A semisimple mod p Langlands correspondence in families for $GL_2(\mathbb{Q}_p)$

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February 28, 2023

Abstract

Let $p \geq 5$ and let $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the center of the mod p pro-p-Iwahori Hecke algebra of $\mathbf{GL}_2(\mathbb{Q}_p)$. Let X be the so-called Emerton-Gee curve parametrizing 2-dimensional mod p semi-simple representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. We construct a natural quotient morphism of schemes $\operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \to X$. We then show that the correspondence between the specialization $\mathcal{M}^{(1)}_z$ of the spherical $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $\mathcal{M}^{(1)}$ from [PS] in closed points $z \in \operatorname{Spec} Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ and the Galois representation $\rho_{x(z)}$ *is* the semi-simple mod p local Langlands correspondence for the group $\mathbf{GL}_2(\mathbb{Q}_p)$.

Contents

1 Introduction

Let F be a non-archimedean local field with residue field $\mathbb{F}_q$ of characteristic p. The conjectured mod p local Langlands correspondence should relate the mod p representation theory of the absolute Galois group of F to the mod p representation theory of the group of F-points of reductive groups defined over F (like $\mathbf{GL}_n(F)$). A mod p local Langlands correspondence (in fact even for p-adic coefficients) has been established for the group $G = \mathbf{GL}_2(\mathbb{Q}_p)$ by Colmez-Dospinescu-Paškūnas [CDP14], building on work of Breuil, Colmez, Emerton, Kisin, Paškūnas and many others (see [Be11] for an overview).

The aforementioned mod p Langlands correspondence for $G = \mathbf{GL}_2(\mathbb{Q}_p)$ is in fact realized through a *functor* defined on a suitable category of smooth representations. In [H16], the author speculates whether, similar to recent attempts of geometrization in the ℓ-adic setting [Hel20, FS21, Z20], also in the mod p setting, the correct formulation of the Langlands correspondence, at least for the group $\mathbf{GL}_n(F)$, might be of geometric nature. He speculates further if there could be a derived equivalence of a certain derived category of modules over $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$ (or a certain DGA version thereof) and a suitable derived category of quasi-coherent sheaves on $\mathcal{X}_{n,\mathrm{red}}$. Here $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$ is the pro-p-Iwahori Hecke algebra of $\mathbf{GL}_n(F)$ with coefficients in an algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$, and $\mathcal{X}_{n,\mathrm{red}}$ is Emerton-Gee's moduli stack 'parametrizing n-dimensional mod p representations of $\mathrm{Gal}(\overline{F}/F)$' [EG19].

In this note, we propose such a geometric formulation of the mod p correspondence for the group $G = \mathbf{GL}_2(\mathbb{Q}_p)$ (for $p \geq 5$), for *semisimple* Hecke modules and Galois representations. Restricting to the semisimple situation simplifies the ultimate goal considerably: on the Hecke side, it allows to work in the non-derived setting, and on the Galois side, it allows to replace the Emerton-Gee stack $\mathcal{X}_{2,\mathrm{red}}$ by an *explicit scheme.*

To be more precise, let $G := \mathbf{GL}_2(F)$ and let $Z(G)$ be its center. Let $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the center of the algebra $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$. In [PS, 7.4.1], we constructed the mod p spherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$. This is a distinguished $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-action on a maximal commutative subring $\mathcal{A}^{(1)}_{\overline{\mathbb{F}}_p}$ of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$, which is a mod p analogue (plus extension to the pro-p Iwahori level) of the classical (anti)spherical module appearing in complex Kazhdan-Lusztig theory [KL87, 3.9]. The quasi-coherent module (associated to) $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$, when specialized at closed points of $\mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ gives rise to a parametrization of *all* irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules [PS, 7.4.9/7.4.15].

We let from now on $F = \mathbb{Q}_p$ with $p \geq 5$. We may then consider the so-called Emerton-Gee moduli curve X, cf. [Em19]. Its connected components X_ζ are indexed by the central characters $\zeta : Z(G) \to \overline{\mathbb{F}}_p^\times$ and parametrize isomorphism classes of two-dimensional semisimple continuous Galois representations over $\overline{\mathbb{F}}_p$ with determinant $\omega\zeta$:

$$X_\zeta(\overline{\mathbb{F}}_p) \cong \left\{\text{semisimple continuous } \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(\overline{\mathbb{F}}_p) \ \text{ with } \det\rho = \omega\zeta\right\}/\sim .$$

Here ω is the mod p cyclotomic character. The curve X_ζ is expected to be the underlying scheme of a ringed moduli space related to the stack of étale (φ,Γ)-modules $\mathcal{X}_2^{\det=\omega\zeta}$ appearing in [EG19] (see also [CEGS19]).

Let $\rho \mapsto \pi(\rho)$ be the semi-simple mod p local Langlands correspondence for the group $G = \mathbf{GL}_2(\mathbb{Q}_p)$, cf. [Be11]. Let $I^{(1)} \subset G$ be the standard pro-p Iwahori subgroup. By work of Ollivier [O09], the functor of $I^{(1)}$-invariants $\pi \mapsto \pi^{I^{(1)}}$ an equivalence of categories between smooth mod p representations of G and $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules, so that the Langlands correspondence may be viewed as a correspondence between Hecke modules and Galois representations. Our main result is the following (cf. Theorem 4.9).

Theorem. *Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. There exists a quotient morphism of $\overline{\mathbb{F}}_p$-schemes*

$$\mathscr{L} : \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \longrightarrow X,$$

with the following property: given a closed point $z \in \mathrm{Spec}\, Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$, the correspondence between the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module $\mathcal{M}^{(1)}_z$, equal to the specialization of $\mathcal{M}^{(1)}$ in the central character z, and the Galois representation $\rho_{x(z)}$, is the semi-simple mod p local Langlands correspondence.

We even show a bit more, namely that the quasi-coherent $\mathcal{O}_X$-module $\mathscr{L}_*\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ equal to the push-forward of $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ *interpolates* the Langlands correspondence: for all $x \in X(\overline{\mathbb{F}}_p)$, one has an isomorphism of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules

$$\left(\mathscr{L}_*\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{\mathcal{O}_X} k(x)\right)^{\mathrm{ss}} = \left(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})} \mathcal{O}_{\mathscr{L}^{-1}(x)}\right)^{\mathrm{ss}} \cong \pi(\rho_x)^{I^{(1)}}.$$

As a byproduct of our constructions, we also obtain a version in families of Paškūnas' parametrization of the blocks of the category $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(\overline{\mathbb{F}}_p[G])$ of locally admissible smooth G-representations over $\overline{\mathbb{F}}_p$ with central character ζ, cf. [Pas13]. See 7.3 for the precise statement.

The construction of the morphism $\mathscr{L}$ is a consequence of our results from [PS] on the geometry of the *generic* pro-p-Iwahori-Hecke algebra (with coefficients in the ring $\mathbb{Z}[\mathbf{q}]$ where $\mathbf{q}$ is a formal variable) for $\mathbf{GL}_2(F)$, specialized at $\mathbf{q} = q = 0 \in \overline{\mathbb{F}}_p$. Let $\widehat{\mathbf{G}}$ be the Langlands dual group of $\mathbf{GL}_2$ over $\overline{\mathbb{F}}_p$, with maximal torus $\widehat{\mathbf{T}}$. We consider the special fibre at $\mathbf{q} = 0$ of the Vinberg fibration $V_{\widehat{\mathbf{T}}} \overset{\mathbf{q}}{\to} \mathbb{A}^1$ associated to $\widehat{\mathbf{T}} \subset \widehat{\mathbf{G}}$ followed by base change to $\overline{\mathbb{F}}_p$. This yields the $\overline{\mathbb{F}}_p$-semigroup scheme

$$V_{\widehat{\mathbf{T}},0} := \mathrm{SingDiag}_{2\times 2} \times_{\overline{\mathbb{F}}_p} \mathbb{G}_m,$$

where $\mathrm{SingDiag}_{2\times 2}$ represents the semigroup of singular diagonal 2×2-matrices over $\overline{\mathbb{F}}_p$, cf. [PS, 7.1]. Let $\mathbb{T}^\vee$ be the finite abelian group dual to $\mathbb{T} = \mathbf{T}(\mathbb{F}_q)$, and consider the extended semigroup

$$V^{(1)}_{\widehat{\mathbf{T}},0} := \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}},0}.$$

It has a natural diagonal W_0-action. In [PS, 7.2.2] we established the mod p pro-p-Iwahori Satake isomorphism

$$\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p} : Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \xrightarrow{\ \sim\ } \mathcal{O}(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)$$

identifying the center $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ with the ring of regular functions on the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$.

In the first section of the present note, we show that the quotient $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ is fibered over the central characters ζ of $\mathbf{GL}_2(F)$. Any fibre $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ is a naturally ordered union of connected components, which generically[1] are equal to two affine lines $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ intersecting at the origin. In the case $F = \mathbb{Q}_p$, the connected component X_ζ of the Emerton-Gee curve X equals a chain of projective lines intersecting successively at 0 and ∞. In this situation, the quotient morphism $\mathscr{L}$ boils down to the toric construction of the projective line: it identifies the open subset $\mathbb{G}_m$ in the "first" irreducible component $\mathbb{A}^1$ of the connected component $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ with the open subset $\mathbb{G}_m$ in "second" irreducible component $\mathbb{A}^1$ of the "next" connected component $\mathbb{A}^1 \cup_0 \mathbb{A}^1$ via the map $z \mapsto z^{-1}$, thus forming a $\mathbb{P}^1$.

In current works in progress, we are generalizing the construction of the "Langlands" morphism $\mathscr{L}$ from $\mathbf{GL}_2$ to the case of an arbitrary connected reductive split group over $\mathbb{Q}_p$, together with a Lubin-Tate variant of it for F not necessarily equal to $\mathbb{Q}_p$.

Notation. We keep the notation from the introduction. In particular, F denotes a local field with ring of integers o_F and residue field $\mathbb{F}_q$ (we switch to $F = \mathbb{Q}_p$ starting from 3). We also let $k := \overline{\mathbb{F}}_q$.

2 Mod p Satake parameters with fixed central character

2.1. Let $\omega : \mathbb{F}_q^\times \to k^\times$ be induced by the inclusion $\mathbb{F}_q \subset k$. Then $(\mathbb{F}_q^\times)^\vee = \langle \omega \rangle$ is a cyclic group of order $q-1$. An element ω^r defines a non-regular character of $\mathbb{T}$:

$$\omega^r(t_1, t_2) := \omega^r(t_1)\omega^r(t_2)$$

for all $(t_1, t_2) \in \mathbb{T} = \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. Composing with multiplication in $\mathbb{T}^\vee$, we get an action of $(\mathbb{F}_q^\times)^\vee$ on $\mathbb{T}^\vee$, which factors on the quotient set $\mathbb{T}^\vee/W_0$:

$$\mathbb{T}^\vee/W_0 \times (\mathbb{F}_q^\times)^\vee \longrightarrow \mathbb{T}^\vee/W_0,\ (\gamma, \omega^r) \mapsto \gamma\omega^r.$$

If $\gamma \in \mathbb{T}^\vee/W_0$ is regular (non-regular), then $\gamma\omega^r$ is regular (non-regular).

2.2. Restricting characters of $\mathbb{T}$ to the subgroup $\mathbb{F}_q^\times \simeq \{\mathrm{diag}(a,a) : a \in \mathbb{F}_q^\times\}$ induces a homomorphism $\mathbb{T}^\vee \to (\mathbb{F}_q^\times)^\vee$ which factors into a restriction map

$$\mathbb{T}^\vee/W_0 \to (\mathbb{F}_q^\times)^\vee,\ \gamma \mapsto \gamma|_{\mathbb{F}_q^\times}.$$

[1]There occur also connected components equal to $\mathbb{A}^1$ corresponding to *non-regular* components of $Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$.

The relation to the $(\mathbb{F}_q^\times)^\vee$-action on the source $\mathbb{T}^\vee/W_0$ is given by the formula

$$(\gamma\omega^r)|_{\mathbb{F}_q^\times} = \gamma|_{\mathbb{F}_q^\times}\ \omega^{2r}.$$

We describe the fibers of the restriction map $\gamma \mapsto \gamma|_{\mathbb{F}_q^\times}$.

Let $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(\omega^{2r})$ be the fibre at a square element ω^{2r}. By the above formula, the action of ω^{-r} on $\mathbb{T}^\vee/W_0$ induces a bijection with the fibre $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(1)$. The fibre

$$(\cdot)|_{\mathbb{F}_q^\times}^{-1}(1) = \{1\otimes 1\} \coprod \{\omega\otimes\omega^{-1}, \omega^2\otimes\omega^{-2}, ..., \omega^{\frac{q-3}{2}}\otimes\omega^{-\frac{q-3}{2}}\} \coprod \{\omega^{\frac{q-1}{2}}\otimes\omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{q+1}{2}$ and, in the above list, we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. The $\frac{q-3}{2}$ elements in the middle of this list, i.e. the W_0-orbits represented by the characters $\omega^r\otimes\omega^{-r}$ for $r = 1, ..., \frac{q-3}{2}$, are all regular W_0-orbits. The two orbits at the two ends of the list are non-regular orbits (note that $\frac{q-1}{2} \equiv -\frac{q-1}{2} \mod (q-1)$). Since the action of ω^{-r} preserves regular (non-regular) orbits, any fibre at a square element (there are $\frac{q-1}{2}$ such fibres) has the same structure.

