Y, \lambda_0) \in \{0,1\}^3} M(\lambda_X, \lambda_Y, \lambda_0)$$

where

$$M(\lambda_X, \lambda_Y, \lambda_0) = \{m \in M \mid m\varepsilon_X = \lambda_X m,\ m\varepsilon_Y = \lambda_Y m,\ m\varepsilon_0 = \lambda_0 m\}.$$

Moreover, since $V_{\widehat{\mathbf{T}},0}$ is commutative, each of these subspaces is in fact a subrepresentation of M.

As $\varepsilon_X \varepsilon_Y = \varepsilon_0 \in V_{\widehat{\mathbf{T}},0}$, we have $M(1,1,0) = 0$. Next, as ε_X is the unit of the monoid $\mathbb{A}^1_X$, if $\lambda_X = 0$ then $\mathrm{Res}^{V_{\widehat{\mathbf{T}},0}}_{\mathbb{A}^1_X} M(\lambda_X, \lambda_Y, \lambda_0)$ must be the null representation, in particular we must have $\lambda_0 = 0$; hence $M(0,0,1) = M(0,1,1) = 0$. Considering ε_Y instead of ε_X, we get similarly that $M(0,0,1) = M(1,0,1) = 0$. Consequently

$$M = M(1,0,0) \bigoplus M(0,1,0) \bigoplus M(1,1,1) \bigoplus M(0,0,0).$$

The restriction $\mathrm{Res}^{V_{\widehat{\mathbf{T}},0}}_{\mathbb{A}^1_X} M(1,0,0)$ is a representation of the monoid $\mathbb{A}^1_X$ where 0 acts by 0, and $\mathrm{Res}^{V_{\widehat{\mathbf{T}},0}}_{\mathbb{A}^1_Y} M(1,0,0)$ is the null representation. Hence, if for $n > 0$ we still denote by X^n the character of $V_{\widehat{\mathbf{T}},0}$ which restricts to the character X^n of $\mathbb{A}^1_X \times \mathbb{G}_m$ and the null map of $\mathbb{A}^1_Y \times \mathbb{G}_m$, then $M(1,0,0)$ decomposes as a sum of weight spaces

$$M(1,0,0) = \oplus_{n>0} M(X^n) := \oplus_{n>0, m\in\mathbb{Z}} M(X^n z_2^m).$$

Similarly
$$M(0,1,0) = \oplus_{n>0} M(Y^n) := \oplus_{n>0, m\in\mathbb{Z}} M(Y^n z_2^m).$$
Finally, $V_{\widehat{\mathbf{T}},0}$ acts through the projection $V_{\widehat{\mathbf{T}},0} \to \mathbb{G}_m$ on
$$M(1,1,1) =: M(1) = \oplus_{m\in\mathbb{Z}} M(z_2^m),$$
and by 0 on
$$M(0,0,0) =: M(0).$$
Thus we have obtained the following

7.1.3. Lemma. *The category* $\mathrm{Rep}(V_{\widehat{\mathbf{T}},0})$ *is semisimple, and there is a ring isomorphism*
$$R(V_{\widehat{\mathbf{T}},0}) \cong \big(\mathbb{Z}[X,Y,z_2^{\pm 1}]/(XY)\big) \times \mathbb{Z}.$$

7.1.4. Next let
$$V_{\widehat{\mathbf{B}},0} = \mathrm{SingUpTriang}_{2\times 2} \times \mathbb{G}_m \subset V_{\mathbf{GL_2},0} = \mathrm{Sing}_{2\times 2}\,\mathbb{G}_m$$
be the subsemigroup scheme of singular upper triangular 2×2-matrices. It contains $V_{\widehat{\mathbf{T}},0}$, and the inclusion $V_{\widehat{\mathbf{T}},0} \subset V_{\widehat{\mathbf{B}},0}$ admits a retraction $V_{\widehat{\mathbf{B}},0} \to V_{\widehat{\mathbf{T}},0}$, namely the specialisation at $\mathbf{q} = 0$ of the retraction 5.2.2.

Let M be an object of $\mathrm{Rep}(V_{\widehat{\mathbf{B}},0})$. Write
$$\mathrm{Res}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}} M = M(1,0,0) \oplus M(0,1,0) \oplus M(1) \oplus M(0).$$
For a subspace $N \subset M$, consider the following property:

(P_N) *the subspace* $N \subset M$ *is a subrepresentation, and* $V_{\widehat{\mathbf{B}},0}$ *acts on* N *through the retraction of* k*-semigroup schemes* $V_{\widehat{\mathbf{B}},0} \to V_{\widehat{\mathbf{T}},0}$.

Let us show that $(\mathrm{P}_{M(0,1,0)})$ is true. Indeed for $m \in M(0,1,0) = \oplus_{n>0} M(Y^n)$, we have
$$m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = (m\varepsilon_Y)\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = m\varepsilon_0 = 0 = m\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$
and
$$m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = (m\varepsilon_Y)\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = m\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$
Next assume $M(0,1,0) = 0$, and let us show that in this case $(\mathrm{P}_{M(0)})$ is true. Indeed for $m \in M(0)$, we have
$$m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = m\Big(\varepsilon_X\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix}\Big) = (m\varepsilon_X)\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = 0,$$
and if we decompose
$$m' := m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = m'_{(1,0,0)} + m'_1 + m'_0 \in M(1,0,0) \oplus M(1) \oplus M(0),$$
then by applying ε_X on the right we see that $0 = m'_{(1,0,0)} + m'_1$ so that $m' \in M(0)$ and hence
$$m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = m\Big(\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix}\varepsilon_Y\Big) = m'\varepsilon_Y = 0.$$
Next assume $M(0,1,0) = M(0) = 0$, and let us show that in this case $(\mathrm{P}_{M(1,0,0)})$ is true. Indeed, let $m \in M(1,0,0) = \oplus_{n>0} M(X^n)$. Then for any $c \in k$,
$$m' := m\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$$

satisfies $m'\varepsilon_X = 0$, $m'\varepsilon_Y = m'$, $m'\varepsilon_0 = 0$, i.e. $m' \in M(0,1,0)$, and hence is equal to 0 by our assumption. It follows that

$$m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = (m\varepsilon_X)\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = m\left(\varepsilon_X\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix}\right) = m\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} = 0 = m\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}.$$

On the other hand, if we decompose

$$m' := m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = m'_{(1,0,0)} + m'_1 \in M(1,0,0) \oplus M(1),$$

then by applying ε_0 on the right we find $0 = m'_1$, i.e. $m' \in M(1,0,0)$ and hence

$$m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = m' = m'\varepsilon_X = m\left(\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix}\varepsilon_X\right) = m\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally assume $M(0,1,0) = M(0) = M(1,0,0) = 0$, and let us show that in this case $(\mathrm{P}_{M(1)})$ is true, i.e. that $V_{\widehat{\mathbf{B}},0}$ acts through the projection $V_{\widehat{\mathbf{B}},0} \to \mathbb{G}_m$ on $M = M(1)$. Indeed for any m we have

$$m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix} = \left(m\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix}\right)\varepsilon_0 = m\left(\begin{pmatrix} x & c \\ 0 & 0 \end{pmatrix}\varepsilon_0\right) = m\varepsilon_0 = m$$

and

$$m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix} = \left(m\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix}\right)\varepsilon_0 = m\left(\begin{pmatrix} 0 & c \\ 0 & y \end{pmatrix}\varepsilon_0\right) = m\varepsilon_0 = m.$$

It follows from the preceding discussion that the irreducible representations of $V_{\widehat{\mathbf{B}},0}$ are the characters, which are inflated from those of $V_{\widehat{\mathbf{T}},0}$ through the retraction $V_{\widehat{\mathbf{B}},0} \to V_{\widehat{\mathbf{T}},0}$. As a consequence, considering the *restriction* and *inflation* functors

$$\mathrm{Res}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}} : \mathrm{Rep}(V_{\widehat{\mathbf{B}},0}) \underset{\longleftarrow}{\longrightarrow} \mathrm{Rep}(V_{\widehat{\mathbf{T}},0}) : \mathrm{Infl}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}},$$

which are exact and compatible with tensor products and units, we get:

7.1.5. Lemma. *The ring homomorphisms*

$$\mathrm{Res}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}} : R(V_{\widehat{\mathbf{B}},0}) \underset{\longleftarrow}{\longrightarrow} R(V_{\widehat{\mathbf{T}},0}) : \mathrm{Infl}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}},$$

are isomorphisms, which are inverse one to the other.

7.1.6. Finally, note that $\varepsilon_0 \in V_{\mathbf{GL_2}}(k)$ belongs to all the left $V_{\mathbf{GL_2}}(k)$-cosets in $V_{\mathbf{GL_2}}(k)$. Hence, by 9.4.3, the catgory $\mathrm{Rep}(V_{\widehat{\mathbf{B}},0})$ is equivalent to the one of induced vector bundles on the semigroupoid flag variety $V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}$:

$$\mathcal{I}nd^{V_{\mathbf{GL_2},0}}_{V_{\widehat{\mathbf{B}},0}} : \mathrm{Rep}(V_{\widehat{\mathbf{B}},0}) \xrightarrow{\sim} \mathcal{C}^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}) \subset \mathcal{C}^{V_{\mathbf{GL_2},0}}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}).$$

7.1.7. Corollary. *We have a ring isomorphism*

$$\mathcal{I}nd^{V_{\mathbf{GL_2},0}}_{V_{\widehat{\mathbf{B}},0}} \circ \mathrm{Infl}^{V_{\widehat{\mathbf{B}},0}}_{V_{\widehat{\mathbf{T}},0}} : R(V_{\widehat{\mathbf{T}},0}) \xrightarrow{\sim} K^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}).$$

7.1.8. Definition. *We call* relevant *the full subcategory*

$$\mathrm{Rep}(V_{\widehat{\mathbf{T}},0})^{\mathrm{rel}} \subset \mathrm{Rep}(V_{\widehat{\mathbf{T}},0})$$

whose objects M satisfy $M(0) = 0$. Correspondingly, we have relevant full subcategories

$$\mathrm{Rep}(V_{\widehat{\mathbf{B}},0})^{\mathrm{rel}} \subset \mathrm{Rep}(V_{\widehat{\mathbf{B}},0}) \quad \textit{and} \quad \mathcal{C}^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0})^{\mathrm{rel}} \subset \mathcal{C}^{V_{\mathbf{GL_2},0}}_{\mathcal{I}nd}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}).$$

7.1.9. Corollary. *We have a ring isomorphism*

$$c_{V_{\mathbf{GL_2}},0} := \mathbb{Z}[X,Y,z_2^{\pm 1}]/(XY) \cong R(V_{\widehat{\mathbf{T}},0})^{\mathrm{rel}} \xrightarrow{\sim} K_{\mathcal{I}nd}^{V_{\mathbf{GL_2}},0}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0})^{\mathrm{rel}},$$

that we call the characteristic isomorphism in the equivariant K-theory of the flag variety $V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0}$.

7.1.10. We have a commutative diagram *specialization at* $\mathbf{q}=0$

$$\begin{array}{ccc} \mathbb{Z}[X,Y,z_2^{\pm 1}] & \xrightarrow[\sim]{c_{V_{\mathbf{GL_2}}}} & K^{V_{\mathbf{GL_2}}}(V_{\mathbf{GL_2}}/V_{\widehat{\mathbf{B}}}) \\ \downarrow & & \downarrow \\ \mathbb{Z}[X,Y,z_2^{\pm 1}]/(XY) & \xrightarrow[\sim]{c_{V_{\mathbf{GL_2}},0}} & K_{\mathcal{I}nd}^{V_{\mathbf{GL_2}},0}(V_{\mathbf{GL_2},0}/V_{\widehat{\mathbf{B}},0})^{\mathrm{rel}}, \end{array}$$

where the vertical right-hand side map is given by restricting equivariant vector bundles to the 0-fiber of $\mathbf{q}: V_{\mathbf{GL_2}} \to \mathbb{A}^1$.

7.2 The mod p Satake and Bernstein isomorphisms

7.2.1. Notation. In the sequel, we will denote by $(\bullet)_{\overline{\mathbb{F}}_q}$ the *specialization at* $\mathbf{q}=q=0$, i.e. the base change functor along the ring morphism

$$\begin{array}{rcl} \mathbb{Z}[\mathbf{q}] & \longrightarrow & \overline{\mathbb{F}}_q =: k \\ \mathbf{q} & \longmapsto & 0. \end{array}$$

Also we fix an embedding $\mu_{q-1} \subset \overline{\mathbb{F}}_q^\times$, so that the above morphism factors through the inclusion $\mathbb{Z}[\mathbf{q}] \subset \tilde{\mathbb{Z}}[\mathbf{q}]$, where $\mathbb{Z} \subset \tilde{\mathbb{Z}}$ is the ring extension considered in 2.2.1.

7.2.2. The mod p Satake and pro-p-Iwahori Satake isomorphisms. Specializing 6.2.6, we get an isomorphism of $\overline{\mathbb{F}}_q$-algebras

$$\mathscr{S}_{\overline{\mathbb{F}}_q} : \mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} \overline{\mathbb{F}}_q[V_{\widehat{\mathbf{T}},0}]^{W_0} = \left(\overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY)\right)^{W_0}.$$

In [H11], Herzig constructed an isomorphism

$$\mathscr{S}_{\mathrm{Her}} : \mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} \overline{\mathbb{F}}_q[\mathbb{X}^\bullet(\widehat{\mathbf{T}})_-] = \overline{\mathbb{F}}_q[e^{(0,1)}, e^{\pm(1,1)}]$$

(this is $\overline{\mathbb{F}}_q \otimes_{\mathbb{Z}} \mathcal{S}'$, with the notation $\mathcal{S}'$ from 6.2.5). They are related by the Steinberg choice of coordinates $z_1 := X+Y$ and z_2 on the quotient $V_{\widehat{\mathbf{T}},0}/W_0$, cf. 6.2.11, i.e. by the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} & \xrightarrow[\sim]{\mathscr{S}_{\overline{\mathbb{F}}_q}} & \left(\overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY)\right)^{W_0} \\ & {\scriptstyle \mathscr{S}_{\mathrm{Her}}}\searrow^{\sim} \quad \nearrow^{\sim}_{e^{(0,1)}\mapsto z_1,\ e^{(1,1)}\mapsto z_2} & \\ & \overline{\mathbb{F}}_q[e^{(0,1)}, e^{\pm(1,1)}]. & \end{array}$$

Specializing 6.2.13, we get an isomorphism of $\overline{\mathbb{F}}_q$-algebras

$$\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_q} : (\mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q})^{W_0} \xrightarrow{\sim} \overline{\mathbb{F}}_q[V^{(1)}_{\widehat{\mathbf{T}},0}]^{W_0} = \left(\overline{\mathbb{F}}_q[\mathbb{T}][X,Y,z_2^{\pm 1}]/(XY)\right)^{W_0}.$$

7.2.3. The mod p Bernstein isomorphism. Specializing 6.1.3, we get an isomorphism of $\overline{\mathbb{F}}_q$-algebras

$$\mathscr{B}^{(1)}_{\overline{\mathbb{F}}_q} : \mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} \overline{\mathbb{F}}_q[V^{(1)}_{\widehat{\mathbf{T}},0}] = \overline{\mathbb{F}}_q[\mathbb{T}][X,Y,z_2^{\pm 1}]/(XY).$$

Moreover, similarly as in 6.1.4 but here using 7.1.5 and 9.5.1, we get the *characteristic isomorphism*

$$c_{V^{(1)}_{\widehat{\mathbf{G}},0}} : R(V^{(1)}_{\widehat{\mathbf{T}},0}) \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}},0}}_{\mathcal{I}nd}(V^{(1)}_{\widehat{\mathbf{G}},0}/V^{(1)}_{\widehat{\mathbf{B}},0}).$$

Whence by 7.1.3 (and recalling 7.1.8) an isomorphism

$$c^{\mathrm{rel}}_{V^{(1)}_{\widehat{\mathbf{G}},0},\overline{\mathbb{F}}_q} \circ \mathscr{B}^{(1)}_{\overline{\mathbb{F}}_q} : \mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}},0}}_{\mathcal{I}nd,\overline{\mathbb{F}}_q}(V^{(1)}_{\widehat{\mathbf{G}},0}/V^{(1)}_{\widehat{\mathbf{B}},0})^{\mathrm{rel}}.$$

Also, specializing 6.1.5, $\mathscr{B}^{(1)}_{\overline{\mathbb{F}}_q}$ splits as a product over $\mathbb{T}^\vee$ of $\overline{\mathbb{F}}_q$-algebras isomorphisms $\mathscr{B}^{\lambda}_{\overline{\mathbb{F}}_q}$, each of them being of the form

$$\mathscr{B}_{1,\overline{\mathbb{F}}_q} : \mathcal{A}_{1,\overline{\mathbb{F}}_q} \xrightarrow{\sim} \overline{\mathbb{F}}_q[V_{\widehat{\mathbf{T}},0}] = \overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY).$$

7.2.4. The mod p central elements embedding. Specializing 6.2.9, we get an embedding of $\overline{\mathbb{F}}_q$-algebras

$$\mathscr{Z}_{1,\overline{\mathbb{F}}_q} : \mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \xrightarrow{\sim} Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q}) \subset \mathcal{A}_{1,\overline{\mathbb{F}}_q} \subset \mathcal{H}_{1,\overline{\mathbb{F}}_q}$$

making the diagram

$$\begin{array}{ccc} \mathcal{A}_{1,\overline{\mathbb{F}}_q} & \xrightarrow[\sim]{\mathscr{B}_{1,\overline{\mathbb{F}}_q}} & \overline{\mathbb{F}}_q[V_{\widehat{\mathbf{T}},0}] = \overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY) \\ {\scriptstyle \mathscr{Z}_{1,\overline{\mathbb{F}}_q}}\uparrow & & \uparrow \\ \mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} & \xrightarrow[\sim]{\mathscr{S}_{\overline{\mathbb{F}}_q}} & \overline{\mathbb{F}}_q[V_{\widehat{\mathbf{T}},0}]^{W_0} = \big(\overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY)\big)^{W_0} \end{array}$$

commutative. Then $\mathscr{Z}_{1,\overline{\mathbb{F}}_q}$ coincides with the central elements construction of Ollivier [O14, Th. 4.3] for the case of $\mathbf{GL_2}$. This follows from the explicit formulas for the values of $\mathscr{Z}_1(\mathbf{q})$ on $T_{(1,0)}$ and $T_{(1,1)}$, cf. 6.2.10.

7.3 The mod p parametrization

7.3.1. Definition. *The category of quasi-coherent modules on the k-scheme $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ will be called the* category of mod p Satake parameters*, and denoted by* $\mathrm{SP}_{\widehat{\mathbf{G}},0}$*:*

$$\mathrm{SP}_{\widehat{\mathbf{G}},0} := \mathrm{QCoh}\big(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0\big).$$

For $\gamma \in \mathbb{T}^\vee/W_0$, we also define $\mathrm{SP}^{\gamma}_{\widehat{\mathbf{G}},0} := \mathrm{QCoh}\big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)$*, where* $V^{\gamma}_{\widehat{\mathbf{T}},0} = \coprod_{\lambda\in\gamma} V_{\widehat{\mathbf{T}},0}$.

