Néron's pairing and relative algebraic equivalence

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Abstract

Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K. Let X_K be a proper smooth and geometrically connected scheme over K. Néron defined a canonical pairing on X_K between 0-cycles of degree zero and divisors which are algebraically equivalent to zero. When X_K is an abelian variety, and if one restricts to those 0-cycles supported on K-rational points, Néron gave an expression of his pairing involving intersection multiplicities on the Néron model A of A_K over R. When X_K is a curve, Gross and Hriljac gave independently an analogous description of Néron's pairing, but for arbitrary 0-cycles of degree zero, by means of intersection theory on a proper flat regular R-model X of X_K.

In this article, we show that these intersection computations are valid for an arbitrary scheme X_K as above and arbitrary 0-cyles of degree zero, by using a proper flat *normal and semi-factorial* model X of X_K over R. When $X_K = A_K$ is an abelian variety, and $X = \overline{A}$ is a semi-factorial compactification of its Néron model A, these computations can be used to study the relative algebraic equivalence on $\overline{A}/R$. We then obtain an interpretation of Grothendieck's duality for the Néron model A, in terms of the Picard functor of $\overline{A}$ over R. Finally, we give an explicit description of Grothendieck's duality pairing when A_K is the Jacobian of a curve of index one.

Key words: Néron's symbol, Picard functor, Néron models, duality, Grothendieck's pairing

Mathematics Subject Classification (2010): 14K30, 14G40, 14K15, 11G10

Contents

1 Introduction

Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K. Let X_K be a proper smooth and geometrically connected scheme over K. Denote by $Z_0^0(X_K)$ the group of 0-cycles of degree zero on X_K, and by $\mathrm{Div}^0(X_K)$ the group of divisors which are algebraically equivalent to zero on X_K. For each $c_K \in Z_0^0(X_K)$ and $D_K \in \mathrm{Div}^0(X_K)$ with disjoint supports, Néron attached a rational number

$$\langle\, c_K \;,\; D_K \,\rangle \in \mathbb{Q},$$

by using the unique (up to constant) Néron function associated to D_K. This defines a bilinear pairing $\langle\;,\;\rangle$: see [N] II 9.3.

Suppose first that $X_K = A_K$ is an abelian variety, and denote by A its Néron model over R. By definition of A, any K-rational point of A_K extends to a section of A over R. Then, if c_K is supported on K-rational points, Néron showed that the pairing attached to A_K can be decomposed as follows:

$$\langle\, c_K \;,\; D_K \,\rangle = i(c_K \;,\; D_K) + j(c_K \;,\; D_K), \tag{1}$$

where $i(c_K \;,\; D_K)$ is the intersection multiplicity $(\overline{c_K}.\overline{D_K}) \in \mathbb{Z}$ of the schematic closures in A, and $j(c_K \;,\; D_K) \in \mathbb{Q}$ depends only on the specialization of c_K on the group Φ_A of connected components of the special fiber A_k: see [N] III 4.1 and [La] 11.5.1.

Suppose now that X_K is a curve, and denote by X a proper flat regular model of X_K over R. Let M be the intersection matrix of the special fiber X_k of X/R: if $\Gamma_1, \dots, \Gamma_\nu$ are the irreducible components of X_k equipped with their reduced scheme structure, the $(i,j)^{\mathrm{th}}$ entry of M is the intersection number $(\Gamma_i \cdot \Gamma_j)$. Let $D_K \in \mathrm{Div}^0(X_K)$ and let $\overline{D_K}$ be its closure in X. Computing the degree $(\overline{D_K}.\Gamma_i)$ of $\overline{D_K}$ along each Γ_i, we get a vector $\rho(\overline{D_K}) \in \mathbb{Z}^\nu$. Next, as a consequence of intersection theory on X, there exists a vector $V \in \mathbb{Q}^\nu$ such that $\rho(\overline{D_K}) = MV$. Denote again by V the $\mathbb{Q}$-linear combination of the Γ_i where the coefficient of Γ_i is the i^{th} entry of V. Then, for any $c_K \in Z_0^0(X_K)$ whose support is disjoint from that of D_K, the following formula holds:

$$\langle\, c_K \;,\; D_K \,\rangle = (\overline{c_K}.\overline{D_K}) + (\overline{c_K}.(-V)), \tag{2}$$

where the second intersection number is defined by $\mathbb{Q}$-linearity from the $(\overline{c_K}.\Gamma_i)$. See [G],[H] and [La2] III 5.2. Now let J_K be the Jacobian of X_K and let J be its Néron model over R. Following the point of view of Bosch and Lorenzini ([BL] 4.3), it results from Raynaud's theory of the Picard functor $\mathrm{Pic}_{X/R}$ ([R] Section 8) that the term $(\overline{c_K}.(-V))$ depends only on the specialization of $(c_K) \in J_K(K)$ into the group of components Φ_J of J_k.

In Section 2, we provide a unified approach to these two descriptions of Néron's pairing. More precisely, for an arbitrary proper geometrically normal and geometrically connected scheme X_K, there always exists some proper flat normal semi-factorial model X of X_K over R ([P] Theorem 2.6). Recall that X/R is *semi-factorial* if the restriction homomorphism on Picard groups $\mathrm{Pic}(X) \to \mathrm{Pic}(X_K)$ is surjective. Note that a regular model is semi-factorial. Using the theory of the Picard functor of semi-factorial models, we define a pairing $[\;,\;]$ on X_K involving intersection multiplicities on X (Definition 2.1.1). It turns out that this pairing depends only on X_K, and *coincides* with Néron's pairing when the latter is defined, that is, when X_K is smooth:

$$\langle\ ,\ \rangle = [\ ,\ ] \tag{3}$$

(Theorem 2.2.1). If $X_K = A_K$ is an abelian variety and $X = \overline{A}$ is a semi-factorial compactification of its Néron model A, then equality (3) provides decomposition (1) for 0-cycles supported on K-rational points. If X_K is a curve and X a proper flat regular model of X_K, then the intersection matrix of X_k is defined, and equality (3) is exactly formula (2).

In Section 3, we consider an abelian variety A_K, with dual A'_K. By definition, the abelian variety A'_K parametrizes the divisors on A_K which are algebraically equivalent to zero, that is, $A'_K = \mathrm{Pic}^0_{A_K/K}$. Now, let A'/R be the Néron model of A'_K, and denote by $(A')^0$ its identity component. By restricting to the generic fiber, the group of sections $(A')^0(R)$ can be viewed as a subgroup of $A'_K(K)$. On the other hand, let $\overline{A}$ be a normal semi-factorial compactification of A, let $\mathrm{Pic}_{\overline{A}/R}$ be its relative Picard functor, and denote by $\mathrm{Pic}^0_{\overline{A}/R}$ the component of the zero section. By restricting to the generic fiber, the group $\mathrm{Pic}^0_{\overline{A}/R}(R)$ can be viewed as a subgroup of $\mathrm{Pic}^0_{A_K/K}(K)$.

In Theorem 3.2.1, we investigate the relationship between the two groups

$$(A')^0(R) \quad \text{and} \quad \mathrm{Pic}^0_{\overline{A}/R}(R) \quad (\text{contained in } A'_K(K) = \mathrm{Pic}^0_{A_K/K}(K)).$$

We show that they are *equal* as soon as the duality conjecture of Grothendieck about A and A' is true ([SGA 7] IX 1.3). More precisely, Grothendieck defined a pairing between the component groups of the special fibers of A and A', and he conjectured that this pairing is perfect. This duality statement has been proved in many situations (e.g., see the introduction of [BL] for a detailed list of the known cases, and also [Loe]), but it remains open in equal characteristic $p > 0$. Here, we give an equivalent formulation of Grothendieck's conjecture, in terms of Cartier divisors on $\overline{A}$. As a consequence, when the conjecture is true, we obtain the equality $(A')^0(R) = \mathrm{Pic}^0_{\overline{A}/R}(R)$. As a Cartier divisor on $\overline{A}$ is said to be algebraically equivalent to zero relative to R if its image into $\mathrm{Pic}_{\overline{A}/R}(R)$ is contained $\mathrm{Pic}^0_{\overline{A}/R}(R)$, the latter equality says that these divisors are parametrized by $(A')^0$. The main ingredients for the proof are a theorem of Bosch and Lorenzini about Néron's and Grothendieck's pairings ([BL] 4.4), and the study of the pairing $[\ ,\ ]$ introduced above, especially for 0-cycles supported on *non-rational* points (Proposition 3.4.2).

In Section 4, we examine the relationship between Néron's and Grothendieck's pairing for the Jacobian of a curve, following Bosch and Lorenzini [BL] 4.6, and Lorenzini [Lor] 3.4. Here we take into account the *index of the curve* (Theorem 4.1.1). As a consequence, we obtain the perfectness of Grothendieck's pairing when this index is prime to the characteristic of the residue field k (Corollary 4.1.2).

Acknoledgements. I would like to thank Michel Raynaud for many enlightening discussions about Néron's and Grothendieck's pairings. I am indebted to Qing Liu for his valuable teaching of intersection theory on relative schemes. I am very grateful to Pascal Autissier, Siegfried Bosch, Dino Lorenzini and Laurent Moret-Bailly for their careful reading and for many helpful comments. I thank the referee for his meticulous reading and for many useful remarks.

2 Néron's pairing and intersection multiplicities

In this article, let us adopt the following terminology: a *divisor* on a scheme will always be a *Cartier divisor*.

2.1 A canonical pairing computed on semi-factorial models

Let R be a discrete valuation ring with fraction field K and residue field k. **We assume R complete and k algebraically closed**. Let X_K be a proper geometrically normal and geometrically connected scheme over K. From [P] Theorem 2.6, there exists a model X/R of X_K, that is, an R-scheme with generic fiber X_K, which is proper, flat, normal and *semi-factorial*: every invertible sheaf on X_K can be extended to an invertible sheaf on X. To each 0-cycle $c_K \in Z_0^0(X_K)$ and divisor $D_K \in \mathrm{Div}^0(X_K)$ with support disjoint from that of c_K, we will attach a number $[c_K\ ,\ D_K]_X \in \mathbb{Q}$ using intersection multiplicities on X. For this purpose, let us first recall some definitions and one result.

Intersection multiplicities. Let X/R be a proper R-scheme. Let c_K be a 0-cycle on the generic fiber X_K. Denote by $\overline{c_K}$ its schematic closure in X: if $c_K = \sum_i n_i[x_{K,i}]$, then $\overline{c_K} = \sum_i n_i[\overline{x_{K,i}}]$, where $\overline{x_{K,i}}$ is the closure in X of the closed point $x_{K,i}$ of X_K. On the other hand, let Δ be a divisor on X whose support does not meet that of c_K. The *intersection multiplicity* $(\overline{c_K}.\Delta)$ of $\overline{c_K}$ and Δ on X is defined as follows. Let x_K be a point of the support of c_K. Let $Z := \overline{x_K}$ be its schematic closure in X. This is an integral scheme, finite and flat over R, which is local because R is henselian. Set $x_k := Z \cap X_k$. If $f \in K(X)$ is a local equation for Δ in a neighborhood of x_k, then $(\overline{c_K}.\Delta)_{x_k}$ is the order of $f|_Z$ at x_k: writing $f|_Z = a/b$ with regular $a, b \in \mathcal{O}(Z)$, then

$$(\overline{c_K}.\Delta)_{x_k} = \mathrm{length}_{\mathcal{O}(Z)}\big(\mathcal{O}(Z)/(a)\big) - \mathrm{length}_{\mathcal{O}(Z)}\big(\mathcal{O}(Z)/(b)\big)$$

([F] page 8). The whole intersection multiplicity $(\overline{c_K}.\Delta)$ is defined by $\mathbb{Z}$-linearity.

Let us also give another description of $(\overline{c_K}.\Delta)_{x_k}$, which will be useful in the sequel. As R is excellent, the normalization $\widetilde{Z} \to Z$ is finite. Moreover, as k is algebraically closed,

$$\mathrm{length}_{\mathcal{O}(Z)}\big(\mathcal{O}(Z)/(a)\big) = \mathrm{length}_R\big(\mathcal{O}(Z)/(a)\big),$$

for any regular $a \in \mathcal{O}(Z)$, and the same formula holds with Z replaced by $\widetilde{Z}$ (*loc. cit.* Appendix A.1.3). But

$$\mathrm{length}_R\big(\mathcal{O}(Z)/(a)\big) = \mathrm{length}_R\big(\mathcal{O}(\widetilde{Z})/(a)\big)$$

for any regular $a \in \mathcal{O}(Z)$ (see [BLR], end of page 237). Thus, if $f \in K(X)$ is a local equation for Δ in a neighborhood of x_k, we have obtained that

$$(\overline{c_K}.\Delta)_{x_k} = \begin{cases} \mathrm{length}_{\mathcal{O}(\widetilde{Z})}\big(\mathcal{O}(\widetilde{Z})/(f)\big) & \text{if } f|_{\widetilde{Z}} \in \mathcal{O}(\widetilde{Z}), \\ -\,\mathrm{length}_{\mathcal{O}(\widetilde{Z})}\big(\mathcal{O}(\widetilde{Z})/(f^{-1})\big) & \text{otherwise.} \end{cases}$$

Relative algebraic equivalence and relative τ-equivalence. ([R] 3.2 d) and [SGA 6] XIII 4) If G is a commutative group scheme locally of finite type over a field, the *identity component* G^0 of G is the open subscheme of G whose underlying topological space is the connected component of the identity element of G. The τ*-component* of G is open subgroup scheme G^τ of G which is the inverse image of the torsion subgroup of G/G^0. When G is a commutative group functor over a scheme T, whose fibers are representable by schemes locally of finite type, the *identity component* (resp. τ*-component*) of G is the subfunctor G^τ of G whose fibers are the G_t^0, $t \in T$ (resp. G_t^τ, $t \in T$). Note that $G^0 \subseteq G^\tau$.

