Exercice 8

Soit $\boldsymbol{u} \in \mathbb{C}^n$ tel que $\|\boldsymbol{u}\|_2 = 1$. On note $\mathbb{H} \in \mathcal{M}_n(\mathbb{C})$ la matrice définie par

$$\mathbb{H} = \mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*.$$

Q. 1

a. Montrer que $\mathbb{H}$ est hermitienne.

b. Montrer que $\mathbb{H}$ est unitaire.

R. 1

a. Cette matrice est hermitienne car

$$\mathbb{H}^* = (\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*)^* = \mathbb{I} - 2(\boldsymbol{u}\boldsymbol{u}^*)^* = \mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^* = \mathbb{H}.$$

b. La matrice $\mathbb{H}$ est unitaire si $\mathbb{H}^*\mathbb{H} = \mathbb{I}$. On a

$$\begin{aligned}\mathbb{H}^*\mathbb{H} = \mathbb{H}\mathbb{H} &= (\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*)(\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*)\\ &= \mathbb{I} - 4\boldsymbol{u}\boldsymbol{u}^* + 4\boldsymbol{u}\boldsymbol{u}^*\boldsymbol{u}\boldsymbol{u}^*.\end{aligned}$$

Or, par hypothèse, on a $\boldsymbol{u}^*\boldsymbol{u} = \|\boldsymbol{u}\|_2 = 1$ et donc

$$\mathbb{H}^*\mathbb{H} = \mathbb{I} - 4\boldsymbol{u}\boldsymbol{u}^* + 4\boldsymbol{u}(\boldsymbol{u}^*\boldsymbol{u})\boldsymbol{u}^* = \mathbb{I}.$$

Soit $\boldsymbol{x} \in \mathbb{K}^n$. On note $\boldsymbol{x}_\parallel = \mathrm{proj}_{\boldsymbol{u}}(\boldsymbol{x}) \stackrel{\text{def}}{=} \langle \boldsymbol{u}, \boldsymbol{x}\rangle \boldsymbol{u}$ et $\boldsymbol{x}_\perp = \boldsymbol{x} - \boldsymbol{x}_\parallel$.

Q. 2

Montrer que

$$\mathbb{H}(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel) = \boldsymbol{x}_\perp - \boldsymbol{x}_\parallel.$$

et

$$\mathbb{H}\boldsymbol{x} = \boldsymbol{x}, \quad si \ \langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0.$$

R. 2

On note que par construction $\langle \boldsymbol{u}, \boldsymbol{x}_\perp \rangle = 0$. En effet, on a

$$\begin{aligned}
\langle \boldsymbol{u}, \boldsymbol{x}_\perp \rangle &= \left\langle \boldsymbol{u}, \boldsymbol{x} - \boldsymbol{x}_\parallel \right\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{x} \rangle - \left\langle \boldsymbol{u}, \boldsymbol{x}_\parallel \right\rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{x} \rangle - \langle \boldsymbol{u}, \langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u} \rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{x} \rangle - \langle \boldsymbol{u}, \boldsymbol{x} \rangle \langle \boldsymbol{u}, \boldsymbol{u} \rangle \\
&= 0 \ \text{ car } \ \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \|\boldsymbol{u}\|_2^2 = 1.
\end{aligned}$$

On a alors

$$\begin{aligned}
\mathbb{H}(\boldsymbol{u})(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel) = (\mathbb{I} - 2\boldsymbol{u}\boldsymbol{u}^*)\,(\boldsymbol{x}_\perp + \boldsymbol{x}_\parallel) &= \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{u} \underbrace{\boldsymbol{u}^*\boldsymbol{x}_\perp}_{=0} - 2\boldsymbol{u}\boldsymbol{u}^*\boldsymbol{x}_\parallel \\
&= \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{u}\boldsymbol{u}^*\big(\langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u}\big) = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\langle \boldsymbol{u}, \boldsymbol{x} \rangle \boldsymbol{u} \underbrace{\boldsymbol{u}^*\boldsymbol{u}}_{=1} \\
&= \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel - 2\boldsymbol{x}_\parallel \\
&= \boldsymbol{x}_\perp - \boldsymbol{x}_\parallel.
\end{aligned}$$

Si $\langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0$ alors $\boldsymbol{x}_\parallel = 0$ et $\boldsymbol{x} = \boldsymbol{x}_\perp$.

References

[1] F. Cuvelier. Analyse numérique I, résolution de systèmes linéaires, méthodes directes, résumé. fichier pdf, `https://www.math.univ-paris13.fr/~cuvelier/docs/Enseignements/MACS1/AnaNumI/25-26/resume_`

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[2] F. Cuvelier. *Analyse numérique élémentaire (version du 29 sep. 2025)*. Polycopié (téléchargement), 2025.