

Propriété

Soient \mathbf{a}, \mathbf{b} deux vecteurs non colinéaires de \mathbb{C}^n avec $\|\mathbf{b}\|_2 = 1$. Soit $\alpha \in \mathbb{C}$ tel que $|\alpha| = \|\mathbf{a}\|_2$ et $\arg \alpha = -\arg \langle \mathbf{a}, \mathbf{b} \rangle [\pi]$. On a alors

$$\mathbb{H} \left(\frac{\mathbf{a} - \alpha \mathbf{b}}{\|\mathbf{a} - \alpha \mathbf{b}\|_2} \right) \mathbf{a} = \alpha \mathbf{b}. \quad (\text{P-1})$$

Proof. Pour simplifier, on note $\mathbb{H} = \mathbb{H}(\mathbf{u})$. Cette matrice est hermitienne car

$$\mathbb{H}^* = \left(\mathbb{I} - 2 \frac{\mathbf{u} \mathbf{u}^*}{\|\mathbf{u}\|_2^2} \right)^* = \mathbb{I} - 2 \frac{(\mathbf{u} \mathbf{u}^*)^*}{\|\mathbf{u}\|_2^2} = \mathbb{I} - 2 \frac{\mathbf{u} \mathbf{u}^*}{\|\mathbf{u}\|_2^2} = \mathbb{H}.$$

Montrons qu'elle est unitaire (i.e. $\mathbb{H}^* \mathbb{H} = \mathbb{I}$). On a

$$\begin{aligned} \mathbb{H}^* \mathbb{H} &= \mathbb{H} \mathbb{H} = \left(\mathbb{I} - 2 \frac{\mathbf{u} \mathbf{u}^*}{\|\mathbf{u}\|_2^2} \right) \left(\mathbb{I} - 2 \frac{\mathbf{u} \mathbf{u}^*}{\|\mathbf{u}\|_2^2} \right) \\ &= \mathbb{I} - 4 \frac{\mathbf{u} \mathbf{u}^*}{\|\mathbf{u}\|_2^2} + 4 \frac{\mathbf{u} \mathbf{u}^* \mathbf{u} \mathbf{u}^*}{\|\mathbf{u}\|_2^2}. \end{aligned}$$

Or $\mathbf{u}^* \mathbf{u} = \|\mathbf{u}\|_2^2$, ce qui donne

$$\mathbb{H}^* \mathbb{H} = \mathbb{I} - 4 \frac{\mathbf{u} \mathbf{u}^*}{\|\mathbf{u}\|_2^2} + 4 \frac{\mathbf{u} (\mathbf{u}^* \mathbf{u}) \mathbf{u}^*}{\|\mathbf{u}\|_2^2} = \mathbb{I}.$$

□

