
 **hypermesh** Matlab toolbox, User's Guide\*  
version 1.0.2

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**Abstract**

This object-oriented Matlab toolbox allows in any dimension  $d$  to generate conforming meshes of hypercubes, hyperrectangles or of any  $d$ -orthotopes by  $p$ -order simplices or orthotopes with their  $m$ -faces. It was created to show the implementation of the algorithms of [1]. The  **hypermesh** toolbox uses Matlab objects and is provided with meshes visualisation tools for dimension less than or equal to 3.

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\* $\text{\LaTeX}$  manual, revision 1.0.2.a, compiled with Matlab 2019a, and toolboxes `fc-hypermesh`[1.0.2], `fc-tools`[0.0.29], `fc-bench`[0.1.1]

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# 1 Introduction

The `hypermesh` toolbox contains a simple class object `OrthMesh` which permits, in any dimension  $d \geq 1$ , to obtain conforming mesh of a  $d$ -orthotope tessellated with  $p$ -order simplices or  $p$ -order orthotopes. Corresponding  $m$ -faces,  $0 \leq m < d$  of the mesh are also provided. The number of  $m$ -faces of a  $d$ -orthotope is

$$E_m^d \stackrel{\text{def}}{=} 2^{d-m} \binom{d}{m} \quad \text{where} \quad \binom{d}{m} = \frac{d!}{m!(d-m)!} \quad (1)$$

Results and vectorized algorithms used in this toolbox are given in [1].

For dimension 1 to 3 and order 1 to 4, orthotope elements and simplicial elements are respectively represented In Table 1 and Table 2. In older toolbox(0.0.x versions) only order 1 was provided.

$d \backslash p$	1	2	3	4
1				
2				
3				

Table 1:  $p$ -order  $d$ -orthotope mesh element in  $\mathbb{R}^d$ . Nodes are the points.


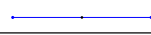

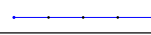
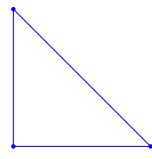
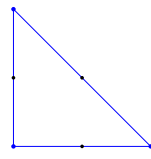
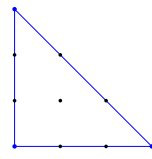
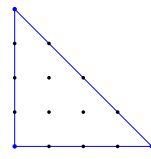
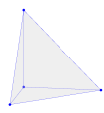
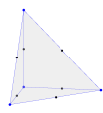

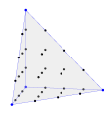
$d \backslash p$	1	2	3	4
1				
2				
3				

Table 2:  $p$ -order  $d$ -simplicial mesh element in  $\mathbb{R}^d$ . Nodes are the points.

In Figure 1 and Figure 3 small meshes of the unit hypercube are given for both tessellations with 1-order orthotopes and 1-order simplices respectively in dimension 2 and 3. From these meshes, all the associated 2-faces meshes are represented in Figure 2 and Figure 4.

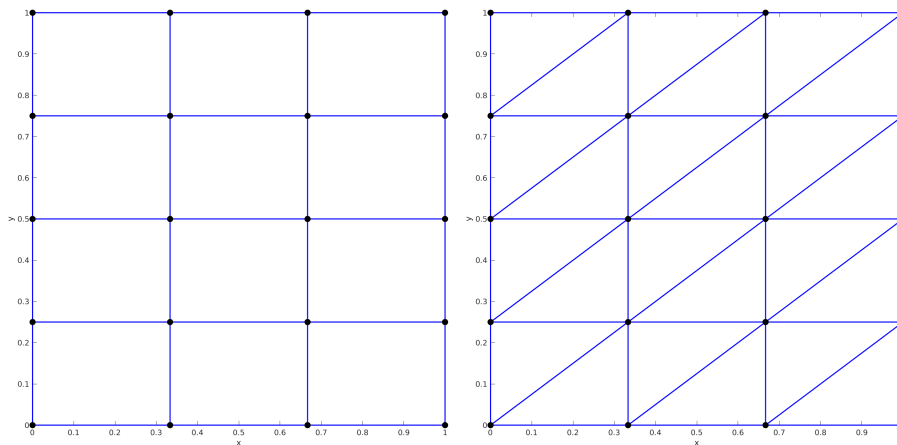


Figure 1: Tessellation samples of  $[0, 1]^2$  with 1-order 2-orthotopes (left) and 1-order 2-simplices (right) where nodes (vertices) of all mesh elements are represented by black points.

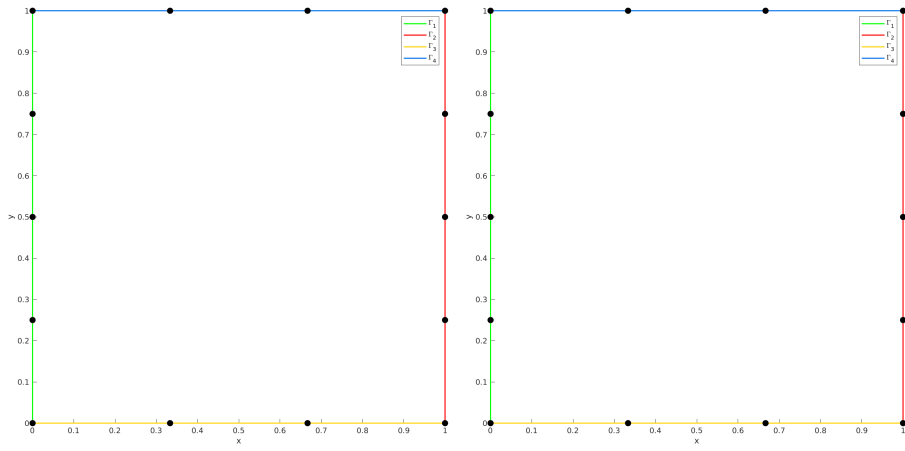


Figure 2: Representation of all the 1-faces meshes with 1-order 1-orthotopes (left) and 1-order 1-simplices (right) obtained from the tessellation samples of the Figure 1

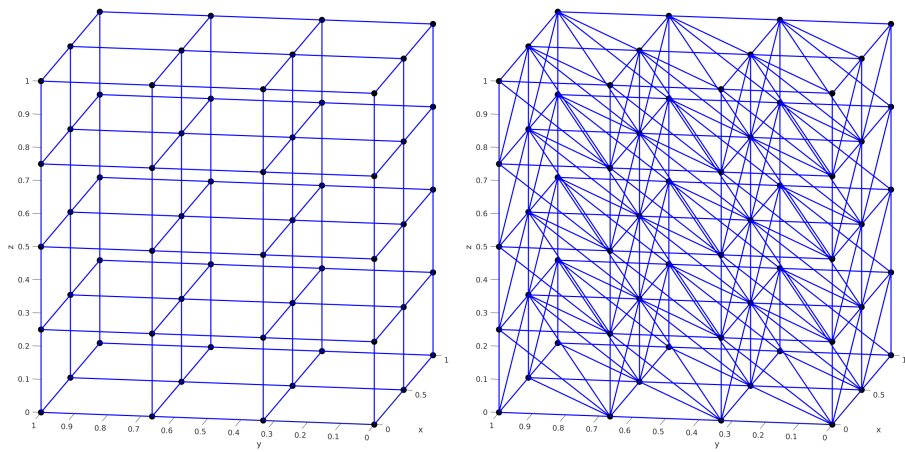


Figure 3: Tessellation samples of  $[0, 1]^3$  with 1-order 3-orthotopes (left) and 1-order 3-simplices (right) where nodes (vertices) of all mesh elements are represented by black spheres.

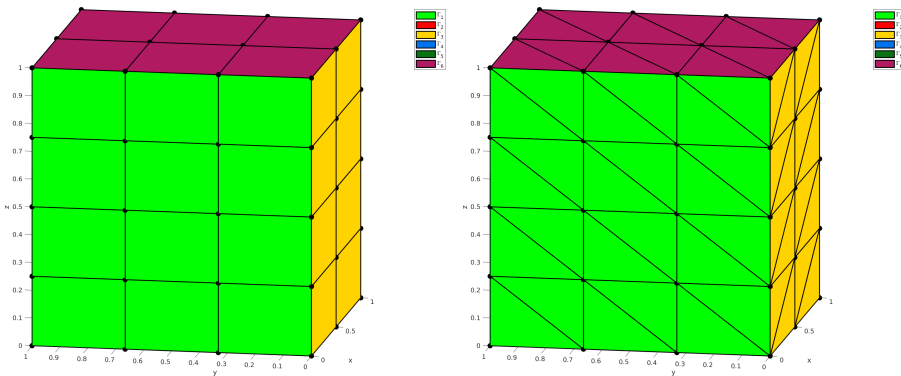


Figure 4: Representation of all the 2-faces meshes with 1-order 2-orthotopes (left) and 1-order 2-simplices (right) obtained from the tessellation samples of the Figure 3

By taking back the meshes in dimension 2 represented in Figure 1 and Figure 2, but this time using 3-order mesh element give the new meshes represented in Figure 5 and Figure 6.

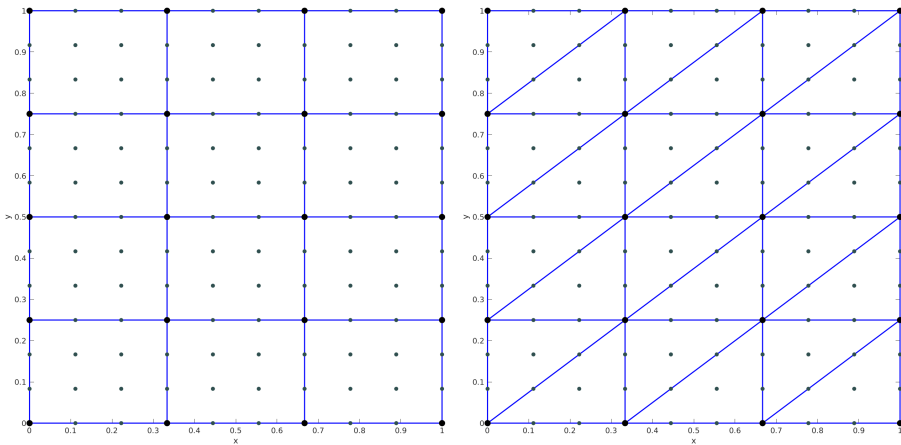


Figure 5: Tessellation samples of  $[0,1]^2$  with 3-order 2-orthotopes (left) and 3-order 2-simplices (right) where nodes of all mesh elements are represented by black (vertices) and grey points.

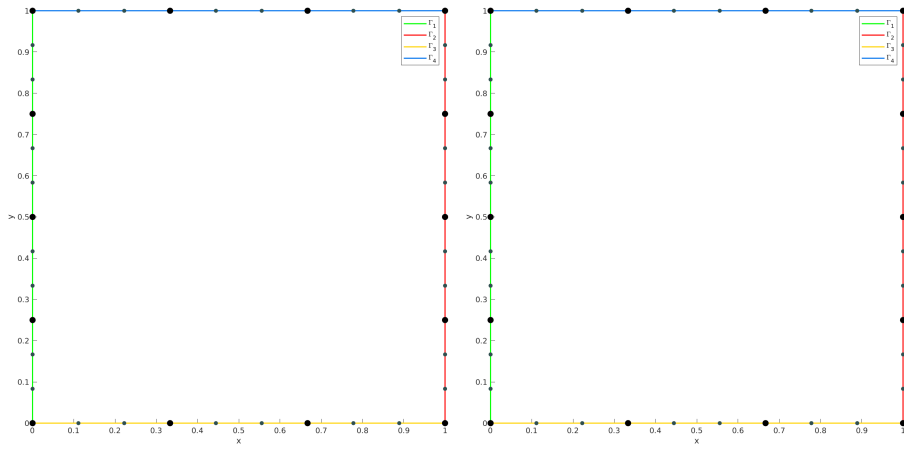


Figure 6: Representation of all the 1-faces meshes with 3-order 1-orthotopes (left) and 3-order 1-simplices (right) obtained from the tessellation samples of the Figure 5

In dimension 3, meshes represented in Figure 3 and Figure 4 are this time tessellated respectively with 3-order orthotopes and 3-order simplices and represented in Figure 7 and Figure 8.