On the other hand, let $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(\omega^{2r-1})$ be the fibre at a non-square element ω^{2r-1}. The action of ω^{-r} induces a bijection with the fibre $(\cdot)|_{\mathbb{F}_q^\times}^{-1}(\omega^{-1})$. The fibre

$$(\cdot)|_{\mathbb{F}_q^\times}^{-1}(\omega^{-1}) = \{1\otimes\omega^{-1}, \omega\otimes\omega^{-2}, ..., \omega^{\frac{q-1}{2}-1}\otimes\omega^{-\frac{q-1}{2}}\}$$

has cardinality $\frac{q-1}{2}$ and we have chosen a representative in $\mathbb{T}^\vee$ for each element in the fibre. All elements of the fibre are regular W_0-orbits. Since the action of ω^{-r} preserves regular (non-regular) orbits, any fibre at a non-square element (there are $\frac{q-1}{2}$ such fibres) has the same structure.

Note that $\frac{q-1}{2}(\frac{q+1}{2} + \frac{q-1}{2}) = \frac{q^2-q}{2}$ is the cardinality of the set $\mathbb{T}^\vee/W_0$.

2.3. Recall the commutative k-semigroup scheme

$$V_{\widehat{\mathbf{T}},0}^{(1)} = \mathbb{T}^\vee \times V_{\widehat{\mathbf{T}},0} = \mathbb{T}^\vee \times \mathrm{SingDiag}_{2\times 2} \times \mathbb{G}_m$$

together with its W_0-action, cf. [PS, 6.2.15]: the natural action of W_0 on the factors $\mathbb{T}^\vee$ and $\mathrm{SingDiag}_{2\times 2}$ and the trivial one on $\mathbb{G}_m$. There is a commuting action of the k-group scheme

$$\mathcal{Z}^\vee := (\mathbb{F}_q^\times)^\vee \times \mathbb{G}_m$$

on $V_{\widehat{\mathbf{T}},0}^{(1)}$: the (constant finite diagonalizable) group $(\mathbb{F}_q^\times)^\vee$ acts only on the factor $\mathbb{T}^\vee$ and in the way described in 2.1; an element $z_0 \in \mathbb{G}_m$ acts trivially on $\mathbb{T}^\vee$, by multiplication with the diagonal matrix $\mathrm{diag}(z_0, z_0)$ on $\mathrm{SingDiag}_{2\times 2}$ and by multiplication with the square z_0^2 on $\mathbb{G}_m$. Therefore the quotient $V_{\widehat{\mathbf{T}},0}^{(1)}/W_0$ inherits a $\mathcal{Z}^\vee$-action. Now, according to [PS, 7.4.7], one has the decomposition

$$V_{\widehat{\mathbf{T}},0}^{(1)}/W_0 = \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\mathrm{reg}}} V_{\widehat{\mathbf{T}},0} \coprod_{\gamma\in(\mathbb{T}^\vee/W_0)_{\text{non-reg}}} V_{\widehat{\mathbf{T}},0}/W_0.$$

Then the $(\mathbb{F}_q^\times)^\vee$-action is by permutations on the index set $\mathbb{T}^\vee/W_0$, i.e. on the set of connected components of $V_{\widehat{\mathbf{T}},0}^{(1)}/W_0$; as observed above, it preserves the subsets of regular and non-regular components. The $\mathbb{G}_m$-action on $V_{\widehat{\mathbf{T}},0}^{(1)}/W_0$ preserves each connected component.

2.4. Recall from [PS, 7.4.7] the spherical map

$$\mathrm{Sph} : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}_{\overline{\mathbb{F}}_q}^{(1)}\text{-modules}\}/\sim.$$

The modules in the image of this map are standard modules of length 1 or 2, cf. [PS, 7.4.9] and [PS, 7.4.15].

Let $(\omega^r, z_0) \in \mathcal{Z}^\vee(k)$. Then recall that the standard $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules and their simple constituents may be 'twisted by the character (ω^r, z_0)' : in the regular case, the actions of X, Y, U^2 get multiplied by z_0, z_0, z_0^2 respectively and the component γ gets multiplied by ω^r, cf. [V04, 2.4]; in the non-regular case, the action of U gets multiplied by z_0, the action of S remains unchanged and the component γ gets multiplied by ω^r, cf. [V04, 1.6]. This gives an action of the group of k-points of $\mathcal{Z}^\vee$ on the standard $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules and their simple constituents.

2.5. Lemma. *The map* Sph *is* $\mathcal{Z}^\vee(k)$*-equivariant.*

Proof. Let $(\omega^r, z_0) \in \mathcal{Z}^\vee(k)$. Let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$. Suppose that γ is regular, choose an ordering $\gamma = (\chi, \chi^s)$ on the set γ and standard coordinates. Then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(v)$ is a simple two-dimensional standard $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. [PS, 7.4.9], i.e. of the form $M(x, y, z_2, \chi)$ [V04, 3.2]. Then

$$\mathrm{Sph}(v.(\omega^r, z_0)) \simeq M(z_0 x, z_0 y, z_0^2 z_2, \chi.\omega^r) \simeq \mathrm{Sph}(v).(\omega^r, z_0).$$

Suppose that $\gamma = \{\chi\}$ is non-regular and choose Steinberg coordinates. (a) If $v \in D(2)_\gamma(k)$, then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(2)(v)$ is a simple two-dimensional $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-module, cf. [PS, 7.4.15], i.e. of the form $M(z_1, z_2, \chi)$ [V04, 3.2]. Then

$$\mathrm{Sph}(v.(\omega^r, z_0)) \simeq M(z_0 z_1, z_0^2 z_2, \chi.\omega^r) \simeq \mathrm{Sph}(v).(\omega^r, z_0).$$

(b) If $v \in D(1)_\gamma(k)$, then the semisimplified module $\mathrm{Sph}(v)^{\mathrm{ss}}$ is the direct sum of the two characters in the spherical pair $\mathrm{Sph}^\gamma(1)(v) = \{(0, z_1), (-1, -z_1)\}$ where $z_2 = z_1^2$. Similarly $\mathrm{Sph}(v.(\omega^r, z_0))^{\mathrm{ss}}$ is the direct sum of the characters $\{(0, z_0 z_1), (-1, -z_0 z_1)\}$ in the component $\gamma.\omega^r$, and hence is isomorphic to $\mathrm{Sph}(v)^{\mathrm{ss}}.(\omega^r, z_0)$. $\square$

2.6. The two canonical projections from $V^{(1)}_{\widehat{\mathbf{T}},0}$ to $\mathbb{T}^\vee$ and $\mathbb{G}_m$ respectively induce two projection morphisms

$$\begin{array}{ccccc} & & V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 & & \\ & \swarrow^{\mathrm{pr}_{\mathbb{T}^\vee/W_0}} & & \searrow^{\mathrm{pr}_{\mathbb{G}_m}} & \\ \mathbb{T}^\vee/W_0 & & & & \mathbb{G}_m. \end{array}$$

Then we may compose the map $\mathrm{pr}_{\mathbb{T}^\vee/W_0}$ with the restriction map $(\cdot)|_{\mathbb{F}_q^\times} : \mathbb{T}^\vee/W_0 \to (\mathbb{F}_q^\times)^\vee$, set

$$\theta := \big((\cdot)|_{\mathbb{F}_q^\times} \circ \mathrm{pr}_{\mathbb{T}^\vee/W_0}\big) \times \mathrm{pr}_{\mathbb{G}_m}$$

and view $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ as fibered over the space $\mathcal{Z}^\vee$:

$$\begin{array}{c} V^{(1)}_{\widehat{\mathbf{T}},0}/W_0 \\ \big\downarrow \theta \\ \mathcal{Z}^\vee. \end{array}$$

The relation to the $\mathcal{Z}^\vee$-action on the source $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ is given by the formula

$$\theta(x.(\omega^r, z_0)) = \theta(x)(\omega^{2r}, z_0^2) = \theta(x)(\omega^r, z_0)^2$$

for $x \in V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ and $(\omega^r, z_0) \in \mathcal{Z}^\vee$. This formula follows from the formula in 2.2 and the definition of the $\mathbb{G}_m$-action in 2.3.

2.7. Definition. *Let* $\zeta \in \mathcal{Z}^\vee$*. The* space of mod p Satake parameters with central character ζ *is the* k*-scheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta := \theta^{-1}(\zeta).$$

2.8. Let $\zeta = (\zeta|_{\mathbb{F}_q^\times}, z_2) \in \mathscr{Z}^\vee(k) = (\mathbb{F}_q^\times)^\vee \times k^\times$. Denote by $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{z_2}$ the fibre of $\mathrm{pr}_{\mathbb{G}_m}$ at $z_2 \in k^\times$. Then by [PS, 7.4.7] we have

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta = \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\mathrm{reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,z_2} \quad \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{non-reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,z_2}/W_0.$$

Recall that the choice of standard coordinates x, y identifies

$$V_{\widehat{\mathbf{T}},0,z_2} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1$$

with two affine lines over k, intersecting at the origin, cf. [PS, 7.4.8]. On the other hand, the choice of the Steinberg coordinate z_1 identifies

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0 \simeq \mathbb{A}^1$$

with a single affine line over k, cf. [PS, 7.4.10].

2.9. Lemma. *Let $\zeta, \eta \in \mathscr{Z}^\vee$. The action of η on $V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ induces an isomorphism of k-schemes $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \simeq (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta\eta^2}$.*

Proof. Follows from the last formula in 2.6. □

3 Mod p Langlands parameters with fixed determinant for $F = \mathbb{Q}_p$

3.1. Notation. In this section, we let $F = \mathbb{Q}_p$ with $p \geq 5$. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ and let $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ be the absolute Galois group. We normalize local class field theory $\mathbb{Q}_p^\times \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)^{\mathrm{ab}}$ by sending p to a geometric Frobenius. In this way, we identify the k-valued smooth characters of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and of $\mathbb{Q}_p^\times$. Finally, $\omega : \mathbb{Q}_p^\times \to k^\times$ denotes the extension of the character $\omega : \mathbb{F}_p^\times \to k^\times$ to $\mathbb{Q}_p^\times$ satisfying $\omega(p) = 1$, and $\mathrm{unr}(x) : \mathbb{Q}_p^\times \to k^\times$ denotes the character trivial on $\mathbb{F}_p^\times$ and sending p to x.

3.2. Let $\zeta : \mathbb{Q}_p^\times \to k^\times$ be a character. Recall from [Em19] that the *Emerton-Gee moduli curve with character* ζ is a certain projective curve X_ζ over k whose points parametrize (isomorphism classes of) two-dimensional semisimple continuous Galois representations over k with determinant $\omega\zeta$:

$$X_\zeta(k) \cong \{\text{semisimple continuous } \rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k) \ \text{ with } \det \rho = \omega\zeta\}/\sim.$$

The curve X_ζ is a chain of projective lines over k of length $\frac{p\pm1}{2}$, whose irreducible components intersect at ordinary double points. The sign ±1 is equal to $-\zeta(-1)$. We refer to ζ in the case $-\zeta(-1) = -1$ resp. $-\zeta(-1) = +1$ as an *even character* resp. *odd character*. There is a finite set of closed points $X_\zeta^{\mathrm{irred}} \subset X_\zeta$ which correspond to the classes of irreducible representations. Its open complement $X_\zeta^{\mathrm{red}} = X_\zeta \setminus X_\zeta^{\mathrm{irred}}$ parametrizes the reducible representations (i.e. direct sums of characters). Let $\eta : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to k^\times$ be a character. Since $\det(\rho \otimes \eta) = (\det \rho)\eta^2$, twisting representations with η induces an isomorphism

$$(\cdot) \otimes \eta : X_\zeta \xrightarrow{\sim} X_{\zeta\eta^2}.$$

Hence one is reduced to consider only two 'basic' cases: the even case where $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = 1$ and the odd case where $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$. Indeed, if $\zeta|_{\mathbb{F}_p^\times} = \omega^r$ for some even r, then choosing η with $\eta(p)^2 = \zeta(p)^{-1}$ and $\eta|_{\mathbb{F}_p^\times} = \omega^{-\frac{r}{2}}$, one finds that $(\zeta\eta^2)(p) = 1$ and $(\zeta\eta^2)|_{\mathbb{F}_p^\times} = 1$; if $\zeta|_{\mathbb{F}_p^\times} = \omega^r$ for some odd r, then choosing η with $\eta(p)^2 = \zeta(p)^{-1}$ and $\eta|_{\mathbb{F}_p^\times} = \omega^{-\frac{r+1}{2}}$, one finds that $(\zeta\eta^2)(p) = 1$ and $(\zeta\eta^2)|_{\mathbb{F}_p^\times} = \omega^{-1}$.

