Irreducible mod p Lubin-Tate (φ, Γ)-modules

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Abstract

Let F be a finite extension of $\mathbb{Q}_p$. We determine the Lubin-Tate (φ, Γ)-modules associated to the absolutely irreducible mod p representations of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$.

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1 Introduction

Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers o_F, residue field $\mathbb{F}_q$ and uniformizer $\pi \in o_F$. Let $\overline{F}$ be an algebraic closure of F and let $\mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of F. By adapting the well-known formalism of Fontaine for the cyclotomic case, Kisin-Ren explained in [KR09] (see also the detailed exposition by Schneider [Sch17]) how to build an equivalence between the category of continuous representations of $\mathrm{Gal}(\overline{F}/F)$ in finitely generated o_F-modules and a category of étale Lubin-Tate (φ, Γ)-modules. Let $k/\mathbb{F}_q$ be a finite extension. Via reduction modulo π and extension of scalars, one deduces an equivalence of categories between smooth representations of $\mathrm{Gal}(\overline{F}/F)$ in finite dimensional k-vector spaces and a category of Lubin-Tate (φ, Γ)-modules over the Laurent series ring $k((t))$. When $F = \mathbb{Q}_p$ and in the cyclotomic case, the (φ, Γ)-modules corresponding to the n-dimensional *absolutely irreducible* mod p Galois representations have been explicitly calculated by Berger [Be10] and then used by him, in the case of $n = 2$, to give a direct proof of the compatibility of Colmez' p-adic local Langlands correspondence with Breuil's mod p correspondence for the group $\mathrm{GL}_2(\mathbb{Q}_p)$ in the irreducible case. In view of extending such results to more general base fields $F \neq \mathbb{Q}_p$, we propose in this note to explicitly calculate the Lubin-Tate (φ, Γ)-modules corresponding to the absolutely irreducible mod p representations of $\mathrm{Gal}(\overline{F}/F)$ for $F \neq \mathbb{Q}_p$, thereby generalizing Berger's result. As a method of proof, we adapt Berger's strategy to the Lubin-Tate setting.

In [GK18] (generalizing [GK16] for $F = \mathbb{Q}_p$) Grosse-Klönne constructs a fully faithful exact functor from the category of so-called supersingular modules for the pro-p Iwahori-Hecke algebra over k of the group $\mathrm{GL}_n(F)$ to the category of Lubin-Tate (φ, Γ)-modules over $k((t))$. It induces a bijection between the absolutely irreducible objects of rang n on both sides. In [PS1] we show, as an application of the results in this note, how to geometrically construct an inverse map to Grosse-Klönne's bijection in the case $n = 2$.

The second author thanks L. Berger for answering some questions on (φ, Γ)-modules.

2 Galois representations and Lubin-Tate (φ, Γ)-modules

Let F_n/F be the unramified extension of degree n. The irreducible smooth $\overline{\mathbb{F}}_q$-representations of $\mathrm{Gal}(\overline{F}/F)$ of dimension n are given by the representations

$$\mathrm{ind}_{\mathrm{Gal}(\overline{F}/F_n)}^{\mathrm{Gal}(\overline{F}/F)}(\chi)$$

smoothly induced from the regular $\overline{\mathbb{F}}_q$-characters χ of $\mathrm{Gal}(\overline{F}/F_n)$. The $\mathrm{Gal}(F_n/F)$-conjugates of χ induce isomorphic representations and there are no other isomorphisms between the representations [V94].

Let $\pi \in o_F$ be a uniformizer and let $q = p^f$. Let $\pi_{nf} \in \overline{F}$ be an element such that $\pi_{nf}^{q^n-1} = -\pi$. We then have Serre's fundamental character of level nf

$$\omega_{nf} : \mathrm{Gal}(\overline{F}/F_n) \longrightarrow \mathbb{F}_{q^n}^\times$$

given by $g \mapsto g(\pi_{nf})/\pi_{nf} \in \mu_{q^n-1}(\overline{F})$ followed by reduction mod π, cf. [Se72]. One has $\omega_{nf}^{\frac{q^n-1}{q-1}} = \omega_f|_{\mathrm{Gal}(\overline{F}/F_n)}$.

