

# vecFEMP1 : eigens BVP

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## 1 Description of the generic problems

The notations of [7] are employed in this section and extended to the vector case.

### 1.1 Scalar boundary value problem

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^d$ ,  $d \geq 1$ . The boundary of  $\Omega$  is denoted by  $\Gamma$ .

We denote by  $\mathcal{L}_{\mathbb{A},\mathbf{b},\mathbf{c},a_0} = \mathcal{L} : H^2(\Omega) \rightarrow L^2(\Omega)$  the second order linear differential operator acting on *scalar fields* defined,  $\forall u \in H^2(\Omega)$ , by

$$\mathcal{L}_{\mathbb{A},\mathbf{b},\mathbf{c},a_0}(u) \stackrel{\text{def}}{=} -\operatorname{div}(\mathbb{A} \nabla u) + \operatorname{div}(\mathbf{b}u) + \langle \nabla u, \mathbf{c} \rangle + a_0 u \quad (1.1)$$

where  $\mathbb{A} \in (L^\infty(\Omega))^{d \times d}$ ,  $\mathbf{b} \in (L^\infty(\Omega))^d$ ,  $\mathbf{c} \in (L^\infty(\Omega))^d$  and  $a_0 \in L^\infty(\Omega)$  are given functions and  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $\mathbb{R}^d$ . We use the same notations as in the chapter 6 of [7] and we note that we can omit either  $\operatorname{div}(\mathbf{b}u)$  or  $\langle \nabla u, \mathbf{c} \rangle$  if  $\mathbf{b}$  and  $\mathbf{c}$  are sufficiently regular functions. We keep both terms with  $\mathbf{b}$  and  $\mathbf{c}$  to

deal with more boundary conditions. It should be also noted that it is important to preserve the two terms  $\mathbf{b}$  and  $\mathbf{c}$  in the generic formulation to enable a greater flexibility in the choice of the boundary conditions.

Let  $\Gamma^D, \Gamma^R$  be open subsets of  $\Gamma$ , possibly empty and  $f \in L^2(\Omega)$ ,  $g^D \in H^{1/2}(\Gamma^D)$ ,  $g^R \in L^2(\Gamma^R)$ ,  $a^R \in L^\infty(\Gamma^R)$  be given data.

A *scalar* boundary value problem is given by



### Scalar BVP 1 : generic problem

Find  $u \in H^2(\Omega)$  such that

$$\mathcal{L}(u) = f \quad \text{in } \Omega, \quad (1.2)$$

$$u = g^D \quad \text{on } \Gamma^D, \quad (1.3)$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = g^R \quad \text{on } \Gamma^R. \quad (1.4)$$

The **conormal derivative** of  $u$  is defined by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} \stackrel{\text{def}}{=} \langle \mathbb{A} \nabla u, \mathbf{n} \rangle - \langle \mathbf{b}u, \mathbf{n} \rangle \quad (1.5)$$

The boundary conditions (1.3) and (1.4) are respectively **Dirichlet** and **Robin** boundary conditions. **Neumann** boundary conditions are particular Robin boundary conditions with  $a^R \equiv 0$ .

## 1.2 Vector boundary value problem

Let  $m \geq 1$  and  $\mathcal{H}$  be the  $m$ -by- $m$  matrix of second order linear differential operators defined by

$$\begin{cases} \mathcal{H} : (H^2(\Omega))^m & \longrightarrow (L^2(\Omega))^m \\ \mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m) & \longmapsto \mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_m) \stackrel{\text{def}}{=} \mathcal{H}(\mathbf{u}) \end{cases} \quad (1.6)$$

where

$$\mathbf{f}_\alpha = \sum_{\beta=1}^m \mathcal{H}_{\alpha,\beta}(\mathbf{u}_\beta), \quad \forall \alpha \in \llbracket 1, m \rrbracket, \quad (1.7)$$

with, for all  $(\alpha, \beta) \in \llbracket 1, m \rrbracket^2$ ,

$$\mathcal{H}_{\alpha,\beta} \stackrel{\text{def}}{=} \mathcal{L}_{\mathbb{A}^{\alpha,\beta}, \mathbf{b}^{\alpha,\beta}, \mathbf{c}^{\alpha,\beta}, a_0^{\alpha,\beta}} \quad (1.8)$$

and  $\mathbb{A}^{\alpha,\beta} \in (L^\infty(\Omega))^{d \times d}$ ,  $\mathbf{b}^{\alpha,\beta} \in (L^\infty(\Omega))^d$ ,  $\mathbf{c}^{\alpha,\beta} \in (L^\infty(\Omega))^d$  and  $a_0^{\alpha,\beta} \in L^\infty(\Omega)$  are given functions. We can also write in matrix form

$$\mathcal{H}(\mathbf{u}) = \begin{pmatrix} \mathcal{L}_{\mathbb{A}^{1,1}, \mathbf{b}^{1,1}, \mathbf{c}^{1,1}, a_0^{1,1}} & \cdots & \mathcal{L}_{\mathbb{A}^{1,m}, \mathbf{b}^{1,m}, \mathbf{c}^{1,m}, a_0^{1,m}} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{\mathbb{A}^{m,1}, \mathbf{b}^{m,1}, \mathbf{c}^{m,1}, a_0^{m,1}} & \cdots & \mathcal{L}_{\mathbb{A}^{m,m}, \mathbf{b}^{m,m}, \mathbf{c}^{m,m}, a_0^{m,m}} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \end{pmatrix}. \quad (1.9)$$

We remark that the  $\mathcal{H}$  operator for  $m = 1$  is equivalent to the  $\mathcal{L}$  operator.

For  $\alpha \in \llbracket 1, m \rrbracket$ , we define  $\Gamma_\alpha^D$  and  $\Gamma_\alpha^R$  as open subsets of  $\Gamma$ , possibly empty, such that  $\Gamma_\alpha^D \cap \Gamma_\alpha^R = \emptyset$ . Let  $\mathbf{f} \in (L^2(\Omega))^m$ ,  $g_\alpha^D \in H^{1/2}(\Gamma_\alpha^D)$ ,  $g_\alpha^R \in L^2(\Gamma_\alpha^R)$ ,  $a_\alpha^R \in L^\infty(\Gamma_\alpha^R)$  be given data.

A *vector* boundary value problem is given by



**Vector BVP 1 : generic problem**

Find  $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m) \in (H^2(\Omega))^m$  such that

$$\mathcal{H}(\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega, \quad (1.10)$$

$$\mathbf{u}_\alpha = g_\alpha^D \quad \text{on } \Gamma_\alpha^D, \forall \alpha \in \llbracket 1, m \rrbracket, \quad (1.11)$$

$$\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \mathbf{u}_\alpha = g_\alpha^R \quad \text{on } \Gamma_\alpha^R, \forall \alpha \in \llbracket 1, m \rrbracket, \quad (1.12)$$

where the  $\alpha$ -th component of the **conormal derivative** of  $\mathbf{u}$  is defined by

$$\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_\alpha}} \stackrel{\text{def}}{=} \sum_{\beta=1}^m \frac{\partial \mathbf{u}_\beta}{\partial n_{\mathcal{H}_{\alpha,\beta}}} = \sum_{\beta=1}^m (\langle \mathbb{A}^{\alpha,\beta} \nabla \mathbf{u}_\beta, \mathbf{n} \rangle - \langle \mathbf{b}^{\alpha,\beta} \mathbf{u}_\beta, \mathbf{n} \rangle). \quad (1.13)$$

The boundary conditions (1.12) are the **Robin** boundary conditions and (1.11) is the **Dirichlet** boundary condition. The **Neumann** boundary conditions are particular Robin boundary conditions with  $a_\alpha^R \equiv 0$ .

In this problem, we may consider on a given boundary some conditions which can vary depending on the component. For example we may have a Robin boundary condition satisfying  $\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_1}} + a_1^R \mathbf{u}_1 = g_1^R$  and a Dirichlet one with  $\mathbf{u}_2 = g_2^D$ . In the following of the report we will solve by a  $\mathbb{P}_1$ -Lagrange finite element method *scalar* BVP (1.2) to (1.4) and *vector* BVP (1.10) to (1.12) without additional restrictive assumption.

## 2 Boundary Value Problems

### 2.1 Main data structures

We suppose that  $\Omega$  is equipped with a mesh  $\mathcal{T}_h$  (locally conforming) where its elements are d-simplices . We denote by  $\Omega_h$  the union of the elements belonging to the mesh,  $\Omega_h = \bigcup_{K \in \mathcal{T}_h} K$ , and by  $\Gamma_h$  its boundary,  $\Gamma_h = \partial \Omega_h$ .


The following data structure is associated to the mesh  $\mathcal{T}_h$  and employs many notations already used in FreeFEM++ (see [5, 6]).

