FC_HYPERMESH package, User's Guide *

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Abstract

This object-oriented Python package allows to mesh any d-orthotopes (hyperrectangle in dimension d) and their m-faces by simplices or orthotopes. It was created to show the implementation of the algorithms of [1]. The FC_HYPERMESH package uses Python objects and is provided with meshes visualisation tools for dimension leather or equal to 3.

Contents

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*Compiled with Python 3.6.0, packages FC_HYPERMESH-0.0.7 and `fc_tools`-dev

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1 Introduction

The FC_HYPERMESH package contains a simple class object `OrthMesh` which permits, in any dimension $\mathrm{d} \geqslant 1$, to obtain a simplicial mesh or orthotope mesh with all their m-faces, $0 \leqslant m < \mathrm{d}$, of a $\mathrm{d} - \textit{orthotope}$. If the matplotlib package is installed, it is also possible with the method function `plotmesh` of the class object `OrthMesh` to represent a mesh or its m-faces for $\mathrm{d} \leqslant 3$.

In the folowing section, the class object `OrthMesh` is presented. Thereafter some warning statements on the memory used by these objects in high dimension are given. Finally computation times for orthotope meshes and simplicial meshes are given in dimension $\mathrm{d} \in [\![1, 5]\!]$.

2 Contents of the fc_hypermesh package

2.1 Class object `EltMesh`

An elementary mesh class object `EltMesh` is used to store only one mesh, the main mesh as well as any of the meshes of the m-faces. This class `EltMesh` also simplify (for me) the codes writing. Its attributes are the following:

- d : space dimension
- m : kind of mesh corresponding to a m-face, 0<=m<=d, m == d for the main mesh.
- **type** : 0 for simplicial mesh or 1 for orthotope mesh.
- nq : number of vertices.
- q : vertices numpy array of dimension d-by-nq
- nme : number of mesh elements
- me : connectivity numpy array of dimension (d+1)-by-nme for simplices elements or 2^d-by-nme for orthotopes elements
- toGlobal : index array linking local array q to the one of the main mesh.
- label : name/number of this elementary mesh
- color : color of this elementary mesh (for plotting purpose)

2.2 Class object `OrthMesh`

The aim of the class object **`OrthMesh`** is to efficiently create an object which contains a mesh of a d-orthotope and all its m-face meshes.

Let the d-orthotope defined by $[a_1, b_1] \times \cdots \times [a_{\mathrm{d}}, b_{\mathrm{d}}]$. The class object **`OrthMesh`** corresponding to this d-orthotope contains the main mesh and all the meshes of its m-faces, $0 \leqslant m < \mathrm{d}$. Its attributes are the following

- d: space dimension
- **`type`**: string 'simplicial' or 'orthotope' mesh
- Mesh: main mesh as an **`EltMesh`** object
- Faces: 2d-list of **`EltMesh`** objects such that Faces[0] is a list of all the meshes of the $(\mathrm{d}-1)$-faces, Faces[1] is a list of all the meshes of the $(\mathrm{d}-2)$-faces, and so on
- box: a d-by-2 numpy array such that box[i−1][0] is a_i value and box[i−1][1] is b_i value.

2.2.1 Constructor

The **`OrthMesh`** constructor is :

$$\mathrm{Oh} = \mathrm{OrthMesh(d,N)}$$

where N is either a 1-by-d array/list such that N[i−1] is the number of discretization for $[a_i, b_i]$ or either an integer if the the number of discretization is the same in all space directions.
Some options are proposed with the constructor:

$$\mathrm{Oh} = \mathrm{OrthMesh(d,N,key{=}value)}$$

- box = value : where value is a d-by-2 list or array such that value[i−1][0] is a_i value and value[i−1][1] is b_i value. Default is $[0, 1]^{\mathrm{d}}$.
- **`type = value`** : The default value for optional key parameter **`type`** is 'simplicial' and ortherwise 'orthotope' can be used.

2.2.2 plotmesh property

$$\mathrm{plotmesh()}$$

Used matplotlib to represents the mesh represented by an **`OrthMesh`** object.
Some options are proposed with this property:

$$\mathrm{plotmesh(key{=}value)}$$

- legend = value : if value is True, a legend is displayed. Default is False.
- m = value : plots all the m-faces of the mesh. Default m = d i.e. the main mesh. (0 ⩽m⩽d)
- ...