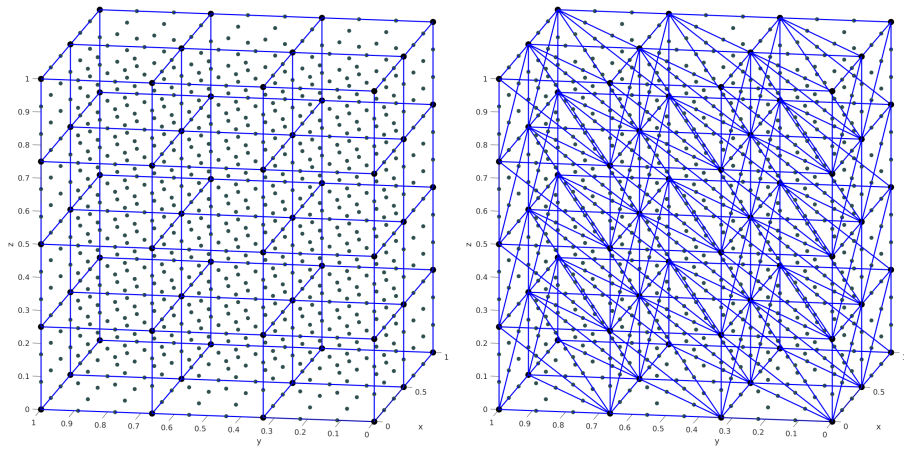


Figure 7: Tessellation samples of  $[0, 1]^3$  with 3-order 3-orthotopes (left) and 3-order 3-simplices (right) where nodes of all mesh elements are represented by black (vertices) and grey spheres.

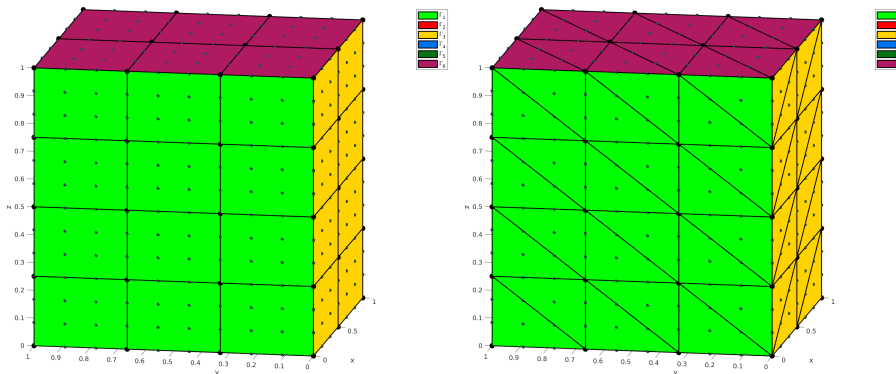


Figure 8: Representation of all the 2-faces meshes with 3-order 2-orthotopes (left) and 3-order 2-simplices (right) obtained from the tessellation samples of the Figure 7

It is also possible with the method function `plotmesh` of the class object `OrthMesh` to represent a mesh or its `m`-faces for  $d \leq 3$ .

This toolbox was tested on various OS with Matlab releases:

Operating system	2015b	2016a	2016b	2017a	2017b	2018a	2018b	2019a
CentOS 7.7.1908	✓	✓	✓	✓	✓	✓	✓	✓
Debian 9.11	✓	✓	✓	✓	✓	✓	✓	✓
Fedora 29	✓	✓	✓	✓	✓	✓	✓	✓
OpenSUSE Leap 15.0	✓	✓	✓	✓	✓	✓	✓	✓
Ubuntu 18.04.3 LTS	✓	✓	✓	✓	✓	✓	✓	✓
MacOS High Sierra 10.13.6	✓	✓	✓	✓	✓	✓	✓	✓
MacOS Mojave 10.14.4	✓	✓	✓	✓	✓	✓	✓	✓
Windows 10 (1909)	✓	✓	✓	✓	✓	✓	✓	✓

It is not compatible with Matlab releases prior to R2015b.

In the following section, the class object `OrthMesh` is presented. Thereafter some warning statements on the memory used by these objects in high dimension are given. Finally computation times for orthotope meshes and simplicial meshes are given in dimension  $d \in \llbracket 1, 5 \rrbracket$ .

## 2 Installation

### 2.1 Installation automatic, all in one (recommended)

For this method, one just has to get/download the install file

```
mfc_hypermesh_install.m
```

or get it on the dedicated web page. Thereafter, it should be run under Matlab. This command downloads, extracts and configures the *fc-hypermesh* and the required *fc-tools* toolboxes in the current directory.

For example, to install this toolbox in `~/Matlab/toolboxes` directory, one has to copy the file `mfc_hypermesh_install.m` in the `~/Matlab/toolboxes` directory. Then in a Matlab terminal run the following commands

```
>> cd ~/Matlab/toolboxes
>> mfc_hypermesh_install
```

This is the output of the `mfc_hypermesh_install` command on a Linux computer:

```
arts of the <fc-hypermesh> Matlab toolbox.
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1- Downloading and extracting the toolboxes
2- Setting the <fc-hypermesh> toolbox
Write in ~/Matlab/toolboxes/fc-hypermesh-full/fc_hypermesh-1.0.2/
  configure_loc.m ...
3- Using toolboxes :
  ->          fc-tools : 0.0.29
  ->          fc-bench : 0.1.1
*** Using instructions
To use the <fc-hypermesh> toolbox:
addpath('~/Matlab/toolboxes/fc-hypermesh-full/fc_hypermesh-1.0.2')
fc_hypermesh.init()

See ~/Matlab/toolboxes/mfc_hypermesh_set.m
```

The complete toolbox (i.e. with all the other needed toolboxes) is stored in the directory `~/Matlab/toolboxes/fc-hypermesh-full` and, for each Matlab session, one has to set the toolbox by:

```
>> addpath('~/Matlab/toolboxes/fc-hypermesh-full/mfc-hypermesh-1.0.2')
>> fc_hypermesh.init()
```

To **uninstall**, one just has to delete directory

```
~/Matlab/toolboxes/fc-hypermesh-full
```

## 2.2 Manual installation

This package uses the `fc_tools` toolbox. So one has to install it as explain in the dedicated web page.

Thereafter, on the `fc_hypermesh` dedicated web page, one can found link to archives ( *zip*, *7z* or *tar.gz* format)

- Downloads an archive and extracts it on a folder, for example `~/Matlab/toolboxes`. The toolbox path is `~/Matlab/toolboxes/mfc-hypermesh-1.0.2`
- Adds the toolbox path in Matlab with `addpath` command.
- Verifies that the `fc_tools` and `fc_bench` toolboxes are in the Matlab path. Otherwise, adds them...

## 3 Using the `fc`hypermesh toolbox

Before using this class it will be necessary to be aware of the memory used by this one. For example, when meshing a 6-dimensional orthotope with 1-order simplices by taking  $N = 10$  intervals in each space direction, gives an `OrthMesh`



object using 20 GB in memory. With 3-order simplices, the `OrthMesh` object use 241 GB in memory!

The memory usage for a  $d$ -dimensional `OrthMesh` object by taking  $N = 10$  intervals in each space direction is given in Table 3 for 1-order elements and in Table 4 for 3-order elements. One can refer to Appendix A for more details.

$d$	<code>OrthMesh</code> (orthotopes)	<code>OrthMesh</code> (simplices)
1	8 B	80 B
2	688 B	2 KB
3	23 KB	96 KB
4	601 KB	4 MB
5	13 MB	288 MB
6	308 MB	20 GB
7	6 GB	1 TB
8	147 GB	145 TB

Table 3: Memory usage of `OrthMesh` object for the tessellation of an orthotope by 1-order orthotopes and by 1-order simplices according to the space dimension  $d$  and with  $N = 10$ .

$d$	<code>OrthMesh</code> (orthotopes)	<code>OrthMesh</code> (simplices)
1	8 B	160 B
2	1 KB	8 KB
3	135 KB	480 KB
4	8 MB	33 MB
5	438 MB	2 GB
6	21 GB	241 GB
7	968 GB	24 TB
8	42 TB	2 661 TB

Table 4: Memory usage of `OrthMesh` object for the tessellation of an orthotope by 3-order orthotopes and by 3-order simplices according to the space dimension  $d$  and with  $N = 10$ .

First of all, the main class object `OrthMesh` is presented. Thereafter some usage samples are given.

### 3.1 Class object `OrthMesh`

The aim of the `OrthMesh` class object is to efficiently create an object which contains a mesh of a  $d$ -orthotope and all its  $m$ -face meshes with either  $p$ -order orthotopes or  $p$ -order simplices. An elementary mesh class object called `EltMesh` is used to store only one mesh, the main mesh as well as any of the  $m$ -face meshes. The `EltMesh` class is described in Appendix B.1.

Let the  $d$ -orthotope defined by  $[a_1, b_1] \times \cdots \times [a_d, b_d]$ . The class object `OrthMesh` corresponding to this  $d$ -orthotope contains the main mesh and all its  $m$ -face meshes,  $0 \leq m < d$ . Its fields are the following

- $d$ : space dimension

- `type`: string `'simplex'` or `'orthotope'` mesh
- `p`, order of the mesh elements
- `Mesh`: main mesh as an `EltMesh` object
- `Faces`: cell array of arrays of `EltMesh` objects such that `Faces{1}` is an array of all the  $(d - 1)$ -face meshes stored as `EltMesh` objects, `Faces{2}` is an array of all the  $(d - 2)$ -face meshes, and so on
- `box`: a  $d$ -by-2 array such that `box(i,1) = ai` and `box(i,2) = bi`.

### 3.1.1 Constructor

```
Oh = fc_hypermesh.OrthMesh(N)
Oh = fc_hypermesh.OrthMesh(d,N)
Oh = fc_hypermesh.OrthMesh(N, key, value, ...)
Oh = fc_hypermesh.OrthMesh(d,N, key, value, ...)
```

#### Description

```
Oh = fc_hypermesh.OrthMesh(N)
```

Generates the `OrthMesh` object `Oh` which contains a 1-order simplicial mesh of the unit  $d$ -orthotope and all its  $m$ -face meshes. `N` is a 1-by- $d$  array used to obtain a regular discretisation with  $N(i) + 1$  points in the  $i$ -th direction.

```
Oh = fc_hypermesh.OrthMesh(d,N)
```

Generates the `OrthMesh` object `Oh` which contains a 1-order simplicial mesh of the unit  $d$ -orthotope and all its  $m$ -face meshes.

- `d` is the space dimension
- `N` is a 1-by- $d$  array used to obtain a regular discretisation with  $N(i) + 1$  points in the  $i$ -th direction. If `N` is a positive integer then `N + 1` points are created in each direction.

```
Oh = fc_hypermesh.OrthMesh(N, key, value, ...)
```

```
Oh = fc_hypermesh.OrthMesh(d,N, key, value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'type'` : used to select the kind of elements used for meshing. The default `value` is `'simplex'` and otherwise `'orthotope'` can be used.

```

Listing 1: : OrthMesh constructor in dimension d=3 (orthotope mesh)
Oh = fc_hypermesh.OrthMesh(3,10, 'type','orthotope')

```

---

Output

```

Oh =
Mesh type : orthotope
dimension : 3
order      : 1
nq         : 1331
nme        : 1000
Number of 2-faces : 6
Number of 1-faces : 12
Number of 0-faces : 8

```

- **'order'** : gives the order of the mesh elements (default is 1).

```

Listing 2: : OrthMesh constructor in dimension d=3 (simplicial mesh)
Oh = fc_hypermesh.OrthMesh(3,[10,20,15], 'order',4)

```

---

Output

```

Oh =
Mesh type : simplex
dimension : 3
order      : 4
nq         : 202581
nme        : 18000
Number of 2-faces : 6
Number of 1-faces : 12
Number of 0-faces : 8

```

- **'box'** : used to specify the  $d$ -orthotope  $[a_1, b_1] \times \dots \times [a_d, b_d]$  by setting **value** as an  $d$ -by-2 array such that  $a_i = \text{value}(i,1)$  and  $b_i = \text{value}(i,2)$ .