Let $\mathcal{I} \subset \mathrm{Gal}(\overline{F}/F)$ be the inertia subgroup and choose an element $\varphi \in \mathrm{Gal}(\overline{F}/F)$ lifting the Frobenius $x \mapsto x^q$ on $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Since $\omega_f : \mathcal{I} \to \mathbb{F}_q^\times$ is surjective [Se72, Prop. 2], we may and will assume $\omega_f(\varphi) = 1$.

A character ω_{nf}^h for $1 \leq h \leq q^n - 2$ is regular if and only if its conjugates $\omega_{nf}^h, \omega_{nf}^{qh}, ..., \omega_{nf}^{q^{n-1}h}$ are all distinct. Equivalently, if and only if h is q-primitive, that is, there is no $d < n$ such that h is a multiple of $(q^n - 1)/(q^d - 1)$. The representation $\mathrm{ind}_{\mathrm{Gal}(\overline{F}/F_n)}^{\mathrm{Gal}(\overline{F}/F)}(\omega_{nf}^h)$ is then defined over $\mathbb{F}_{q^n}$. It has a basis $\{v_0, ..., v_{n-1}\}$ of eigenvectors for the characters $\omega_{nf}^h, \omega_{nf}^{qh}, ..., \omega_{nf}^{q^{n-1}h}$ of $\mathrm{Gal}(\overline{F}/F_n)$ such that $\varphi(e_i) = e_{i-1}$ and $\varphi(e_0) = e_{n-1}$. In particular, its determinant coincides with ω_f^h on the subgroup $\mathrm{Gal}(\overline{F}/F_n)$ and takes φ to $(-1)^{n-1}$.

For $\lambda \in \overline{\mathbb{F}}_q^\times$, let μ_λ be the unramified character of $\mathrm{Gal}(\overline{F}/F)$ sending φ to λ^{-1}. Fix δ with $\delta^n = (-1)^{n-1}$. The representation

$$\mathrm{ind}(\omega_{nf}^h) := (\mathrm{ind}_{\mathrm{Gal}(\overline{F}/F_n)}^{\mathrm{Gal}(\overline{F}/F)}(\omega_{nf}^h)) \otimes \mu_\delta$$

is then uniquely determined by the two conditions

$$\det \mathrm{ind}(\omega_{nf}^h) = \omega_f^h \quad \text{and} \quad \mathrm{ind}(\omega_{nf}^h)|_{\mathcal{I}} = \omega_{nf}^h \oplus \omega_{nf}^{qh} \oplus ... \oplus \omega_{nf}^{q^{n-1}h}.$$

Let $k/\mathbb{F}_q$ be a finite extension. Every absolutely irreducible smooth k-representation of $\mathrm{Gal}(\overline{F}/F)$ of dimension n is isomorphic to $\mathrm{ind}(\omega_{nf}^h) \otimes \mu_\lambda$ for a q-primitive $1 \leq h \leq q^n - 2$ and a scalar $\lambda \in \overline{\mathbb{F}}_q^\times$ such that $\lambda^n \in k^\times$ and one has

$$\mathrm{ind}(\omega_{nf}^h) \otimes \mu_\lambda \simeq \mathrm{ind}(\omega_{nf}^{\tilde{h}}) \otimes \mu_{\tilde{\lambda}}$$

if and only if $\mathrm{Gal}(F_n/F).\omega_{nf}^h = \mathrm{Gal}(F_n/F).\omega_{nf}^{\tilde{h}}$ and $\lambda^n = \tilde{\lambda}^n$.

The theory of Lubin-Tate (φ, Γ)-modules and their relation to Galois representations is developed in [KR09] and [Sch17]. Let F_ϕ be a Lubin-Tate group for π, with Frobenius power series $\phi(t) \in o_F[[t]]$. The corresponding ring homomorphism $o_F \to \mathrm{End}(F_\phi)$ is denoted by $a \mapsto [a](t) = at + ...$ In particular, $[\pi](t) = \phi(t)$. Let F_∞/F be the extension generated by all torsion points of F_ϕ and let

$$H_F := \mathrm{Gal}(\overline{F}/F_\infty) \quad \text{and} \quad \Gamma := \mathrm{Gal}(\overline{F}/F)/H_F = \mathrm{Gal}(F_\infty/F).$$

Let $z = (z_j)_{j \geq 0}$ be a o_F-generator of the Tate module of F_ϕ. In particular, for $j \geq 0$

$$z_j = [\pi](z_{j+1}) \equiv z_{j+1}^q \mod \pi$$

and $N_{F(z_1)/F}(-z_1) = \pi$. This implies

$$z_{j+1}^q = z_j(1 + O(\pi)) \text{ for } j \geq 1 \text{ and } z_1^{q-1} = -\pi(1 + O(z_1)).$$

The Galois action on the generator z is given by a character $\chi_L : \mathrm{Gal}(\overline{F}/F) \to o_F^\times$, which is surjective and has kernel H_F. One has $\omega_f \equiv \chi_L \bmod \pi$.

