FC-VFEM$\mathbb{P}_1$ Matlab toolbox, User's Guide [1]

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Abstract

FC-VFEM$\mathbb{P}_1$ is an object-oriented Matlab toolbox dedicated to solve scalar or vector boundary value problem (BVP) by $\mathbb{P}^1$-Lagrange finite element methods in any space dimension. It integrates the FC-SIMESH toolbox which allows a great flexibility in graphical representations of the meshes and datas on the meshes.

This toolbox also contains the techniques of vectorization presented in [2] and extended in [1] and allows good performances when using finite elements methods.

Contents

Chapter 1

Generic Boundary Value Problems

The notations of [4] are employed in this section and extended to the vector case.

1.1 Scalar boundary value problem

Let Ω be a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$. The boundary of Ω is denoted by Γ.

We denote by $\mathcal{L}_{\mathbb{A},\boldsymbol{b},\boldsymbol{c},a_0} = \mathcal{L} : \mathrm{H}^2(\Omega) \longrightarrow L^2(\Omega)$ the second order linear differential operator acting on *scalar fields* defined, $\forall u \in \mathrm{H}^2(\Omega)$, by

$$\mathcal{L}_{\mathbb{A},\boldsymbol{b},\boldsymbol{c},a_0}(u) \stackrel{\text{def}}{=} -\operatorname{div}(\mathbb{A}\boldsymbol{\nabla} u) + \operatorname{div}(\boldsymbol{b}u) + \langle \boldsymbol{\nabla} u, \boldsymbol{c}\rangle + a_0 u \tag{1.1}$$

where $\mathbb{A} \in (L^\infty(\Omega))^{d\times d}$, $\boldsymbol{b} \in (L^\infty(\Omega))^d$, $\boldsymbol{c} \in (L^\infty(\Omega))^d$ and $a_0 \in L^\infty(\Omega)$ are given functions and $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^d$. We use the same notations as in the chapter 6 of [4] and we note that we can omit either $\operatorname{div}(\boldsymbol{b}u)$ or $\langle \boldsymbol{\nabla} u, \boldsymbol{c}\rangle$ if $\boldsymbol{b}$ and $\boldsymbol{c}$ are sufficiently regular functions. We keep both terms with $\boldsymbol{b}$ and $\boldsymbol{c}$ to deal with more boundary conditions. It should be also noted that it is important to preserve the two terms $\boldsymbol{b}$ and $\boldsymbol{c}$ in the generic formulation to enable a greater flexibility in the choice of the boundary conditions.

Let Γ^D, Γ^R be open subsets of Γ, possibly empty and $f \in L^2(\Omega)$, $g^D \in \mathrm{H}^{1/2}(\Gamma^D)$, $g^R \in L^2(\Gamma^R)$, $a^R \in L^\infty(\Gamma^R)$ be given data.

A *scalar* boundary value problem is given by

Scalar BVP 1 : generic problem

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = f \quad \text{in } \Omega, \tag{1.2}$$

$$u = g^D \quad \text{on } \Gamma^D, \tag{1.3}$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = g^R \quad \text{on } \Gamma^R. \tag{1.4}$$

The **conormal derivative** of u is defined by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} \stackrel{\text{def}}{=} \langle \mathbb{A}\boldsymbol{\nabla} u, \boldsymbol{n}\rangle - \langle \boldsymbol{b}u, \boldsymbol{n}\rangle \tag{1.5}$$

The boundary conditions (1.3) and (1.4) are respectively **Dirichlet** and **Robin** boundary conditions. **Neumann** boundary conditions are particular Robin boundary conditions with $a^R \equiv 0$.

To have an outline of the FC-VFEM$\mathbb{P}_1$ toolbox, a first and simple problem is quickly present. Explanations will be given in next sections.

The problem to solve is the Laplace problem for a condenser.

Usual BVP 1 : 2D condenser problem

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$-\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.6}$$
$$u = 0 \quad \text{on } \Gamma_1, \tag{1.7}$$
$$u = -12 \quad \text{on } \Gamma_{98}, \tag{1.8}$$
$$u = 12 \quad \text{on } \Gamma_{99}, \tag{1.9}$$

where Ω and its boundaries are given in Figure 1.1.

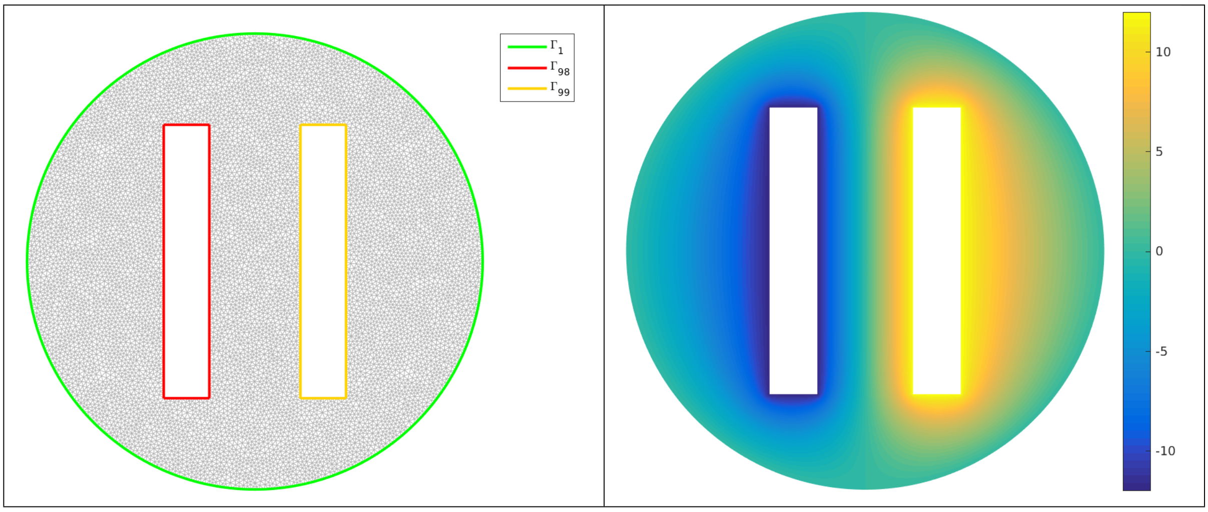


Figure 1.1: 2D condenser mesh and boundaries (left) and numerical solution (right)

The problem (1.6)-(1.9) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

Scalar BVP 2 : 2D condenser problem

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = f \quad \text{in } \Omega,$$
$$u = g^D \quad \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99}.$$

where $\mathcal{L} := \mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}$, $f \equiv 0$, and

$$g^D := 0 \text{ on } \Gamma_1, \; g^D := -12 \text{ on } \Gamma_{98}, \; g^D := +12 \text{ on } \Gamma_{99}$$

In Listing 19 a complete code is given to solve this problem.

```
meshfile=gmsh.buildmesh2d('condenser',10); % generate mesh
Th=siMesh(meshfile); % read mesh
Lop=Loperator(2,2,{1,0;0,1},[],[],[]);
pde=PDEelt(Lop);
bvp=BVP(Th,pde);
bvp.setDirichlet( 1, 0.);
bvp.setDirichlet( 98, -12.);
bvp.setDirichlet( 99, +12.);
U=bvp.solve();
% Graphic parts
figure(1)
Th.plotmesh('color',0.7*[1,1,1])
hold on
Th.plotmesh('d',1,'Linewidth',2,'legend',true)
```

```
axis off,axis image
figure(2)
Th.plot(U,'edgecolor','none','facecolor','interp')
axis off,axis image;colorbar
```

Listing 1.1: Complete Matlab code to solve the 2D condenser problem with graphical representations

Obviously, more complex problems will be studied in section ?? and complete explanations on the code will be given in next sections. Previously, the vector BVP is formally presented with an application.

1.2 Vector boundary value problem

Let $m \geqslant 1$ and $\mathcal{H}$ be the m-by-m matrix of second order linear differential operators defined by

$$\begin{cases} \mathcal{H} & : & \left(\mathrm{H}^2(\Omega)\right)^m & \longrightarrow & \left(L^2(\Omega)\right)^m \\ & & \boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) & \longmapsto & \boldsymbol{f} = (\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m) \stackrel{\text{def}}{=} \mathcal{H}(\boldsymbol{u}) \end{cases} \tag{1.10}$$

where

$$\boldsymbol{f}_\alpha = \sum_{\beta=1}^{m} \mathcal{H}_{\alpha,\beta}(\boldsymbol{u}_\beta), \quad \forall \alpha \in [\![1, m]\!], \tag{1.11}$$

with, for all $(\alpha, \beta) \in [\![1, m]\!]^2$,

$$\mathcal{H}_{\alpha,\beta} \stackrel{\text{def}}{=} \mathcal{L}_{\mathbb{A}^{\alpha,\beta}, \boldsymbol{b}^{\alpha,\beta}, \boldsymbol{c}^{\alpha,\beta}, a_0^{\alpha,\beta}} \tag{1.12}$$

and $\mathbb{A}^{\alpha,\beta} \in (L^\infty(\Omega))^{d \times d}$, $\boldsymbol{b}^{\alpha,\beta} \in (L^\infty(\Omega))^d$, $\boldsymbol{c}^{\alpha,\beta} \in (L^\infty(\Omega))^d$ and $a_0^{\alpha,\beta} \in L^\infty(\Omega)$ are given functions. We can also write in matrix form

$$\mathcal{H}(\boldsymbol{u}) = \begin{pmatrix} \mathcal{L}_{\mathbb{A}^{1,1}, \boldsymbol{b}^{1,1}, \boldsymbol{c}^{1,1}, a_0^{1,1}} & \cdots & \mathcal{L}_{\mathbb{A}^{1,m}, \boldsymbol{b}^{1,m}, \boldsymbol{c}^{1,m}, a_0^{1,m}} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{\mathbb{A}^{m,1}, \boldsymbol{b}^{m,1}, \boldsymbol{c}^{m,1}, a_0^{m,1}} & \cdots & \mathcal{L}_{\mathbb{A}^{m,m}, \boldsymbol{b}^{m,m}, \boldsymbol{c}^{m,m}, a_0^{m,m}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_m \end{pmatrix}. \tag{1.13}$$

We remark that the $\mathcal{H}$ operator for $m = 1$ is equivalent to the $\mathcal{L}$ operator.

For $\alpha \in [\![1, m]\!]$, we define Γ_α^D and Γ_α^R as open subsets of Γ, possibly empty, such that $\Gamma_\alpha^D \cap \Gamma_\alpha^R = \emptyset$. Let $\boldsymbol{f} \in (L^2(\Omega))^m$, $g_\alpha^D \in \mathrm{H}^{1/2}(\Gamma_\alpha^D)$, $g_\alpha^R \in L^2(\Gamma_\alpha^R)$, $a_\alpha^R \in L^\infty(\Gamma_\alpha^R)$ be given data.

A *vector* boundary value problem is given by

Vector BVP 1 : generic problem

Find $\boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) \in (\mathrm{H}^2(\Omega))^m$ such that

$$\mathcal{H}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.14}$$

$$\boldsymbol{u}_\alpha = g_\alpha^D \quad \text{on } \Gamma_\alpha^D, \ \forall \alpha \in [\![1, m]\!], \tag{1.15}$$

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \boldsymbol{u}_\alpha = g_\alpha^R \quad \text{on } \Gamma_\alpha^R, \ \forall \alpha \in [\![1, m]\!], \tag{1.16}$$

where the α-th component of the **conormal derivative** of $\boldsymbol{u}$ is defined by

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} \stackrel{\text{def}}{=} \sum_{\beta=1}^{m} \frac{\partial \boldsymbol{u}_\beta}{\partial n_{\mathcal{H}_{\alpha,\beta}}} = \sum_{\beta=1}^{m} \left(\left\langle \mathbb{A}^{\alpha,\beta} \nabla \boldsymbol{u}_\beta, \boldsymbol{n} \right\rangle - \left\langle \boldsymbol{b}^{\alpha,\beta} \boldsymbol{u}_\beta, \boldsymbol{n} \right\rangle \right). \tag{1.17}$$

The boundary conditions (1.16) are the **Robin** boundary conditions and (1.15) is the **Dirichlet** boundary condition. The **Neumann** boundary conditions are particular Robin boundary conditions with $a_\alpha^R \equiv 0$.

In this problem, we may consider on a given boundary some conditions which can vary depending on the component. For example we may have a Robin boundary condition satisfying $\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_1}} + a_1^R \boldsymbol{u}_1 = g_1^R$ and a Dirichlet one with $\boldsymbol{u}_2 = g_2^D$.

To have an outline of the FC-VFEM$\mathbb{P}_1$ toolbox, a second and simple problem is quickly present.

Usual vector BVP 1 : 2D simple vector problem

Find $\boldsymbol{u} = (u_1, u_2) \in (\mathrm{H}^2(\Omega))^2$ such that

$$-\Delta u_1 + u_2 = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.18}$$

$$-\Delta u_2 + u_1 = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.19}$$

$$(u_1, u_2) = (0, 0) \quad \text{on } \Gamma_1, \tag{1.20}$$

$$(u_1, u_2) = (-12., +12.) \quad \text{on } \Gamma_{98}, \tag{1.21}$$

$$(u_1, u_2) = (+12., -12.) \quad \text{on } \Gamma_{99}, \tag{1.22}$$

where Ω and its boundaries are given in Figure 1.1.