Mesh structure associated to $\mathcal{T}_h$	
d	: integer space dimension
$n_q$	: integer number of vertices
$n_{me}$	: integer number of elements (d-simplices )
$n_{be}$	: integer number of boundary elements ((d-1)-simplices )
q	: d-by- $n_q$ array of reals array of vertices coordinates
me	: (d+1)-by- $n_{me}$ array of integers connectivity array for <b>mesh elements</b>
be	: d-by- $n_{be}$ array of integers connectivity array for <b>boundary elements</b>
bel	: 1-by- $n_{be}$ array of integers array of boundary elements labels
vols	: 1-by- $n_{me}$ array of reals array of mesh elements volumes

More precisely

- $q(\nu, j)$  is the  $\nu$ -th coordinate of the  $j$ -th vertex,  $\nu \in \{1, \dots, d\}$ ,  $j \in \{1, \dots, n_q\}$ . The  $j$ -th vertex will be also denoted by  $q^j = q(:, j)$ .
- $me(\beta, k)$  is the storage index of the  $\beta$ -th vertex of the  $k$ -th element (d-simplex ), in the array  $q$ , for  $\beta \in \{1, \dots, d + 1\}$  and  $k \in \{1, \dots, n_{me}\}$ . So  $q(:, me(\beta, k))$  represents the coordinates of the  $\beta$ -th vertex of the  $k$ -th mesh element.
- $be(\beta, l)$  is the storage index of the  $\beta$ -th vertex of the  $l$ -th boundary element ((d-1)-simplex ), in the array  $q$ , for  $\beta \in \{1, \dots, d\}$  and  $l \in \{1, \dots, n_{be}\}$ . So  $q(:, be(\beta, l))$  represents the coordinates of the  $\beta$ -th vertex of the  $l$ -th boundary element.
- $vols(k)$  is the volume of the  $k$ -th d-simplex .

A PDE structure contains all the data necessary to solve a scalar or vector BVP is described by

 <b>PDE structure</b>	
d	: integer space dimension
m	: integer operator dimension (1 for $\mathcal{L}$ operator)
op	: operator $\mathcal{L}$ operator or $\mathcal{H}$ operator
f	: 1-by-m array of functions
$\mathcal{T}_h$	: mesh structure
$\mathcal{B}_h$	: 1-by-nlab array of boundary mesh structures
labels	: 1-by-nlab array of integers Boundary mesh labels
nlab	: integer number of boundary labels
bclR	: m-by-nlab array of <code>BCrobin</code> structures
bclD	: m-by-nlab array of <code>BCdirichlet</code> structures

## 2.2 First level functions or commonly used functions

We briefly describe the main functions that will be used in the sequel.

$\mathcal{T}_h \leftarrow \text{GETMESH}(\text{FileName})$  : to define the mesh  $\mathcal{T}_h$  by reading a 2d or 3d mesh from the file `FileName`.

$\mathcal{T}_h \leftarrow \text{HYPERCUBE}(d, N, < \text{trans} = \Phi >)$  : to define the mesh  $\mathcal{T}_h$  as the unit hypercube  $[0, 1]^d$ . There are  $N(i)$  (or  $N$  if  $N$  is a scalar) points in each direction and the mesh of the hypercube contains  $\prod_{i=1}^d N(i)$  points. Optional parameter `trans` set the displacement vector of mesh transformation  $\Phi(\mathbf{q}) = [\Phi_1(\mathbf{q}), \dots, \Phi_d(\mathbf{q})]$ . The  $2^d$  faces of the hypercube (before transformation) have an unique label :  $\forall i \in \llbracket 1, d \rrbracket$ , the faces  $x_i \equiv 0$  and  $x_i \equiv 1$  are respectively of label  $2i - 1$  and  $2i$ .

$\text{Lop} \leftarrow \text{LOPERATOR}(d, \mathbb{A}, \mathbf{b}, \mathbf{c}, a_0)$  : to initialize the operator  $\mathcal{L}$  in dimension  $d$  given by (1.1) :  $\text{Lop} \leftarrow \mathcal{L}_{\mathbb{A}, \mathbf{b}, \mathbf{c}, a_0}$ .

$\text{Hop} \leftarrow \text{HOPERATOR}(d, m)$  : to initialize the operator  $\mathcal{H}$  given by (1.6) verifying  $\mathcal{H}_{\alpha, \beta} = 0, \forall (\alpha, \beta) \in \llbracket 1, m \rrbracket^2$ . Each operator  $\mathcal{H}_{\alpha, \beta}$  corresponds to  $\text{Hop.H}(\alpha, \beta)$  and can be initialized by the function `LOPERATOR`

$\text{pde} \leftarrow \text{INITPDE}(\text{Op}, \mathcal{T}_h)$  : to initialize a PDE structure from an operator (either  $\mathcal{L}$ -operator or  $\mathcal{H}$ -operator) and a mesh. Default boundary conditions are homogeneous generalized Neumann.

$\text{pde} \leftarrow \text{SETBC\_PDE}(\text{pde}, \text{label}, \text{comps}, \text{type}, \mathbf{g}, \mathbf{ar})$  : to define or modify the boundary conditions on the boundary  $\Gamma_{\text{label}}$  on the mesh  $\text{pde}.\mathcal{T}_h$  for components of index `comps` (in the scalar case `comps`  $\equiv 1$ ). For a scalar PDE, we have for example

- Dirichlet condition :  $\mathbf{u}_2 = g$  on  $\Gamma_{11}$ , then  
`pde ← SETBC_PDE(pde, 11, 2, 'Dirichlet', g, ∅)`
- generalized Neumann condition :  $\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_3}} = g$  on  $\Gamma_{12}$ , then  
`pde ← SETBC_PDE(pde, 12, 3, 'Neumann', g, ∅)`
- generalized Robin condition :  $\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_2}} + a_2^R \mathbf{u}_2 = g$  on  $\Gamma_{13}$ , then  
`pde ← SETBC_PDE(pde, 13, 2, 'Robin', g, a_2^R)`


$\mathbf{x} \leftarrow \text{SOLVEPDE}(\text{pde})$  : to solve by  $\mathbb{P}_1$ -Lagrange finite elements the partial differential equation defined by the structure PDE. This function returns the solution  $\mathbf{x}()$

### 3 Eigenvalue problems

We want to solve eigenvalue problems coming from scalar or vector BVP's.

#### 3.1 Scalar case

The eigenvalue problems associated with *scalar* BVP (1.2)-(1.4) can be written as

 **Scalar EBVP 1 : generic problem**

Find  $(\lambda, u) \in \mathbb{K} \times H^2(\Omega)$  such that

$$\mathcal{L}(u) = \lambda \mathcal{B}(u) \quad \text{in } \Omega, \quad (3.1)$$

$$u = 0 \quad \text{on } \Gamma^D, \quad (3.2)$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = 0 \quad \text{on } \Gamma^R. \quad (3.3)$$

where  $\mathcal{B} = \mathcal{L}_{0_{d \times d}, \mathbf{0}_d, \tilde{\mathbf{c}}, \tilde{a}_0}$ .


We briefly describe the main function that will be used to solve eigenvalues problems.

$(\mathbf{U}, \boldsymbol{\lambda}) \leftarrow \text{EIGSPDE}(\text{pde}, \mathcal{B}, N_e)$  : compute the  $N_e$  smallest eigenvalues in magnitude and the corresponding eigenvectors.  $\boldsymbol{\lambda}(i)$  is the  $i$ -th eigenvalue and the  $i$ -th eigenvector is given by  $\mathbf{U}(:, i)$ . The pde variable is defined using functions `INITPDE` and `SETBC_PDE`.

$(\mathbf{U}, \boldsymbol{\lambda}) \leftarrow \text{EIGSPDE}(\text{pde}, \mathcal{B}, N_e, \sigma)$  : compute the  $N_e$  eigenvalues closest to sigma.

### 3.1.1 2D Laplace eigenvalues problem with Dirichlet boundary condition

We want to solve the eigenvalue problem given by (3.4)-(3.5).


 **Usual EBVP 1 : 2D Laplace with Dirichlet boundary condition**

Find  $(\lambda, u) \in \mathbb{K} \times H^2(\Omega)$  such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad (3.4)$$

$$u = 0 \quad \text{on } \Gamma, \quad (3.5)$$

The problem (3.4)-(3.5) can be equivalently write as the *Scalar* EBVP 1:

 **Scalar EBVP 2 : 2D Laplace with Dirichlet boundary condition**

Find  $(\lambda, u) \in \mathbb{K} \times H^2(\Omega)$  such that

$$\mathcal{L}(u) = \lambda \mathcal{B}(u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \Gamma^D = \Gamma,$$

where  $\mathcal{L} = \mathcal{L}_{\mathbb{I},0,0,0}$ ,  $\mathcal{B} = \mathcal{L}_{\mathbb{O},0,0,1}$ .