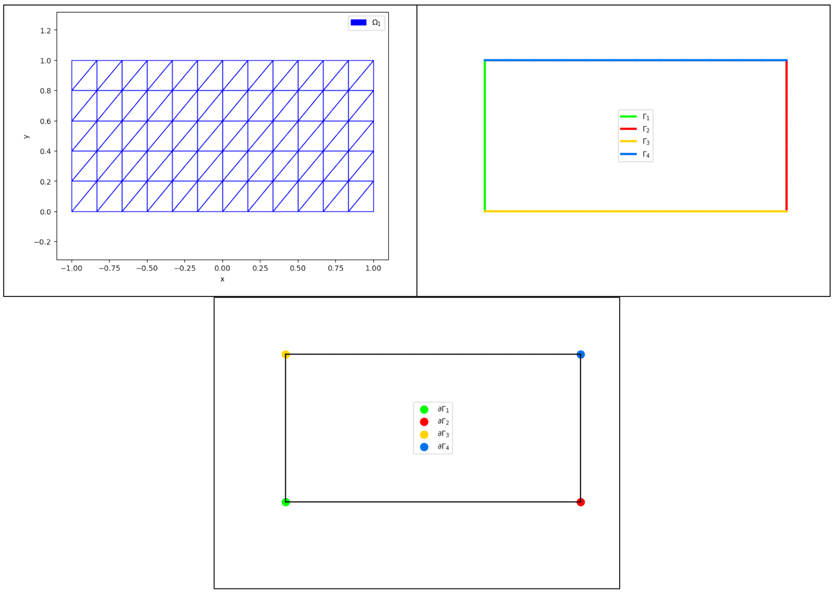
3 Using the fc_hypermesh package

In all the next examples, the following code is previously load:

```
import matplotlib.pyplot as plt
from fc_tools.colors import str2rgb
from fc_hypermesh.OrthMesh import OrthMesh
from fc_tools.graphics import DisplayFigures,set_axes_equal
```

3.1 2d-orthotope meshing by simplices

In Listing 1, an `OrthMesh` object is built under Python for the orthotope $[-1,1] \times [0,1]$ with simplicial elements and $\mathbf{N} = (12, 5)$. The main mesh and all the m-face meshes of the resulting object are plotted.

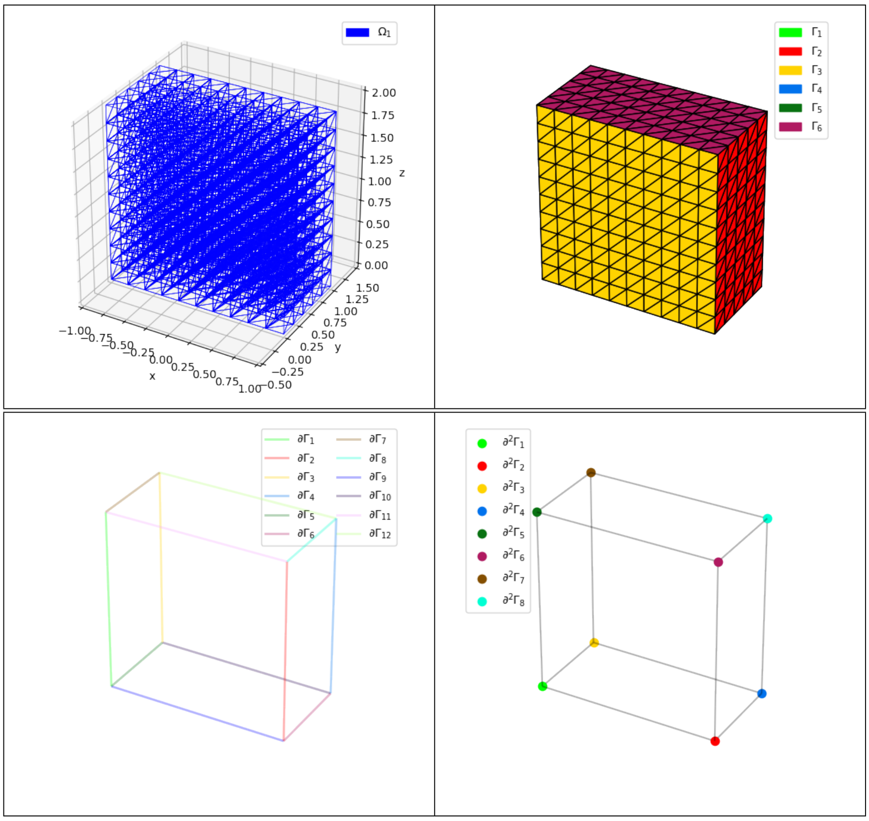



```
oTh=OrthMesh(2,[12,5],type='simplicial',box=[[-1,1],[0,1]])
plt.figure(1)
oTh.plotmesh(legend=True)
set_axes_equal()
plt.figure(2)
oTh.plotmesh(m=1,legend=True,linewidth=3)
plt.axis('off')
set_axes_equal()
plt.figure(3)
oTh.plotmesh(m=1,color='black')
oTh.plotmesh(m=0,legend=True,s=105)
plt.axis('off')
set_axes_equal()
DisplayFigures()
```