7.3.2. Similarly to the generic case 6.3, the mod p pro-p-Iwahori Satake isomorphism induces an equivalence of categories

$$S : \mathrm{Mod}(Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})) \xrightarrow{\sim} \mathrm{SP}_{\widehat{\mathbf{G}},0},$$

that will be referred to as the *functor of mod p Satake parameters*, and which decomposes as a product over the finite set $\mathbb{T}^\vee/W_0$:

$$S = \prod_\gamma S^\gamma : \prod_\gamma \mathrm{Mod}(Z(\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q})) \xrightarrow{\sim} \prod_\gamma \mathrm{SP}^{\gamma}_{\widehat{\mathbf{G}},0} \simeq \prod_{\gamma\ \mathrm{reg}} \mathrm{QCoh}(V_{\widehat{\mathbf{T}},0}) \prod_{\gamma\ \text{non-reg}} \mathrm{QCoh}(V_{\widehat{\mathbf{T}},0}/W_0).$$

For $\gamma = \{1\}$ and using 7.2.4 we get an equivalence

$$S^{\{1\}} : \mathrm{Mod}(\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}) \xrightarrow{\sim} \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}},0} = \mathrm{QCoh}(V_{\widehat{\mathbf{T}},0}/W_0).$$

Note that under this equivalence, the characters $\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q} \to \overline{\mathbb{F}}_q$ correspond to the skyscraper sheaves on $V_{\widehat{\mathbf{T}},0}/W_0$, and hence to its k-points. Choosing the Steinberg coordinates (z_1, z_2) on the k-scheme $V_{\widehat{\mathbf{T}},0}/W_0$, they may also be regarded as the k-points of $\mathrm{Spec}(k[\mathbb{X}^\bullet(\widehat{\mathbf{T}})_-])$, which are precisely the mod p Satake parameters defined by Herzig in [H11].

7.3.3. Definition. *The category of quasi-coherent modules on the k-scheme $V^{(1)}_{\widehat{\mathbf{T}},0}$ will be called the* category of mod p Bernstein parameters*, and denoted by* $\mathrm{BP}_{\widehat{\mathbf{G}},0}$*:*

$$\mathrm{BP}_{\widehat{\mathbf{G}},0} := \mathrm{QCoh}\left(V^{(1)}_{\widehat{\mathbf{T}},0}\right).$$

7.3.4. Similarly to the generic case 6.3, the inclusion $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q} \supset \mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q}$ together with the mod p Bernstein isomorphism define a *functor of mod p Bernstein parameters*

$$B : \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \longrightarrow \mathrm{BP}_{\widehat{\mathbf{G}},0}\,.$$

Moreover the category $\mathrm{BP}_{\widehat{\mathbf{G}},0}$ decomposes as a product over the finite group $\mathbb{T}^\vee$:

$$\mathrm{BP}_{\widehat{\mathbf{G}},0} = \prod_\lambda \mathrm{BP}^\lambda_{\widehat{\mathbf{G}},0} = \prod_\lambda \mathrm{QCoh}(V_{\widehat{\mathbf{T}},0}).$$

7.3.5. Notation. Let $\pi : V^{(1)}_{\widehat{\mathbf{T}},0} \to V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ be the canonical projection.

7.3.6. Definition. *The* mod p parametrization functor *is the functor*

$$P := S \circ \mathrm{Res}^{\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}}_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})} = \pi_* \circ B :$$

$$\begin{array}{c} \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \\ \big\downarrow \\ \mathrm{SP}_{\widehat{\mathbf{G}},0}\,. \end{array}$$

7.3.7. The functor P decomposes as a product over the finite set $\mathbb{T}^\vee/W_0$:

$$P = \prod_\gamma P^\gamma : \prod_\gamma \mathrm{Mod}(\mathcal{H}^\gamma_{\overline{\mathbb{F}}_q}) \xrightarrow{\ \sim\ } \prod_\gamma \mathrm{SP}^\gamma_{\widehat{\mathbf{G}},0}\,.$$

In the case of the trivial orbit $\gamma := \{1\}$, $P^{\{1\}}$ factors as

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{H}^{\{1\}}_{\overline{\mathbb{F}}_q}) & & \\ {\scriptstyle \mathrm{Res}^{\mathcal{H}^{\{1\}}_{\overline{\mathbb{F}}_q}}_{\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}}}\big\downarrow & \searrow^{P^{\{1\}}} & \\ \mathrm{Mod}(\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}) & \xrightarrow[\sim]{S^{\{1\}}} & \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}},0}\,. \end{array}$$

7.4 The mod p antispherical module

7.4.1. Definition. *We call*

$$\mathscr{A}^{(1)}_{\overline{\mathbb{F}}_q} : \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q} \longrightarrow \mathrm{End}_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})}(\mathcal{A}^{(1)}_{\overline{\mathbb{F}}_q})$$

the mod p antispherical representation*, and the corresponding left $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}$ the* mod p antispherical module.

7.4.2. Proposition.

1. *The mod p antispherical representation is faithful.*

2. *The mod p Bernstein parameter of the antispherical module is the structural sheaf:*

$$B(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}) = \mathcal{O}_{V^{(1)}_{\widehat{\mathbf{T}},0}}.$$

3. *The mod p Satake parameter of the antispherical module is the $R_{\overline{\mathbb{F}}_q}(V^{(1)}_{\widehat{\mathbf{T}},0})^{\mathrm{rel},W_0}$-module of the relevant induced $V^{(1)}_{\widehat{\mathbf{G}},0}$-equivariant $K_{\overline{\mathbb{F}}_q}$-theory of the flag variety of $V^{(1)}_{\widehat{\mathbf{G}},0}$:*

$$c^{\mathrm{rel}}_{V^{(1)}_{\widehat{\mathbf{G}},0},\overline{\mathbb{F}}_q} : S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}) \xrightarrow{\sim} K^{V^{(1)}_{\widehat{\mathbf{G}},0}}_{\mathcal{I}nd,\overline{\mathbb{F}}_q}(V^{(1)}_{\widehat{\mathbf{G}},0}/V^{(1)}_{\widehat{\mathbf{B}},0})^{\mathrm{rel}}.$$

Proof. Part 1. follows from 3.3.3 and 4.3.5, part 2. from the property *(i)* in 3.3.1 and 4.3.1, and part 3. from the characteristic isomorphism in 7.2.3. □

7.4.3. Corollary. *The diagram*

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) & \xrightarrow{\ B\ } & \mathrm{BP}_{\widehat{\mathbf{G}},0} \\ {\scriptstyle \mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q}\otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})}\bullet}\ \uparrow & & \uparrow\ {\scriptstyle \pi^*} \\ \mathrm{Mod}(Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})) & \xrightarrow[\sim]{\ S\ } & \mathrm{SP}_{\widehat{\mathbf{G}},0} \end{array}$$

is commutative.

7.4.4. Definition. *The* mod p antispherical functor *is the functor*

$$\mathrm{ASph} := (\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})} \bullet) \circ S^{-1} :$$

$$\mathrm{SP}_{\widehat{\mathbf{G}},0} \longrightarrow \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}).$$

7.4.5. Corollary. *The diagram*

$$\begin{array}{ccccc} & & & & \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \\ & & \overset{\mathrm{ASph}}{\nearrow} & & \downarrow\ {\scriptstyle P} \\ \mathrm{SP}_{\widehat{\mathbf{G}},0} & \xrightarrow[\pi^*]{} & \mathrm{BP}_{\widehat{\mathbf{G}},0} & \xrightarrow[\pi_*]{} & \mathrm{SP}_{\widehat{\mathbf{G}},0} \end{array}$$

is commutative.

7.4.6. The antispherical functor ASph decomposes as a product of functors ASph^{γ} for $\gamma \in \mathbb{T}^{\vee}/W_0$, and accordingly the previous diagram decomposes over $\mathbb{T}^{\vee}/W_0$. In particular for $\gamma = \{1\}$ we have the commutative diagram

$$\begin{array}{ccccccc} & & & & \mathrm{Mod}(\mathcal{H}^{\{1\}}_{\overline{\mathbb{F}}_q}) & & \\ & & \overset{\mathrm{ASph}^{\{1\}}}{\nearrow} & & \downarrow\ {\scriptstyle P^{\{1\}}} & \overset{\mathrm{Res}^{\mathcal{H}^{\{1\}}_{\overline{\mathbb{F}}_q}}_{\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}}}{\searrow} & \\ \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}},0} & \xrightarrow[\pi^*]{} & \mathrm{BP}^{\{1\}}_{\widehat{\mathbf{G}},0} & \xrightarrow[\pi_*]{} & \mathrm{SP}^{\{1\}}_{\widehat{\mathbf{G}},0} & \xleftarrow[S^{\{1\}}]{\sim} & \mathrm{Mod}(\mathcal{H}^{\mathrm{sph}}_{\overline{\mathbb{F}}_q}). \end{array}$$

7.4.7. Now, identifying the k-points of the k-scheme $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ with the skyscraper sheaves on it, the antispherical functor ASph induces a map

$$\mathrm{ASph} : \big(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\text{-modules}\}.$$

Considering the decomposition of $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ into its connected components, cf. 6.2.14,

$$V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 = \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)} V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0 \simeq \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{reg}}} V_{\widehat{\mathbf{T}},0} \quad \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{non-reg}}} V_{\widehat{\mathbf{T}},0}/W_0,$$

the antispherical map decomposes as a disjoint union of maps

$$\mathrm{ASph}^\gamma : \big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \simeq V_{\widehat{\mathbf{T}},0}(k) \longrightarrow \{\text{left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\text{-modules}\} \qquad \text{for } \gamma \text{ regular},$$

$$\mathrm{ASph}^\gamma : \big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \simeq (V_{\widehat{\mathbf{T}},0}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\text{-modules}\} \qquad \text{for } \gamma \text{ non-regular}.$$

7.4.8. In the regular case, we make the standard choice of coordinates

$$V_{\widehat{\mathbf{T}},0}(k) = \Big(\{(x,0) \mid x\in k\} \coprod_{(0,0)} \{(0,y) \mid y \in k\}\Big) \times \{z_2 \in k^\times\}$$

and we identify $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}$ with $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$ using 3.1.4. A point $v \in V_{\widehat{\mathbf{T}},0}(k)$ corresponds to a character

$$\theta_v : Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q}) \simeq \overline{\mathbb{F}}_q[X,Y,z_2^{\pm 1}]/(XY) \longrightarrow \overline{\mathbb{F}}_q,$$

and then $\mathrm{ASph}^\gamma(v)$ identifies with the central reduction

$$\mathcal{A}_{2,\theta_v} := \mathcal{A}_{2,\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{2,\overline{\mathbb{F}}_q}),\theta_v} \overline{\mathbb{F}}_q$$

of the mod p regular antispherical representation $\mathscr{A}_{2,\overline{\mathbb{F}}_q}$ specializing 3.3.1. The latter being an isomorphism by 3.3.3, so is

$$\mathscr{A}_{2,\theta_v} : \mathcal{H}_{2,\theta_v} \xrightarrow{\ \sim\ } \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{2,\theta_v}).$$

Consequently $\mathcal{H}_{2,\theta_v}$ is a matrix algebra and $\mathcal{A}_{2,\theta_v}$ is the unique simple finite dimensional left $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module with central character θ_v, up to isomorphism. It is the *standard module with character* θ_v, with *standard basis* $\{\varepsilon_1,\varepsilon_2\}$ (in particular its $\overline{\mathbb{F}}_q$-dimension is 2). Conversely, any simple finite dimensional $\mathcal{H}_{2,\overline{\mathbb{F}}_q}$-module has a central character, by Schur's lemma.

Following [V04], a central character θ is called *supersingular* if $\theta(X+Y)=0$, and the standard module with character θ is called *supersingular* if θ is. Since $XY=0$, one has $\theta(X+Y)=0$ if and only if $\theta(X)=\theta(Y)=0$.

7.4.9. Theorem. *Let $\gamma\in\mathbb{T}^\vee/W_0$ regular. Then the antispherical map induces a bijection*

$$\mathrm{ASph}^\gamma : \big(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0\big)(k) \xrightarrow{\ \sim\ } \{\textit{simple finite dimensional left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\textit{-modules}\}/\sim.$$

The singular locus of the parametrizing k-scheme $V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0$ is given by $(0,0)\times\mathbb{G}_m \subset V_{\widehat{\mathbf{T}},0}$ in the standard coordinates, and its k-points correspond to the supersingular Hecke modules through the correspondence ASph^γ.

7.4.10. In the non-regular case, we make the Steinberg choice of coordinates

$$(V_{\widehat{\mathbf{T}},0}/W_0)(k) = \{z_1 \in k\} \times \{z_2 \in k^\times\}$$

and we identify $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}$ with $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$ using 4.1.4. A point $v \in (V_{\widehat{\mathbf{T}},0}/W_0)(k)$ corresponds to a character

$$\theta_v : Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q}) \simeq \overline{\mathbb{F}}_q[z_1, z_2^{\pm 1}] \longrightarrow \overline{\mathbb{F}}_q,$$

and then $\mathrm{ASph}^{\gamma}(v)$ identifies with the central reduction

$$\mathcal{A}_{1,\theta_v} := \mathcal{A}_{1,\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q}),\theta_v} \overline{\mathbb{F}}_q$$

of the mod p non-regular antispherical representation $\mathscr{A}_{1,\overline{\mathbb{F}}_q}$ specializing 4.3.1.

Now recall from [V04, 1.4] the classification of the simple finite dimensional $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$-modules: they are the characters and the simple standard modules. The characters

$$\mathcal{H}_{1,\overline{\mathbb{F}}_q} = \overline{\mathbb{F}}_q[S, U^{\pm 1}] \longrightarrow \overline{\mathbb{F}}_q^{\times}$$

are parametrized by the set $\{0, -1\} \times \overline{\mathbb{F}}_q^{\times}$ via evaluation on the elements S and U. On the other hand, given $v = (z_1, z_2) \in k \times k^{\times} = \overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q^{\times}$, a *standard module with character θ_v* over $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$ is defined to be a module of type

$$M_2(z_1, z_2) := \overline{\mathbb{F}}_q m \oplus \overline{\mathbb{F}}_q Um, \qquad Sm = -m, \qquad SUm = z_1 m, \qquad U^2 m = z_2 m$$

(in particular its $\overline{\mathbb{F}}_q$-dimension is 2). The center $Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q})$ acts on $M_2(z_1, z_2)$ by the character θ_v. In particular such a module is uniquely determined by its central character. It is simple if and only if $z_2 \neq z_1^2$. It is called *supersingular* if $z_1 = 0$.

7.4.11. Lemma. *Set*

$$\mathscr{A}_{1,\theta_v} := \mathscr{A}_{1,\overline{\mathbb{F}}_q} \otimes_{Z(\mathcal{H}_{1,\overline{\mathbb{F}}_q}),\theta_v} \overline{\mathbb{F}}_q : \mathcal{H}_{1,\theta_v} \longrightarrow \mathrm{End}_{\overline{\mathbb{F}}_q}(\mathcal{A}_{1,\theta_v}).$$

- *Assume $z_2 \neq z_1^2$. Then $\mathscr{A}_{1,\theta_v}$ is an isomorphism, and the $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$-module $\mathcal{A}_{1,\theta_v}$ is isomorphic to the simple standard module $M_2(z_1, z_2)$.*
- *Assume $z_2 = z_1^2$. Then $\mathscr{A}_{1,\theta_v}$ has a 1-dimensional kernel, and the $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$-module $\mathcal{A}_{1,\theta_v}$ is a non-split extension of the character $(0, z_1)$ by the character $(-1, -z_1)$.*

Proof. The proof of Proposition 4.3.3 shows that $\mathcal{H}_{1,\theta_v}$ has an $\overline{\mathbb{F}}_q$-basis given by the elements $1, S, U, SU$, and that their images

$$1,\ \mathscr{A}_{1,\theta_v}(S),\ \mathscr{A}_{1,\theta_v}(U),\ \mathscr{A}_{1,\theta_v}(S)\mathscr{A}_{1,\theta_v}(U)$$

by $\mathscr{A}_{1,\theta_v}$ are linearly independent over $\overline{\mathbb{F}}_q$ if and only if $z_1^2 - z_2 \neq 0$.

If $z_2 \neq z_1^2$, then $\mathscr{A}_{1,\theta_v}$ is injective, and hence bijective since $\dim_{\overline{\mathbb{F}}_q} \mathcal{A}_{1,\theta_v} = 2$ from 4.2.1. Moreover $S \cdot Y = -Y$ and $U \cdot Y = (z_1^2 - z_2) - z_1 Y$ and so $SUY = S((z_1^2 - z_2) - z_1 Y) = S(-z_1 Y) = z_1 Y$, so that

$$\mathcal{A}_{1,\theta_v} = \overline{\mathbb{F}}_q Y \oplus \overline{\mathbb{F}}_q U \cdot Y = M_2(z_1, z_2).$$

If $z_2 = z_1^2$, then the proof of Proposition 4.3.3 shows that $\mathscr{A}_{1,\theta_v}$ has a 1-dimensional kernel which is the $\overline{\mathbb{F}}_q$-line generated by $-z_1(1 + S) + U + SU$. Moreover $\overline{\mathbb{F}}_q Y \subset \mathcal{A}_{1,\theta_v}$ realizes the character $(-1, -z_1)$ of $\mathcal{H}_{1,\overline{\mathbb{F}}_q}$, and $\mathcal{A}_{1,\theta_v}/\overline{\mathbb{F}}_q Y \simeq \overline{\mathbb{F}}_q 1$ realizes the character $(0, z_1)$. Finally the 0-eigenspace of S in $\mathcal{A}_{1,\theta_v}$ is $\overline{\mathbb{F}}_q 1$, which is not U-stable, so that the character $(0, z_1)$ does not lift in $\mathcal{A}_{1,\theta_v}$. $\square$

7.4.12. Geometrically, the function $z_2 - z_1^2$ on $V_{\widehat{\mathbf{T}},0}/W_0$ defines a family of parabolas

$$\begin{array}{c} V_{\widehat{\mathbf{T}},0}/W_0, \\ \Big\downarrow {\scriptstyle z_2 - z_1^2} \\ \mathbb{A}^1 \end{array}$$

whose parameter is 4Δ, where Δ is the discriminant of the parabola. Then the locus of $V_{\widehat{\mathbf{T}},0}/W_0$ where $z_2 = z_1^2$ corresponds to the parabola at 0, having vanishing discriminant (at least if $p \neq 2$).

7.4.13. Definition. *We will say that a pair of characters of* $\mathcal{H}_{1,\overline{\mathbb{F}}_q} = \overline{\mathbb{F}}_q[S, U^{\pm 1}] \to \overline{\mathbb{F}}_q^{\times}$ *is* antispherical *if there exists* $z_1 \in \overline{\mathbb{F}}_q^{\times}$ *such that, after evaluating on* (S, U)*, it is equal to*

$$\{(0, z_1), (-1, -z_1)\}.$$

7.4.14. Note that the set of characters $\mathcal{H}_{1,\overline{\mathbb{F}}_q} \to \overline{\mathbb{F}}_q^{\times}$ is the disjoint union of the antispherical pairs, by the very definition.