The algorithm using the toolbox to find eigenvalues and eigenvectors of this problem is the following :

---

**Algorithm 3.1** 2D Laplace eigenvalues problem with Dirichlet boundary condition

---

```

1:  $\mathcal{T}_h \leftarrow \text{GETMESH}(\dots)$  ▷ Load mesh
2:  $\mathcal{L} \leftarrow \text{L OPERATOR}(\mathbb{I}, \mathbf{0}, \mathbf{0}, \mathbf{0})$  ▷ Laplacian operator
3: pde  $\leftarrow \text{INITPDE}(\mathcal{L}, \mathcal{T}_h)$ 
4: for  $i \leftarrow 1$  to pde.nlab do ▷ Set Dirichlet boundaries conditions  $u = 0$  on  $\Gamma$ 
5:   pde  $\leftarrow \text{SETBC\_PDE}(\text{pde}, \text{pde.labels}(i), 1, \text{'Dirichlet'}, 0.)$ 
6: end for
7:  $\mathcal{B} \leftarrow \text{L OPERATOR}(\mathbb{O}, \mathbf{0}, \mathbf{0}, 1)$  ▷ set  $\mathcal{B}$  operator
8:  $(\mathbf{U}, \boldsymbol{\lambda}) \leftarrow \text{EIGSPDE}(\text{pde}, \mathcal{B}, 24)$ 

```

---

Listing 1: 2D Laplace eigenvalues problem with Dirichlet boundary condition : Matlab/Octave code

```

1 fprintf('1. Read mesh\n');
2 Th=GetMesh2DOpt(MeshFile,'format','gmsh');
3 fprintf('2. Set eigenvalues problem\n');
4 Lop=Operator(Th,d,{1,0;0,1},[],[],[]);
5 pde=initPDE(Lop,Th);
6 Bop=Operator(Th,d,[],[],1);
7 for i=1:pde.nlab
8   pde=setBC_PDE(pde,pde.labels(i),1,'Dirichlet',0);
9 end
10 fprintf('3. Solve eigenvalues problem\n');
11 [U,lambda]=eigsPDE(pde,Bop,NumEigs);

```

Listing 2: 2D Laplace eigenvalues problem with Dirichlet boundary condition : Python code

```

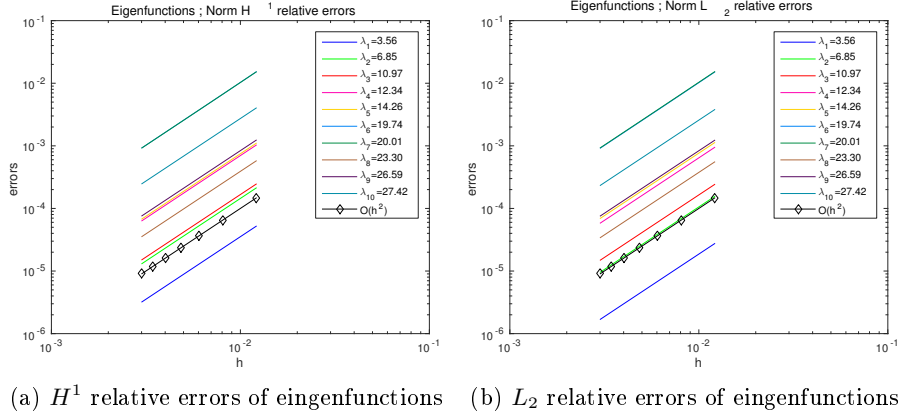
1 # Set functions
2 from numpy import sin, cos
3 u=lambda x,y: sin(x)*sin(y);
4 # exact solution
5 f=lambda x,y: 4*sin(x)*sin(y);
6 # RHS
7 dudx=lambda x,y: cos(x)*sin(y); # dx(u)
8 dudy=lambda x,y: sin(x)*cos(y); # dy(u)
9 p=lambda x,y: -2*sin(x)*sin(y); # p=Laplacien(u)
10 dpdx=lambda x,y: -2*cos(x)*sin(y); # dx(p)
11 dpdy=lambda x,y: -2*cos(y)*sin(x); # dy(p)
12 print('1. Reading of the mesh')

```

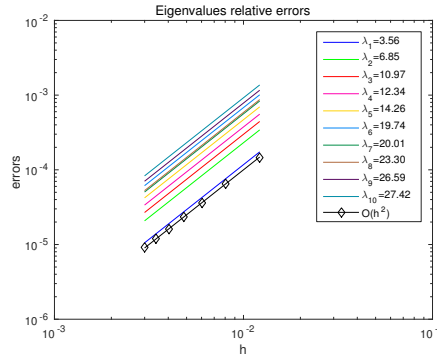
**Application on the rectangle**  $\Omega = [0, L] \times [0, H]$ . The eigenvalues and the associated eigenfunctions are given by

$$\lambda_{k,l} = \left(\frac{k\pi}{L}\right)^2 + \left(\frac{l\pi}{H}\right)^2, \quad u_{k,l}(x, y) = \sin\left(\frac{k\pi}{L}x\right) \sin\left(\frac{l\pi}{H}y\right), \quad \forall (k, l) \in \mathbb{N}^* \times \mathbb{N}^*.$$

We represent in Figure 1 ...



(a)  $H^1$  relative errors of eigenfunctions (b)  $L_2$  relative errors of eigenfunctions



(c) relative errors of eigenvalues

Figure 1: eigenvalues and eigenfunctions : order computation for 2D Laplace with Dirichlet boundary condition on rectangle

We represent in Figure 2 eigenvectors associated to the first sixteen smallest magnitude eigenvalues.

**Application on the unit disk.** Let  $\Omega \subset \mathbb{R}^2$  be the unit disk. We represent in Figure 3 eigenvectors associated to the first twenty-four smallest magnitude eigenvalues.

**Application on the L-shape domain.** Results can be found in [8], Figure 1 page 4 and [3]. From [4] section 6.52 page 122 or [9] Table 1 page 1088, we have

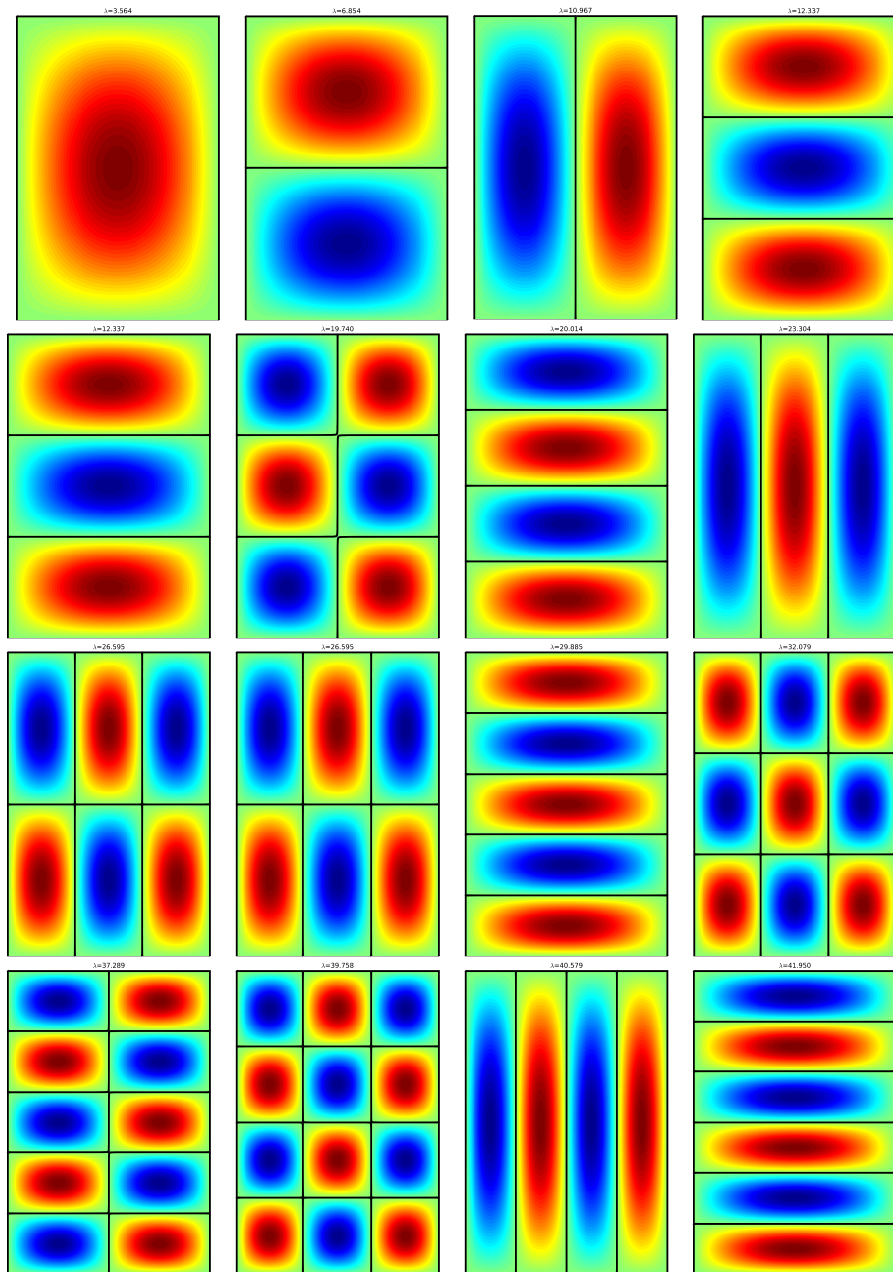


Figure 2: 2D Laplace in rectangle  $[0, 2] \times [0, 3]$  with Dirichlet boundary conditions : eigenvectors of the smallest magnitude eigenvalues