3.3. We make explicit some structure elements of X_ζ in the even case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = 1$. Every irreducible component of X_ζ is isomorphic to $\mathbb{P}^1$ and there are $\frac{p-1}{2}$ components. They are labelled by pairs of Serre weights of the following form:

$$\begin{array}{ccc} \mathrm{Sym}^0 & | & \mathrm{Sym}^{p-3}\otimes\det \\ \mathrm{Sym}^2\otimes\det^{-1} & | & \mathrm{Sym}^{p-5}\otimes\det^2 \\ \mathrm{Sym}^4\otimes\det^{-2} & | & \mathrm{Sym}^{p-7}\otimes\det^3 \\ \vdots & \vdots & \vdots \\ \mathrm{Sym}^{p-3}\otimes\det^{\frac{p+1}{2}} & | & \mathrm{Sym}^0\otimes\det^{\frac{p-1}{2}}. \end{array}$$

The component with label " $\mathrm{Sym}^0\,|\,\mathrm{Sym}^{p-3}\otimes\det$ " intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^2\otimes\det^{-1}, \mathrm{Sym}^{p-3}\otimes\det\}$. The component with label " $\mathrm{Sym}^2\otimes\det^{-1}\ |\ \mathrm{Sym}^{p-5}\otimes\det^2$ " intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^4\otimes\det^{-2}, \mathrm{Sym}^{p-5}\otimes\det^2\}$. Continuing in this way, one finds $\frac{p-3}{2}$ points of X_ζ^{irred}, which correspond to the $\frac{p-3}{2}$ double points of the chain X_ζ. There are two more points in X_ζ^{irred}: they are smooth points, each one lies on one of the two 'exterior' components and corresponds there to the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}\}$ and $\{\mathrm{Sym}^0\otimes\det^{\frac{p-1}{2}}, \mathrm{Sym}^{p-1}\otimes\det^{\frac{p-1}{2}}\}$ respectively. So X_ζ^{irred} has cardinality $\frac{p+1}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate z_1 on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $z_1 = \pm 1$ *the four exceptional points* of X_ζ.

3.4. We make explicit some structure elements of X_ζ in the odd case $\zeta(p) = 1$ and $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$. Every irreducible component of X_ζ is isomorphic to $\mathbb{P}^1$ and there are $\frac{p+1}{2}$ components. They are labelled by pairs of Serre weights of the following form:

$$\begin{array}{ccc} \mathrm{Sym}^{p-2} & | & \text{"}\,\mathrm{Sym}^{-1}\,\text{"} \\ \mathrm{Sym}^{p-4}\otimes\det & | & \mathrm{Sym}^1\otimes\det^{-1} \\ \mathrm{Sym}^{p-6}\otimes\det^2 & | & \mathrm{Sym}^3\otimes\det^{-2} \\ \vdots & \vdots & \vdots \\ \mathrm{Sym}^1\otimes\det^{\frac{p-3}{2}} & | & \mathrm{Sym}^{p-4}\otimes\det^{\frac{p+1}{2}} \\ \text{"}\,\mathrm{Sym}^{-1}\otimes\det^{\frac{p-1}{2}}\,\text{"} & | & \mathrm{Sym}^{p-2}\otimes\det^{\frac{p-1}{2}}. \end{array}$$

The component with label " $\mathrm{Sym}^{p-2}\ |$ " Sym^{-1} "" intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^1\otimes\det^{-1}, \mathrm{Sym}^{p-2}\}$. The component with label " $\mathrm{Sym}^{p-4}\otimes\det\ |\ \mathrm{Sym}^1\otimes\det^{-1}$ " intersects the next component at the point of X_ζ^{irred} parametrizing the irreducible Galois representation whose associated Serre weights are $\{\mathrm{Sym}^3\otimes\det^{-2}, \mathrm{Sym}^{p-4}\otimes\det\}$. Continuing in this way, one finds $\frac{p-1}{2}$ points of X_ζ^{irred}, which correspond to the $\frac{p-1}{2}$ double points of the chain X_ζ. There are no more points in X_ζ^{irred} and X_ζ^{irred} has cardinality $\frac{p-1}{2}$. Suppose we are on one of the two exterior components $\mathbb{P}^1$. There is a canonical affine coordinate t on the open complement of the double point, identifying this open complement with $\mathbb{A}^1$. We call the four points where $t = \pm 2$ *the four exceptional points* of X_ζ. [2]

3.5. Definition. *The category of quasi-coherent modules on the Emerton-Gee moduli curve X_ζ will be called the* category of mod p Langlands parameters with determinant $\omega\zeta$, *and denoted by* $\mathrm{LP}_{\widehat{\mathbf{G}},0,\omega\zeta}$*:*

$$\mathrm{LP}_{\widehat{\mathbf{G}},0,\omega\zeta} := \mathrm{QCoh}(X_\zeta).$$

[2] The Galois representations living on the two exterior components in the odd case are *unramified* (up to twist), i.e. of type $\rho = \begin{pmatrix} \mathrm{unr}(x) & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$ and t equals the 'trace of Frobenius' $x + x^{-1}$. Hence $t = \pm 2$ if and only if $x = \pm 1$.

4 A semisimple mod p Langlands correspondence in families for $F = \mathbb{Q}_p$

4.1. Let us consider W to be a subgroup of G, by sending s to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and by identifying the group Λ with a subgroup of T via $(1,0) \mapsto \mathrm{diag}(\varpi^{-1}, 1)$ and $(0,1) \mapsto \mathrm{diag}(1, \varpi^{-1})$. We obtain for example (recall that $u = (1,0)s \in W$)

$$u = \begin{pmatrix} 0 & \varpi^{-1} \\ 1 & 0 \end{pmatrix}, \quad u^{-1} = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}, \quad us = \begin{pmatrix} \varpi^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad su = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^{-1} \end{pmatrix}.$$

Moreover, $u^2 = \mathrm{diag}(\varpi^{-1}, \varpi^{-1})$.[3] Since

$$\begin{pmatrix} 0 & \varpi^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} d & \varpi^{-1} c \\ \varpi b & a \end{pmatrix}$$

the element $u \in G$ normalizes the group $I^{(1)}$.

4.2. Let $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ be the category of smooth G-representations over k. Taking $I^{(1)}$-invariants yields a functor $\pi \mapsto \pi^{I^{(1)}}$ from $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ to the category $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})$. If $F = \mathbb{Q}_p$, it induces a bijection between the irreducible G-representations and the irreducible $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules, under which supersingular representations correspond to supersingular Hecke modules [V04].

For future reference, let us recall the $I^{(1)}$-invariants for some classes of representations. If $\pi = \mathrm{Ind}_B^G(\chi)$ is a principal series representation with $\chi = \chi_1 \otimes \chi_2$, then $\pi^{I^{(1)}}$ is a standard module in the component $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$.

In the regular case, one chooses the ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ on the set γ and standard coordinates x, y. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(0, \chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}})$$

In the non-regular case, one has

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(\chi(su), \chi(u^2), \chi|_{\mathbb{T}}) = M(\chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}}).$$

These standard modules are irreducible if and only if $\chi \neq \chi^s$ [V04, 4.2/4.3].[4]

Let $F = \mathbb{Q}_p$. If $\pi = \pi(r, 0, \eta)$ is a standard supersingular representation with parameter $r = 0, ..., p-1$ and a character $\eta : \mathbb{Q}_p^\times \to k^\times$, then $\pi^{I^{(1)}}$ is a supersingular module in the component $\gamma = \{\chi, \chi^s\}$ represented by the character $\chi := (\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$, cf. [Br07, 5.1/5.3]. If π is the trivial representation $\mathbb{1}$ or the Steinberg representation St, then $\gamma = 1$ and $\pi^{I^{(1)}}$ is the character $(0,1)$ or $(-1,-1)$ respectively.

4.3. Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$. Since $u \in G$ normalizes the group $I^{(1)}$, one has $I^{(1)} u I^{(1)} = u I^{(1)}$. It follows that the convolution action of the Hecke operator U (resp. U^2) on $\pi^{I^{(1)}}$ is therefore induced by the action of u (resp. u^2 on π). Similarly, the group $I^{(1)}$ is normalized by the Iwahori subgroup I and $I/I^{(1)} \simeq \mathbb{T}$. It follows that the convolution action of the operators $T_t, t \in \mathbb{T}$ on $\pi^{I^{(1)}}$ is the factorization of the $\mathbf{T}(o_F)$-action on π.

4.4. We identify $F^\times$ with the center $Z(G)$ via $a \mapsto \mathrm{diag}(a, a)$. A (smooth) character

$$\zeta : Z(G) = F^\times \longrightarrow k^\times$$

is determined by its value $\zeta(\varpi^{-1}) \in k^\times$ and its restriction $\zeta|_{o_F^\times}$. Since the latter is trivial on the subgroup $1 + \varpi o_F$, we may view it as a character of $\mathbb{F}_q^\times$; we will write $\zeta|_{\mathbb{F}_q^\times}$ for this restriction in

[3] Note that our element u equals the element u^{-1} in [Be11],[Br07] and [V04].

[4] Our formulas differ from [V04, 4.2/4.3] by $\chi(\cdot) \leftrightarrow \chi(\cdot)^{-1}$, since we are working with left modules; also compare with the explicit calculation with right convolution given in [V04, Appendix A.5].

the following. Thus the group of characters of $Z(G)$ gets identified with the group of k-points of the group scheme $\mathscr{Z}^\vee = (\mathbb{F}_q^\times)^\vee \times \mathbb{G}_m$:

$$Z(G)^\vee \xrightarrow{\sim} \mathscr{Z}^\vee(k),\ \zeta \mapsto (\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1})).$$

4.5. Lemma. *Suppose that $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$ has a central character $\zeta : Z(G) \to k^\times$. Then the Satake parameter $S(\pi^{I^{(1)}})$ of $\pi^{I^{(1)}} \in \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q})$ has central character ζ, i.e. it is supported on the closed subscheme*

$$(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{(\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1}))} \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0.$$

Proof. If M is any $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-module, then

$$M = \bigoplus_{\gamma \in \mathbb{T}^\vee/W_0} \varepsilon_\gamma M = \bigoplus_{\gamma \in \mathbb{T}^\vee/W_0} \oplus_{\lambda \in \gamma} \varepsilon_\lambda M,$$

and $\mathbb{T} \subset \overline{\mathbb{F}}_q[\mathbb{T}] \subset \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ acts on $\varepsilon_\lambda M$ through the character $\lambda : \mathbb{T} \to \mathbb{F}_q^\times$. Now if $M = \pi^{I^{(1)}}$, then the $\mathbb{T}$-action on M is the factorization of the $\mathbf{T}(o_F)$-action on π, cf. 4.3. In particular, the restriction of the $\mathbb{T}$-action along the diagonal inclusion $\mathbb{F}_q^\times \subset \mathbb{T}$ is the factorization of the action of the central subgroup $o_F^\times \subset Z(G)$ on π, which is given by $\zeta|_{o_F^\times}$ by assumption. Hence

$$\varepsilon_\gamma M \neq 0 \quad \Longrightarrow \quad \forall \lambda \in \gamma,\ \lambda|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times} \text{ i.e. } \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}.$$

Moreover, the element $u^2 = \mathrm{diag}(\varpi^{-1}, \varpi^{-1}) \in Z(G)$ acts on π by multiplication by $\zeta(\varpi^{-1})$ by assumption. Therefore, by 4.3, the Hecke operator $z_2 := U^2 \in \mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$ acts on $\pi^{I^{(1)}}$ by multiplication by $\zeta(\varpi^{-1})$. Thus we have obtained that $S(\pi^{I^{(1)}})$ is supported on

$$\coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\mathrm{reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,\zeta(\varpi^{-1})} \coprod_{\gamma \in (\mathbb{T}^\vee/W_0)_{\text{non-reg}}, \gamma|_{\mathbb{F}_q^\times} = \zeta|_{\mathbb{F}_q^\times}} V_{\widehat{\mathbf{T}},0,\zeta(\varpi^{-1})}/W_0 = (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{(\zeta|_{\mathbb{F}_q^\times}, \zeta(\varpi^{-1}))}.$$

□

Next, recall the twisting action of the group $\mathscr{Z}^\vee(k)$ on the standard $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules and their simple constituents 2.4.

4.6. Proposition. *Let $\pi \in \mathrm{Mod}^{\mathrm{ladm}}(k[G])$ be irreducible or a reducible principal series representation. Let $\eta : F^\times \to k^\times$ be a character. Then*

$$(\pi \otimes \eta)^{I^{(1)}} = \pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times}, \eta(\varpi^{-1}))$$

as $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules.

Proof. An irreducible locally admissible representation, being a finitely generated $k[G]$-module, is admissible [Em10, 2.2.19]. A principal series representation (irreducible or not) is always admissible [Em10, 4.1.7]. The list of irreducible admissible smooth G-representations is given in [H11b, Thm. 1.1]. There are four families: principal series representations, supersingular representations, characters and twists of the Steinberg representation.