Let $Z \to T$ be a proper morphism of schemes. Then the fibers of the Picard functor $\mathrm{Pic}_{Z/T}$ are representable by schemes locally of finite type ([Mur] and [O]).

Let $\mathcal{L}$ be an invertible $\mathcal{O}_Z$-module. The sheaf $\mathcal{L}$ is said to be *algebraically equivalent to zero relative to* T if its image into $\mathrm{Pic}_{Z/T}(T)$ belongs to the subgroup $\mathrm{Pic}^0_{Z/T}(T)$, that is $\mathcal{L}_t \in \mathrm{Pic}^0_{Z_t/t}(t)$ for all $t \in T$. When there is no ambiguity about the base scheme T, we will just say that $\mathcal{L}$ is *algebraically equivalent to zero*. Similarly, the sheaf $\mathcal{L}$ is said to be τ*-equivalent to zero relative to* T if its image into $\mathrm{Pic}_{Z/T}(T)$ belongs to the subgroup $\mathrm{Pic}^\tau_{Z/T}(T)$, that is $\mathcal{L}_t \in \mathrm{Pic}^\tau_{Z_t/t}(t)$ for all $t \in T$. If D is a divisor on Z, it is *algebraically equivalent to zero* (resp. τ*-equivalent to zero*) *relative to* T if the associated invertible sheaf $\mathcal{O}_Z(D)$ is. We denote by $\mathrm{Div}^0(Z/T)$ (resp. $\mathrm{Div}^\tau(Z/T)$) the group of divisors on Z which are algebraically equivalent to zero (resp. τ-equivalent to zero) relative to T. Then $\mathrm{Div}^0(Z/T) \subseteq \mathrm{Div}^\tau(Z/T)$.

Relative algebraic equivalence and semi-factoriality. Let X/R be a proper flat *semi-factorial* R-scheme. Suppose that the generic fiber X_K is geometrically normal and geometrically connected. Its Picard variety $\mathrm{Pic}^0_{X_K/K,\mathrm{red}}$ is then an abelian variety ([FGA VI] 3.2). Let A/R be its Néron model, and let n be the exponent of the component group of the special fiber of A. In this situation, Corollary 3.14 of [P] can be read as follows: for any divisor D_K on X_K which is algebraically equivalent to zero, there exists a divisor Δ on X which is algebraically equivalent to zero relative to R and whose generic fiber Δ_K is equal to nD_K.

Definition 2.1.1. *Let X_K be a proper, geometrically normal and geometrically connected scheme over K. Let X/R be a proper, flat, normal and semi-factorial model of X_K over R.*

Consider $c_K \in Z_0^0(X_K)$ and $D_K \in \mathrm{Div}^\tau(X_K)$ with disjoint supports. Let $\overline{c_K}$ be the schematic closure of c_K in X. Choose any $(n,\Delta) \in (\mathbb{Z}\setminus\{0\}) \times \mathrm{Div}^\tau(X/R)$ such that $\Delta_K = nD_K$. Then set

$$[c_K\ ,\ D_K]_X := \frac{1}{n}(\overline{c_K}.\Delta) \quad \in \mathbb{Q}\,.$$

This definition makes sense because $\frac{1}{n}\Delta \in \mathrm{Div}^\tau(X/R) \otimes_{\mathbb{Z}} \mathbb{Q}$ is uniquely determined by D_K, up to a rational multiple of the principal divisor X_k. Indeed, if (n', Δ') is another choice in Definition 2.1.1, then the divisor $n'\Delta - n\Delta'$ is τ-equivalent to zero on X and equal to zero on X_K. Thus, as X is normal, this difference is a rational multiple of X_k ([R] 6.4.1 3)). Now note that $(\overline{c_K}.X_k)$ is equal to the degree of c_K, which is zero, so that $\frac{1}{n}(\overline{c_K}.\Delta) = \frac{1}{n'}(\overline{c_K}.\Delta')$.

Next, one checks easily that the symbol $[\ ,\ ]_X$ is *bilinear* (in its definition domain). To prove that this pairing does not depend on the choice of X, we will use the following lemma.

We will denote by $(\cdot)_*$ and $(\cdot)^*$ the push-forward of cycles and the pull-back of divisors respectively (see [F] 20.1.3, and [Li] 7.1.29, 7.1.33, 7.1.34, respectively).

Lemma 2.1.2. *Let X and X' be integral schemes, proper over R. Let $\varphi : X \to X'$ be an R-morphism. Let $c_K \in Z_0^0(X_K)$ and let $\overline{c_K}$ be its schematic closure in X. Let Δ' be a divisor on X' whose support does not meet that of $(\varphi_K)_* c_K$. Then the following projection formula holds:*

$$\overline{c_K}.\varphi^*\Delta' = \varphi_*\overline{c_K}.\Delta'.$$

In particular, let X and X' be proper, flat, normal and semi-factorial schemes over R, with geometrically normal and geometrically connected generic fibers, so that $[\ ,\ ]_X$ and $[\ ,\ ]_{X'}$ are defined. Let $\varphi : X \to X'$ be an R-morphism. Let $c_K \in Z_0^0(X_K)$, and let $D'_K \in \mathrm{Div}^\tau(X'_K)$ whose support does not meet that of $(\varphi_K)_* c_K$. Then the following equality holds:

$$[c_K\ ,\ (\varphi_K)^* D'_K]_X = [(\varphi_K)_* c_K\ ,\ D'_K]_{X'}.$$

Proof. Let us first note that the divisors $\varphi^*\Delta'$ (and $(\varphi_K)^*D'_K$) are well-defined. Indeed, as φ is proper, its image Y is a closed subset of X'. Endow Y with its reduced scheme structure. As X is reduced, φ factors through Y:

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[dr, "\varphi"'] & Y \arrow[d, hook, "\iota"] \\ & X'. \end{tikzcd}$$

Now, by hypothesis, the support of Δ' is disjoint from that of $(\varphi_K)_*c_K$. In particular, Y is not contained in the support of Δ'. So the pull-back $\iota^*\Delta'$ is well-defined. Next, X and Y being integral and ψ dominant, $\varphi^*\Delta' := \psi^*(\iota^*\Delta')$ is well-defined.

Let us now recall the proof of the projection formula $\overline{c_K}.\varphi^*\Delta' = \varphi_*\overline{c_K}.\Delta'$. Let x_K be a closed point of the support of c_K, let Z be its schematic closure in X, set $x_k := Z \cap X_k$ and let $\widetilde{Z}$ be the normalization of Z. The reduced scheme $V := \varphi(Z)$ is the schematic closure of $\varphi(x_K)$ and we have $\varphi(x_k) = V \cap X_k$. Denote by $\widetilde{V}$ the normalization of V. The morphism φ induces a finite surjective morphism $Z \to V$, which in turn induces a finite surjective morphism $\widetilde{Z} \to \widetilde{V}$ (R is excellent). Let f' be a local equation of Δ' at $\varphi(x_k)$. Suppose for example that $f'|_{\widetilde{V}} \in \mathcal{O}(\widetilde{V})$. The equality $(\varphi_*\overline{c_K}.\Delta')_{x_k} = (\overline{c_K}.\varphi^*\Delta')_{x_k}$ to be proved can be written as

$$[K(Z) : K(V)] \cdot \text{length}\left(\mathcal{O}(\widetilde{V})/(f')\right) = \text{length}\left(\mathcal{O}(\widetilde{Z})/(\varphi^* f')\right).$$

But $[K(Z) : K(V)]$ is equal to the ramification index of the discrete valuation rings extension $\mathcal{O}(\widetilde{V}) \to \mathcal{O}(\widetilde{Z})$. Consequently, the above formula is true.

Now, when the pairings $[\ ,\ ]_X$ and $[\ ,\ ]_{X'}$ are defined, the projection formula can be written as the equality $[c_K\ ,\ (\varphi_K)^*D'_K]_X = [(\varphi_K)_*c_K\ ,\ D'_K]_{X'}$. Indeed, let Δ' be a divisor which is τ-equivalent to zero on X' and let n' be a non-zero integer such that $(\Delta')_K = n'D'_K$. The direct image $\varphi_*\overline{c_K}$ of the schematic closure of c_K coincides with the schematic closure of $(\varphi_K)_*c_K$. Thus, by definition,

$$n'[(\varphi_K)_*c_K\ ,\ D'_K]_{X'} = \varphi_*\overline{c_K}.\Delta'.$$

The divisor $\varphi^*\Delta'$ is τ-equivalent to zero on X, and satisfies $(\varphi^*\Delta')_K = n'(\varphi_K)^*D'_K$. Hence, by definition,

$$n'[c_K\ ,\ (\varphi_K)^*D'_K]_X = \overline{c_K}.\varphi^*\Delta'.$$

□

In the situation of Definition 2.1.1, let X' be another proper flat normal semi-factorial R-model of X_K. Consider the graph Γ of the rational map $X \dashrightarrow X'$ induced by the identity on the generic fibers. By definition, this is the schematic closure of the graph of the identity morphism $X_K \to X'_K$ in $X \times_R X'$. In particular, this is a closed subscheme of $X \times_R X'$, proper and flat over R, with generic fiber isomorphic to X_K. Applying Theorem 2.6 of [P], we can find an R-scheme $\widetilde{X}$ which is proper flat normal and semi-factorial, together with an R-morphism $\widetilde{X} \to \Gamma$ which is an isomorphism on the generic fibers. Composing with the two projections from $X \times_R X'$ to X and X', we get arrows

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr] & \\ X & & X' \end{tikzcd}$$

which are isomorphisms on the generic fibers. Now, Lemma 2.1.2 shows that the pairings $[\ ,\ ]_X$ and $[\ ,\ ]_{X'}$ both coincide with $[\ ,\ ]_{\widetilde{X}}$. In conclusion, the pairing $[\ ,\ ]_X$ depends only on X_K, and not on the choice of X.

Let us summarize the above considerations :

Proposition 2.1.3. *Let X_K be a proper, geometrically normal and geometrically connected scheme over K. There exists a pairing*

$$[\ ,\ ] : Z_0^0(X_K) \times \mathrm{Div}^\tau(X_K) \longrightarrow \mathbb{Q},$$

defined for the pairs (c_K, D_K) such that the supports of c_K and D_K are disjoint, and which can be computed as follows.

Let X/R be any proper flat normal and semi-factorial model of X_K over R. Let $\overline{c_K}$ be the schematic closure of c_K in X. Choose $(n, \Delta) \in (\mathbb{Z}\setminus\{0\}) \times \mathrm{Div}^\tau(X/R)$ such that $\Delta_K = nD_K$. Then we have

$$[c_K\ ,\ D_K] = \frac{1}{n}(\overline{c_K}.\Delta) \quad \in \mathbb{Q}\,.$$

2.2 Comparison with Néron's pairing

As before, let R be a complete discrete valuation ring with fraction field K and algebraically closed residue field k. Let X_K be a proper smooth and geometrically connected scheme over K. Let v be the *normalized* valuation on K, which maps any uniformizing element of R to $1 \in \mathbb{Z}$. We fix an algebraic closure $\overline{K}$ of K, and we still denote by v the unique valuation on $\overline{K}$ extending v. Néron attached to X_K a pairing $\langle\ ,\ \rangle$ with respect to the valuation v ([N] II 9.3). This is a pairing

$$\langle\ ,\ \rangle : Z_0^0(X_K) \times \mathrm{Div}^\tau(X_K) \longrightarrow \mathbb{R},$$

defined for (c_K, D_K) when the supports of c_K and D_K are disjoint (the definition of Néron's pairing is briefly reviewed at the beginning of the proof of Theorem 2.2.1). Actually, Néron considers divisors belonging to the subgroup $\mathrm{Div}^0(X_K) \subseteq \mathrm{Div}^\tau(X_K)$ consisting of divisors which are algebraically equivalent to zero on X_K. However, the group $(\mathbb{R}, +)$ being divisible, the real number $\langle\, c_K,\ D_K\,\rangle$ is naturally defined when D_K is only τ-equivalent to zero. Néron shows in *loc.cit* III 4.2 that the pairing takes values in $\mathbb{Q}$. This fact will be recovered and made more precise below (Corollary 2.2.2).

Our goal in this subsection is to prove the following common generalization of Néron [N] III 4.1, Gross [G], Hriljac [H], Lang [La2] III 5.2 and Bosch-Lorenzini [BL] 4.3, over a complete discrete valuation ring R with algebraically closed residue field k and fraction field K.

Theorem 2.2.1. *For every proper, smooth and geometrically connected scheme over K, the pairing $[\ ,\ ]$ defined in Proposition 2.1.3 coincides with Néron's pairing $\langle\ ,\ \rangle$ defined in [N] II.9 Theorem 3.*

In particular, the pairing $[\ ,\ ]$ generalizes Néron's pairing to K-schemes which are proper geometrically normal and geometrically connected, but not necessarily smooth.

Before proving the theorem, let us note the following consequence of Proposition 2.1.3.