Listing 1: 2D simplicial `OrthMesh` object with Python 3.6.0, main mesh (upper left), 1-face meshes (upper right), and 0-face meshes (bottom)

3.2 3d-orthotope meshing by simplices

In Listing 2, an `OrthMesh` object is built under Python for the orthotope $[-1,1] \times [0,1] \times [0,2]$ with simplicial elements and $\mathbf{N} = (10,5,10)$. The main mesh and all the m-face meshes of the resulting object are plotted.

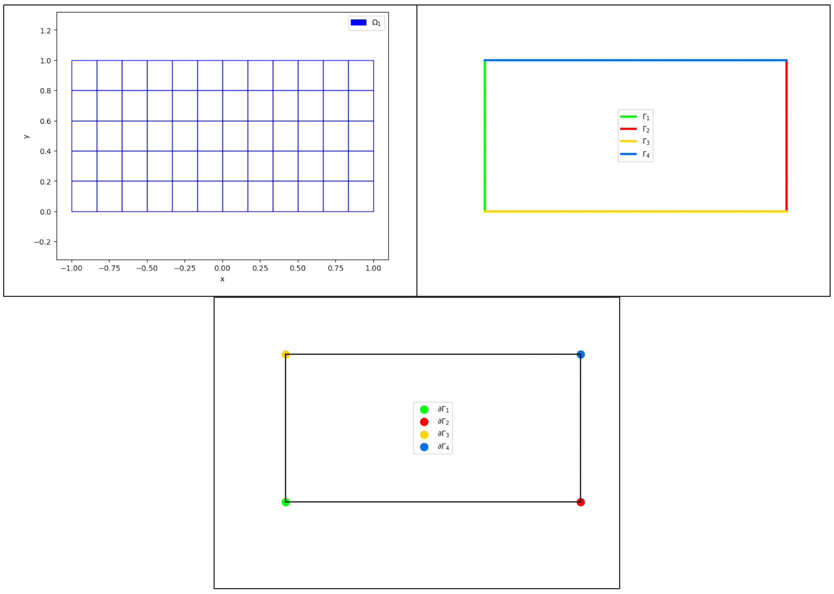



```
oTh=OrthMesh(3,[10,5,10],box=[[-1,1],[0,1],[0,2]])
plt.figure(1)
oTh.plotmesh(legend=True,linewidth=0.5)
set_axes_equal()
plt.figure(2)
oTh.plotmesh(m=2,legend=True,edgecolor=[0,0,0])
plt.axis('off')
set_axes_equal()
plt.figure(3)
oTh.plotmesh(m=2,edgecolor=[0,0,0],color='none')
oTh.plotmesh(m=1,legend=True,linewidth=2,alpha=0.3)
plt.axis('off')
set_axes_equal()
plt.figure(4)
oTh.plotmesh(m=1,color='black',alpha=0.3)
oTh.plotmesh(m=0,legend=True,s=55)
set_axes_equal()
plt.axis('off')
```

Listing 2: 3D simplicial `OrthMesh` object with Python 3.6.0, main mesh (upper left), 2-face meshes (upper right), 1-face meshes (bottom left) and 0-face meshes (bottom right)

3.3 2d-orthotope meshing by orthotopes

In Listing 3, an `OrthMesh` object is built under Python for the orthotope $[-1, 1] \times [0, 1]$ with orthotope elements and $\mathbf{N} = (10, 5, 10)$. The main mesh and all the m-face meshes of the resulting object are plotted.