We first suppose that π is a principal series representation (irreducible or not), i.e. of the form $\mathrm{Ind}_B^G(\chi)$ with a character $\chi = \chi_1 \otimes \chi_2$. Then $\pi \otimes \eta \simeq \mathrm{Ind}_B^G(\chi_1\eta \otimes \chi_2\eta)$. We use the results from 4.2. The modules $\pi^{I^{(1)}}$ and $(\pi \otimes \eta)^{I^{(1)}}$ are standard modules in the components $\gamma := \{\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}}\}$ and $\gamma(\eta|_{\mathbb{F}_q^\times})$ respectively. Suppose that γ is regular. We choose the ordering $(\chi|_{\mathbb{T}}, \chi^s|_{\mathbb{T}})$ and standard coordinates x, y. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}} = M(0, \chi_2(\varpi^{-1}), \chi_1(\varpi^{-1})\chi_2(\varpi^{-1}), \chi|_{\mathbb{T}})$$

and

$$\mathrm{Ind}_B^G(\chi_1\eta\otimes\chi_2\eta)^{I^{(1)}}=M(0,\chi_2(\varpi^{-1})\eta(\varpi^{-1}),\chi_1(\varpi^{-1})\chi_2(\varpi^{-1})\eta(\varpi^{-2}),(\chi|_{\mathbb{T}}).(\eta|_{\mathbb{F}_q^\times})).$$

This shows $(\pi\otimes\eta)^{I^{(1)}}=\pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times},\eta(\varpi^{-1}))$ in the regular case. Suppose that γ is non-regular. Then

$$\mathrm{Ind}_B^G(\chi)^{I^{(1)}}=M(\chi_2(\varpi^{-1}),\chi_1(\varpi^{-1})\chi_2(\varpi^{-1}),\chi|_{\mathbb{T}})$$

and

$$\mathrm{Ind}_B^G(\chi_1\eta\otimes\chi_2\eta)^{I^{(1)}}=M(\chi_2(\varpi^{-1})\eta(\varpi^{-1}),\chi_1(\varpi^{-1})\chi_2(\varpi^{-1})\eta(\varpi^{-2}),(\chi|_{\mathbb{T}}).(\eta|_{\mathbb{F}_q^\times})).$$

This shows $(\pi\otimes\eta)^{I^{(1)}}=\pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times},\eta(\varpi^{-1}))$ in the non-regular case.

We now treat the case where π is a character or a twist of the Steinberg representation. Consider the exact sequence

$$1\to\mathbb{1}\to\mathrm{Ind}_B^G(1)\to\mathrm{St}\to 1.$$

According to [V04, 4.4] the sequence of invariants

$$(S):1\to\mathbb{1}^{I^{(1)}}\to\mathrm{Ind}_B^G(1)^{I^{(1)}}\to\mathrm{St}^{I^{(1)}}\to 1$$

is still exact and $\mathbb{1}^{I^{(1)}}$ resp. $\mathrm{St}^{I^{(1)}}$ is the trivial character $(0,1)$ resp. sign character $(-1,-1)$ in the Iwahori component $\gamma=1$. Tensoring the first exact sequence with η produces the exact sequence

$$1\to\eta\to\mathrm{Ind}_B^G(1)\otimes\eta\to\mathrm{St}\otimes\eta\to 1.$$

Since the restriction $\eta|_{o_F^\times}$ is trivial on $1+\varpi o_F$, one has $(\eta\circ\det)|_{I^{(1)}}=1$ and so, as a sequence of k-vector spaces with k-linear maps, the sequence of invariants

$$1\to\eta^{I^{(1)}}\to(\mathrm{Ind}_B^G(1)\otimes\eta)^{I^{(1)}}\to(\mathrm{St}\otimes\eta)^{I^{(1)}}\to 1$$

coincides with the sequence (S). It is therefore an exact sequence of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules, with outer terms being characters of $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$. From the discussion above, we deduce

$$(\mathrm{Ind}_B^G(1)\otimes\eta)^{I^{(1)}}=\mathrm{Ind}_B^G(1)^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times},\eta(\varpi)^{-1})=M(\eta(\varpi^{-1}),\eta(\varpi^{-2}),1.(\eta|_{\mathbb{F}_q^\times})).$$

It follows then from [V04, 1.1] that $\eta^{I^{(1)}}$ must be the trivial character $(0,\eta(\varpi^{-1}))$ in the component $1.(\eta|_{\mathbb{F}_q^\times})$ and $(\mathrm{St}\otimes\eta)^{I^{(1)}}$ must be the sign character $(-1,-\eta(\varpi^{-1}))$ in the component $1.(\eta|_{\mathbb{F}_q^\times})$. This implies

$$\eta^{I^{(1)}}=\mathbb{1}^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times},\eta(\varpi)^{-1})\quad\text{and}\quad(\mathrm{St}\otimes\eta)^{I^{(1)}}=\mathrm{St}^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times},\eta(\varpi)^{-1}).$$

This proves the claim in the cases $\pi=\mathbb{1}$ or $\pi=\mathrm{St}$. If, more generally, $\pi=\eta'$ is a general character of G, then

$$(\pi\otimes\eta)^{I^{(1)}}=(\eta'\eta)^{I^{(1)}}=\mathbb{1}^{I^{(1)}}.((\eta'\eta)|_{\mathbb{F}_q^\times},(\eta'\eta)(\varpi)^{-1})=\pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times},\eta(\varpi)^{-1}).$$

On the other hand, if $\pi=\mathrm{St}\otimes\eta'$ is a twist of Steinberg, then

$$(\pi\otimes\eta)^{I^{(1)}}=(\mathrm{St}\otimes(\eta'\eta))^{I^{(1)}}=\mathrm{St}^{I^{(1)}}.((\eta'\eta)|_{\mathbb{F}_q^\times},(\eta'\eta)(\varpi)^{-1})=\pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times},\eta(\varpi)^{-1}).$$

It remains to treat the case where π is a supersingular representation. In this case $\pi\otimes\eta$ is also supersingular and the two modules $\pi^{I^{(1)}}$ and $(\pi\otimes\eta)^{I^{(1)}}$ are supersingular $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}$-modules [V04, 4.9]. Let γ be the component of the module $\pi^{I^{(1)}}$. By 4.3, the component of $(\pi\otimes\eta)^{I^{(1)}}$ equals $\gamma(\eta|_{\mathbb{F}_q^\times})$. Moreover, if U^2 acts on $\pi^{I^{(1)}}$ via the scalar $z_2\in k^\times$, then U^2 acts on $(\pi\otimes\eta)^{I^{(1)}}$ via $z_2(\eta\circ\det)(u^2)=z_2\eta(\varpi)^{-2}$, cf. 4.3. Since the supersingular modules are uniquely characterized by their component and their U^2-action, we obtain $(\pi\otimes\eta)^{I^{(1)}}=\pi^{I^{(1)}}.(\eta|_{\mathbb{F}_q^\times},\eta(\varpi)^{-1})$, as claimed. $\square$

4.7. Let $F = \mathbb{Q}_p$ with $p \geq 5$. We let $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$ be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}(k[G])$ consisting of locally admissible representations having central character ζ. By work of Paškūnas [Pas13], the blocks b of the category $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$, defined as certain equivalence classes of simple objects, can be parametrized by the set of isomorphism classes $[\rho]$ of semisimple continuous Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$ having determinant $\det \rho = \omega\zeta$, i.e. by the k-points of X_ζ. There are three types of blocks. Blocks of type 1 are supersingular blocks. Each such block contains only one irreducible G-representation, which is supersingular. Blocks of type 2 contain only two irreducible representations. These two representations are two generic principal series representations of the form $\mathrm{Ind}^G_B(\chi_1 \otimes \chi_2\omega^{-1})$ and $\mathrm{Ind}^G_B(\chi_2 \otimes \chi_1\omega^{-1})$ (where $\chi_1\chi_2 \neq 1, \omega^{\pm 1}$). There are four blocks of type 3 which correspond to the four exceptional points. In the even case, each such block contains only three irreducible representations. These representations are of the form $\eta, \mathrm{St} \otimes \eta$ and $\mathrm{Ind}^G_B(\omega \otimes \omega^{-1}) \otimes \eta$. In the odd case, each block of type 3 contains only one irreducible representation. It is of the form $\mathrm{Ind}^G_B(\chi \otimes \chi\omega^{-1})$.

4.8. Let $F = \mathbb{Q}_p$ with $p \geq 5$. Paškūnas' parametrization $[\rho] \mapsto b_{[\rho]}$ is compatible with Breuil's semisimple mod p local Langlands correspondence

$$\rho \mapsto \pi(\rho)$$

for the group G [Br07, Be11], in the sense that if ρ has determinant $\omega\zeta$, then the simple constituents of the G-representation $\pi(\rho)$ lie in the block $b_{[\rho]}$ of $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$.

The correspondence and the parametrizations (for varying ζ) commute with twists: for a character $\eta : \mathbb{Q}_p^\times \to k^\times$, $\pi(\rho \otimes \eta) = \pi(\rho) \otimes \eta$ and $b_{[\rho]} \otimes \eta = b_{[\rho\otimes\eta]}$.

4.9. Theorem. *Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in \mathcal{Z}^\vee(k)$ under the identification $\mathcal{Z}(G)^\vee \cong \mathcal{Z}^\vee(k)$ from 4.4. Let $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ be the space of mod p Satake parameters with central character ζ and X_ζ be the moduli space of mod p Langlands parameters with determinant $\omega\zeta$.*

There exists a finite morphism of k-schemes

$$L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$$

such that the quasi-coherent $\mathcal{O}_{X_\zeta}$-module

$$L_{\zeta *} S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})|_{(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta}$$

equal to the push-forward along L_ζ of the restriction to $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$ of the Satake parameter of the mod p antispherical module $\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p}$ interpolates the $I^{(1)}$-invariants of the semisimple mod p Langlands correspondence

$$\begin{array}{ccccc} X_\zeta(k) & \longrightarrow & \mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G]) & \longrightarrow & \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \\ x & \longmapsto & \pi(\rho_x) & \longmapsto & \pi(\rho_x)^{I^{(1)}}, \end{array}$$

in the sense that for all $x \in X_\zeta(k)$,

$$\Big(\big(L_{\zeta *} S(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p})|_{(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta}\big) \otimes_{\mathcal{O}_{X_\zeta}} k(x) \Big)^{\mathrm{ss}} = \Big(\mathcal{M}^{(1)}_{\overline{\mathbb{F}}_p} \otimes_{Z(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})} (\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p})^{-1}(\mathcal{O}_{L_\zeta^{-1}(x)}) \Big)^{\mathrm{ss}} \cong \pi(\rho_x)^{I^{(1)}}$$

in $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$.

4.10. The connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are either regular and then of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$, or non-regular and then of type $\mathbb{A}^1$. The morphism L_ζ appearing in the theorem depends on the choice of an order of the two affine lines in each regular component. It is surjective and quasi-finite. Moreover, writing L^γ_ζ for its restriction to the connected component $(V^\gamma_{\widehat{\mathbf{T}},0}/W_0)_\zeta \subset (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, one has:

(e) *Even case.* All connected components are of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$, except for the two 'exterior' components which are of type $\mathbb{A}^1$. L_ζ^γ is an open immersion for any γ.

(o) *Odd case.* All connected components are of type $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. L_ζ is an open immersion on all connected components, except for the two 'exterior' ones. On an 'exterior' component γ, the restriction of L_ζ^γ to one irreducible component $\mathbb{A}^1$ is an open immersion, and its restriction to the open complement $\mathbb{G}_m$ is a degree 2 finite flat covering of its image, with branched locus equal to the intersection of this image with the exceptional locus of X_ζ.

4.11. Note that the semisimple mod p Langlands correspondence associates with any semisimple $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$ a semisimple smooth G-representation $\pi(\rho)$ of length $1, 2$ or 3, hence whose semisimple $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$-module of $I^{(1)}$-invariants $\pi(\rho)^{I^{(1)}}$ has length $1, 2$ or 3. On the other hand, the antispherical map

$$\mathrm{Sph} : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)(k) \longrightarrow \{\text{left } \mathcal{H}_{\overline{\mathbb{F}}_q}^{(1)}\text{-modules}\}$$

has an image consisting of $\mathcal{H}_{\overline{\mathbb{F}}_q}^{(1)}$-modules are of length 1 or 2, cf. [PS, 7.4.9] and [PS, 7.4.15]. Theorem 4.9 combined with the properties 4.10 of the morphism L_ζ provide the following case-by-case elucidation of the $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$-modules $\pi(\rho)^{I^{(1)}}$.

4.12. Corollary. *Let $x \in X_\zeta(k)$, corresponding to $\rho_x : \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \widehat{\mathbf{G}}(k)$. Then the $\mathcal{H}_{\overline{\mathbb{F}}_p}^{(1)}$-module $\pi(\rho)^{I^{(1)}}$ admits the following explicit description.*

(i) If $x \in X_\zeta^{irred}(k)$, then the fibre $L_\zeta^{-1}(x) = \{v\}$ has cardinality 1 and

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v).$$

It is irreducible and supersingular.