Corollary 2.2.2. *Let X_K be a proper, geometrically normal and geometrically connected scheme over K. Let n be the exponent of the component group of the special fiber of the Néron model of the Picard variety $A_K = \mathrm{Pic}^0_{X_K/K,\mathrm{red}}$. Then Néron's pairing on $Z_0^0(X_K) \times \mathrm{Div}^0(X_K)$ takes values in $(1/n)\,\mathbb{Z}$.*

Proof. As recalled before Definition 2.1.1, the exponent n has the following property: for any $D_K \in \mathrm{Div}^0(X_K)$ and any proper flat normal and semi-factorial model X of X_K, there exists $\Delta \in \mathrm{Div}^0(X/R)$ such that $\Delta_K = nD_K$. In particular, for any $D_K \in \mathrm{Div}^0(X_K)$, we can choose this integer n, together with a divisor $\Delta \in \mathrm{Div}^0(X/R)$, to compute

$$[c_K \ , \ D_K] = \frac{1}{n}(\overline{c_K}.\Delta) \quad \in \frac{1}{n}\mathbb{Z}\,.$$

Now Theorem 2.2.1 asserts that $\langle\, c_K \ , \ D_K \,\rangle = [c_K \ , \ D_K]$. □

Corollary 2.2.2 provides a refinement of [N] III 4.2. More precisely, Néron shows that the pairing

$$\langle \ , \ \rangle : Z_0^0(X_K) \times \mathrm{Div}^0(X_K) \longrightarrow \mathbb{R}$$

takes values in $(1/2n'ab)\,\mathbb{Z}$, where n', a and b are defined as follows. The integer n' is the exponent of the component group of the special fiber of the Néron model of the Albanese variety A'_K of X_K. Conjecturally, n' is equal to n (see Subsection 3.1). Next, a is the smallest positive integer such that there exists a map $h : X_K \to A'_K$ from X_K to its Albanese variety, with the property that for any divisor $D_K \in \mathrm{Div}^0(X_K)$, there exists a divisor $W_K \in \mathrm{Div}^0(A'_K)$ such that h^*W_K is linearly equivalent to aD_K. We can have $a > 1$ if $X_K(K)$ is empty. Finally, b is the smallest degree of a polarization of the Albanese variety A'_K.

In [MT] (1.5) and (2.3), or [La] 11.5.1-11.5.2, it is proved that $\langle\, c_K \ , \ D_K \,\rangle$ belongs to $(1/n')\,\mathbb{Z}$ when X_K is an abelian variety and if c_K is supported on rational points. This statement is also a consequence of [BL] 4.4. Moreover, note that Néron's pairing can take the value $1/n$, for instance when X_K is an elliptic curve (see [BL] Example 5.8).

Let us go back to Theorem 2.2.1. To prove the theorem, we will use the characterization of Néron's pairing given in [La] 11.3.2 and that we recall now.

An element c_K of $Z_0^0(X_K)$ can be written uniquely as a difference of two positive 0-cycles with disjoint supports: $c_K = c_K^+ - c_K^-$. Denoting by deg the degree of a 0-cycle, let us set

$$\deg^+ c_K := \deg(c_K^+) = \deg(c_K^-) \geq 0.$$

Lemma 2.2.3. ([La] 11.3.2) *Suppose that for each projective smooth and geometrically connected scheme X_K over K, we are given a bilinear pairing*

$$\begin{array}{rcl} Z_0^0(X_K) \times \mathrm{Div}^0(X_K) & \longrightarrow & \mathbb{R} \\ (c_K \ , \ D_K) & \longmapsto & \delta(c_K \ , \ D_K) \end{array}$$

such that the following properties are true:

1. *If D_K is a principal divisor on X_K, then $\delta(c_K \ , \ D_K) = 0$.*

2. *Let $\varphi_K : X_K \to X'_K$ be a K-morphism. For all $c_K \in Z_0^0(X_K)$, and for all $D'_K \in \mathrm{Div}^0(X'_K)$ whose support does not meet that of the 0-cycle $(\varphi_K)_* c_K$, the following equality holds*

$$\delta(c_K \ , \ (\varphi_K)^* D'_K) = \delta((\varphi_K)_* c_K \ , \ D'_K).$$

3. *For $D_K \in \mathrm{Div}^0(X_K)$ fixed and $\deg^+ c_K$ bounded, the values $\delta(c_K \ , \ D_K)$ are bounded.*

Then $\delta(c_K \ , \ D_K) = 0$ for all c_K, D_K and X_K.

Remark 2.2.4. In the statement of [La] 11.3.2, one reads 'projective variety V over K' instead of 'projective smooth and geometrically connected scheme X_K over K'. According to the general conventions of [La] page 21, a 'variety over K' is a 'geometrically integral scheme of finite type over K'. However, the given proof of [La] 11.3.2 works if and only if the Albanese variety of each V is an abelian variety. The latter is true, for example, if each V is geometrically normal, or if each V is smooth. For our purposes, namely the proof of Theorem 2.2.1, we need the version of the lemma where all the V are smooth.

Proof of Theorem 2.2.1. Starting from the existence of *Néron functions* on a proper smooth and geometrically connected K-scheme X_K ([N] II 8.2), let us recall the definition of Néron's pairing. Let

$$c_K = \sum_i n_i[x_{K,i}] \in Z_0^0(X_K)$$

and $D_K \in \mathrm{Div}^0(X_K)$ whose support $\mathrm{Supp}(D_K)$ does not contain any of the $x_{K,i}$. Let $\lambda_{D_K} : (X_K - \mathrm{Supp}(D_K))(\overline{K}) \to \mathbb{R}$ be a Néron function associated to D_K. For each i, the scheme $x_{K,i} \otimes_K \overline{K}$ is supported on some $\overline{K}$-points $x_{\overline{K},j_i}$, $j_i = 1, \ldots, s_i$, where s_i is the separable degree of $K(x_{K,i})/K$. Denoting by l_i the inseparable degree of $K(x_{K,i})/K$, then

$$\lambda_{D_K}(x_{K,i}) := \sum_{j_i=1}^{s_i} l_i \lambda_{D_K}(x_{\overline{K},j_i}) \quad \text{and} \quad \langle\, c_K \ , \ D_K \,\rangle := \sum_i n_i \lambda_{D_K}(x_{K,i}).$$

The real number $\langle\, c_K \ , \ D_K \,\rangle$ is well-defined because λ_{D_K} is unique up to constant and c_K has degree zero.

Comparison of the pairings for a principal divisor D_K.

Let us keep the previous notation, and suppose that $D_K = \mathrm{div}_{X_K} f$ for a non-zero $f \in K(X_K)$. Let $z \in (X_K - \mathrm{Supp}(\mathrm{div}_{X_K} f))(\overline{K})$, mapping to a closed point $x_K \in X_K$. The evaluation of f at z is defined by the pull-back $z^* : \mathcal{O}_{X_K,x_K} \to \overline{K}$, that is, $f(z) := z^* f$. The formula $\lambda_f(z) = v(f(z))$ then defines a Néron function for the divisor $\mathrm{div}_{X_K} f$.

Fix an i. There is a 1-1 correspondence between the $x_{\overline{K},j_i}$ and the K-embeddings of the residue field extension $K(x_{K,i})/K$ into $\overline{K}/K$. By pulling back the valuation v, each of these embeddings induces a valuation on $K(x_{K,i})$. However, as R is complete, these valuations are equal to the unique valuation on $K(x_{K,i})$ which extends the normalized valuation on K, and that we can also denote by v. Consequently,

$$\lambda_f(x_{K,i}) = \sum_{j_i=1}^{s_i} l_i v(f(x_{K,i})) = [K(x_{K,i}) : K] v(f(x_{K,i}))$$

where $f(x_{K,i})$ is the image of f by the canonical surjection $\mathcal{O}_{X_K,x_{K,i}} \to K(x_{K,i})$.

Now, take the schematic closure Z_i of $x_{K,i}$ in X, denote by $\widetilde{Z_i}$ its normalization and set $x_{k,i} := X_k \cap Z_i$. The ring $\mathcal{O}(\widetilde{Z_i})$ is a discrete valuation ring with fraction field $K(x_{K,i})$. So it is precisely the valuation ring of v in $K(x_{K,i})$. As k is algebraically closed, its ramification index over R is equal to $[K(x_{K,i}) : K]$. From this observation, we get

$$v(f(x_{K,i})) = \begin{cases} 1/[K(x_{K,i}) : K]\, \mathrm{length}_{\mathcal{O}(\widetilde{Z_i})} \left(\mathcal{O}(\widetilde{Z_i})/(f) \right) & \text{if } f|_{\widetilde{Z_i}} \in \mathcal{O}(\widetilde{Z_i}), \\ -1/[K(x_{K,i}) : K]\, \mathrm{length}_{\mathcal{O}(\widetilde{Z_i})} \left(\mathcal{O}(\widetilde{Z_i})/(f^{-1}) \right) & \text{otherwise.} \end{cases}$$

We have thus obtained $[K(x_{K,i}) : K] v(f(x_{K,i})) = (\overline{c_K}.\mathrm{div}_X f)_{x_{k,i}}$ (recall the beginning of Subsection 2.1). But $\mathrm{div}_X f$ is a divisor on X which is τ-equivalent to zero and extends $\mathrm{div}_{X_K} f$. The desired equality $\langle\, c_K \ , \ \mathrm{div}_{X_K} f \,\rangle = [c_K \ , \ \mathrm{div}_{X_K} f]$ follows.

Functoriality of the pairing [,].

Let $\varphi_K : X_K \to X'_K$ be a K-morphism of proper smooth and geometrically connected schemes over K. Let us show that for all $c_K \in Z_0^0(X_K)$, and for all $D'_K \in \mathrm{Div}^\tau(X'_K)$ whose support does not meet that of the 0-cycle $(\varphi_K)_* c_K$, the following equality holds

$$[c_K \ , \ (\varphi_K)^* D'_K] = [(\varphi_K)_* c_K \ , \ D'_K].$$

Let X/R (resp. X'/R) be a proper flat normal semi-factorial model of X_K (resp. X'_K). Consider the graph Γ of the rational map $X \dashrightarrow X'$ defined by φ_K. Applying Theorem 2.6 of [P] to Γ, we obtain a proper flat normal semi-factorial $\widetilde{X}/R$ and R-morphisms

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X & & X' \end{tikzcd}$$

such that on the generic fibers, α is an isomorphism and β coincides with φ_K. In particular, the pairing [,] for X_K can be computed on $\widetilde{X}$, and the desired functoriality follows from Lemma 2.1.2 applied to β.

The pairing $\delta(\ ,\ )$.

At this point, we recall that for any proper smooth and geometrically connected scheme X_K over K, there exists a non-zero integer a and a map $X_K \to A'_K$ from X_K to its Albanese variety, with the property that for any divisor $D_K \in \mathrm{Div}^\tau(X_K)$, there exists a divisor $W_K \in \mathrm{Div}^0(A'_K)$ such that $h^* W_K$ is well-defined and linearly equivalent to aD_K ([N] II 2.1). Let $c_K \in Z_0^0(X_K)$ and $D_K \in \mathrm{Div}^\tau(X_K)$ with disjoint supports. Keep the previous notation. After *moving* W_K *on the projective smooth scheme* A'_K if necessary (e.g. [Li] 9.1.11), we can assume that the support of $h^* W_K$ does not meet that of c_K. Then, using the functoriality of [,], we can write

$$a[c_K \ , \ D_K] = [c_K \ , \ h^* W_K] + [c_K \ , \ \mathrm{div}_{X_K} f] = [h_* c_K \ , \ W_K] + [c_K \ , \ \mathrm{div}_{X_K} f]$$

for some non-zero $f \in K(X_K)$. By definition, Néron's pairing has the same functoriality property as [,]. And we have seen that both pairings coincide for principal divisors. Consequently, as A'_K is projective smooth geometrically connected over K, Theorem 2.2.1 is proved if we know that both pairings coincide on such schemes. So, until the end of the proof, we will only consider the pairings for projective smooth geometrically connected schemes. Furthermore, by $\mathbb{Z}$-linearity, we can only consider divisors which are algebraically equivalent to zero.

Now, both [,] and $\langle \ , \ \rangle$ are bilinear in their definition domain, and they coincide for principal divisors. Using a *moving lemma on the projective smooth scheme* X_K, we see that

$$\delta(c_K \ , \ D_K) := \langle\, c_K \ , \ D_K \,\rangle - [c_K \ , \ D_K]$$

is well-defined on the *whole* product $Z_0^0(X_K) \times \mathrm{Div}^0(X_K)$. And conditions 1 and 2 of Lemma 2.2.3 are satisfied by δ.

Condition 3 of 2.2.3 is satisfied by $\delta(\ ,\ )$.

Denote by $\overline{R}$ the valuation ring of v in $\overline{K}$.

Fix $D_K \in \mathrm{Div}^0(X_K)$. Let $(n, \Delta) \in (\mathbb{Z} \setminus \{0\}) \times \mathrm{Div}^\tau(X_K)$ satisfying $\Delta_K = nD_K$. Represent the divisor Δ by a family $(U_t, g_t)_{t=1,\ldots,m}$, where the U_t are affine open subsets of X and the g_t are rational functions on X. Let E_t be the set of $\overline{K}$-points of X_K which extend to $\overline{R}$-points of U_t. As X is proper over R, we see that $X(\overline{K}) = \cup_{t=1}^m E_t$. The family $(U_{t,K}, g_t)_{t=1,\ldots,m}$ represents the divisor nD_K on X_K.

Let us choose a Néron function λ_{nD_K} on X_K. By definition, we can find some v-continuous locally bounded functions $\alpha_t : U_{t,K}(\overline{K}) \to \mathbb{R}$ such that

$$\lambda_{nD_K}(z) = v(g_t(z)) + \alpha_t(z)$$

for all $z \in (U_{t,K} - \mathrm{Supp}(D_K))(\overline{K})$. As E_t is bounded in $U_t(\overline{K})$ (by construction), the function α_t is bounded on E_t.