The problem (1.18)-(1.22) can be equivalently expressed as the vector BVP (1.2)-(1.4) :

Vector BVP 2 : 2D simple vector problem

Find $\boldsymbol{u} = (u_1, u_2) \in (\mathrm{H}^2(\Omega))^2$ such that

$$\mathcal{H}(\boldsymbol{u}) = \boldsymbol{f} \qquad \text{in } \Omega,$$

$$u_1 = g_1^D \qquad \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99},$$

$$u_2 = g_2^D \qquad \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99},$$

where

$$\mathcal{H} := \begin{pmatrix} \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},0} & \mathcal{L}_{\mathbb{O},\boldsymbol{O},\boldsymbol{O},1} \\ \mathcal{L}_{\mathbb{O},\boldsymbol{O},\boldsymbol{O},1} & \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},0} \end{pmatrix}, \text{ as } \mathcal{H}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\Delta & 1 \\ 1 & -\Delta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$f \equiv 0,$$

and

$$g_1^D = g_2^D := 0 \text{ on } \Gamma_1, \; g_1^D := -12, \; g_2^D := +12 \text{ on } \Gamma_{98}, \; g_1^D := +12, \; g_2^D := -12 \text{ on } \Gamma_{99}$$

In Listing 21 a complete code is given to solve this problem. Numerical solutions are given in Figure 1.2.

```
meshfile=gmsh.buildmesh2d('condenser',10); % generate mesh
Th=siMesh(meshfile); % read mesh
Hop=Hoperator(2,2,2);
Hop.set([1,2],[1,2],Loperator(2,2,{1,[];[],1},[],[],[]));
Hop.set([1,2],[2,1],Loperator(2,2,[],[],[],1));
pde=PDEelt(Hop);
bvp=BVP(Th,pde);
bvp.setDirichlet( 1, 0.,1:2);
bvp.setDirichlet( 98, {-12,+12},1:2);
bvp.setDirichlet( 99, {+12,-12},1:2);
U=bvp.solve('split',true);
% Graphic parts
figure(1)
Th.plot(U{1})
axis image;axis off;shading interp
colorbar
figure(2);
Th.plot(U{2})
axis image;axis off;shading interp
colorbar
```

Listing 1.2: Complete Matlab code to solve the funny 2D vector problem with graphical representations

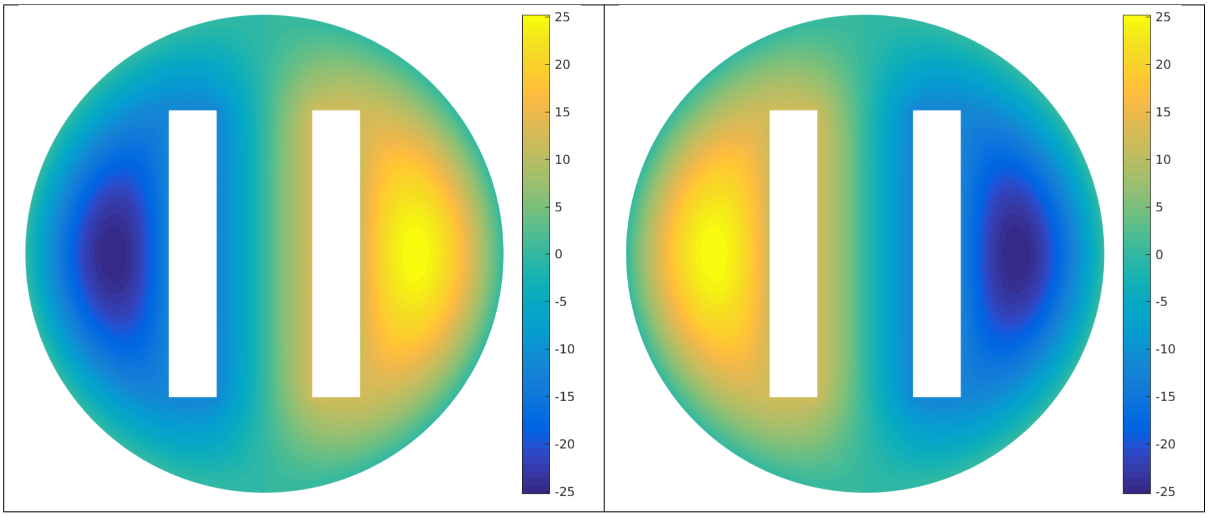


Figure 1.2: Funny vector BVP, u_1 numerical solution (left) and u_2 numerical solution (right)

Obviously, more complex problems will be studied in section **??** and complete explanations on the code will be given in next sections.

In the following of the report we will solve by a $\mathbb{P}^1$-Lagrange finite element method *scalar* B.V.P. (1.2) to (1.4) and *vector* B.V.P. (1.14) to (1.16) without additional restrictive assumption.

Chapter 2

Matlab objects

2.1 Fdata object

This object is used to create the datas associated with the scalar boundary value problem (1.2)-(1.4) or vector boundary value problem (1.14)-(1.16).

2.2 Loperator object

The object Loperator is used to create the operator $\mathcal{L}_{\mathbb{A},\boldsymbol{b},\boldsymbol{c},a_0}$ defined in (1.1). Its main properties are

Properties of Loperator object

d	:	integer, space dimension.
A	:	array of d-by-d cells. Used to store the $\mathbb{A}$ functions such that A{i,j} $\leftarrow \mathbb{A}_{i,j}$. Each cell contains a Fdata object or is empty for 0 value.
b	:	array of d-by-1 cells. Used to store the $\boldsymbol{b}$ functions such that b{i} $\leftarrow \boldsymbol{b}_i$. Each cell contains a Fdata object or is empty for 0 value.
c	:	array of d-by-1 cells. Used to store the $\boldsymbol{c}$ functions such that c{i} $\leftarrow \boldsymbol{c}_i$. Each cell contains a Fdata object or is empty for 0 value.
a0	:	a Fdata object or empty for 0 value Used to store the a_0 function such that a0 $\leftarrow a_0$.
order	:	integer order of the operator : 2 if A is not empty, 1 if A is empty and b or c not empty, 0 if A, b and c are empty.

2.2.1 Constructor

Its contructor are

```
obj=Loperator()
obj=Lopertor(dim,d,A,b,c,a0)
```

Description

obj=Loperator() create an empty operator.

obj=Loperator(dim,d,A,b,c,a0) ...

-
-
-

Samples

$-\Delta u := \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},0}$

in $\mathbb{R}$ Lop=Loperator(1,1,{1},[],[],[])
in $\mathbb{R}^2$ Lop=Loperator(2,2,{1,[];[],1},[],[],[])
in $\mathbb{R}^3$ Lop=Loperator(3,3,{1,[],[];[],1,[];[],[],1},[],[],[])
⋮

$-\Delta u + u := \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},1}$

in $\mathbb{R}$ Lop=Loperator(1,1,{1},[],[],1)
in $\mathbb{R}^2$ Lop=Loperator(2,2,{1,[];[],1},[],[],1)
in $\mathbb{R}^3$ Lop=Loperator(3,3,{1,[],[];[],1,[];[],[],1},[],[],1)
⋮

In R^2, $-\Delta u + (1 + cos(x + y))u := \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},(x,y)\mapsto(1+cos(x+y))}$

Lop=Loperator(2,2,{1,[];[],1},[],[],@(x,y)1+cos(x+y))

2.2.2 Methods

apply function

2.3 Hoperator object

The object Hoperator is used to create the operator $\mathcal{H}$ defined in (1.10). Its main properties are

Properties of Hoperator object

d	:	integer, space dimension.
m	:	integer
H	:	array of d-by-d cells. Used to store the $\mathcal{H}$ operators such that H{i,j} $\leftarrow \mathcal{H}_{i,j}$, $\forall i,j \in [\![1, m]\!]$. Each cell contains a Loperator object or an empty value.

2.3.1 Constructor

Its contructor are

```
obj=Hoperator()
obj=Hoperator(d,s,m)
```

Description

`obj=Hoperator()` create an empty operator with all dimensions set to 0.

`obj=Hopertor(d,s,m)` create an empty/null operator with the given dimensions.

-
-
-

Samples

In $\mathbb{R}^2$, with $\boldsymbol{u} = (u_1, u_2)$ the operator $\mathcal{H}$ defined by

$$\mathcal{H}(\boldsymbol{u}) \stackrel{\text{def}}{=} \begin{pmatrix} -\Delta u_1 + u_2 \\ u_1 - \Delta u_2 \end{pmatrix}$$

could be written as

$$\mathcal{H}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\Delta & 1 \\ 1 & -\Delta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

and then

$$\mathcal{H} = \begin{pmatrix} \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},0} & \mathcal{L}_{\mathbb{O},\boldsymbol{O},\boldsymbol{O},1} \\ \mathcal{L}_{\mathbb{O},\boldsymbol{O},\boldsymbol{O},1} & \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},0} \end{pmatrix}$$

```
Hop=Hoperator(2,2,2);
Lop1=Loperator(2,2,{1,[];[],1},[],[],[]);
Lop2=Loperator(2,2,[],[],[],1);
Hop.set(1,1,Lop1);Hop.set(2,2,Lop1);
Hop.set(1,2,Lop2);Hop.set(2,1,Lop2);
```

or

```
Hop=Hoperator(2,2,2);
Hop.set([1,2],[1,2],Loperator(2,2,{1,[];[],1},[],[],[]));
Hop.set([1,2],[2,1],Loperator(2,2,[],[],[],1));
```

2.3.2 Methods

set function

zeros function

opStiffElas function

2.4 PDEelt object

This object is used to create the scalar PDE (1.2) or the vector PDE (1.14):

$$\mathcal{L}(u) = f \quad \text{or} \quad \mathcal{H}(\boldsymbol{u}) = \boldsymbol{f}.$$

Its main properties are

Properties of PDEelt object

d	:	integer, space dimension.
m	:	integer
Op	:	Loperator or Hoperator object.
f	:	(cells of) Fdata object or empty. Used to store the right-hand side of the PDE. If Op is an Loperator object then f is an Fdata object or is empty. If Op is an Hoperator object then f is a cell array of Op.m Fdata object or empty value.

Its contructor are

```
obj=PDEelt()
obj=PDEelt(Op)
obj=PDEelt(Op,f)
```

Description

obj=PDEelt() create an empty object.

obj=PDEelt(Op) create the PDE with $f \equiv 0$: i.e. `Op(u)=0`

obj=PDEelt(Op,f) create the PDE `Op(u)=f`. If Op is an Hoperator object then f must be a cell array of length Hoperator.m.

Samples

In $\mathbb{R}^2$, $-\Delta u + u = f$, with $f(x, y) = x \sin(x + y)$

```
Lop=Loperator(2,2,{1,[];[],1},[],[],1);
f=@(x,y) x.*sin(x+y);
pde=PDEelt(Lop,f);
```

The f function must be written in a vectorized form.

2.5 BVP object

The object BVP is used to create a scalar boundary value problem (1.2)-(1.4) or a vector boundary value problem (1.14)-(1.16). The usage of this object is strongly correlated with good comprehension of the FC-SIMESH toolbox and and more particularly with the siMesh object.

The properties of the object BVP are

Properties of BVP object

d	:	integer, space dimension.
m	:	integer, system of m PDEs.
Th	:	a siMesh object
pdes	:	Th.nsTh-by-1 cell array. Used to store the PDE associated with each submesh Th.sTh{i}. If pdes{i} is empty then there is no PDE defined on Th.sTh{i}.

2.5.1 Constructor

Its contructor are

```
obj=BVP()
obj=BVP(Th,pde)
obj=BVP(Th,pde,labels)
```

Description

obj=BVP() create an empty BVP object.

obj=BVP(Th,pde) create a BVP object with PDE's defined by pde object on all submeshes of index Th.find(pde.d) i.e. on all submeshes such that Th.sTh{i}==pde.d. By default, homogeneous Neumann boundary conditions are set on all *boundaries*.

obj=BVP(Th,pde,labels) similar to previous one except among the selected objects are choosen those with label (Th.sTh{i}.label) in labels array. By default, homogeneous Neumann boundary conditions are set on all *boundaries*.

2.5.2 Main methods

Let bvp be a BVP object.

setPDE function

```
bvp.setPDE(d,label,pde)
```

Description

bvp.setPDE(d,label,pde) associated the pde object with the i-th submesh such that i=bvp.Th.find(d,label) If i exists then bvp.pdes{i} is set to pde.

setDirichlet function

```
bvp.setDirichlet(label,g)
bvp.setDirichlet(label,g,Lm)
```

Description

bvp.setDirichlet(label,g) for scalar B.V.P., sets Dirichlet boundary condition

$$u = \text{g}, \quad \text{on } \Gamma_{\text{label}}$$

and for vector B.V.P., sets Dirichlet boundary condition

$$u_i = \text{g}\{\text{i}\}, \forall i \in [\![1, m]\!] \text{ on } \Gamma_{\text{label}}.$$

bvp.setDirichlet(label,g,Lm) for vector B.V.P., sets Dirichlet boundary condition

$$u_{\text{Lm(i)}} = \text{g}\{\text{i}\}, \forall i \in [\![1, \text{length}(\text{Lm})]\!] \text{ on } \Gamma_{\text{label}}.$$

setRobin function

```
bvp.setRobin(label,gr,ar)
bvp.setRobin(label,gr,ar,Lm)
```

Description

bvp.setRobin(label,gr,ar) for scalar B.V.P., sets Robin boundary condition (1.4)

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + \mathrm{ar}\, u = \mathrm{gr}, \quad \text{on } \Gamma_{\mathrm{label}}.$$

For vector B.V.P., sets Robin boundary condition (1.16)

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_i}} + \mathrm{ar}\{\mathrm{i}\}\boldsymbol{u}_i = \mathrm{gr}\{\mathrm{i}\}, \quad \forall i \in [\![1, m]\!] \text{ on } \Gamma_{\mathrm{label}}.$$

bvp.setRobin(label,gr,ar,Lm) for vector B.V.P., sets Robin boundary condition (1.16) :
$\forall i \in [\![1, \mathrm{length}(\mathrm{Lm})]\!]$, let $\alpha = \mathrm{Lm}(\mathrm{i})$ then

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} + \mathrm{ar}\{\mathrm{i}\}\boldsymbol{u}_\alpha = \mathrm{gr}\{\mathrm{i}\}, \ \text{on } \Gamma_{\mathrm{label}}.$$

solve function

```
x=bvp.solve()
x=bvp.solve(key,value,...)
```

Description

x=bvp.solve()) uses P_1-Lagrange finite elements method to solve the B.V.P. described by the bvp object.

x=bvp.solve(key,value,...)