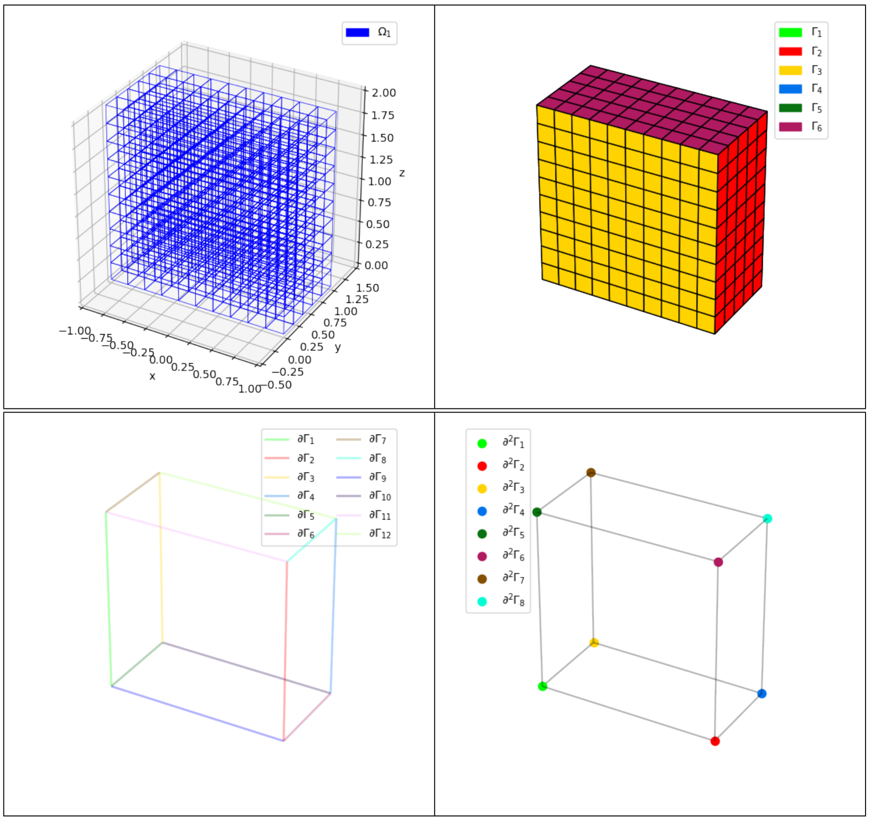



```
oTh=OrthMesh(2,[12,5],type='orthotope',box=[[-1,1],[0,1]])
plt.figure(1)
oTh.plotmesh(legend=True)
set_axes_equal()
plt.figure(2)
oTh.plotmesh(m=1,legend=True,linewidth=3)
plt.axis('off')
set_axes_equal()
plt.figure(3)
oTh.plotmesh(m=1,color='black')
oTh.plotmesh(m=0,legend=True,s=105)
plt.axis('off')
set_axes_equal()
```

Listing 3: 2D orthotope `OrthMesh` object with Python 3.6.0, main mesh (upper left), 1-face meshes (upper right), and 0-face meshes (bottom)

3.4 3d-orthotope meshing by orthotopes

In Listing 4, an `OrthMesh` object is built under Python for the orthotope $[-1, 1] \times [0, 1] \times [0, 2]$ with orthotope elements and $\mathbf{N} = (10, 5, 10)$. The main mesh and all the m-face meshes of the resulting object are plotted.