Let $c_K = \sum_i n_i[x_{K,i}] \in Z_0^0(X_K)$ whose support does not meet that of D_K. Fix an i, let Z_i be the schematic closure of $x_{K,i}$ in X, set $x_{k,i} := X_k \cap Z_i$ and let t_i be such that $Z_i \subset U_{t_i}$. The same local computation as in the case of a principal divisor shows that

$$(\overline{c_K}.\Delta)_{x_{k,i}} = [K(x_{K,i}) : K]v(g_{t_i}(x_{K,i})) = \sum_{j_i=1}^{s_i} l_i v(g_{t_i}(x_{K,i})).$$

On the other hand, keeping the same notation as in the beginning of the proof,

$$\langle\, c_K \ , \ nD_K \,\rangle = \sum_i n_i \sum_{j_i=1}^{s_i} l_i \lambda_{nD_K}(x_{\overline{K},j_i}).$$

Consequently,

$$n\delta(c_K \ , \ D_K) = \sum_i n_i \sum_{j_i=1}^{s_i} l_i \alpha_{t_i}(x_{\overline{K},j_i}).$$

By construction, the $\overline{K}$-point $x_{\overline{K},j_i}$ of X_K belongs to E_{t_i}. Denoting by $|\cdot|$ the usual absolute value on $\mathbb{R}$, and setting

$$B := \max_{t=1,\dots,m}(\sup_{E_t}|\alpha_t|) \quad \in \mathbb{R},$$

we obtain

$$|\delta(c_K \ , \ D_K)| \le \frac{1}{|n|}\sum_i |n_i|[K(x_{K,i}) : K]B = \frac{2B}{|n|}\deg^+ c_K.$$

As the divisor D_K is fixed, the numbers n and B are fixed, and so the right-hand side of the above inequality is bounded if $\deg^+ c_K$ is. $\square$

Let us note the following properties of the pairing $[\ ,\ ]$, and consequently of Néron's pairing.

Proposition 2.2.5. *Let X_K be a proper, geometrically normal and geometrically connected scheme over K. Let $c_K \in Z_0^0(X_K)$ and let $D_K \in \mathrm{Div}^\tau(X_K)$ with disjoint supports. If c_K or D_K is rationally equivalent to zero, then $[c_K \ , \ D_K] \in \mathbb{Z}$.*

Proof. The case where D_K is rationally equivalent to zero follows directly from the definition of $[\ ,\ ]$: if $D_K = \mathrm{div}_K f$ with $f \in K(X_K) \setminus \{0\}$, then $[c_K \ , \ \mathrm{div}_K f] = (c_K.\mathrm{div} f) \in \mathbb{Z}$.

Let us now suppose that c_K is rationally equivalent to zero. As $[\ ,\ D_K]$ is $\mathbb{Z}$-linear, we have to show that if $c_K = (\varphi_K)_*\mathrm{div}_{C_K} f$ for some K-morphism

$$\varphi_K : C_K \longrightarrow X_K$$

from a proper normal connected curve C_K to X_K, and some non-zero $f \in K(C_K)$, then

$$[c_K \ , \ D_K] \in \mathbb{Z}\,.$$

As R is excellent, there exists a proper flat *regular* model C/R of C_K. On the other hand, let us consider a proper flat normal semi-factorial model X/R of X_K. After replacing C by a desingularization of the graph of the rational map $C \dashrightarrow X$ induced by φ_K, we can suppose that φ_K extends to an R-morphism $\varphi : C \to X$. If Δ is a divisor on X which is τ-equivalent to zero and such that $\Delta_K = nD_K$ for some integer $n \neq 0$, then

$$[c_K \ , \ D_K] := \frac{1}{n}\big(\overline{(\varphi_K)_* \mathrm{div}_{C_K} f}.\Delta\big) = \frac{1}{n}\big(\overline{\mathrm{div}_{C_K} f}.\varphi^*\Delta\big)$$

by the projection formula (Lemma 2.1.2). Let us write

$$\mathrm{div}_C f = \overline{\mathrm{div}_{C_K} f} - V \quad \text{and} \quad \varphi^*\Delta = \overline{(\varphi_K)^*\Delta_K} - W$$

for some vertical divisors V and W on C/R. Denote by $\Gamma_1, \ldots, \Gamma_\nu$ the reduced irreducible components of C_k, by M the intersection matrix associated to C_k (as defined in the introduction), and by $\rho : \mathrm{Pic}(C) \to \mathbb{Z}^\nu$ the degree homomorphism $(E) \mapsto (E \cdot \Gamma_i)_{i=1,\ldots,\nu}$. Following [BLR] 9.2/13, the divisor E on the *R-curve* C is algebraically equivalent to zero if and only if (E) belongs to the kernel of ρ. Therefore the τ-equivalence relation and the algebraic equivalence relation on C/R are the same, and the linear equivalence classes of $\varphi^*\Delta$ and $\mathrm{div}_C f$ belongs to the kernel of ρ. Thus we get:

$$\rho(\overline{\mathrm{div}_{C_K} f}) = \rho(V) = MV \quad \text{and} \quad \rho(\overline{(\varphi_K)^*\Delta_K}) = \rho(W) = MW,$$

where we have identified a vertical divisor on C/R with an element of $\mathbb{Z}^\nu$. Next, we use that the matrix M is *symmetric* to obtain

$$(\overline{\mathrm{div}_{C_K} f}.W) = {}^tW\rho(\overline{\mathrm{div}_{C_K} f}) = {}^tWMV = {}^tVMW = (\overline{(\varphi_K)^*\Delta_K}.V).$$

Then it follows that

$$[c_K \ , \ D_K] = \frac{1}{n}\big(\overline{(\varphi_K)^*\Delta_K}.\mathrm{div}_C f\big) = \big(\overline{(\varphi_K)^*D_K}.\mathrm{div}_C f\big) \in \mathbb{Z}\,.$$

□

Remark 2.2.6. Let us keep the notation of the proof of 2.2.5. If the curve C_K is *geometrically* normal and *geometrically* connected, the pairing $[\ ,\ ]$ is defined on C_K and

$$\big(\overline{(\varphi_K)^*D_K}.\mathrm{div}_C f\big) = [(\varphi_K)^*D_K \ , \ \mathrm{div}_{C_K} f].$$

In other words, in this case, the proof consists in using the functoriality of the pairing $[\ ,\ ]$, then showing that it is *symmetric* for curves, and finally to apply the definition of the pairing for a principal divisor. The symmetry property of Néron's pairing $\langle\ ,\ \rangle$ for such a curve is well known: e.g. see [La] 11.3.6 et 11.3.7. But here, there is no reason for the curve C_K coming from the rational equivalence relation to satisfy the above *geometric* hypotheses. So we could not use directly the properties of the pairing $\langle\ ,\ \rangle$. However, over an excellent discrete valuation ring, there is no need of these geometric hypotheses on C_K for the existence of the regular model C/R. So we have been able to prove the proposition for the pairing $[\ ,\ ]$, and thus also for Néron's pairing $\langle\ ,\ \rangle$ thanks to Theorem 2.2.1.

3 Duality and algebraic equivalence for models of abelian varieties

3.1 Grothendieck's duality for Néron models

Let us recall here Grothendieck's duality theory for Néron models of abelian varieties, as developed in [SGA 7] VII-VIII-IX.

Let R be a discrete valuation ring with perfect residue field k and fraction field K. Let A_K be an abelian variety over K, with dual A'_K. Let A/R, A'/R be the Néron models of A_K, A'_K, and Φ_A, $\Phi_{A'}$ be the étale k-group schemes of connected components of the special fibers A_k, A'_k.

By definition, the abelian variety A'_K represents the identity component $\mathrm{Pic}^0_{A_K/K}$ of the Picard functor of A_K, and the canonical isomorphism $A'_K = \mathrm{Pic}^0_{A_K/K}$ is given by the Poincaré sheaf $\mathcal{P}_K$ on $A_K \times_K A'_K$ birigidified along the unit sections of A_K and A'_K. Now, this sheaf is canonically endowed with the structure of a *biextension of* (A_K, A'_K) *by* $\mathbb{G}_{m,K}$ (*loc. cit.* VII 2.9.5). Then the duality theory for Néron models is to understand how this biextension *extends* at the level of Néron models. For this, Grothendieck attached to $\mathcal{P}_K$ a canonical pairing

$$\langle\ ,\ \rangle : \Phi_A \times_k \Phi_{A'} \longrightarrow \mathbb{Q}/\mathbb{Z},$$

which measures the obstruction to extending $\mathcal{P}_K$ as a biextension of (A, A') by $\mathbb{G}_{m,R}$. The duality statement is: *this pairing is a perfect duality* (*loc. cit.* IX 1.3). As mentioned in the introduction, it has been proved in various situations, including the semi-stable case (Grothendieck *loc. cit.* IX 11.4 and Werner [W]) and the mixed characteristic case (Bégueri [Beg]). In general, the duality statement remains a conjecture.

3.2 Duality and Picard functor

Keep the notation of the previous subsection. From [P] Corollary 2.23, it is always possible to find an *R-compactification of A*, that is, an open R-immersion of A into a proper R-scheme $\overline{A}$ with dense image, such that $\overline{A}/R$ is flat, $\overline{A}$ is normal and the canonical map $\mathrm{Pic}(\overline{A}) \to \mathrm{Pic}(A)$ is surjective. Note that, in particular, $\overline{A}/R$ is semi-factorial: the map $\mathrm{Pic}(A) \to \mathrm{Pic}(A_K)$ is surjective because A is regular, so that $\mathrm{Pic}(\overline{A}) \to \mathrm{Pic}(A_K)$ is surjective by composition. As $\overline{A}/R$ is proper, it makes sense to consider the notion of algebraic equivalence on $\overline{A}$ relative to R using the identity component of the Picard functor $\mathrm{Pic}_{\overline{A}/R}$, as defined in Subsection 2.1. Our goal in this section is to understand the duality from the point of view of algebraic equivalence, starting from the canonical isomorphism $A'_K = \mathrm{Pic}^0_{A_K/K}$. To do this, we need the following notions.

$\mathbb{Q}$-divisors and relative τ-equivalence. Let Z be a *normal* locally noetherian scheme, so that the canonical homomorphism from the group of divisors on Z into that of 1-codimensional cycles is *injective* ([EGA IV]$_4$ 21.6.9 (i)). A 1-codimensional cycle C on Z is said to be a *$\mathbb{Q}$-divisor* if there exists $n \in \mathbb{Z}\setminus\{0\}$ such that nC is a divisor.

Let $Z \to T$ be a proper morphism of schemes, with Z locally noetherian and normal. A $\mathbb{Q}$-divisor C on Z is said to be *τ-equivalent to zero relative to T* (or *τ-equivalent to zero* if there is no ambiguity on the base scheme T) if there exists $n \in \mathbb{Z}\setminus\{0\}$ such that nC is a divisor on Z which is τ-equivalent to zero relative to T (see 2.1). The group of classes of $\mathbb{Q}$-divisors on Z which are τ-equivalent to zero relative to T, modulo the principal divisors, will be denoted by $\mathrm{Pic}^{\mathbb{Q},\tau}(Z/T)$.

When $Z = \overline{A}$, the restriction to the generic fiber induces an injective morphism

$$\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R) \hookrightarrow \mathrm{Pic}^{\tau}_{A_K/K}(K) = \mathrm{Pic}^0_{A_K/K}(K) = A'_K(K).$$

The fact that $\mathrm{Pic}^{\tau}_{A_K/K}(K) = \mathrm{Pic}^0_{A_K/K}(K)$ can be found in [Mum] (v) p. 75. To see that the above morphism is injective, let (C) be in its kernel. After modifying C by a principal divisor if necessary, we can assume that $C_K = 0$, that is, the support of C is contained in the special fiber $\overline{A}_k$ of $\overline{A}/R$. Let n be a non-zero integer such that nC is a divisor on $\overline{A}$ which is τ-equivalent to zero relative to R. As $\overline{A}_k$ admits at least one irreducible component Γ with multiplicity 1 (the component containing the unit element of A_k), the vertical divisor nC is principal ([R] 6.4.1 3)). In other words, there exists an integer m such that $nC = m\mathrm{div}(\pi)$, where π is a uniformizing element of R. Taking the associated cycles, and comparing the coefficients of Γ, we obtain that n divides m. Consequently, the $\mathbb{Q}$-divisor C is a principal divisor, whence the injectivity.

By definition, the group $\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R)$ contains the group $\mathrm{Pic}^0(\overline{A}/R)$ of classes of divisors on $\overline{A}$ which are algebraically equivalent to zero relative to R, modulo principal divisors. Now, when R is complete with algebraically closed residue field, we know from [P] Corollary 3.14 that the image of the composition

$$\mathrm{Pic}^0(\overline{A}/R) \hookrightarrow \mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R) \hookrightarrow A'_K(K)$$

contains the subgroup $(A')^0(R)$ of $A'_K(K)$.

Conversely, we will show that Grothendieck's duality statement for A and A' is equivalent to the following assertion: *the image of* $\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R) \hookrightarrow A'_K(K)$ *is contained in the subgroup* $(A')^0(R)$.