- 'solver' :
- 'split' :
- 'local' :
- 'perm' :

Chapter 3

Scalar boundary value problems

3.1 Poisson BVP's

The generic problem to solve is the following

Usual BVP 2 : Poisson problem

Find $u \in \mathrm{H}^1(\Omega)$ such that

$$-\Delta u = f \quad \text{in } \Omega \subset \mathbb{R}^{\dim}, \tag{3.1}$$

$$u = g_D \quad \text{on } \Gamma_D, \tag{3.2}$$

$$\frac{\partial u}{\partial n} + a_R\, u = g_R \quad \text{on } \Gamma_R, \tag{3.3}$$

where $\Omega \subset \mathbb{R}^{\dim}$ with $\partial\Omega = \Gamma_D \cup \Gamma_R$ and $\Gamma_D \cap \Gamma_R = \emptyset$.

The Laplacian operator Δ can be rewritten according to a $\mathcal{L}$ operator defined in (1.1) and we have

$$-\Delta \stackrel{\text{def}}{=} -\sum_{i=1}^{\dim} \frac{\partial^2}{\partial {x_i}^2} = \mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}. \tag{3.4}$$

The conormal derivative $\frac{\partial u}{\partial n_{\mathcal{L}}}$ of this $\mathcal{L}$ operator is given by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} \stackrel{\text{def}}{=} \langle \mathbb{A}\,\boldsymbol{\nabla}\, u, \boldsymbol{n}\rangle - \langle \boldsymbol{b}u, \boldsymbol{n}\rangle = \frac{\partial u}{\partial n}. \tag{3.5}$$

We now will see how to implement different Poisson's BVP while using the FC-VFEM$\mathbb{P}_1$ toolbox.

3.1.1 2D Poisson BVP with Dirichlet boundary conditions on the unit square

Let Ω be the unit square with the associated mesh obtain from HyperCube function (see section ?? for explanation and Figure ?? for a mesh sample) by the command

```
Th=fc_simesh.HyperCube(2,50);
```

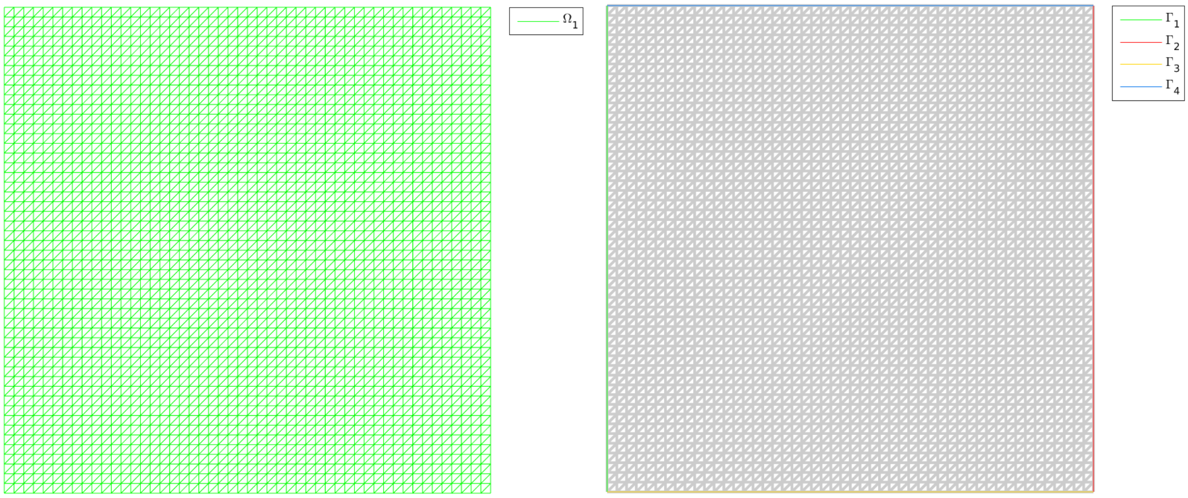


Figure 3.1: 2D hypercube (left) and its boundaries (right)

We choose the problem to have exact solution

$$u_{\text{ex}}(x, y) = \cos(x - y)\sin(x + y) + e^{(-x^2 - y^2)}.$$

So we set $f = -\Delta u_{\text{ex}}$ i.e.

$$f(x, y) = -4\,x^2 e^{(-x^2-y^2)} - 4\,y^2 e^{(-x^2-y^2)} + 4\,\cos(x - y)\sin(x + y) + 4\,e^{(-x^2-y^2)}.$$

On all the 4 boundaries we set a Dirichlet boundary conditions (and so $\Gamma_R = \emptyset$) :

$$u = u_{\text{ex}}, \text{ on } \Gamma_D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4.$$

So this problem can be written as the scalar BVP 5

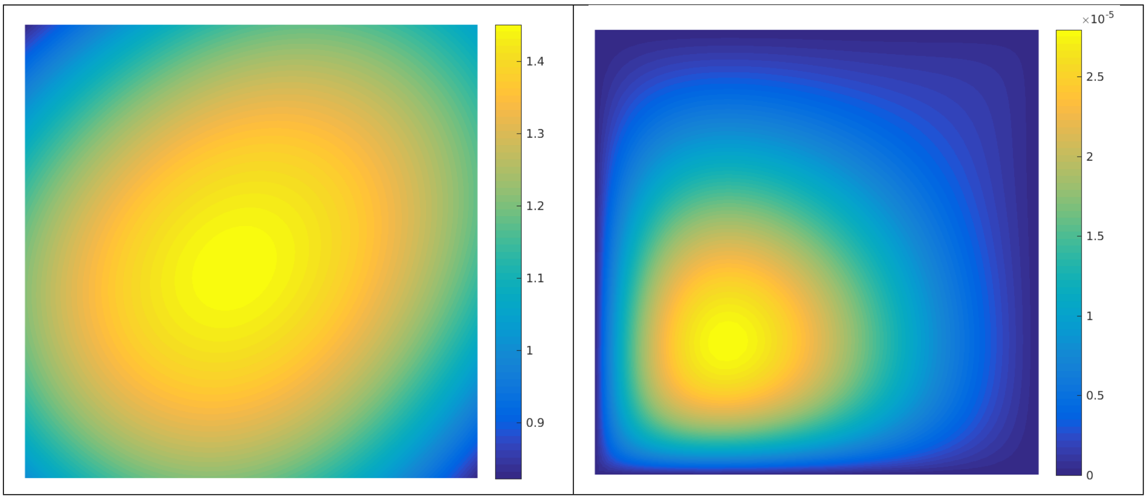
Scalar BVP 3 : 2D Poisson BVP with Dirichlet boundary conditions

Find $u \in \mathrm{H}^1(\Omega)$ such that

$$\mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}(u) = f \text{ in } \Omega = [0, 1]^2, \quad (3.6)$$

$$u = u_{\text{ex}} \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \quad (3.7)$$

In Listing 9, we give the complete code to solve this problem with FC-VFEM$\mathbb{P}_1$ toolbox.




```
1 uex=@(x,y) cos(x-y).*sin(x+y)+exp(-(x.^2+y.^2));
2 f=@(x,y) -4*x.^2.*exp(-x.^2-y.^2) - 4*y.^2.*exp(-x.^2-y.^2) + 4*cos(x-y).*sin(x+y) + ...
      4*exp(-x.^2-y.^2);
3 Th=fc_simesh.HyperCube(2,50);
4 Lop=Loperator(2,2,{1,0;0,1},[],[],[]);
5 pde=PDEelt(Lop,f);
6 bvp=BVP(Th,pde);
7 for lab=1:4, bvp.setDirichlet( lab, uex);end
8 U=bvp.solve();
```

Listing 3.1: Poisson 2D BVP with Dirichlet boundary conditions : numerical solution (left) and error (right)

In line **??** we set the Dirichlet boundary conditions and in line **??** we solve the BVP.

3.1.2 2D Poisson BVP with mixed boundary conditions

Let Ω be the unit square with the associated mesh obtain from HYPERCUBE function (see section **??** for explanation and Figure **??** for a mesh sample)

We choose the problem to have exact solution

$$u_{\rm ex}(x, y) = \cos(2\,x + y)\,.$$

So we set $f = -\Delta u_{\rm ex}$ i.e.

$$f(x, y) = 5\,\cos(2\,x + y)\,.$$

On boundary labels 1 and 2 we set a Dirichlet boundary conditions :

$$u = u_{\rm ex},\ \text{on}\ \Gamma^D = \Gamma_1 \cup \Gamma_2.$$

On boundary label 3, we choose a Robin boundary condition with $a^R(x, y) = x^2 + y^2 + 1$. So we have

$$\frac{\partial u}{\partial n} + a^R u = g^R,\ \text{on}\ \Gamma^R = \Gamma_3$$

with $g^R = \left(x^2 + y^2 + 1\right)\cos(2\,x + y) + \sin(2\,x + y)\,.$

On boundary label 4, we choose a Newmann boundary condition. So we have

$$\frac{\partial u}{\partial n} = g^N,\ \text{on}\ \Gamma^N = \Gamma_4$$

with $g^N = -\sin(2\,x + y)\,.$ this can be also written in the form of a Robin condition with $aR = 0$

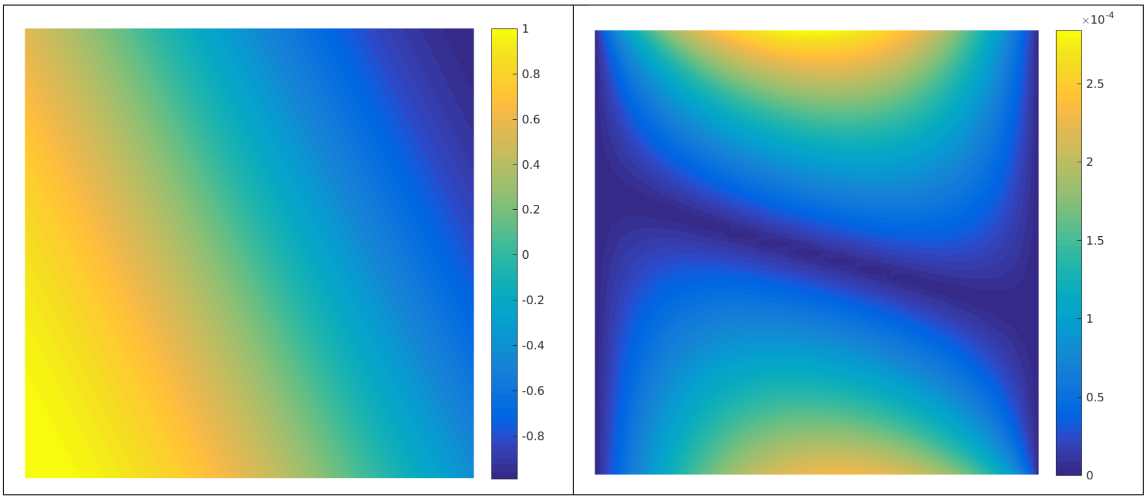
So this problem can be written as the scalar BVP 5

Scalar BVP 4 : 2D Poisson BVP with Dirichlet boundary conditions

Find $u \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned}
\mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}(u) &= f \quad \text{in } \Omega = [0,1]^2, &(3.8)\\
u &= u_{\text{ex}} \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, &(3.9)\\
\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u &= g^R \quad \text{on } \Gamma_3, &(3.10)\\
\frac{\partial u}{\partial n_{\mathcal{L}}} &= g^N \quad \text{on } \Gamma_4, &(3.11)\\
& &(3.12)
\end{aligned}$$

In Listing 14, we give the complete code to solve this problem with FC-VFEM$\mathbb{P}_1$ toolbox.




```
1  uex=@(x,y) cos(2*x+y);
2  f=@(x,y) 5*cos(2*x+y);
3  gradu={@(x,y) -2*sin(2*x+y), @(x,y) -sin(2*x+y)};
4  ar3=@(x,y) 1+x.^2+y.^2;
5  Th=fc_simesh.HyperCube(2,50);
6  Lop=Loperator(2,2,{1,0;0,1},[],[],[]);
7  pde=PDEelt(Lop,f);
8  bvp=BVP(Th,pde);
9  bvp.setDirichlet( 1, uex);
10 bvp.setDirichlet( 2, uex);
11 bvp.setRobin( 3, @(x,y) -gradu{2}(x,y)+ar3(x,y).*uex(x,y),ar3);
12 bvp.setRobin( 4, gradu{2},[]);
13 U=bvp.solve();
```

Listing 3.2: Poisson 2D BVP with mixed boundary conditions : numerical solution (left) and error (right)

We set respectively in lines 11 and 12, the Robin and the Neumann boundary conditions by using SETROBIN member function of BVP class.

3.1.3 3D Poisson BVP with mixed boundary conditions

Let Ω be the unit cube with the associated mesh obtain from HYPERCUBE function (see section ?? for explanation and Figure ?? for a mesh sample)

We choose the problem to have exact solution

$$u_{\text{ex}}(x, y, y) = \cos(4\,x - 3\,y + 5\,z)\,.$$

So we set $f = -\Delta u_{\text{ex}}$ i.e.

$$f(x, y, z) = 50\,\cos(4\,x - 3\,y + 5\,z)\,.$$

On boundary labels $1, 3, 5$ we set a Dirichlet boundary conditions :

$$u = u_{\rm ex}, \text{ on } \Gamma^D = \Gamma_1 \cup \Gamma_3 \cup \Gamma_5.$$

On boundary label 2, we choose a Robin boundary condition with $a^R(x, y) = 1$. So we have

$$\frac{\partial u}{\partial n} + a^R u = g^R, \text{ on } \Gamma^R = \Gamma_2 \cup \Gamma_4$$

with $g^R(x, y, z) = \cos(4\,x - 3\,y + 5\,z) - 4\,\sin(4\,x - 3\,y + 5\,z)$, on Γ_2 and $g^R(x, y, z) = \cos(4\,x - 3\,y + 5\,z) + 3\,\sin(4\,x - 3\,y + 5\,z)$, on Γ_4.

On boundary label 6, we choose a Newmann boundary condition. So we have

$$\frac{\partial u}{\partial n} = g^N, \text{ on } \Gamma^N = \Gamma_6$$

with $g^N = -5\,\sin(4\,x - 3\,y + 5\,z)$. this can be also written in the form of a Robin condition with $aR = 0$ on Γ_6.