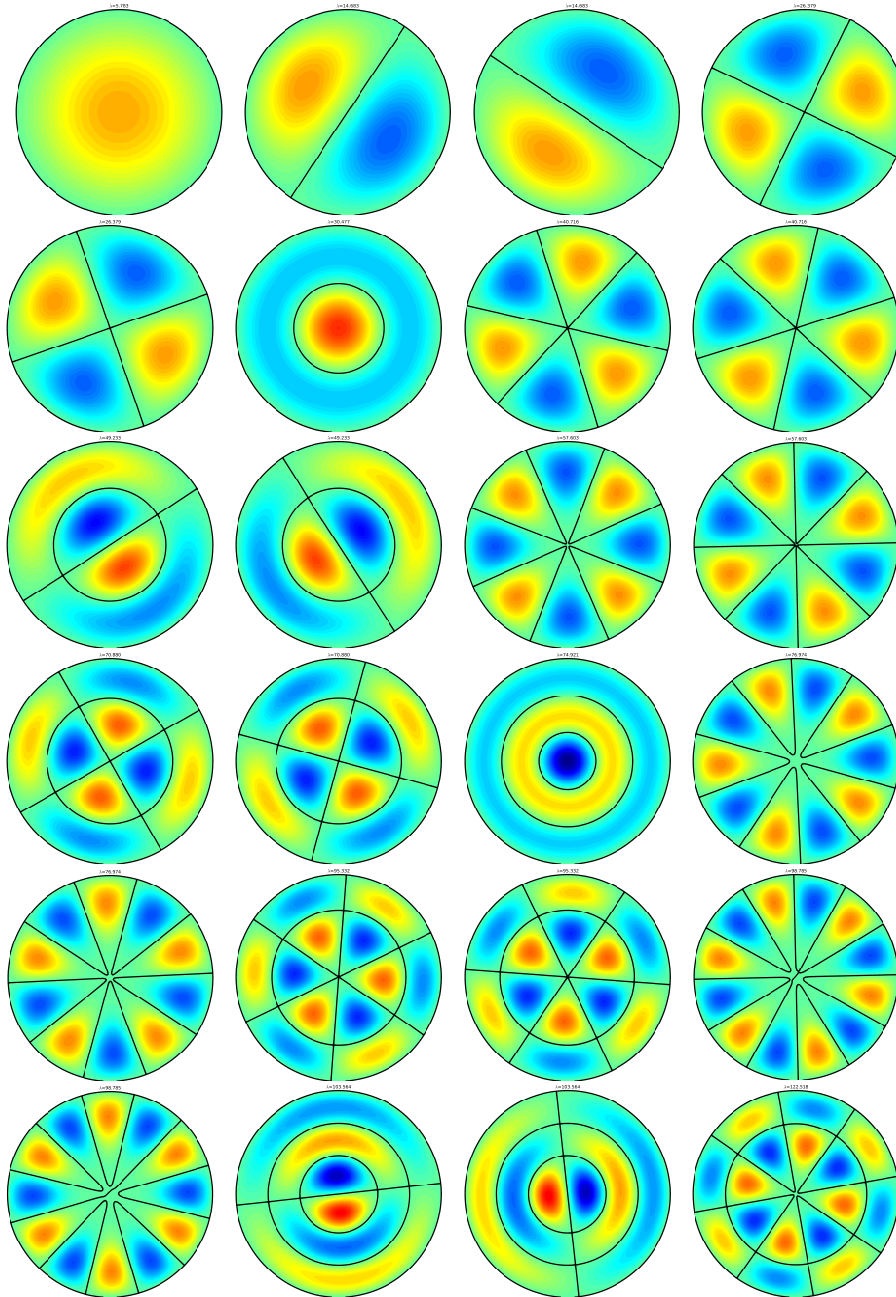


Figure 3: 2D Laplace in unit disk with Dirichlet boundary conditions : eigenvectors of the smallest magnitude eigenvalues

the bounds to the first ten eigenvalues of the L-shaped Laplacian problem is given Table 1. We also give the computed values from a L-shaped mesh with  $n_q = 357991$ ,  $n_{me} = 713580$  and  $h \approx 0.0052$ .

We represent in Figure 4 eigenvectors associated to the first twenty-four smallest

$n$	bounds of $\lambda_n$ from [9]	computed here
1	$9.63972384_{04}^{44}$	9.640649039039735
2	$15.19725192_{59}^{66}$	15.197407727841403
3	$2\pi^2 = 19.739208802178$	19.739472143639855
4	$29.52148111_{38}^{42}$	29.522069934441525
5	$31.9126359_{37}^{59}$	31.915429982722600
6	$41.4745098_{66}^{92}$	41.477254867865106
7	$44.9484877_{77}^{82}$	44.949850369893589
8	$5\pi^2 = 49.34802200544$	49.349664612562883
9	$5\pi^2 = 49.34802200544$	49.349665685274879
10	$56.7096098_{18}^{90}$	56.713551009263014

Table 1: Bounds to the first ten eigenvalues of the L-shaped Laplacian problem

magnitude eigenvalues and in Figure 5 the eigenvectors associated with the four eigenvalues nearest 250 and 493.

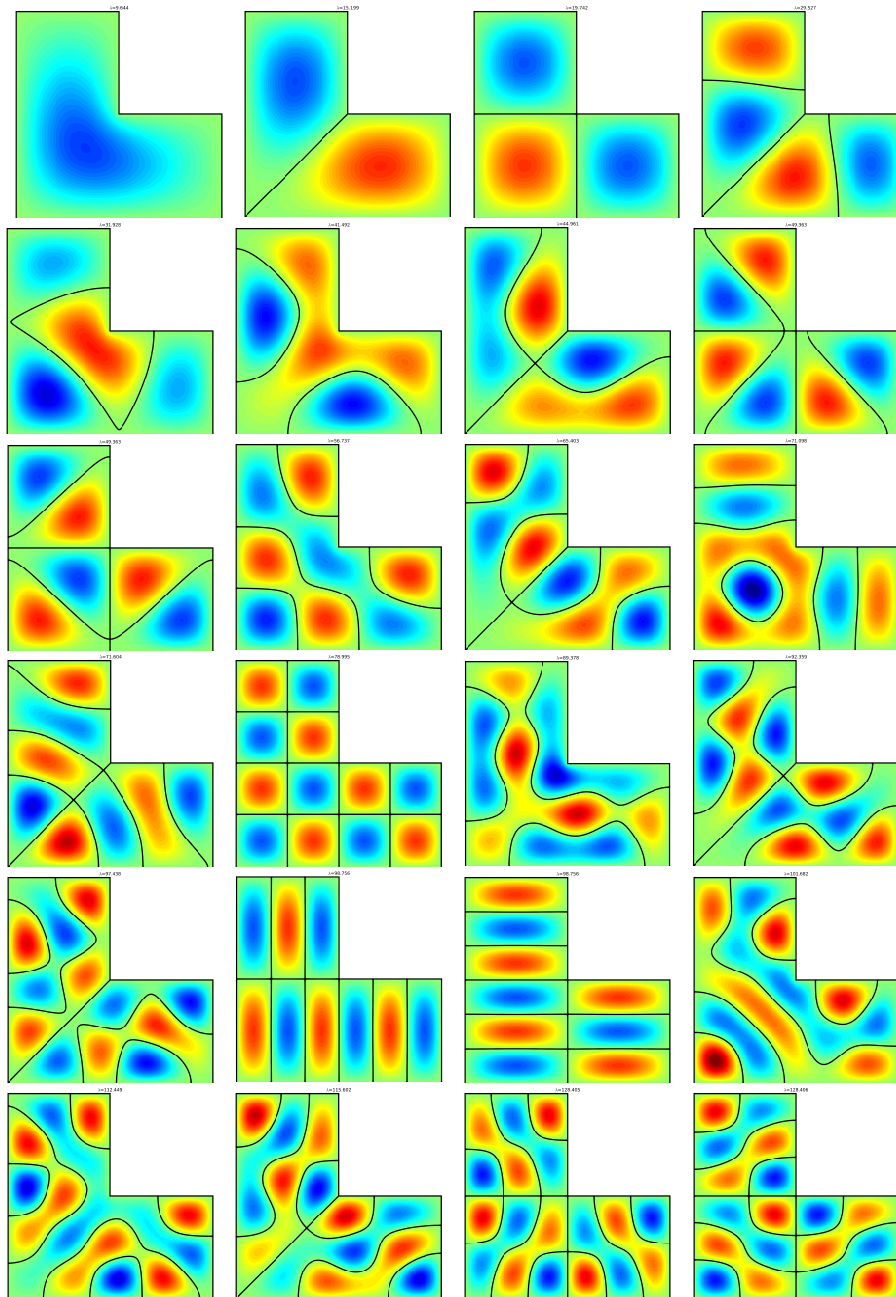


Figure 4: 2D Laplace with Dirichlet boundary conditions : eigenvectors of the smallest magnitude eigenvalues

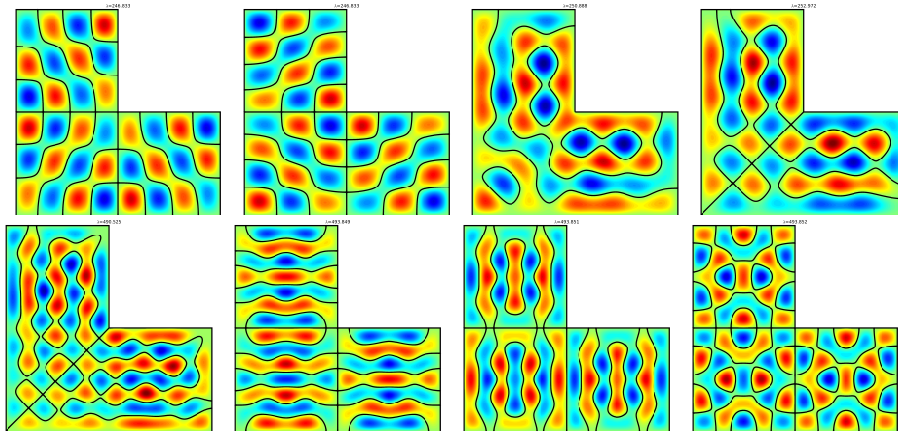


Figure 5: 2D Laplace with Dirichlet boundary conditions : eigenvectors of the eigenvalues near  $\lambda_{50} = 250.78548$  (multiplicity 1) and  $\lambda_{104} = 493.48022$  (multiplicity 3)

### 3.1.2 2D Laplace eigenvalues problem with mixed boundary conditions

We want to solve the eigenvalue problem given by (3.6)-(3.9).