```

Listing 3: : OrthMesh constructor in dimension d=3 (simplicial mesh)
Oh = fc_hypermesh.OrthMesh(3,10, 'box',[-1 1;-2 2;0 3])

```

---

Output

```

Oh =
Mesh type : simplex
dimension : 3
order      : 1
nq         : 1331
nme        : 6000
Number of 2-faces : 6
Number of 1-faces : 12
Number of 0-faces : 8

```

- **'m\_min'** : used to only build the  $m$ -Faces for  $m$  in  $[[m\_min, d]$ . Default **value** is 0.

```
Listing 4: : OrthMesh constructor in dimension d=3 (simplicial mesh)
Oh = fc_hypermesh.OrthMesh(3,10, 'm_min',2)
```

---

Output

```
Oh =
Mesh type : simplex
dimension : 3
order      : 1
nq         : 1331
nme        : 6000
Number of 2-faces : 6
```

- **'mapping'** : used to apply on the mesh a mapping function given by a function handle.

```
Listing 5: : OrthMesh constructor in dimension d=3 (simplicial mesh)
mfun =@(q) [q(1,:)+sin(q(2,:));q(2,:);q(3,:)];
Oh = fc_hypermesh.OrthMesh(3,10, 'mapping',mfun)
```

---

Output

```
Oh =
Mesh type : simplex
dimension : 3
order      : 1
nq         : 1331
nme        : 6000
Number of 2-faces : 6
Number of 1-faces : 12
Number of 0-faces : 8
```

### 3.1.2 Access to OrthMesh's fields

In all examples given in this section, **Oh** is the **OrthMesh** object given by

```
Oh = fc_hypermesh.OrthMesh(3,10, 'order',2)
```

---

Output

```
Oh =
Mesh type : simplex
dimension : 3
order      : 2
nq         : 9261
nme        : 6000
Number of 2-faces : 6
Number of 1-faces : 12
Number of 0-faces : 8
```

It's a 3 dimensional mesh of the unit cube tessellated with 2-order simplices where their vertices are in  $(i/10, j/10, k/10)$  for all  $(i, j, k) \in \llbracket 0, 10 \rrbracket$ . The main mesh given as an **EltMesh** object is **Oh.Mesh**

```

Th=Oh.Mesh
-----
Output

Th =
EltMesh object
  type : simplex
  order : 2
  d : 3
  m : 3
  q : (3,9261)
  me : (10,6000)

```

The  $k^{\text{th}}$   $m$ -faces of `Oh` stored as an `EltMesh` object is given by `Oh.Faces{Oh.d-m}(k)`.

```

Fh=Oh.Faces{1}(3)
-----
Output

Fh =
EltMesh object
  type : simplex
  order : 2
  d : 3
  m : 2
  q : (3,441)
  me : (6,200)

```

One can easily access to each field of an `EltMesh` object (see description in Appendix B.1). For example, to access the nodes array and the connectivity array of the `Th` `EltMesh` object we do respectively `Th.q` and `Th.me`. We also have the following link between global nodes array `Th.q` and local nodes array `Fh.q`:

$$\text{Th.q}(:, \text{Fh.toGlobal}) == \text{Fh.q}$$

or more generally, for all  $m$  in  $\llbracket 0, Oh.d \rrbracket$  and for all  $k$  in  $\llbracket 1, E_m^d \rrbracket$

$$\text{Oh.Mesh.q}(:, \text{Oh.Faces}\{Oh.d-m\}(k).toGlobal) == \text{Oh.Faces}\{Oh.d-m\}(k).q$$

where  $E_m^d$  is defined in (1).

### 3.1.3 plotmesh method

The `plotmesh()` member function can be used to represent the mesh given by an `OrthMesh` object if the space dimension is less than or equal to 3.

#### Syntaxe

```

obj.plotmesh()
obj.plotmesh(key, value, ...)

```

#### Description

```

obj.plotmesh()

```

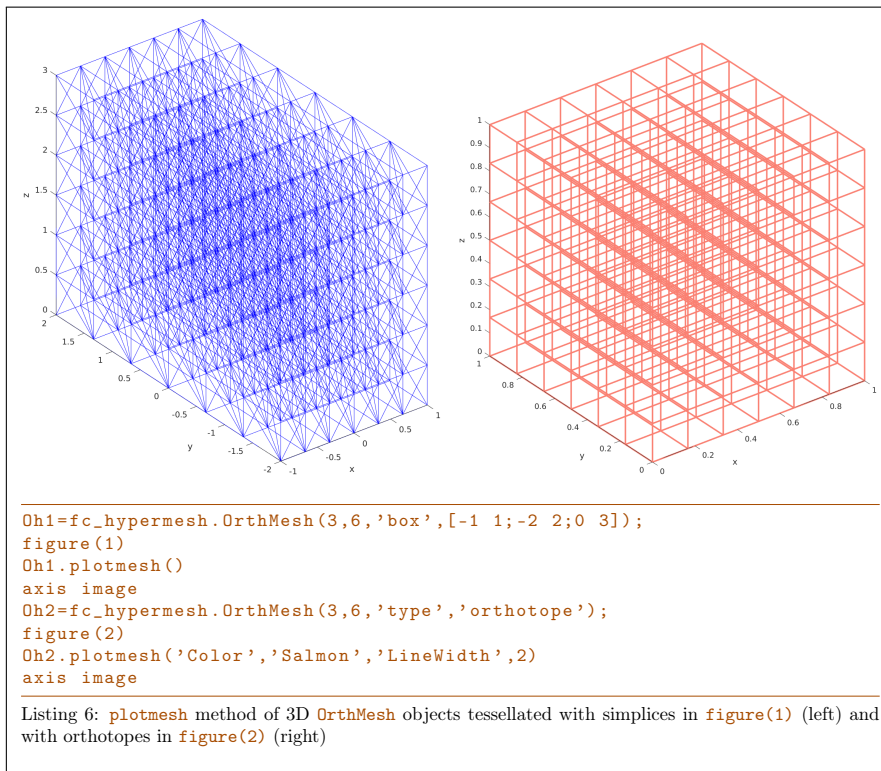
```
obj.plotmesh(key, value, ...)
```

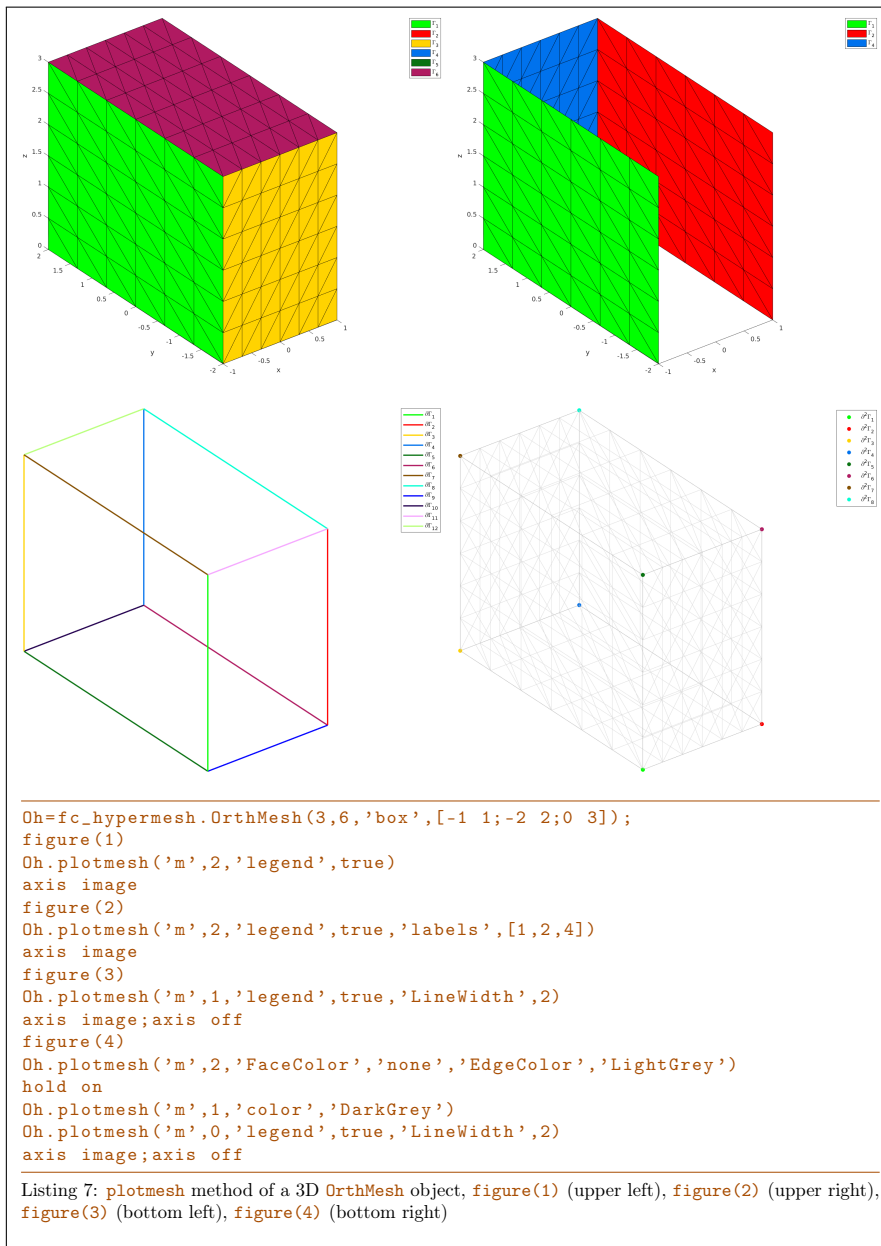
Some optional `key/value` pairs arguments are available with `key`:

- `'legend'` : if `value` is `True`, a legend is displayed. Default is `False`.
- `'m'` : plots all the `m`-faces of the mesh. Default `m = d` i.e. the main mesh. ( $0 \leq m \leq d$ )
- `'labels'` : plot all the `m`-faces of the mesh with number/label in `value` list
- `'color'` : use to specify the color to use.
- ...

Other `key/value` pairs arguments can be used depending of `obj.d` and `obj.m` values and they are those of the plotting function used:

- with `obj.d=3` and `obj.m=3` , `patch` function is used;
- with `obj.d=3` and `obj.m=2` , `trimesh` function is used for simplicial mesh and `patch` function is used for orthotope mesh;
- with `obj.d=3` and `obj.m=1` , `line` function is used;
- with `obj.d=3` and `obj.m=0` , `scatter3` function is used;
- with `obj.d=2` and `obj.m=2` , `triplot` function is used for simplicial mesh and `patch` function is used for orthotope mesh;
- with `obj.d=2` and `obj.m=1` , `line` function is used;
- with `obj.d=2` and `obj.m=0` , `scatter` function is used;
- with `obj.d=1` and `obj.m=1` , `line` function is used;
- with `obj.d=1` and `obj.m=0` , `scatter` function is used;





### 3.1.4 `plotnodes` method

The `plotnodes()` member function can be used to represent nodes of the mesh given by an `OrthMesh` object if the space dimension `d` is less than or equal to 3.

#### Syntaxe