So this problem can be written as the scalar BVP 5

***Scalar* BVP 5 : 3D Poisson BVP with mixed boundary conditions**

Find $u \in \mathrm{H}^1(\Omega)$ such that

$$\mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}(u) = f \text{ in } \Omega = [0,1]^3, \tag{3.13}$$

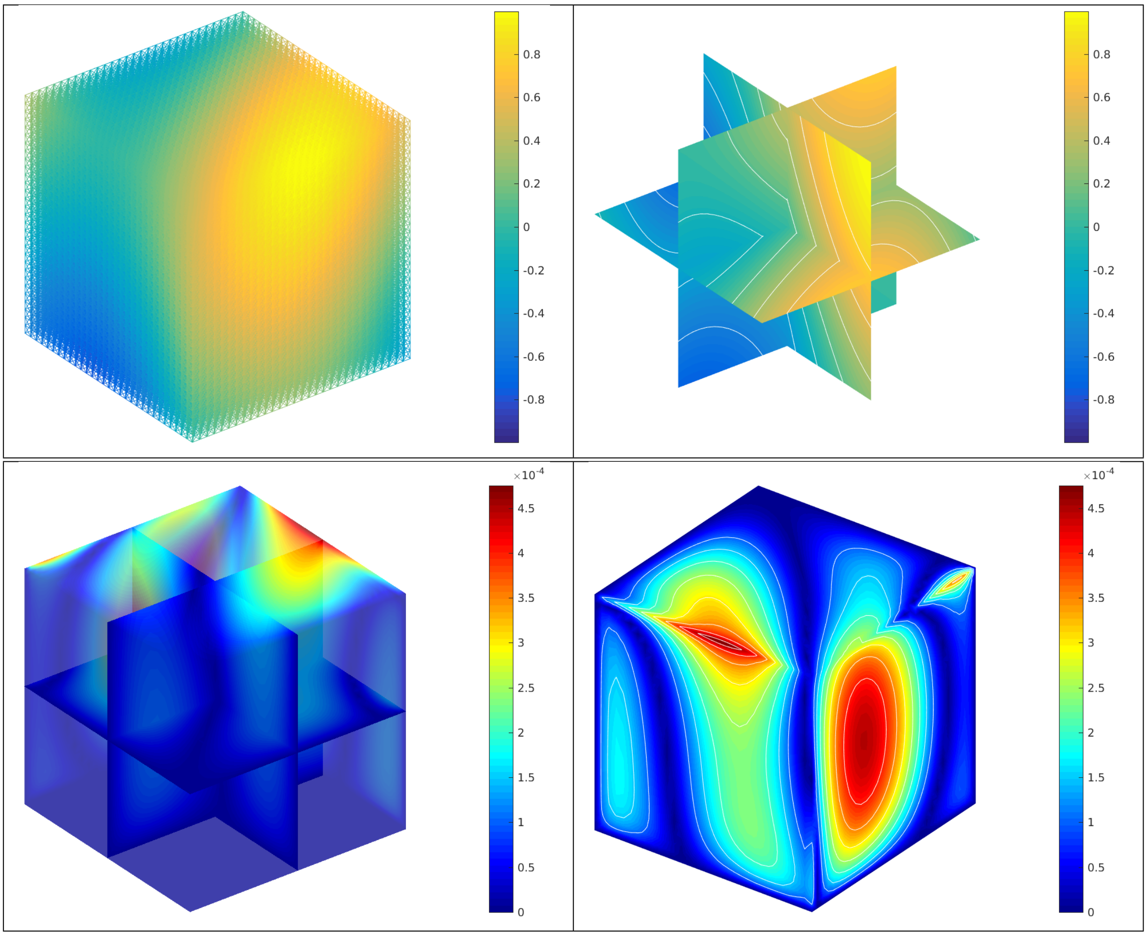
$$u = u_{\rm ex} \text{ on } \Gamma_1 \cup \Gamma_3 \cup \Gamma_5, \tag{3.14}$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = g^R \text{ on } \Gamma_2 \cup \Gamma_4, \tag{3.15}$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} = g^N \text{ on } \Gamma_6, \tag{3.16}$$

(3.17)

In Listing 16, we give the complete code to solve this problem with FC-VFEM$\mathbb{P}_1$ toolbox.




```
uex=@(x,y,z) cos(2*x−y−z).*sin(x−2*y+z);
f=@(x,y,z) 6*cos(x − 2*y + z).*sin(2*x − y − z) + 12*cos(2*x − y − z).*sin(x − 2*y + z);
ar=1;
gradu={@(x,y,z) cos(2*x − y − z).*cos(x − 2*y + z) − 2*sin(2*x − y − z).*sin(x − 2*y + z), ...
          @(x,y,z) −2*cos(2*x − y − z).*cos(x − 2*y + z) + sin(2*x − y − z).*sin(x − 2*y + z), ...
          @(x,y,z) cos(2*x − y − z).*cos(x − 2*y + z) + sin(2*x − y − z).*sin(x − 2*y + z)};
Th=fc_simesh.HyperCube(3,30);
Lop=Loperator(3,3,{1,0,0;0,1,0;0,0,1},[],[],[]);
pde=PDEelt(Lop,f);
bvp=BVP(Th,pde);
for lab=[1,3,5], bvp.setDirichlet( lab, uex);end
bvp.setRobin(2,@(x,y,z) gradu{1}(x,y,z)+ar*uex(x,y,z),ar);
bvp.setRobin(4,@(x,y,z) gradu{2}(x,y,z)+ar*uex(x,y,z),ar);
bvp.setRobin(6,@(x,y,z) gradu{3}(x,y,z),[]);
U=bvp.solve();
```

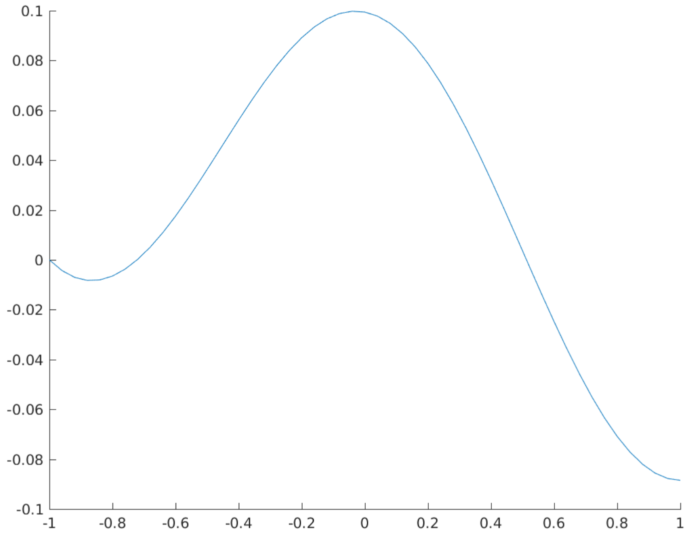
Listing 3.3: 3D Poisson BVP with mixed boundary conditions : numerical solution (upper) and error (bottom)

3.1.4 1D BVP : just for fun

Let Ω be the interval $[a, b]$ we want to solve the following PDE

$$-u''(x) + c(x)u(x) = f(x) \quad \forall x \in]a, b[$$

with the Dirichlet boundary condition $u(a) = 0$ and the homgeneous Neumann boundary condition on b




```
f=@(x) cos(pi*x);
c=@(x) 1+(x-1).^2;
a=-1;b=1;
Th=fc_simesh.HyperCube(1,50,'trans',@(x) a + (b-a)*x);
Lop=Loperator(1,1,{1},[],[],c);
pde=PDEelt(Lop,f);
bvp=BVP(Th,pde);
bvp.setDirichlet( 1, 0);
U=bvp.solve();
```

Listing 3.4: 1D BVP with mixed boundary conditions

3.2 Stationary convection-diffusion problem

3.2.1 Stationary convection-diffusion problem in 2D

The 2D problem to solve is the following

***Usual* BVP 3 : 2D stationary convection-diffusion problem**

Find $u \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} -\operatorname{div}(\alpha \nabla u) + \langle \boldsymbol{V}, \nabla u \rangle + \beta u &= f \quad \text{in } \Omega \subset \mathbb{R}^2, && (3.18) \\ u &= 4 \quad \text{on } \Gamma_2, && (3.19) \\ u &= -4 \quad \text{on } \Gamma_4, && (3.20) \\ u &= 0 \quad \text{on } \Gamma_{20} \cup \Gamma_{21}, && (3.21) \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_1 \cup \Gamma_3 \cup \Gamma_{10} && (3.22) \end{aligned}$$

where Ω and its boundaries are given in Figure **??**. This problem is well posed if $\alpha(\boldsymbol{x}) > 0$ and $\beta(\boldsymbol{x}) \geqslant 0$.
We choose α, $\boldsymbol{V}$, β and f in Ω as :

$$\begin{aligned} \alpha(\boldsymbol{x}) &= 0.1 + (x_1 - 0.5)^2, \\ \boldsymbol{V}(\boldsymbol{x}) &= (-10x_2, 10x_1)^t, \\ \beta(\boldsymbol{x}) &= 0.01, \\ f(\boldsymbol{x}) &= -200\exp(-10((x_1 - 0.75)^2 + x_2^2)). \end{aligned}$$

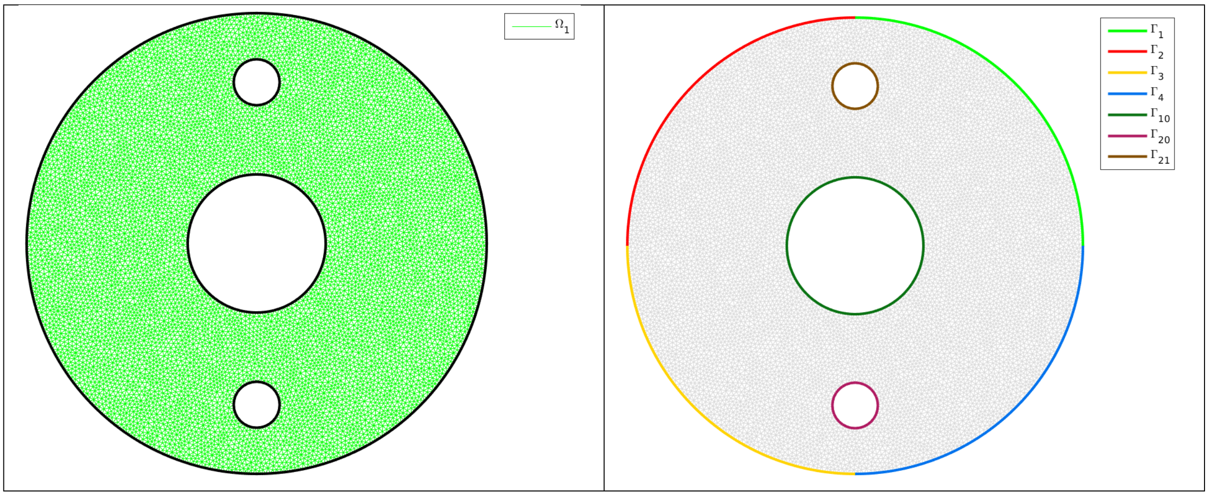


Figure 3.2: 2D stationary convection-diffusion BVP : mesh (left) and boundaries (right)

The problem (3.18)-(3.22) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

Scalar BVP 6 : 2D stationary convection-diffusion problem

Find $u \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} \mathcal{L}(u) &= f && \text{in } \Omega, \\ u &= g^D && \text{on } \Gamma^D, \\ \frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u &= g^R && \text{on } \Gamma^R. \end{aligned}$$

where

- $\mathcal{L} := \mathcal{L}_{\alpha\mathbb{I},\mathbf{0},\boldsymbol{V},\beta}$, and then the conormal derivative of u is given by

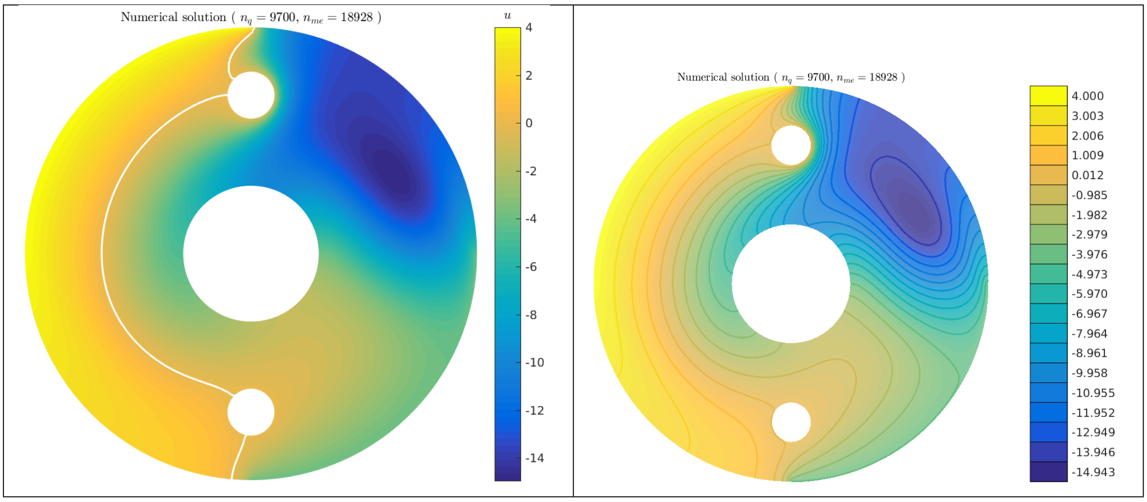
$$\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \boldsymbol{\nabla} u, \boldsymbol{n} \rangle - \langle \boldsymbol{b} u, \boldsymbol{n} \rangle = \alpha \frac{\partial u}{\partial n}.$$

- $\Gamma^D = \Gamma_2 \cup \Gamma_4 \cup \Gamma_{20} \cup \Gamma_{21}$ and $\Gamma^R = \Gamma_1 \cup \Gamma_3 \cup \Gamma_{10}$
- $g^D := 4$ on Γ_2, and $g^D := -4$ on Γ_4 and $g^D := 0$ on $\Gamma_{20} \cup \Gamma_{21}$
- $a^R = g^R := 0$ on Γ^R.