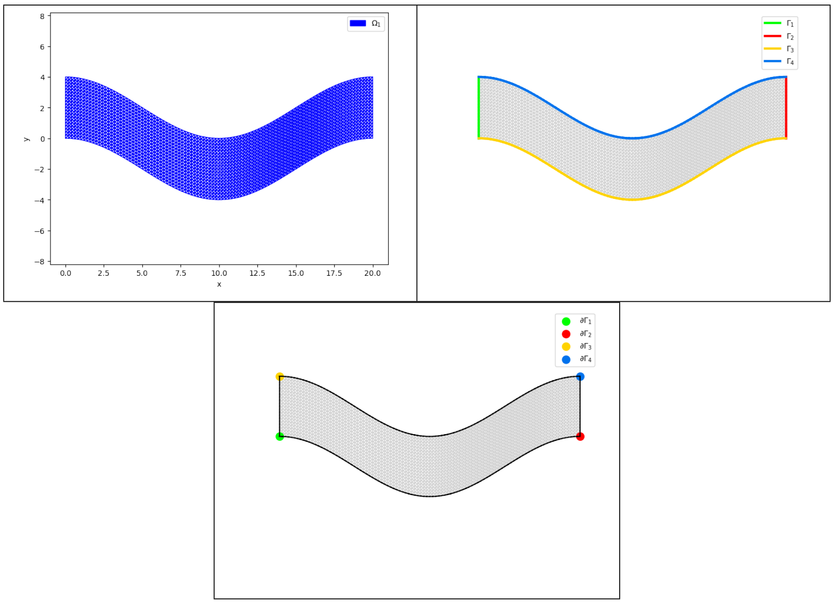

```python
oTh=OrthMesh(3,[10,5,10],type='orthotope',
      box=[[-1,1],[0,1],[0,2]])
plt.figure(1)
oTh.plotmesh(legend=True,linewidth=0.5)
set_axes_equal()
plt.figure(2)
oTh.plotmesh(m=2,legend=True,edgecolor=[0,0,0])
plt.axis('off')
set_axes_equal()
plt.figure(3)
oTh.plotmesh(m=2,edgecolor=[0,0,0],color='none')
oTh.plotmesh(m=1,legend=True,linewidth=2,alpha=0.3)
plt.axis('off')
set_axes_equal()
plt.figure(4)
oTh.plotmesh(m=1,color='black',alpha=0.3)
oTh.plotmesh(m=0,legend=True,s=55)
set_axes_equal()
plt.axis('off')
```

Listing 4: 3D orthotope `OrthMesh` object with Python 3.6.0, main mesh (upper left), 2-face meshes (upper right), 1-face meshes (bottom left) and 0-face meshes (bottom right)

3.5 Mapping of a 2d-orthotope meshing by simplices

For example, the following 2D geometrical transformation allows to deform the reference unit hypercube.

$$\begin{array}{ccc} [0,1]\times[0,1] & \longrightarrow & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longrightarrow & F(x,y) = \begin{pmatrix} 20\,x \\ 2\,(2\,y - 1 + \cos(2\pi x)) \end{pmatrix} \end{array}$$



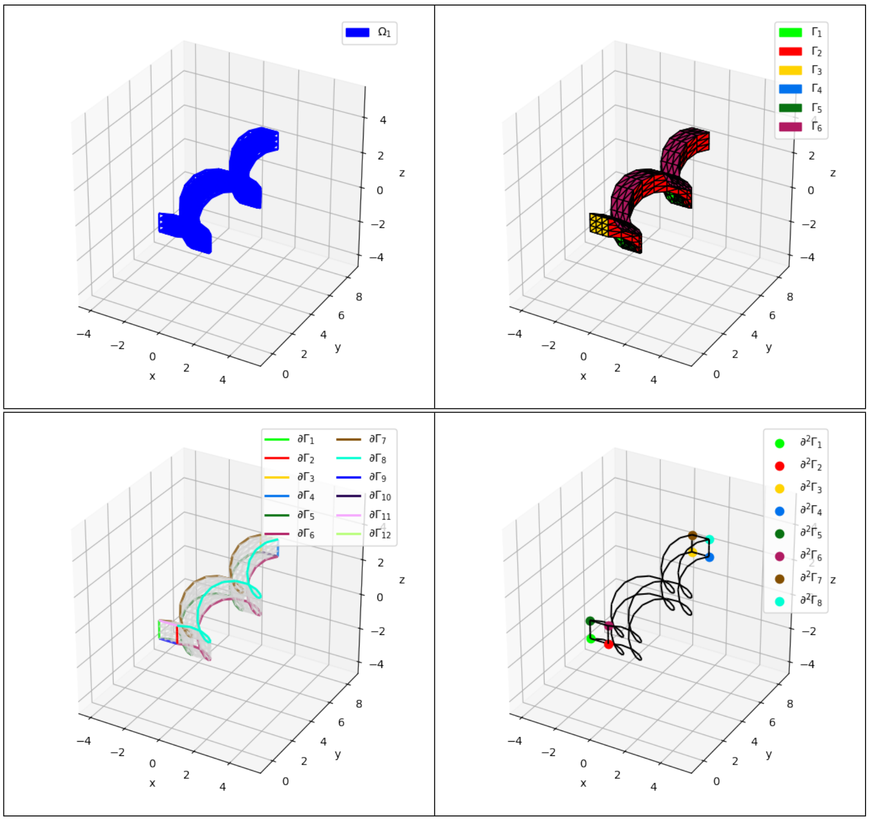
```python
import numpy as np
trans=lambda q:
    np.array([20*q[0],2*(2*q[1]-1+np.cos(2*np.pi*q[0]))])
oTh=OrthMesh(2,[100,20],type='simplicial',mapping=trans)
plt.figure(1)
oTh.plotmesh(legend=True)
plt.axis('equal')
plt.figure(2)
oTh.plotmesh(color='lightgray')
oTh.plotmesh(m=1,legend=True,linewidth=3)
plt.axis('equal')
plt.axis('off')
plt.figure(3)
oTh.plotmesh(color='lightgray')
oTh.plotmesh(m=1,color='black')
oTh.plotmesh(m=0,legend=True,s=105)
plt.axis('equal')
plt.axis('off')
```