(ii) If $x \in X_\zeta^{red}(k) \setminus \{$the four exceptional points$\}$, then $L_\zeta^{-1}(x) = \{v_1, v_2\}$ has cardinality 2 and

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2).$$

It has length 2.

(iiie) If $x \in X_\zeta^{red}(k)$ is exceptional in the even case, then $L_\zeta^{-1}(x) = \{v_1, v_2\}$ has cardinality 2 and

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v_1)^{\mathrm{ss}} \oplus \mathrm{Sph}(v_2).$$

It has length 3.

(iiio) If $x \in X_\zeta^{red}(k)$ is exceptional in the odd case, then $L_\zeta^{-1}(x) = \{v\}$ has cardinality 1 and

$$\pi(\rho_x)^{I^{(1)}} \simeq \mathrm{Sph}(v) \oplus \mathrm{Sph}(v).$$

It has length 2.

4.13. Now we proceed to the proof of 4.9, 4.10 and 4.12.

We start by defining the morphism L_ζ at the level of k-points. Let $v \in (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^\vee/W_0$.

1. Suppose that γ is regular. Then $\mathrm{Sph}(v) = \mathrm{Sph}^\gamma(v)$ is a simple two-dimensional $\mathcal{H}_{\overline{\mathbb{F}}_p}^\gamma$-module, cf. [PS, 7.4.9]. Let $\pi \in \mathrm{Mod}^{\mathrm{sm}}(k[G])$ be the simple module, unique up to isomorphism, such that $\pi^{I^{(1)}} \simeq \mathrm{Sph}^\gamma(v)$, cf. 4.2. Then $\pi \in \mathrm{Mod}_\zeta^{\mathrm{ladm}}(k[G])$ with

$$\zeta = (\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) = (\gamma|_{\mathbb{F}_p^\times}, z_2)$$

by 4.5. Let b be the block of $\mathrm{Mod}_{\zeta}^{\mathrm{ladm}}(k[G])$ which contains π. We define $L_{\zeta}(v)$ to be the point of $X_{\zeta}(k)$ which corresponds to b.

2. Suppose that γ is non-regular.

(a) If $v \in D(2)_{\gamma}(k)$, then $\mathrm{Sph}(v) = \mathrm{Sph}^{\gamma}(2)(v)$ is a simple two-dimensional $\mathcal{H}^{\gamma}_{\mathbb{F}_p}$-module, cf. [PS, 7.4.15]. As in the regular case, there is a simple module π, unique up to isomorphism, such that $\pi^{I^{(1)}} \simeq \mathrm{Sph}^{\gamma}(2)(v)$. It has central character $\zeta = (\gamma|_{\mathbb{F}_p^{\times}}, z_2)$ and there is a block b of $\mathrm{Mod}_{\zeta}^{\mathrm{ladm}}(k[G])$ which contains π. We define $L_{\zeta}(v)$ to be the point of $X_{\zeta}(k)$ which corresponds to b.

(b) If $v \in D(1)_{\gamma}(k)$, then $\mathrm{Sph}(v)^{\mathrm{ss}}$ is the direct sum of the two characters forming the anti-spherical pair $\mathrm{Sph}^{\gamma}(1)(v) = \{(0, z_1), (-1, -z_1)\}$ where $z_2 = z_1^2$, cf. [PS, 7.4.15]. As in the regular case, there are two simple modules π_1 and π_2, unique up to isomorphism, such that $\pi_1^{I^{(1)}} \simeq (0, z_1)$ and $\pi_2^{I^{(1)}} \simeq (-1, -z_1)$ and π_1, π_2 have central character $\zeta = (\gamma|_{\mathbb{F}_p^{\times}}, z_2)$. Moreover, we claim that there is a unique block b of $\mathrm{Mod}_{\zeta}^{\mathrm{ladm}}(k[G])$ which contains both π_1 and π_2. Indeed, if $\gamma = \{1 \otimes 1\}$ and $z_1 = 1$, then $\pi_1 = \mathbb{1}$ and $\pi_2 = \mathrm{St}$, cf. 4.2. Then by 4.6 it follows more generally that if $\gamma = \{\omega^r \otimes \omega^r\}$, then $\pi_1 = \eta$ and $\pi_2 = \mathrm{St} \otimes \eta$ with $\eta = (\eta|_{\mathbb{F}_p^{\times}}, \eta(p^{-1})) := (\omega^r, z_1)$. Consequently π_1, π_2 are contained in a unique block b of type 3, cf. 4.7. We define $L_{\zeta}(v)$ to be the point of $X_{\zeta}(k)$ which corresponds to b.

Thus we have a well-defined map of sets $L_{\zeta} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}(k) \longrightarrow X_{\zeta}(k)$.

We show property (i) of 4.12. Let $x \in X_{\zeta}^{\mathrm{irred}}(k)$ and suppose $L_{\zeta}(v) = x$. Then b_x is a supersingular block, contains a unique irreducible representation π, which is supersingular, and $\pi = \pi(\rho_x)$, cf. 4.7-4.8. By definition of L_{ζ}, one has $\mathrm{Sph}(v) \simeq \pi^{I^{(1)}}$. Since the spherical map Sph is $1:1$ over supersingular modules, cf. [PS, 7.4.9] and [PS, 7.4.15], such a preimage v of x exists and is uniquely determined by x. Summarizing, we have $L_{\zeta}^{-1}(x) = \{v\}$ and $\mathrm{Sph}(v) \simeq \pi(\rho_x)^{I^{(1)}}$. This is property (i).

As a next step, we take a second character $\eta : \mathbb{Q}_p^{\times} \to k^{\times}$ and show that the diagram

$$\begin{array}{ccc} (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}(k) & \xrightarrow{L_{\zeta}} & X_{\zeta}(k) \\ {\scriptstyle \cdot\eta}\downarrow\simeq & & \simeq\downarrow{\scriptstyle (\cdot)\otimes\eta} \\ (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta\eta^2}(k) & \xrightarrow{L_{\zeta\eta^2}} & X_{\zeta\eta^2}(k) \end{array}$$

commutes. Here, the vertical arrows are the bijections coming from 2.9 and 3.2. To verify the commutativity, let $v \in (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}(k)$ and let its connected component be indexed by $\gamma \in \mathbb{T}^{\vee}/W_0$. Suppose that γ is regular or that γ is non-regular with $v \in D(2)_{\gamma}(k)$. Let π be the simple G-module with $\pi^{I^{(1)}} \simeq \mathrm{Sph}(v)$ and let $b_{[\rho]}$ be the block corresponding to the point $L_{\zeta}(v)$. By the equivariance property 2.5, one has $\mathrm{Sph}(v.\eta) \simeq \mathrm{Sph}(v).\eta$. Taking $I^{(1)}$-invariants is compatible with twist, cf. 4.6, and so $L_{\zeta\eta^2}(v.\eta)$ corresponds to the block which contains the representation $\pi \otimes \eta$, i.e. to $b_{[\rho]} \otimes \eta = b_{[\rho\otimes\eta]}$, cf. 4.8, and so $L_{\zeta\eta^2}(v.\eta) = [\rho \otimes \eta] = L_{\zeta}(v).\eta$.

If $v \in D(1)_{\gamma}(k)$, let π_1 and π_2 be the simple modules such that $(\pi_1 \oplus \pi_2)^{I^{(1)}} \simeq \mathrm{Sph}^{\gamma}(v)^{\mathrm{ss}}$. As before, we conclude from $\mathrm{Sph}(v.\eta)^{\mathrm{ss}} \simeq \mathrm{Sph}(v)^{\mathrm{ss}} \otimes \eta$ that $L_{\zeta\eta^2}(v.\eta)$ corresponds to the block which contains $\pi_1 \otimes \eta$ and $\pi_2 \otimes \eta$ and that $L_{\zeta\eta^2}(v.\eta) = L_{\zeta}(v).\eta$. The commutativity of the diagram is proved.

Thus, we are reduced to prove that the map L_{ζ} comes from a morphism of k-schemes satisfying 4.9 and the remaining parts of 4.12 in the two basic cases of a character ζ such that $\zeta(p^{-1}) = 1$ and $\zeta|_{\mathbb{F}_p^{\times}} \in \{1, \omega^{-1}\}$. This is established in the next two subsections.

5 The morphism L_{ζ} in the basic even case

Let $\zeta : \mathbb{Q}_p^{\times} \to k^{\times}$ be the trivial character. Here we show that the map of sets $L_{\zeta} : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_{\zeta}(k) \to X_{\zeta}(k)$ that we have defined in 4.13 satisfies properties (ii) and (iiie) of 4.12, and we define a morphism of k-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides with the previous map of sets at the level of k-points. By construction, it will have the properties 4.10. This will complete the proof of 4.12, 4.10 and 4.9 in the case of an even character.

5.1. We verify the properties (ii) and (iiie). We work over an irreducible component $\mathbb{P}^1$ with label " $\mathrm{Sym}^r \otimes \det^a \mid \mathrm{Sym}^{p-3-r} \otimes \det^{r+1+a}$ " where $0 \le r \le p-3$ and $0 \le a \le p-2$, cf. 3.3. On this component, we choose an affine coordinate x around the double point having $\mathrm{Sym}^r \otimes \det^a$ as one of its Serre weights. Away from this point, we have $x \neq 0$ and the corresponding Galois representation has the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{r+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^a$. By [Be11, 1.3] or [Br07, 4.11], we have

$$\pi(\rho_x) = \pi(r, x, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], x^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}} =: \pi_1 \oplus \pi_2$$

where $[p-3-r]$ denotes the unique integer in $\{0, ..., p-2\}$ which is congruent to $p-3-r$ modulo $p-1$. Now suppose that $L_\zeta(v) = x$. We distinguish two cases.

1. *The generic case* $0 < r < p-3$. In this case, the point x lies on one of the 'interior' components of the chain X_ζ, which has no exceptional points. The length of $\pi(\rho_x)$ is 2. Indeed, $\pi_1 = \pi(r, x, \eta)$ and $\pi_2 = \pi(p-3-r, x^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations [Br07, Thm. 4.4]. The block b_x is of type 2 and contains only these two irreducible representations, cf. 4.7-4.8. We may write

$$\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$$

with $\chi = \mathrm{unr}(x) \otimes \omega^r\, \mathrm{unr}(x^{-1})$, according to [Br07, Rem. 4.4(ii)]. By our assumptions on r, the character $\chi|_{\mathbb{T}} = 1 \otimes \omega^r$ is regular (i.e. different from its s-conjugate). We conclude from 4.6 and 4.2 that $\pi_1^{I^{(1)}}$ is a simple 2-dimensional standard module in the regular component represented by the character $(1 \otimes \omega^r).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^r \in \mathbb{T}^\vee$. Similarly, we may write

$$\pi_2 = \mathrm{Ind}_B^G(\chi) \otimes \omega^{r+1}\eta$$

where now $\chi = \mathrm{unr}(x^{-1}) \otimes \omega^{p-3-r}\, \mathrm{unr}(x)$. By our assumptions on r, the character $\chi|_{\mathbb{T}} = 1 \otimes \omega^{p-3-r}$ is regular and we conclude, as above, that the $I^{(1)}$-invariants $\pi_2^{I^{(1)}}$ form a simple 2-dimensional standard module in the regular component represented by the character $(\eta|_{\mathbb{F}_p^\times})\omega^{r+1} \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{r+1}\omega^{p-3-r} \in \mathbb{T}^\vee$. Note that the component of $\pi_1^{I^{(1)}}$ is different from the component of $\pi_2^{I^{(1)}}$, by our assumptions on r.

We conclude from $L_\zeta(v) = x$ that either $\mathrm{Sph}(v) = \pi_1^{I^{(1)}}$ or $\mathrm{Sph}(v) = \pi_2^{I^{(1)}}$. Since for γ regular, the map Sph^γ is a bijection onto all simple $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-modules, cf. [PS, 7.4.9], one finds that $L_\zeta^{-1}(x) = \{v_1, v_2\}$ has cardinality 2 and

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_x)^{I^{(1)}}.$$

This settles property (ii) of 4.12 in the generic case.

2. *The boundary cases* $r \in \{0, p-3\}$. In this case, the point x lies on one of the two 'exterior' components of X_ζ. On such a component, we will denote the variable x rather by z_1, which is the notation[5] which we used already in 3.3.