```
obj.plotnodes()
```



```
obj.plotnodes(key, value, ...)
```

## Description

```
obj.plotnodes()
```

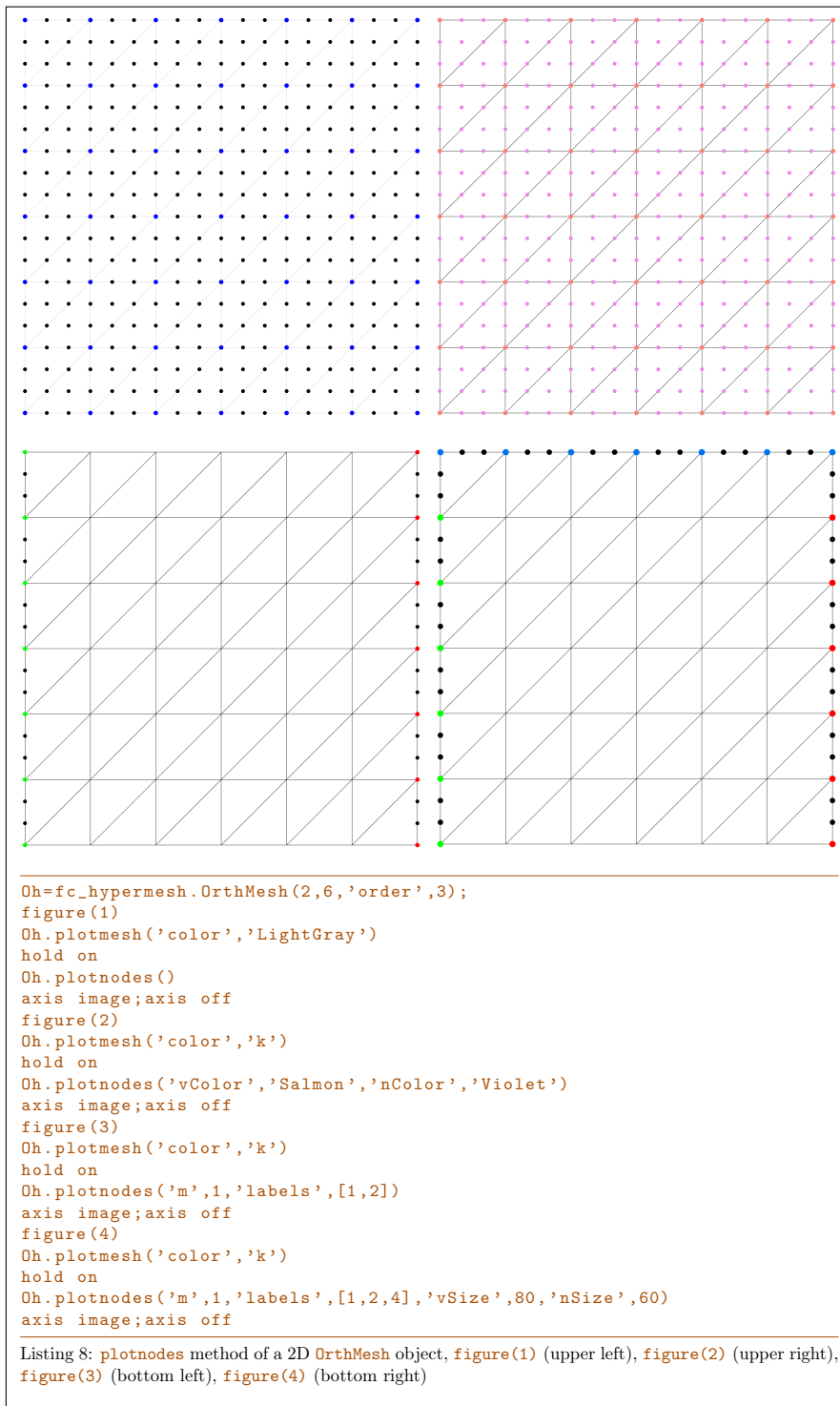
Uses `scatter` function (1D and 2D) and `scatter3` function (3D) to represent nodes of the mesh as points. Vertices of the mesh elements are also nodes and they are distinguishable from others nodes.

```
obj.plotnodes(key, value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'m'` : plots all the nodes of the `m`-faces of the mesh. Default `m = d` i.e. the main mesh. ( $0 \leq m \leq d$ )
- `'labels'` : plot all the nodes of the `m`-faces of the mesh with number/label in `value` list
- `'vcolor'` : use to specify the point color for the mesh vertices. Default is `obj.color`.
- `'vsize'` : use to specify the point size for the mesh vertices. Default is `40`.
- `'ncolor'` : use to specify the color of the nodes (not vertices) of the mesh elements. Default is `'k'` (ie. black).
- `'nsize'` : use to specify the size of the nodes (not vertices) of the mesh elements. Default is `30`.

Other `key/value` pairs arguments can be used: they are those of the `scatter` and `scatter3` function used.



### 3.1.5 `plotnodesNumber` method

The `plotnodesNumber()` member function can be used to display nodes index/number of the mesh given by an `OrtMesh` object if the space dimension `d` is less than or equal to 3.

#### Syntaxe

```
obj.plotnodesNumber()  
obj.plotnodesNumber(key, value, ...)
```

#### Description

```
obj.plotnodesNumber()
```

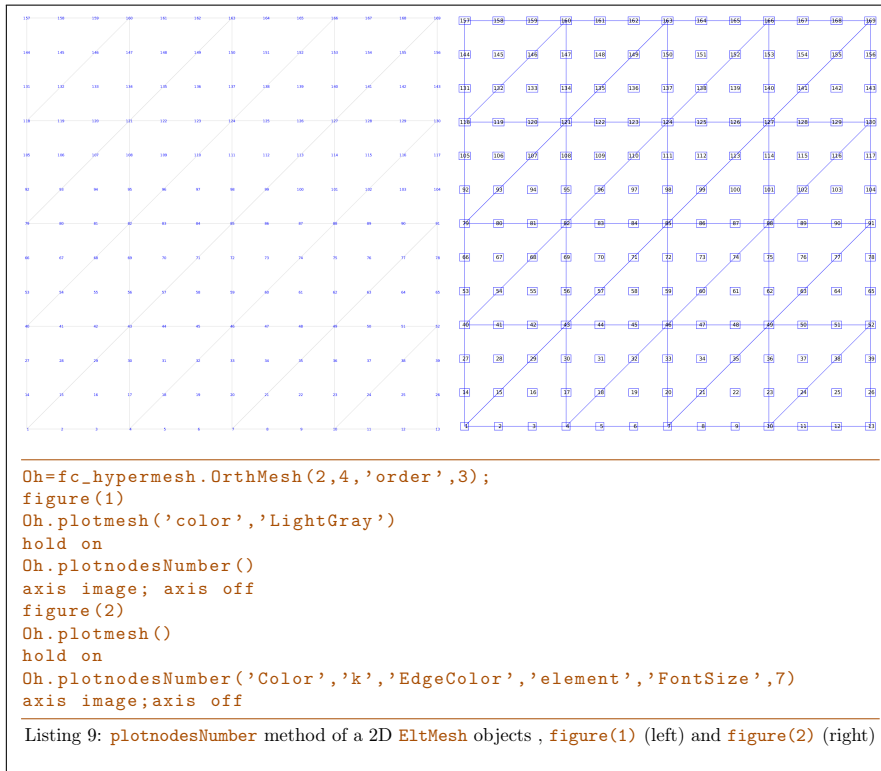
Uses `fc_hypermesh.plotnodesNumber` function to represent node numbers.

```
obj.plotnodesNumber(key, value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'m'` : plots all the nodes index/number of the `m`-faces of the mesh. Default `m = d` i.e. the main mesh. ( $0 \leq m \leq d$ )
- `'labels'` : plot all the nodes index/number of the `m`-faces of the mesh with number/label in `value` list
- `'color'` : use to specify text color. Default is the color element.
- `'EdgeColor'` : use to specify the color of box outline. Default is `'none'` (i.e. no box outline).
- `'BackgroundColor'` : use to specify text background color. Default is `'w'` (i.e. white).
- `'margin'` : use to specify the margin. Default is `1`.
- `'HorizontalAlignment'` : Default is `'center'`.
- `'VerticalAlignment'` : Default is `'middle'`.
- `'clipping'` : Default is `'on'`.

Other `key/value` pairs arguments can be used: they are those of the `text` function used. For the color options `'Color'`, `'EdgeColor'` and `'BackgroundColor'`, one can use `'element'` as `value` to set color to the element color.



### 3.1.6 `ploteltsNumber` method

The `ploteltsNumber()` member function can be used to display elements index/number of the mesh given by an `OrthMesh` object if the space dimension `d` is less than or equal to 3.

#### Syntaxe

```

obj.ploteltsNumber()
obj.ploteltsNumber(key, value, ...)

```

#### Description

```
obj.ploteltsNumber()
```

Uses `fc_hypermesh.ploteElementsNumber` function to represent node numbers.

```
obj.ploteltsNumber(key, value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'m'` : plots all the elements index/number of the `m`-faces of the mesh. Default `m = d` i.e. the main mesh. ( $0 \leq m \leq d$ )
- `'labels'` : plot all the elements index/number of the `m`-faces of the mesh with number/label in `value` list.

- `'color'` : use to specify text color. Default is the color element.
- `'EdgeColor'` : use to specify the color of box outline. Default is `'none'` (i.e. no box outline).
- `'BackgroundColor'` : use to specify text background color. Default is `'w'` (i.e. white).
- `'margin'` : use to specify the margin. Default is `1` .
- `'HorizontalAlignment'` : Default is `'center'` .
- `'VerticalAlignment'` : Default is `'middle'` .
- `'clipping'` : Default is `'on'` .
- `'vLineColor'` : Draw lines between vertices and barycenter of the mesh elements. Default is `''` (no lines).
- `'vLineStyle'` : Select lines type. Default is `':'` (dotted lines).
- `'vLineWidth'` : Set lines width. Default is `0.5` .

Other `key/value` pairs arguments can be used: they are those of the `text` function used. For the color options `'Color'`, `'EdgeColor'`, `'BackgroundColor'` and `'vLineColor'`, one can use `'element'` as `value` to set color to the element color.