Listing 5: Mapping of a 2D simplicial `OrthMesh` object with Python 3.6.0, main mesh (upper left), 1-face meshes (upper right), and 0-face meshes (bottom)

3.6 3d-orthotope meshing by orthotopes

For example, the following 3D geometrical transformation allows to deform the reference unit hypercube.

$$
\begin{array}{rcl}
[0,1]\times[0,1]\times[0,1] & \longrightarrow & \mathbb{R}^2 \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} & \longrightarrow & F(x,y,y) = \begin{pmatrix} x+\sin(4\pi y) \\ 10y \\ z+\cos(4\pi y) \end{pmatrix}
\end{array}
$$




```
import numpy as np
trans=lambda q: np.array([q[0]+np.sin(4*np.pi*q[1]),
    10*q[1]-1, q[2]+np.cos(4*np.pi*q[1])])
oTh=OrthMesh(3,[3,25,3],type='simplicial',mapping=trans)
plt.figure(1)
oTh.plotmesh(legend=True)
set_axes_equal()
plt.figure(2)
oTh.plotmesh(m=2,legend=True,edgecolor=[0,0,0])
set_axes_equal()
plt.figure(3)
oTh.plotmesh(m=2,edgecolor='lightgray',facecolor=None,alpha=0.3)
oTh.plotmesh(m=1,legend=True,linewidth=2)
set_axes_equal()
plt.figure(4)
oTh.plotmesh(m=1,color='black')
oTh.plotmesh(m=0,legend=True,s=55)
set_axes_equal()
```

Listing 6: Mapping of a 3D orthotope `OrthMesh` object with Python 3.6.0, main mesh (upper left), 2-face meshes (upper right), 1-face meshes (bottom left) and 0-face meshes (bottom right)

4 Memory consuming

Take care when using theses codes with memory consuming : the number of points $\mathrm{n_q}$ and the number of elements increases exponentially according to the space dimension d. If $(N+1)$ points are taken in each space direction, we have

$$\mathrm{n_q} = (N+1)^d, \quad \text{for both tessellation and triangulation}$$

and

$$\begin{aligned} \mathrm{n_{me}} &= N^d, & \text{for tessellation by orthotopes} \\ \mathrm{n_{me}} &= d!N^d, & \text{for tessellation by simplices.} \end{aligned}$$

If the array q is stored as *double* (8 octets) then

$$\text{mem. size of q} = \mathrm{d} \times \mathrm{n_q} \times 8 \text{ octets}$$

and if the array me as *int* (4 octets) then

$$\text{mem. size of me} = \begin{cases} 2^{\mathrm{d}} \times \mathrm{n_{me}} \times 4 \text{ octets} & \text{(tessellation by orthotopes)} \\ (\mathrm{d}+1) \times \mathrm{n_{me}} \times 4 \text{ octets} & \text{(tessellation by simplices)} \end{cases}$$