7.4.15. Theorem. *Let* $\gamma \in \mathbb{T}^{\vee}/W_0$ *non-regular. Consider the decomposition*

$$V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0 = D(2)_{\gamma} \cup D(1)_{\gamma}$$

where $D(1)_{\gamma}$ *is the closed subscheme defined by the parabola* $z_2 = z_1^2$ *in the Steinberg coordinates* z_1, z_2 *and* $D(2)_{\gamma}$ *is the open complement. Then the antispherical map induces bijections*

$$\mathrm{ASph}^{\gamma}(2) : D(2)_{\gamma}(k) \xrightarrow{\sim} \{\textit{simple 2-dimensional left } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\textit{-modules}\}/\sim$$

$$\mathrm{ASph}^{\gamma}(1) : D(1)_{\gamma}(k) \xrightarrow{\sim} \{\textit{antispherical pairs of characters of } \mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_q}\}/\sim .$$

The branch locus of the covering

$$V_{\widehat{\mathbf{T}},0} \longrightarrow V_{\widehat{\mathbf{T}},0}/W_0 \simeq V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0$$

is contained in $D(2)_{\gamma}$*, with equation* $z_1 = 0$ *in Steinberg coordinates, and its* k*-points correspond to the supersingular Hecke modules through the correspondence* $\mathrm{ASph}^{\gamma}(2)$*.*

7.4.16. Remark. The matrices of S, U and $S_0 = USU^{-1}$ in the $\overline{\mathbb{F}}_q$-basis $\{1, Y\}$ of the supersingular module $\mathcal{A}_{1,\theta_v} \cong M_2(0, z_2)$ are

$$S = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & -z_2 \\ -1 & 0 \end{pmatrix}, \quad S_0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The two characters of the *finite subalgebra* $\overline{\mathbb{F}}_q[S]$ corresponding to $S \mapsto 0$ and $S \mapsto -1$ are realized by 1 and Y. From the matrix of S_0, we see in fact that the whole *affine subalgebra* $\overline{\mathbb{F}}_q[S_0, S]$ acts on 1 and Y via the two *supersingular affine characters*, which by definition are the characters different from the trivial character $(S_0, S) \mapsto (0, 0)$ and the sign character $(S_0, S) \mapsto (-1, -1)$.

7.4.17. Finally, let v be any k-point of the parametrizing space $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. As a particular case of 7.4.5, the Bernstein parameter of the antispherical module $\mathrm{ASph}(v)$ is the structure sheaf of the fiber of the quotient map π at v, and its Satake parameter is the underlying k-vector space:

$$B(\mathrm{ASph}(v)) = \mathcal{O}_{\pi^{-1}(v)} \quad \text{and} \quad S(\mathrm{ASph}(v)) = \pi_* \mathcal{O}_{\pi^{-1}(v)}.$$

8 The theory at q $= q = 0$: Semisimple Langlands correspondence

We keep the notation introduced in the preceding section. In particular, $k = \overline{\mathbb{F}}_q$.

8.1 Mod p Satake parameters with fixed central character

8.1.1. Let $\omega : \mathbb{F}_q^{\times} \to k^{\times}$ be induced by the inclusion $\mathbb{F}_q \subset k$. Then $(\mathbb{F}_q^{\times})^{\vee} = \langle \omega \rangle$ is a cyclic group of order $q - 1$. An element ω^r defines a non-regular character of $\mathbb{T}$:

$$\omega^r(t_1, t_2) := \omega^r(t_1)\omega^r(t_2)$$

for all $(t_1, t_2) \in \mathbb{T} = \mathbb{F}_q^{\times} \times \mathbb{F}_q^{\times}$. Composing with multiplication in $\mathbb{T}^{\vee}$, we get an action of $(\mathbb{F}_q^{\times})^{\vee}$ on $\mathbb{T}^{\vee}$, which factors on the quotient set $\mathbb{T}^{\vee}/W_0$:

$$\mathbb{T}^{\vee}/W_0 \times (\mathbb{F}_q^{\times})^{\vee} \longrightarrow \mathbb{T}^{\vee}/W_0, \ (\gamma, \omega^r) \mapsto \gamma\omega^r.$$

If $\gamma \in \mathbb{T}^{\vee}/W_0$ is regular (non-regular), then $\gamma\omega^r$ is regular (non-regular).

8.1.2. Restricting characters of $\mathbb{T}$ to the subgroup $\mathbb{F}_q^\times \simeq \{\mathrm{diag}(a,a) : a \in \mathbb{F}_q^\times\}$ induces a homomorphism $\mathbb{T}^\vee \to (\mathbb{F}_q^\times)^\vee$ which factors into a restriction map

$$\mathbb{T}^\vee/W_0 \to (\mathbb{F}_q^\times)^\vee, \ \gamma \mapsto \gamma|_{\mathbb{F}_q^\times}.$$

The relation to the $(\mathbb{F}_q^\times)^\vee$-action on the source $\mathbb{T}^\vee/W_0$ is given by the formula

$$(\gamma\omega^r)|_{\mathbb{F}_q^\times} = \gamma|_{\mathbb{F}_q^\times}\ \omega^{2r}.$$

We describe the fibers of the restriction map $\gamma \mapsto \gamma|_{\mathbb{F}_q^\times}$.

Let $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(\omega^{2r})$ be the fibre at a square element ω^{2r}. By the above formula, the action of ω^{-r} on $\mathbb{T}^\vee/W_0$ induces a bijection with the fibre $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(1)$. The fibre

$$(\cdot)|_{\mathbb{F}_q^\times}^{-1}(1) = \{1\otimes 1\} \coprod \{\omega\otimes\omega^{-1}, \omega^2\otimes\omega^{-2}, ..., \omega^{\frac{q-3}{2}}\otimes\omega^{-\frac{q-3}{2}}\} \coprod \{\omega^{\frac{q-1}{2}}\otimes\omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{q+1}{2}$ and, in the above list, we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. The $\frac{q-3}{2}$ elements in the middle of this list, i.e. the W_0-orbits represented by the characters $\omega^r\otimes\omega^{-r}$ for $r = 1, ..., \frac{q-3}{2}$, are all regular W_0-orbits. The two orbits at the two ends of the list are non-regular orbits (note that $\frac{q-1}{2} \equiv -\frac{q-1}{2} \mod (q-1)$). Since the action of ω^{-r} preserves regular (non-regular) orbits, any fibre at a square element (there are $\frac{q-1}{2}$ such fibres) has the same structure.

On the other hand, let $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(\omega^{2r-1})$ be the fibre at a non-square element ω^{2r-1}. The action of ω^{-r} induces a bijection with the fibre $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(\omega^{-1})$. The fibre

$$(\cdot)|_{\mathbb{F}_q^\times}^{-1}(\omega^{-1}) = \{1\otimes\omega^{-1}, \omega\otimes\omega^{-2}, ..., \omega^{\frac{q-1}{2}-1}\otimes\omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{q-1}{2}$ and we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. All elements of the fibre are regular W_0-orbits. Since the action of ω^{-r} preserves regular (non-regular) orbits, any fibre at a non-square element (there are $\frac{q-1}{2}$ such fibres) has the same structure.

Note that $\frac{q-1}{2}(\frac{q+1}{2} + \frac{q-1}{2}) = \frac{q^2-q}{2}$ is the cardinality of the set $\mathbb{T}^\vee/W_0$.

8.1.3. Recall the commutative k-semigroup scheme

$$V^{(1)}_{\widehat{\mathbf{T}},0} = \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}},0} = \mathbb{T}^\vee \times \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m$$

together with its W_0-action, cf. 6.2.14: the natural action of W_0 on the factors $\mathbb{T}^\vee$ and $\mathrm{SingDiag}_{2\times 2}$ and the trivial one on $\mathbb{G}_m$. There is a commuting action of the k-group scheme

$$\mathscr{Z}^\vee := (\mathbb{F}_q^\times)^\vee \times \mathbb{G}_m$$

on $V^{(1)}_{\widehat{\mathbf{T}},0}$: the (constant finite diagonalizable) group $(\mathbb{F}_q^\times)^\vee$ acts only on the factor $\mathbb{T}^\vee$ and in the way described in 8.1.1; an element $z_0 \in \mathbb{G}_m$ acts trivially on $\mathbb{T}^\vee$, by multiplication with the diagonal matrix $\mathrm{diag}(z_0, z_0)$ on $\mathrm{SingDiag}_{2\times 2}$ and by multiplication with the square z_0^2 on $\mathbb{G}_m$. Therefore the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ inherits a $\mathscr{Z}^\vee$-action. Now, according to 7.4.7, one has the decomposition

$$V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 = \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\mathrm{reg}}} V_{\widehat{\mathbf{T}},0} \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{non-reg}}} V_{\widehat{\mathbf{T}},0}/W_0.$$

Then the $(\mathbb{F}_q^\times)^\vee$-action is by permutations on the index set $\mathbb{T}^\vee/W_0$, i.e. on the set of connected components of $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$; as observed above, it preserves the subsets of regular and non-regular components. The $\mathbb{G}_m$-action on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ preserves each connected component.

8.1.4. Recall from 7.4.7 the antispherical map

$$\mathrm{ASph} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\text{-modules}\}/\sim .$$

The $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules in the image of this map are of length 1 or 2, cf. 7.4.9 and 7.4.15. We write $\mathrm{ASph}(v)^{\mathrm{ss}}$ for the semisimplification of the module $\mathrm{ASph}(v)$, for $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k)$.

Let $(\omega^r, z_0) \in \mathscr{Z}^\vee(k)$. Recall that the standard or irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules may be 'twisted by the character (ω^r, z_0)' : in the regular case, the actions of X, Y, U^2 get multiplied by z_0, z_0, z_0^2 respectively and the component γ gets multiplied by ω^r, cf. [V04, 2.4]; in the non-regular case, the action of U gets multiplied by z_0, the action of S remains unchanged and the component γ gets multiplied by ω^r, cf. [V04, 1.6]. This gives an action of the group of k-points of $\mathscr{Z}^\vee$ on the standard or irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules. It extends to an action on semisimple $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules.

8.1.5. Lemma. *The map* $\mathrm{ASph}(-)^{\mathrm{ss}}$ *is* $\mathscr{Z}^\vee(k)$*-equivariant.*

Proof. Let $(\omega^r, z_0) \in \mathscr{Z}^\vee(k)$. Let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$. Suppose that γ is regular, choose an ordering $\gamma = (\chi, \chi^s)$ on the set γ and standard coordinates. Then $\mathrm{ASph}(v) = \mathrm{ASph}^\gamma(v)$ is a simple two-dimensional standard $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. 7.4.9, i.e. of the form $M(x, y, z_2, \chi)$ [V04, 3.2]. Then

$$\mathrm{ASph}(v.(\omega^r, z_0)) \simeq M(z_0 x, z_0 y, z_0^2 z_2, \chi.\omega^r) \simeq \mathrm{ASph}(v).(\omega^r, z_0).$$

Suppose that $\gamma = \{\chi\}$ is non-regular and choose Steinberg coordinates. (a) If $v \in D(2)_\gamma(k)$, then $\mathrm{ASph}(v) = \mathrm{ASph}^\gamma(2)(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. 7.4.15, i.e. of the form $M(z_1, z_2, \chi)$ [V04, 3.2]. Then

$$\mathrm{ASph}(v.(\omega^r, z_0)) \simeq M(z_0 z_1, z_0^2 z_2, \chi.\omega^r) \simeq \mathrm{ASph}(v).(\omega^r, z_0).$$

(b) If $v \in D(1)_\gamma(k)$, then the semisimplified module $\mathrm{ASph}(v)^{\mathrm{ss}}$ is the direct sum of the two characters in the antispherical pair $\mathrm{ASph}^\gamma(1)(v) = \{(0, z_1), (-1, -z_1)\}$ where $z_2 = z_1^2$. Similarly $\mathrm{ASph}(v.(\omega^r, z_0))^{\mathrm{ss}}$ is the direct sum of the characters $\{(0, z_0 z_1), (-1, -z_0 z_1)\}$ in the component $\gamma.\omega^r$, and hence is isomorphic to $\mathrm{ASph}(v)^{\mathrm{ss}}.(\omega^r, z_0)$. □

8.1.6. The two canonical projections from $V^{(1)}_{\widehat{\mathbf{T}},0}$ to $\mathbb{T}^\vee$ and $\mathbb{G}_m$ respectively induce two projection morphisms

$$\begin{array}{ccccc} & & V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 & & \\ & \swarrow^{\mathrm{pr}_{\mathbb{T}^\vee/W_0}} & & \searrow^{\mathrm{pr}_{\mathbb{G}_m}} & \\ \mathbb{T}^\vee/W_0 & & & & \mathbb{G}_m. \end{array}$$

Then we may compose the map $\mathrm{pr}_{\mathbb{T}^\vee/W_0}$ with the restriction map $(\cdot)|_{\mathbb{F}_q^\times} : \mathbb{T}^\vee/W_0 \to (\mathbb{F}_q^\times)^\vee$, set

$$\theta := \big((\cdot)|_{\mathbb{F}_q^\times} \circ \mathrm{pr}_{\mathbb{T}^\vee/W_0}\big) \times \mathrm{pr}_{\mathbb{G}_m}$$

and view $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ as fibered over the space $\mathscr{Z}^\vee$:

$$\begin{array}{c} V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \\ \big\downarrow \theta \\ \mathscr{Z}^\vee. \end{array}$$

The relation to the $\mathcal{Z}^\vee$-action on the source $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ is given by the formula

$$\theta(x.(\omega^r, z_0)) = \theta(x)(\omega^{2r}, z_0^2) = \theta(x)(\omega^r, z_0)^2$$

for $x \in V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ and $(\omega^r, z_0) \in \mathcal{Z}^\vee$. This formula follows from the formula in 8.1.2 and the definition of the $\mathbb{G}_m$-action in 8.1.3.

8.1.7. Definition. *Let* $\zeta \in \mathcal{Z}^\vee$. *The* space of mod p Satake parameters with central character ζ *is the* k*-scheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta := \theta^{-1}(\zeta).$$

8.1.8. Let $\zeta = (\zeta|_{\mathbb{F}_q^\times}, z_2) \in \mathcal{Z}^\vee(k) = (\mathbb{F}_q^\times)^\vee \times k^\times$. Denote by $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{z_2}$ the fibre of $\mathrm{pr}_{\mathbb{G}_m}$ at $z_2 \in k^\times$. Then by 7.4.7 we have

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta = \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\mathrm{reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,z_2} \quad \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{non-reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,z_2}/W_0.$$

Recall that the choice of standard coordinates x, y identifies

$$V_{\widehat{\mathbf{T}},0,z_2} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1$$

with two affine lines over k, intersecting at the origin, cf. 7.4.8. On the other hand, the choice of the Steinberg coordinate z_1 identifies

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0 \simeq \mathbb{A}^1$$

with a single affine line over k, cf. 7.4.10.

8.1.9. Lemma. *Let* $\zeta, \eta \in \mathcal{Z}^\vee$. *The action of* η *on* $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ *induces an isomorphism of* k*-schemes* $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \simeq (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta\eta^2}$.

Proof. Follows from the last formula in 8.1.6. □

8.2 Mod p Langlands parameters with fixed determinant for $F = \mathbb{Q}_p$

8.2.1. Notation. In this section, we let $F = \mathbb{Q}_p$ with $p \geq 5$. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ and let $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the absolute Galois group. We normalize local class field theory $\mathbb{Q}_p^\times \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}$ by sending p to a geometric Frobenius. In this way, we identify the k-valued smooth characters of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and of $\mathbb{Q}_p^\times$. Finally, $\omega : \mathbb{Q}_p^\times \to k^\times$ denotes the extension of the character $\omega : \mathbb{F}_p^\times \to k^\times$ to $\mathbb{Q}_p^\times$ satisfying $\omega(p) = 1$, and $\mathrm{unr}(x) : \mathbb{Q}_p^\times \to k^\times$ denotes the character trivial on $\mathbb{F}_p^\times$ and sending p to x.

8.2.2. Let $\zeta : \mathbb{Q}_p^\times \to k^\times$ be a character. Recall from [Em19] that the *Emerton-Gee moduli curve with character* ζ is a certain projective curve X_ζ over k whose points parametrize (isomorphism classes of) two-dimensional semisimple continuous Galois representations over k with determinant $\omega\zeta$:

$$X_\zeta(k) \cong \{\text{semisimple continuous } \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k) \ \text{ with } \det \rho = \omega\zeta\}/\sim.$$

It is expected to serve as a moduli space for the stack of étale (φ, Γ)-modules $(\mathcal{X}_2^{\det = \omega\zeta})_{\mathrm{red}}$ appearing in [EG19] (see also [CEGS19]) for $K = \mathbb{Q}_p$ (in their notation).

The curve X_ζ is a chain of projective lines over k of length $\frac{p\pm1}{2}$, whose irreducible components intersect at ordinary double points. The sign ±1 is equal to $-\zeta(-1)$. We refer to ζ in the case $-\zeta(-1) = -1$ resp. $-\zeta(-1) = +1$ as an *even character* resp. *odd character*. There is a finite set of closed points $X_\zeta^{\mathrm{irred}} \subset X_\zeta$ which correspond to the classes of irreducible representations. Its open complement $X_\zeta^{\mathrm{red}} = X_\zeta \setminus X_\zeta^{\mathrm{irred}}$ parametrizes the reducible representations (i.e. direct sums of characters). Let $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to k^\times$ be a character. Since $\det(\rho \otimes \eta) = (\det \rho)\eta^2$, twisting representations with η induces an isomorphism

$$(\cdot) \otimes \eta : X_\zeta \xrightarrow{\sim} X_{\zeta\eta^2}.$$

Hence one is reduced to consider only two 'basic' cases: the even case where $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = 1$ and the odd case where $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$. Indeed, if $\zeta|_{\mathbb{F}_p^\times} = \omega^r$ for some even r, then choosing η with $\eta(p)^2 = \zeta(p)^{-1}$ and $\eta|_{\mathbb{F}_p^\times} = \omega^{-\frac{r}{2}}$, one finds that $(\zeta\eta^2)(p) = 1$ and $(\zeta\eta^2)|_{\mathbb{F}_p^\times} = 1$; if $\zeta|_{\mathbb{F}_p^\times} = \omega^r$ for some odd r, then choosing η with $\eta(p)^2 = \zeta(p)^{-1}$ and $\eta|_{\mathbb{F}_p^\times} = \omega^{-\frac{r+1}{2}}$, one finds that $(\zeta\eta^2)(p) = 1$ and $(\zeta\eta^2)|_{\mathbb{F}_p^\times} = \omega^{-1}$.

8.2.3. We make explicit some structure elements of X_ζ in the even case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = 1$. Every irreducible component of X_ζ is isomorphic to $\mathbb{P}^1$ and there are $\frac{p-1}{2}$ components. They are labelled by pairs of Serre weights of the following form:

$$\begin{array}{ccc}
\mathrm{Sym}^0 & | & \mathrm{Sym}^{p-3} \otimes \det \\
\mathrm{Sym}^2 \otimes \det^{-1} & | & \mathrm{Sym}^{p-5} \otimes \det^2 \\
\mathrm{Sym}^4 \otimes \det^{-2} & | & \mathrm{Sym}^{p-7} \otimes \det^3 \\
\vdots & \vdots & \vdots \\
\mathrm{Sym}^{p-3} \otimes \det^{\frac{p+1}{2}} & | & \mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}}.
\end{array}$$

The component with label "$\mathrm{Sym}^0 \,|\, \mathrm{Sym}^{p-3} \otimes \det$" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^2 \otimes \det^{-1}, \mathrm{Sym}^{p-3} \otimes \det\}$. The component with label "$\mathrm{Sym}^2 \otimes \det^{-1} \,|\, \mathrm{Sym}^{p-5} \otimes \det^2$" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^4 \otimes \det^{-2}, \mathrm{Sym}^{p-5} \otimes \det^2\}$. Continuing in this way, one finds $\frac{p-3}{2}$ points of X_ζ^{irred}, which correspond to the $\frac{p-3}{2}$ double points of the chain X_ζ. There are two more points in X_ζ^{irred}: they are smooth points, each one lies on one of the two 'exterior' components and corresponds there to the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}\}$ and $\{\mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}}, \mathrm{Sym}^{p-1} \otimes \det^{\frac{p-1}{2}}\}$ respectively. So X_ζ^{irred} has cardinality $\frac{p+1}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate z_1 on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $z_1 = \pm 1$ *the four exceptional points* of X_ζ.