(a) Suppose that $z_1 \neq \pm 1$. The length of $\pi(\rho_{z_1})$ is 2. Indeed, as in the generic case, $\pi_1 = \pi(r, z_1, \eta)$ and $\pi_2 = \pi(p-3-r, z_1^{-1}, \omega^{r+1}\eta)$ are two irreducible principal series representations. The block b_{z_1} is of type 2 and contains only these two irreducible representations. It follows, as above, that their invariants $\pi_1^{I^{(1)}}$ and $\pi_2^{I^{(1)}}$ are simple 2-dimensional standard modules, in the components represented by $(\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^r \in \mathbb{T}^\vee$ and $(\eta|_{\mathbb{F}_p^\times})\omega^{r+1} \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{r+1}\omega^{p-3-r} \in \mathbb{T}^\vee$ respectively. Since $r \in \{0, p-3\}$, one of these components is regular, the other non-regular. In

[5] The reason for this notation will become clear in the discussion of the non-regular case in 5.2.

particular, the two components are different. We conclude from $L_\zeta(v) = z_1$ that either $\mathrm{Sph}(v) = \pi_1^{I^{(1)}}$ or $\mathrm{Sph}(v) = \pi_2^{I^{(1)}}$. Since for non-regular γ, the map $\mathrm{Sph}^\gamma(2)$ is a bijection from $D(2)_\gamma(k)$ onto all simple standard $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-modules, cf. [PS, 7.4.15], we may conclude as in the generic case: $L_\zeta^{-1}(z_1) = \{v_1, v_2\}$ has cardinality 2 and

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_{z_1})^{I^{(1)}}.$$

This settles property 4.12 (ii) in the remaining case $z_1 \neq \pm 1$.

(b) Suppose now that $z_1 = \pm 1$, i.e. we are at one of the four exceptional points. We will verify property (iiie). The length of $\pi(\rho_{z_1})$ is 3. Indeed, the representation $\pi(0, \pm 1, \eta)$ is a twist of the representation $\pi(0, 1, 1)$ (note that $\pi(r, z_1, \eta) \simeq \pi(r, -z_1, \mathrm{unr}(-1)\eta)$ according to [Br07, Rem. 4.4(v)]), which itself is an extension of $\mathbb{1}$ by St, cf. [Br07, Thm. 4.4(iii)]. As in the case (a), the representation $\pi_2 = \pi(p-3, \pm 1, \omega\eta)$ is an irreducible principal series representation. The block b_{z_1} is of type 3 and contains only these three irreducible representations. The invariants $\pi_1^{I^{(1)}}$ form a direct sum of two spherical characters in a non-regular component γ, whereas the invariants $\pi_2^{I^{(1)}}$ form a simple standard module in a regular component, as before. Since for non-regular γ, the map $\mathrm{Sph}^\gamma(1)$ is a bijection from $D(1)_\gamma(k)$ onto all spherical pairs of characters of $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$, cf. [PS, 7.4.15], we may conclude that $L_\zeta^{-1}(z_1) = \{v_1, v_2\}$ has cardinality 2 with $v_1 \in D(1)_\gamma(k)$ and $\mathrm{Sph}^\gamma(1)(v_1)^{\mathrm{ss}} = \pi_1^{I^{(1)}}$. In particular,

$$\mathrm{Sph}(v_1)^{\mathrm{ss}} \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_x)^{I^{(1)}}.$$

This settles property 4.12 (iiie).

5.2. We define a morphism of k-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides on k-points with the map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$. We work over a connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, indexed by some $\gamma \in \mathbb{T}^\vee/W_0$. Let v be a k-point of this component.

Since $\zeta|_{\mathbb{F}_p^\times} = 1$, the connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are indexed by the fibre $(\cdot)|^{-1}_{\mathbb{F}_p^\times}(1)$. This fibre consists of the $\frac{p-3}{2}$ regular components, represented by the characters of $\mathbb{T}$

$$\chi_k = \omega^k \otimes \omega^{-k}$$

for $k = 1, ..., \frac{p-3}{2}$, and of the two non-regular components, given by χ_0 and $\chi_{\frac{p-1}{2}}$, cf. 2.2. We distinguish two cases. Note that $z_2 = \zeta(p^{-1}) = 1$.

1. *The regular case* $0 < k < \frac{p-1}{2}$. We fix the order $\gamma = (\chi_k, \chi_k^s)$ on the set γ and choose the standard coordinates x, y. According to [PS, 7.4.8], our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\widehat{\mathbf{T}},0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$

Suppose that $v = (0, 0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r, \eta) = \mathrm{ind}(\omega_2^{r+1}) \otimes \eta$, in the notation of [Be11, 1.3], whence $L_\zeta(v) = [\rho(r, \eta)]$. According to 4.2, the component of the Hecke module $\pi(r, 0, \eta)^{I^{(1)}}$ is given by $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times})$. Setting $\eta|_{\mathbb{F}_p^\times} = \omega^a$, this implies $(\omega^r \otimes 1) \cdot (\eta|_{\mathbb{F}_p^\times}) = \omega^{r+a} \otimes \omega^a = \chi_k$ and hence $a = -k$ and $r = 2k$. Therefore the Serre weights of the irreducible representation $\rho(r, \eta)$ are $\{\mathrm{Sym}^{2k} \otimes \det^{-k}, \mathrm{Sym}^{p-1-2k} \otimes \det^k\}$, cf. [Br07, 1.9].

Comparing these pairs of Serre weights with the list 3.3 shows that the $\frac{p-3}{2}$ points

$$\{\text{origin } (0, 0) \text{ on the component } (\chi_k, \chi_k^s)\}$$

for $0 < k < \frac{p-1}{2}$ are mapped successively to the $\frac{p-3}{2}$ double points of the chain X_ζ.

Fix $0 < k < \frac{p-1}{2}$ and consider the double point

$$Q = L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, Q lies on the irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^{2k} \otimes \det^{-k}$ (i.e. on the component " $\mathrm{Sym}^{2k} \otimes \det^{-k} \mid \mathrm{Sym}^{p-3-2k} \otimes \det^{k+1}$"). We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will also call x (there will be no risk of confusion with the standard coordinate above!). Away from Q, the affine coordinate $x \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{2k+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{-k}$. As we have seen above, $\pi(\rho_x) = \pi(2k, x, \eta) \oplus \pi(p-3-2k, x^{-1}, \omega^{r+1}\eta) =: \pi_1 \oplus \pi_2$. Moreover, $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(x) \otimes \omega^{2k}\,\mathrm{unr}(x^{-1})$. Since

$$(1 \otimes \omega^{2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{-k} \otimes \omega^k = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that

$$\pi_1^{I^{(1)}} = M(0, x, 1, \chi_k^s)$$

is a simple 2-dimensional standard module. Note that $M(0, x, 1, \chi_k^s) = M(x, 0, 1, \chi_k)$ according to [V04, Prop. 3.2].

Now suppose that $v = (x, 0), x \neq 0$, denotes a point on the x-line of $\mathbb{A}_k^1 \cup_0 \mathbb{A}_k^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(x, 0, 1, \chi_k)$. By our discussion, the point $L_\zeta((x, 0))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_x]$, cf. 4.8, it follows that

$$L_\zeta((x,0)) = [\rho_x] = x \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since $(0,0)$ maps to Q, i.e. to the point at $x = 0$, the map L_ζ identifies the whole affine x-line $\mathbb{A}^1 = \{(x, 0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine x-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point Q lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of Q, i.e. the weight $\mathrm{Sym}^{p-1-2k} \otimes \det^k$. We fix an affine coordinate y on this $\mathbb{P}^1$ around Q. Away from Q, the coordinate $y \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(y)\omega^{p-2k} & 0 \\ 0 & \mathrm{unr}(y^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the first case, $\pi(\rho_y)$ contains $\pi_1 := \pi(p-1-2k, y, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ as a direct summand, where now $\chi = \mathrm{unr}(y) \otimes \omega^{p-1-2k}\,\mathrm{unr}(y^{-1})$. Since

$$(1 \otimes \omega^{p-1-2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^k \otimes \omega^{-k} = \chi_k \in \mathbb{T}^\vee,$$

we deduce, as above, that $\pi_1^{I^{(1)}} = M(0, y, 1, \chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v = (0, y), y \neq 0$, denotes a point on the y-line of $\mathbb{A}_k^1 \cup_0 \mathbb{A}_k^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0, y, 1, \chi_k)$. By our discussion, the point $L_\zeta((0, y))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_y]$, cf. 4.8, it follows that

$$L_\zeta((0,y)) = [\rho_y] = y \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since $(0,0)$ maps to Q, i.e. to the point at $y = 0$, the map L_ζ identifies the whole affine y-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine y-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each regular connected component of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$ in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

2. *The non-regular case* $k \in \{0, \frac{p-1}{2}\}$. We choose the Steinberg coordinate z_1. According to [PS, 7.4.10], our non-regular connected component identifies with an affine line :

$$V_{\widehat{\mathbf{T}},0,z_2}/W_0 \simeq \mathbb{A}^1.$$

Suppose that $v = (0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r, 0, \eta)$ be the corresponding supersingular representation so that $L_\zeta(v) = [\rho(r, \eta)]$. Exactly as in the regular case, we may conclude that the Serre weights of the irreducible representation $\rho(r, \eta)$ are $\{\mathrm{Sym}^{2k} \otimes \det^{-k}, \mathrm{Sym}^{p-1-2k} \otimes \det^k\}$. For the two values of $k = 0$ and $k = \frac{p-1}{2}$ we find $\{\mathrm{Sym}^0, \mathrm{Sym}^{p-1}\}$ and $\{\mathrm{Sym}^0 \otimes \det^{\frac{p-1}{2}}, \mathrm{Sym}^{p-1} \otimes \det^{\frac{p-1}{2}}\}$ respectively. Comparing with the list 3.3 shows that the 2 points

$$\{\text{origin } (0) \text{ on the component } (\chi_k = \chi_k^s)\}$$

for $k \in \{0, \frac{p-1}{2}\}$ are mapped to the 2 smooth points in X_ζ^{irred}, which lie on the two 'exterior' components of X_ζ, cf. 3.3.

Fix $k \in \{0, \frac{p-1}{2}\}$ and consider the point

$$Q = L_\zeta(\text{origin } (0) \text{ on the component } \gamma = (\chi_k = \chi_k^s)).$$

As we have just seen, Q lies on an 'exterior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^0 \otimes \det^k$. We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we call z_1 (there will be no risk of confusion with the Steinberg coordinate above!). Away from Q, the affine coordinate $z_1 \neq 0$ parametrizes Galois representations of the form

$$\rho_{z_1} = \begin{pmatrix} \mathrm{unr}(z_1)\omega & 0 \\ 0 & \mathrm{unr}(z_1^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^k$. As in the regular case, $\pi(\rho_{z_1}) = \pi(0, z_1, \eta)^{\mathrm{ss}} \oplus \pi(p-3, z_1^{-1}, \omega\eta)^{\mathrm{ss}}$. Moreover, $\pi(0, z_1, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(z_1) \otimes \mathrm{unr}(z_1^{-1})$ [6]. Since

$$(1 \otimes 1).(\eta|_{\mathbb{F}_p^\times}) = \omega^k \otimes \omega^k = \chi_k = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the non-regular case of 4.2 that $\pi(0, z_1, \eta)^{I^{(1)}} = M(z_1, 1, \chi_k)$ is a 2-dimensional standard module. Moreover, the standard module is simple if and only if $\chi \neq \chi^s$, i.e. if and only if $z_1 \neq \pm 1$.

Now let $v = z_1 \neq 0$ denote a nonzero point on our connected component $\mathbb{A}^1 = V_{\widehat{\mathbf{T}},0,1}/W_0$. Suppose that $z_1 \neq \pm 1$, i.e. $v \in D(2)_\gamma$. In particular, $\mathrm{Sph}(v) = M(z_1, 1, \gamma)$ is irreducible. By our discussion, the point $L_\zeta(z_1)$ corresponds to the block (a block of type 2) which contains $\pi(0, z_1, \eta)$. Suppose that $z_1 = \pm 1$, i.e. $v \in D(1)_\gamma$. In particular, $\mathrm{Sph}^{\mathrm{ss}}(v) = M(z_1, 1, \chi_k)^{\mathrm{ss}}$ and again, $L_\zeta(z_1)$ corresponds to the block (now a block of type 3) which contains the simple constituents of $\pi(0, z_1, \eta)^{\mathrm{ss}}$. In both cases, we conclude

$$L_\zeta(z_1) = [\rho_{z_1}] = z_1 \in \mathbb{G}_m \subset \mathbb{P}^1 \subset X_\zeta.$$

Since (0) maps to Q, i.e. to the point at $z_1 = 0$, the map L_ζ identifies the whole z_1-line $\mathbb{A}^1 = V_{\widehat{\mathbf{T}},0,1}/W_0$ with the z_1-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each non-regular connected component of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$ in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

6 The morphism L_ζ in the basic odd case

Let $\zeta := \omega^{-1} : \mathbb{Q}_p^\times \to k^\times$. Here we show that the map of sets $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta(k) \to X_\zeta(k)$ that we have defined in 4.13 satisfies properties (ii) and (iiio) of 4.12, and we define a morphism of k-schemes $L_\zeta : (V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides with the previous map of sets at the level of k-points. By construction, it will have the properties 4.10. This will complete the proof of 4.12, 4.10 and 4.9 in the case of an odd character.

[6] The representations $\pi(0, z_1, \eta)$ constitute the *unramified* principal series of G.

6.1. We verify properties (ii) and (iiio). We work over an irreducible component $\mathbb{P}^1$ with label "$\mathrm{Sym}^r \otimes \det^a \mid \mathrm{Sym}^{p-3-r} \otimes \det^{r+1+a}$" where $1 \le r \le p-2$ and $0 \le a \le p-2$, cf. 3.4. We distinguish two cases.