The power series ring $o_F[[t]]$ has a Frobenius endomorphism and a Γ-action via $\varphi(f)(t) = f([\pi](t))$ and $(\gamma f)(t) = f([\chi_L(\gamma)](t))$ for $f(t) \in o_F[[t]]$. Via reduction mod π, these actions induce a Frobenius action and a Γ-action on $\mathbb{F}_q[[t]]$ and its quotient field $\mathbb{F}_q((t))$. This allows to introduce an abelian tensor category of étale Lubin-Tate (φ, Γ)-modules over $\mathbb{F}_q((t))$. It turns out to be canonically equivalent to the category of continuous finite-dimensional $\mathbb{F}_q$-representations of $\mathrm{Gal}(\overline{F}/F)$, cf. [KR09, 1.6], [Sch17, 3.2.7].

To explain the functor from (φ, Γ)-modules to Galois representations, we denote by $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$ and choose an embedding $\overline{F} \subseteq \mathbb{C}_p$. Recall that the tilt $\mathbb{C}_p^\flat$ of the perfectoid field $\mathbb{C}_p$ is an algebraically closed and perfect complete non-archimedean field of characteristic p. Its valuation ring $o_{\mathbb{C}_p^\flat}$ is given by the projective limit $\varprojlim_{x \mapsto x^q} o_{\mathbb{C}_p}/\pi o_{\mathbb{C}_p}$ and its residue field is $\overline{\mathbb{F}}_q$. There is a unique multiplicative section

$$s : \overline{\mathbb{F}}_q \longrightarrow o_{\mathbb{C}_p^\flat}, a \mapsto (\tau(a) \bmod \pi, \tau(a^{q^{-1}}) \bmod \pi, \tau(a^{q^{-2}}) \bmod \pi, ...)$$

where τ denotes the Teichmüller map $\overline{\mathbb{F}}_q \to o_{\mathbb{C}_p}$. There is an inclusion

$$\mathbb{F}_q((t)) \xrightarrow{\subset} \mathbb{C}_p^\flat, \; t \mapsto (..., z_j \bmod \pi, ...)$$

and one has $\mathbb{C}_p^\flat = o_{\mathbb{C}_p^\flat}[1/t]$. The field $\mathbb{C}_p^\flat$ is endowed with a continuous action of $\mathrm{Gal}(\overline{F}/F)$ and a Frobenius φ_q, which raises any element to its q-th power. We let $\mathbb{F}_q((t))^{\mathrm{sep}}$ denote the separable algebraic closure of $\mathbb{F}_q((t))$ inside $\mathbb{C}_p^\flat$. The field $\mathbb{F}_q((t))$ and its separable closure $\mathbb{F}_q((t))^{\mathrm{sep}}$ inherit the Frobenius action and the commuting $\mathrm{Gal}(\overline{F}/F)$-action from $\mathbb{C}_p^\flat$ and there is an isomorphism

$$H_F \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{F}_q((t))^{\mathrm{sep}}/\mathbb{F}_q((t))).$$

The functor $\mathscr{V}$ from (φ, Γ)-modules to Galois representations is then given by

$$D \rightsquigarrow \mathscr{V}(D) := (\mathbb{F}_q((t))^{\mathrm{sep}} \otimes_{\mathbb{F}_q((t))} D)^{\varphi=1}$$

where $\mathrm{Gal}(\overline{F}/F)$ acts diagonally (and via its projection to Γ on the second factor).

Now suppose that $k/\mathbb{F}_q$ is a finite extension. One can consider a k-representation of $\mathrm{Gal}(\overline{F}/F)$ as an $\mathbb{F}_q$-representation with a k-linear structure. Similarly, one may introduce (φ, Γ)-modules over $k((t)) = k \otimes_{\mathbb{F}_q} \mathbb{F}_q((t))$, where k has the trivial Frobenius and Γ-action. The functor $\mathscr{V}$ then restricts to an equivalence of categories between étale (φ, Γ)-modules over $k((t))$ and continuous finite-dimensional k-representations of $\mathrm{Gal}(\overline{F}/F)$.