8.2.4. We make explicit some structure elements of X_ζ in the odd case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$. Every irreducible component of X_ζ is isomorphic to $\mathbb{P}^1$ and there are $\frac{p+1}{2}$ components. They are labelled by pairs of Serre weights of the following form:

$$\begin{array}{ccc}
\mathrm{Sym}^{p-2} & | & \text{"}\mathrm{Sym}^{-1}\text{"} \\
\mathrm{Sym}^{p-4} \otimes \det & | & \mathrm{Sym}^1 \otimes \det^{-1} \\
\mathrm{Sym}^{p-6} \otimes \det^2 & | & \mathrm{Sym}^3 \otimes \det^{-2} \\
\vdots & \vdots & \vdots \\
\mathrm{Sym}^1 \otimes \det^{\frac{p-3}{2}} & | & \mathrm{Sym}^{p-4} \otimes \det^{\frac{p+1}{2}} \\
\text{"}\mathrm{Sym}^{-1} \otimes \det^{\frac{p-1}{2}}\text{"} & | & \mathrm{Sym}^{p-2} \otimes \det^{\frac{p-1}{2}}.
\end{array}$$

The component with label "$\mathrm{Sym}^{p-2} \,|\,$ "Sym^{-1}"" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^1 \otimes \det^{-1}, \mathrm{Sym}^{p-2}\}$. The component with label "$\mathrm{Sym}^{p-4} \otimes \det \,|\, \mathrm{Sym}^1 \otimes \det^{-1}$" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^3 \otimes \det^{-2}, \mathrm{Sym}^{p-4} \otimes \det\}$. Continuing in this way, one finds $\frac{p-1}{2}$ points of X_ζ^{irred}, which correspond to the $\frac{p-1}{2}$ double points of the chain X_ζ. There are no more points in X_ζ^{irred} and X_ζ^{irred} has cardinality $\frac{p-1}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate t on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $t = \pm 2$ *the four exceptional points* of X_ζ. [2]

8.2.5. Definition. *The category of quasi-coherent modules on the Emerton-Gee moduli curve X_ζ will be called the* category of mod p Langlands parameters with determinant $\omega\zeta$, *and denoted by* $\mathrm{LP}_{\widehat{\mathbf{G}},0,\omega\zeta}$*:*

$$\mathrm{LP}_{\widehat{\mathbf{G}},0,\omega\zeta} := \mathrm{QCoh}(X_\zeta).$$

8.3 A semisimple mod p Langlands correspondence in families for $F = \mathbb{Q}_p$

8.3.1. Let us consider W to be a subgroup of G, by sending s to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and by identifying the group Λ with a subgroup of T via $(1,0) \mapsto \mathrm{diag}(\varpi^{-1}, 1)$ and $(0,1) \mapsto \mathrm{diag}(1, \varpi^{-1})$. We obtain for example (recall that $u = (1,0)s \in W$)

$$u = \begin{pmatrix} 0 & \varpi^{-1} \\ 1 & 0 \end{pmatrix}, \quad u^{-1} = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}, \quad us = \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad su = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^{-1} \end{pmatrix}.$$

Moreover, $u^2 = \mathrm{diag}(\varpi^{-1}, \varpi^{-1})$.[3] Since

$$\begin{pmatrix} 0 & \varpi^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} d & \varpi^{-1}c \\ \varpi b & a \end{pmatrix}$$

the element $u \in G$ normalizes the group $I^{(1)}$.

8.3.2. Let $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ be the category of smooth G-representations over k. Taking $I^{(1)}$-invariants yields a functor $\pi \mapsto \pi^{I^{(1)}}$ from $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ to the category $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})$. If $F = \mathbb{Q}_p$, it induces a bijection between the irreducible G-representations and the irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules, under which supersingular representations correspond to supersingular Hecke modules [V04].

For future reference, let us recall the $I^{(1)}$-invariants for some classes of representations. If $\pi = \mathrm{Ind}_B^G(\chi)$ is a principal series representation with $\chi = \chi_1 \otimes \chi_2$, then $\pi^{I^{(1)}}$ is a standard module in the component $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$.

In the regular case, one chooses the ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ on the set γ and standard coordinates x, y. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(0, \chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}})$$

In the non-regular case, one has

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(\chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(\chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}}).$$

These standard modules are irreducible if and only if $\chi \neq \chi^s$ [V04, 4.2/4.3].[4]

Let $F = \mathbb{Q}_p$. If $\pi = \pi(r, 0, \eta)$ is a standard supersingular representation with parameter $r = 0, ..., p-1$ and central character $\eta : \mathbb{Q}_p^\times \to k^\times$, then $\pi^{I^{(1)}}$ is a supersingular module in the component $\gamma = \{\chi, \chi^s\}$ represented by the character $\chi := (\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$, cf. [Br07, 5.1/5.3]. If π is the trivial representation $\mathbb{1}$ or the Steinberg representation St, then $\gamma = 1$ and $\pi^{I^{(1)}}$ is the character $(0,1)$ or $(-1,-1)$ respectively.

[2]The Galois representations living on the two exterior components in the odd case are *unramified* (up to twist), i.e. of type $\rho = \begin{pmatrix} \mathrm{unr}(x) & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$ and t equals the 'trace of Frobenius' $x + x^{-1}$. Hence $t = \pm 2$ if and only if $x = \pm 1$.

[3]Note that our element u equals the element u^{-1} in [Be11],[Br07] and [V04].

[4]Our formulas differ from [V04, 4.2/4.3] by $\chi(\cdot) \leftrightarrow \chi(\cdot)^{-1}$, since we are working with left modules; also compare with the explicit calculation with right convolution given in [V04, Appendix A.5].

8.3.3. Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$. Since $u \in G$ normalizes the group $I^{(1)}$, one has $I^{(1)}uI^{(1)} = uI^{(1)}$. It follows that the convolution action of the Hecke operator U (resp. U^2) on $\pi^{I^{(1)}}$ is therefore induced by the action of u (resp. u^2 on π). Similarly, the group $I^{(1)}$ is normalized by the Iwahori subgroup I and $I/I^{(1)} \simeq \mathbb{T}$. It follows that the convolution action of the operators $T_t, t \in \mathbb{T}$ on $\pi^{I^{(1)}}$ is the factorization of the $\mathbf{T}(o_F)$-action on π.

8.3.4. We identify $F^\times$ with the center $Z(G)$ via $a \mapsto \mathrm{diag}(a,a)$. A (smooth) character

$$\zeta : Z(G) = F^\times \longrightarrow k^\times$$

is determined by its value $\zeta(\varpi^{-1}) \in k^\times$ and its restriction $\zeta|_{o_F^\times}$. Since the latter is trivial on the subgroup $1 + \varpi o_F$, we may view it as a character of $\mathbb{F}_q^\times$; we will write $\zeta|_{\mathbb{F}_q^\times}$ for this restriction in the following. Thus the group of characters of $Z(G)$ gets identified with the group of k-points of the group scheme $\mathscr{Z}^\vee = (\mathbb{F}_q^\times)^\vee \times \mathbb{G}_m$:

$$Z(G)^\vee \xrightarrow{\sim} \mathscr{Z}^\vee(k),\ \zeta \mapsto (\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1})).$$

8.3.5. Lemma. *Suppose that $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$ has a central character $\zeta : Z(G) \to k^\times$. Then the Satake parameter $S(\pi^{I^{(1)}})$ of $\pi^{I^{(1)}} \in \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})$ has central character ζ, i.e. it is supported on the closed subscheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{(\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1}))} \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0.$$

Proof. If M is any $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-module, then

$$M = \bigoplus_{\gamma \in \mathbb{T}^\vee/W_0} \varepsilon_\gamma M = \bigoplus_{\gamma \in \mathbb{T}^\vee/W_0} \oplus_{\lambda\in\gamma} \varepsilon_\lambda M,$$

and $\mathbb{T} \subset \overline{\mathbb{F}}_q[\mathbb{T}] \subset \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ acts on $\varepsilon_\lambda M$ through the character $\lambda : \mathbb{T} \to \mathbb{F}_q^\times$. Now if $M = \pi^{I^{(1)}}$, then the $\mathbb{T}$-action on M is the factorization of the $\mathbf{T}(o_F)$-action on π, cf. 8.3.3. In particular, the restriction of the $\mathbb{T}$-action along the diagonal inclusion $\mathbb{F}_q^\times \subset \mathbb{T}$ is the factorization of the action of the central subgroup $o_F^\times \subset Z(G)$ on π, which is given by $\zeta|_{o_F^\times}$ by assumption. Hence

$$\varepsilon_\gamma M \neq 0 \quad \Longrightarrow \quad \forall \lambda \in \gamma,\ \lambda|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times} \text{ i.e. } \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}.$$

Moreover, the element $u^2 = \mathrm{diag}(\varpi^{-1}, \varpi^{-1}) \in Z(G)$ acts on π by multiplication by $\zeta(\varpi^{-1})$ by assumption. Therefore, by 8.3.3, the Hecke operator $z_2 := U^2 \in \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ acts on $\pi^{I^{(1)}}$ by multiplication by $\zeta(\varpi^{-1})$. Thus we have obtained that $S(\pi^{I^{(1)}})$ is supported on

$$\coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\mathrm{reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,\zeta(\varpi^{-1})} \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{non-reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,\zeta(\varpi^{-1})}/W_0 = (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{(\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1}))}.$$

□

Next, recall the twisting action of the group $\mathscr{Z}^\vee(k)$ on the standard $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules and their simple constituents 8.1.4.

8.3.6. Proposition. *Let $\pi \in \mathrm{Mod}^{\mathrm{ladm}}(k[G])$ be irreducible or a reducible principal series representation. Let $\eta : F^\times \to k^\times$ be a character. Then*

$$(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi^{-1}))$$

as $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules.

Proof. An irreducible locally admissible representation, being a finitely generated $k[G]$-module, is admissible [Em10, 2.2.19]. A principal series representation (irreducible or not) is always admissible [Em10, 4.1.7]. The list of irreducible admissible smooth G-representations is given in [H11b, Thm. 1.1]. There are four families: principal series representations, supersingular representations, characters and twists of the Steinberg representation.

We first suppose that π is a principal series representation (irreducible or not), i.e. of the form $\mathrm{Ind}_B^G(\chi)$ with a character $\chi = \chi_1 \otimes \chi_2$. Then $\pi \otimes \eta \simeq \mathrm{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)$. We use the results from 8.3.2. The modules $\pi^{I^{(1)}}$ and $(\pi \otimes \eta)^{I^{(1)}}$ are standard modules in the components $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$ and $\gamma(\eta|_{\mathbb{F}_q^\times})$ respectively. Suppose that γ is regular. We choose the ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ and standard coordinates x, y. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}})$$

and

$$\mathrm{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)^{I^{(1)}} = M(0, \chi_2(\varpi^{-1})\eta(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1})\eta(\varpi^{-2}), (\chi|_{\mathbb{T}}).(\eta|_{\mathbb{F}_q^\times})).$$

This shows $(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi^{-1}))$ in the regular case. Suppose that γ is non-regular. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(\chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}})$$

and

$$\mathrm{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)^{I^{(1)}} = M(\chi_2(\varpi^{-1})\eta(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1})\eta(\varpi^{-2}), (\chi|_{\mathbb{T}}).(\eta|_{\mathbb{F}_q^\times})).$$

This shows $(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi^{-1}))$ in the non-regular case.

We now treat the case where π is a character or a twist of the Steinberg representation. Consider the exact sequence

$$1 \to \mathbb{1} \to \mathrm{Ind}_B^G(1) \to \mathrm{St} \to 1.$$

According to [V04, 4.4] the sequence of invariants

$$(S) : 1 \to \mathbb{1}^{I^{(1)}} \to \mathrm{Ind}_B^G(1)^{I^{(1)}} \to \mathrm{St}^{I^{(1)}} \to 1$$

is still exact and $\mathbb{1}^{I^{(1)}}$ resp. $\mathrm{St}^{I^{(1)}}$ is the trivial character $(0, 1)$ resp. sign character $(-1, -1)$ in the Iwahori component $\gamma = 1$. Tensoring the first exact sequence with η produces the exact sequence

$$1 \to \eta \to \mathrm{Ind}_B^G(1) \otimes \eta \to \mathrm{St} \otimes \eta \to 1.$$

Since the restriction $\eta|_{o_F^\times}$ is trivial on $1 + \varpi o_F$, one has $(\eta \circ \det)|_{I^{(1)}} = 1$ and so, as a sequence of k-vector spaces with k-linear maps, the sequence of invariants

$$1 \to \eta^{I^{(1)}} \to (\mathrm{Ind}_B^G(1) \otimes \eta)^{I^{(1)}} \to (\mathrm{St} \otimes \eta)^{I^{(1)}} \to 1$$

coincides with the sequence (S). It is therefore an exact sequence of $\mathcal{H}_{\overline{\mathbb{F}}_q}^{(1)}$-modules, with outer terms being characters of $\mathcal{H}_{\overline{\mathbb{F}}_q}^{(1)}$. From the discussion above, we deduce

$$(\mathrm{Ind}_B^G(1) \otimes \eta)^{I^{(1)}} = \mathrm{Ind}_B^G(1)^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi)^{-1}) = M(\eta(\varpi^{-1}), \eta(\varpi^{-2}), 1.(\eta|_{\mathbb{F}_q^\times})).$$

It follows then from [V04, 1.1] that $\eta^{I^{(1)}}$ must be the trivial character $(0, \eta(\varpi^{-1}))$ in the component $1.(\eta|_{\mathbb{F}_q^\times})$ and $(\mathrm{St} \otimes \eta)^{I^{(1)}}$ must be the sign character $(-1, -\eta(\varpi^{-1}))$ in the component $1.(\eta|_{\mathbb{F}_q^\times})$. This implies

$$\eta^{I^{(1)}} = \mathbb{1}^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi)^{-1}) \quad \text{and} \quad (\mathrm{St} \otimes \eta)^{I^{(1)}} = \mathrm{St}^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi)^{-1}).$$

This proves the claim in the cases $\pi = \mathbb{1}$ or $\pi = \mathrm{St}$. If, more generally, $\pi = \eta'$ is a general character of G, then

$$(\pi \otimes \eta)^{I^{(1)}} = (\eta'\eta)^{I^{(1)}} = \mathbb{1}^{I^{(1)}}.((\eta'\eta)|_{\mathbb{F}_q^\times}, (\eta'\eta)(\varpi)^{-1}) = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi)^{-1}).$$

On the other hand, if $\pi = \mathrm{St} \otimes \eta'$ is a twist of Steinberg, then

$$(\pi \otimes \eta)^{I^{(1)}} = (\mathrm{St} \otimes (\eta'\eta))^{I^{(1)}} = \mathrm{St}^{I^{(1)}}.((\eta'\eta)|_{\mathbb{F}_q^\times}, (\eta'\eta)(\varpi)^{-1}) = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi)^{-1}).$$

It remains to treat the case where π is a supersingular representation. In this case $\pi \otimes \eta$ is also supersingular and the two modules $\pi^{I^{(1)}}$ and $(\pi \otimes \eta)^{I^{(1)}}$ are supersingular $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules [V04, 4.9]. Let γ be the component of the module $\pi^{I^{(1)}}$. By 8.3.3, the component of $(\pi \otimes \eta)^{I^{(1)}}$ equals $\gamma(\eta|_{\mathbb{F}_q^\times})$. Moreover, if U^2 acts on $\pi^{I^{(1)}}$ via the scalar $z_2 \in k^\times$, then U^2 acts on $(\pi \otimes \eta)^{I^{(1)}}$ via $z_2(\eta \circ \det)(u^2) = z_2 \eta(\varpi)^{-2}$, cf. 8.3.3. Since the supersingular modules are uniquely characterized by their component and their U^2-action, we obtain $(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi)^{-1})$, as claimed. □

8.3.7. Let $F = \mathbb{Q}_p$ with $p \geq 5$. We let $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ consisting of locally admissible representations having central character ζ. By work of Paškūnas [Pas13], the blocks b of the category $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$, defined as certain equivalence classes of simple objects, can be parametrized by the set of isomorphism classes $[\rho]$ of semisimple continuous Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$ having determinant $\det \rho = \omega\zeta$, i.e. by the k-points of X_ζ. There are three types of blocks. Blocks of type 1 are supersingular blocks. Each such block contains only one irreducible G-representation, which is supersingular. Blocks of type 2 contain only two irreducible representations. These two representations are two generic principal series representations of the form $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2\omega^{-1})$ and $\mathrm{Ind}_B^G(\chi_2 \otimes \chi_1\omega^{-1})$ (where $\chi_1\chi_2 \neq 1, \omega^{\pm 1}$). There are four blocks of type 3 which correspond to the four exceptional points. In the even case, each such block contains only three irreducible representations. These representations are of the form $\eta, \mathrm{St} \otimes \eta$ and $\mathrm{Ind}_B^G(\omega \otimes \omega^{-1}) \otimes \eta$. In the odd case, each block of type 3 contains only one irreducible representation. It is of the form $\mathrm{Ind}_B^G(\chi \otimes \chi\omega^{-1})$.

8.3.8. Let $F = \mathbb{Q}_p$ with $p \geq 5$. Paškūnas' parametrization $[\rho] \mapsto b_{[\rho]}$ is compatible with Breuil's semisimple mod p local Langlands correspondence

$$\rho \mapsto \pi(\rho)$$

for the group G [Br07, Be11], in the sense that if ρ has determinant $\omega\zeta$, then the simple constituents of the G-representation $\pi(\rho)$ lie in the block $b_{[\rho]}$ of $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$.

The correspondence and the parametrizations (for varying ζ) commute with twists: for a character $\eta : \mathbb{Q}_p^\times \to k^\times$, $\pi(\rho \otimes \eta) = \pi(\rho) \otimes \eta$ and $b_{[\rho]} \otimes \eta = b_{[\rho \otimes \eta]}$.

8.3.9. Theorem. *Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in \mathcal{Z}^\vee(k)$ under the identification $\mathcal{Z}(G)^\vee \cong \mathcal{Z}^\vee(k)$ from 8.3.4. Let $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ be the space of mod p Satake parameters with central character ζ and X_ζ be the moduli space of mod p Langlands parameters with determinant $\omega\zeta$.*

There exists a morphism of k-schemes

$$L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$$

such that the quasi-coherent $\mathcal{O}_{X_\zeta}$-module

$$L_{\zeta *} S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})|_{(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta}$$

equal to the push-forward along L_ζ of the restriction to $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ of the Satake parameter of the mod p antispherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$, cf. 7.4.2 3., interpolates the $I^{(1)}$-invariants of the semisimple mod p Langlands correspondence

$$\begin{array}{ccccc} X_\zeta(k) & \longrightarrow & \mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G]) & \longrightarrow & \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \\ x & \longmapsto & \pi(\rho_x) & \longmapsto & \pi(\rho_x)^{I^{(1)}}, \end{array}$$

in the sense that for all $x \in X_\zeta(k)$,

$$\Big(\big(L_{\zeta *}S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})|_{(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta}\big) \otimes_{\mathcal{O}_{X_\zeta}} k(x)\Big)^{\mathrm{ss}} = \Big(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})} (\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p})^{-1}(\mathcal{O}_{L_\zeta^{-1}(x)})\Big)^{\mathrm{ss}} \cong \pi(\rho_x)^{I^{(1)}}$$

in $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$.