 **Usual EBVP 2 : 2D Laplace with mixed boundary condition**

Find  $(\lambda, u) \in \mathbb{K} \times H^2(\Omega)$  such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad (3.6)$$

$$\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \Gamma^a, \quad (3.7)$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma^b, \quad (3.8)$$

$$u = 0 \quad \text{on } \Gamma^c, \quad (3.9)$$

The problem (3.6)-(3.9) can be equivalently write as the *Scalar* EBVP 1:

### Scalar EBVP 3 : 2D Laplace with mixed boundary condition

Find  $(\lambda, u) \in \mathbb{K} \times H^2(\Omega)$  such that

$$\begin{aligned} \mathcal{L}(u) &= \lambda \mathcal{B}(u) && \text{in } \Omega, \\ \frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u &= 0 && \text{on } \Gamma^R = \Gamma^a \cup \Gamma^b, \\ u &= 0 && \text{on } \Gamma^D = \Gamma^c, \end{aligned}$$

where  $\mathcal{L} = \mathcal{L}_{\mathbb{I}, \mathbf{0}, \mathbf{0}, \mathbf{0}}$  (and then  $\frac{\partial u}{\partial n_{\mathcal{L}}} = \frac{\partial u}{\partial n}$ ),  $\mathcal{B} = \mathcal{L}_{\mathbb{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}}$ ,  $a^R = \alpha \delta_{\Gamma^b}$

The algorithm using the toolbox to find eigenvalues and eigenvectors of this problem is the following :

---

**Algorithm 3.2** 2D Laplace eigenvalues problem with mixed boundary condition

---

```

1:  $\mathcal{T}_h \leftarrow \text{GETMESH}(\dots)$  ▷ Load mesh
2:  $\mathcal{L} \leftarrow \text{LOPERATOR}(\mathbb{I}, \mathbf{0}, \mathbf{0}, \mathbf{0})$  ▷ Laplacian operator
3:  $\text{pde} \leftarrow \text{INITPDE}(\mathcal{L}, \mathcal{T}_h)$ 
4:  $\text{pde} \leftarrow \text{SETBC\_PDE}(\text{pde}, c, 1, \text{'Dirichlet'}, 0.)$ 
5:  $\text{pde} \leftarrow \text{SETBC\_PDE}(\text{pde}, a, 1, \text{'Robin'}, 0., \alpha)$ 
6:  $\mathcal{B} \leftarrow \text{LOPERATOR}(\mathbb{0}, \mathbf{0}, \mathbf{0}, \mathbf{1})$  ▷ set  $\mathcal{B}$  operator
7:  $(\mathbf{U}, \boldsymbol{\lambda}) \leftarrow \text{EIGSPDE}(\text{pde}, \mathcal{B}, 24)$ 

```

---

**Application on the disk with 5 holes domain.** Let  $\Gamma_1$  be the unit disk,  $\Gamma_{10}$  be the disk with center point  $(0, 0)$  and radius 0.3. Let  $\Gamma_{20}, \Gamma_{21}, \Gamma_{22}$  and  $\Gamma_{23}$  be the disks with radius 0.1 and respectively with center point  $(0, -0.7)$ ,  $(0, 0.7)$ ,  $(-0.7, 0)$  and  $(0.7, 0)$ . The domain  $\Omega \subset \mathbb{R}^2$  is defined as the inner of  $\Gamma_1$  and the outer of all other disks (see Figure 6).

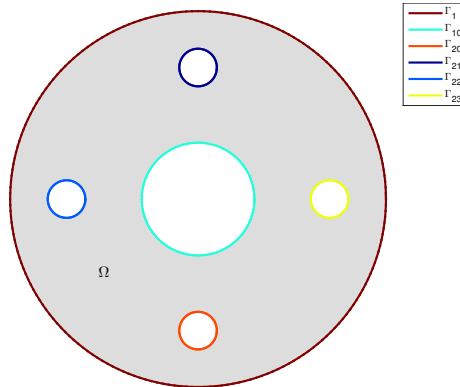


Figure 6: Domain and boundaries

We want to solve the eigenvalue problem given by (3.10)-(3.13)).

### 2D Laplace eigenvalues problem with mixed boundary conditions

Find  $(\lambda, u) \in \mathbb{K} \times H^2(\Omega)$  such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad (3.10)$$

$$\frac{\partial u}{\partial n} + 10u = 0 \quad \text{on } \Gamma_{22} \cup \Gamma_{23}. \quad (3.11)$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_{20} \cup \Gamma_{21}, \quad (3.12)$$

$$u = 0 \quad \text{on } \Gamma_1 \cup \Gamma_{10}. \quad (3.13)$$

So we have,  $\Gamma^D = \Gamma_1 \cup \Gamma_{10}$ ,  $\Gamma^R = \bigcup_{i=20}^{23} \Gamma_i$ , and  $a^R = 10\delta_{\Gamma_{22} \cup \Gamma_{23}}$ . We give in Listing 5 and 6 the corresponding Matlab/Octave and Python codes.

Listing 3: 2D Laplace eigenvalues problem with mixed boundary conditions : Matlab/Octave code

```

1 fprintf('1. Read mesh\n');
2 Th=GetMesh2DOpt(MeshFile,'format','gmsh');
3 fprintf('2. Set eigenvalues problem\n');
4 Lop=Loperator(Th,d,{1,0;0,1},[],[],[]);
5 pde=initPDE(Lop,Th);
6 Bop=Loperator(Th,d,[],[],[]);
7 pde=setBC_PDE(pde,1,1,'Dirichlet',0);
8 pde=setBC_PDE(pde,10,1,'Dirichlet',0);
9 pde=setBC_PDE(pde,22,1,'Robin',0,10);
10 pde=setBC_PDE(pde,23,1,'Robin',0,10);
11 fprintf('3. Solve eigenvalues problem\n');
12 [U,lambda]=eigsPDE(pde,Bop,NumEigs);

```

Listing 4: 2D Laplace eigenvalues problem with Dirichlet boundary condition : Python code

```

1 # Set functions
2 from numpy import sin,cos
3 u=lambda x,y: sin(x)*sin(y);
4 # exact solution
5 f=lambda x,y: 4*sin(x)*sin(y);
6 # RHS
7 dudx=lambda x,y: cos(x)*sin(y); # dx(u)
8 dudy=lambda x,y: sin(x)*cos(y); # dy(u)
9 p=lambda x,y: -2*sin(x)*sin(y); # p=Laplacien(u)
10 dpdx=lambda x,y: -2*cos(x)*sin(y); # dx(p)
11 dpdy=lambda x,y: -2*cos(y)*sin(x); # dy(p)
12
13 print('1. Reading of the mesh')

```

We represent in Figure 8 the twenty four first eigenvectors.