3 The main result

We fix once and for all an element $y \in \mathbb{F}_q((t))^{\mathrm{sep}}$ such that

$$y^{(q^n-1)/(q-1)} = t.$$

For $g \in \mathrm{Gal}(\overline{F}/F)$, the power series

$$f_g(t) = \chi_L(g)t/g(t) \in 1 + (\pi)[[t]]$$

depends only on the class of g in Γ. The same is true for its mod π reduction $\overline{f}_g(t) = \omega_f(g)t/g(t)$. Note that the formula $f_g^s(t)$ defines an element of $o_F[[t]]$ for any $s \in \mathbb{Z}_p$.

3.1. Lemma. *One has* $g(y) = y\omega_{nf}^q(g)\overline{f}_g^{-\frac{q-1}{q^n-1}}(t)$ *in* $\mathbb{F}_q((t))^{\mathrm{sep}}$ *for all* $g \in \mathrm{Gal}(\overline{F}/F_n)$.

Proof. This is a version of [Be10, Lem. 2.1.3]. Let $j \geq 1$ and choose $\pi_{nf,j} \in o_{\mathbb{C}_p}$ such that

$$\pi_{nf,j}^{\frac{q^n-1}{q-1}} = z_j.$$

We write π_j for $\pi_{nf,j}$ in the following calculations. Let $g \in \mathrm{Gal}(\overline{F}/F_n)$. Then

$$(g(\pi_j)/\pi_j)^{\frac{q^n-1}{q-1}} = g(z_j)/z_j = \chi_L(g) f_g^{-1}(z_j)$$

and so the quotient of $g(\pi_j)/\pi_j$ by $f_g^{-\frac{q-1}{q^n-1}}(z_j)$ is a certain $\frac{q^n-1}{q-1}$-th root of $\chi_L(g)$. Since exponentiation with $\frac{q^n-1}{q-1} \in \mathbb{Z}_p^\times$ is surjective on the subgroup $1 + (\pi) \subset o_F^\times$ we may write this root as $\tau(\omega_{nf,j}(g))$, with an element $\omega_{nf,j}(g) \in \mathbb{F}_{q^n}^\times$, and arrive at

$$g(\pi_j)/\pi_j = \tau(\omega_{nf,j}(g)) f_g^{-\frac{q-1}{q^n-1}}(z_j).$$

The map $g \mapsto \omega_{nf,j}(g)$ is a character of the group $\mathrm{Gal}(\overline{F}/F_n)$, since

$$\omega_{nf,j}(g) \equiv g(\pi_j)/\pi_j \mod \mathfrak{m}_{\mathbb{C}_p}$$

in the field $\overline{\mathbb{F}}_q = o_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p}$ and this element is fixed by $\mathrm{Gal}(\overline{F}/F_n)$. Moreover, this character does not depend on the choice of π_j: a different choice π_j' differs from π_j by a $\frac{q^n-1}{q-1}$-th root of unity, i.e. by an element of F_n. Hence $g(\pi_j')/\pi_j' = g(\pi_j)/\pi_j$. By this independence, we see (using the element π_{j+1}^q as an alternative choice for π_j) that

$$\omega_{nf,j+1}^q = \omega_{nf,j} \text{ for } j \geq 1.$$

Moreover, $\pi_{nf,1}^{q^n-1} = z_1^{q-1} = -\pi(1 + O(z_1))$ and so $(\pi_{nf}/\pi_{nf,1})^{q^n-1} \equiv 1 \mod \mathfrak{m}_{\mathbb{C}_p}$. The quotient $\pi_{nf}/\pi_{nf,1} \mod \mathfrak{m}_{\mathbb{C}_p}$ is therefore fixed by $\mathrm{Gal}(\overline{F}/F_n)$, in other words

$$g(\pi_{nf,1})/\pi_{nf,1} \equiv g(\pi_{nf})/\pi_{nf} \mod \mathfrak{m}_{\mathbb{C}_p}$$