8.3.10. The connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are either regular and then of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$, or non-regular and then of type $\mathbb{A}^1$. The morphism L_ζ appearing in the theorem depends on the choice of an order of the two affine lines in each regular component. It is surjective and quasi-finite. Moreover, writing L^γ_ζ for its restriction to the connected component $(V^\gamma_{\widehat{\mathbf{T}},0}/W_0)_\zeta \subset (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, one has:

(e) *Even case.* All connected components are of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$, except for the two 'exterior' components which are of type $\mathbb{A}^1$. L^γ_ζ is an open immersion for any γ.

(o) *Odd case.* All connected components are of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. L_ζ is an open immersion on all connected components, except for the two 'exterior' ones. On an 'exterior' component γ, the restriction of L^γ_ζ to one irreducible component $\mathbb{A}^1$ is an open immersion, and its restriction to the open complement $\mathbb{G}_m$ is a degree 2 finite flat covering of its image, with branched locus equal to the intersection of this image with the exceptional locus of X_ζ.

8.3.11. Note that the semisimple mod p Langlands correspondence associates with any semisimple $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$ a semisimple smooth G-representation $\pi(\rho)$ of length $1, 2$ or 3, hence whose semisimple $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module of $I^{(1)}$-invariants $\pi(\rho)^{I^{(1)}}$ has length $1, 2$ or 3. On the other hand, the antispherical map

$$\mathrm{ASph} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}\text{-modules}\}$$

has an image consisting of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules are of length 1 or 2, cf. 7.4.9 and 7.4.15. Theorem 8.3.9 combined with the properties 8.3.10 of the morphism L_ζ provide the following case-by-case elucidation of the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules $\pi(\rho)^{I^{(1)}}$.

8.3.12. Corollary. *Let* $x \in X_\zeta(k)$*, corresponding to* $\rho_x : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$*. Then the* $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$*-module* $\pi(\rho)^{I^{(1)}}$ *admits the following explicit description.*

(i) If $x \in X^{irred}_\zeta(k)$*, then the fibre* $L^{-1}_\zeta(x) = \{v\}$ *has cardinality* 1 *and*

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{ASph}(v).$$

It is irreducible and supersingular.

(ii) If $x \in X^{red}_\zeta(k) \setminus \{$*the four exceptional points*$\}$*, then* $L^{-1}_\zeta(x) = \{v_1, v_2\}$ *has cardinality* 2 *and*

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{ASph}(v_1) \oplus \mathrm{ASph}(v_2).$$

It has length 2.

(iiie) If $x \in X^{red}_\zeta(k)$ *is exceptional in the even case, then* $L^{-1}_\zeta(x) = \{v_1, v_2\}$ *has cardinality* 2 *and*

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{ASph}(v_1)^{\mathrm{ss}} \oplus \mathrm{ASph}(v_2).$$

It has length 3.

(iiio) If $x \in X^{red}_\zeta(k)$ *is exceptional in the odd case, then* $L^{-1}_\zeta(x) = \{v\}$ *has cardinality* 1 *and*

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{ASph}(v) \oplus \mathrm{ASph}(v).$$

It has length 2.

8.3.13. Now we proceed to the proof of 8.3.9, 8.3.10 and 8.3.12.

We start by defining the morphism L_ζ at the level of k-points. Let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$.

1. Suppose that γ is regular. Then $\mathrm{ASph}(v) = \mathrm{ASph}^\gamma(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\mathbb{F}_p}$-module, cf. 7.4.9. Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$ be the simple module, unique up to isomorphism, such that $\pi^{I^{(1)}} \simeq \mathrm{ASph}^\gamma(v)$, cf. 8.3.2. Then $\pi \in \mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ with

$$\zeta = (\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) = (\gamma|_{\mathbb{F}_p^\times}, z_2)$$

by 8.3.5. Let b be the block of $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ which contains π. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to b.

2. Suppose that γ is non-regular.

(a) If $v \in D(2)_\gamma(k)$, then $\mathrm{ASph}(v) = \mathrm{ASph}^\gamma(2)(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\mathbb{F}_p}$-module, cf. 7.4.15. As in the regular case, there is a simple module π, unique up to isomorphism, such that $\pi^{I^{(1)}} \simeq \mathrm{ASph}^\gamma(2)(v)$. It has central character $\zeta = (\gamma|_{\mathbb{F}_p^\times}, z_2)$ and there is a block b of $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ which contains π. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to b.

(b) If $v \in D(1)_\gamma(k)$, then $\mathrm{ASph}(v)^{\mathrm{ss}}$ is the direct sum of the two characters forming the antispherical pair $\mathrm{ASph}^\gamma(1)(v) = \{(0, z_1), (-1, -z_1)\}$ where $z_2 = z_1^2$, cf. 7.4.15. As in the regular case, there are two simple modules π_1 and π_2, unique up to isomorphism, such that $\pi_1^{I^{(1)}} \simeq (0, z_1)$ and $\pi_2^{I^{(1)}} \simeq (-1, -z_1)$ and π_1, π_2 have central character $\zeta = (\gamma|_{\mathbb{F}_p^\times}, z_2)$. Moreover, we claim that there is a unique block b of $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ which contains both π_1 and π_2. Indeed, if $\gamma = \{1 \otimes 1\}$ and $z_1 = 1$, then $\pi_1 = \mathbb{1}$ and $\pi_2 = \mathrm{St}$, cf. 8.3.2. Then by 8.3.6 it follows more generally that if $\gamma = \{\omega^r \otimes \omega^r\}$, then $\pi_1 = \eta$ and $\pi_2 = \mathrm{St} \otimes \eta$ with $\eta = (\eta|_{\mathbb{F}_p^\times}, \eta(p^{-1})) := (\omega^r, z_1)$. Consequently π_1, π_2 are contained in a unique block b of type 3, cf. 8.3.7. We define $L_\zeta(v)$ to be the point of $X_\zeta(k)$ which corresponds to b.

Thus we have a well-defined map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$.

We show property (i) of 8.3.12. Let $x \in X^{\mathrm{irred}}_\zeta(k)$ and suppose $L_\zeta(v) = x$. Then b_x is a supersingular block, contains a unique irreducible representation π, which is supersingular, and $\pi = \pi(\rho_x)$, cf. 8.3.7-8.3.8. By definition of L_ζ, one has $\mathrm{ASph}(v) \simeq \pi^{I^{(1)}}$. Since the antispherical map ASph is $1:1$ over supersingular modules, cf. 7.4.9 and 7.4.15, such a preimage v of x exists and is uniquely determined by x. Summarizing, we have $L_\zeta^{-1}(x) = \{v\}$ and $\mathrm{ASph}(v) \simeq \pi(\rho_x)^{I^{(1)}}$. This is property (i).

As a next step, we take a second character $\eta : \mathbb{Q}_p^\times \to k^\times$ and show that the diagram

$$\begin{array}{ccc} (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) & \xrightarrow{L_\zeta} & X_\zeta(k) \\ {\scriptstyle \cdot\eta}\downarrow\simeq & & \simeq\downarrow{\scriptstyle (\cdot)\otimes\eta} \\ (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta\eta^2}(k) & \xrightarrow{L_{\zeta\eta^2}} & X_{\zeta\eta^2}(k) \end{array}$$

commutes. Here, the vertical arrows are the bijections coming from 8.1.9 and 8.2.2. To verify the commutativity, let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$. Suppose that γ is regular or that γ is non-regular with $v \in D(2)_\gamma(k)$. Let π be the simple G-module with $\pi^{I^{(1)}} \simeq \mathrm{ASph}(v)$ and let $b_{[\rho]}$ be the block corresponding to the point $L_\zeta(v)$. By the equivariance property 8.1.5, one has $\mathrm{ASph}(v.\eta) \simeq \mathrm{ASph}(v).\eta$. Taking $I^{(1)}$-invariants is compatible with twist, cf. 8.3.6, and so $L_{\zeta\eta^2}(v.\eta)$ corresponds to the block which contains the representation $\pi \otimes \eta$, i.e. to $b_{[\rho]} \otimes \eta = b_{[\rho\otimes\eta]}$, cf. 8.3.8, and so $L_{\zeta\eta^2}(v.\eta) = [\rho \otimes \eta] = L_\zeta(v).\eta$.

If $v \in D(1)_\gamma(k)$, let π_1 and π_2 be the simple modules such that $(\pi_1 \oplus \pi_2)^{I^{(1)}} \simeq \mathrm{ASph}^\gamma(v)^{\mathrm{ss}}$. As before, we conclude from $\mathrm{ASph}(v.\eta)^{\mathrm{ss}} \simeq \mathrm{ASph}(v)^{\mathrm{ss}} \otimes \eta$ that $L_{\zeta\eta^2}(v.\eta)$ corresponds to the block which contains $\pi_1 \otimes \eta$ and $\pi_2 \otimes \eta$ and that $L_{\zeta\eta^2}(v.\eta) = L_\zeta(v).\eta$. The commutativity of the diagram is proved.

Thus, we are reduced to prove that the map L_ζ comes from a morphism of k-schemes satisfying 8.3.9 and the remaining parts of 8.3.12 in the two basic cases of a character ζ such that $\zeta(p^{-1}) = 1$ and $\zeta|_{\mathbb{F}_p^\times} \in \{1, \omega^{-1}\}$. This is established in the next two subsections.

8.4 The morphism L_ζ in the basic even case

Let $\zeta : \mathbb{Q}_p^\times \to k^\times$ be the trivial character. Here we show that the map of sets $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k) \to X_\zeta(k)$ that we have defined in 8.3.13 satisfies properties (ii) and (iiie) of 8.3.12, and we define a morphism of k-schemes $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides with the previous map of sets at the level of k-points. By construction, it will have the properties 8.3.10. This will complete the proof of 8.3.12, 8.3.10 and 8.3.9 in the case of an even character.

8.4.1. We verify the properties (ii) and (iiie). We work over an irreducible component $\mathbb{P}^1$ with label " $\mathrm{Sym}^r \otimes \det^a \mid \mathrm{Sym}^{p-3-r} \otimes \det^{r+1+a}$ " where $0 \le r \le p-3$ and $0 \le a \le p-2$, cf. 8.2.3. On this component, we choose an affine coordinate x around the double point having $\mathrm{Sym}^r \otimes \det^a$ as one of its Serre weights. Away from this point, we have $x \neq 0$ and the corresponding Galois representation has the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{r+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^a$. By [Be11, 1.3] or [Br07, 4.11], we have

$$\pi(\rho_x) = \pi(r, x, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], x^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}} =: \pi_1 \oplus \pi_2$$

where $[p-3-r]$ denotes the unique integer in $\{0, ..., p-2\}$ which is congruent to $p-3-r$ modulo $p-1$. Now suppose that $L_\zeta(v) = x$. We distinguish two cases.

1. *The generic case* $0 < r < p-3$. In this case, the point x lies on one of the 'interior' components of the chain X_ζ, which has no exceptional points. The length of $\pi(\rho_x)$ is 2. Indeed, $\pi_1 = \pi(r, x, \eta)$ and $\pi_2 = \pi(p-3-r, x^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations [Br07, Thm. 4.4]. The block b_x is of type 2 and contains only these two irreducible representations, cf. 8.3.7-8.3.8. We may write

$$\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$$

with $\chi = \mathrm{unr}(x) \otimes \omega^r\,\mathrm{unr}(x^{-1})$, according to [Br07, Rem. 4.4(ii)]. By our assumptions on r, the character $\chi|_{\mathbb{T}} = 1 \otimes \omega^r$ is regular (i.e. different from its s-conjugate). We conclude from 8.3.6 and 8.3.2 that $\pi_1^{I^{(1)}}$ is a simple 2-dimensional standard module in the regular component represented by the character $(1 \otimes \omega^r).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^r \in \mathbb{T}^\vee$. Similarly, we may write

$$\pi_2 = \mathrm{Ind}_B^G(\chi) \otimes \omega^{r+1}\eta$$

where now $\chi = \mathrm{unr}(x^{-1}) \otimes \omega^{p-3-r}\,\mathrm{unr}(x)$. By our assumptions on r, the character $\chi|_{\mathbb{T}} = 1 \otimes \omega^{p-3-r}$ is regular and we conclude, as above, that the $I^{(1)}$-invariants $\pi_2^{I^{(1)}}$ form a simple 2-dimensional standard module in the regular component represented by the character $(\eta|_{\mathbb{F}_p^\times})\omega^{r+1} \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{r+1}\omega^{p-3-r} \in \mathbb{T}^\vee$. Note that the component of $\pi_1^{I^{(1)}}$ is different from the component of $\pi_2^{I^{(1)}}$, by our assumptions on r.

We conclude from $L_\zeta(v) = x$ that either $\mathrm{ASph}(v) = \pi_1^{I^{(1)}}$ or $\mathrm{ASph}(v) = \pi_2^{I^{(1)}}$. Since for γ regular, the map ASph^γ is a bijection onto all simple $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-modules, cf. 7.4.9, one finds that $L_\zeta^{-1}(x) = \{v_1, v_2\}$ has cardinality 2 and

$$\mathrm{ASph}(v_1) \oplus \mathrm{ASph}(v_2) \simeq \pi(\rho_x)^{I^{(1)}}.$$

This settles property (ii) of 8.3.12 in the generic case.

2. *The boundary cases* $r \in \{0, p-3\}$. In this case, the point x lies on one of the two 'exterior' components of X_ζ. On such a component, we will denote the variable x rather by z_1, which is the notation[5] which we used already in 8.2.3.

(a) Suppose that $z_1 \neq \pm 1$. The length of $\pi(\rho_{z_1})$ is 2. Indeed, as in the generic case, $\pi_1 = \pi(r, z_1, \eta)$ and $\pi_2 = \pi(p-3-r, z_1^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations. The block b_{z_1} is of type 2 and contains only these two irreducible representations. It follows, as above, that their invariants $\pi_1^{I^{(1)}}$ and $\pi_2^{I^{(1)}}$ are simple 2-dimensional standard modules, in the components represented by $(\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^r \in \mathbb{T}^\vee$ and $(\eta|_{\mathbb{F}_p^\times})\omega^{r+1} \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{r+1}\omega^{p-3-r} \in \mathbb{T}^\vee$ respectively. Since $r \in \{0, p-3\}$, one of these components is regular, the other non-regular. In particular, the two components are different. We conclude from $L_\zeta(v) = z_1$ that either $\mathrm{ASph}(v) = \pi_1^{I^{(1)}}$ or $\mathrm{ASph}(v) = \pi_2^{I^{(1)}}$. Since for non-regular γ, the map $\mathrm{ASph}^\gamma(2)$ is a bijection from $D(2)_\gamma(k)$ onto all simple standard $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-modules, cf. 7.4.15, we may conclude as in the generic case: $L_\zeta^{-1}(z_1) = \{v_1, v_2\}$ has cardinality 2 and

$$\mathrm{ASph}(v_1) \oplus \mathrm{ASph}(v_2) \simeq \pi(\rho_{z_1})^{I^{(1)}}.$$

This settles property 8.3.12 (ii) in the remaining case $z_1 \neq \pm 1$.

(b) Suppose now that $z_1 = \pm 1$, i.e. we are at one of the four exceptional points. We will verify property (iiie). The length of $\pi(\rho_{z_1})$ is 3. Indeed, the representation $\pi(0, \pm 1, \eta)$ is a twist of the representation $\pi(0, 1, 1)$ (note that $\pi(r, z_1, \eta) \simeq \pi(r, -z_1, \mathrm{unr}(-1)\eta)$ according to [Br07, Rem. 4.4(v)]), which itself is an extension of $\mathbb{1}$ by St, cf. [Br07, Thm. 4.4(iii)]. As in the case (a), the representation $\pi_2 = \pi(p-3, \pm 1, \omega\eta)$ is an irreducible principal series representation. The block b_{z_1} is of type 3 and contains only these three irreducible representations. The invariants $\pi_1^{I^{(1)}}$ form a direct sum of two antispherical characters in a non-regular component γ, whereas the invariants $\pi_2^{I^{(1)}}$ form a simple standard module in a regular component, as before. Since for non-regular γ, the map $\mathrm{ASph}^\gamma(1)$ is a bijection from $D(1)_\gamma(k)$ onto all antispherical pairs of characters of $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$, cf. 7.4.15, we may conclude that $L_\zeta^{-1}(z_1) = \{v_1, v_2\}$ has cardinality 2 with $v_1 \in D(1)_\gamma(k)$ and $\mathrm{ASph}^\gamma(1)(v_1)^{\mathrm{ss}} = \pi_1^{I^{(1)}}$. In particular,

$$\mathrm{ASph}(v_1)^{\mathrm{ss}} \oplus \mathrm{ASph}(v_2) \simeq \pi(\rho_x)^{I^{(1)}}.$$

This settles property 8.3.12 (iiie).

8.4.2. We define a morphism of k-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides on k-points with the map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$. We work over a connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, indexed by some $\gamma \in \mathbb{T}^\vee/W_0$. Let v be a k-point of this component.

Since $\zeta|_{\mathbb{F}_p^\times} = 1$, the connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are indexed by the fibre $(\cdot)|^{-1}_{\mathbb{F}_p^\times}(1)$. This fibre consists of the $\frac{p-3}{2}$ regular components, represented by the characters of $\mathbb{T}$

$$\chi_k = \omega^k \otimes \omega^{-k}$$

for $k = 1, ..., \frac{p-3}{2}$, and of the two non-regular components, given by χ_0 and $\chi_{\frac{p-1}{2}}$, cf. 8.1.2. We distinguish two cases. Note that $z_2 = \zeta(p^{-1}) = 1$.

1. *The regular case* $0 < k < \frac{p-1}{2}$. We fix the order $\gamma = (\chi_k, \chi_k^s)$ on the set γ and choose the standard coordinates x, y. According to 7.4.7, our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\widehat{\mathbf{T}},0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$

Suppose that $v = (0,0)$ is the origin, so that $\mathrm{ASph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r, \eta) = \mathrm{ind}(\omega_2^{r+1}) \otimes \eta$, in the notation of [Be11, 1.3], whence $L_\zeta(v) = [\rho(r, \eta)]$. According to 8.3.2,

[5] The reason for this notation will become clear in the discussion of the non-regular case in 8.4.2.

the component of the Hecke module $\pi(r,0,\eta)^{I^{(1)}}$ is given by $(\omega^r\otimes 1)\cdot(\eta|_{\mathbb{F}_p^\times})$. Setting $\eta|_{\mathbb{F}_p^\times}=\omega^a$, this implies $(\omega^r\otimes 1)\cdot(\eta|_{\mathbb{F}_p^\times})=\omega^{r+a}\otimes\omega^a=\chi_k$ and hence $a=-k$ and $r=2k$. Therefore the Serre weights of the irreducible representation $\rho(r,\eta)$ are $\{\mathrm{Sym}^{2k}\otimes\det^{-k},\mathrm{Sym}^{p-1-2k}\otimes\det^{k}\}$, cf. [Br07, 1.9].

Comparing these pairs of Serre weights with the list 8.2.3 shows that the $\frac{p-3}{2}$ points

$$\{\text{origin } (0,0) \text{ on the component } (\chi_k,\chi_k^s)\}$$

for $0<k<\frac{p-1}{2}$ are mapped successively to the $\frac{p-3}{2}$ double points of the chain X_ζ.