1. *The generic case* $r \ne p-2$. In this case, the irreducible component of X_ζ we consider is an 'interior' component and has no exceptional points. On this component, we choose an affine coordinate x around the double point having $\mathrm{Sym}^r \otimes \det^a$ as one of its Serre weights. Away from this point, we have $x \ne 0$ and the corresponding Galois representation has the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{r+1} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^a$. As before, we have

$$\pi(\rho_x) = \pi(r, x, \eta)^{\mathrm{ss}} \oplus \pi([p-3-r], x^{-1}, \omega^{r+1}\eta)^{\mathrm{ss}}.$$

The length of $\pi(\rho_x)$ is 2. Indeed, by our assumptions on r, the principal series representations $\pi(r, x, \eta)$ and $\pi(p-3-r, x^{-1}, \omega^{r+1}\eta)$ are irreducible and the block b_x contains only these two irreducible representations. We may follow the argument of the generic case of 5.1 word for word and deduce property 4.12 (ii).

2. *The two boundary cases* $r = p-2$. In this case, the irreducible component is one of the two 'exterior' components with labels "$\mathrm{Sym}^{p-2} \mid$ "Sym^{-1}"" or ""$\mathrm{Sym}^{-1} \det^{\frac{p-1}{2}}$" $\mid \mathrm{Sym}^{p-2} \det^{\frac{p-1}{2}}$". Points of the open locus X_ζ^{red} lying on such a component correspond to twists of unramified Galois representations of the form

$$\rho_{x+x^{-1}} = \begin{pmatrix} \mathrm{unr}(x) & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = 1$ or $\eta = \omega^{\frac{p-1}{2}}$. Let us concentrate on one of the two components, i.e. let us fix η.

Mapping an unramified Galois representation $\rho_{x+x^{-1}}$ to $t := x + x^{-1} \in k$ identifies this open locus with the t-line $\mathbb{A}^1 \subset \mathbb{P}^1$. We have

$$\pi(\rho_t) = \pi(p-2, x, \eta)^{\mathrm{ss}} \oplus \pi(p-2, x^{-1}, \eta)^{\mathrm{ss}} =: \pi_1 \oplus \pi_2$$

since $[p-3-(p-2)] = p-2$ (indeed, $p-3-(p-2) = -1 \equiv p-2 \mod (p-1)$). The length of $\pi(\rho_t)$ is 2. Indeed, $\pi_1 = \pi(p-2, x, \eta)$ and $\pi_2 = \pi(p-2, x^{-1}, \eta)$ are two irreducible principal series representations and the block b_t contains only these two irreducible representations. They are isomorphic if and only if $x = \pm 1$, i.e. if and only if $t = \pm 2$ is an exceptional point. In this case, b_t contains only one irreducible representation and is of type 3, otherwise it is of type 2.

We may write

$$\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$$

with $\chi = \mathrm{unr}(x) \otimes \omega^{p-2}\,\mathrm{unr}(x^{-1})$. Similarly for π_2. The character $\chi|_{\mathbb{F}_p^\times} = 1 \otimes \omega^{p-2}$ is regular (i.e. different from its s-conjugate) and we are in the regular case of 4.2. We conclude that $\pi_1^{I^{(1)}} = M(0, x, 1, (1 \otimes \omega^{p-2}).\eta)$ and $\pi_2^{I^{(1)}} = M(0, x^{-1}, 1, (1 \otimes \omega^{p-2}).\eta)$ are both simple 2-dimensional standard modules in the regular component γ represented by the character $(1 \otimes \omega^{p-2}).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} \in \mathbb{T}^\vee$. They are isomorphic if and only if $t = \pm 2$. We choose an order $\gamma = ((\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2}, (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} \otimes (\eta|_{\mathbb{F}_p^\times}))$ on the set γ. Then from $L_\zeta(v) = t$ we get that either $\mathrm{Sph}^\gamma(v) = \pi_1^{I^{(1)}}$ or $\mathrm{Sph}^\gamma(v) = \pi_2^{I^{(1)}}$. Since for regular γ, the map Sph^γ is a bijection onto all simple $\mathcal{H}^\gamma_{\overline{\mathbb{F}}_p}$-modules, cf. [PS, 7.4.9], one finds that $L_\zeta^{-1}(t) = \{v_1, v_2\}$ has cardinality 2 if $t \ne \pm 2$ and then

$$\mathrm{Sph}(v_1) \oplus \mathrm{Sph}(v_2) \simeq \pi(\rho_t)^{I^{(1)}}.$$

This settles property 4.12 (ii). In turn, if $t = \pm 2$ is an exceptional point, then $L_\zeta^{-1}(t) = \{v\}$ has cardinality 1 and

$$\mathrm{Sph}(v) \oplus \mathrm{Sph}(v) \simeq \pi(\rho_t)^{I^{(1)}}.$$

This settles property 4.12 (iiio).

6.2. We define a morphism of k-schemes $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \longrightarrow X_\zeta$ which coincides on k-points with the map of sets $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta(k) \longrightarrow X_\zeta(k)$. We work over a connected component of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$, indexed by some $\gamma \in \mathbb{T}^\vee/W_0$. Let v be a k-point of this component.

Since $\zeta|_{\mathbb{F}_p^\times} = \omega^{-1}$, the connected components of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ are indexed by the fibre $(\cdot)|^{-1}_{\mathbb{F}_p^\times}(\omega^{-1})$. This fibre consists of the $\frac{p-1}{2}$ regular components, represented by the characters

$$\chi_k = \omega^{k-1} \otimes \omega^{-k}$$

for $k = 1, ..., \frac{p-1}{2}$, cf. 2.2. Recall that $z_2 = \zeta(p) = 1$.

Fix an order $\gamma = (\chi_k, \chi_k^s)$ on the set γ and choose standard coordinates x, y. According to [PS, 7.4.8], our regular connected component identifies with two affine lines intersecting at the origin:

$$V_{\widehat{\mathbf{T}},0,1} \simeq \mathbb{A}^1 \cup_0 \mathbb{A}^1.$$

Suppose that $v = (0,0)$ is the origin, so that $\mathrm{Sph}(v)$ is a supersingular module. Let $\pi(r,0,\eta)$ be the corresponding supersingular representation. It corresponds to the irreducible Galois representation $\rho(r,\eta)$, in the notation of [Be11, 1.3], whence $L_\zeta(v) = [\rho(r,\eta)]$. According to 4.2, the component of $\pi(r,0,\eta)^{I^{(1)}}$ is given by $(\omega^r \otimes 1)\cdot(\eta|_{\mathbb{F}_p^\times})$. Setting $\eta|_{\mathbb{F}_p^\times} = \omega^a$, this implies $(\omega^r \otimes 1)\cdot(\eta|_{\mathbb{F}_p^\times}) = \omega^{r+a} \otimes \omega^a = \chi_k$ and hence $a = -k$ and $r = 2k-1$. The Serre weights of the irreducible representation $\rho(r,\eta)$ are therefore $\{\mathrm{Sym}^{2k-1} \otimes \det^{-k}, \mathrm{Sym}^{p-2k} \otimes \det^{k-1}\}$, cf. [Br07, 1.9].

Comparing these pairs of Serre weights with the list 3.4 shows that the $\frac{p-1}{2}$ points

$$\{\text{origin } (0,0) \text{ on the component } (\chi_k, \chi_k^s)\}$$

for $k = 1, ..., \frac{p-1}{2}$ are mapped successively to the $\frac{p-1}{2}$ double points of the chain X_ζ. We distinguish two cases.

1. The generic case $1 < k < \frac{p-1}{2}$. In this case, the argument proceeds as in the regular case of 5.2. Consider the double point

$$Q = L_\zeta(\text{origin } (0,0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, Q lies on an 'interior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^{2k-1} \otimes \det^{-k}$. We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will also call x. Away from Q, the affine coordinate $x \neq 0$ parametrizes Galois representations of the form

$$\rho_x = \begin{pmatrix} \mathrm{unr}(x)\omega^{2k} & 0 \\ 0 & \mathrm{unr}(x^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{-k}$. As we have seen above, $\pi(\rho_x) = \pi(2k-1, x, \eta) \oplus \pi(p-3-2k+1, x^{-1}, \omega^{2k}\eta) =: \pi_1 \oplus \pi_2$. Moreover, $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(x) \otimes \omega^{2k-1}\,\mathrm{unr}(x^{-1})$. Since

$$(1 \otimes \omega^{2k-1}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{-k} \otimes \omega^{k-1} = \chi_k^s \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that $\pi_1^{I^{(1)}} = M(0, x, 1, \chi_k^s)$ is a simple 2-dimensional standard module. Note that $M(0,x,1,\chi_k^s) = M(x,0,1,\chi_k)$ according to [V04, Prop. 3.2].

Now suppose that $v = (x,0)$, $x \neq 0$, denotes a nonzero point on the x-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(x,0,1,\chi_k)$. Our discussion shows that the point $L_\zeta((x,0))$ corresponds to the block which contains π_1. Since π_1 lies in the block parametrized by $[\rho_x]$, cf. 4.8, it follows that

$$L_\zeta((x,0)) = [\rho_x] = x \in \mathbb{G}_m \subset \mathbb{P}^1.$$

Since $(0,0)$ maps to Q, i.e. to the point at $x = 0$, the map L_ζ identifies the whole affine x-line $\mathbb{A}^1 = \{(x,0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine x-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point Q also lies on the irreducible component whose labelling includes the other weight of Q, i.e. the weight $\mathrm{Sym}^{p-2k} \otimes \det^{k-1}$. We fix an affine coordinate y on this $\mathbb{P}^1$ around Q. Away from Q, the coordinate $y \neq 0$ parametrizes Galois representations of the form

$$\rho_y = \begin{pmatrix} \mathrm{unr}(y)\omega^{p-2k+1} & 0 \\ 0 & \mathrm{unr}(y^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta := \omega^{k-1}$. As in the first case, $\pi(\rho_y)$ contains $\pi_1 := \pi(p-2k, y, \eta) = \mathrm{Ind}_B^G(\chi) \otimes \eta$ as a direct summand, where now $\chi = \mathrm{unr}(y) \otimes \omega^{p-2k}\,\mathrm{unr}(y^{-1})$. Since

$$(1 \otimes \omega^{p-2k}).(\eta|_{\mathbb{F}_p^\times}) = \omega^{k-1} \otimes \omega^{-k} = \chi_k \in \mathbb{T}^\vee,$$

we deduce from the regular case of 4.2 that $\pi_1^{I^{(1)}} = M(0, y, 1, \chi_k)$ is a simple 2-dimensional standard module.

Now suppose that $v = (0, y)$, $y \neq 0$, denotes a nonzero point on the y-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0, y, 1, \chi_k)$. Our discussion shows that the point $L_\zeta((0, y))$ corresponds to the block which contains π_1, parametrized by $[\rho_y]$. Hence

$$L_\zeta((0, y)) = [\rho_y] = y \in \mathbb{G}_m \subset \mathbb{P}^1.$$

Since $(0, 0)$ maps to Q, i.e. to the point at $y = 0$, the map L_ζ identifies the whole y-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the affine y-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

In this way, we get an open immersion of each connected component $(V^{\gamma}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ of $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$ such that $\gamma = (\chi_k, \chi_k^s)$ with $1 < k < \frac{p-1}{2}$, in the scheme X_ζ, which coincides on k-points with the restriction of the map of sets L_ζ.

2. The two boundary cases $k \in \{1, \frac{p-1}{2}\}$. Consider the double point

$$Q = L_\zeta(\text{origin } (0, 0) \text{ on the component } \gamma = (\chi_k, \chi_k^s)).$$

As we have just seen, Q lies on an 'interior' irreducible component $\mathbb{P}^1$ whose label includes the weight $\mathrm{Sym}^1 \otimes \det^{-1}$ (for $k = 1$) or the weight $\mathrm{Sym}^1 \otimes \det^{\frac{p-3}{2}}$ (for $k = \frac{p-1}{2}$). We fix an affine coordinate on this $\mathbb{P}^1$ around Q, which we will call z. Away from Q, the coordinate $z \neq 0$ parametrizes Galois representations of the form

$$\rho_z = \begin{pmatrix} \mathrm{unr}(z)\omega^2 & 0 \\ 0 & \mathrm{unr}(z^{-1}) \end{pmatrix} \otimes \eta$$

with $\eta = \omega^{-1}$ or $\eta = \omega^{\frac{p-3}{2}}$.

Let $k = 1$, i.e. $\eta = \omega^{-1}$. Following the argument in the generic case word for word, we may conclude that L_ζ identifies the x-line $\mathbb{A}^1 = \{(x, 0) : x \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the z-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

Let $k = \frac{p-1}{2}$, i.e. $\eta = \omega^{\frac{p-3}{2}}$. As in the generic case, we may conclude that L_ζ identifies the y-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ with the z-line $\mathbb{A}^1 \subset \mathbb{P}^1 \subset X_\zeta$.

