Exercice 8

Soit $\boldsymbol{u} \in \mathbb{C}^n$ tel que $\|\boldsymbol{u}\|_2 = 1$. On note $\mathbb{H} \in \mathcal{M}_n(\mathbb{C})$ la matrice définie par

$$\mathbb{H} = \mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*.$$

Q. 1

a. Montrer que $\mathbb{H}$ est hermitienne.

b. Montrer que $\mathbb{H}$ est unitaire.

R. 1

a. Cette matrice est hermitienne car

$$\mathbb{H}^* = (\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*)^* = \mathbb{I} - 2(\boldsymbol{u}\boldsymbol{u}^*)^* = \mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^* = \mathbb{H}.$$

b. La matrice $\mathbb{H}$ est unitaire si $\mathbb{H}^*\mathbb{H} = \mathbb{I}$. On a

$$\begin{aligned} \mathbb{H}^*\mathbb{H} = \mathbb{H}\mathbb{H} &= (\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*)(\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*) \\ &= \mathbb{I} - 4\boldsymbol{u}\boldsymbol{u}^* + 4\boldsymbol{u}\boldsymbol{u}^*\boldsymbol{u}\boldsymbol{u}^*. \end{aligned}$$

Or, par hypothèse, on a $\boldsymbol{u}^*\boldsymbol{u} = \|\boldsymbol{u}\|_2 = 1$ et donc

$$\mathbb{H}^*\mathbb{H} = \mathbb{I} - 4\boldsymbol{u}\boldsymbol{u}^* + 4\boldsymbol{u}(\boldsymbol{u}^*\boldsymbol{u})\boldsymbol{u}^* = \mathbb{I}.$$

Soit $\boldsymbol{x} \in \mathbb{K}^n$. On note $\boldsymbol{x}_{\|} = \operatorname{proj}_{\boldsymbol{u}}(\boldsymbol{x}) \stackrel{\text{def}}{=} \langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u}$ et $\boldsymbol{x}_{\perp} = \boldsymbol{x} - \boldsymbol{x}_{\|}$.

Q. 2

Montrer que

$$\mathbb{H}(\boldsymbol{x}_{\perp} + \boldsymbol{x}_{\|}) = \boldsymbol{x}_{\perp} - \boldsymbol{x}_{\|}.$$

et

$$\mathbb{H}\boldsymbol{x} = \boldsymbol{x}, \quad \textit{si } \langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0.$$

R. 2

On note que par construction $\langle \boldsymbol{u}, \boldsymbol{x}_\perp \rangle = 0$. En effet, on a

$$\begin{aligned}
\langle \boldsymbol{u}, \boldsymbol{x}_\perp \rangle &= \left\langle \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{x}_\parallel \right\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{x} \rangle - \left\langle \boldsymbol{u}, \boldsymbol{x}_\parallel \right\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{x} \rangle - \langle \boldsymbol{u}, \langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u} \rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{x} \rangle - \langle \boldsymbol{u}, \boldsymbol{x} \rangle \langle \boldsymbol{u}, \boldsymbol{u} \rangle \\
&= 0 \quad \text{car} \quad \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \|\boldsymbol{u}\|_2^2 = 1.
\end{aligned}$$

On a alors

$$\begin{aligned}
\mathbb{H}(\boldsymbol{u})(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel) = (\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*)\,(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel) &= \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{u}\,\underbrace{\boldsymbol{u}^*\boldsymbol{x}_\perp}_{=0} - 2\boldsymbol{u}\boldsymbol{u}^*\boldsymbol{x}_\parallel \\
&= \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{u}\boldsymbol{u}^*\big(\langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u}\big) = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u}\,\underbrace{\boldsymbol{u}^*\boldsymbol{u}}_{=1} \\
&= \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{x}_\parallel \\
&= \boldsymbol{x}_\perp - \boldsymbol{x}_\parallel.
\end{aligned}$$

Si $\langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0$ alors $\boldsymbol{x}_\parallel = 0$ et $\boldsymbol{x} = \boldsymbol{x}_\perp$.