Exercice

Soient $\boldsymbol{a}$ et $\boldsymbol{b}$ deux vecteurs non colinéaires de $\mathbb{C}^n$ avec $\|\boldsymbol{b}\|_2 = 1$. On va chercher $\alpha \in \mathbb{C}$ et $\boldsymbol{u} \in \mathbb{C}^n$ vérifiant

$$\mathbb{H}(\boldsymbol{u})\boldsymbol{a} = \alpha\boldsymbol{b}. \tag{P-1}$$

Q. 1 1. *Montrer que si* α *vérifie* (P-1) *alors* $|\alpha| = \|\boldsymbol{a}\|_2$.

2. *Montrer que si* $\arg\alpha = -\arg(\langle \boldsymbol{a}, \boldsymbol{b} \rangle)\ [\pi]$ *alors* $\alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \in \mathbb{R}$.

Q. 2 *Soient* α *et* $\boldsymbol{u}$ *vérifiant* (P-1).

1. *Montrer que*

$$|\langle \boldsymbol{u}, \boldsymbol{a} \rangle|^2 = \frac{\langle \boldsymbol{a}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle}{2} \tag{P-2}$$

2. *Montrer que si* $\arg\alpha = -\arg(\langle \boldsymbol{a}, \boldsymbol{b} \rangle)\ [\pi]$ *alors* $\langle \boldsymbol{a}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \in \mathbb{R}^{*+}$.

3. *En déduire que*

$$\boldsymbol{u} = \frac{1}{2\lambda}(\boldsymbol{a} - \alpha\boldsymbol{b}), \quad \textit{avec } \lambda = \pm\left(\frac{\langle \boldsymbol{a}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle}{2}\right)^{1/2} \tag{P-3}$$

Correction Exercice On pose $\mathbb{H} = \mathbb{H}(\boldsymbol{u})$ pour alléger les notations.

Q. 1 1. On a

$$\begin{aligned}
\|\boldsymbol{a}\|_2^2 = \langle \boldsymbol{a}, \boldsymbol{a} \rangle &= \langle \mathbb{H}^*\mathbb{H}\boldsymbol{a}, \boldsymbol{a} \rangle \quad \text{car } \mathbb{H} \text{ unitaire} \\
&= \langle \mathbb{H}\boldsymbol{a}, \mathbb{H}\boldsymbol{a} \rangle \quad \text{par définition du produit scalaire} \\
&= \|\mathbb{H}\boldsymbol{a}\|_2^2 = \|\alpha\boldsymbol{b}\|_2^2 = |\alpha|^2 \|\boldsymbol{b}\|_2^2 = |\alpha|^2.
\end{aligned}$$

2. On a par définition de l'argument $\alpha = |\alpha| e^{\imath \arg\alpha}$ et $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = |\langle \boldsymbol{a}, \boldsymbol{b} \rangle| e^{\imath \arg(\langle \boldsymbol{a}, \boldsymbol{b} \rangle)}$ ce qui donne

$$\alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle = |\alpha| |\langle \boldsymbol{a}, \boldsymbol{b} \rangle| e^{\imath(\arg\alpha + \arg(\langle \boldsymbol{a}, \boldsymbol{b} \rangle))} \tag{P-4}$$

et donc $\alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle$ est réel si $\arg \alpha + \arg(\langle \boldsymbol{a}, \boldsymbol{b} \rangle) = 0\ [\pi]$.

Q. 2 1. On a

$$\begin{aligned}\mathbb{H}(\boldsymbol{u})\boldsymbol{a} = \alpha \boldsymbol{b} &\iff (\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*)\boldsymbol{a} = \alpha \boldsymbol{b} \\ &\iff \boldsymbol{a} - 2\boldsymbol{u}(\boldsymbol{u}^*\boldsymbol{a}) = \alpha \boldsymbol{b}\end{aligned}$$

et donc

$$\boldsymbol{a} - 2 \langle \boldsymbol{u}, \boldsymbol{a} \rangle \boldsymbol{u} = \alpha \boldsymbol{b} \tag{P-5}$$

En effectuant le produit scalaire avec $\boldsymbol{a}$ de cette dernière équation, on obtient

$$\langle \boldsymbol{a}, \boldsymbol{a} \rangle - 2 \langle \boldsymbol{u}, \boldsymbol{a} \rangle \langle \boldsymbol{a}, \boldsymbol{u} \rangle = \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle$$

ce qui prouve (P-2).