Theorem 3.2.1. *Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K. Let A_K be an abelian variety over K, with dual A'_K. Let A (resp. A') be the Néron model of A_K (resp. A'_K) over R. Let $\overline{A}$ be a proper flat normal model of A_K over R, equipped with a dense open R-immersion $A \to \overline{A}$, such that the induced map $\mathrm{Pic}(\overline{A}) \to \mathrm{Pic}(A)$ is surjective. Let $\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R)$ be the group of $\mathbb{Q}$-divisors on $\overline{A}$ which are τ-equivalent to zero relative to R, modulo the principal divisors. Then, the duality statement recalled in 3.1 is equivalent to the following:*

The image of the restriction map $\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R) \hookrightarrow A'_K(K)$ is contained in the subgroup $(A')^0(R)$.

Let $\mathrm{Pic}^0(\overline{A}/R)$ be the group of divisors on $\overline{A}$ which are algebraically equivalent to zero relative to R, modulo the principal divisors. Then, when the duality statement is true, the inclusion $\mathrm{Pic}^0(\overline{A}/R) \hookrightarrow \mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R)$ is an equality, and there is a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}^0(A_K) \arrow[r, "\sim"] & A'_K(K) \\
\mathrm{Pic}^0(\overline{A}/R) \arrow[r, "\sim"] \arrow[u, hook] & (A')^0(R) \arrow[u, hook]
\end{tikzcd}$$

where the two vertical maps are injective, and the two horizontal maps are bijective.

See the end of Subsection 3.4 for the proof.

Remark 3.2.2. With the notation of Theorem 3.2.1, the canonical morphisms of abstract groups

$$\mathrm{Pic}^0(A_K) \longrightarrow \mathrm{Pic}^0_{A_K/K}(K), \quad \mathrm{Pic}^0(\overline{A}/R) \longrightarrow \mathrm{Pic}^0_{\overline{A}/R}(R)$$

are isomorphisms. For the second one, note that $\mathrm{Pic}_{\overline{A}/R}$ can be defined using the étale topology, and that R is strictly henselian. Note also that, when $\overline{A}$ is locally factorial (e.g. regular), the group $\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R)$ coincides with the group $\mathrm{Pic}^\tau(\overline{A}/R)$ of divisors on $\overline{A}$ which are τ-equivalent to zero relative to R, modulo the principal divisors, which in turn can be identified with the group $\mathrm{Pic}^\tau_{\overline{A}/R}(R)$.

The last assertion of Theorem 3.2.1 provides a refinement of [P] Corollary 3.14 in the present case $X = \overline{A}$. Here, when Grothendieck's duality holds, we obtain a *necessary and sufficient condition* for an invertible sheaf which is algebraically equivalent to zero on A_K to extend into an invertible sheaf on $\overline{A}$ which is algebraically equivalent to zero relative to R: the corresponding point $a'_K \in A'_K(K)$ must extend in the identity component of A'. Thus, conjecturally, *the group* $(A')^0(R)$ *parametrizes the invertible sheaves on* $\overline{A}$ *which are algebraically equivalent to zero relative to* R.

To make the link between Grothendieck's duality for A and A', and algebraic equivalence on $\overline{A}$, we need some preparation about *non-rational* 0-cycles on A_K, especially those which are supported on *inseparable* points over K.

3.3 About non-rational 0-cycles on abelian varieties

Let K be a field, and denote by $\overline{K}$ its algebraic closure. Let A_K be an abelian variety over K. Let d be a positive integer and let $\mathrm{Hilb}^d_{A_K/K}$ be the Hilbert scheme of points of degree d on A_K. The Grothendieck-Deligne norm map

$$\sigma_d : \mathrm{Hilb}^d_{A_K/K} \longrightarrow A_K^{(d)}$$

defined in [SGA 4] XVII page 184 (see also [BLR] pages 252-254) maps $\mathrm{Hilb}^d_{A_K/K}$ to the d-fold *symmetric* product $A_K^{(d)}$. On the other hand, the map

$$A_K^d \longrightarrow A_K, \quad (x_1, \ldots, x_d) \longmapsto x_1 + \cdots + x_d,$$

induces a map

$$m_d : A_K^{(d)} \longrightarrow A_K.$$

Let us set

$$\mathcal{S}_d := m_d \circ \sigma_d : \mathrm{Hilb}^d_{A_K/K} \longrightarrow A_K.$$

Let $a_K \in A_K$ be a closed point of degree d, that is to say, the residue field extension $K(a_K)/K$ has degree d. It corresponds to a rational point

$$h(a_K) \in \mathrm{Hilb}^d_{A_K/K}(K).$$

We will need an explicit description of its image $\mathcal{S}_d(h(a_K)) \in A_K(K)$, when considered as an element of $A_{\overline{K}}(\overline{K})$.

Let us consider the artinian $\overline{K}$-scheme $a_K \otimes_K \overline{K}$. It is supported on some $a_j \in A_{\overline{K}}(\overline{K})$, $j = 1, \ldots, s$, where s is the separable degree of $K(a_K)/K$. The length of each local component of $a_K \otimes_K \overline{K}$ is equal to the inseparable degree of $K(a_K)/K$, and will be denoted by l. So the effective 0-cycle associated to $a_K \otimes_K \overline{K}$ is

$$\sum_{j=1}^{s} l[a_j] \quad \in Z_0(A_{\overline{K}}).$$

We are going to show that

$$\mathcal{S}_d(h(a_K)) = \sum_{j=1}^{s} l a_j \in A_{\overline{K}}(\overline{K}).$$

Note that, in particular, this will show that the right-hand-side of the equality belongs to $A_K(K)$.

Lemma 3.3.1. *Let C be an artinian algebra over an algebraically closed field $\overline{K}$. Let $C_1, \ldots, C_s$ be the local components of C, with respective lengths $l_1, \ldots, l_s$, and let $u_j : C_j \to \overline{K}$ be the canonical surjection from C_j to its residue field. Then, for all*

$$c = (c_1, \ldots, c_s) \in C = C_1 \times \cdots \times C_s,$$

the following formula holds for the norm of c over $\overline{K}$:

$$N_{C/\overline{K}}(c) = \prod_{i=1}^{s} (u_j(c_j))^{l_j}.$$

Proof. We can assume that C is local, with length l. Let $\mathfrak{m}$ be the maximal ideal of C. Let n be the smallest integer such that $\mathfrak{m}^n = 0$. Choose a basis $\mathcal{E} = \mathcal{E}_0 \coprod \ldots \coprod \mathcal{E}_{n-1}$ of C over $\overline{K}$ which is adapted to the filtration

$$0 = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \mathfrak{m} \subset C,$$

that is, $\mathcal{E}_i$ is contained in $\mathfrak{m}^i \setminus \mathfrak{m}^{i+1}$ and induces a basis of the $\overline{K}$-vector space $\mathfrak{m}^i \,/\, \mathfrak{m}^{i+1}$.

Fix $c \in C$ and let M be the matrix of multiplication-by-c in the basis $\mathcal{E}$. Write $c = \lambda + \epsilon$ with $\lambda \in \overline{K}$ and $\epsilon \in \mathfrak{m}$. Then M is a $l \times l$ lower triangular matrix, with all diagonal entries equal to λ. Hence $N_{C/\overline{K}}(c) = \lambda^l$, as required. □

Let us use the lemma to compute $\sigma_d(h(a_K))$, considered as an element of $A^{(d)}_{\overline{K}}(\overline{K})$. Let C be the $\overline{K}$-algebra of global sections of the scheme $a_K \otimes_K \overline{K}$. Set $\mathrm{TS}^d_{\overline{K}}(C) := (C^{\otimes d})^{\mathfrak{S}_d} \subseteq C^{\otimes d}$ where $\mathfrak{S}_d$ is the symmetric group acting on $C^{\otimes d}$ by permuting factors. By definition, the point $\sigma_d(h(a_K)) \in (a_K \otimes_K \overline{K})^{(d)}(\overline{K}) \subset A^{(d)}_{\overline{K}}(\overline{K})$ corresponds to the unique $\overline{K}$-algebra homomorphism

$$\mathrm{TS}^d_{\overline{K}}(C) \longrightarrow \overline{K}, \quad c^{\otimes d} \longmapsto N_{C/\overline{K}}(c).$$

Now, from Lemma 3.3.1, this homomorphism is induced by the point

$$\big(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_s, \ldots, a_s\big) \in A^d_{\overline{K}}(\overline{K}),$$

where a_j is repeated l times.

Next, the element $\mathcal{S}_d(h(a_K)) \in A_{\overline{K}}(\overline{K})$ is just the sum

$$m_d(\sigma_d(h(a_K))) = \sum_{j=1}^{s} l a_j \in A_{\overline{K}}(\overline{K}),$$

as claimed.

Notation 3.3.2. *The above K-morphisms $\mathcal{S}_d$ induce a homomorphism*

$$\mathcal{S} : Z_0(A_K) \longrightarrow A_K(K)$$

from the group of 0-cycles on A_K to that of K-rational points: if $a_K \in A_K$ is a closed point of degree d, defining $h(a_K) \in \mathrm{Hilb}^d_{A_K/K}(K)$, then $\mathcal{S}([a_K]) := \mathcal{S}_d(h(a_K))$.

We will also need to 'translate divisors on A_K by non-rational points'.

Let $\mathrm{Div}_{A_K/K}$ be the scheme of relative effective divisors on A_K ([FGA VI] 4.1). Fix a positive integer d and consider the map

$$A^d_K \times_K \mathrm{Div}_{A_K/K} \longrightarrow \mathrm{Div}_{A_K/K}$$

which is given by the functorial formula

$$\big((a_1,\ldots,a_d)\ ,\ D\big) \longmapsto D_{a_1} + \cdots + D_{a_d},$$

where D_a is obtained from D by translation by the section a. By symmetry, it induces a map

$$A_K^{(d)} \times_K \mathrm{Div}_{A_K/K} \quad \longrightarrow \quad \mathrm{Div}_{A_K/K}\,.$$

By composing with the norm map σ_d, the latter gives rise to a map

$$\mathrm{Hilb}^d_{A_K/K} \times_K \mathrm{Div}_{A_K/K} \quad \longrightarrow \quad \mathrm{Div}_{A_K/K}\,.$$

Let $a_K \in A_K$ be a closed point of degree d and let D_K be an effective divisor on A_K. Denote by $(D_K)_{a_K} \in \mathrm{Div}_{A_K/K}(K)$ the image of $(h(a_K), D_K)$ by the previous arrow. As above, write

$$\sum_{r=1}^{d} [a_{\overline{K},r}]$$

for the 0-cycle associated to $a_K \otimes_K \overline{K}$. In this expression, repetitions are allowed. Then, using the above computation of $\sigma_d(h(a_K))$, we see that $(D_K)_{a_K}$, as an element of the group $\mathrm{Div}_{A_{\overline{K}}/\overline{K}}(\overline{K})$, is equal to

$$\sum_{r=1}^{d} (D_{\overline{K}})_{a_{\overline{K},r}},$$

where $D_{\overline{K}}$ denotes the pull-back of D_K on $A_{\overline{K}}$. When a_K is étale over K, it is easy to see that the latter divisor descends on A_K. But this turns out to be true in general because of the above construction. Moreover, this description shows that the formation of $(D_K)_{a_K}$ is additive in D_K. We can thus associate a divisor $(D_K)_{a_K}$ on A_K to *any* divisor D_K in the following way: identifying divisors on A_K with 1-codimensional cycles, first use the above to define $(D_K)_{a_K}$ when D_K is a prime cycle, and then extend by $\mathbb{Z}$-linearity.

Notation 3.3.3. *If c_K is a 0-cycle on A_K and D_K a divisor on A_K, define the divisor $(D_K)_{c_K}$ on A_K by $\mathbb{Z}$-linearity from the above situation where c_K is a closed point.*

3.4 Relative algebraic equivalence on semi-factorial compactifications

Our goal in this subsection is to prove Theorem 3.2.1. So, until the end of the subsection, **we fix a complete discrete valuation ring R with algebraically closed residue field k and fraction field K.**

The starting point is the link between Grothendieck's pairing and Néron's pairing, which has been established by Bosch and Lorenzini: Grothendieck's pairing is the *specialization* of Néron's pairing.

Theorem 3.4.1. ([BL] 4.4) *Keep the notation of Theorem 3.2.1. Moreover, let Φ_A (resp. $\Phi_{A'}$) be the group of connected components of A_k (resp. A'_k). On the one hand, consider Grothendieck's pairing [SGA 7] IX 1.3*

$$\langle\ ,\ \rangle : \Phi_A \times \Phi_{A'} \longrightarrow \mathbb{Q}\,/\,\mathbb{Z},$$

and on the other hand, consider Néron's pairing [N] II 9.3

$$\langle\ ,\ \rangle : Z_0^0(A_K) \times \mathrm{Div}^0(A_K) \longrightarrow \mathbb{Q}$$

(defined for (c_K, D_K) when the supports of c_K and D_K are disjoint).

Let $(a, a') \in \Phi_A \times \Phi_{A'}$. Fix a point $a_K \in A_K(K)$ specializing to a, and a divisor $D'_K \in \mathrm{Div}^0(A_K)$ whose image in $A'_K(K)$ specializes to a'. Assume that a_K and 0_K do not belong to the support of D'_K. Then

$$\langle\, a \ ,\ a' \,\rangle = -\langle\ [a_K] - [0_K]\ ,\ D'_K \,\rangle \quad \mathrm{mod}\, \mathbb{Z}\,.$$

The next proposition is a key result about the pairing $[\ ,\ ]$ defined in Proposition 2.1.3.