for all $g \in \mathrm{Gal}(\overline{F}/F_n)$ and so

$$\omega_{nf,1} = \omega_{nf}.$$

Now recall that there is an isomorphism $\varprojlim_{x \mapsto x^q} o_{\mathbb{C}_p} \simeq o_{\mathbb{C}_p^\flat}$ of multiplicative monoids given by reduction modulo π. We use the notation $u = (u^{(j)})$ for elements in the projective limit $\varprojlim_{x \mapsto x^q} o_{\mathbb{C}_p}$. The element $y \in o_{\mathbb{C}_p^\flat}$ is given by $(..., \pi_j \bmod \pi o_{\mathbb{C}_p}, ...)$. Its preimage $(y^{(j)})$ under the above isomorphism is therefore given by $y^{(j)} = \lim_{m \to \infty} \pi_{j+m}^{q^m}$. By the same argument, the preimage of the element $\overline{f}_g^{-\frac{q-1}{q^n-1}}(t)$ has coordinates

$$\overline{f}_g^{-\frac{q-1}{q^n-1}}(t)^{(j)} = \lim_{m \to \infty} (f_g^{-\frac{q-1}{q^n-1}}(z_{j+m}))^{q^m}.$$

The composite map $s : \overline{\mathbb{F}}_q \to o_{\mathbb{C}_p^\flat} \simeq \varprojlim_{x \mapsto x^q} o_{\mathbb{C}_p}$, which we also denote by s, is given by $a \mapsto (\tau(a), \tau(a^{q^{-1}}), \tau(a^{q^{-2}}), ...)$. Since

$$s(\omega_{nf}(g)^q)^{(j)} = \tau(\omega_{nf}(g)^{q^{-j+1}}) = \tau(\omega_{nf,j}(g)),$$

we may put everything together and obtain

$$\frac{g(y^{(j)})}{y^{(j)}} = \lim_{m \to \infty} (\frac{g(\pi_{j+m})}{\pi_{j+m}})^{q^m} = \tau(\omega_{nf,j}(g)) \lim_{m \to \infty} (f_g^{-\frac{q-1}{q^n-1}}(z_{j+m}))^{q^m} = s(\omega_{nf}(g)^q)^{(j)} \overline{f}_g^{-\frac{q-1}{q^n-1}}(t)^{(j)}.$$

Reducing this equation modulo π yields the assertion of the lemma. □

We now consider the (φ, Γ)-modules associated to the irreducible Galois representations of the form $\mathrm{ind}(\omega_{nf}^h)$.

3.2. Theorem. *The Lubin-Tate (φ,Γ)-module associated to an irreducible Galois representation of the form* $\mathrm{ind}(\omega_{nf}^h)$ *is defined over the ring* $\mathbb{F}_q((t))$ *and admits a basis* $e_0, e_1, ..., e_{n-1}$ *in which*

$$\gamma(e_j) = \overline{f}_\gamma(t)^{hq^j(q-1)/(q^n-1)} e_j$$

for all $\gamma \in \Gamma$ *and* $\varphi(e_j) = e_{j+1}$ *and* $\varphi(e_{n-1}) = (-1)^{n-1} t^{-h(q-1)} e_0$.

Proof. Let D be the (φ,Γ)-module described in the statement and let $W = \mathscr{V}(D)$. With $x = t^h e_0 \wedge ... \wedge e_{n-1}$, one has

$$\varphi(x) = \varphi(t)^h (-1)^{n-1} t^{-h(q-1)} e_1 \wedge ... \wedge e_{n-1} \wedge e_0 = t^{qh-h(q-1)} e_0 \wedge ... \wedge e_{n-1} = x.$$

Moreover,

$$\gamma(t)^h \prod_{j=0}^{n-1} \overline{f}_\gamma^{hq^j(q-1)/(q^n-1)}(t) = (\omega_f(\gamma)t/\overline{f}_\gamma(t))^h \overline{f}_\gamma^{h(q-1)/(q^n-1)\sum_{j=0}^{n-1} q^j} = \omega_f(\gamma)^h t^h$$

which implies $\gamma(x) = \omega_f(\gamma)^h x$ for all $\gamma \in \Gamma$. So $\det W = \omega_f^h$. Put $k := \mathbb{F}_{q^n}$ as a coefficient field, i.e. endowed with the trivial Frobenius action. To complete the proof, it remains to check that the restriction of $k \otimes_{\mathbb{F}_q} W$ to the inertia subgroup $\mathcal{I}$ is given by $\omega_{nf}^h \oplus \omega_{nf}^{qh} \oplus ... \oplus \omega_{nf}^{q^{n-1}h}$. There is a ring isomorphism