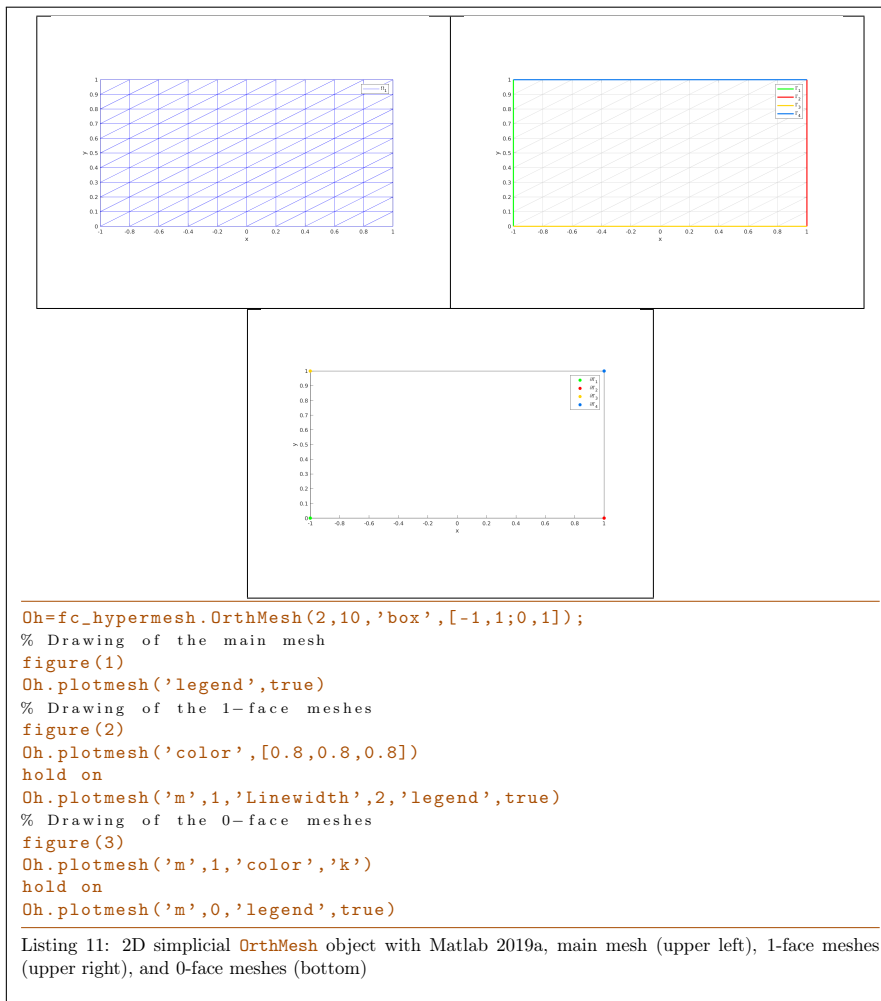


### 3.2 2d-orthotope meshed by simplices

In Listing 1, an `OrthMesh` object is built under Matlab by using command

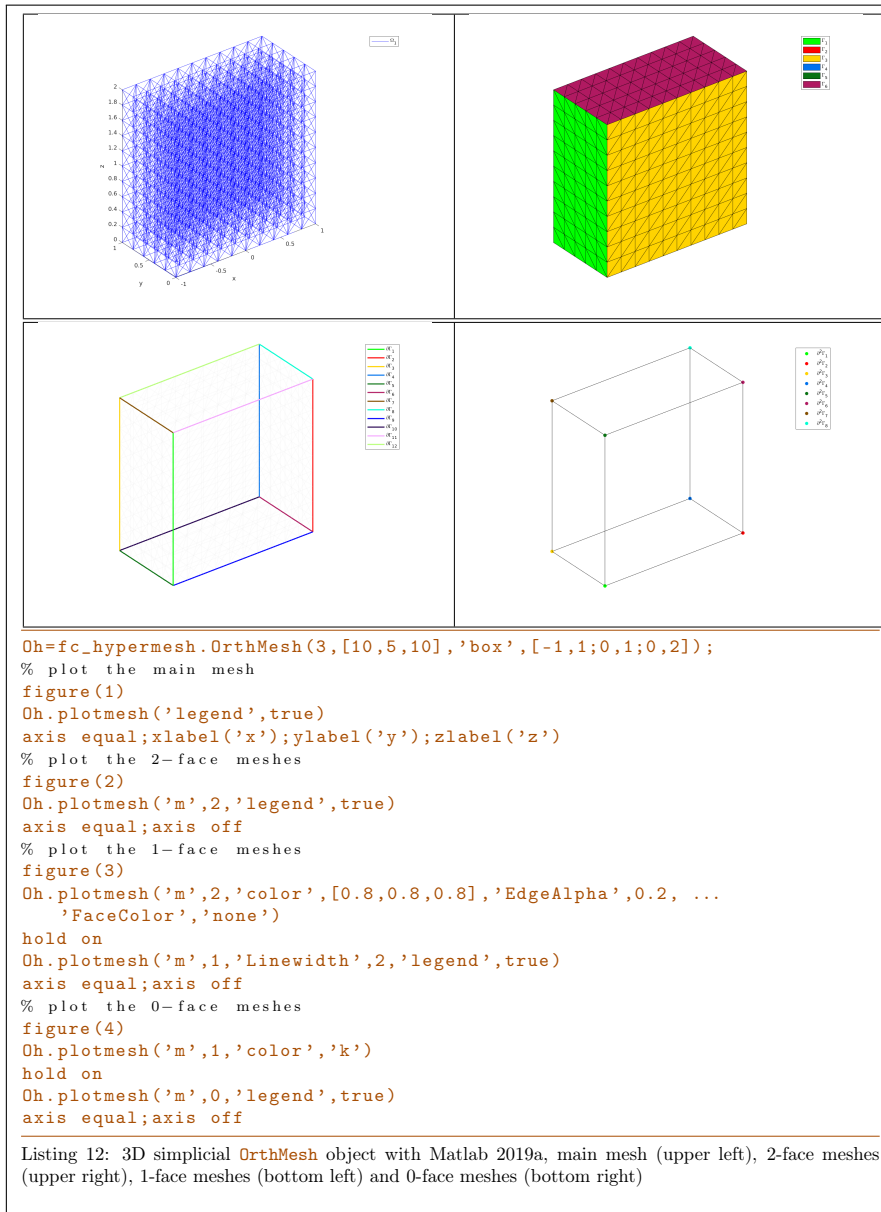
```
Oh=fc_hypermesh.OrthMesh(2,10,'box',[-1,1;0,1]);
```

So the `Oh` object is the tessellations of the orthotope  $[-1, 1] \times [0, 1]$  with simplicial elements. In each direction  $10 + 1 (= 11!)$  points are taken. So we have  $11^2$  vertices in this mesh. The main mesh and all the `m`-face meshes of the resulting object are plotted by using `plotmesh` method.



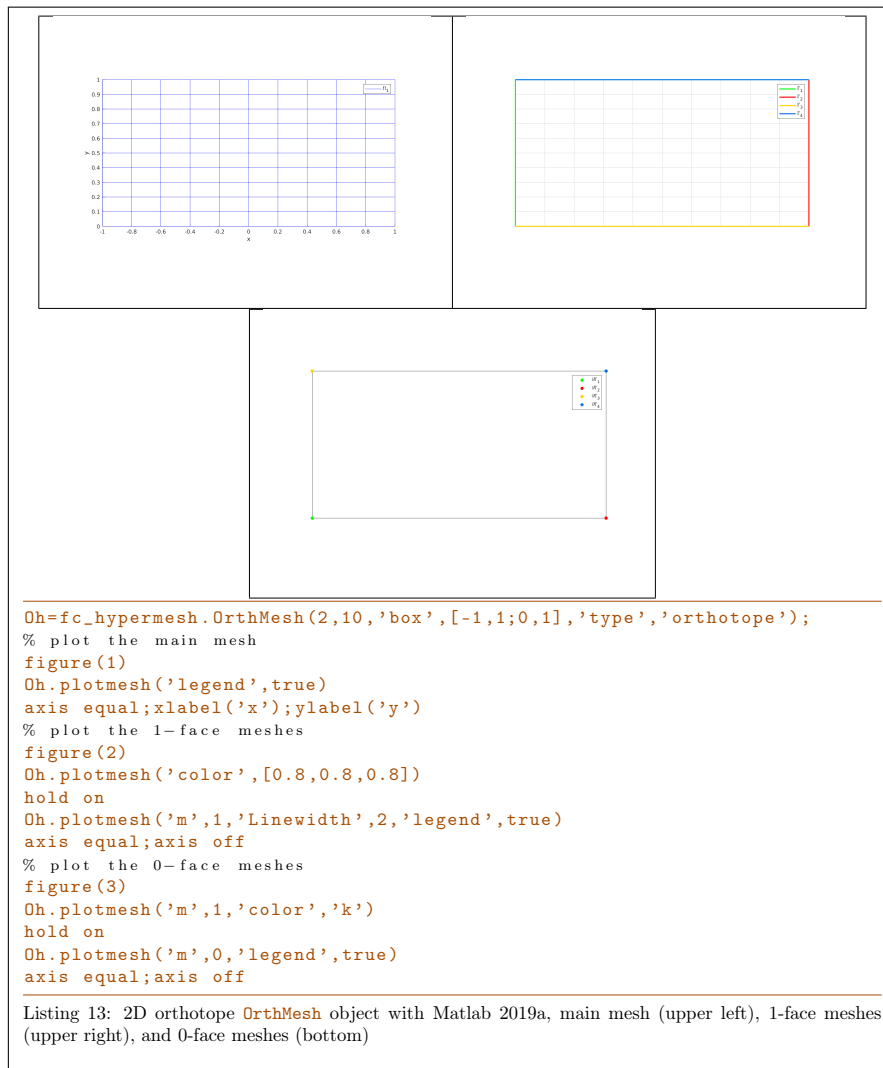
### 3.3 3d-orthotope meshed by simplices

In Listing 1, an `OrthMesh` object is built under Matlab for the orthotope  $[-1, 1] \times [0, 1] \times [0, 2]$  with simplicial elements and  $N=[10, 5, 10]$ . The main mesh and all the `m`-face meshes of the resulting object are plotted.



### 3.4 2d-orthotope meshed by orthotopes

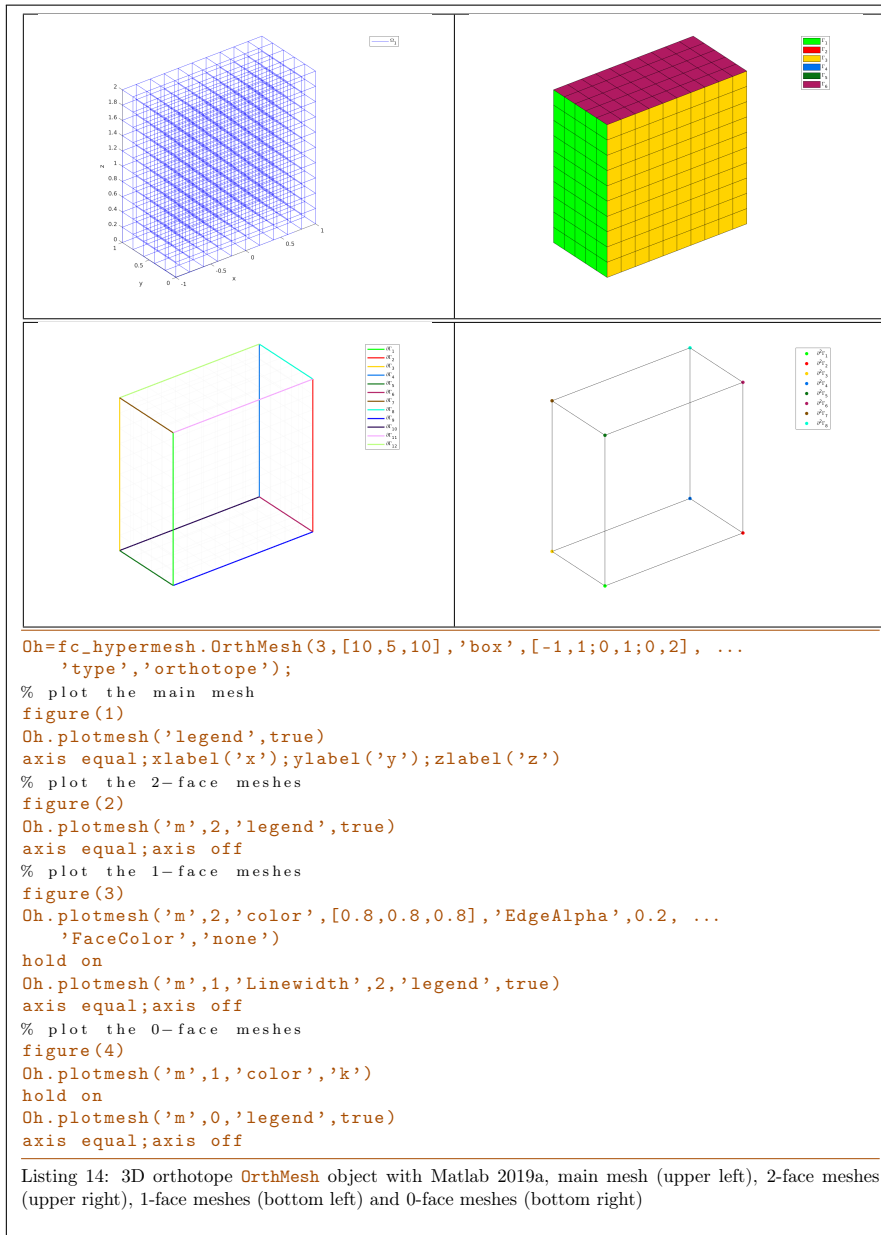
In Listing 1, an `OrthMesh` object is built under Matlab for the orthotope  $[-1, 1] \times [0, 1] \times [0, 2]$  with orthotope elements and  $\mathbf{N}=[10, 5, 10]$ . The main mesh and all the `m`-face meshes of the resulting object are plotted.



### 3.5 3d-orthotope meshed by orthotopes

In Listing 1, an `OrthMesh` object is built under Matlab for the orthotope  $[-1, 1] \times [0, 1] \times [0, 2]$  with orthotope elements and  $N=[10, 5, 10]$ . The main mesh and all the `m`-face meshes of the resulting object are plotted.