Fix $0<k<\frac{p-1}{2}$ and consider the double point

$$Q=L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma=(\chi_k,\chi_k^s)).$$

As we have just seen, Q lies on the irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^{2k}\otimes\det^{-k}$ (i.e. on the component " $\mathrm{Sym}^{2k}\otimes\det^{-k}$ | $\mathrm{Sym}^{p-3-2k}\otimes\det^{k+1}$"). We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will also call x (there will be no risk of confusion with the standard coordinate above!). Away from Q, the affine coordinate $x\neq 0$ parametrizes Galois representations of the form

$$\rho_x=\begin{pmatrix}\mathrm{unr}(x)\omega^{2k+1} & 0\\ 0 & \mathrm{unr}(x^{-1})\end{pmatrix}\otimes\eta$$

with $\eta:=\omega^{-k}$. As we have seen above, $\pi(\rho_x)=\pi(2k,x,\eta)\oplus\pi(p-3-2k,x^{-1},\omega^{r+1}\eta)=:\pi_1\oplus\pi_2$. Moreover, $\pi_1=\mathrm{Ind}_B^G(\chi)\otimes\eta$ with $\chi=\mathrm{unr}(x)\otimes\omega^{2k}\,\mathrm{unr}(x^{-1})$. Since

$$(1\otimes\omega^{2k}).(\eta|_{\mathbb{F}_p^\times})=\omega^{-k}\otimes\omega^k=\chi_k^s\in\mathbb{T}^\vee,$$

we deduce from the regular case of 8.3.2 that

$$\pi_1^{I^{(1)}}=M(0,x,1,\chi_k^s)$$

is a simple 2-dimensional standard module. Note that $M(0,x,1,\chi_k^s)=M(x,0,1,\chi_k)$ according to [V04, Prop. 3.2].

Now suppose that $v=(x,0),x\neq 0$, denotes a point on the x-line of $\mathbb{A}_k^1\cup_0\mathbb{A}_k^1$. In particular, $\mathrm{ASph}^\gamma(v)=M(x,0,1,\chi_k)$. By our discussion, the point $L_\zeta((x,0))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_x]$, cf. 8.3.8, it follows that

$$L_\zeta((x,0))=[\rho_x]=x\in\mathbb{G}_m\subset\mathbb{P}^1\subset X_\zeta.$$

Since $(0,0)$ maps to Q, i.e. to the point at $x=0$, the map L_ζ identifies the whole affine x-line $\mathbb{A}^1=\{(x,0):x\in k\}\subset V_{\widehat{\mathbf{T}},0,1}$ with the affine x-line $\mathbb{A}^1\subset\mathbb{P}^1\subset X_\zeta$.

On the other hand, the double point Q lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of Q, i.e. the weight $\mathrm{Sym}^{p-1-2k}\otimes\det^k$. We fix an affine coordinate y on this $\mathbb{P}^1$ around Q. Away from Q, the coordinate $y\neq 0$ parametrizes Galois representations of the form

$$\rho_x=\begin{pmatrix}\mathrm{unr}(y)\omega^{p-2k} & 0\\ 0 & \mathrm{unr}(y^{-1})\end{pmatrix}\otimes\eta$$

with $\eta:=\omega^k$. As in the first case, $\pi(\rho_y)$ contains $\pi_1:=\pi(p-1-2k,y,\eta)=\mathrm{Ind}_B^G(\chi)\otimes\eta$ as a direct summand, where now $\chi=\mathrm{unr}(y)\otimes\omega^{p-1-2k}\,\mathrm{unr}(y^{-1})$. Since

$$(1\otimes\omega^{p-1-2k}).(\eta|_{\mathbb{F}_p^\times})=\omega^k\otimes\omega^{-k}=\chi_k\in\mathbb{T}^\vee,$$

we deduce, as above, that $\pi_1^{I^{(1)}}=M(0,y,1,\chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v=(0,y),y\neq 0$, denotes a point on the y-line of $\mathbb{A}_k^1\cup_0\mathbb{A}_k^1$. In particular, $\mathrm{ASph}^\gamma(v)=M(0,y,1,\chi_k)$. By our discussion, the point $L_\zeta((0,y))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_y]$, cf. 8.3.8, it follows that

$$L_\zeta((0,y))=[\rho_y]=y\in\mathbb{G}_m\subset\mathbb{P}^1\subset X_\zeta.$$

Since $(0,0)$ maps to Q, i.e. to the point at $y=0$, the map L_ζ identifies the whole affine y-line $\mathbb{A}^1 = \{(0,y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine y-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each regular connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

2. *The non-regular case* $k \in \{0, \frac{p-1}{2}\}$. We choose the Steinberg coordinate z_1. According to 7.4.7, our non-regular connected component identifies with an affine line :

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0 \simeq \mathbb{A}^1.$$

Suppose that $v=(0)$ is the origin, so that $\mathrm{ASph}(v)$ is a supersingular module. Let $\pi(r,0,\eta)$ be the corresponding supersingular representation so that $L_\zeta(v) = [\rho(r,\eta)]$. Exactly as in the regular case, we may conclude that the Serre weights of the irreducible representation $\rho(r,\eta)$ are $\{\mathrm{Sym}^{2k} \otimes \det^{-k}, \mathrm{Sym}^{p-1-2k} \otimes \det^k\}$. For the two values of $k=0$ and $k=\frac{p-1}{2}$ we find $\{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}\}$ and $\{\mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}}, \mathrm{Sym}^{p-1} \otimes \det^{\frac{p-1}{2}}\}$ respectively. Comparing with the list 8.2.3 shows that the 2 points

$$\{\text{origin } (0) \text{ on the component } (\chi_k = \chi_k^s)\}$$

for $k \in \{0, \frac{p-1}{2}\}$ are mapped to the 2 smooth points in X_ζ^{irred}, which lie on the two 'exterior' components of X_ζ, cf. 8.2.3.

Fix $k \in \{0, \frac{p-1}{2}\}$ and consider the point

$$Q = L_\zeta(\text{origin } (0) \text{ on the component } \gamma = (\chi_k = \chi_k^s)).$$

As we have just seen, Q lies on an 'exterior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^0 \otimes \det^k$. We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we call z_1 (there will be no risk of confusion with the Steinberg coordinate above!). Away from Q, the affine coordinate $z_1 \neq 0$ parametrizes Galois representations of the form

$$\rho_{z_1} = \begin{pmatrix} \mathrm{unr}(z_1)\omega & 0 \\ 0 & \mathrm{unr}(z_1^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the regular case, $\pi(\rho_{z_1}) = \pi(0,z_1,\eta)^{\mathrm{ss}} \oplus \pi(p-3, z_1^{-1}, \omega\eta)^{\mathrm{ss}}$. Moreover, $\pi(0,z_1,\eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(z_1) \otimes \mathrm{unr}(z_1^{-1})$ [6]. Since

$$(1 \otimes 1).(\eta|_{\mathbb{F}_p^\times}) = \omega^k \otimes \omega^k = \chi_k = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the non-regular case of 8.3.2 that $\pi(0,z_1,\eta)^{I^{(1)}} = M(z_1,1,\chi_k)$ is a 2-dimensional standard module. Moreover, the standard module is simple if and only if $\chi \neq \chi^s$, i.e. if and only if $z_1 \neq \pm 1$.

Now let $v = z_1 \neq 0$ denote a nonzero point on our connected component $\mathbb{A}^1 = V_{\widehat{\mathbf{T}},0,1}/W_0$. Suppose that $z_1 \neq \pm 1$, i.e. $v \in D(2)_\gamma$. In particular, $\mathrm{ASph}(v) = M(z_1,1,\gamma)$ is irreducible. By our discussion, the point $L_\zeta(z_1)$ corresponds to the block (a block of type 2) which contains $\pi(0,z_1,\eta)$. Suppose that $z_1 = \pm 1$, i.e. $v \in D(1)_\gamma$. In particular, $\mathrm{ASph}^{\mathrm{ss}}(v) = M(z_1,1,\chi_k)^{\mathrm{ss}}$ and again, $L_\zeta(z_1)$ corresponds to the block (now a block of type 3) which contains the simple constituents of $\pi(0,z_1,\eta)^{\mathrm{ss}}$. In both cases, we conclude

$$L_\zeta(z_1) = [\rho_{z_1}] = z_1 \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since (0) maps to Q, i.e. to the point at $z_1 = 0$, the map L_ζ identifies the whole z_1-line $\mathbb{A}^1 = V_{\widehat{\mathbf{T}},0,1}/W_0$ with the z_1-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each non-regular connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

[6] The representations $\pi(0,z_1,\eta)$ constitute the *unramified* principal series of G.

8.5 The morphism L_ζ in the basic odd case

Let $\zeta := \omega^{-1} : \mathbb{Q}_p^\times \to k^\times$. Here we show that the map of sets $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k) \to X_\zeta(k)$ that we have defined in 8.3.13 satisfies properties (ii) and (iiio) of 8.3.12, and we define a morphism of k-schemes $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides with the previous map of sets at the level of k-points. By construction, it will have the properties 8.3.10. This will complete the proof of 8.3.12, 8.3.10 and 8.3.9 in the case of an odd character.

8.5.1. We verify properties (ii) and (iiio). We work over an irreducible component $\mathbb{P}^1$ with label " $\mathrm{Sym}^r \otimes \det^a$ | $\mathrm{Sym}^{p-3-r} \otimes \det^{r+1+a}$ " where $1 \le r \le p-2$ and $0 \le a \le p-2$, cf. 8.2.4. We distinguish two cases.

1. *The generic case $r \neq p-2$.* In this case, the irreducible component of X_ζ we consider is an 'interior' component and has no exceptional points. On this component, we choose an affine coordinate x around the double point having $\mathrm{Sym}^r \otimes \det^a$ as one of its Serre weights. Away from this point, we have $x \neq 0$ and the corresponding Galois representation has the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{r+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^a$. As before, we have

$$\pi(\rho_x) = \pi(r, x, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], x^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}}.$$

The length of $\pi(\rho_x)$ is 2. Indeed, by our assumptions on r, the principal series representations $\pi(r, x, \eta)$ and $\pi(p-3-r, x^{-1}, \omega^{r+1}\eta)$ are irreducible and the block b_x contains only these two irreducible representations. We may follow the argument of the generic case of 8.4.1 word for word and deduce property 8.3.12 (ii).

2. *The two boundary cases $r = p-2$.* In this case, the irreducible component is one of the two 'exterior' components with labels " Sym^{p-2} | " Sym^{-1} "" or "" $\mathrm{Sym}^{-1} \det^{\frac{p-1}{2}}$ " | $\mathrm{Sym}^{p-2} \det^{\frac{p-1}{2}}$ ". Points of the open locus X_ζ^{red} lying on such a component correspond to twists of unramified Galois representations of the form

$$\rho_{x+x^{-1}} = \begin{pmatrix} \mathrm{unr}(x) & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = 1$ or $\eta = \omega^{\frac{p-1}{2}}$. Let us concentrate on one of the two components, i.e. let us fix η.

Mapping an unramified Galois representation $\rho_{x+x^{-1}}$ to $t := x + x^{-1} \in k$ identifies this open locus with the t-line $\mathbb{A}^1 \subset \mathbb{P}^1$. We have

$$\pi(\rho_t) = \pi(p-2, x, \eta)^{\mathrm{ss}} \oplus \pi(p-2, x^{-1}, \eta)^{\mathrm{ss}} =: \pi_1 \oplus \pi_2$$

since $[p-3-(p-2)] = p-2$ (indeed, $p-3-(p-2) = -1 \equiv p-2 \mod (p-1)$). The length of $\pi(\rho_t)$ is 2. Indeed, $\pi_1 = \pi(p-2, x, \eta)$ and $\pi_2 = \pi(p-2, x^{-1}, \eta)$ are two irreducible principal series representations and the block b_t contains only these two irreducible representations. They are isomorphic if and only if $x = \pm 1$, i.e. if and only if $t = \pm 2$ is an exceptional point. In this case, b_t contains only one irreducible representation and is of type 3, otherwise it is of type 2.

We may write

$$\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$$

with $\chi = \mathrm{unr}(x) \otimes \omega^{p-2}\,\mathrm{unr}(x^{-1})$. Similarly for π_2. The character $\chi|_{\mathbb{F}_p^\times} = 1 \otimes \omega^{p-2}$ is regular (i.e. different from its s-conjugate) and we are in the regular case of 8.3.2. We conclude that $\pi_1^{I^{(1)}} = M(0, x, 1, (1 \otimes \omega^{p-2}).\eta)$ and $\pi_2^{I^{(1)}} = M(0, x^{-1}, 1, (1 \otimes \omega^{p-2}).\eta)$ are both simple 2-dimensional standard modules in the regular component γ represented by the character $(1 \otimes \omega^{p-2}).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} \in \mathbb{T}^\vee$. They are isomorphic if and only if $t = \pm 2$. We choose an order $\gamma = ((\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2}, (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} \otimes (\eta|_{\mathbb{F}_p^\times}))$ on the set γ. Then from $L_\zeta(v) = t$ we get that either $\mathrm{ASph}^\gamma(v) = \pi_1^{I^{(1)}}$ or $\mathrm{ASph}^\gamma(v) = \pi_2^{I^{(1)}}$. Since for regular γ, the map ASph^γ is a bijection onto all simple $\mathcal{H}^{\gamma}_{\mathbb{F}_p}$-modules, cf. 7.4.9, one finds that $L_\zeta^{-1}(t) = \{v_1, v_2\}$ has cardinality 2 if $t \neq \pm 2$ and then

$$\mathrm{ASph}(v_1) \oplus \mathrm{ASph}(v_2) \simeq \pi(\rho_t)^{I^{(1)}}.$$

This settles property 8.3.12 (ii). In turn, if $t = \pm 2$ is an exceptional point, then $L_\zeta^{-1}(t) = \{v\}$ has cardinality 1 and

$$\mathrm{ASph}(v) \oplus \mathrm{ASph}(v) \simeq \pi(\rho_t)^{I^{(1)}}.$$

This settles property 8.3.12 (iiio).

8.5.2. We define a morphism of k-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides on k-points with the map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$. We work over a connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, indexed by some $\gamma \in \mathbb{T}^\vee/W_0$. Let v be a k-point of this component.

Since $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$, the connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are indexed by the fibre $(\cdot)|^{-1}_{\mathbb{F}_p^\times}(\omega^{-1})$. This fibre consists of the $\frac{p-1}{2}$ regular components, represented by the characters

$$\chi_k = \omega^{k-1} \otimes \omega^{-k}$$

for $k = 1, ..., \frac{p-1}{2}$, cf. 8.1.2. Recall that $z_2 = \zeta(p) = 1$.

Fix an order $\gamma = (\chi_k, \chi_k^s)$ on the set γ and choose standard coordinates x, y. According to 7.4.7, our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\widehat{\mathbf{T}},0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$

Suppose that $v = (0,0)$ is the origin, so that $\mathrm{ASph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r, \eta)$, in the notation of [Be11, 1.3], whence $L_\zeta(v) = [\rho(r, \eta)]$. According to 8.3.2, the component of $\pi(r, 0, \eta)^{I^{(1)}}$ is given by $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$. Setting $\eta|_{\mathbb{F}_p^\times} = \omega^a$, this implies $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times}) = \omega^{r+a} \otimes \omega^a = \chi_k$ and hence $a = -k$ and $r = 2k - 1$. The Serre weights of the irreducible representation $\rho(r, \eta)$ are therefore $\{\mathrm{Sym}^{2k-1} \otimes \det^{-k}, \mathrm{Sym}^{p-2k} \otimes \det^{k-1}\}$, cf. [Br07, 1.9].

Comparing these pairs of Serre weights with the list 8.2.4 shows that the $\frac{p-1}{2}$ points

$$\{\text{origin } (0,0) \text{ on the component } (\chi_k, \chi_k^s)\}$$

for $k = 1, ..., \frac{p-1}{2}$ are mapped successively to the $\frac{p-1}{2}$ double points of the chain X_ζ. We distinguish two cases.

1. The generic case $1 < k < \frac{p-1}{2}$. In this case, the argument proceeds as in the regular case of 8.4.2. Consider the double point

$$Q = L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, Q lies on an 'interior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^{2k-1} \otimes \det^{-k}$. We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will also call x. Away from Q, the affine coordinate $x \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{2k} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{-k}$. As we have seen above, $\pi(\rho_x) = \pi(2k-1, x, \eta) \oplus \pi(p-3-2k+1, x^{-1}, \omega^{2k}\eta) =: \pi_1 \oplus \pi_2$. Moreover, $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(x) \otimes \omega^{2k-1}\,\mathrm{unr}(x^{-1})$. Since

$$(1 \otimes \omega^{2k-1}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{-k} \otimes \omega^{k-1} = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the regular case of 8.3.2 that $\pi_1^{I^{(1)}} = M(0, x, 1, \chi_k^s)$ is a simple 2-dimensional standard module. Note that $M(0, x, 1, \chi_k^s) = M(x, 0, 1, \chi_k)$ according to [V04, Prop. 3.2].

Now suppose that $v = (x, 0)$, $x \neq 0$, denotes a nonzero point on the x-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{ASph}^\gamma(v) = M(x, 0, 1, \chi_k)$. Our discussion shows that the point $L_\zeta((x, 0))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_x]$, cf. 8.3.8, it follows that

$$L_\zeta((x,0)) = [\rho_x] = x \in \mathbb{G}_m \subset \mathbb{P}^1.$$

Since $(0,0)$ maps to Q, i.e. to the point at $x = 0$, the map L_ζ identifies the whole affine x-line $\mathbb{A}^1 = \{(x,0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine x-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point Q also lies on the irreducible component whose labelling includes the other weight of Q, i.e. the weight $\mathrm{Sym}^{p-2k} \otimes \det^{k-1}$. We fix an affine coordinate y on this $\mathbb{P}^1$ around Q. Away from Q, the coordinate $y \neq 0$ parametrizes Galois representations of the form

$$\rho_y = \begin{pmatrix} \mathrm{unr}(y)\omega^{p-2k+1} & 0 \\ 0 & \mathrm{unr}(y^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{k-1}$. As in the first case, $\pi(\rho_y)$ contains $\pi_1 := \pi(p-2k, y, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ as a direct summand, where now $\chi = \mathrm{unr}(y) \otimes \omega^{p-2k}\,\mathrm{unr}(y^{-1})$. Since

$$(1 \otimes \omega^{p-2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{k-1} \otimes \omega^{-k} = \chi_k \in \mathbb{T}^\vee,$$

we deduce from the regular case of 8.3.2 that $\pi_1^{I^{(1)}} = M(0, y, 1, \chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v = (0, y)$, $y \neq 0$, denotes a nonzero point on the y-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{ASph}^\gamma(v) = M(0, y, 1, \chi_k)$. Our discussion shows that the point $L_\zeta((0, y))$ corresponds to the block which contains π_1, parametrized by $[\rho_y]$. Hence

$$L_\zeta((0,y)) = [\rho_y] = y \in \mathbb{G}_m \subset \mathbb{P}^1.$$

Since $(0,0)$ maps to Q, i.e. to the point at $y = 0$, the map L_ζ identifies the whole y-line $\mathbb{A}^1 = \{(0,y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine y-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each connected component $(V^\gamma_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ such that $\gamma = (\chi_k, \chi_k^s)$ with $1 < k < \frac{p-1}{2}$, in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

2. The two boundary cases $k \in \{1, \frac{p-1}{2}\}$. Consider the double point

$$Q = L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, Q lies on an 'interior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^1 \otimes \det^{-1}$ (for $k = 1$) or the weight $\mathrm{Sym}^1 \otimes \det^{\frac{p-3}{2}}$ (for $k = \frac{p-1}{2}$). We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will call z. Away from Q, the coordinate $z \neq 0$ parametrizes Galois representations of the form

$$\rho_z = \begin{pmatrix} \mathrm{unr}(z)\omega^2 & 0 \\ 0 & \mathrm{unr}(z^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^{-1}$ or $\eta = \omega^{\frac{p-3}{2}}$.