$$k \otimes_{\mathbb{F}_q} \mathbb{F}_q((t))^{\mathrm{sep}} \xrightarrow{\simeq} \prod_{j=0}^{n-1} \mathbb{F}_q((t))^{\mathrm{sep}},\ x \otimes z \mapsto (\varphi_q^j(x) z)$$

where φ_q is the q-Frobenius on k. The induced Frobenius and $\mathrm{Gal}(\overline{F}/F_n)$-action on $\prod_{j=0}^{n-1} \mathbb{F}_q((t))^{\mathrm{sep}}$ are given as

$$\begin{aligned} \varphi((x_0, ..., x_{n-1})) &= (\varphi_q(x_{n-1}), \varphi_q(x_0), ..., \varphi_q(x_{n-2})) \\ g((x_0, ..., x_{n-1})) &= (g(x_0), ..., g(x_{n-1})). \end{aligned}$$

Choose $\alpha \in \overline{\mathbb{F}}_q \subset \mathbb{F}_q((t))^{\mathrm{sep}}$ such that $\alpha^{q^n-1} = (-1)^{n-1}$ and define the elements

$$\begin{aligned} v_0 &= (\alpha y^h, 0, ..., 0) e_0 + (0, \alpha^q y^{qh}, 0, ..., 0) e_1 + ... + (0, ..., 0, \alpha^{q^{n-1}} y^{q^{n-1}h}) e_{n-1} \\ v_1 &= (0, \alpha y^h, 0, ..., 0) e_0 + (0, 0, \alpha^q y^{qh}, 0, ..., 0) e_1 + ... + (\alpha^{q^{n-1}} y^{q^{n-1}h}, 0, ..., 0) e_{n-1} \\ &\vdots \\ v_{n-1} &= (0, ...0, \alpha y^h) e_0 + (\alpha^q y^{qh}, 0, ..., 0) e_1 + ... + (0, ..., \alpha^{q^{n-1}} y^{q^{n-1}h}, 0) e_{n-1}. \end{aligned}$$

By definition of D, the vectors e_i form a $\mathbb{F}_q((t))$-basis for D and it follows easily that the vectors v_i form a $k \otimes_{\mathbb{F}_q} \mathbb{F}_q((t))^{\mathrm{sep}}$-basis for $k \otimes_{\mathbb{F}_q} (\mathbb{F}_q((t))^{\mathrm{sep}} \otimes_{\mathbb{F}_q((t))} D)$. Moreover,

$$\begin{aligned} \varphi((0, ..., 0, \alpha^{q^{n-1}} y^{q^{n-1}h}) e_{n-1}) &= (\alpha^{q^n} y^{q^n h}, 0, ..., 0) \varphi(e_{n-1}) \\ &= (\alpha^{q^n} y^{q^n h}, 0, ..., 0)) (-1)^{n-1} t^{-h(q-1)} e_0 = (\alpha y^h, 0, ..., 0) e_0 \end{aligned}$$

since $\alpha^{q^n} = (-1)^{n-1}\alpha$ and $y^{q^n} t^{1-q} = y$. This means

$$\begin{aligned} \varphi(v_0) &= (0, \alpha^q y^{qh}, 0, ..., 0) \varphi(e_0) + (0, 0, \alpha^{q^2} y^{q^2 h}, 0, ..., 0) \varphi(e_1) + ... + (\alpha^{q^n} y^{q^n h}, 0, ..., 0) \varphi(e_{n-1}) \\ &= (0, \alpha^q y^{qh}, 0, ..., 0) e_1 + (0, 0, \alpha^{q^2} y^{q^2 h}, 0, ..., 0) e_2 + ... + (\alpha y^h, 0, ..., 0) e_0 \\ &= v_0. \end{aligned}$$

Similarly, one shows $\varphi(v_j) = v_j$ for $j \geq 1$, so that

$$v_0, ..., v_{n-1} \in k \otimes_{\mathbb{F}_q} (\mathbb{F}_q((t))^{\mathrm{sep}} \otimes_{\mathbb{F}_q((t))} D)^{\varphi=1} = k \otimes_{\mathbb{F}_q} \mathscr{V}(D) = k \otimes_{\mathbb{F}_q} W.$$

