Exo 3 (TD 4)

$a, b, c, d \in \mathbb{R}$ $\quad ad-bc>0$ et $c \neq 0$

$$f(x) := \frac{ax+b}{cx+d}$$

1. $\mathcal{D}_f = \mathbb{R} \setminus \{-\frac{d}{c}\}$ ie $cx+d \neq 0 \Leftrightarrow x = \frac{-d}{c}$ car $c \neq 0$

2. $f(x) = \frac{u(x)}{v(x)}$ avec $u(x) = ax+b$ $\quad u'(x) = a$

$v(x) = cx+d$ $\quad v'(x) = c$

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)}$$

$$= \frac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2} > 0 \quad \text{car } ad-bc>0$$

Calculons les limites de f en $-\infty$, $+\infty$, et en $-\frac{d}{c}$

- $$\lim_{x \to -\infty} \frac{ax+b}{cx+d} = \lim_{x \to -\infty} \frac{x(a+\frac{b}{x})}{x(c+\frac{d}{x})} = \lim_{x \to -\infty} \frac{a+\frac{b}{x}}{c+\frac{d}{x}} = \frac{a}{c}$$

on peut écrire ici directement

$$\lim_{x \to -\infty} \frac{ax+b}{cx+d} = \lim_{x \to -\infty} \frac{ax}{cx} = \frac{a}{c}$$

- $$\lim_{x \to +\infty} \frac{ax+b}{cx+d} = \lim_{x \to +\infty} \frac{ax}{cx} = \frac{a}{c}$$

- $$\lim_{\substack{x \to -\frac{d}{c} \\ x > -\frac{d}{c}}} \frac{ax+b}{cx+d} = \, ?$$

on a $ax+b \xrightarrow[x \to -\frac{d}{c}]{} -\frac{ad}{c} + b = \frac{bc-ad}{c}$

* Si $c>0$ alors $\frac{bc-ad}{c} < 0$ car $ad-bc>0$

et $x > -\frac{d}{c} \Leftrightarrow cx > -d$ $(c>0)$

$\Leftrightarrow cx+d>0$

D'où $$\lim_{\substack{x \to -\frac{d}{c} \\ x > -\frac{d}{c}}} \frac{ax+b}{cx+d} = \frac{bc-ad}{c} \times \lim_{\substack{x \to -\frac{d}{c} \\ x > -\frac{d}{c}}} \frac{1}{\underbrace{cx+d}_{\to 0^+}} = \underbrace{\frac{bc-ad}{c}}_{<0} \times (+\infty)$$

$= -\infty$

* Si $c<0$ alors $\frac{bc-ad}{c} > 0$

et $x > -\frac{d}{c} \Leftrightarrow cx < -d$ $(c<0)$

$\Leftrightarrow cx+d<0$

D'où $$\lim_{\substack{x \to -\frac{d}{c} \\ x > -\frac{d}{c}}} \frac{ax+b}{cx+d} = \frac{bc-ad}{c} \lim_{\substack{x \to -d/c \\ x > -d/c}} \frac{1}{\underbrace{cx+d}_{\to 0^-}} = \underbrace{\frac{bc-ad}{c}}_{>0} \times (-\infty)$$

$= -\infty$

Donc $$\lim_{\substack{x \to -d/c \\ x > -d/c}} \frac{ax+b}{cx+d} = -\infty \quad \forall c \neq 0$$

A vous de faire $$\lim_{\substack{x \to -d/c \\ x < -d/c}} \frac{ax+b}{cx+d}$$