The algorithm using the toolbox for solving (3.18)-(3.22) is the following:

Algorithm 1 Stationary convection-diffusion problem in 2D

$\mathcal{T}_h \leftarrow$ siMesh(...) ▷ Get mesh
$\alpha \leftarrow (x, y) \longmapsto 0.1 + (y - 0.5)(y - 0.5)$
$\beta \leftarrow 0.01$
$f \leftarrow (x, y) \longmapsto -200 e^{-10((x-0.75)^2 + y^2)}$
Lop $\leftarrow$ Loperator$\left(2, 2, \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \mathbf{0}, \begin{pmatrix} -10y \\ 10x \end{pmatrix}, \beta\right)$
pde $\leftarrow$ PDEelt(Lop, f)
bvp $\leftarrow$ BVP($\mathcal{T}_h$, pde)
bvp.setDirichlet(2, 4.0) ▷ Set `'Dirichlet'` condition on Γ_2
bvp.setDirichlet(4, −4.0) ▷ Set `'Dirichlet'` condition on Γ_4
bvp.setDirichlet(20, 0.0) ▷ Set `'Dirichlet'` condition on Γ_{20}
bvp.setDirichlet(21, 0.0) ▷ Set `'Dirichlet'` condition on Γ_{21}
$\boldsymbol{u} \leftarrow$ bvp.solve()




```
af=@(x,y) 0.1+y.^2;
Vx=@(x,y) -10*y;Vy=@(x,y) 10*x;
b=0.01;g2=4;g4=-4;
f=@(x,y) -200.0*exp(-((x-0.75).^2+y.^2)/(0.1));
Th=siMesh(meshfile);
Lop=Loperator(Th.dim,Th.d,{af,[];[],af},[],{Vx,Vy},b);
pde=PDEelt(Lop,f);
bvp=BVP(Th,pde);
bvp.setDirichlet(2, g2);
bvp.setDirichlet(4, g4);
bvp.setDirichlet(20, 0.);
bvp.setDirichlet(21, 0.);
```

Listing 3.5: Setting the 2D stationary convection-diffusion BVP and representation of the numerical solution

The numerical solution for a given mesh is shown on figures of Listing ??

3.2.2 Stationary convection-diffusion problem in 3D

Let $A = (x_A, y_A) \in \mathbb{R}^2$ and $\mathcal{C}_A^r([z_{min}, z_{max}])$ be the right circular cylinder along $z-$axis ($z \in [z_{min}, z_{max}]$) with bases the circles of radius r and center (x_A, y_A, z_{min}) and (x_A, y_A, z_{max}).

Let Ω be the cylinder defined by

$$\Omega = \mathcal{C}^1_{(0,0)}([0,3]) \backslash \{\mathcal{C}^{0.3}_{(0,0)}([0,3]) \cup \mathcal{C}^{0.1}_{(0,-0.7)}([0,3]) \cup \mathcal{C}^{0.1}_{(0,0.7)}([0,3])\}.$$

We respectively denote by Γ_{1000} and Γ_{1001} the $z = 0$ and $z = 3$ bases of Ω.

Γ_1, Γ_{10}, Γ_{20} and Γ_{21} are respectively the curved surfaces of cylinders $\mathcal{C}^1_{(0,0)}([0,3])$, $\mathcal{C}^{0.3}_{(0,0)}([0,3])$, $\mathcal{C}^{0.1}_{(0,-0.7)}([0,3])$ and $\mathcal{C}^{0.1}_{(0,0.7)}([0,3])$.

The domain Ω and its boundaries are represented in Figure ??.

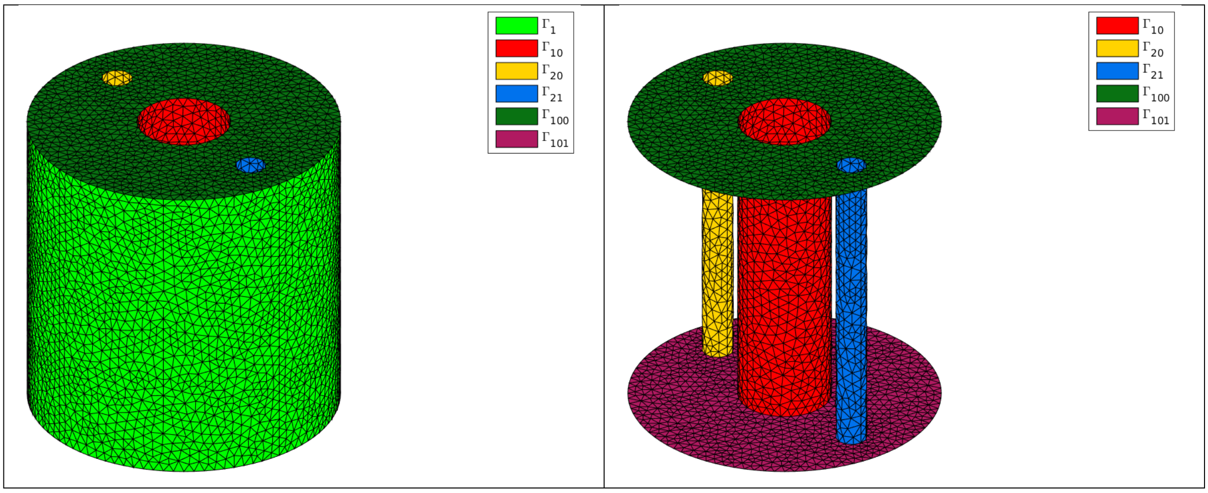


Figure 3.3: 3D stationary convection-diffusion BVP : all boundaries (left) and boundaries without Γ_1 (right)

The 3D problem to solve is the following

***Usual* BVP 4 :**

3D problem : Stationary convection-diffusion Find $u \in \mathrm{H}^2(\Omega)$ such that

$$-\operatorname{div}(\alpha \nabla u) + \langle \boldsymbol{V}, \nabla u\rangle + \beta u = f \quad \text{in } \Omega \subset \mathbb{R}^3, \tag{3.23}$$

$$\alpha \frac{\partial u}{\partial n} + a_{20} u = g_{20} \quad \text{on } \Gamma_{20}, \tag{3.24}$$

$$\alpha \frac{\partial u}{\partial n} + a_{21} u = g_{21} \quad \text{on } \Gamma_{21}, \tag{3.25}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma^N \tag{3.26}$$

where $\Gamma^N = \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{1000} \cup \Gamma_{1001}$. This problem is well posed if $\alpha(\boldsymbol{x}) > 0$ and $\beta(\boldsymbol{x}) \geqslant 0$. We choose $a_{20} = a_{21} = 1$, $g_{21} = -g_{20} = 0.05$ $\beta = 0.01$ and :

$$\begin{aligned}
\alpha(\boldsymbol{x}) &= 0.7 + \boldsymbol{x}_3/10, \\
\boldsymbol{V}(\boldsymbol{x}) &= (-10x_2, 10x_1, 10x_3)^t, \\
f(\boldsymbol{x}) &= -800\exp(-10((x_1 - 0.65)^2 + x_2^2 + (x_3 - 0.5)^2)) \\
&\quad +800\exp(-10((x_1 + 0.65)^2 + x_2^2 + (x_3 - 0.5)^2)).
\end{aligned}$$

The problem (3.23)-(3.26) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

***Scalar* BVP 7 :**

3D stationary convection-diffusion problem as a *scalar* BVP Find $u \in \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = f \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = g^R \quad \text{on } \Gamma^R.$$

where

- $\mathcal{L} := \mathcal{L}_{\alpha\mathbb{I},\boldsymbol{0},\boldsymbol{V},\beta}$, and then the conormal derivative of u is given by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A}\nabla u, \boldsymbol{n}\rangle - \langle \boldsymbol{b}u, \boldsymbol{n}\rangle = \alpha\frac{\partial u}{\partial n}.$$

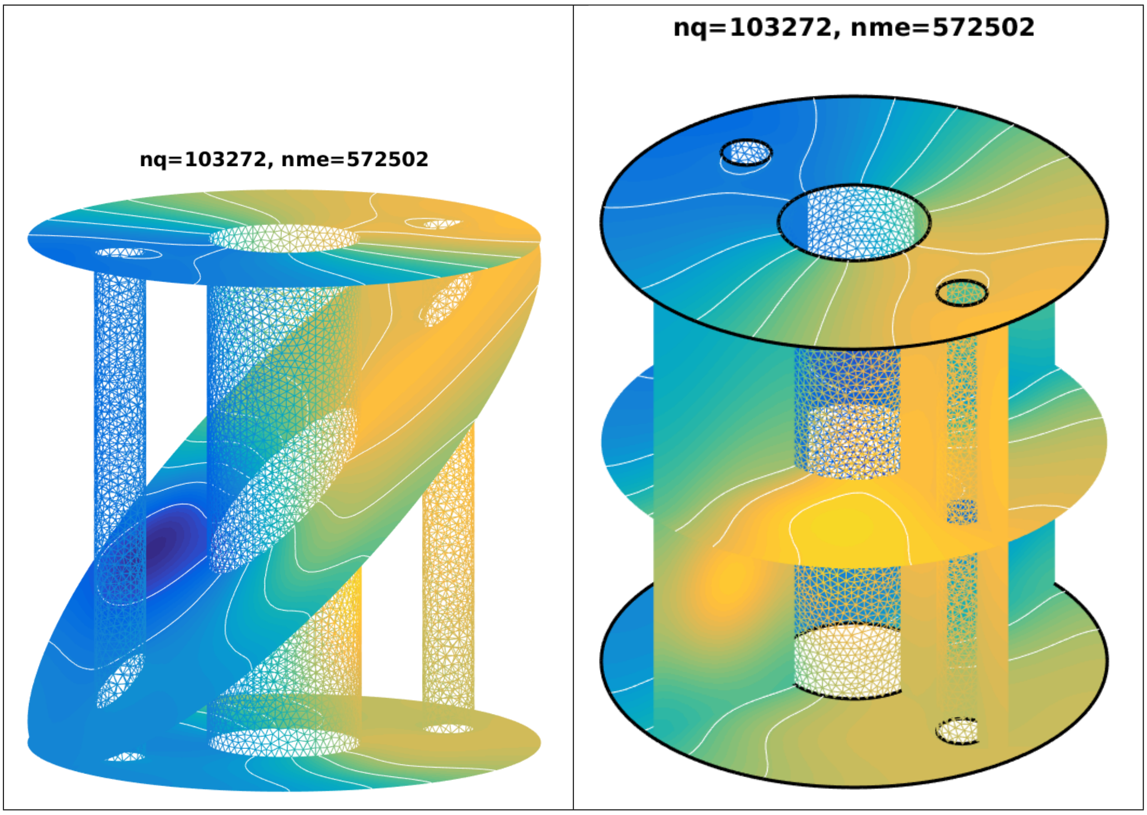
- $\Gamma^R = \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{20} \cup \Gamma_{21} \cup \Gamma_{1000} \cup \Gamma_{1001}$ (and $\Gamma^D = \varnothing$)

-

$$a^R = \begin{cases} 0 & \text{on } \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{1000} \cup \Gamma_{1001} \\ 1 & \text{on } \Gamma_{20} \cup \Gamma_{21} \end{cases}$$

$$g^R = \begin{cases} 0 & \text{on } \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{1000} \cup \Gamma_{1001} \\ 0.05 & \text{on } \Gamma_{21}, \\ -0.05 & \text{on } \Gamma_{20} \end{cases}$$

We give respectively in Listing 11 the corresponding Matlab codes and the numerical solution for a more refined mesh.




```
af=@(x,y,z) 0.7+ z/10;
beta=0.01;
V={@(x,y,z) -10*y ,@(x,y,z) 10*x ,@(x,y,z) 10*z};
f=@(x,y,z) -800*exp(-10*((x-0.65).^2+y.^2+(z-0.5).^2)) + ...
      800*exp(-10*((x+0.65).^2+y.^2+(z-0.5).^2));
Th=siMesh(meshfile);
Lop=Loperator(Th.dim,Th.d,{af,[],[];[],af,[];[],[],af},[],V,beta);
pde=PDEelt(Lop,f);
bvp=BVP(Th,pde);
bvp.setRobin(20,0.05,1);
bvp.setRobin(21,-0.05,1);
```

Listing 3.6: Setting the 3D stationary convection-diffusion BVP and representation of the numerical solution

3.3 2D electrostatic BVPs

In this sample, we shall discuss electrostatic solutions for current flow in resistive media. Consider a region Ω of contiguous solid and/or liquid conductors. Let $\boldsymbol{j}$ be the current density in A/m^2. It's satisfy

$$\operatorname{div} \boldsymbol{j} = 0, \quad \text{in } \Omega. \tag{3.27}$$

$$\boldsymbol{j} = \sigma \boldsymbol{E}, \quad \text{in } \Omega. \tag{3.28}$$

where σ is the local electrical conductivity and $\boldsymbol{E}$ the local electric field.

The electric field can be written as a gradient of a scalar potential

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi, \quad \text{in } \Omega. \tag{3.29}$$

Combining all these equations leads to Laplace's equation

$$\operatorname{div}(\sigma \boldsymbol{\nabla}\varphi) = 0 \tag{3.30}$$

In the resistive model, a good conductor has high value of σ and a good insulator has $0 < \sigma\mu 1$.

