

Propriété

Soient $\mathbf{x} \in \mathbb{K}^n$ et $\mathbf{u} \in \mathbb{K}^n$, $\|\mathbf{u}\|_2 = 1$. On note $\mathbf{x}_{\parallel} = \text{proj}_{\mathbf{u}}(\mathbf{x}) \stackrel{\text{def}}{=} \langle \mathbf{u}, \mathbf{x} \rangle \mathbf{u}$ et $\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\parallel}$. On a alors

$$\mathbb{H}(\mathbf{u})(\mathbf{x}_{\perp} + \mathbf{x}_{\parallel}) = \mathbf{x}_{\perp} - \mathbf{x}_{\parallel}. \quad (\text{P-1})$$

et

$$\mathbb{H}(\mathbf{u})\mathbf{x} = \mathbf{x}, \quad \text{si } \langle \mathbf{x}, \mathbf{u} \rangle = 0. \quad (\text{P-2})$$

Proof. On note que par construction $\langle \mathbf{u}, \mathbf{x}_{\perp} \rangle = 0$. On a

$$\begin{aligned} \mathbb{H}(\mathbf{u})(\mathbf{x}_{\perp} + \mathbf{x}_{\parallel}) &= (\mathbb{I} - 2\mathbf{u}\mathbf{u}^*)(\mathbf{x}_{\perp} + \mathbf{x}_{\parallel}) = \mathbf{x}_{\perp} + \mathbf{x}_{\parallel} - 2\mathbf{u} \underbrace{\mathbf{u}^*\mathbf{x}_{\perp}}_{=0} - 2\mathbf{u}\mathbf{u}^*\mathbf{x}_{\parallel} \\ &= \mathbf{x}_{\perp} + \mathbf{x}_{\parallel} - 2\mathbf{u}\mathbf{u}^*\mathbf{u} \langle \mathbf{u}, \mathbf{x} \rangle = \mathbf{x}_{\perp} + \mathbf{x}_{\parallel} - 2\mathbf{u} \underbrace{\mathbf{u}^*\mathbf{u}}_{=1} \mathbf{u}^*\mathbf{x} \\ &= \mathbf{x}_{\perp} + \mathbf{x}_{\parallel} - 2\mathbf{u}\mathbf{u}^*\mathbf{x} = \mathbf{x}_{\perp} + \mathbf{x}_{\parallel} - 2\mathbf{x}_{\parallel} \\ &= \mathbf{x}_{\perp} - \mathbf{x}_{\parallel}. \end{aligned}$$

Si $\langle \mathbf{x}, \mathbf{u} \rangle = 0$ alors $\mathbf{x}_{\parallel} = 0$ et $\mathbf{x} = \mathbf{x}_{\perp}$. □