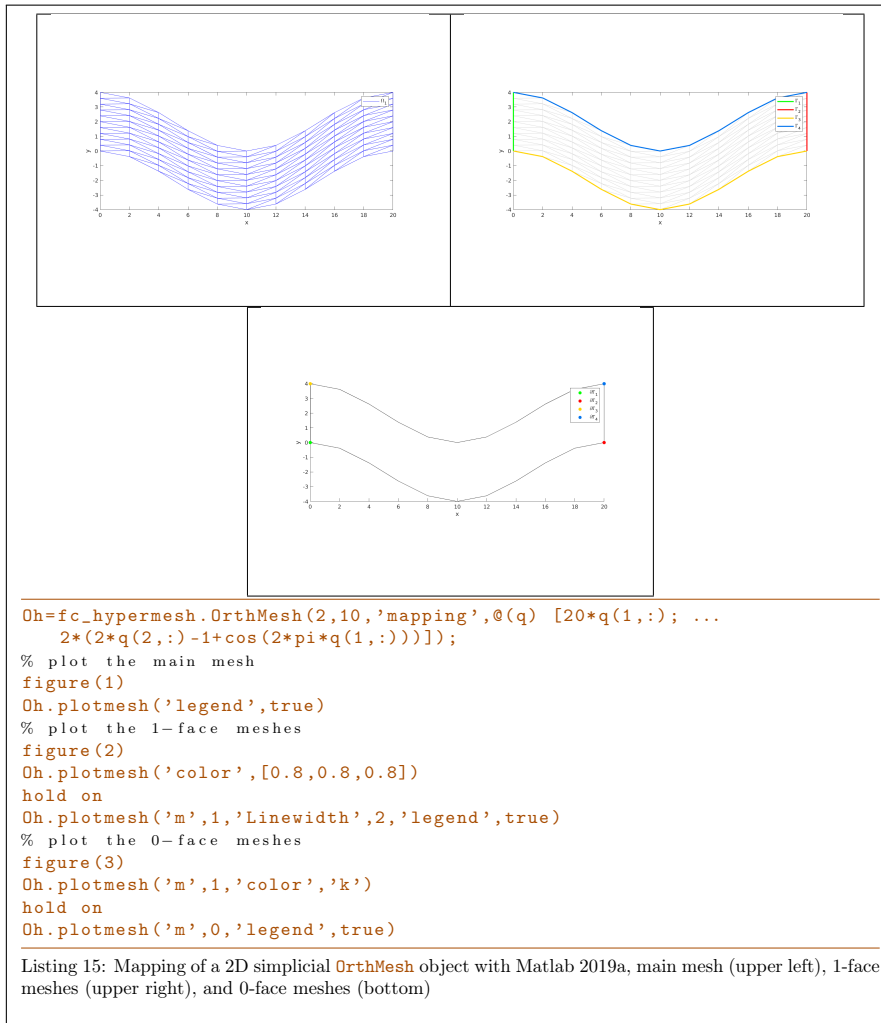


### 3.6 Mapping of a 2d-orthotope meshed by simplices

For example, the following 2D geometrical transformation allows to deform the reference unit hypercube.

$$[0, 1] \times [0, 1] \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow F(x, y) = \begin{pmatrix} 20x \\ 2(2y - 1 + \cos(2\pi x)) \end{pmatrix}$$

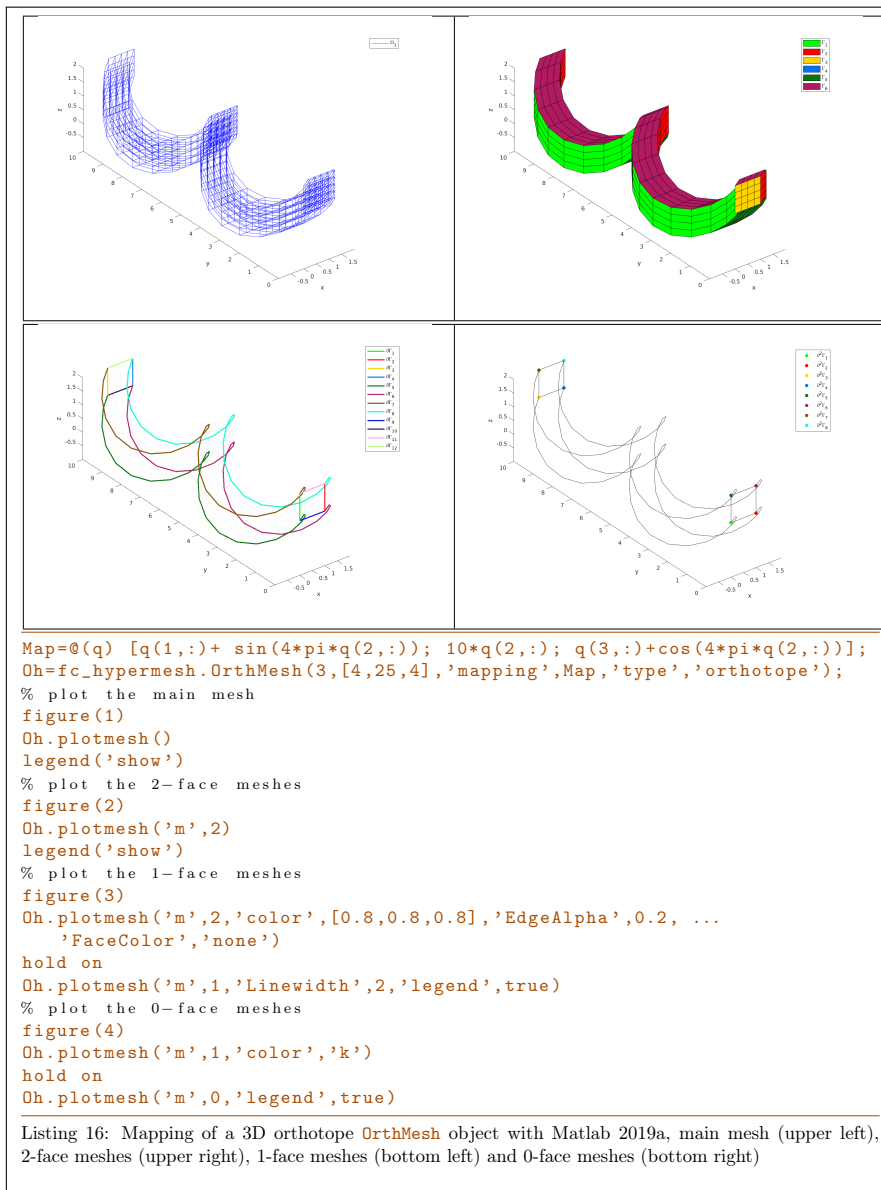


### 3.7 3d-orthotope meshed by orthotopes

For example, the following 3D geometrical transformation allows to deform the reference unit hypercube.

$$[0, 1] \times [0, 1] \times [0, 1] \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow F(x, y, y) = \begin{pmatrix} x + \sin(4\pi y) \\ 10y \\ z + \cos(4\pi y) \end{pmatrix}$$



## 4 Benchmarking

### 4.1 `fc_hypermesh.bench` function

The `fc_hypermesh.bench` function can be used to obtain computational times of the `OrthMesh` constructor.

#### Syntaxe

```
fc_hypermesh.bench(d, LN)
fc_hypermesh.bench(d, LN, key, value, ...)
```

## Description

```
fc_bench.bench(d, LN)
```

displays computational times of the `OrthMesh` constructor tessellated with simplices (default) as follows

```
ts=tic();Oh=fc_hypermesh.OrthMesh(d, LN(:,i));tcpu=toc(ts);
```

for each `i` in `1:size(LN,2)`.

- `d`, space dimension.
- `LN` is either a 1-by-`n` or a `d`-by-`n` array of positive integers.

```
fc_bench.bench(d, LN, key, value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'type'` : string `'simplex'` (default) or `'orthotope'` to specify kind of tessellation.
- `'order'` : positive integer (1 by default) to specify order of the mesh elements.
- `'box'` : `d`-by-2 array of double

## 4.2 Examples

Listing 17: : Computational times of `OrthMesh` constructor in dimension `d=3` where mesh elements are 1-order simplices

```
fc_hypermesh.bench(3, 25:25:175, 'box', [-1 1; -1 1; -1 1], 'type', 'simplex')
```

Output

```
# BENCH in dimension 3 with simplex mesh
#d: 3
#order: 1
#type: simplex
#box: [-1 1; -1 1; -1 1]
#desc:  N      nq      nme time(s)
      25    17576    93750  0.447
      50   132651   750000  0.135
      75   438976   2531250  0.220
     100  1030301   6000000  0.394
     125  2000376  11718750  0.732
     150  3442951  20250000  1.191
     175  5451776  32156250  2.262
```

Listing 18: : Computaionnal times of `OrthMesh` constructor in dimension `d=3` where mesh elements are 3-order simplices

```
fc_hypermesh.bench(3,10:10:100,'box',[-1 1;-1 1;-1 1], ...
    'type','simplex','order',3)
```

Output

```
# BENCH in dimension 3 with simplex mesh
#d: 3
#order: 3
#type: simplex
#box: [-1 1; -1 1; -1 1]
#desc:  N      nq      nme time(s)
        10     29791     6000  0.459
        20    226981    48000  0.110
        30    753571   162000  0.131
        40   1771561   384000  0.212
        50   3442951   750000  0.354
        60   5929741  1296000  0.560
        70   9393931  2058000  0.853
        80  13997521  3072000  1.237
        90  19902511  4374000  1.722
       100  27270901  6000000  2.354
```

Listing 19: : Computaionnal times of `OrthMesh` constructor in dimension `d=5` where mesh elements are 1-order orthotopes

```
fc_hypermesh.bench(5,[5:5:25,27],'type','orthotope','box',[-1 1;-1 ...
    1;-1 1;-1 1;-1 1])
```

Output

```
# BENCH in dimension 5 with orthotope mesh
#d: 5
#order: 1
#type: orthotope
#box: [-1 1; -1 1; -1 1; -1 1; -1 1]
#desc:  N      nq      nme time(s)
        5     7776     3125  0.641
       10    161051   100000  0.350
       15   1048576   759375  0.469
       20   4084101  3200000  0.793
       25  11881376  9765625  1.610
       27  17210368 14348907  2.187
```

Listing 20: : Computaionnal times of `OrthMesh` constructor in dimension `d=5` where mesh elements are 3-order orthotopes

```
fc_hypermesh.bench(5,2:2:12,'type','orthotope','order',3,'box',[-1 ...
    1;-1 1;-1 1;-1 1;-1 1])
```

Output