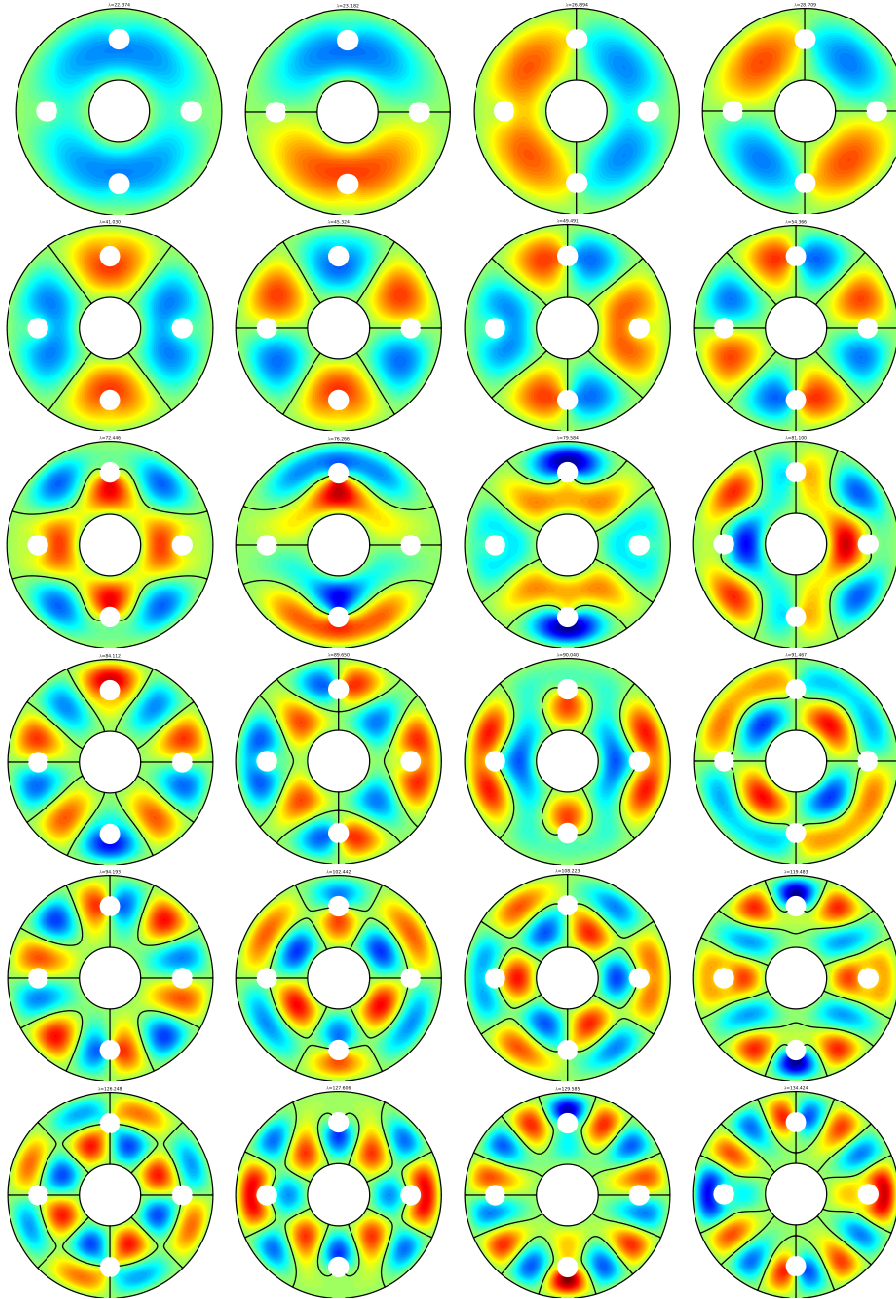



Figure 7: 2D Laplace with mixed boundary conditions : eigenvectors of the smallest magnitude eigenvalues

### 3.1.3 Other 2D eigenvalues problems with Dirichlet boundary condition

**Convection-Diffusion on the L-shaped domain.** We want to solve the eigenvalue problem given by

 **Usual EBVP 3 : 2D Convection-Diffusion eigenvalues problem with Dirichlet boundary condition**

Find  $(\lambda, u) \in \mathbb{K} \times H^2(\Omega)$  such that

$$\begin{aligned} -\Delta u + \boldsymbol{\beta} \cdot \nabla u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

with constant convection parameter  $\boldsymbol{\beta} \in \mathbb{R}^2$ .

From [4] section 6.52 page 122 the eigenvalues of *Usual EBVP 3* are  $\lambda_i^\beta = |\boldsymbol{\beta}|/4 + \lambda_i$  where  $\lambda_i$  are the eigenvalues of the L-shaped Laplacian problem with Dirichlet boundary condition (the ten first are given in Table 1). We have for example

$$\begin{aligned} \lambda_i^\beta &\approx |\boldsymbol{\beta}|/4 + 9.63972, & \lambda_3^\beta &= |\boldsymbol{\beta}|/4 + 2\pi^2 \\ \lambda_5^\beta &\approx |\boldsymbol{\beta}|/4 + 31.912636, & \lambda_8^\beta &= \lambda_9^\beta = |\boldsymbol{\beta}|/4 + 5\pi^2 \\ \lambda_{20}^\beta &\approx |\boldsymbol{\beta}|/4 + 101.60529, & \lambda_{50}^\beta &\approx |\boldsymbol{\beta}|/4 + 250.78548. \end{aligned}$$

We give in Listing 5 and 6 the corresponding Matlab/Octave and Python codes.

Listing 5: 2D Laplace eigenvalues problem with mixed boundary conditions : **Matlab/Octave code**

```

1 fprintf('1. Read mesh\n');
2 Th=GetMesh2DOpt(MeshFile,'format','gmsh');
3 fprintf('2. Set eigenvalues problem\n');
4 beta={3,0};
5 Lop=Loperator(Th.d,{1,0;0,1},[],beta,[]);
6 pde=initPDE(Lop,Th);
7 Bop=Loperator(Th.d,[],[],[],1);
8 for i=1:pde.nlab
9     pde=setBC_PDE(pde,pde.labels(i),1, ...
10                 'Dirichlet', 0);
11 end
12 fprintf('3. Solve eigenvalues problem\n');
13 [U,lambd]=eigsPDE(pde,Bop,NumEigs);

```

Listing 6: 2D Laplace eigenvalues problem with Dirichlet boundary condition : **Python code**

```

1 # Set functions
2 from numpy import sin,cos
3 u=lambda x,y: sin(x)*sin(y);
4 # exact solution
5 f=lambda x,y: 4*sin(x)*sin(y);
6 # RHS
7 dudx=lambda x,y: cos(x)*sin(y); # dx(u)
8 dudy=lambda x,y: sin(x)*cos(y); # dy(u)
9 p=lambda x,y: -2*sin(x)*sin(y); # p=Laplacien(u)
10 dpdx=lambda x,y: -2*cos(x)*sin(y); # dx(p)
11 dpdy=lambda x,y: -2*cos(y)*sin(x); # dy(p)
12 print('1. Reading of the mesh')

```

We give the computed values from a L-shaped mesh with  $n_q = 357991$ ,  $n_{me} = 713580$  and  $h \approx 0.0052$ .

We represent in Figure 8 the twenty four first eigenvectors with  $\boldsymbol{\beta} = (3, 0)$ .

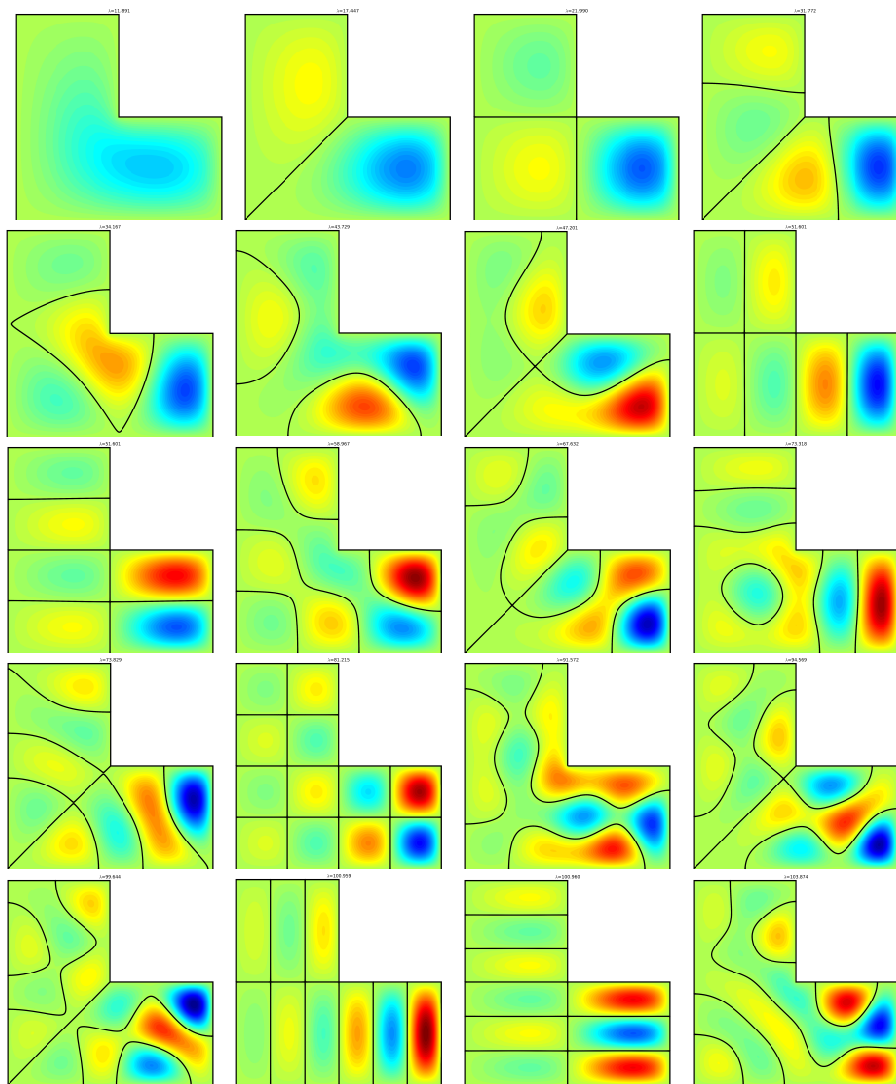



Figure 8: 2D Laplace with mixed boundary conditions : eigenvectors of the smallest magnitude eigenvalues