Now if $g \in \mathrm{Gal}(\overline{F}/F_n)$, then $g(y) = y\omega_{nf}^q(g)c_g$ with $c_g := \overline{f}_g^{-\frac{q-1}{q^n-1}}(t)$ by lemma 3.1 and $g(e_j) = c_g^{-q^j h} e_j$ by definition of D. Hence

$$g(y)^{q^j h} g(e_j) = (y\omega_{nf}^q(g))^{q^j h} e_j.$$

If $g \in \mathcal{I}$, then $g(\alpha) = \alpha$ and then altogether

$$\begin{aligned} g(v_0) &= (\alpha g(y)^h, 0, ...)g(e_0) + (0, \alpha^q g(y)^{qh}, 0, ...)g(e_1) + ... + (0, ..., \alpha^{q^{n-1}} g(y)^{q^{n-1}h})g(e_{n-1}) \\ &= \omega_{nf}^{qh}(g) \cdot ((\alpha y^h, 0, ...)e_0 + (0, \alpha^q y^{qh}, 0, ...)e_1 + ... + (0, ..., \alpha^{q^{n-1}} y^{q^{n-1}h})e_{n-1}) \\ &= \omega_{nf}^{qh}(g) \cdot v_0, \end{aligned}$$

where $\cdot$ refers to the left k-structure of $\prod_{j=0}^{n-1} \mathbb{F}_q((t))^{\mathrm{sep}}$. Similarly, one shows $g(v_j) = \omega_{nf}^{q^{1-j}h}(g)v_j$ for all $j \geq 1$ and $g \in \mathcal{I}$. Since $\omega_{nf}^{q^n} = \omega_{nf}$ and hence $\omega_{nf}^{q^{1-j}h} = \omega_{nf}^{q^{n+1-j}h}$, this proves that the restriction of $k \otimes_{\mathbb{F}_q} W$ to $\mathcal{I}$ is given by the sum of the characters $\omega_{nf}^h, \omega_{nf}^{qh}, ..., \omega_{nf}^{q^{n-1}h}$. $\square$

As explained above, one may pass from irreducible representations of the form $\mathrm{ind}(\omega_{nf}^h)$ to the general case by twisting with characters. Note that any character $\mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{F}}_q^\times$ can be written in the form $\omega_f^s \mu_\lambda$, for $1 \leq s \leq q-1$ and $\lambda \in \overline{\mathbb{F}}_q^\times$.

3.3. Lemma. *Let $k/\mathbb{F}_q$ be a finite extension. The (φ, Γ)-module associated a Galois character of the form $\omega_f^s \mu_\lambda$ with $\lambda \in k^\times$ admits a basis e such that $\varphi(e) = \lambda \cdot e$ and $\gamma(e) = \omega_f^s(\gamma) \cdot e$ for all $\gamma \in \Gamma$.*

Proof. Since the functor $\mathscr{V}$ preserves the tensor product, we may discuss the two characters ω_f^s and μ_λ separately. For the twists by a character of Γ, such as ω_f^s, see [SV16, Remark 4.6]. So let $V = \mu_\lambda = k$ and let

$$D(V) = (\mathbb{F}_q((t))^{\mathrm{sep}} \otimes_{\mathbb{F}_q} V)^{H_F}$$

be the associated (φ, Γ)-module. It is instructive to check the case $k = \mathbb{F}_q$ first. Here, we choose $\beta \in \overline{\mathbb{F}}_q$ with $\beta^{q-1} = \lambda$ and put $e = \beta \otimes 1 \in \mathbb{F}_q((t))^{\mathrm{sep}} \otimes_{\mathbb{F}_q} V$. Since $\beta \neq 0$, we have $e \neq 0$. Moreover, $\mathcal{I}$ acts trivial on e and for $\varphi \in \mathrm{Gal}(\overline{F}/F)$ we find

$$\varphi(e) = \varphi(\beta) \otimes \varphi(1) = \beta^q \otimes \lambda^{-1} = \beta\lambda \otimes \lambda^{-1} = \beta \otimes 1 = e.$$

Hence e is indeed $\mathrm{Gal}(\overline{F}/F)$-invariant. Moreover, if ϕ denotes the Frobenius endomorphism on $D(V)$ we have

$$\phi(e) = \phi(\beta) \otimes 1 = \beta^q \otimes 1 = \lambda\beta \otimes 1 = \lambda e.$$