```
# BENCH in dimension 5 with orthotope mesh
#d: 5
#order: 3
#type: orthotope
#box: [-1 1; -1 1; -1 1; -1 1; -1 1]
#desc:  N      nq      nme time(s)
        2    16807      32  0.644
        4   371293    1024  0.337
        6   2476099    7776  0.529
        8   9765625   32768  0.989
       10  28629151  100000  2.268
       12  69343957  248832  4.755
```

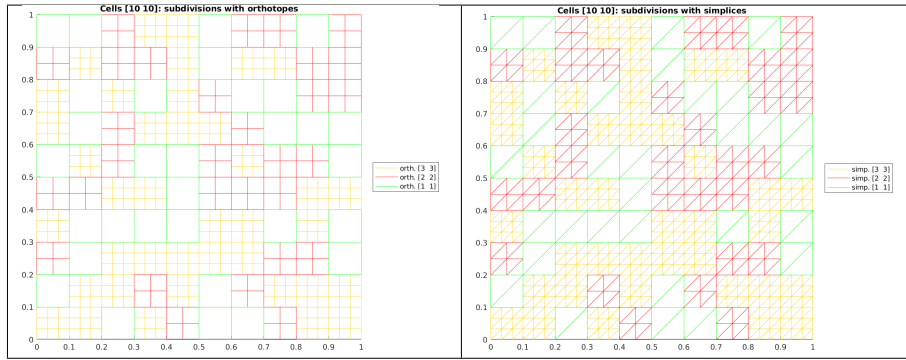


Figure 9: regular refinement of a 2D-grid with orthotopes (left) and simplices (right)

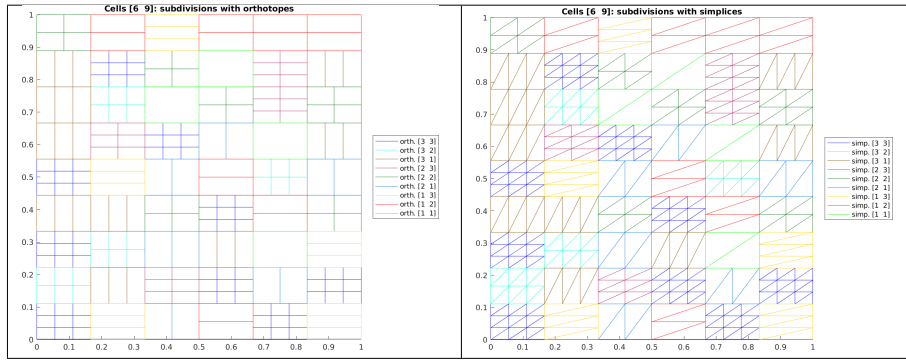


Figure 10: no regular refinement of a 2D-grid with orthotopes (left) and simplices (right)

## 5 Mesh refinement

### 5.1 Non-conforming mesh refinement

In this part we propose a preliminary refinement of a regular  $d$ -grid. We want to generate a refinement of some cells of this regular grid with  $d$ -simplices and/or  $d$ -orthotopes: the mesh obtained is therefore non-conforming.

In Figure 9, a 2D-grid with 10-by-10 cells is refined with regular subcells (i.e. same number of discretisation in each directory) made of simplices or orthotopes. For example a cell subdivided in 3-by-3 orthotopes is denoted by `orth[3 3]` and this same cell subdivided in 3-by-3 simplices (in fact  $3 \times 3 \times 2$  simplices) is denoted by `simp[3 3]`. In Figure 10, a 2D-grid with 10-by-10 cells (left) and with 6-by-9 cells (right) is refined with subcells (not necessarily with same number of discretisation in each directory). In Figure 11, a 2D-grid with 10-by-10 cells (left) is refined with regular subcells made of orthotopes or simplices. On the right, a 2D-grid with 6-by-9 cells is is refined with not necessarily regular subcells made of orthotopes or simplices refined with subcells respectively.

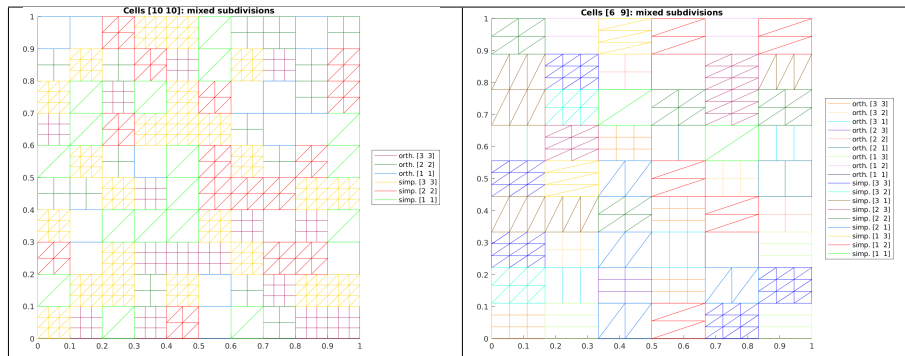


Figure 11: Regular (left) and not regular (right) refinement of a 2D-grid with orthotopes and simplices

### 5.1.1 Refinement function

The refinement of an `OrthMesh` object whose elements are orthotopes is obtained by using the `fc_hypermesh.refinement.refine` Matlabfunction.

```

scs = refine(Oh, ndiscells)
scs = refine(Oh, ndiscells, type)
scs = refine(Oh, ndiscells, types)

```

This function returns cells array of structures. Each array entry contains all cells refined with a `type` of element (simplicial or orthotope) and selected numbers of discretisation in each axis direction. Each structure (called *subcells structure*) has the fields

- `d` : space dimension
- `type` : `type` of element: `'simplicial'` or `'orthotope'`
- `n` : 1-by-`d` array corresponding to selected numbers of discretisation in each axis direction
- `nq` : number of vertices
- `nme` : number of mesh elements
- `q` : `d`-by-`nq` vertices array
- `me` : `p`-by-`nme` connectivity array ( $p = d + 1$  for simplices and  $p = 2^d$  for orthotopes)
- `ncell` : contains all the number of the cells grid which are refined.

### Description

```
scs = fc_hypermesh.refinement.refine(Oh,ndiscells)
```

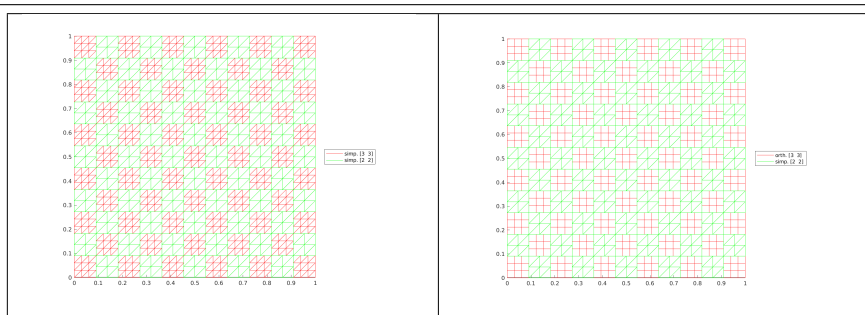
The first input `Oh` is an `OrthMesh` object whose elements are orthotopes. `ndiscells` is an 1-by-`Oh.Mesh.nme` array (for regular refinement) or a `d`-by-`Oh.Mesh.nme` array. `ndiscells(:,k)` are the numbers of discretisation in each axis direction. The output is a cells array of *subcells structure* where all subcells `type` are `'orthotope'`

```
scs = fc_hypermesh.refinement.refine(Oh,ndiscells,type)
```

Same as previous one if `type` is `'orthotope'`. Otherwise `type` is `'simplicial'` and all subcells `type` of the output are `'simplicial'`

```
[scsimp,scorth] = fc_hypermesh.refinement.refine(Oh,ndiscells,types)
```

The input `types` is an 1-by-`Oh.Mesh.nme` array: if `types(k)` is equal to 0 then the  $k$ -th mesh element (cell) of the `OrthMesh` is refined with simplices otherwise with orthotopes.



```
d=2; % space dimension
N=11; % or N=[10,10]
Oh=fc_hypermesh.OrthMesh(d,N,'type','orthotope');
nme=Oh.Mesh.nme; % Number of Mesh elements
ndiscells=2*ones(1,nme);
ndiscells(1:2:nme)=3;
sc1=fc_hypermesh.refinement.refine(Oh,ndiscells,'simplex');
figure(1)
fc_hypermesh.refinement.plotsubcells(sc1)
set(legend(),'Location','eastoutside');axis image

types=zeros(1,nme);ndiscells=2*ones(1,nme);
types(1:2:nme)=1;ndiscells(1:2:nme)=3;
sc2=fc_hypermesh.refinement.refine(Oh,ndiscells,types);
figure(2)
fc_hypermesh.refinement.plotsubcells(sc2)
set(legend(),'Location','eastoutside');axis image
```

Listing 21: 2D orthotope `OrthMesh` refinement with Matlab 2019a, figure 1 (left) and figure 2 (right)

More examples are provided by scripts:

- `fc_hypermesh.refinement.demo01`
- ...
- `fc_hypermesh.refinement.demo10`



# Appendices

## A Memory consuming

Beware when using these codes of memory consuming : the number of points  $nq$  and the number of elements quickly increase according to the space dimension  $d$ . When meshing a  $d$ -orthotope with  $p$ -order mesh elements by taking  $N$  intervals in each space direction, we have a number of nodes given by

$$nq = (p * N + 1)^d, \text{ for both tessellation and triangulation}$$

and the number of mesh elements is

$$\begin{aligned} nme &= N^d, & \text{for tessellation by orthotopes} \\ nme &= d!N^d, & \text{for tessellation by simplices.} \end{aligned}$$

If the array  $q$  is stored as *double* (8 bytes) then

$$\text{mem. size of } q = d \times nq \times 8 \text{ bytes}$$

and if the array  $me$  as *int* (4 bytes) then

$$\text{mem. size of } me = \begin{cases} (p + 1)^d \times nme \times 4 \text{ bytes} & \text{(tessellation by orthotopes)} \\ \frac{(d+p)!}{d!p!} \times nme \times 4 \text{ bytes} & \text{(tessellation by simplices)} \end{cases}$$

With  $N = 10$ ,  $d \in \llbracket 1, 8 \rrbracket$  and order  $p = 1$ , the values of  $nq$  and  $nme$  are given in Table 5. The memory usage for the corresponding array  $q$  and array  $me$  is available in Table 6. In Table ??, the memory usage for the array  $q$  and array  $me$  for  $N = 10$ ,  $d = 3$  and order  $p \in \llbracket 1, 8 \rrbracket$ .

$d$	$nq = (N + 1)^d$	$nme = N^d$ (orthotopes)	$nme = d!N^d$ (simplices)
1	11	10	10
2	121	100	200
3	1 331	1 000	6 000
4	14 641	10 000	240 000
5	161 051	100 000	12 000 000
6	1 771 561	1 000 000	720 000 000
7	19 487 171	10 000 000	50 400 000 000
8	214 358 881	100 000 000	4 032 000 000 000

Table 5: Number of vertices  $nq$  and number of elements  $nme$  for the tessellation of an orthotope by orthotopes and by simplices according to the space dimension  $d$  and with  $N = 10$ .

<b>d</b>	<b>q</b>	<b>me</b> (orthotopes)	<b>me</b> (simplices)
1	88 B	80 B	80 B
2	1 KB	1 KB	2 KB
3	31 KB	32 KB	96 KB
4	468 KB	640 KB	4 MB
5	6 MB	12 MB	288 MB
6	85 MB	256 MB	20 GB
7	1 GB	5 GB	1 TB
8	13 GB	102 GB	145 TB

Table 6: Memory usage of the array **q** and the array **me** for the tessellation of an orthotope by 1-order orthotopes and by 1-order simplices according to the space dimension **d** and with  $N = 10$ .

<b>d</b>	<b>q</b>	<b>me</b> (orthotopes)	<b>me</b> (simplices)
1	248 B	160 B	160 B
2	15 KB	6 KB	8 KB
3	714 KB	256 KB	480 KB
4	29 MB	10 MB	33 MB
5	1 GB	409 MB	2 GB
6	42 GB	16 GB	241 GB
7	1 TB	655 GB	24 TB
8	54 TB	26 TB	2 661 TB

Table 7: Memory usage of the array **q** and the array **me** for the tessellation of an orthotope by 3-order orthotopes and by 3-order simplices according to the space dimension **d** and with  $N = 10$ .

## B Low level codes

### B.1 Class object `EltMesh`

The elementary mesh class object called `EltMesh` is used to store only one mesh, the main mesh as well as any of the **m**-face meshes. This class `EltMesh` also simplifies (for me) the codes writing and its fields are the following:

- **d**, space dimension
- **m**, kind of mesh ( $m = d$  for the main mesh), tessellated with **m**-simplices or **m**-orthotopes.
- **order**, order **p** of the mesh elements
- **type**, 0 for simplicial mesh or 1 for orthotope mesh
- **nq**, number of nodes
- **q**, nodes array of dimension **d**-by-**nq**
- **nme**, number of mesh elements

- `me`, connectivity array of dimension  $\frac{(d+p)!}{d!p!}$ -by-`nme` for `p`-order simplices elements or  $(p+1)^d$ -by-`nme` for `p`-order orthotopes elements
- `toGlobal`, index array linking local array `q` to the one of the main mesh
- `label`, name/number of this elementary mesh
- `color`, color of this elementary mesh (for plotting purpose)

### B.1.1 Constructor

```
Eh = fc_hypermesh.EltMesh(m,q,me)
Eh = fc_hypermesh.EltMesh(m,q,me, ...
    key,value,...)
```

### Description

```
Eh = fc_hypermesh.EltMesh(m,q,me)
```

Generates the `EltMesh` object `Eh` which contains a 1-order `m`-simplicial mesh by default. The space dimension `d` is `size(q,1)`.

- `m`, kind of mesh: tessellated with `m`-simplices.
- `q` is the nodes array of dimension `d`-by-`nq`.
- `me` is the connectivity array of dimension  $(d+1)$ -by-`nme` corresponding to 1-order simplices elements.

Listing 22: `EltMesh` constructor in dimension `d=2` (simplicial mesh)

---

```
[q,me]=fc_hypermesh.CG.TessSim(10^3*ones(1,2));
Eh=fc_hypermesh.EltMesh(2,q,me)
```

---

Output

```
Eh =
EltMesh object
  type : simplex
  order : 1
    d : 2
    m : 2
    q : (2,1002001)
    me : (3,2000000)
```

```
Eh = fc_hypermesh.EltMesh(m,q,me, key,value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'type'` : used to select the kind of elements used. The default `value` is `'simplex'` and otherwise `'orthotope'` can be used.
- `'order'` : gives the order of the mesh elements (default is 1).