$n$	bounds of $\lambda_n^\beta$ from [9]	computed here
1	$ \beta /4 + 9.63972384_{04}^{44} \approx 11.88972384$	11.890593787727795
2	$ \beta /4 + 15.19725192_{59}^{66} \approx 17.44725192$	17.447318816724781
3	$ \beta /4 + 2\pi^2 = 21.989208802178716$	21.989355694703399
4	$ \beta /4 + 29.52148111_{38}^{42} \approx 31.77148111$	31.771894040076543
5	$ \beta /4 + 31.9126359_{37}^{59} \approx 34.1626359$	34.165239540278506
6	$ \beta /4 + 41.4745098_{66}^{92} \approx 43.7245099$	43.727006318946891
7	$ \beta /4 + 44.9484877_{77}^{82} \approx 47.19848777$	47.199580624022417
8	$ \beta /4 + 5\pi^2 = 51.598022005446794$	51.599278693930657
9	$ \beta /4 + 5\pi^2 = 51.598022005446794$	51.599458742449400
10	$ \beta /4 + 56.7096098_{18}^{90} \approx 58.9596098$	58.963209967358949
20	$ \beta /4 + 101.60529 \approx 103.85529$	103.8646014207418
50	$ \beta /4 + 250.78548 \approx 253.03548$	253.0805934769167

Table 2: Eigenvalues of the L-shaped Convection-Diffusion problem with  $\beta = (3, 0)$ .

### 3.2 Vector case

The eigenvalue problems associated with *vector* BVP (1.10)-(1.12) can be written as

 **Vector eigenvalue problems**

Find  $\lambda \in \mathbb{K}$  and  $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m) \in (H^2(\Omega))^m$  such that

$$\mathcal{H}(\mathbf{u}) = \lambda \mathcal{B}(\mathbf{u}) \quad \text{in } \Omega, \quad (3.14)$$

$$\mathbf{u}_\alpha = 0 \quad \text{on } \Gamma_\alpha^D, \quad \forall \alpha \in \llbracket 1, m \rrbracket, \quad (3.15)$$

$$\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \mathbf{u}_\alpha = 0 \quad \text{on } \Gamma_\alpha^R, \quad \forall \alpha \in \llbracket 1, m \rrbracket, \quad (3.16)$$

where  $\mathcal{B}$  is a given  $\mathcal{H}$ -operator.

In most cases  $\mathcal{B}$  is the identity operator ( $\mathcal{B}$  is a diagonal  $\mathcal{H}$ -operator with  $\mathcal{B}_{\alpha,\alpha} = \mathcal{L}_{\mathbb{O}_{d \times d}, \mathbf{0}_d, \mathbf{0}_d, 1}, \forall \alpha \in \llbracket 1, m \rrbracket$ ).

#### 3.2.1 Elasticity problem

Let  $d = 2$  or  $d = 3$ . We consider here Hooke's law in linear elasticity, under small strain hypothesis (see for example [2]).

For a sufficiently regular vector field  $\mathbf{u} = (u_1, \dots, u_d) : \Omega \rightarrow \mathbb{R}^d$ , we define the linearized strain tensor  $\underline{\epsilon}$  by

$$\underline{\epsilon}(\mathbf{u}) = \frac{1}{2} (\nabla(\mathbf{u}) + \nabla^t(\mathbf{u})).$$

We set  $\underline{\epsilon} = (\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})^t$  in 2d and  $\underline{\epsilon} = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{13})^t$  in 3d, with  $\epsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ . Then the Hooke's law writes

$$\underline{\sigma} = \mathbb{C} \underline{\epsilon},$$

where  $\underline{\sigma}$  is the elastic stress tensor and  $\mathbb{C}$  the elasticity tensor.

The material is supposed to be isotropic. Thus the elasticity tensor  $\mathbb{C}$  is only defined by the Lamé parameters  $\lambda$  and  $\mu$ , which satisfy  $\lambda + \mu > 0$ . We also set  $\gamma = 2\mu + \lambda$ . For  $d = 2$  or  $d = 3$ ,  $\mathbb{C}$  is given by

$$\mathbb{C} = \begin{pmatrix} \lambda \mathbb{1}_2 + 2\mu \mathbb{I}_2 & 0 \\ 0 & \mu \end{pmatrix}_{3 \times 3} \quad \text{or} \quad \mathbb{C} = \begin{pmatrix} \lambda \mathbb{1}_3 + 2\mu \mathbb{I}_3 & 0 \\ 0 & \mu \mathbb{I}_3 \end{pmatrix}_{6 \times 6},$$

respectively, where  $\mathbb{1}_d$  is a  $d$ -by- $d$  matrix of ones, and  $\mathbb{I}_d$  the  $d$ -by- $d$  identity matrix.

For dimension  $d = 2$  or  $d = 3$ , we have:

$$\sigma_{\alpha\beta}(\mathbf{u}) = 2\mu \epsilon_{\alpha\beta}(\mathbf{u}) + \lambda \operatorname{tr}(\boldsymbol{\epsilon}(\mathbf{u})) \delta_{\alpha\beta} \quad \forall \alpha, \beta \in \llbracket 1, d \rrbracket$$

The problem to solve is the following



#### Usual EBVP 4 : Elasticity problem



Find  $(k, \mathbf{u}) = \mathbb{K} \times \mathbf{H}^2(\Omega)^d$  such that

$$-\operatorname{div}(\boldsymbol{\sigma}(\mathbf{u})) = k\mathbf{u}, \quad \text{in } \Omega \subset \mathbb{R}^d, \quad (3.17)$$

$$\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma^R, \quad (3.18)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma^D. \quad (3.19)$$

We recall the following lemma (see [1])

**Lemma 1.** Let  $\mathcal{H}^\sigma$  be the  $\mathcal{H}$ -operator defined in (1.6) by

$$\mathcal{H}_{\alpha,\beta} = \mathcal{L}_{\mathbb{A}^{\alpha,\beta}, \mathbf{0}, \mathbf{0}, \mathbf{0}}, \quad \forall (\alpha, \beta) \in \llbracket 1, d \rrbracket^2 \quad (3.20)$$

with

$$(\mathbb{A}^{\alpha,\beta})_{k,l} = \mu \delta_{\alpha\beta} \delta_{kl} + \mu \delta_{k\beta} \delta_{l\alpha} + \lambda \delta_{k\alpha} \delta_{l\beta}, \quad \forall (k, l) \in \llbracket 1, d \rrbracket^2. \quad (3.21)$$

Then, we have

$$\mathcal{H}^\sigma(\mathbf{u}) = -\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) \quad (3.22)$$

and,  $\forall \alpha \in \llbracket 1, d \rrbracket$ ,

$$\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}^\sigma}} = (\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n})_\alpha. \quad (3.23)$$

The matrices  $\mathbb{A}^{\alpha,\beta}$  of previous lemma are explicitly given by

- for  $d = 2$ ,

$$\mathbb{A}^{1,1} = \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix}, \quad \mathbb{A}^{1,2} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}, \quad \mathbb{A}^{2,1} = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix}, \quad \mathbb{A}^{2,2} = \begin{pmatrix} \mu & 0 \\ 0 & \gamma \end{pmatrix}$$

- for  $d = 3$ ,

$$\begin{aligned} \mathbb{A}^{1,1} &= \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, & \mathbb{A}^{1,2} &= \begin{pmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbb{A}^{1,3} &= \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix} \\ \mathbb{A}^{2,1} &= \begin{pmatrix} 0 & \mu & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbb{A}^{2,2} &= \begin{pmatrix} \mu & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \mu \end{pmatrix}, & \mathbb{A}^{2,3} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \mu & 0 \end{pmatrix}, \\ \mathbb{A}^{3,1} &= \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, & \mathbb{A}^{3,2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \lambda & 0 \end{pmatrix}, & \mathbb{A}^{3,3} &= \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \end{aligned}$$

So the elasticity problem (3.17) to (3.19) can be equivalently written as :

**Vector EBVP 1 : Linear elasticity in dimension  $d = 2$  or  $d = 3$**   
 Find  $(k, \mathbf{u}) \in \mathbb{K} \times (\mathbb{H}^2(\Omega))^d$  such that

$$\mathcal{H}^\sigma(\mathbf{u}) = k\mathcal{B}^\sigma(\mathbf{u}), \quad \text{in } \Omega, \quad (3.24)$$

$$\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}^\sigma}} = 0, \quad \text{on } \Gamma_\alpha^R = \Gamma^R, \quad \forall \alpha \in \llbracket 1, d \rrbracket \quad (3.25)$$

$$\mathbf{u}_\alpha = 0, \quad \text{on } \Gamma_\alpha^D = \Gamma^D, \quad \forall \alpha \in \llbracket 1, d \rrbracket. \quad (3.26)$$

with  $\mathcal{B}_{\alpha,\beta}^\sigma = \delta_{\alpha,\beta} \mathcal{L}_{\mathbf{0},\mathbf{0},1}$ .