Let $k = 1$, i.e. $\eta = \omega^{-1}$. Following the argument in the generic case word for word, we may conclude that L_ζ identifies the x-line $\mathbb{A}^1 = \{(x,0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the z-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

Let $k = \frac{p-1}{2}$, i.e. $\eta = \omega^{\frac{p-3}{2}}$. As in the generic case, we may conclude that L_ζ identifies the y-line $\mathbb{A}^1 = \{(0,y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the z-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point Q lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of Q, i.e. the weight Sym^{p-2} (for $k = 1$) or the weight $\mathrm{Sym}^{p-2} \otimes \det^{\frac{p-1}{2}}$ (for $k = \frac{p-1}{2}$). These are the two 'exterior' components. Points of the open locus X_ζ^{red} lying on such a component correspond to unramified (up to twist) Galois representations of the form

$$\rho_t = \begin{pmatrix} \mathrm{unr}(z) & 0 \\ 0 & \mathrm{unr}(z^{-1}) \end{pmatrix} \otimes \eta$$

where $\eta = 1$ (for $k = 1$) or $\eta = \omega^{\frac{p-1}{2}}$ (for $k = \frac{p-1}{2}$) and with $t = z + z^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1$. As in the boundary case of 8.5.1, we have $\pi(\rho_t) = \pi(p-2, z, \eta) \oplus \pi(p-2, z^{-1}, \eta) =: \pi_1 \oplus \pi_2$ and these are irreducible principal series representations. We may write $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(z) \otimes \omega^{p-2}\,\mathrm{unr}(z^{-1})$. The character $\chi|_{\mathbb{F}_p^\times} = 1 \otimes \omega^{p-2}$ is regular (i.e. different from its s-conjugate) and we are in the regular case of 8.3.2. We conclude that

$$\pi_1^{I^{(1)}} = M(0, z, 1, (1 \otimes \omega^{p-2}).\eta)$$

is a simple 2-dimensional standard module in the regular component represented by the character

$$(1 \otimes \omega^{p-2}).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{-1} \in \mathbb{T}^\vee.$$

This latter character equals χ_1 for $\eta = 1$ and $(\chi_{\frac{p-1}{2}})^s$ for $\eta = \omega^{\frac{p-1}{2}}$ (indeed, note that $\frac{p-1}{2} \equiv -\frac{p-1}{2}$ mod $p-1$).

Now suppose that $k = 1$, i.e. $\eta = 1$. Let $v = (0, y)$, $y \neq 0$, be a nonzero point on the y-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{ASph}^\gamma(v) = M(0, y, 1, \chi_1)$. Our discussion shows that the point $L_\zeta((0, y))$ corresponds to the block which contains π_1, i.e. which is parametrized by $[\rho_t]$. It follows that

$$L_\zeta((0, y)) = [\rho_t] = t = y + y^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1.$$

Since $(0, 0)$ maps to Q, i.e. to the point at $t = \infty$, the map of sets L_ζ maps the k-points of the whole affine y-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ to the k-points of the whole 'left exterior' component $\mathbb{P}^1 \subset X_\zeta$ via the formula

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ y & \longmapsto & \begin{cases} y + y^{-1} & \text{if } y \neq 0 \\ \infty = Q & \text{if } y = 0. \end{cases} \end{array}$$

This formula is algebraic: indeed, for $y \in \mathbb{A}^1 \setminus \{\pm i\}$ (where $\pm i$ are the roots of the polynomial $f(y) = y^2 + 1$), we have $y + y^{-1} \neq 0$ and $(y + y^{-1})^{-1} = y/(y^2 + 1)$, which is equal to 0 at $y = 0$. Moreover, it glues at the origin $(0, 0)$ with the open immersion of the x-line of $V_{\widehat{\mathbf{T}},0,1} = \mathbb{A}^1 \cup_0 \mathbb{A}^1$ in X_ζ defined above, since both map $(0, 0)$ to Q. We take the resulting morphism of k-schemes $\mathbb{A}^1 \cup_0 \mathbb{A}^1 \to X_\zeta$ as the definition of L_ζ on the connected component $(V_{\widehat{\mathbf{T}},0}^{(\chi_1, \chi_1^s)}/W_0)_\zeta$ of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$. Note that its restriction to the open subset $\{y \neq 0\}$ in the y-line $\mathbb{A}^1$ is the morphism $\mathbb{G}_m \to \mathbb{A}^1$ corresponding to the ring extension

$$k[t] \longrightarrow k[y, y^{-1}] = k[t][y]/(y^2 - ty + 1),$$

and that the discriminant $t^2 - 4$ of $y^2 - ty + 1 \in k[t][y]$ vanishes precisely at the two exceptional points $t = \pm 2$.

Suppose $k = \frac{p-1}{2}$, i.e. $\eta = \omega^{\frac{p-1}{2}}$. Let $v = (x, 0)$, $x \neq 0$, denote a nonzero point on the x-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular,

$$\mathrm{ASph}^\gamma(v) = M(0, x, 1, (\chi_{\frac{p-1}{2}})^s) = M(x, 0, 1, \chi_{\frac{p-1}{2}}).$$

Our discussion shows that the point $L_\zeta((x, 0))$ corresponds to the block which contains π_1, i.e. which is parametrized by $[\rho_t]$. It follows that $L_\zeta((x, 0)) = [\rho_t] = t = x + x^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1$. Since $(0, 0)$ maps to the point Q at $t = \infty$, the map of sets L_ζ maps the k-points of the whole affine x-line $\mathbb{A}^1 = \{(x, 0) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ to the k-points of the whole 'right exterior' component $\mathbb{P}^1 \subset X_\zeta$ via the formula

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ x & \longmapsto & \begin{cases} x + x^{-1} & \text{if } x \neq 0 \\ \infty = Q & \text{if } x = 0. \end{cases} \end{array}$$

This formula is algebraic. Moreover, it glues at the origin $(0, 0)$ with the open immersion of the y-line of $V_{\widehat{\mathbf{T}},0,1} = \mathbb{A}^1 \cup_0 \mathbb{A}^1$ in X_ζ defined above, since both map $(0, 0)$ to Q. We take the resulting morphism of k-schemes $\mathbb{A}^1 \cup_0 \mathbb{A}^1 \to X_\zeta$ as the definition of L_ζ on the connected component $(V_{\widehat{\mathbf{T}},0}^{(\chi_{\frac{p-1}{2}}, (\chi_{\frac{p-1}{2}})^s)}/W_0)_\zeta$ of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$.

8.6 A mod p Langlands parametrization in families for $F = \mathbb{Q}_p$

In this subsection we continue to assume that $F = \mathbb{Q}_p$ with $p \geq 5$.

8.6.1. Recall the mod p parametrization functor $P : \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \to \mathrm{SP}_{\widehat{\mathbf{G}},0}$ from 7.3.6. For $\zeta \in \mathcal{Z}^\vee(k)$, let $\mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the full subcategory of $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ whose objets are the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules whose Satake parameter is supported on the closed subscheme $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. A $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module M lies in the category $\mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ if and only if: M is only supported in γ-components where $\gamma|_{\mathbb{F}_p^\times} = \zeta|_{\mathbb{F}_p^\times}$ and the operator U^2 acts on M via the $\mathbb{G}_m$-part of ζ. Set $\mathrm{SP}_{\widehat{\mathbf{G}},0,\zeta} := \mathrm{QCoh}((V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta)$, the category of quasi-coherent modules on the k-scheme $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$. Then P induces a *mod p ζ-parametrization functor*

$$P_\zeta : \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \longrightarrow \mathrm{SP}_{\widehat{\mathbf{G}},0,\zeta}\,.$$

For $\zeta \in \mathcal{Z}^\vee(k)$, also recall the category $\mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta} := \mathrm{QCoh}(X_\zeta)$ of mod p Langlands parameters with determinant $\omega\zeta$ from 8.2.5; it induces the functor

$$L_{\zeta*} : \mathrm{SP}_{\widehat{\mathbf{G}},0,\zeta} \longrightarrow \mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta}$$

push-forward along the k-morphism $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \to X_\zeta$ from 8.3.9.

Finally recall that for $\zeta \in \mathcal{Z}^\vee(k)$, the functor of $I^{(1)}$-invariants $(\cdot)^{I^{(1)}} : \mathrm{Mod}^{\mathrm{sm}}(k[G]) \to \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ induces a functor

$$(\cdot)^{I^{(1)}}_\zeta : \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \to \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}),$$

by 8.3.5.

8.6.2. Definition. *Let* $\zeta \in \mathcal{Z}^\vee(k)$. *The* mod p ζ-Langlands parametrization functor *is the functor*

$$\mathrm{L}_\zeta\mathrm{P}_\zeta := L_{\zeta*} \circ P_\zeta : \quad \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \longrightarrow \mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta}\,.$$

Identifying ζ with a central character of G, the functor $\mathrm{L}_\zeta\mathrm{P}_\zeta$ *extends to the category* $\mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G])$ *by precomposing with the functor* $(\cdot)^{I^{(1)}}_\zeta : \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \to \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$*:*

$$\mathrm{L}_\zeta\mathrm{P}_\zeta \circ (\cdot)^{I^{(1)}}_\zeta : \quad \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \longrightarrow \mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta}\,.$$

8.6.3. Theorem. *Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in \mathcal{Z}^\vee(k)$ under the identification $\mathcal{Z}(G)^\vee \cong \mathcal{Z}^\vee(k)$ from 8.3.4.*

The mod p ζ-Langlands parametrization functor $\mathrm{L}_\zeta\mathrm{P}_\zeta$ *interpolates the Langlands parametrization of the blocks of the category* $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$*, cf. 8.3.7 : for all $x \in X_\zeta(k)$ and for all $\pi \in b_{[\rho_x]}$,*

$$\mathrm{L}_\zeta\mathrm{P}_\zeta(\pi^{I^{(1)}}) = \begin{cases} i_{x*}(\pi^{I^{(1)}}) & \text{if } x \text{ is not an exceptional point in the odd case} \\ i_{x*}(\pi^{I^{(1)}})^{\oplus 2} & \text{otherwise} \end{cases} \in \mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta}$$

where $i_x : \mathrm{Spec}(k) \to X_\zeta$ is the k-point x.

Proof. By definition of a block of a category as a certain equivalence class of simple objects [Pas13], if $\pi \in b_{[\rho_x]}$ then in particular π is simple. Then $\pi^{I^{(1)}}$ is simple too, and hence has a central character. Therefore $P_\zeta(\pi^{I^{(1)}})$ is the underlying k-vector space of $\pi^{I^{(1)}}$ supported at the k-point $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ corresponding to its central character under the isomorphism $\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p}$, which lies on some connected component γ. Suppose $\dim_k(\pi^{I^{(1)}}) = 2$. Then $\pi^{I^{(1)}}$ is isomorphic to the simple standard module of $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$ with central character v, i.e. to $\mathrm{ASph}^\gamma(v)$, and hence $L_\zeta(v) = x$ by definition of the map of sets $L_\zeta(k)$. Suppose $\dim_k(\pi^{I^{(1)}}) = 1$. Then $\pi^{I^{(1)}}$ is one of the two antispherical characters of $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$ whose restriction to the center $Z(\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p})$ is equal to v, i.e. it is one of the simple constituents of $(\mathrm{ASph}^\gamma(v))^{\mathrm{ss}}$, and hence again $L_\zeta(v) = x$ by definition of the map of sets $L_\zeta(k)$. Now if x is not an exceptional point in an odd case, then L_ζ is an open immersion at v, and otherwise it has ramification index 2 at v. The theorem follows. □

9 Appendix: Virtual quotients for actions of semigroups

A *semigroup* is a set equipped with an internal law which is *associative*. If the law admits a (necessary unique) identity element then the semigroup is a *monoid*, and if furthermore every element is invertible then it is a group. These set theoretic notions induce corresponding notions for set-valued functors on a given category, in particular on the category of schemes. Using the Yoneda embedding, we get the notions of a semigroup scheme, monoid scheme and group scheme (over a fixed base scheme).

In this appendix, we consider the following setup. We fix a base scheme S and let (Sch/S) be the category of schemes over S. We fix a semigroup scheme G over S and a subsemigroup scheme $B \subset G$ (i.e. a subsemigroup functor which is representable by a scheme). We denote by $\alpha_{G,G} : G \times G \to G$ the law of G (resp. $\alpha_{B,B} : B \times B \to B$ the law of B). If G is a monoid we denote by e_G its identity section and then we suppose that $B \subset G$ is a submonoid: $e_B := e_G \in B$. If G is a group then we suppose that $B \subset G$ is a subgroup, and denote by $i_G : G \to G$ the inverse map of G (resp. $i_B : B \to B$ the inverse map of B).

9.1 Virtual quotients

Recall that an *S-space in groupoids* is a pair of sheaves of sets (R, U) on (Sch/S) with five morphisms s, t, e, c, i (source, target, identity, composition, inversion)

$$R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{\ e\ } R \qquad\qquad R \times_{s,U,t} R \xrightarrow{\ c\ } R \qquad\qquad R \xrightarrow{\ i\ } R$$

satifying certain natural compatibilities. Given a groupoid space, one defines the fibered groupoid over (Sch/S) to be the category $[R, U]'$ over (Sch/S) whose objects resp. morphisms over a scheme T are the elements of the set $U(T)$ resp. $R(T)$. Given a morphisms $f : T' \to T$ in (Sch/S) one defines the pull-back functor $f^* : [R, U]'(T) \to [R, U]'(T')$ using the maps $U(T) \to U(T')$ and $R(T) \to R(T')$. An equivalent terminology for 'fibered groupoid over (Sch/S)' is 'prestack over S', and given a Grothendieck topology on (Sch/S), one can associate a stack to a prestack; in the case of the prestack $[R, U]'$, the associated stack is denoted by $[R, U]$.

If X is a scheme equipped with a (right) action of a group scheme B, one takes $U = X$, $R = X \times B$, and let s be the action of the group and $t = p_1$ be the first projection. Then c is the product in the group and e, i are defined by means of the identity and the inverse of B. By definition, the quotient stack $[X/B]$ is the stack $[X \times B, X]$. For all of this, we refer to [LM00, (2.4.3)].

In the context of semigroups, we adopt the same point of view, however, the maps e and i are missing. This leads to the following definition.

9.1.1. Definition. *The* virtual quotient *associated to the inclusion of semigroups $B \subset G$ is the* *semigroupoid consisting of the source and target maps* $\alpha_{G,B} := \alpha_{G,G}|_{G\times B}$ *and first projection* p_1

$$G \times B \underset{p_1}{\overset{\alpha_{G,B}}{\rightrightarrows}} G$$

together with the composition

$$\begin{array}{rcl} c : (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) & \longrightarrow & G \times B \\ \big((g, b), (gb, b')\big) & \longmapsto & (g, bb'). \end{array}$$

We denote it by G/B.

9.1.2. Saying that these data define a semigroupoid means that they satisfy the following axioms:

(0) $\alpha_{G,B} \circ c = \alpha_{G,B} \circ p_2$ and $p_1 \circ c = p_1 \circ p_1$ where we have denoted the two projections $(G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \to G \times B$ by p_1, p_2 ;

(i) (associativity) the two composed maps

$$(G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \underset{\mathrm{id}_{G\times B} \times c}{\overset{c\times \mathrm{id}_{G\times B}}{\rightrightarrows}} (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \xrightarrow{\ c\ } (G \times B)$$

are equal.

9.1.3. If $B \subset G$ is an inclusion of monoids, then G/B becomes a monoidoid thanks to the additional datum of the identity map

$$\varepsilon : G \xrightarrow{\mathrm{id}_G \times e_B} G \times B.$$

This means that the following additional axioms are satisfied:

(0)' $\alpha_{G,B} \circ (\mathrm{id}_G \times e_B) = p_1 \circ (\mathrm{id}_G \times e_B) = \mathrm{id}_G$;

(ii) (identity element) the two composed maps

$$G \times B = (G \times B)_{\alpha_{G,B}} \times_G G = G \times_{G\ p_1} (G \times B) \underset{\mathrm{id}_{G\times B} \times \varepsilon}{\overset{\varepsilon\times \mathrm{id}_{G\times B}}{\rightrightarrows}} (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \xrightarrow{\ c\ } (G \times B)$$

are equal.

9.1.4. If $B \subset G$ is an inclusion of groups, then G/B becomes a groupoid thanks to the additional datum of the inverse map

$$i : G \times B \xrightarrow{\alpha_{G,B}\times i_B} G \times B.$$

This means that the following additional axioms are satisfied:

(0)" $\alpha_{G,B} \circ (\alpha_{G,B} \times i_B) = p_1$ and $p_1 \circ (\alpha_{G,B} \times i_B) = \alpha_{G,B}$;

(iii) (inverse) the two diagrams

$$\begin{array}{ccc} G \times B & \xrightarrow{(\alpha_{G,B}\times i_B)\times \mathrm{id}_{G\times B}} & (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \\ \downarrow \alpha_{G,B} & & \downarrow c \\ G & \xrightarrow{\mathrm{id}_G \times e_B} & G \times B \end{array}$$

$$\begin{array}{ccc} G \times B & \xrightarrow{\mathrm{id}_{G\times B} \times(\alpha_{G,B}\times i_B)} & (G \times B)_{\alpha_{G,B}} \times_{G\ p_1} (G \times B) \\ \downarrow p_1 & & \downarrow c \\ G & \xrightarrow{\mathrm{id}_G \times e_B} & G \times B \end{array}$$

are commutative.

9.2 Categories on the virtual quotient

Let $\mathcal{C}$ be a category fibered over (Sch/S).

9.2.1. Definition. *The* (fiber of the) category $\mathcal{C}$ over G/B *is the category* $\mathcal{C}(G/B)$ *defined by:*

(Obj) an object of $\mathcal{C}(G/B)$ *is a couple* $(\mathcal{F}, \phi_B)$ *where* $\mathcal{F}$ *is an object of* $\mathcal{C}(G)$ *and*

$$\phi_B : p_1^*\mathcal{F} \longrightarrow \alpha_{G,B}^*\mathcal{F}$$

is a morphism in $\mathcal{C}(G \times B)$ *satisfying the following cocycle condition: considering the maps*

$$G \times B \times B \longrightarrow G$$

$$p_1 = p_1 \circ (\mathrm{id}_G \times \alpha_{B,B}) = p_1 \circ p_{12}$$
$$q := \alpha_{G,B} \circ (\mathrm{id}_G \times \alpha_{B,B}) = \alpha_{G,B} \circ (\alpha_{G,B} \times \mathrm{id}_B)$$
$$r := p_1 \circ (\alpha_{G,B} \times \mathrm{id}_B) = \alpha_{G,B} \circ p_{12},$$

the diagram in $\mathcal{C}(G \times B \times B)$

$$\begin{array}{ccc} p_1^*\mathcal{F} & \xrightarrow{(\mathrm{id}_G \times \alpha_{B,B})^*\phi_B} & q^*(\mathcal{F}, \phi_B) \\ & {\scriptstyle p_{12}^*\phi_B} \searrow \quad \nearrow {\scriptstyle (\alpha_{G,B} \times \mathrm{id}_B)^*\phi_B} & \\ & r^*\mathcal{F} & \end{array}$$

is commutative ;

(Hom) a morphism $(\mathcal{F}^1, \phi_B^1) \to (\mathcal{F}^2, \phi_B^2)$ *in* $\mathcal{C}(G/B)$ *is a morphism* $\varphi : \mathcal{F}^1 \to \mathcal{F}^2$ *in* $\mathcal{C}(G)$ *such that the diagram in* $\mathcal{C}(G \times B)$

$$\begin{array}{ccc} p_1^*\mathcal{F}^1 & \xrightarrow{p_1^*\varphi} & p_1^*\mathcal{F}^2 \\ {\scriptstyle \phi_B^1}\downarrow & & \downarrow{\scriptstyle \phi_B^2} \\ \alpha_{G,B}^*\mathcal{F}^1 & \xrightarrow{\alpha_{G,B}^*\varphi} & \alpha_{G,B}^*\mathcal{F}^2 \end{array}$$

is commutative.