Listing 23: `EltMesh` constructor in dimension `d=3` (orthotope mesh)

```
[q,me]=fc_hypermesh.CG.TessHyp(3:5,2);  
Eh=fc_hypermesh.EltMesh(3,q,me,'type','orthotope','order',2)
```

Output

```
Eh =  
  
EltMesh object  
  type : orthotope  
  order : 2  
  d : 3  
  m : 3  
  q : (3,693)  
  me : (27,60)
```

### B.1.2 `plotmesh` method

The `plotmesh()` member function can be used to represent the mesh given by an `EltMesh` object if the space dimension `d` is less than or equal to 3.

#### Syntaxe

```
obj.plotmesh()  
obj.plotmesh(key, value, ...)
```

#### Description

```
obj.plotmesh()
```

```
obj.plotmesh(key, value, ...)
```

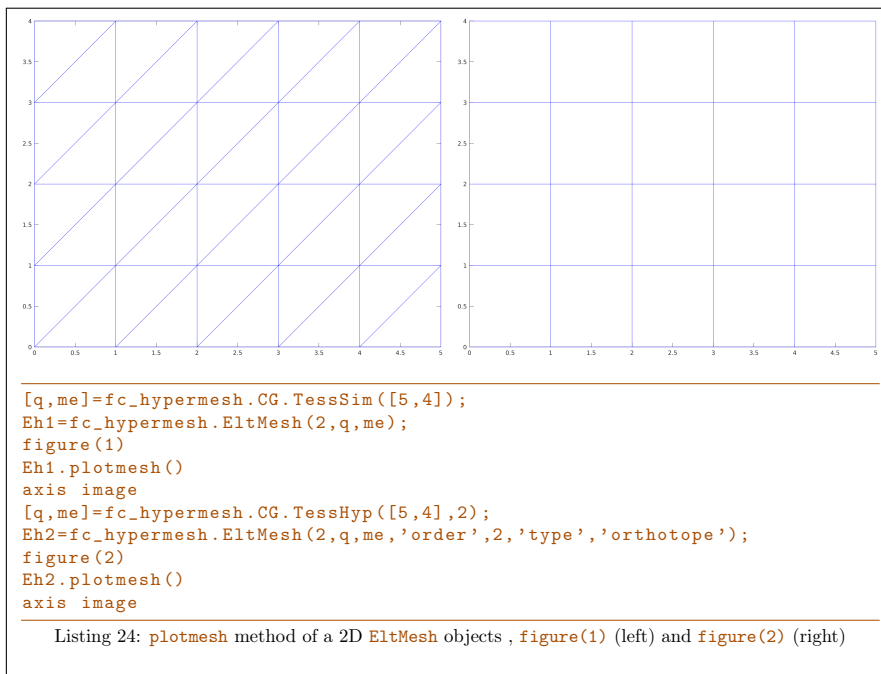
Some optional `key/value` pairs arguments are available with `key`:

- `'color'` : use to specify the color of the mesh. Otherwise `obj.color` is used.

Other `key/value` pairs arguments can be used depending of `obj.d` and `obj.m` values and they are those of the plotting function used:

- with `obj.d=3` and `obj.m=3`, `patch` function is used;
- with `obj.d=3` and `obj.m=2`, `trimesh` function is used for simplicial mesh and `patch` function is used for orthotope mesh;
- with `obj.d=3` and `obj.m=1`, `line` function is used;
- with `obj.d=3` and `obj.m=0`, `scatter3` function is used;
- with `obj.d=2` and `obj.m=2`, `triplot` function is used for simplicial mesh and `patch` function is used for orthotope mesh;
- with `obj.d=2` and `obj.m=1`, `line` function is used;
- with `obj.d=2` and `obj.m=0`, `scatter` function is used;

- with `obj.d=1` and `obj.m=1`, `line` function is used;
- with `obj.d=1` and `obj.m=0`, `scatter` function is used;



### B.1.3 plotnodes method

The `plotnodes()` member function can be used to represent nodes of the mesh given by an `EltMesh` object if the space dimension `d` is less than or equal to 3.

#### Syntaxe

```

obj.plotnodes()
obj.plotnodes(key, value, ...)

```

#### Description

```
obj.plotnodes()
```

Uses `scatter` function (1D and 2D) and `scatter3` function (3D) to represent nodes of the mesh as points. Vertices of the mesh elements are also nodes and they are distinguishable from others nodes.

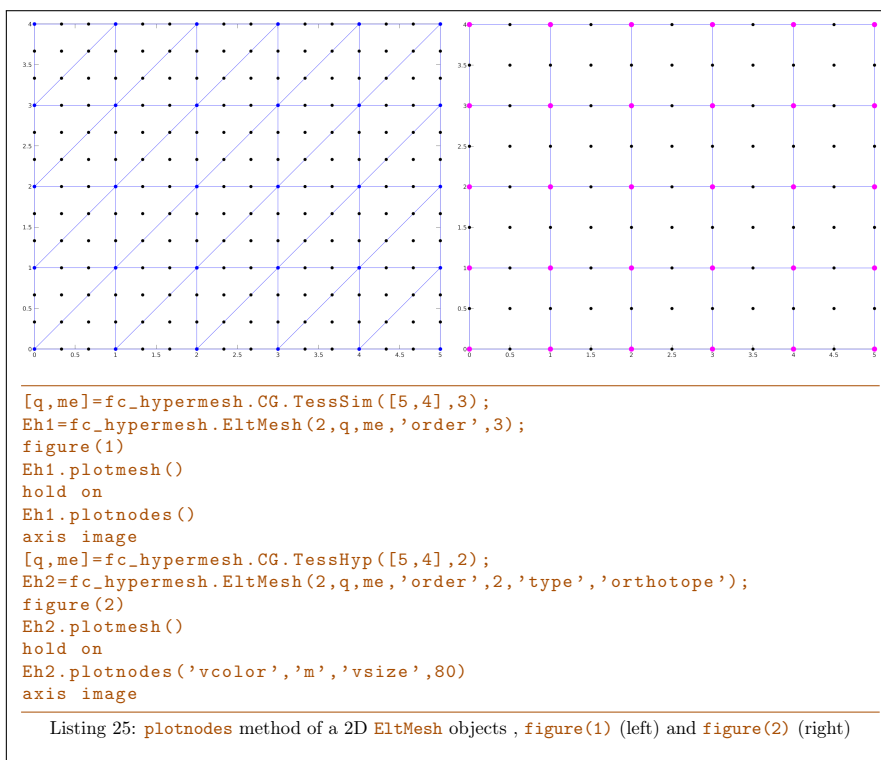
```
obj.plotnodes(key, value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'vcolor'`: use to specify the point color for the mesh vertices. Default is `obj.color`.

- `'vsize'` : use to specify the point size for the mesh vertices. Default is 40 .
- `'ncolor'` : use to specify the color of the nodes (not vertices) of the mesh elements. Default is `'k'` (ie. black).
- `'nsize'` : use to specify the size of the nodes (not vertices) of the mesh elements. Default is 30 .

Other `key/value` pairs arguments can be used: they are those of the `scatter` and `scatter3` function used.



#### B.1.4 `plotnodesNumber` method

The `plotnodesNumber()` member function can be used to display node numbers of the mesh given by an `EltMesh` object if the space dimension `d` is less than or equal to 3.

#### Syntaxe

```

obj.plotnodesNumber()
obj.plotnodesNumber(key, value, ...)

```

#### Description

```
obj.plotnodes()
```

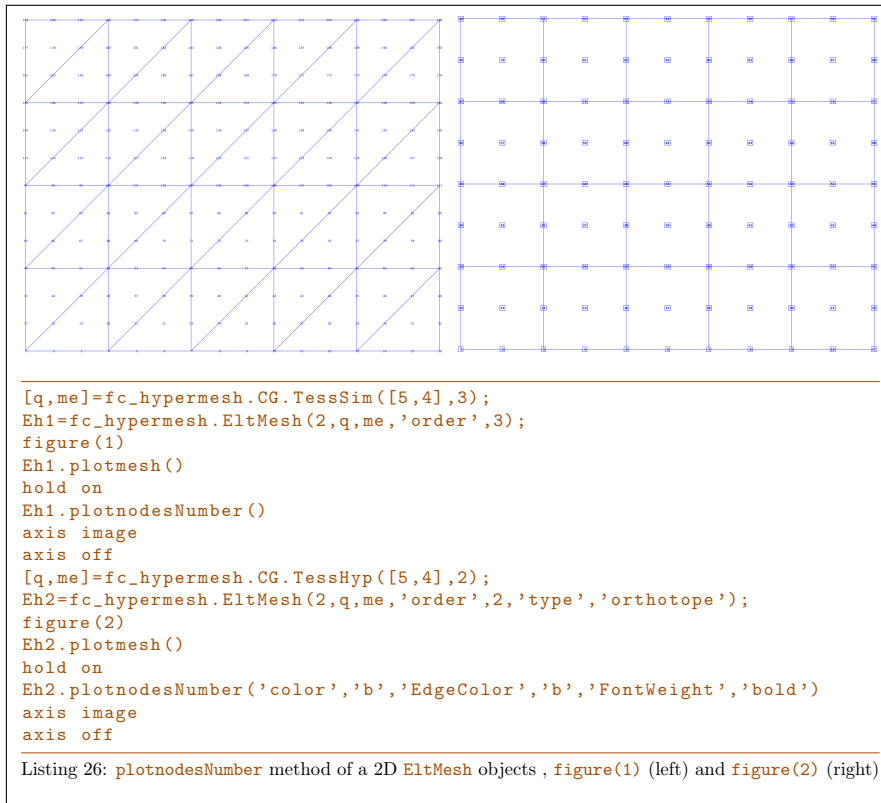
Uses `fc_hypermesh.plotnodesNumber` function to represent node numbers.

```
obj.plotnodes(key, value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'BackgroundColor'` : use to specify text background color. Default is `'w'` (i.e. white).
- `'margin'` : use to specify the margin. Default is `1` .
- `'HorizontalAlignment'` : Default is `'center'` .
- `'VerticalAlignment'` : Default is `'middle'` .
- `'clipping'` : Default is `'on'` .

Other `key/value` pairs arguments can be used: they are those of the `text` function used.



### B.1.5 `ploteltsNumber` method

The `ploteltsNumber()` member function can be used to display element numbers of the mesh given by an `EltMesh` object if the space dimension `d` is less than or equal to 3.

## Syntaxe

```
obj.ploteltsNumber()  
obj.ploteltsNumber(key, value, ...)
```

## Description

```
obj.ploteltsNumber()
```

Uses `fc_hypermesh.ploteElementsNumber` function to represent node numbers.

```
obj.ploteltsNumber(key, value, ...)
```

Some optional `key/value` pairs arguments are available with `key`:

- `'Color'` : use to specify text color. Default is `'r'` (i.e. red).
- `'BackgroundColor'` : use to specify text background color. Default is `'w'` (i.e. white).
- `'margin'` : use to specify the margin. Default is `1`.
- `'HorizontalAlignment'` : Default is `'center'`.
- `'VerticalAlignment'` : Default is `'middle'`.
- `'clipping'` : Default is `'on'`.
- `'vLineColor'` : Draw lines between vertices and barycenter of the mesh elements. Default is `''` (no lines).
- `'vLineStyle'` : Select lines type. Default is `':'` (dotted lines).
- `'vLineWidth'` : Set lines width. Default is `0.5`.

Other `key/value` pairs arguments can be used: they are those of the `text` function used.





## H References

- [1] F. Cuvelier. Vectorized algorithms for regular and conforming tessellations of d-orthotopes and their faces with high-order orthotopes or simplicial elements, March 2019. working paper or preprint.