2. On a montré en Q.1 que $\alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \in \mathbb{R}$ et donc $\langle \boldsymbol{a}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \in \mathbb{R}$. Il reste donc à montrer que $\langle \boldsymbol{a}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle > 0$.

   - Si $\arg \alpha = -\arg(\langle \boldsymbol{a}, \boldsymbol{b} \rangle) + \pi\ [2\pi]$, alors de (P-4) on obtient $\alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \leqslant 0$ et donc $\langle \boldsymbol{a}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \geqslant \|\boldsymbol{a}\|_2 > 0$ car $\boldsymbol{a} \neq 0$.
   - Si $\arg \alpha = -\arg(\langle \boldsymbol{a}, \boldsymbol{b} \rangle)\ [2\pi]$, alors de (P-4) on obtient $\alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \geqslant 0$.
     Comme les vecteurs $\boldsymbol{a}$ et $\boldsymbol{b}$ ne sont pas colinéaires, on a inégalité stricte dans Cauchy-Schwarz :

     $$|\langle \boldsymbol{a}, \boldsymbol{b} \rangle| < \|\boldsymbol{a}\|_2 \|\boldsymbol{b}\|_2 = \|\boldsymbol{a}\|_2 .$$

     On obtient donc

     $$0 \leqslant \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \leqslant |\alpha| |\langle \boldsymbol{a}, \boldsymbol{b} \rangle| < |\alpha| \|\boldsymbol{a}\|_2 = \|\boldsymbol{a}\|_2^2$$

     Attention, dans ce cas $\langle \boldsymbol{a}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle$ peut-être très petit.

3. De (P-5), on en déduit immédiatement (P-3).
   Vérifions que $\|\boldsymbol{u}\|_2 = 1$. On a

   $$\|\boldsymbol{u}\|_2^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \frac{1}{4|\lambda|^2} \langle \boldsymbol{a} - \alpha \boldsymbol{b}, \boldsymbol{a} - \alpha \boldsymbol{b} \rangle$$

Or

$$\begin{aligned}
\langle \boldsymbol{a} - \alpha \boldsymbol{b}, \boldsymbol{a} - \alpha \boldsymbol{b} \rangle = \langle \boldsymbol{a}, \boldsymbol{a} \rangle - \overline{\alpha} \langle \boldsymbol{b}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle + |\alpha|^2 \langle \boldsymbol{b}, \boldsymbol{b} \rangle &= \|\boldsymbol{a}\|_2^2 - \overline{\alpha} \langle \boldsymbol{b}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle + |\alpha|^2 \\
&= 2 \|\boldsymbol{a}\|_2^2 - \overline{\alpha} \langle \boldsymbol{b}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \\
&= 2 \|\boldsymbol{a}\|_2^2 - 2\alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle \quad \text{car } \alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle = \overline{(\alpha \langle \boldsymbol{a}, \boldsymbol{b} \rangle)} = \overline{\alpha} \langle \boldsymbol{b}, \boldsymbol{a} \rangle \in \mathbb{R}
\end{aligned}$$

De plus

$$\begin{aligned}
4|\lambda|^2 &= 2(\langle \boldsymbol{a}, \boldsymbol{a} \rangle - \alpha \langle \boldsymbol{b}, \boldsymbol{a} \rangle) \in \mathbb{R} \\
&= 2 \|\boldsymbol{a}\|_2^2 - 2\alpha \langle \boldsymbol{b}, \boldsymbol{a} \rangle
\end{aligned}$$

$\diamond$