9.2.2. If $B \subset G$ is an inclusion of monoids, then an object of $\mathcal{C}(G/B)$ is a couple $(\mathcal{F}, \phi_B)$ as in 9.2.1 which is required to satisfy the additional condition that the morphism in $\mathcal{C}(G)$

$$\varepsilon^*(\phi_B) := (\mathrm{id}_G \times e_B)^*\phi_B : \mathcal{F} \longrightarrow \mathcal{F}$$

is equal to the identity. Homomorphisms in $\mathcal{C}(G/B)$ remain the same as in the case of semigroups.

9.2.3. If $B \subset G$ is an inclusion of groups, then given an object $(\mathcal{F}, \phi_B)$ of $\mathcal{C}(G/B)$ as in 9.2.2, the morphism ϕ_B in $\mathcal{C}(G \times B)$ is automatically an isomorphism, whose inverse is equal to $i^*(\phi_B) := (\alpha_{G,B} \times i_B)^*(\phi_B)$. The category $\mathcal{C}(G/B)$ coincides therefore with the category attached to the underlying inclusion of monoids.

9.3 Equivariant categories on the virtual quotient

9.3.1. By taking the direct product $\mathrm{id}_G \times \bullet$ of all the maps appearing in the definition 9.1.1 of the semigroupoid G/B, we get a semigroupoid $G \times G/B$, whose source and target maps are

$$(G \times G) \times B \underset{p_1}{\overset{\alpha_{G \times G,B}}{\rightrightarrows}} G \times G.$$

Then given $\mathcal{C}$ we define the category $\mathcal{C}(G \times G/B)$ exactly as we defined the category $\mathcal{C}(G/B)$, but now using the semigroupoid $G \times G/B$ instead of G/B. Applying once more $\mathrm{id}_G \times \bullet$, we also get the semigroupoid $G \times G \times G/B$ with source and target maps

$$(G \times G \times G) \times B \underset{p_1}{\overset{\alpha_{G\times G\times G,B}}{\rightrightarrows}} G \times G \times G,$$

and then the category $\mathcal{C}(G \times G \times G/B)$.

9.3.2. A morphism $f : G \times G \to G$ is *B-equivariant* if the diagram

$$\begin{array}{ccc} (G \times G) \times B & \xrightarrow{f \times \mathrm{id}_B} & G \times B \\ {\scriptstyle \alpha_{G\times G,B}} \downarrow & & \downarrow {\scriptstyle \alpha_{G,B}} \\ G \times G & \xrightarrow{\quad f \quad} & G \end{array}$$

commutes. Then there is a well-defined *pull-back functor*

$$f^* : \mathcal{C}(G/B) \longrightarrow \mathcal{C}(G \times G/B),$$

given by the rules $(\mathcal{F}, \phi_B) \mapsto (f^*\mathcal{F}, (f \times \mathrm{id}_B)^*\phi_B)$ and $\varphi \mapsto f^*\varphi$. One defines similarly the B-equivariant morphisms $f : G \times G \times G \to G \times G$ and the associated pull-back functors $f^* : \mathcal{C}(G \times G/B) \to \mathcal{C}(G \times G \times G/B)$.

9.3.3. With this preparation, we will now be able to define the G-equivariant version of the category $\mathcal{C}(G/B)$. It relies on the semigroupoid $G\backslash G$ consisting of the source and target maps

$$G \times G \underset{p_2}{\overset{\alpha_{G,G}}{\rightrightarrows}} G$$

together with the composition

$$\begin{array}{rcl} (G \times G)_{\alpha_{G,G}} \times_{G\ p_2} (G \times G) & \longrightarrow & G \times G \\ \big((g_1, g_0), (g_2, g_1g_0)\big) & \longmapsto & (g_2g_1, g_0). \end{array}$$

Note that the source and target maps $\alpha_{G,G}$ and p_2 are B-equivariant.

9.3.4. Definition. *The* (G-)equivariant (fiber of the) category $\mathcal{C}$ over G/B *is the category* $\mathcal{C}^G(G/B)$ *defined by:*

(Obj) an object of $\mathcal{C}^G(G/B)$ is a triple $(\mathcal{F}, \phi_B, {}_G\phi)$ where $(\mathcal{F}, \phi_B)$ is an object of $\mathcal{C}(G/B)$ and

$${}_G\phi : p_2^*(\mathcal{F}, \phi_B) \longrightarrow \alpha_{G,G}^*(\mathcal{F}, \phi_B)$$

is an isomorphism *in $\mathcal{C}(G \times G/B)$ satisfying the following cocycle condition: considering the B-equivariant maps*

$$G \times G \times G \longrightarrow G$$

$$p_3$$

$$q := \alpha_{G,G} \circ (\alpha_{G,G} \times \mathrm{id}_G) = \alpha_{G,G} \circ (\mathrm{id}_G \times \alpha_{G,G})$$

$$r := p_2 \circ (\mathrm{id}_G \times \alpha_{G,G}) = \alpha_{G,G} \circ p_{23},$$

and the B-equivariant maps $\alpha_{G,G} \times \mathrm{id}_G$, p_{23}, $\mathrm{id}_G \times \alpha_{G,G}$ from $G \times G \times G$ to $G \times G$, the diagram in $\mathcal{C}(G \times G \times G/B)$

$$\begin{array}{ccc} p_3^*(\mathcal{F}, \phi_B) & \xrightarrow{\quad (\alpha_{G,G} \times \mathrm{id}_G)^* {}_G\phi \quad} & q^*(\mathcal{F}, \phi_B) \\ & {\scriptstyle p_{23}^* {}_G\phi} \searrow \qquad \nearrow {\scriptstyle (\mathrm{id}_G \times \alpha_{G,G})^* {}_G\phi} & \\ & r^*(\mathcal{F}, \phi_B) & \end{array}$$

is commutative ;

(Hom) a morphism $(\mathcal{F}^1, \phi_B^1, {}_G\phi^1) \to (\mathcal{F}^2, \phi_B^2, {}_G\phi^2)$ *in* $\mathcal{C}^G(G/B)$ *is a morphism* $\varphi : (\mathcal{F}^1, \phi_B^1) \to (\mathcal{F}^2, \phi_B^2)$ *in* $\mathcal{C}(G/B)$ *such that the diagram in* $\mathcal{C}(G \times G/B)$

$$\begin{array}{ccc} p_2^*(\mathcal{F}^1, \phi_B^1) & \xrightarrow{p_2^*\varphi} & p_2^*(\mathcal{F}^2, \phi_B^2) \\ {\scriptstyle {}_G\phi^1}\downarrow & & \downarrow{\scriptstyle {}_G\phi^2} \\ \alpha_{G,G}^*(\mathcal{F}^1, \phi_B^1) & \xrightarrow{\alpha_{G,G}^*\varphi} & \alpha_{G,G}^*(\mathcal{F}^2, \phi_B^2) \end{array}$$

is commutative (which by definition means that the diagram in $\mathcal{C}(G \times G)$

$$\begin{array}{ccc} p_2^*\mathcal{F}^1 & \xrightarrow{p_2^*\varphi} & p_2^*\mathcal{F}^2 \\ {\scriptstyle {}_G\phi^1}\downarrow & & \downarrow{\scriptstyle {}_G\phi^2} \\ \alpha_{G,G}^*\mathcal{F}^1 & \xrightarrow{\alpha_{G,G}^*\varphi} & \alpha_{G,G}^*\mathcal{F}^2 \end{array}$$

is commutative).

9.3.5. If $B \subset G$ is an inclusion of monoids, then an object of $\mathcal{C}^G(G/B)$ is a triple $(\mathcal{F}, \phi_B, {}_G\phi)$ as in 9.3.4, where now the object $(\mathcal{F}, \phi_B)$ of $\mathcal{C}(G/B)$ is as in 9.2.2, which is required to satisfy the additional condition that the morphism in $\mathcal{C}(G)$

$$(e_G \times \mathrm{id}_G)^* {}_G\phi : \mathcal{F} \longrightarrow \mathcal{F}$$

is equal to the identity. Homomorphisms in $\mathcal{C}^G(G/B)$ remain the same as in the case of semigroups.

9.3.6. As in the non-equivariant setting, cf. 9.2.3, if $B \subset G$ is an inclusion of groups, then the category $\mathcal{C}^G(G/B)$ coincides with the category attached to the underlying inclusion of monoids.

9.4 Induction of representations

From now on, the fixed base scheme is a field k and $\mathcal{C}$ is the fibered category of vector bundles.

9.4.1. Definition. *The category* $\mathrm{Rep}(B)$ *of* right representations of the k-semigroup scheme B on finite dimensional k-vector spaces *is defined as follows:*

(Obj) an object of $\mathrm{Rep}(B)$ *is a couple* $(M, \alpha_{M,B})$ *where* M *is a finite dimensional* k*-vector space and*

$$\alpha_{M,B} : M \times B \longrightarrow M$$

is a morphism of k*-schemes such that*

$$\forall (m, b_1, b_2) \in M \times B \times B, \quad \alpha_{M,B}(\alpha_{M,B}(m, b_1), b_2) = \alpha_{M,B}(m, \alpha_{B,B}(b_1, b_2)).$$

(Hom) a morphism $(M^1, \alpha_{M^1,B}) \to (M^2, \alpha_{M^2,B})$ *in* $\mathrm{Rep}(B)$ *is a* k*-linear map* $f : M^1 \to M^2$ *such that*

$$\forall (m, b) \in M^1 \times B, \quad f(\alpha_{M_1,B}(m, b)) = \alpha_{M_2,B}(f(m), b).$$

9.4.2. We define an *induction functor*

$$\mathcal{I}nd_B^G : \mathrm{Rep}(B) \longrightarrow \mathcal{C}^G(G/B)$$

as follows. Let $(M, \alpha_{M,B})$ be an object of $\mathrm{Rep}(B)$. Set $\mathcal{F} := G \times M \in \mathcal{C}(G)$. There are canonical identifications $p_1^*\mathcal{F} = G \times M \times B$ and $\alpha_{G,B}^*\mathcal{F} = G \times B \times M$ in $\mathcal{C}(G \times B)$. Set

$$\begin{array}{rcl} \phi_B : G \times M \times B & \longrightarrow & G \times B \times M \\ (g, m, b) & \mapsto & (g, b, \alpha_{M,B}(m, b)). \end{array}$$

Then $(\mathcal{F}, \phi_B)$ is an object of $\mathcal{C}(G/B)$. Next, there are canonical identifications $p_2^*\mathcal{F} = G \times G \times M$ and $\alpha_{G,G}^*\mathcal{F} = G \times G \times M$ in $\mathcal{C}(G \times G)$. Set

$$_G\phi := \mathrm{id}_{G\times G\times M}\,.$$

Then $_G\phi$ is an isomorphism $p_2^*(\mathcal{F}, \phi_B) \to \alpha_{G,G}^*(\mathcal{F}, \phi_B)$ in $\mathcal{C}(G \times G/B)$, and $((\mathcal{F}, \phi_B), {_G\phi})$ is an object of $\mathcal{C}^G(G/B)$.

Let $f : (M^1, \alpha_{M^1,B}) \to (M^2, \alpha_{M^2,B})$ be a morphism in $\mathrm{Rep}(B)$. Then

$$\mathrm{id}_G \times f : \mathcal{F}^1 = G \times M^1 \longrightarrow \mathcal{F}^2 = G \times M^2$$

defines a morphism $\varphi : ((\mathcal{F}^1, \phi_B^1), {_G\phi^1}) \to ((\mathcal{F}^2, \phi_B^2), {_G\phi^2})$ in $\mathcal{C}^G(G/B)$.

These assignments are functorial.

9.4.3. Lemma. *The functor $\mathcal{I}nd_B^G$ is faithful. Suppose moreover that the k-semigroup scheme G has the following property:*

There exists a k-point of G which belongs to all the $G(\overline{k})$-left cosets in $G(\overline{k})$, and the underlying k-scheme of G is locally of finite type.

Then the functor $\mathcal{I}nd_B^G$ is fully faithful.

Proof. Faithfulness is obvious. Now let $\varphi : \mathcal{I}nd_B^G(M^1) = G \times M^1 \to \mathcal{I}nd_B^G(M^2) = G \times M^2$. The compatibilty of φ with $_G\phi^i$, $i = 1, 2$, reads as

$$\mathrm{id}_G \times \varphi = \alpha_{G,G}^*\varphi : G \times G \times M^1 \longrightarrow G \times G \times M^2.$$

For $g \in G(\overline{k})$, denote by $\phi_g : M^1_{\overline{k}} \to M^2_{\overline{k}}$ the fiber of φ over g. Taking the fiber at (g', g) in the above equality implies that $\varphi_g = \varphi_{g'g}$ for all $g, g' \in G(\overline{k})$, i.e. φ_g depends only on the left coset $G(\overline{k})g$, hence is independent of g if all the left cosets share a common point. Assuming that such a point exists and is defined over k, let $f : M^1 \to M^2$ be the corresponding k-linear endomorphism. Then $\varphi - \mathrm{id}_G \times f$ is a linear morphism between two vector bundles on G, which vanishes on each geometric fiber. Then it follows from Nakayama's Lemma that $\varphi - \mathrm{id}_G \times f = 0$ on G, at least if the latter is locally of finite type over k. ☐

9.4.4. Definition. *When the functor $\mathcal{I}nd_B^G$ is fully faithful, we call its essential image the* category of induced vector bundles on G/B*, and denote it by $\mathcal{C}^G_{\mathcal{I}nd}(G/B)$:*

$$\mathcal{I}nd_B^G : \mathrm{Rep}(B) \xrightarrow{\sim} \mathcal{C}^G_{\mathcal{I}nd}(G/B) \subset \mathcal{C}^G(G/B).$$

9.4.5. If $B \subset G$ is an inclusion of monoids, then an object of $\mathrm{Rep}(B)$ is a couple $(M, \alpha_{M,B})$ as in 9.4.1 which is required to satisfy the additional condition that the k-morphism

$$\alpha_{M,B} \circ (\mathrm{id}_M \times e_B) : M \longrightarrow M$$

is equal to the identity. Homomorphisms in $\mathrm{Rep}(B)$ remain the same as in the case of semigroups.

In particular, comparing with 9.3.5, the same assignments as in the case of semigroups define an induction functor

$$\mathcal{I}nd_B^G : \mathrm{Rep}(B) \longrightarrow \mathcal{C}^G(G/B).$$

Now set $e := e_B = e_G \in B(k) \subset G(k)$, the identity element. We define a functor *fiber at e*

$$\mathrm{Fib}_e : \mathcal{C}^G(G/B) \longrightarrow \mathrm{Rep}(B)$$

as follows. Let $(\mathcal{F}, \phi_B, {_G\phi})$ be an object of $\mathcal{C}^G(G/B)$. Set $M := \mathcal{F}|_e$, a finite dimensional k-vector space. There are canonical identifications $(p_1^*\mathcal{F})|_{e\times B} = M \times B$, $(\alpha_{G,B}^*\mathcal{F})|_{e\times B} = (\alpha_{G,G}^*\mathcal{F})|_{B\times e} = \mathcal{F}|_B$ and $(p_2^*\mathcal{F})|_{B\times e} = B \times M$. Set

$$\alpha_{M,B} : M \times B \xrightarrow{\phi_B|_{e\times B}} \mathcal{F}|_B \xleftarrow[\sim]{_G\phi|_{B\times e}} B \times M \xrightarrow{p_2} M.$$

Then $(M, \alpha_{M,B})$ is an object of $\mathrm{Rep}(B)$.

Let $\varphi : (\mathcal{F}^1, \phi^1_B, {}_G\phi^1) \to (\mathcal{F}^2, \phi^2_B, {}_G\phi^2)$ be a morphism in $\mathcal{C}^G(G/B)$. Then

$$f = \varphi_e : \mathcal{F}^1|_e = M^1 \longrightarrow \mathcal{F}^2|_e = M^2$$

defines a morphism $(M^1, \alpha_{M^1,B}) \to (M^2, \alpha_{M^2,B})$ in $\mathrm{Rep}(B)$.

These assignments are functorial.

9.4.6. Proposition. *For an inclusion of k-monoid schemes $B \subset G$ with unit e, the functors $\mathcal{I}nd^G_B$ and Fib_e are equivalences of categories, which are quasi-inverse one to the other.*

Proof. Left to the reader. □

9.4.7. Analogous to the property 9.3.6 for equivariant vector bundles, we have that if $B \subset G$ is an inclusion of groups, then given an object $(M, \alpha_{M,B})$ of $\mathrm{Rep}(B)$, the right B-action on M defined by $\alpha_{M,B}$ factors automatically through the k-group scheme opposite to the one of k-linear automorphisms of M, the inverse of $\alpha_{M,B}(\bullet, b)$ being equal to $\alpha_{M,B}(\bullet, i_B(b))$ for all $b \in B$. The category $\mathrm{Rep}(B)$ coincides therefore with the category attached to the underlying monoid of B.

In particular, we have the functors $\mathcal{I}nd^G_B$ and Fib_e attached to the underlying inclusion of monoids $B \subset G$, for which Proposition 9.4.6 holds.

9.5 Grothendieck rings of equivariant vector bundles

9.5.1. For a k-semigroup scheme B, the category $\mathrm{Rep}(B)$ is abelian k-linear symmetric monoidal with unit. Hence, for an inclusion of k-semigroup schemes $B \subset G$ such that the functor $\mathcal{I}nd^G_B$ is fully faithful, the essential image $\mathcal{C}^G_{\mathcal{I}nd}(G/B)$ has the same structure. In particular, it is an abelian category whose Grothendieck group $K^G_{\mathcal{I}nd}(G/B)$ is a commutative ring, which is isomorphic to the ring $R(B)$ of right representations of the k-semigroup scheme B on finite dimensional k-vector spaces:

$$\mathcal{I}nd^G_B : R(B) \xrightarrow{\sim} K^G_{\mathcal{I}nd}(G/B).$$

9.5.2. If $B \subset G$ is an inclusion of monoids, then it follows from 9.4.6 that the category $\mathcal{C}^G(G/B)$ is abelian k-linear symmetric monoidal with unit. In particular, it is an abelian category whose Grothendieck group $K^G(G/B)$ is a commutative ring, which is isomorphic to the ring $R(B)$ of right representations of the k-monoid scheme B on finite dimensional k-vector spaces:

$$\mathcal{I}nd^G_B : R(B) \xrightarrow{\sim} K^G(G/B).$$

9.5.3. If $B \subset G$ is an inclusion of groups, then 9.5.2 applies to the underlying inclusion of monoids.

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