Propriété

Soient $\boldsymbol{x} \in \mathbb{K}^n$ et $\boldsymbol{u} \in \mathbb{K}^n$, $\|\boldsymbol{u}\|_2 = 1$. On note $\boldsymbol{x}_\parallel = \mathrm{proj}_{\boldsymbol{u}}(\boldsymbol{x}) \stackrel{\text{def}}{=} \langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u}$ et $\boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{x}_\parallel$. On a alors

$$\mathbb{H}(\boldsymbol{u})(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel) = \boldsymbol{x}_\perp - \boldsymbol{x}_\parallel. \tag{P-1}$$

et

$$\mathbb{H}(\boldsymbol{u})\boldsymbol{x} = \boldsymbol{x}, \quad \text{si } \langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0. \tag{P-2}$$

Proof. On note que par construction $\langle \boldsymbol{u}, \boldsymbol{x}_\perp \rangle = 0$. On a

$$\begin{aligned}
\mathbb{H}(\boldsymbol{u})(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel) &= \left(\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*\right)\left(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel\right) = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{u}\underbrace{\boldsymbol{u}^*\boldsymbol{x}_\perp}_{=0} - 2\boldsymbol{u}\boldsymbol{u}^*\boldsymbol{x}_\parallel \\
&= \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{u}\boldsymbol{u}^*\boldsymbol{u}\langle \boldsymbol{u}, \boldsymbol{x} \rangle = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{u}\underbrace{\boldsymbol{u}^*\boldsymbol{u}}_{=1}\boldsymbol{u}^*\boldsymbol{x} \\
&= \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{u}\boldsymbol{u}^*\boldsymbol{x} = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{x}_\parallel \\
&= \boldsymbol{x}_\perp - \boldsymbol{x}_\parallel.
\end{aligned}$$

Si $\langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0$ alors $\boldsymbol{x}_\parallel = 0$ et $\boldsymbol{x} = \boldsymbol{x}_\perp$. $\square$