Proposition 3.4.2. *Let X_K be a proper geometrically normal and geometrically connected scheme over K. Let X be a proper flat normal semi-factorial model of X_K over R. Let ν be the number of irreducible components of the special fiber X_k. There exist some 0-cycles of degree zero $c_{K,1}, \ldots, c_{K,\nu}$ on X_K, with the following property:*

If D_K is a divisor on X_K which is τ-equivalent to zero, whose support is disjoint from those of the $c_{K,i}$, and if $[c_{K,i}\ ,\ D_K]$ is an integer for all $i = 1, \ldots, \nu$, then there exists a $\mathbb{Q}$-divisor on X which is τ-equivalent to zero relative R, with generic fiber D_K.

Proof. Let U be the open subset of X consisting of the regular points. As X is normal, for any irreducible closed subset C of codimension 1 in X, the intersection $C \cap U$ is a dense open subset of C. Furthermore, for any 1-codimensional cycle C on X, the restriction $C|_U$ is a *divisor* on U.

Next, let $\Gamma_1, \ldots, \Gamma_\nu$ be the reduced irreducible components of X_k. Let $\xi_1, \ldots, \xi_\nu$ be the generic points of $\Gamma_1, \ldots, \Gamma_\nu$. Set $d_i := \mathrm{length}(\mathcal{O}_{X_k,\xi_i})$. From [R] 7.1.2, there exists, for all $i = 1, \ldots, \nu$, an R-immersion $u_i : Z_i \to U$, with Z_i finite and flat over R, with rank d_i, such that $u_{i,k}(Z_{i,k})$ is a point $x_{i,k}$ of Γ_i. Then the intersection multiplicity of Z_i and $\Gamma_j \cap U$ is equal to 1 if $i = j$, and 0 otherwise. In particular, the generic fiber of Z_i is a closed point $x_{K,i} \in U_K$ of degree d_i. Moreover, as Z_i is proper over R, the immersion $Z_i \to X$ is closed. Finally, setting $d := \gcd(d_i, i = 1, \ldots, \nu)$, an appropriate $\mathbb{Z}$-linear combination of the $x_{K,i}$ provides a 0-cycle c_K on X_K of degree d. We set

$$c_{K,i} := [x_{K,i}] - \frac{d_i}{d} c_K \quad \in Z_0^0(X_K).$$

Let $D_K \in \mathrm{Div}^\tau(X_K)$ whose support is disjoint from those of the $c_{K,i}$. Choose $\Delta \in \mathrm{Div}^\tau(X/R)$ with a non-zero integer n such that $\Delta_K = nD_K$. Denoting by $\overline{D_K}$ the schematic closure of D_K in X, we can view Δ as a 1-codimensional cycle on X, and write

$$\Delta = n\overline{D_K} + \sum_{i=1}^{\nu} n_i \Gamma_i$$

for some integers $n_1, \ldots, n_\nu$. Set $V := \sum_{i=1}^{\nu} n_i \Gamma_i$. As the schematic closures $\overline{c_{K,i}}$ of the $c_{K,i}$ in X are contained in U (by construction), the following computation is valid:

$$\begin{aligned}
\overline{c_{K,i}}.\Delta &= n(\overline{c_{K,i}}.\overline{D_K}) + (\overline{x_{K,i}}.V) - \frac{d_i}{d}(\overline{c_K}.V) \\
&= n(\overline{c_{K,i}}.\overline{D_K}) + n_i - \frac{d_i}{d}(\overline{c_K}.V).
\end{aligned}$$

Assume that $[c_{K,i}\ ,\ D_K]$ belongs to $\mathbb{Z}$. Then, the left-hand side of the above equality belongs to $n\,\mathbb{Z}$. Consequently, there exists $r_i \in \mathbb{Z}$ such that

$$nr_i = n_i - \frac{d_i}{d}(\overline{c_K}.V).$$

Now, consider the vertical cycle (with integral coefficients)

$$W := (\overline{c_K}.V)\frac{1}{d}[X_k].$$

By definition,

$$V - W = n\sum_{i=1}^{\nu} r_i\Gamma_i, \quad \text{that is,} \quad \Delta - W = n(\overline{D_K} - \sum_{i=1}^{\nu} r_i\Gamma_i).$$

The cycle $D := \overline{D_K} - \sum_{i=1}^{\nu} r_i\Gamma_i$ is equal to D_K on the generic fiber. This is a $\mathbb{Q}$-divisor on X which is τ-equivalent to zero because dnD is a divisor on X which is τ-equivalent to zero. $\square$

Keep the notation of Proposition 3.4.2. Even if X/R admits a section, so that d is equal to 1, the closed point $x_{K,i}$ is *not* rational as soon as the special fiber X_k is not reduced at the generic point of the irreducible component Γ_i. Therefore, if we want to combine Theorem 3.4.1 and Proposition 3.4.2 when $X = \overline{A}$ (notation of Theorem 3.2.1), we need to compare the values of Néron's pairing on the *abelian variety* A_K for 0-cycles which are supported on non-rational points, with its values for 0-cycles of the form $[a_K] - [0_K]$, with $a_K \in A_K(K)$. Here we will use the constructions of Subsection 3.3, together with some biduality argument. To take care of the conditions on supports involved in the computations of Néron's pairings, let us first note the following lemma.

Lemma 3.4.3. *Let A_K be an abelian variety over K with dual A'_K. Let $a'_K \in A'_K(K)$ and let $\mathcal{E}$ be a finite set of closed points of A_K. Then there exists a Poincaré divisor on $A_K \times_K A'_K$, that is, a divisor such that the invertible sheaf $\mathcal{O}_{A_K\times_K A'_K}(P)$ is a Poincaré sheaf which is birigidified along $0_K \in A_K(K)$ and $0'_K \in A'_K(K)$, satisfying the following conditions:*

1. *$P_{0_K} := P|_{0_K\times_K A'_K}$ and $P_{0'_K} := P|_{A_K\times_K 0'_K}$ are well-defined and equal to zero;*
2. *$P_{a'_K} := P|_{A_K\times_K a'_K}$ is well-defined, and its support does not meet $\mathcal{E}$;*
3. *For all $a_K \in \mathcal{E}$, $P_{a_K} := P|_{a_K\times_K A'_K}$ is well-defined, and its support does not meet $\{0'_K, a'_K\}$.*

Proof. Consider the finite set $\mathcal{F}$ whose elements are the following closed points of the product $A_K \times_K A'_K$:

$$a_K \times_K 0'_K \text{ or } a_K \times_K a'_K, \text{ with } a_K \in (\{0_K\} \coprod \mathcal{E}).$$

Let $\mathcal{P}$ be a Poincaré sheaf on $A_K \times_K A'_K$, birigidified along $0_K \in A_K(K)$ and $0'_K \in A'_K(K)$. Choose an arbitrary divisor Q such that $\mathcal{O}_{A_K\times_K A'_K}(Q) \simeq \mathcal{P}$. Using a moving lemma on the product $A_K \times_K A'_K$ if necessary ([Li] 9.1.11), one can assume that the support of Q is disjoint from the finite set $\mathcal{F}$. As $0_K \times_K 0'_K \in \mathcal{F}$, the divisors $Q|_{0_K\times_K A'_K}$ and $Q|_{A_K\times_K 0'_K}$ are well-defined, and are principal. Then

$$P := Q - p_2^*(Q|_{0_K\times_K A'_K}) - p_1^*(Q|_{A_K\times_K 0'_K})$$

(where $p_1 : A_K \times_K A'_K \to A_K$ and $p_2 : A_K \times_K A'_K \to A'_K$ are the projections) is a Poincaré divisor again.

Now, let $a_K \in (\{0_K\} \coprod \mathcal{E})$. Then $a_K \times_K a'_K$ does not belong to the support $\mathrm{Supp}(Q)$ of Q because $a_K \times_K a'_K \in \mathcal{F}$. Next, $a_K \times_K a'_K \notin \mathrm{Supp}(p_2^*(Q|_{0_K\times_K A'_K}))$: indeed, $0_K \times_K a'_K \in \mathcal{F}$ by definition, hence $0_K \times_K a'_K \notin \mathrm{Supp}(Q)$, and consequently $a'_K \notin \mathrm{Supp}(Q|_{0_K\times_K A'_K})$. Finally $a_K \times_K a'_K \notin \mathrm{Supp}(p_1^*(Q|_{A_K\times_K 0'_K}))$, because otherwise $a_K \in \mathrm{Supp}(Q|_{A_K\times_K 0'_K})$ and $a_K \times_K 0'_K \in \mathrm{Supp}(Q)$, which is not the case because $a_K \times_K 0'_K \in \mathcal{F}$. We have thus shown that the point $a_K \times_K a'_K$ does not belong to the support of P. Similarly, the point $a_K \times_K 0'_K$ does not belong to the support of P. In conclusion:

1. $P|_{0_K \times_K A'_K}$ and $P|_{A_K \times_K 0'_K}$ are well-defined, and are equal to zero, by definition of P;

2. $P|_{A_K \times_K a'_K}$ is well-defined, and its support does not meet $\mathcal{E}$, because $a_K \times_K a'_K \notin \mathrm{Supp}(P)$ for all $a_K \in \mathcal{E}$;

3. $P|_{a_K \times_K A'_K}$ is well-defined for all $a_K \in \mathcal{E}$, and its support does not meet $\{0'_K, a'_K\}$, because $a_K \times_K a'_K \notin \mathrm{Supp}(P)$ and $a_K \times_K 0'_K \notin \mathrm{Supp}(P)$ for all $a_K \in \mathcal{E}$.

□

We can now proceed to the announced comparison of some values of Néron's pairing.

Proposition 3.4.4. *Let A_K be an abelian variety with dual A'_K. Let $c_K \in Z_0^0(A_K)$ and $D'_K \in \mathrm{Div}^0(A_K)$. Assume that the support of D'_K is disjoint from that of c_K and that of $[\mathcal{S}(c_K)] - [0_K]$ (Notation 3.3.2). Then the following relation between values of Néron's pairing on A_K is true:*

$$\langle\, c_K \ , \ D'_K \,\rangle \equiv \langle\ [\mathcal{S}(c_K)] - [0_K] \ , \ D'_K \,\rangle \quad \mathrm{mod}\, \mathbb{Z}\,.$$

Proof. Let $a'_K \in A'_K(K)$ corresponding to D'_K. Let $\mathcal{E}$ be a finite set of closed points of A_K, containing the supports of c_K and $[\mathcal{S}(c_K)] - [0_K]$. From Lemma 3.4.3, there exists a Poincaré divisor P satisfying the following conditions:

1. $P_{0_K} := P|_{0_K \times_K A'_K}$ and $P_{0'_K} := P|_{A_K \times_K 0'_K}$ are well-defined and equal to zero;

2. $P_{a'_K} := P|_{A_K \times_K a'_K}$ is well-defined, and its support does not meet $\mathcal{E}$;

3. $P_{a_K} := P|_{a_K \times_K A'_K}$ is well-defined for all $a_K \in \mathcal{E}$, and its support does not meet $\{0'_K, a'_K\}$.

Then, the divisors D'_K and $P_{a'_K}$ are linearly equivalent. Consequently, we can assume $D'_K = P_{a'_K}$ (Proposition 2.2.5).

Write $c_K = c_K^+ - c_K^-$ where c_K^+ and c_K^- are positive 0-cycles with disjoint supports. Let L/K be a finite field extension such that

$$c_K^+ \otimes_K L = \sum_{r=1}^{d} [a_{r,+}] \quad \text{and} \quad c_K^- \otimes_K L = \sum_{r=1}^{d} [a_{r,-}]$$

where $d := \deg c_K^+ = \deg c_K^-$ and with $a_{r,+}$, $a_{r,-}$ in $A_L(L)$ (repetitions allowed). Computing Néron's pairings over K and over L with normalized valuations, we get

$$\langle\, c_K \ , \ P_{a'_K} \,\rangle_{A_K} = \frac{1}{e_L} \langle \sum_{r=1}^{d} [a_{r,+}] - \sum_{r=1}^{d} [a_{r,-}] \ , \ (P_L)_{a'_L} \,\rangle_{A_L},$$

where P_L is the pull-back of P over L, the point $a'_L \in A'_L(L)$ is the image of $a'_K \in A'_K(K)$ by the inclusion $A'_K(K) \subseteq A'_L(L)$, and e_L is the ramification index of L/K. As $(P_L)_{0'_L} = 0$, the *reciprocity law* for Néron's pairing ([La] 11.4.2) [1] asserts

[1] Here we use the reciprocity law in the case where the divisorial correspondence is the Poincaré divisor P_L. By using a definition of Néron's pairing relying on the Poincaré biextension (see [Z] §5 or [MT] §2), the reciprocity law for P_L is a direct consequence of the *biduality* of abelian varieties.

that the right-hand side of the equality is equal to the (well-defined) quantity

$$\frac{1}{e_L}\,\langle\ [a'_L]-[0'_L]\ ,\ \sum_{r=1}^{d}(P_L)_{a_{r,+}}-\sum_{r=1}^{d}(P_L)_{a_{r,-}}\ \rangle_{A'_L}\,.$$

Now, with Notation 3.3.3, the divisor $\sum_{r=1}^{d}(P_L)_{a_{r,+}}-\sum_{r=1}^{d}(P_L)_{a_{r,-}}$ is precisely the pull-back over L of the divisor P_{c_K} on A'_K. Furthermore, as the Poincaré map

$$A_L(L)\longrightarrow \mathrm{Pic}^0_{A'_L/L}(L)$$

is a group homomorphism, the divisors P_{c_K} and $P_{S(c_K)}$ are linearly equivalent on A'_L, and thus on A'_K (because $\mathrm{Pic}^0_{A'_K/K}(K)$ is contained in $\mathrm{Pic}^0_{A'_K/K}(L)$). Let $f\in K(A'_K)$ be such that $P_{c_K}-P_{S(c_K)}=\mathrm{div}(f)$. As the normalized valuation on K takes values in $\mathbb{Z}$, the (well-defined) pairing

$$\frac{1}{e_L}\,\langle\ [a'_L]-[0'_L]\ ,\ (\mathrm{div}(f))_L\ \rangle_{A'_L}=\langle\ [a'_K]-[0'_K]\ ,\ \mathrm{div}(f)\ \rangle_{A'_K}$$

is an *integer*. Consequently,

$$\langle\, c_K\ ,\ P_{a'_K}\,\rangle_{A_K}\equiv\langle\ [a'_K]-[0'_K]\ ,\ P_{\mathcal{S}(c_K)}\,\rangle_{A'_K}\quad \mathrm{mod}\,\mathbb{Z}\,.$$

As $P_{0_K}=0$ and $P_{0'_K}=0$, we conclude by using once again the reciprocity law. □

We can now interpret Grothendieck's obstruction (Subsection 3.1) in terms of relative algebraic equivalence.