Now suppose that $k = \mathbb{F}_{q^n}$ for some n and $\lambda \in k^\times$. We use the ring isomorphism

$$k \otimes_{\mathbb{F}_q} \mathbb{F}_q((t))^{\mathrm{sep}} \xrightarrow{\simeq} \prod_{j=0}^{n-1} \mathbb{F}_q((t))^{\mathrm{sep}},\ x \otimes z \mapsto (\varphi_q^j(x)z)$$

where φ_q is the q-Frobenius on k. It is $\mathrm{Gal}(\overline{F}/F_n)$-equivariant, where the Galois action on the right-hand side is componentwise (see proof of the above theorem). By the normal basis theorem, there is $x \in k^\times$ such that its conjugates $\varphi_q^j(x)$ are linearly independent over $\mathbb{F}_q$. The j-th copy $\mathbb{F}_q((t))^{\mathrm{sep}}$ in the above product has therefore a $\mathbb{F}_q((t))^{\mathrm{sep}}$-basis element $e_j := \varphi_q^j(x) \in k = V$ on which $\mathcal{I}$ acts trivial and on which the element $\varphi^n \in \mathrm{Gal}(\overline{F}/F_n)$ acts by λ^{-n}. Choose $\beta \in \overline{\mathbb{F}}_q$ such that $\beta^{q^n-1} = \lambda^n$ and put $v_j = \beta e_j$. Then $\mathcal{I}$ obviously acts trivial on v_j and the same holds for φ^n, since

$$\varphi^n(v_j) = \varphi^n(\beta)\varphi^n(e_j) = \beta^{q^n}\lambda^{-n}e_j = \beta\lambda^n\lambda^{-n}e_j = v_j.$$

Hence, $\mathcal{I}$ and φ^n act trivial on $(v_j) \in \prod_{j=0}^{n-1} \mathbb{F}_q((t))^{\mathrm{sep}}$ and then also on its preimage $v = x \otimes \beta \in k \otimes_{\mathbb{F}_q} \mathbb{F}_q((t))^{\mathrm{sep}}$. Note that $v \neq 0$ since $x, \beta \neq 0$. Write $N = \prod_{j=0}^{n-1} \varphi^j$ and $e = N(v)$. Then e

is fixed by $\mathcal{I}$ (since $\mathcal{I}$ is normalized by the φ^j) and is fixed by φ by construction. Hence, e is $\mathrm{Gal}(\overline{F}/F)$-invariant. Note that $e \neq 0$, since $e = N(x) \otimes N(\beta)$ and $N(x), N(\beta) \neq 0$ and so e is indeed a basis element of $D(V)$ on which Γ acts trivial. Finally, write $e = \sum_{j=0}^{n-1} \varphi_q^j(x) \otimes z_j$ with $z_j \in \mathbb{F}_q((t))^{\mathrm{sep}}$. The Frobenius endomorphism ϕ on $D(V)$ satisfies

$$\phi(e) = \sum_j \varphi_q^j(x) \otimes \varphi(z_j) = \varphi(\sum_j \varphi^{-1}(\varphi_q^j(x)) \otimes z_j) = \varphi(\sum_j \lambda\varphi_q^j(x) \otimes z_j) = \lambda\varphi(e) = \lambda e.$$

□

3.4. Corollary. *Let $k/\mathbb{F}_q$ be a finite extension. The (φ, Γ)-module associated to an irreducible Galois representation of the form $(\mathrm{ind}(\omega_{nf}^h)) \otimes \omega_f^s \mu_\lambda$, for $1 \leq s \leq q-1$ and $\lambda^n \in k^\times$, is defined over the ring $k((t))$ and admits a basis $e_0, e_1, ..., e_{n-1}$ in which*

$$\gamma(e_j) = \omega_f(\gamma)^s \overline{f}_\gamma(t)^{hq^j(q-1)/(q^n-1)} e_j$$

for all $\gamma \in \Gamma$ and $\varphi(e_j) = \lambda e_{j+1}$ and $\varphi(e_{n-1}) = (-1)^{n-1} t^{-h(q-1)} \lambda e_0$.

Proof. This follows from the preceding lemma and the theorem. □

Since $\omega_{nf}^{\frac{q^n-1}{q-1}} = \omega_f$, every irreducible representation of $\mathrm{Gal}(\overline{F}/F)$ of dimension n is therefore isomorphic to $\mathrm{ind}(\omega_{nf}^h) \otimes \omega_f^s \mu_\lambda$ for $1 \leq s \leq q-1$, a scalar $\lambda \in \overline{\mathbb{F}}_q^\times$ and a q-primitive $1 \leq h \leq \frac{q^n-1}{q-1} - 1$.

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