---

**Algorithm 3.3** 2D elasticity
 

---

```

1:  $\mathcal{T}_h \leftarrow \text{GETMESH}(\dots)$ 
2:  $\lambda \leftarrow \frac{E\nu}{(1+\nu)(1-2\nu)}$ 
3:  $\mu \leftarrow \frac{E}{2(1+\nu)}$ 
4:  $\text{Hop} \leftarrow \text{INITHOPERATOR}(2, 2)$ 
5:  $\text{Hop}(1, 1) \leftarrow \text{LOPERATOR}(2, [2\mu + \lambda, 0; 0, \mu], \mathbf{0}, \mathbf{0}, 0)$ 
6:  $\text{Hop}(2, 1) \leftarrow \text{LOPERATOR}(2, [0, \lambda; \mu, 0], \mathbf{0}, \mathbf{0}, 0)$ 
7:  $\text{Hop}(1, 2) \leftarrow \text{LOPERATOR}(2, [0, \mu; \lambda, 0], \mathbf{0}, \mathbf{0}, 0)$ 
8:  $\text{Hop}(2, 2) \leftarrow \text{LOPERATOR}(2, [\mu, 0; 0, 2\mu + \lambda], \mathbf{0}, \mathbf{0}, 0)$ 
9:  $\text{pde} \leftarrow \text{INITPDE}(\text{Hop}, \mathcal{T}_h)$ 
10:  $\text{Bop} \leftarrow \text{INITHOPERATOR}(2, 2)$ 
11:  $\text{Hop}(1, 1) \leftarrow \text{LOPERATOR}(2, \mathbf{0}, \mathbf{0}, \mathbf{0}, 1)$ 
12:  $\text{Hop}(2, 2) \leftarrow \text{LOPERATOR}(2, \mathbf{0}, \mathbf{0}, \mathbf{0}, 1)$ 
13:  $\text{pde} \leftarrow \text{SETBC\_PDE}(\text{pde}, \Gamma^D, 1 : 2, \text{'Dirichlet'}, 0)$ 
14:  $\text{pde} \leftarrow \text{SETBC\_PDE}(\text{pde}, \Gamma^R, 1 : 2, \text{'Neumann'}, 0)$ 
15:  $(\mathbf{U}, \mathbf{k}) \leftarrow \text{EIGSPDE}(\text{pde}, \mathcal{B}, 12)$ 

```

---

**Application on the unit square with Dirichlet boundary condition**

The physical parameters are  $E = 21.5e4$  and  $\nu = 0.45$ . We take  $\Gamma^D = \Gamma$  and thus  $\Gamma^R = \emptyset$ . For each eigenfunction  $\mathbf{u}$ , we represent  $\sqrt{\mathbf{u}_1^2 + \mathbf{u}_2^2}$  in Figure 9 for the first twelve smallest magnitude eigenvalues. The mesh parameters are  $n_q = 90000$ ,  $n_{me} = 178802$  and  $h = 0.0047$ .

**Application on the bar  $[0, 7] \times [-1, 1]$  with mixed boundary conditions**

The physical parameters are  $E = 21.5e4$  and  $\nu = 0.45$ . We take  $\Gamma^D = \Gamma^1$  and  $\Gamma^R = \Gamma^2 \cup \Gamma^3 \cup \Gamma^4$ . For each eigenfunction  $\mathbf{u}$ , we represent  $\sqrt{\mathbf{u}_1^2 + \mathbf{u}_2^2}$  in Figure 10 for the first twelve smallest magnitude eigenvalues. The mesh parameters are  $n_q = 140000$ ,  $n_{me} = 278202$  and  $h = .$

**Application on a bar with 4 holes and with mixed boundary conditions**

The domain  $\Omega \subset \mathbb{R}^2$  is given in Figure 11. The physical parameters are  $E = 21.5e4$  and  $\nu = 0.45$ . We take  $\Gamma^D = \Gamma^1$  and  $\Gamma^R = \Gamma^R \setminus \Gamma^D$ . For each eigenfunction  $\mathbf{u}$ , we represent  $\sqrt{\mathbf{u}_1^2 + \mathbf{u}_2^2}$  in Figure 12 for the first twelve smallest magnitude eigenvalues. The mesh parameters are  $n_q = 141554$ ,  $n_{me} = 280746$  and  $h = .$

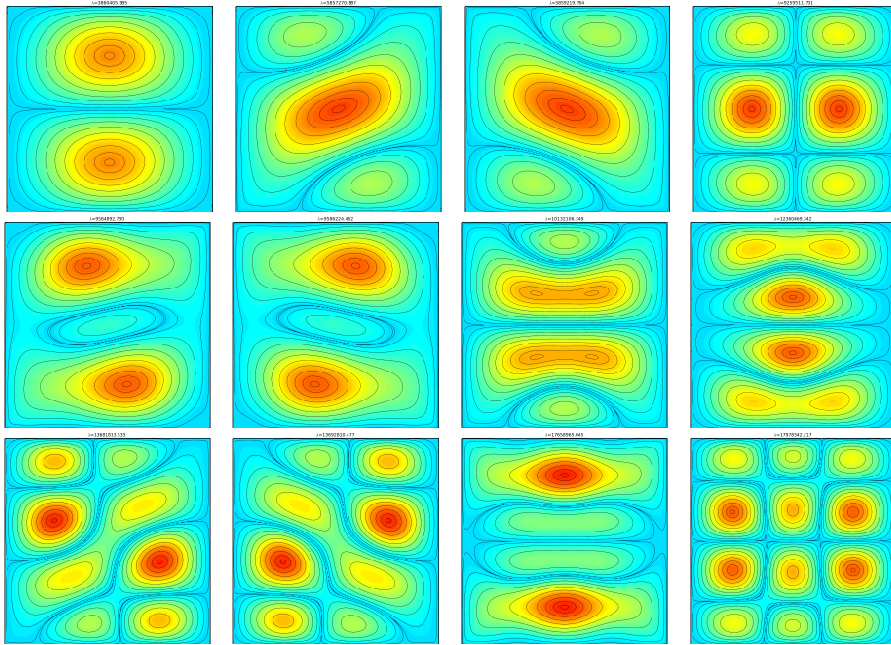


Figure 9: 2D linear elasticity on the unit square with Dirichlet boundary condition : euclidean norm of eigenfunctions associated with the smallest magnitude eigenvalues

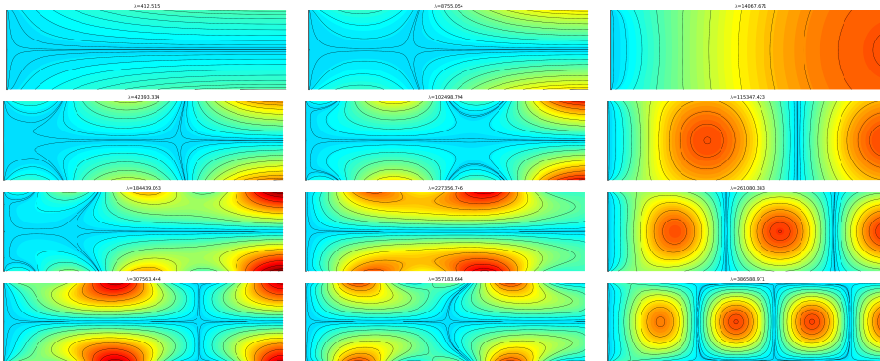


Figure 10: 2D linear elasticity on the bar  $[0, 7] \times [-1, 1]$  with mixed boundary conditions : euclidean norm of eigenfunctions associated with the smallest magnitude eigenvalues

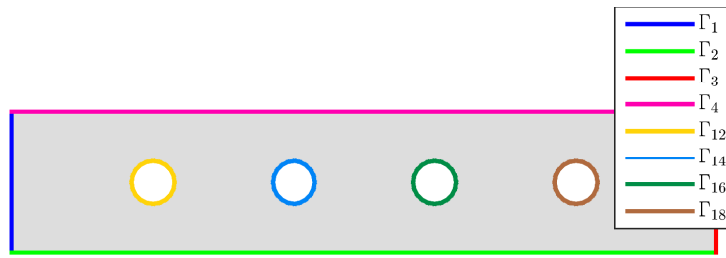


Figure 11: bar with 4 holes : domain and boundaries

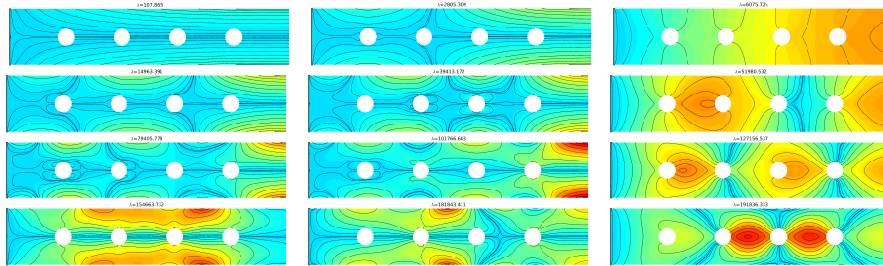


Figure 12: 2D linear elasticity on the bar  $[0, 7] \times [-1, 1]$  with mixed boundary conditions : euclidean norm of eigenfunctions associated with the smallest magnitude eigenvalues

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