Theorem 3.4.5. *Keep the notation of Theorem 3.2.1. Moreover, let Φ_A (resp. $\Phi_{A'}$) be the group of connected components of A_k (resp. A'_k).*

Let $a'\in\Phi_{A'}$. Lift a' to a point $a'_K\in A'_K(K)$, representing the linear equivalence class of a divisor D'_K on A_K. Then the homomorphism

$$\langle\ ,\ a'\,\rangle:\Phi_A\longrightarrow\mathbb{Q}\,/\,\mathbb{Z}$$

induced by Grothendieck's pairing is identically zero if and only if D'_K can be extended to a $\mathbb{Q}$-divisor on $\overline{A}$ which is τ-equivalent to zero relative to R.

Proof. Suppose that the obstruction $\langle\ ,\ a'\,\rangle$ vanishes. Choose 0-cycles of degree zero $c_{K,1},\dots,c_{K,\nu}$ on A_K satisfying the conclusion of Proposition 3.4.2 when applied to the model $\overline{A}/R$ of A_K. To prove that D'_K extends to a $\mathbb{Q}$-divisor on $\overline{A}$ which is τ-equivalent to zero, we can replace D'_K by any divisor on A_K which is linearly equivalent to D'_K. In particular, using moving lemma [Li] 9.1.11, we can assume that the support of D'_K does not meet the finite set

$$\{0_K,S(c_{K,1}),\dots,S(c_{K,\nu})\}\coprod_{i=1}^{\nu}\mathrm{Supp}(c_{K,i}).$$

Then, as $\langle\ ,\ a'\,\rangle=0$, we get from Bosch-Lorenzini's Theorem 3.4.1 that

$$\langle\ [\mathcal{S}(c_{K,i})]-[0_K]\ ,\ D'_K\,\rangle\in\mathbb{Z}$$

for all $i=1,\dots,\nu$. Proposition 3.4.4 and Theorem 2.2.1 then imply that

$$[c_{K,i}\ ,\ D'_K]\in\mathbb{Z}$$

for all $i=1,\dots,\nu$. Due to the choice of the $c_{K,i}$, the divisor D'_K can then be extended to a $\mathbb{Q}$-divisor on $\overline{A}$ which is τ-equivalent to zero.

Conversely, suppose that there is a $\mathbb{Q}$-divisor D' on $\overline{A}$ which is τ-equivalent to zero, with generic fiber D'_K. To prove that $\langle\ ,\ a'\,\rangle = 0$, we can assume that 0_K does not belong to the support of D'_K, by adding to D' the divisor of a rational function on $\overline{A}$ if needed. Let n' be a non-zero integer such that $\Delta' := n'D'$ is a divisor on $\overline{A}$ which is τ-equivalent to zero. For each $a_K \in A_K(K)$ which is not in the support of D'_K, we get:

$$[\ [a_K] - [0_K]\ ,\ D'_K] = \frac{1}{n'}\big([\overline{a_K}] - [\overline{0_K}].\Delta'\big) = \big([\overline{a_K}] - [\overline{0_K}].D'\big) \in \mathbb{Z}\,.$$

The first equality holds by definition of the pairing $[\ ,\ ]$, and the second one is true because $[\overline{a_K}] - [\overline{0_K}]$ is contained in the regular locus of $\overline{A}$. Now observe that an element $a \in \Phi_A$ can always be lifted to a point $a_K \in A_K(K)$ which is not in the support of D'_K. Thus, it follows from Theorem 2.2.1 and Bosch-Lorenzini's Theorem 3.4.1 that the obstruction $\langle\ ,\ a'\,\rangle$ vanishes. $\square$

Proof of Theorem 3.2.1. By biduality of abelian varieties, Grothendieck's duality statement is equivalent to the following: the obstruction $\langle\ ,\ a'\,\rangle$ vanishes if and only if $a' = 0$.

Suppose that this assertion is true. Let $(C) \in \mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R)$ and let a'_K be its canonical image in $A'_K(K)$. By Theorem 3.4.5, the obstruction $\langle\ ,\ a'\,\rangle$ vanishes. Hence $a' = 0$, that is, $a_{K'} \in (A')^0(R)$.

Conversely, suppose that the canonical image of $\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R)$ in $A'_K(K)$ is contained in $(A')^0(R)$. Let $a' \in \Phi_{A'}$, and assume that the corresponding obstruction $\langle\ ,\ a'\,\rangle$ vanishes. Choose a lifting $a'_K \in A'_K(K)$ of a'. Then, by Theorem 3.4.5, the point a'_K belongs to the image of $\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R)$. In particular, it belongs to $(A')^0(R)$, and $a' = 0$.

Thus, we have proved that Grothendieck's conjecture is equivalent to the fact that the image of $\mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R)$ in $A'_K(K)$ is contained in $(A')^0(R)$. Now suppose that the conjecture is true. Then, from [P] Corollary 3.14, we obtain isomorphisms

$$\mathrm{Pic}^0(\overline{A}/R) \xrightarrow{\sim} \mathrm{Pic}^{\mathbb{Q},\tau}(\overline{A}/R) \xrightarrow{\sim} (A')^0(R).$$

The last assertion of Theorem 3.2.1 follows. $\square$

4 Grothendieck's pairing for Jacobians

4.1 Statement of the results

Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K. Let X_K be a proper smooth geometrically connected curve over K, and let $J_K := \mathrm{Pic}^0_{X_K/K}$ be its Jacobian. Denote by J (resp. J') the Néron model of J_K (resp. J'_K) over R, and Φ_J (resp. $\Phi_{J'}$) the group of connected components of the special fiber of J/R (resp. J'/R). Theorems 3.4.1 and 2.2.1 describe Grothendieck's pairing associated to J_K in terms of intersection multiplicities on some compactification $\overline{J}$ of J. It is natural to wonder if these computations can be replaced by intersection computations *on a proper flat regular model X of X_K*.

Assume that $X_K(K)$ is nonempty. In this case, the curve X_K can be *embedded into* J_K, and can be used to define a classical theta divisor on J_K. Then, using Theorem 3.4.1, Bosch and Lorenzini described Grothendieck's pairing associated to J_K in terms of the Néron pairing *on* X_K, and so in terms of intersection multiplicities on X, thanks to Gross's and Hriljac's Theorems [G] and [H]. Their precise result is as follows. Let M be the intersection matrix of the special fiber of X/R: if $\Gamma_1, \ldots, \Gamma_\nu$ are the irreducible components of X_k equipped with their reduced scheme structure, the $(i,j)^{\text{th}}$ entry of M is the intersection number $(\Gamma_i \cdot \Gamma_j)$. Denote by Φ_M the torsion part of the cokernel of $M : \mathbb{Z}^\nu \to \mathbb{Z}^\nu$. According to Raynaud's work on the sheaf $\mathrm{Pic}_{X/S}$, there is a canonical isomorphism $\Phi_J = \Phi_M$ (see [BLR] 9.6/1). Now, on the product $\Phi_M \times \Phi_M$, there is the canonical pairing

$$\begin{array}{rcll} \langle\ ,\ \rangle_M : \Phi_M \times \Phi_M & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \\ (\overline{T}, \overline{T'}) & \longmapsto & ({}^tS/n)M(S'/n') & \mod \mathbb{Z} \end{array}$$

for any $n, n' \in \mathbb{Z}\backslash\{0\}$ and $S, S' \in \mathbb{Z}^\nu$ such that $MS = nT$, $MS' = n'T'$. Now let $(a, a') \in \Phi_J \times \Phi_{J'}$. By identifying J_K and J'_K with the help of the opposite of the canonical principal polarization defined by a theta divisor, Grothendieck's pairing of a and a' can be computed by the formula

$$\langle\, a\ ,\ a'\,\rangle = \langle\, a\ ,\ a'\,\rangle_M$$

([BL] Theorem 4.6).

Now assume that $X_K(K)$ is empty. Choosing a field extension L/K such that $X_K(L)$ is nonempty, one can consider a theta divisor on J_L, and it is a classical fact that the associated canonical principal polarization is defined over K. Using its opposite, one can still identify Φ_J with $\Phi_{J'}$, and thus $\Phi_{J'}$ with Φ_M (as k is algebraically closed, the identification $\Phi_J = \Phi_M$ holds without assuming that $X_K(K)$ is nonempty). Then the authors of [BL] ask if both pairings $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_M$ still coincide in this situation (*loc. cit.* Remark 4.9). In [Lor] Theorem 3.4, Lorenzini gives a positive answer to this question when the special fiber of X/R admits two irreducible components C_i and C_j with multiplicities d_i and d_j such that $(C_i \cdot C_j) > 0$ and $\gcd(d_i, d_j) = 1$. Here we show that this result still holds if we only assume that *the global* gcd *of the multiplicities of the irreducible components of X_k is equal to* 1. Note that, due to the hypotheses on R and on X, this global gcd coincides with the *index* of the curve X_K, that is, the smallest positive degree of a divisor on X_K ([R] 7.1.6 1)).

Theorem 4.1.1. *Let R be a complete discrete valuation ring with algebraically closed residue field k and fraction field K. Let X_K be a proper smooth geometrically connected curve over K, with index d. Let J_K be the Jacobian of X_K, identified with its dual using the opposite of its canonical principal polarization. Let X/R be a proper flat regular model of X_K. The following relation between Grothendieck's pairing for J_K and the above pairing defined by the intersection matrix M of X_k is true:*

$$d\,\langle\, a\ ,\ a'\,\rangle = d\,\langle\, a\ ,\ a'\,\rangle_M\,.$$

In particular, we get the following partial answers to Grothendieck's conjecture [SGA 7] IX 1.3 in this case:

Corollary 4.1.2. *Keep the notation of Theorem 4.1.1. Then:*

- *The kernel of Grothendieck's pairing for J_K is killed by d.*
- *If d is prime to the characteristic of k, then Grothendieck's pairing for J_K is perfect.*

Proof. From [BL] Theorem 1.3, the pairing $\langle\ ,\ \rangle_M$ is a perfect duality. So the first point follows directly from Theorem 4.1.1. For the second point, denote by p the characteristic of k. Then Grothendieck's pairing is perfect when restricted to the prime-to-p parts of the component groups: [SGA 7] IX 11.3 and [Ber] Theorem 1. Consequently, the second point follows again from the perfectness of $\langle\ ,\ \rangle_M$ and Theorem 4.1.1. $\square$

4.2 Proof of Theorem 4.1.1

Here are two lemmas to prepare the proof of the theorem.

Recall that, as R is complete with algebraically closed residue field, a classical result of Lang asserts that the Brauer group of K is zero, whence $\mathrm{Pic}^0(X_K) = J_K(K)$.