For $N = 10$ and $\mathrm{d} \in [\![1, 8]\!]$, the values of $\mathrm{n_q}$ and $\mathrm{n_{me}}$ are given in Table 1. The memory usage for the corresponding array q and array me is available in Table 2.

d	$\mathrm{n_q} = (N+1)^{\mathrm{d}}$	$\mathrm{n_{me}} = N^{\mathrm{d}}$ (orthotopes)	$\mathrm{n_{me}} = \mathrm{d}!N^{\mathrm{d}}$ (simplices)
1	11	10	10
2	121	100	200
3	1 331	1 000	6 000
4	14 641	10 000	240 000
5	161 051	100 000	12 000 000
6	1 771 561	1 000 000	720 000 000
7	19 487 171	10 000 000	50 400 000 000
8	214 358 881	100 000 000	4 032 000 000 000

Table 1: Number of vertices $\mathrm{n_q}$ and number of elements $\mathrm{n_{me}}$ for the tessellation of an orthotope by orthotopes and by simplices according to the space dimension d and with $N = 10$.

d	q	me (orthotopes)	me (simplices)
1	88 o	80 o	80 o
2	1 ko	1 ko	2 ko
3	31 ko	32 ko	96 ko
4	468 ko	640 ko	4 Mo
5	6 Mo	12 Mo	288 Mo
6	85 Mo	256 Mo	20 Go
7	1 Go	5 Go	1 612 Go
8	13 Go	102 Go	145 152 Go

Table 2: Memory usage of the array q and the array me for the tessellation of an orthotope by orthotopes and by simplices according to the space dimension d and with $N = 10$.

5 Benchmarks

For all the following tables, the computational costs of the `OrthMesh` constructor are given for the orthotope $[-1,1]^{\mathrm{d}}$ under Python 3.6.0. The computations were done on a laptop with `Core i7-4800MQ` processor and 16Go of RAM under Ubuntu 14.04 LTS (64bits).

In the following pages, computational costs of the `OrthMesh` constructor will be done by using bench_gen function. As sample, we give an example with output. Thereafter, all the output will be presented in tabular form.

```python
from fc_hypermesh import bench_gen
bench_gen(3,'simplicial',[[-1,1],[-1,1],[-1,1]],
    range(20,170,20))
```

Listing 7: bench sample

Output

```
# BENCH in dimension 3 with simplicial mesh
#d: 3
#type: simplicial
#box: [[-1, 1], [-1, 1], [-1, 1]]
#desc:  N        nq        nme    time(s)
       20      9261      48000     0.237
       40     68921     384000     0.256
       60    226981    1296000     0.312
       80    531441    3072000     0.410
      100   1030301    6000000     0.537
      120   1771561   10368000     0.741
      140   2803221   16464000     1.044
      160   4173281   24576000     1.414
```

5.1 Tessellation by orthotopes

```python
from fc_hypermesh import  bench_gen
bench_gen(2,'orthotope',[[-1,1],[-1,1]],range(1000,6000,1000))
```

Listing 8: Tessellation of $[-1,1]^2$ by orthotopes

N	$\mathrm{n_q}$	$\mathrm{n_{me}}$	Python
1000	1 002 001	1 000 000	0.191
2000	4 004 001	4 000 000	0.292
3000	9 006 001	9 000 000	0.485
4000	16 008 001	16 000 000	0.764
5000	25 010 001	25 000 000	1.176

Table 3: Tessellation of $[-1,1]^2$ by orthotopes

```
from fc_hypermesh import bench_gen
bench_gen(3,'orthotope',[[-1,1],[-1,1],[-1,1]],
    range(50,400,50))
```

Listing 9: Tessellation of $[-1,1]^3$ by orthotopes

N	$\mathrm{n_q}$	$\mathrm{n_{me}}$	Python
50	132 651	125 000	0.227
100	1 030 301	1 000 000	0.298
150	3 442 951	3 375 000	0.461
200	8 120 601	8 000 000	0.76
250	15 813 251	15 625 000	1.245
300	27 270 901	27 000 000	2.107
350	43 243 551	42 875 000	3.215