Material	$\rho(\Omega.m)$ at $20^\circ C$	$\sigma(S/m)$ at $20^\circ C$
Carbon (graphene)	1.00×10^{-8}	1.00×10^{8}
Gold	2.44×10^{-8}	4.10×10^{8}
Drinking water	2.00×10^{1} to 2.00×10^{3}	5.00×10^{-4} to 5.00×10^{-2}
Silicon	6.40×10^{2}	1.56×10^{-3}
Glass	1.00×10^{11} to 1.00×10^{15}	10^{-15} to 10^{-11}
Air	1.30×10^{16} to 3.30×10^{16}	3×10^{-15} to 8×10^{-15}

As example, we use the mesh obtain with `gmsh` from `square4holes6dom.geo` file represented in Figure 3.4

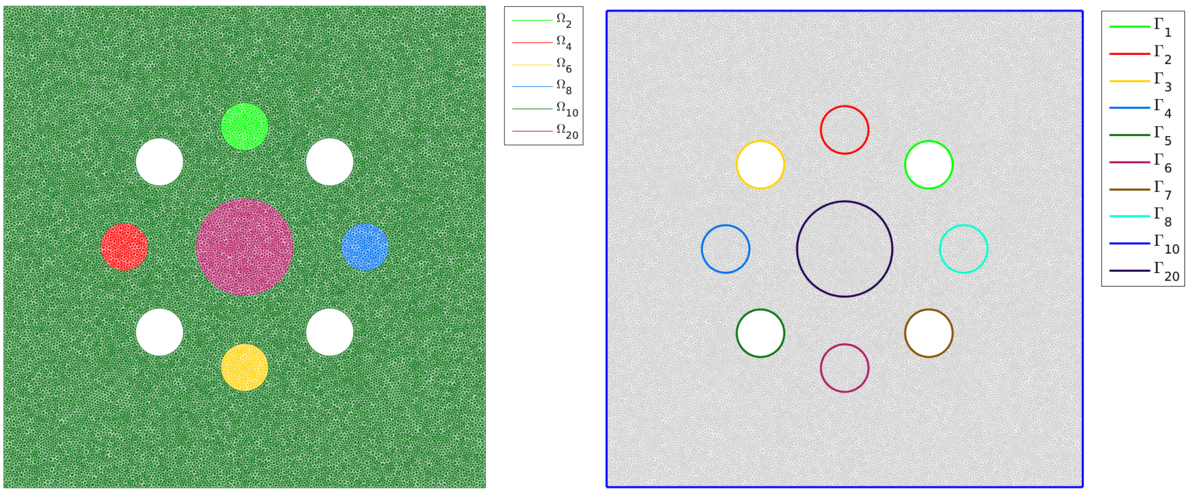


Figure 3.4: Mesh from `square4holes6dom.geo`, domains representation (left) and boundaries (right)

We have two resistive medias

$$\Omega_a = \Omega_{10} \quad \text{and} \quad \Omega_b = \Omega_{20} \cup \Omega_2 \cup \Omega_4 \cup \Omega_6 \cup \Omega_8.$$

In Ω_a and Ω_b the local electrical conductivity are respectively given by

$$\sigma = \begin{cases} \sigma_a & = & 10^4, & \text{in } \Omega_a \\ \sigma_b & = & 10^{-4} & \text{in } \Omega_a \end{cases}$$

We solve the following BVP

Usual BVP 5 : 2D electrostatic problem

Find $\varphi \in \mathrm{H}^1(\Omega)$ such that

$$\operatorname{div}(\sigma \boldsymbol{\nabla}\varphi) = 0 \quad \text{in } \Omega, \tag{3.31}$$

$$\varphi = 0 \quad \text{on } \Gamma_3 \cup \Gamma_7, \tag{3.32}$$

$$\varphi = 12 \quad \text{on } \Gamma_1 \cup \Gamma_5, \tag{3.33}$$

$$\sigma \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \Gamma_{10}. \tag{3.34}$$

The problem (3.31)-(3.34) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

$\mathcal{S}calar$ BVP 8 : 2D electrostatic problem

Find $\varphi \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} \mathcal{L}(\varphi) &= 0 && \text{in } \Omega, \\ \varphi &= g^D && \text{on } \Gamma^D, \\ \frac{\partial \varphi}{\partial n_{\mathcal{L}}} + a^R \varphi &= g^R && \text{on } \Gamma^R. \end{aligned}$$

where

- $\mathcal{L} := \mathcal{L}_{\sigma \mathbb{I}, \mathbf{0}, \boldsymbol{V}, \beta}$, and then the conormal derivative of φ is given by
$$\frac{\partial \varphi}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \boldsymbol{\nabla} \varphi, \boldsymbol{n} \rangle - \langle \boldsymbol{b} \varphi, \boldsymbol{n} \rangle = \sigma \frac{\partial \varphi}{\partial n}.$$
- $\Gamma^D = \Gamma_1 \cup \Gamma_3 \cup \Gamma_5 \cup \Gamma_7$ and $\Gamma^R = \Gamma_{10}$. The other borders should not be used to specify boundary conditions: they do not intervene in the variational formulation and in the physical problem!
- $g^D := 0$ on $\Gamma_3 \cup \Gamma_7$, and $g^D := 12$ on $\Gamma_1 \cup \Gamma_5$.
- $a^R = g^R := 0$ on Γ^R.

To write this problem properly with FC-VFEM$\mathbb{P}_1$ toolbox, we split (3.31) in two parts

$$\begin{aligned} \operatorname{div}(\sigma_a \boldsymbol{\nabla} \varphi) &= 0 && \text{in } \Omega_a \\ \operatorname{div}(\sigma_b \boldsymbol{\nabla} \varphi) &= 0 && \text{in } \Omega_b \end{aligned}$$

and we set these PDEs on each domains. This is done in Matlab Listing 3.7.

Listing 3.7: **Setting the 2D electrostatic BVP, Matlab code**

```
Th=siMesh(meshfile,'dim',2,'format','gmsh');
Lop=Loperator(dim,d,{sigma2,0;0,sigma2},[],[],[]);
pde=PDEelt(Lop);
bvp=BVP(Th,pde);
Lop=Loperator(dim,d,{sigma1,0;0,sigma1},[],[],[]);
pde=PDEelt(Lop);
bvp.setPDE(2,10,pde);
bvp.setDirichlet( 1, 12);
bvp.setDirichlet( 3, 0);
bvp.setDirichlet( 5, 12);
bvp.setDirichlet( 7, 0);
```

We show in Figures 3.5 and 3.6 respectively the potential φ and the norm of the electric field $\boldsymbol{E}$.

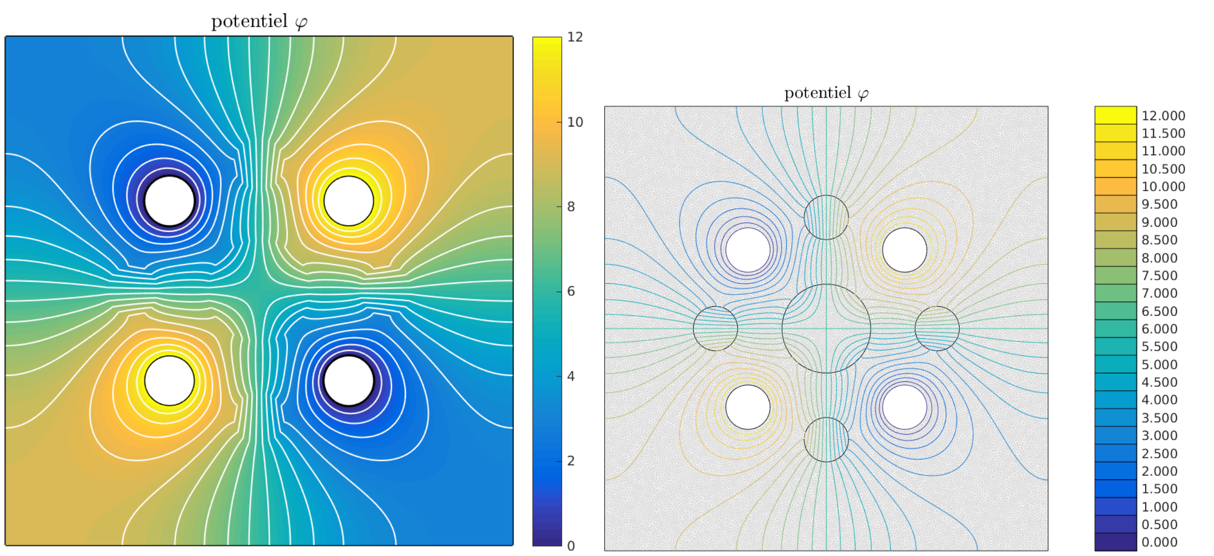


Figure 3.5: Test 1, potential φ

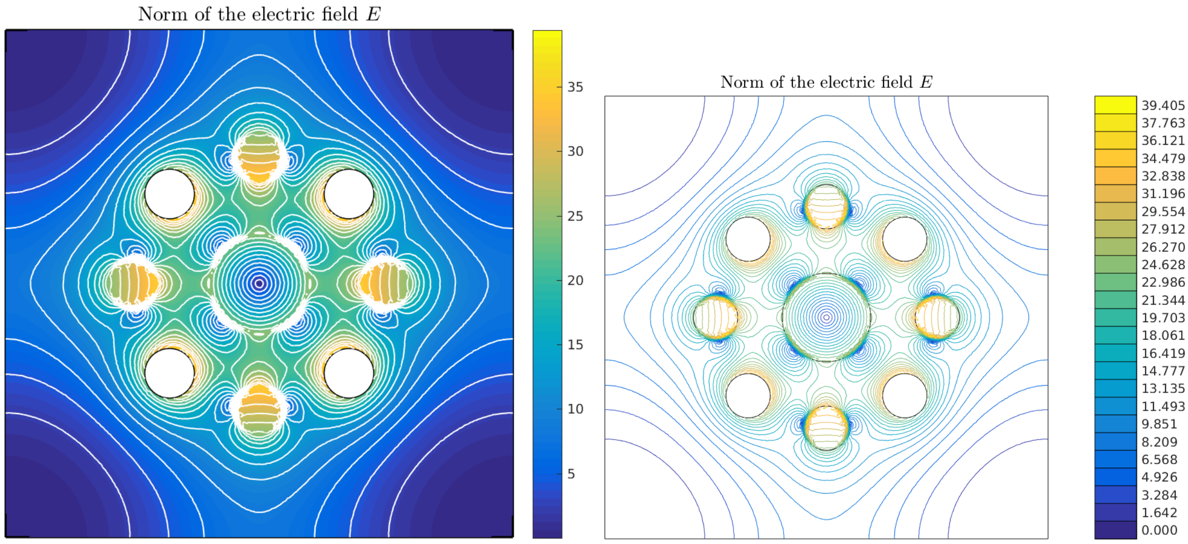


Figure 3.6: Test 1, norm of the electrical field $\boldsymbol{E}$

Chapter 4

Vector boundary value problems

4.1 Elasticity problem

4.1.1 General case ($d = 2, 3$)

We consider here Hooke's law in linear elasticity, under small strain hypothesis (see for example [3]).

For a sufficiently regular vector field $\boldsymbol{u} = (u_1, \dots, u_d) : \Omega \to \mathbb{R}^d$, we define the linearized strain tensor $\underline{\boldsymbol{\epsilon}}$ by

$$\underline{\boldsymbol{\epsilon}}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}(\boldsymbol{u}) + \boldsymbol{\nabla}^t(\boldsymbol{u})\right).$$

We set $\underline{\boldsymbol{\epsilon}} = (\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})^t$ in 2d and $\underline{\boldsymbol{\epsilon}} = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{13})^t$ in 3d, with $\epsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. Then the Hooke's law writes

$$\underline{\boldsymbol{\sigma}} = \mathbb{C}\underline{\boldsymbol{\epsilon}},$$

where $\underline{\boldsymbol{\sigma}}$ is the elastic stress tensor and $\mathbb{C}$ the elasticity tensor.

The material is supposed to be isotropic. Thus the elasticity tensor $\mathbb{C}$ is only defined by the Lamé parameters λ and μ, which satisfy $\lambda + \mu > 0$. We also set $\gamma = 2\mu + \lambda$. For $d = 2$ or $d = 3$, $\mathbb{C}$ is given by

$$\mathbb{C} = \begin{pmatrix} \lambda \mathbb{1}_2 + 2\mu \mathbb{I}_2 & 0 \\ 0 & \mu \end{pmatrix}_{3\times 3} \quad \text{or} \quad \mathbb{C} = \begin{pmatrix} \lambda \mathbb{1}_3 + 2\mu \mathbb{I}_3 & 0 \\ 0 & \mu \mathbb{I}_3 \end{pmatrix}_{6\times 6},$$

respectively, where $\mathbb{1}_d$ is a d-by-d matrix of ones, and $\mathbb{I}_d$ the d-by-d identity matrix.

For dimension $d = 2$ or $d = 3$, we have:

$$\boldsymbol{\sigma}_{\alpha\beta}(\boldsymbol{u}) = 2\mu\,\boldsymbol{\epsilon}_{\alpha\beta}(\boldsymbol{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{u}))\delta_{\alpha\beta} \quad \forall \alpha, \beta \in [\![1, d]\!]$$

The problem to solve is the following

Usual vector BVP 2 : Elasticity problem

Find $\boldsymbol{u} = \mathrm{H}^2(\Omega)^d$ such that

$$-\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u})) = \boldsymbol{f}, \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{4.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma^R, \tag{4.2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma^D. \tag{4.3}$$

Now, with the following lemma, we obtain that this problem can be rewritten as the vector BVP defined by (1.14) to (1.16).