Lemma 4.2.1. *Let $a, a' \in \Phi_J = \Phi_M$, and choose divisors D_K, D'_K on X_K with disjoint supports, such that $a_K := (D_K)$, $a'_K := (D'_K) \in J_K(K) = \mathrm{Pic}^0(X_K)$ specialize to a, a'. The relationship between the pairing $\langle\ ,\ \rangle_M$ and Néron's pairing on X_K is given by:*

$$\langle\, a,\ a'\,\rangle_M = -\,\langle\, D_K,\ D'_K\,\rangle \quad \mathrm{mod}\,\mathbb{Z}\,.$$

Proof. This is an immediate consequence of the definitions, and of the description of Néron's pairing for the curve X_K in terms of intersection multiplicities on X. Indeed, let $\rho : \mathrm{Pic}(X) \to \mathbb{Z}^\nu$ be the degree morphism $(Z) \mapsto (Z \cdot \Gamma_i)_{i=1,\dots,\nu}$. Denote by $\overline{D_K}$ the schematic closure of D_K in X. By definition of Raynaud's isomorphism $\Phi_J = \Phi_M$, the image of $\rho(\overline{D_K}) \in \mathbb{Z}^\nu$ in $\mathbb{Z}^\nu\,/\,\mathrm{Im}\,M$ is contained in the torsion part Φ_M, and the resulting element is precisely the image of $a \in \Phi_J$ under the isomorphism. In particular, there are $n, n' \in \mathbb{Z}\backslash\{0\}$ and $S, S' \in \mathbb{Z}^\nu$ such that $MS = n\rho(\overline{D_K})$, $MS' = n'\rho(\overline{D'_K})$, and by definition of the symmetric pairing $\langle\ ,\ \rangle_M$, we get

$$\langle\, a,\ a'\,\rangle_M = ({}^tS'/n')\rho(\overline{D_K}) \quad \mathrm{mod}\,\mathbb{Z}\,.$$

Under the identification $\oplus_{i=1}^{\nu}\,\mathbb{Z}\,\Gamma_i \simeq \mathbb{Z}^\nu$, the right-hand side can also be written as an intersection multiplicity:

$$\langle\, a,\ a'\,\rangle_M = \frac{1}{n'}(\overline{D_K}.S') = -\frac{1}{n'}\big(\overline{D_K}.(n'\overline{D'_K} - S')\big) \quad \in \mathbb{Q}\,/\,\mathbb{Z}\,.$$

Now, the equality $MS' = n'\rho(\overline{D'_K})$ means that the divisor $n'\overline{D'_K} - S'$ on X is algebraically equivalent to zero relative to R ([BLR] 9.2/13). Applying Theorem 2.2.1 to the curve X_K, we conclude that

$$\langle\, a,\ a'\,\rangle_M = -[D_K\ ,\ D'_K] = -\,\langle\, D_K\ ,\ D'_K\,\rangle \quad \in \mathbb{Q}\,/\,\mathbb{Z}\,.$$

□

Next, the index d of X_K divides $g-1$ where g is the genus of X_K ([R] 9.5.1). Let us fix a divisor E of degree d on X_K, and consider the linear equivalence class of divisors of degree $g-1$ given by $t_K := (g-1)d^{-1}(E) \in \mathrm{Pic}^{g-1}_{X_K/K}(K)$. The canonical image of the $(g-1)$-fold symmetric product $X_K^{(g-1)}$ in $\mathrm{Pic}^{g-1}_{X_K/K}$ can be translated by t_K to a divisor on J_K, that we will denote by Θ. Then, by extending K and reducing to the case where $X_K(K)$ is nonempty, one sees that the canonical principal polarization φ of J_K can be written explicitly here as $\varphi(z) = -(\Theta_z - \Theta)$, where Θ_z is obtained from Θ by translation by the point z. On the other hand, denoting by Δ the diagonal of $X_K \times_K X_K$, the divisor $d\Delta - E \times_K X_K$ on $X_K \times_K X_K$ defines an element of $\mathrm{Pic}^0_{X_K/K}(X_K)$, hence a K-morphism $h : X_K \to \mathrm{Pic}^0_{X_K/K} = J_K$.

Lemma 4.2.2. *The following diagram of K-morphisms is commutative:*

$$\begin{tikzcd} & J'_K \arrow[dr, "h^*"] & \\ J_K \arrow[ur, "-\varphi"] \arrow[rr, "d"] & & J_K. \end{tikzcd}$$

The commutativity can be stated as follows. Let $z \in J_K(\overline{K})$. Let Z be any divisor of degree 0 on $X_{\overline{K}}$, whose linear equivalence class (Z) corresponds to z via the canonical isomorphism $\mathrm{Pic}^0(X_{\overline{K}}) = J_K(\overline{K})$. Then the following relation holds:

$$h^*(\Theta_z - \Theta) = d(Z) \in \mathrm{Pic}^0(X_{\overline{K}}) = J_K(\overline{K}).$$

In particular, there is a nonempty open subset U_K of J_K such that $h^*\Theta_z$ is a well-defined divisor on X_K for all $z \in U_K(K)$, and whose degree does not depend on the point z.

Proof. To check that the diagram is commutative, one can replace K by its algebraic closure, and so we can assume that K is algebraically closed. As the pull-back by the multiplication-by-d on J_K acts as multiplication-by-d on the group $\mathrm{Pic}^0(J_K)$, the lemma then follows from the classical situation where X_K can be embedded into J_K using a rational point of X_K. □

Proof of Theorem 4.1.1. Let $(a, a') \in \Phi_J \times \Phi_J$. Choose a point $a_K \in J_K(K)$ which specializes to $a \in \Phi_J$. The point a_K corresponds, under the equality $J_K(K) = \mathrm{Pic}^0(X_K)$, to the linear equivalence class of a divisor $D(a)_K$ of degree 0 on X_K. Write $D(a)_K = D(a)_K^+ - D(a)_K^-$ with $D(a)_K^+$ and $D(a)_K^-$ positive with disjoint supports. Let L/K be a finite field extension such that

$$D(a)_K^+ \otimes_K L = \sum_{r=1}^{\alpha} [a_{r,+}] \quad \text{and} \quad D(a)_K^- \otimes_K L = \sum_{r=1}^{\alpha} [a_{r,-}]$$

where $\alpha := \deg D(a)_K^+ = \deg D(a)_K^-$ and with $a_{r,+}$, $a_{r,-}$ in $X_L(L)$ (repetitions allowed).

Next, still denoting by U_K the open subset of J_K provided by Lemma 4.2.2, one can find a'_K and z_K in $U_K(K)$ specializing respectively to a' and 0 in Φ_J, and such that

$$\begin{aligned} da_K,\ 0_K &\notin \mathrm{Supp}(\Theta_{a'_K} - \Theta_{z_K}) \\ \bar{a}_{r,+},\ \bar{a}_{r,-} &\notin \mathrm{Supp}\left((\Theta_{a'_K} - \Theta_{z_K})_L\right) \quad \forall r = 1, \ldots, \alpha, \end{aligned}$$

where $\bar{a}_{r,+} := h(a_{r,+})$ and $\bar{a}_{r,-} := h(a_{r,-})$. The points a'_K and z_K correspond to the classes of some divisors $D(a')_K$ and $D(0)_K$ on X_K, under the identification $J_K(K) = \mathrm{Pic}^0(X_K)$. From Lemma 4.2.2, we get:

$$h^*(\Theta_{a'_K} - \Theta_{z_K}) = d(D(a')_K - D(0)_K) = d(a'_K - z_K)$$

in $\mathrm{Pic}^0(X_K) = J_K(K)$. And by construction, the K-point $d(a'_K - z_K)$ of J_K specializes to $da' \in \Phi_J$. As a consequence, Lemma 4.2.1 provides the formula:

$$\langle\, a\ ,\ da' \,\rangle_M \quad = \quad -\langle\, D(a)_K\ ,\ h^*(\Theta_{a'_K} - \Theta_{z_K}) \,\rangle_{X_K} \quad \mathrm{mod}\, \mathbb{Z}$$

(note that $h^*(\Theta_{a'_K} - \Theta_{z_K})$ is a well-defined divisor, and not only a class, because $a'_K, z_K \in U_K(K)$).

Still working with normalized valuations to compute Néron's pairing, and using functoriality, we obtain:

$$\langle\, a\ ,\ da' \,\rangle_M \quad = \quad -\frac{1}{e_L} \langle \sum_{r=1}^{\alpha} [\bar{a}_{r,+}] - [\bar{a}_{r,-}]\ ,\ (\Theta_{a'_K} - \Theta_{z_K})_L \,\rangle_{J_L} \quad \mathrm{mod}\, \mathbb{Z}\,.$$

where e_L is the ramification index of L/K. Then we apply the reciprocity law for Néron's pairing with the divisorial correspondence $(\delta^*\Theta - p_1^*\Theta - p_2^*\Theta)_L$, where δ, p_1 and $p_2 : J_K \times_K J_K \to J_K$ are the difference map and the two projections, to get:

$$\langle\, a\ ,\ da' \,\rangle_M \quad = \quad -\frac{1}{e_L} \langle\ [a'_L] - [z_L]\ ,\ \sum_{r=1}^{\alpha} (\Theta_L)^-_{\bar{a}_{r,+}} - (\Theta_L)^-_{\bar{a}_{r,-}} \,\rangle_{J_L} \quad \mathrm{mod}\, \mathbb{Z}\,.$$

Here $(\Theta_L)^-$ stands for $[-1]^*(\Theta_L)$.

Now, with Notation 3.3.3, the divisor $\sum_{r=1}^{\alpha}(\Theta_L)^-_{\bar{a}_{r,+}} - (\Theta_L)^-_{\bar{a}_{r,-}}$ is the pull-back on J_L of the divisor $(\Theta^-)_{h_* D(a)}$ defined on J_K. On the other hand,

$$\begin{aligned} \sum_{r=1}^{\alpha} \bar{a}_{r,+} - \bar{a}_{r,-} &= \sum_{r=1}^{\alpha} (d[a_{r,+}] - E_L) - (d[a_{r,-}] - E_L) \\ &= d(D(a)_L) \in J_K(L) \\ &= da_K \in J_K(K). \end{aligned}$$

Therefore the theorem of the square on J_L shows that the two divisors $(\Theta^-)_{h_* D(a)}$ and $\Theta^-_{da_K} - \Theta^-$ on J_K are linearly equivalent over L, hence also over K ($J'_K(K)$ injects into $J'_L(L)$). From this observation, and the fact that the normalized valuation on K takes values in $\mathbb{Z}$, we deduce that

$$\langle\, a \;,\; da' \,\rangle_M \quad = \quad -\langle\, [a'_K] - [z_K] \;,\; \Theta^-_{da_K} - \Theta^- \,\rangle_{J_K} \quad \bmod \mathbb{Z}\,.$$

Applying once more the reciprocity law, we find

$$\langle\, a \;,\; da' \,\rangle_M \quad = \quad -\langle\, [da_K] - [0_K] \;,\; \Theta_{a'_K} - \Theta_{z_K} \,\rangle \quad \bmod \mathbb{Z}\,.$$

Finally, note that $(\Theta_{a'_K} - \Theta_{z_K}) = -\varphi(a'_K - z_K) \in J'(K)$ and $a'_K - z_K$ specializes to $a' \in \Phi_J$. Consequently, if we use $-\varphi$ to identify J_K with its dual, Theorem 3.4.1 tells us that

$$-\langle\, [da_K] - [0_K] \;,\; \Theta_{a'_K} - \Theta_{z_K} \,\rangle \quad = \quad \langle\, da \;,\; a' \,\rangle \quad \bmod \mathbb{Z}\,.$$

Whence

$$\langle\, a \;,\; da' \,\rangle_M \quad = \quad \langle\, da \;,\; a' \,\rangle,$$

as claimed. □

References

[Beg] L. Bégueri, *Dualité sur un corps local à corps résiduel algébriquement clos*, Mém Soc. Math. Fr. **108** (1980), fasc. 4.

[Ber] A. Bertapelle, *On perfectness of Grothendieck's pairing for the ℓ-parts of component groups*, J. Reine Angew. Math. **538** (2001), 223-236.

[BL] S. Bosch, D. Lorenzini, *Grothendieck's pairing on components groups of Jacobians*, Invent. Math. **148** (2002), 353-396.

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Erg. Math. 3. Folge **21**, Springer-Verlag, 1990.

[EGA IV] A. Grothendieck et J. Dieudonné, *Étude locale des schémas et des morphismes de schémas*, Publ. Math. IHES **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[F] W. Fulton, *Intersection Theory*, Springer Verlag, 1984. Second Edition 1998.

[FGA VI] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique. VI. Les schémas de Picard : propriétés générales*, Exp. No. 236, Séminaire Bourbaki, t. **14**, 1961-1962, SMF, Paris, 1995, 221-243.

[G] B. Gross, *Local heights on curves*, in: *Arithmetic Geometry*, G. Cornell, J. Silverman, Eds, Springer-Verlag, 1986.

[H] P. Hriljac, *Heights and Arakelov intersection theory*, Amer. J. Math. **107** (1985), 23-38.

[La] S. Lang, *Fundamentals of diophantine geometry*, Springer-Verlag, 1983.

[La2] S. Lang, *Introduction to Arakelov therory*, Springer-Verlag, 1988.

[Li] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics **6**, Oxford University Press, paperback new edition (2006).

[Loe] K. Loerke, *Reduction of abelian varieties and Grothendieck's pairing*, preprint arxiv 0909.4425v1.pdf.

[Lor] D. Lorenzini, *Grothendieck's pairing for Jacobians and base change*, J. Number Theory **128** (2008), 1448-1457.

[MT] B. Mazur, J. Tate, *Canonical height pairings via biextensions*, Arithmetic and geometry, Vol. 1, Progr. Math. **35**, Birkhäuser Boston, Boston, MA, 1983, 195-237.

[Mum] D. Mumford, *Abelian varieties*, 2nd edition, Oxford Univ. Press, Oxford, 1974.

[Mur] J. P. Murre, *On contravariant functors from the category of preschemes over a field into the category of abelian groups*, Publ. Math. IHES **23** (1964), 5-43.

[N] A. Néron, *Quasi-fonctions et hauteurs sur les variétés abéliennes*, Ann. Math. **82** (1965), 249-331.

[O] F. Oort, *Sur le schéma de Picard*, Bull. Soc. Math. Fr. **90** (1966), 119-127.

[P] C. Pépin, *Modèles semi-factoriels et modèles de Néron*, to appear in Math. Ann.

[R] M. Raynaud, *Spécialisation du foncteur de Picard*, Publ. Math. IHES **38** (1970), 27-76.

[SGA 4] A. Grothendieck et al., *Théorie des topos et cohomologie étale*, Lect. Notes Math. **269, 270, 305**, Springer, Berlin-Heidelberg-New York, 1972-1973.

[SGA 6] A. Grothendieck et al., *Théorie des intersections et Théoréme de Riemann-Roch*, Lect. Notes Math. **225**, Springer, Berlin-Heidelberg-New York, 1971.

[SGA 7] A. Grothendieck et al., *Groupes de monodromie en géométrie algébrique* SGA 7, I, Springer-Verlag LNM **288** (1972).

[W] A. Werner, *On Grothendieck's pairing on component groups in the semistable reduction case*, J. reine angew. Math. **436** (1997), 205-215.

[Z] Ju. G. Zahrin, *Néron pairing and quasicharacters*, Math. USSR Izv. **6** No. 3 (1972), 491-503.

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