On the other hand, the double point Q lies also on the irreducible component $\mathbb{P}^1$ whose labelling includes the other weight of Q, i.e. the weight Sym^{p-2} (for $k = 1$) or the weight $\mathrm{Sym}^{p-2} \otimes \det^{\frac{p-1}{2}}$ (for $k = \frac{p-1}{2}$). These are the two 'exterior' components. Points of the open locus X_ζ^{red} lying on such a component correspond to unramified (up to twist) Galois representations of the form

$$\rho_t = \begin{pmatrix} \mathrm{unr}(z) & 0 \\ 0 & \mathrm{unr}(z^{-1}) \end{pmatrix} \otimes \eta$$

where $\eta = 1$ (for $k = 1$) or $\eta = \omega^{\frac{p-1}{2}}$ (for $k = \frac{p-1}{2}$) and with $t = z + z^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1$. As in the boundary case of 6.1, we have $\pi(\rho_t) = \pi(p-2, z, \eta) \oplus \pi(p-2, z^{-1}, \eta) =: \pi_1 \oplus \pi_2$ and these are irreducible principal series representations. We may write $\pi_1 = \mathrm{Ind}_B^G(\chi) \otimes \eta$ with $\chi = \mathrm{unr}(z) \otimes \omega^{p-2}\,\mathrm{unr}(z^{-1})$. The character $\chi|_{\mathbb{F}_p^\times} = 1 \otimes \omega^{p-2}$ is regular (i.e. different from its s-conjugate) and we are in the regular case of 4.2. We conclude that

$$\pi_1^{I^{(1)}} = M(0, z, 1, (1 \otimes \omega^{p-2}).\eta)$$

is a simple 2-dimensional standard module in the regular component represented by the character

$$(1 \otimes \omega^{p-2}).(\eta|_{\mathbb{F}_p^\times}) = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{p-2} = (\eta|_{\mathbb{F}_p^\times}) \otimes (\eta|_{\mathbb{F}_p^\times})\omega^{-1} \in \mathbb{T}^\vee.$$

This latter character equals χ_1 for $\eta = 1$ and $(\chi_{\frac{p-1}{2}})^s$ for $\eta = \omega^{\frac{p-1}{2}}$ (indeed, note that $\frac{p-1}{2} \equiv -\frac{p-1}{2}$ mod $p-1$).

Now suppose that $k = 1$, i.e. $\eta = 1$. Let $v = (0, y)$, $y \neq 0$, be a nonzero point on the y-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular, $\mathrm{Sph}^\gamma(v) = M(0, y, 1, \chi_1)$. Our discussion shows that the point $L_\zeta((0, y))$ corresponds to the block which contains π_1, i.e. which is parametrized by $[\rho_t]$. It follows that

$$L_\zeta((0, y)) = [\rho_t] = t = y + y^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1.$$

Since $(0, 0)$ maps to Q, i.e. to the point at $t = \infty$, the map of sets L_ζ maps the k-points of the whole affine y-line $\mathbb{A}^1 = \{(0, y) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ to the k-points of the whole 'left exterior' component $\mathbb{P}^1 \subset X_\zeta$ via the formula

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ y & \longmapsto & \left\{ \begin{array}{ll} y + y^{-1} & \text{if } y \neq 0 \\ \infty = Q & \text{if } y = 0. \end{array} \right. \end{array}$$

This formula is algebraic: indeed, for $y \in \mathbb{A}^1 \setminus \{\pm i\}$ (where $\pm i$ are the roots of the polynomial $f(y) = y^2 + 1$), we have $y + y^{-1} \neq 0$ and $(y + y^{-1})^{-1} = y/(y^2 + 1)$, which is equal to 0 at $y = 0$. Moreover, it glues at the origin $(0, 0)$ with the open immersion of the x-line of $V_{\widehat{\mathbf{T}},0,1} = \mathbb{A}^1 \cup_0 \mathbb{A}^1$ in X_ζ defined above, since both map $(0, 0)$ to Q. We take the resulting morphism of k-schemes $\mathbb{A}^1 \cup_0 \mathbb{A}^1 \to X_\zeta$ as the definition of L_ζ on the connected component $(V_{\widehat{\mathbf{T}},0}^{(\chi_1,\chi_1^s)}/W_0)_\zeta$ of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$. Note that its restriction to the open subset $\{y \neq 0\}$ in the y-line $\mathbb{A}^1$ is the morphism $\mathbb{G}_m \to \mathbb{A}^1$ corresponding to the ring extension

$$k[t] \longrightarrow k[y, y^{-1}] = k[t][y]/(y^2 - ty + 1),$$

and that the discriminant $t^2 - 4$ of $y^2 - ty + 1 \in k[t][y]$ vanishes precisely at the two exceptional points $t = \pm 2$.

Suppose $k = \frac{p-1}{2}$, i.e. $\eta = \omega^{\frac{p-1}{2}}$. Let $v = (x, 0)$, $x \neq 0$, denote a nonzero point on the x-line of $\mathbb{A}^1 \cup_0 \mathbb{A}^1$. In particular,

$$\mathrm{Sph}^\gamma(v) = M(0, x, 1, (\chi_{\frac{p-1}{2}})^s) = M(x, 0, 1, \chi_{\frac{p-1}{2}}).$$

Our discussion shows that the point $L_\zeta((x, 0))$ corresponds to the block which contains π_1, i.e. which is parametrized by $[\rho_t]$. It follows that $L_\zeta((x, 0)) = [\rho_t] = t = x + x^{-1} \in \mathbb{A}^1 \subset \mathbb{P}^1$. Since $(0, 0)$ maps to the point Q at $t = \infty$, the map of sets L_ζ maps the k-points of the whole affine x-line $\mathbb{A}^1 = \{(x, 0) : y \in k\} \subset V_{\widehat{\mathbf{T}},0,1}$ to the k-points of the whole 'right exterior' component $\mathbb{P}^1 \subset X_\zeta$ via the formula

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \\ x & \longmapsto & \left\{ \begin{array}{ll} x + x^{-1} & \text{if } x \neq 0 \\ \infty = Q & \text{if } x = 0. \end{array} \right. \end{array}$$

This formula is algebraic. Moreover, it glues at the origin $(0, 0)$ with the open immersion of the y-line of $V_{\widehat{\mathbf{T}},0,1} = \mathbb{A}^1 \cup_0 \mathbb{A}^1$ in X_ζ defined above, since both map $(0, 0)$ to Q. We take the resulting morphism of k-schemes $\mathbb{A}^1 \cup_0 \mathbb{A}^1 \to X_\zeta$ as the definition of L_ζ on the connected component $(V_{\widehat{\mathbf{T}},0}^{(\chi_{\frac{p-1}{2}},(\chi_{\frac{p-1}{2}})^s)}/W_0)_\zeta$ of $(V_{\widehat{\mathbf{T}},0}^{(1)}/W_0)_\zeta$.

7 A mod p Langlands parametrization in families for $F = \mathbb{Q}_p$

In this subsection we continue to assume that $F = \mathbb{Q}_p$ with $p \geq 5$.

7.1. Recall the mod p parametrization functor $P : \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \to \mathrm{SP}_{\widehat{\mathbf{G}},0}$ from [PS, 7.3.6]. For $\zeta \in \mathscr{Z}^\vee(k)$, let $\mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ be the full subcategory of $\mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ whose objets are the $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-modules whose Satake parameter is supported on the closed subscheme $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \subset V^{(1)}_{\widehat{\mathbf{T}},0}/W_0$. A $\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}$-module M lies in the category $\mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ if and only if: M is only supported in γ-components where $\gamma|_{\mathbb{F}_p^\times} = \zeta|_{\mathbb{F}_p^\times}$ and the operator U^2 acts on M via the $\mathbb{G}_m$-part of ζ. Set $\mathrm{SP}_{\widehat{\mathbf{G}},0,\zeta} := \mathrm{QCoh}((V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta)$, the category of quasi-coherent modules on the k-scheme $(V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta$. Then P induces a *mod p ζ-parametrization functor*

$$P_\zeta : \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}) \longrightarrow \mathrm{SP}_{\widehat{\mathbf{G}},0,\zeta}\,.$$

For $\zeta \in \mathscr{Z}^\vee(k)$, also recall the category $\mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta} := \mathrm{QCoh}(X_\zeta)$ of mod p Langlands parameters with determinant $\omega\zeta$ from 3.5; it induces the functor

$$L_{\zeta *} : \mathrm{SP}_{\widehat{\mathbf{G}},0,\zeta} \longrightarrow \mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta}$$

push-forward along the k-morphism $L_\zeta : (V^{(1)}_{\widehat{\mathbf{T}},0}/W_0)_\zeta \to X_\zeta$ from 4.9.

Finally recall that for $\zeta \in \mathscr{Z}^\vee(k)$, the functor of $I^{(1)}$-invariants $(\cdot)^{I^{(1)}} : \mathrm{Mod}^{\mathrm{sm}}(k[G]) \to \mathrm{Mod}(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$ induces a functor

$$(\cdot)^{I^{(1)}}_\zeta : \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \to \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p}),$$

by 4.5.

7.2. Definition. *Let $\zeta \in \mathscr{Z}^\vee(k)$. The* mod p ζ-Langlands parametrization functor *is the functor* $\mathrm{L}_\zeta\mathrm{P}_\zeta := L_{\zeta *} \circ P_\zeta$:

$$\begin{array}{c} \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_q}) \\ \Big\downarrow \\ \mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta}\,. \end{array}$$

Identifying ζ with a central character of G, the functor $\mathrm{L}_\zeta\mathrm{P}_\zeta$ extends to the category $\mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G])$ by precomposing with the functor $(\cdot)^{I^{(1)}}_\zeta : \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \to \mathrm{Mod}_\zeta(\mathcal{H}^{(1)}_{\overline{\mathbb{F}}_p})$. This gives the functor $\mathrm{L}_\zeta\mathrm{P}_\zeta \circ (\cdot)^{I^{(1)}}_\zeta$:

$$\begin{array}{c} \mathrm{Mod}^{\mathrm{sm}}_\zeta(k[G]) \\ \Big\downarrow \\ \mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta}\,. \end{array}$$

7.3. Theorem. *Suppose $F = \mathbb{Q}_p$ with $p \geq 5$. Fix a character $\zeta : Z(G) = \mathbb{Q}_p^\times \to k^\times$, corresponding to a point $(\zeta|_{\mathbb{F}_p^\times}, \zeta(p^{-1})) \in \mathscr{Z}^\vee(k)$ under the identification $\mathcal{Z}(G)^\vee \cong \mathscr{Z}^\vee(k)$ from 4.4.*

The mod p ζ-Langlands parametrization functor $\mathrm{L}_\zeta\mathrm{P}_\zeta$ interpolates the Langlands parametrization of the blocks of the category $\mathrm{Mod}^{\mathrm{ladm}}_\zeta(k[G])$, cf. 4.7 : for all $x \in X_\zeta(k)$ and for all $\pi \in b_{[\rho_x]}$,

$$\mathrm{L}_\zeta\mathrm{P}_\zeta(\pi^{I^{(1)}}) = \begin{cases} i_{x*}(\pi^{I^{(1)}}) & \text{if } x \text{ is not an exceptional point in the odd case} \\ i_{x*}(\pi^{I^{(1)}})^{\oplus 2} & \text{otherwise} \end{cases} \in \mathrm{LP}_{\widehat{\mathbf{G}},0,\zeta}$$

where $i_x : \mathrm{Spec}(k) \to X_\zeta$ is the k-point x.

Proof. By definition of a block of a category as a certain equivalence class of simple objects [Pas13], if $\pi \in b_{[\rho_x]}$ then in particular π is simple. Then $\pi^{I^{(1)}}$ is simple too, and hence has a central character. Therefore $P_\zeta(\pi^{I^{(1)}})$ is the underlying k-vector space of $\pi^{I^{(1)}}$ supported at the k-point $v \in (V^{(1)}_{\hat{\mathbf{T}},0}/W_0)_\zeta$ corresponding to its central character under the isomorphism $\mathscr{S}^{(1)}_{\overline{\mathbb{F}}_p}$, which lies on some connected component γ. Suppose $\dim_k(\pi^{I^{(1)}}) = 2$. Then $\pi^{I^{(1)}}$ is isomorphic to the simple standard module of $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p}$ with central character v, i.e. to $\mathrm{Sph}^\gamma(v)$, and hence $L_\zeta(v) = x$ by definition of the map of sets $L_\zeta(k)$. Suppose $\dim_k(\pi^{I^{(1)}}) = 1$. Then $\pi^{I^{(1)}}$ is one of the two spherical characters of $\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p}$ whose restriction to the center $Z(\mathcal{H}^{\gamma}_{\overline{\mathbb{F}}_p})$ is equal to v, i.e. it is one of the simple constituents of $(\mathrm{Sph}^\gamma(v))^{\mathrm{ss}}$, and hence again $L_\zeta(v) = x$ by definition of the map of sets $L_\zeta(k)$. Now if x is not an exceptional point in an odd case, then L_ζ is an open immersion at v, and otherwise it has ramification index 2 at v. The theorem follows. □

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