Lemme 4.1

Let $\mathcal{H}$ be the d-by-d matrix of the second order linear differential operators defined in (1.10) where $\mathcal{H}_{\alpha,\beta} = \mathcal{L}_{\mathbb{A}^{\alpha,\beta},\mathbf{0},\mathbf{0},0}$, $\forall(\alpha,\beta) \in [\![1,d]\!]^2$, with

$$(\mathbb{A}^{\alpha,\beta})_{k,l} = \mu\delta_{\alpha\beta}\delta_{kl} + \mu\delta_{k\beta}\delta_{l\alpha} + \lambda\delta_{k\alpha}\delta_{l\beta}, \ \forall(k,l) \in [\![1,d]\!]^2. \tag{4.4}$$

then

$$\mathcal{H}(\boldsymbol{u}) = -\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}) \tag{4.5}$$

and, $\forall\alpha \in [\![1,d]\!]$,

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} = (\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{n})_\alpha. \tag{4.6}$$

The proof is given in appendix ??. So we obtain

***Vector* BVP 3 : Elasticity problem with $\mathcal{H}$ operator in dimension $d = 2$ or $d = 3$**

Let $\mathcal{H}$ be the d-by-d matrix of the second order linear differential operators defined in (1.10) where $\forall(\alpha,\beta) \in [\![1,d]\!]^2$, $\mathcal{H}_{\alpha,\beta} = \mathcal{L}_{\mathbb{A}^{\alpha,\beta},\mathbf{0},\mathbf{0},0}$, with

- for $d = 2$,

$$\mathbb{A}^{1,1} = \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix}, \ \mathbb{A}^{1,2} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}, \ \mathbb{A}^{2,1} = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix}, \ \mathbb{A}^{2,2} = \begin{pmatrix} \mu & 0 \\ 0 & \gamma \end{pmatrix}$$

- for $d = 3$,

$$\begin{aligned}
\mathbb{A}^{1,1} &= \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, & \mathbb{A}^{1,2} &= \begin{pmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbb{A}^{1,3} &= \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix} \\
\mathbb{A}^{2,1} &= \begin{pmatrix} 0 & \mu & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \mathbb{A}^{2,2} &= \begin{pmatrix} \mu & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \mu \end{pmatrix}, & \mathbb{A}^{2,3} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \mu & 0 \end{pmatrix}, \\
\mathbb{A}^{3,1} &= \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, & \mathbb{A}^{3,2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \lambda & 0 \end{pmatrix}, & \mathbb{A}^{3,3} &= \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \gamma \end{pmatrix}.
\end{aligned}$$

The elasticity problem (4.1) to (4.3) can be rewritten as :
Find $\boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d) \in (\mathrm{H}^2(\Omega))^d$ such that

$$\mathcal{H}(\boldsymbol{u}) = \boldsymbol{f}, \quad \text{in } \Omega, \tag{4.7}$$

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} = 0, \quad \text{on } \Gamma_\alpha^R = \Gamma^R, \ \forall\alpha \in [\![1,d]\!] \tag{4.8}$$

$$\boldsymbol{u}_\alpha = 0, \quad \text{on } \Gamma_\alpha^D = \Gamma^D, \ \forall\alpha \in [\![1,d]\!]. \tag{4.9}$$

4.1.2 2D example

For example, in 2d, we want to solve the elasticity problem (4.1) to (4.3) where Ω and its boundaries are given in Figure 4.1.

The material's properties are given by Young's modulus E and Poisson's coefficient ν. As we use plane strain hypothesis, Lame's coefficients verify

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\,\nu}{(1+\nu)(1-2\nu)}, \quad \gamma = 2\mu + \lambda$$

The material is rubber so that $E = 21.10^5$Pa and $\nu = 0.45$. We also have $\boldsymbol{f} = \boldsymbol{x} \mapsto (0,-1)^t$ and we choose $\Gamma^R = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$, $\Gamma^D = \Gamma^4$.

We give in Listing 4.1 the corresponding Matlab codes.

Figure 4.1: Domain and boundaries

Listing 4.1: **2D elasticity, Matlab code**

```
Th=fc_simesh.HyperCube(dim,[10*N,N],'trans',@(q) [20*q(1,:);-1+2*q(2,:)]);
gamma=lambda+2*mu;
Hop=Hoperator(dim,dim,dim);
Hop.set(1,1,Loperator(dim,dim,{gamma,[];[],mu},[],[],[]));
Hop.set(1,2,Loperator(dim,dim,{[],lambda;mu,[]},[],[],[]));
Hop.set(2,1,Loperator(dim,dim,{[],mu;lambda,[]},[],[],[]));
Hop.set(2,2,Loperator(dim,dim,{mu,[];[],gamma},[],[],[]));
pde=PDEelt(Hop,{0,-1});
bvp=BVP(Th,pde);
bvp.setDirichlet(1,0.,1:2);
U=bvp.solve('split',true);
```

One can also use the Matlab function HOPERATOR.STIFFELAS to build the elasticity operator :

```
Hop=Hoperator.StiffElas(dim,lambda,mu);
```

For a given mesh, its displacement scaled by a factor 50 is shown on Figure 4.2

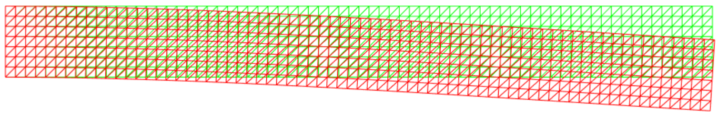


Figure 4.2: Mesh displacement scaled by a factor 50 for the 2D elasticity problem

4.1.3 3D example

Let $\Omega = [0,5] \times [0,1] \times [0,1] \subset \mathbb{R}^3$. The boundary of Ω is made of six faces and each one has a unique label : 1 to 6 respectively for faces $x_1 = 0$, $x_1 = 5$, $x_2 = 0$, $x_2 = 1$, $x_3 = 0$ and $x_3 = 1$. We represent them in Figure 4.3.

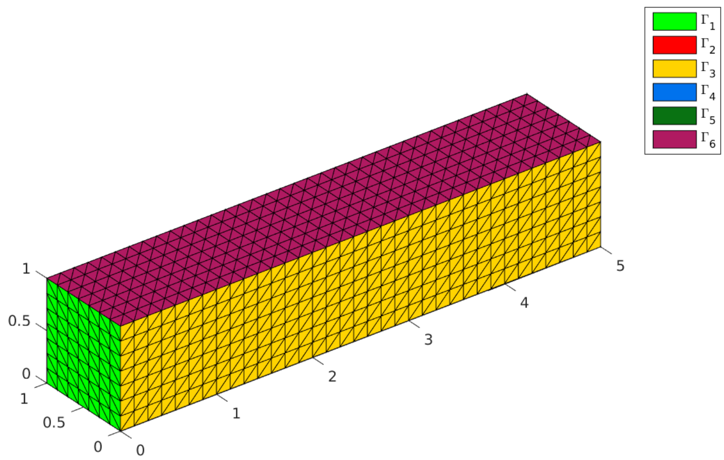


Figure 4.3: Domain and boundaries

We want to solve the elasticity problem (4.1) to (4.3) with $\Gamma^D = \Gamma_1$, $\Gamma^N = \bigcup_{i=2}^{6} \Gamma_i$ and $\boldsymbol{f} = \boldsymbol{x} \mapsto (0,0,-1)^t$.

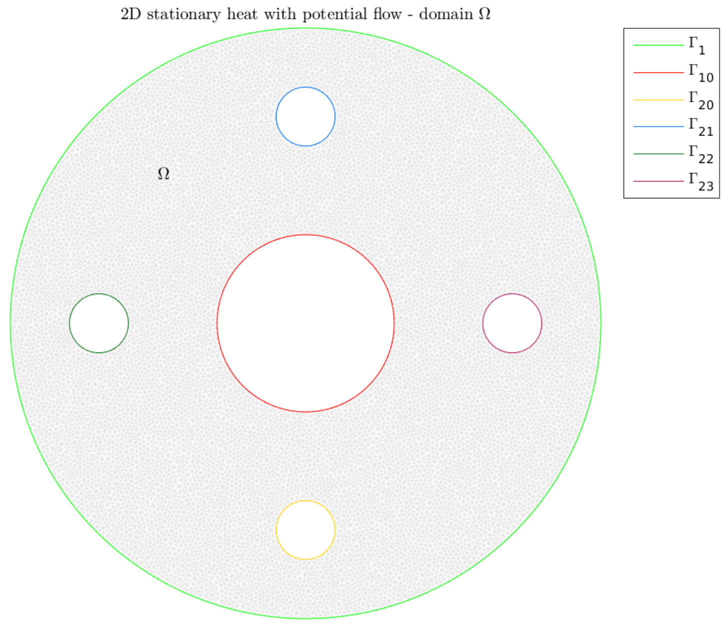


Figure 4.5: Domain and boundaries

We give in Listing 4.2 the corresponding Matlab code using function HOPERATOR.STIFFELAS .

Listing 4.2: **3D elasticity, Matlab code**

```
Th=fc_simesh.HyperCube(dim,[L*N,N,N],'trans',@(q) [L*q(1,:);q(2,:);q(3,:)]);
Hop=Hoperator();
Hop.opStiffElas(dim,lambda,mu);
pde=PDEelt(Hop,{0,0,-1});
bvp=BVP(Th,pde);
fprintf('2.b Solving 3D elasticity BVP\n')
```

The displacement scaled by a factor 2000 for a given mesh is shown on Figure 4.4.

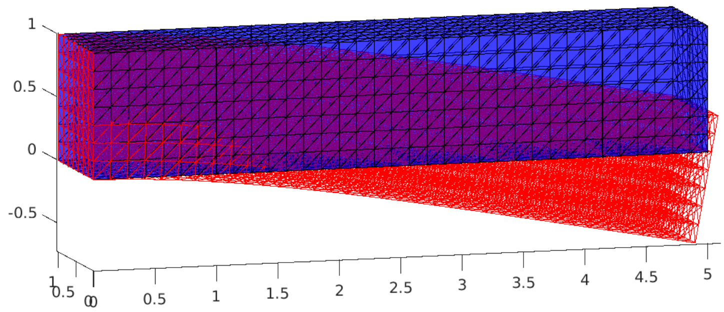


Figure 4.4: Result for the 3D elasticity problem

4.2 Stationary heat with potential flow in 2D

Let Γ_1 be the unit circle, Γ_{10} be the circle with center point $(0,0)$ and radius 0.3. Let Γ_{20}, Γ_{21}, Γ_{22} and Γ_{23} be the circles with radius 0.1 and respectively with center point $(0,-0.7)$, $(0,0.7)$, $(-0.7,0)$ and $(0.7,0)$. The domain $\Omega \subset \mathbb{R}^2$ is defined as the inner of Γ_1 and the outer of all other circles (see Figure 4.5).

The 2D problem to solve is the following

Usual BVP 6 : 2D problem : stationary heat with potential flow

Find $u \in H^2(\Omega)$ such that

$$\begin{aligned} -\operatorname{div}(\alpha \nabla u) + \langle \boldsymbol{V}, \nabla u \rangle + \beta u &= 0 \ \text{ in } \Omega \subset \mathbb{R}^2, &(4.10)\\ u &= 20 * \boldsymbol{x}_2 \ \text{ on } \Gamma_{21}, &(4.11)\\ u &= 0 \ \text{ on } \Gamma_{22} \cup \Gamma_{23}, &(4.12)\\ \frac{\partial u}{\partial n} &= 0 \ \text{ on } \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{20} &(4.13) \end{aligned}$$

where Ω and its boundaries are given in Figure 4.5. This problem is well posed if $\alpha(\boldsymbol{x}) > 0$ and $\beta(\boldsymbol{x}) \geqslant 0$.
We choose α and β in Ω as :

$$\begin{aligned} \alpha(\boldsymbol{x}) &= 0.1 + \boldsymbol{x}_2^2, \\ \beta(\boldsymbol{x}) &= 0.01 \end{aligned}$$

The potential flow is the velocity field $\boldsymbol{V} = \nabla \phi$ where the scalar function ϕ is the velocity potential solution of the 2D BVP (4.14)-(4.17)

Usual BVP 7 : 2D velocity potential BVP

Find $\phi \in H^2(\Omega)$ such that

$$\begin{aligned} -\Delta \phi &= 0 \ \text{ in } \Omega, &(4.14)\\ \phi &= -20 \ \text{ on } \Gamma_{21}, &(4.15)\\ \phi &= 20 \ \text{ on } \Gamma_{20}, &(4.16)\\ \frac{\partial \phi}{\partial n} &= 0 \ \text{ on } \Gamma_1 \cup \Gamma_{23} \cup \Gamma_{22} &(4.17) \end{aligned}$$

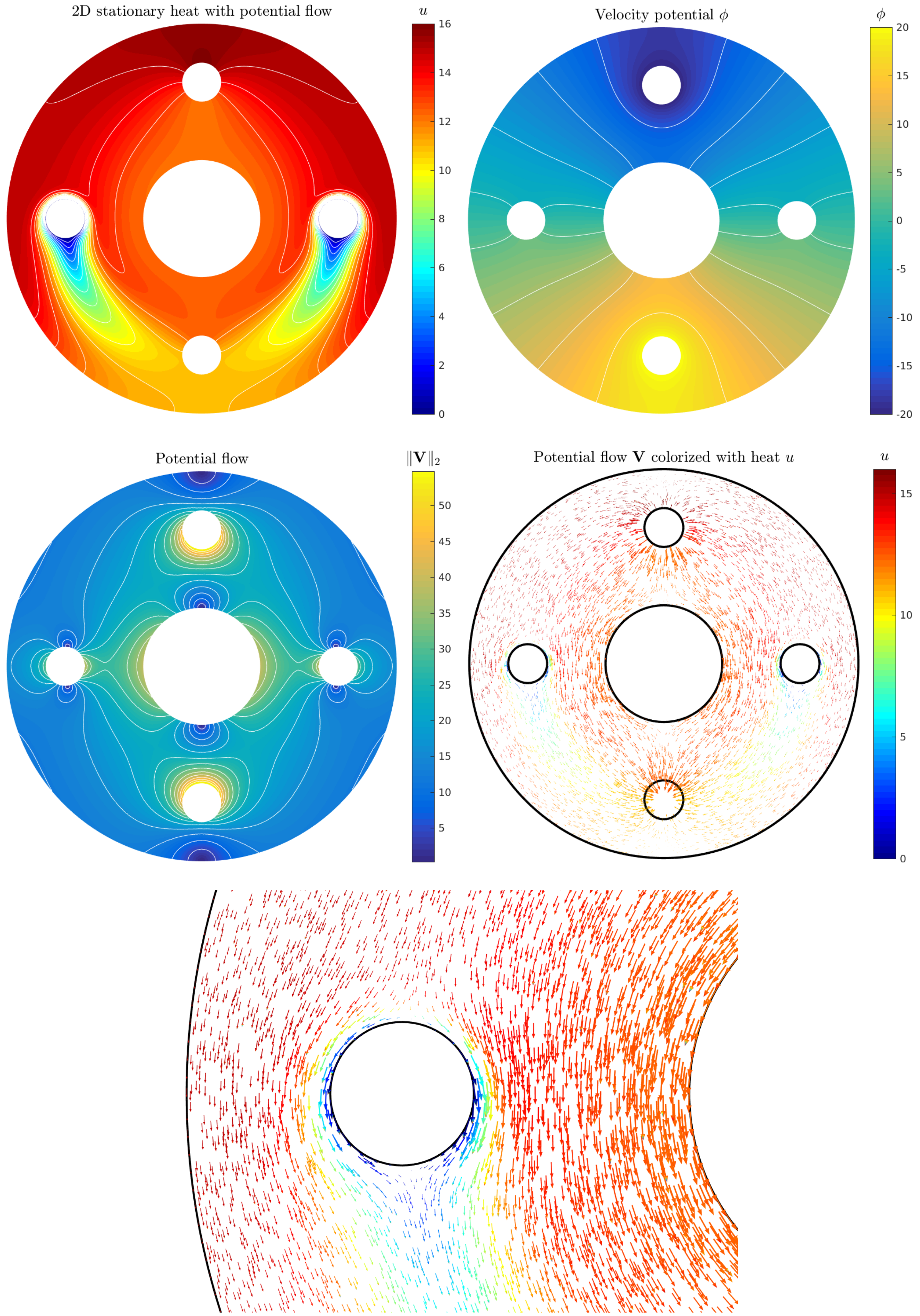
Then the potential flow $\boldsymbol{V}$ is *solution* of (4.18)

Usual vector BVP 3 : 2D potential flow

Find $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2) \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\boldsymbol{V} = \nabla \phi \ \text{ in } \Omega, \qquad (4.18)$$

For a given mesh, the numerical result for heat u is represented in Figure **??**, velocity potential ϕ and potential flow $\boldsymbol{V}$ are shown on Figure **??**.