Table 4: Tessellation of $[-1,1]^3$ by orthotopes

```
from fc_hypermesh import bench_gen
bench_gen(4,'orthotope',[[-1,1],[-1,1],[-1,1],[-1,1]],
    [10,20,30,40,50,62])
```

Listing 10: Tessellation of $[-1,1]^4$ by orthotopes

N	$\mathrm{n_q}$	$\mathrm{n_{me}}$	Python
10	14 641	10 000	0.326
20	194 481	160 000	0.336
30	923 521	810 000	0.439
40	2 825 761	2 560 000	0.684
50	6 765 201	6 250 000	1.135
62	15 752 961	14 776 336	2.007

Table 5: Tessellation of $[-1,1]^4$ by orthotopes

```
from fc_hypermesh import bench_gen
bench_gen(5,'orthotope',[[-1,1],[-1,1],[-1,1],[-1,1],[-1,1]],
    [5,10,15,20,25,27])
```

Listing 11: Tessellation of $[-1,1]^5$ by orthotopes

N	$\mathrm{n_q}$	$\mathrm{n_{me}}$	Python
5	7 776	3 125	0.587
10	161 051	100 000	0.608
15	1 048 576	759 375	0.878
20	4 084 101	3 200 000	1.378
25	11 881 376	9 765 625	2.895
27	17 210 368	14 348 907	3.863

Table 6: Tessellation of $[-1,1]^5$ by orthotopes

5.2 Tessellation by simplices

```
from fc_hypermesh import bench_gen
bench_gen(2,'simplicial',[[-1,1],[-1,1]],range(1000,6000,1000))
```

Listing 12: Tessellation of $[-1,1]^2$ by simplices

N	$\mathrm{n_q}$	$\mathrm{n_{me}}$	Python
1000	1 002 001	2 000 000	0.216
2000	4 004 001	8 000 000	0.424
3000	9 006 001	18 000 000	0.78
4000	16 008 001	32 000 000	1.262
5000	25 010 001	50 000 000	1.96

Table 7: Tessellation of $[-1,1]^2$ by simplices

```
from fc_hypermesh import bench_gen
bench_gen(3,'simplicial',[[-1,1],[-1,1],[-1,1]],
    range(40,190,20))
```

Listing 13: Tessellation of $[-1,1]^3$ by simplices

N	$\mathrm{n_q}$	$\mathrm{n_{me}}$	Python
40	68 921	384 000	0.238
60	226 981	1 296 000	0.308
80	531 441	3 072 000	0.409
100	1 030 301	6 000 000	0.541
120	1 771 561	10 368 000	0.739
140	2 803 221	16 464 000	1.075
160	4 173 281	24 576 000	1.524
180	5 929 741	34 992 000	1.954

Table 8: Tessellation of $[-1,1]^3$ by simplices

```
from fc_hypermesh import bench_gen
bench_gen(4,'simplicial',[[-1,1],[-1,1],[-1,1],[-1,1]],
    [10,20,25,30,35,40])
```

Listing 14: Tessellation of $[-1,1]^4$ by simplices

N	$\mathrm{n_q}$	$\mathrm{n_{me}}$	Python
10	14 641	240 000	0.392
20	194 481	3 840 000	0.674
25	456 976	9 375 000	1.212
30	923 521	19 440 000	1.986
35	1 679 616	36 015 000	3.284

Table 9: Tessellation of $[-1,1]^4$ by simplices

```
from fc_hypermesh import bench_gen
bench_gen(5,'simplicial',[[-1,1],[-1,1],[-1,1],[-1,1],[-1,1]],
    range(2,14,2))
```

Listing 15: Tessellation of $[-1, 1]^5$ by simplices

N	n_q	n_{me}	Python
2	243	3 840	0.492
4	3 125	122 880	0.486
6	16 807	933 120	0.624
8	59 049	3 932 160	0.801
10	161 051	12 000 000	1.492
12	371 293	29 859 840	3.313

Table 10: Tessellation of $[-1, 1]^5$ by simplices

References

[1] F. Cuvelier and G. Scarella. Vectorized algorithms for regular tessellations of d-orthotopes and their faces. http://www.math.univ-paris13.fr/ cuvelier/-docs/reports/HyperMesh/HyperMesh_0.0.4.pdf, 2016.