Now we will present two manners of solving these problems using FC-VFEM$\mathbb{P}_1$ codes.

4.2.1 Method 1 : split in three parts

The 2D potential velocity problem (4.14)-(4.17) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

Scalar BVP 9 : 2D potential velocity

Find $\phi \in H^2(\Omega)$ such that

$$\begin{aligned} \mathcal{L}(\phi) &= f && \text{in } \Omega, \\ \phi &= g^D && \text{on } \Gamma^D, \\ \frac{\partial \phi}{\partial n_{\mathcal{L}}} + a^R \phi &= g^R && \text{on } \Gamma^R. \end{aligned}$$

where

- $\mathcal{L} := \mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}$, and then the conormal derivative of ϕ is given by

$$\frac{\partial \phi}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \nabla \phi, \boldsymbol{n} \rangle - \langle \boldsymbol{b}\phi, \boldsymbol{n} \rangle = \frac{\partial \phi}{\partial n}.$$

- $f(\boldsymbol{x}) := 0$
- $\Gamma^D = \Gamma_{20} \cup \Gamma_{21}$
- $\Gamma^R = \Gamma_1 \cup \Gamma_{23} \cup \Gamma_{22}$
- $g^D := 20$ on Γ_{20}, and $g^D := -20$ on Γ_{21}
- $g^R = a^R := 0$ on Γ^R. (Neumann boundary condition)

The code using the toolbox for solving (4.14)-(4.17) is given in Listing 4.6.

Listing 4.3: **Stationary heat with potential flow in 2D, Matlab code (method 1)**

```
d=2;
Lop=Loperator(d,d,{1,[];[],1},[],[],[]);
bvpPotential=BVP(Th,PDEelt(Lop));
bvpPotential.setDirichlet(20,20);
bvpPotential.setDirichlet(21,−20);
phi=bvpPotential.solve();
```

Now to compute $\boldsymbol{V}$, we can write the potential flow problem (4.18) with $\mathcal{H}$-operators as

$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{pmatrix} = \mathcal{B} \begin{pmatrix} \phi \\ \phi \end{pmatrix}$$

where

$$\mathcal{B} = \begin{pmatrix} \mathcal{L}_{\mathbb{O}_2,\mathbf{0}_2,(1,0)^t,1} & 0 \\ 0 & \mathcal{L}_{\mathbb{O}_2,\mathbf{0}_2,(0,1)^t,0} \end{pmatrix}$$

The code using the toolbox for solving this problem is given in Listing 4.6.

Listing 4.4: **Stationary heat with potential flow in 2D, Matlab code (method 1)**

```
Hop=Hoperator(Th.dim,d,d);
Hop.H{1,1}=Loperator(d,d,[],[],{1,0},[]);
Hop.H{2,2}=Loperator(d,d,[],[],{0,1},[]);
V=Hop.apply(Th,{phi,phi});
```

Obviously, one can compute separately $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$.

Finally, the stationary heat BVP (4.10)-(4.13) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

Usual BVP 8 : 2D stationary heat

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$\begin{aligned} \mathcal{L}(u) &= f && \text{in } \Omega, \\ u &= g^D && \text{on } \Gamma^D, \\ \frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u &= g^R && \text{on } \Gamma^R. \end{aligned}$$

where

- $\mathcal{L} := \mathcal{L}_{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \mathbf{0}, \boldsymbol{V}, \beta}$, and then the conormal derivative of u is given by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \boldsymbol{\nabla} u, \boldsymbol{n} \rangle - \langle \boldsymbol{b} u, \boldsymbol{n} \rangle = \alpha \frac{\partial u}{\partial n}.$$

- $f := 0$
- $\Gamma^D = \Gamma_{21} \cup \Gamma_{22} \cup \Gamma_{23}$
- $\Gamma^R = \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{20}$
- $g^D(x, y) := 20y$ on Γ_{21}, and $g^D := 0$ on $\Gamma_{22} \cup \Gamma_{23}$
- $g^R := 0$ and $a^R := 0$ on Γ^R

The code using the toolbox FC-VFEM$\mathbb{P}_1$ for solving (4.10)-(4.13) is given in Listing 4.6.

Listing 4.5: **Stationary heat with potential flow in 2D, Matlabcode (method 1)**

```
Lop=Loperator(d,d,{af,[];[],af},[],V,b);
bvpHeat=BVP(Th,PDEelt(Lop));
bvpHeat.setDirichlet(21,gD);
bvpHeat.setDirichlet(22, 0);
bvpHeat.setDirichlet(23, 0);
u=bvpHeat.solve();
```

4.2.2 Method 2 : have fun with $\mathcal{H}$-operators

We can merged velocity potential BVP (4.14)-(4.17) and potential flow to obtain the new BVP

Usual vector BVP 4 : Velocity potential and potential flow in 2D

Find $\phi \in \mathrm{H}^2(\Omega)$ and $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ such that

$$-\left(\frac{\partial \boldsymbol{V}_1}{\partial x} + \frac{\partial \boldsymbol{V}_2}{\partial y}\right) = 0 \quad \text{in } \Omega, \tag{4.19}$$

$$\boldsymbol{V}_1 - \frac{\partial \phi}{\partial x} = 0 \quad \text{in } \Omega, \tag{4.20}$$

$$\boldsymbol{V}_2 - \frac{\partial \phi}{\partial y} = 0 \quad \text{in } \Omega, \tag{4.21}$$

$$\phi = -20 \quad \text{on } \Gamma_{21}, \tag{4.22}$$

$$\phi = 20 \quad \text{on } \Gamma_{20}, \tag{4.23}$$

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_{23} \cup \Gamma_{22} \tag{4.24}$$

We can also replace (4.19) by $-\Delta\phi = 0$.

Let $\boldsymbol{w} = \begin{pmatrix} \phi \\ \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{pmatrix}$, the previous problem (4.19)-(4.24) can be equivalently expressed as the vector BVP (1.14)-(1.16) :

Vector BVP 4 : Velocity potential and potential flow in 2D

Find $\boldsymbol{w} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3) \in (\mathrm{H}^2(\Omega))^3$ such that

$$\mathcal{H}(\boldsymbol{w}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{4.25}$$

$$\boldsymbol{w}_\alpha = g_\alpha^D \quad \text{on } \Gamma_\alpha^D,\ \forall \alpha \in [\![1,3]\!], \tag{4.26}$$

$$\frac{\partial \boldsymbol{w}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \boldsymbol{w}_\alpha = g_\alpha^R \quad \text{on } \Gamma_\alpha^R,\ \forall \alpha \in [\![1,3]\!], \tag{4.27}$$

where $\Gamma_\alpha^R = \Gamma_\alpha^D = \varnothing$ for all $\alpha \in \{2,3\}$ (no boundary conditions on $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$) and

- $\mathcal{H}$ is the 3-by-3 operator defined by

$$\mathcal{H} = \begin{pmatrix} 0 & \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_1,\boldsymbol{0},0} & \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_2,\boldsymbol{0},0} \\ \mathcal{L}_{\mathbb{O},\boldsymbol{0},-\boldsymbol{e}_1,0} & \mathcal{L}_{\mathbb{O},\boldsymbol{0},\boldsymbol{0},1} & 0 \\ \mathcal{L}_{\mathbb{O},\boldsymbol{0},-\boldsymbol{e}_2,0} & 0 & \mathcal{L}_{\mathbb{O},\boldsymbol{0},\boldsymbol{0},1} \end{pmatrix}$$

its conormal derivative are given by

$$\frac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{1,1}}} = 0, \qquad \frac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{1,2}}} = \boldsymbol{w}_2 \boldsymbol{n}_1, \qquad \frac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{1,3}}} = \boldsymbol{w}_3 \boldsymbol{n}_2,$$
$$\frac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{2,1}}} = 0, \qquad \frac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{2,2}}} = 0, \qquad \frac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{2,3}}} = 0$$
$$\frac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{3,1}}} = 0, \qquad \frac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{3,2}}} = 0, \qquad \frac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{3,3}}} = 0.$$

So we obtain

$$\frac{\partial \boldsymbol{w}}{\partial n_{\mathcal{H}_1}} \stackrel{\mathsf{def}}{=} \sum_{\alpha=1}^{3} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{1,\alpha}}} = \langle \boldsymbol{V}, \boldsymbol{n} \rangle = \frac{\partial \phi}{\partial \boldsymbol{n}}, \tag{4.28}$$

and

$$\frac{\partial \boldsymbol{w}}{\partial n_{\mathcal{H}_2}} = \frac{\partial \boldsymbol{w}}{\partial n_{\mathcal{H}_3}} := 0. \tag{4.29}$$

From (4.29), we cannot impose boundary conditions on components 2 and 3.

- $\boldsymbol{f} := \boldsymbol{0}$
- $\Gamma_1^D = \Gamma_{20} \cup \Gamma_{21}$ and $\Gamma_1^R = \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{22} \cup \Gamma_{23}$
- $g_1^D := 20$ on Γ_{20}, and $g_1^D := -20$ on Γ_{21}
- $g_1^R = a_1^R := 0$ on Γ_1^R

The solution of this vector BVP is obtain by using the Matlab code is given by Listing 4.6.

Listing 4.6: **Stationary heat with potential flow in 2D, Matlab code (method 1)**

```
d=2;
Hop=Hoperator(d,d,3);
Hop.set(1,2,Loperator(d,d,[],{-1,0},[],[]));
Hop.set(1,3,Loperator(d,d,[],{0,-1},[],[]));
Hop.set(2,1,Loperator(d,d,[],[],{-1,0},[]));
Hop.set(2,2,Loperator(d,d,[],[],[],1));
Hop.set(3,1,Loperator(d,d,[],[],{0,-1},[]));
Hop.set(3,3,Loperator(d,d,[],[],[],1));
bvpFlow=BVP(Th,PDEelt(Hop));
bvpFlow.setDirichlet(20,20,1);
bvpFlow.setDirichlet(21,-20,1);
U=bvpFlow.solve('split',true);
```

4.3 Stationary heat with potential flow in 3D

Let $\Omega \subset \mathbb{R}^3$ be the cylinder given in Figure 4.6.

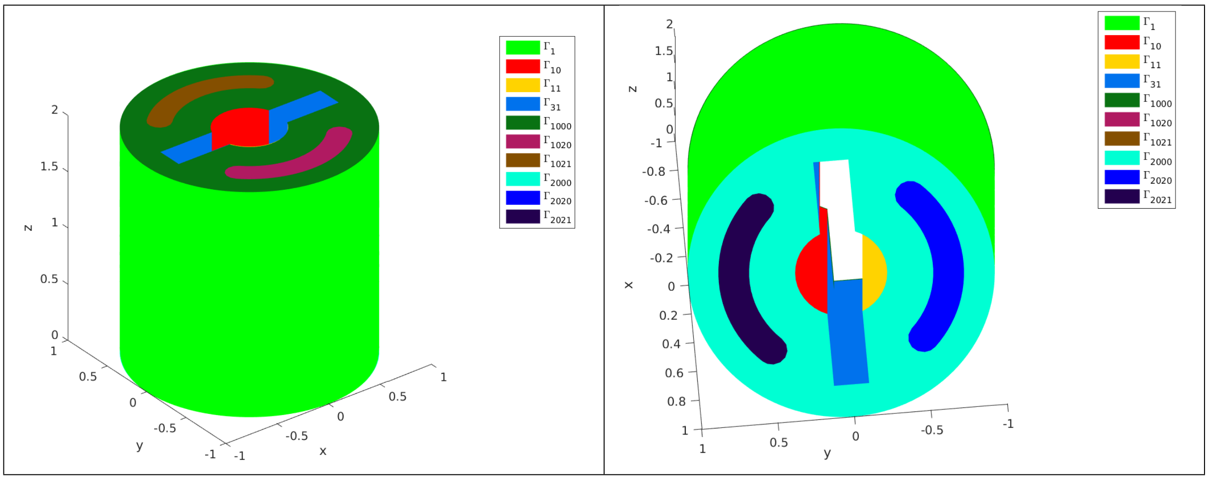


Figure 4.6: Stationary heat with potential flow : 3d mesh

The bottom and top faces of the cylinder are respectively $\Gamma_{1000} \cup \Gamma_{1020} \cup \Gamma_{1021}$ and $\Gamma_{2000} \cup \Gamma_{2020} \cup \Gamma_{2021}$. The hole surface is $\Gamma_{10} \cup \Gamma_{11} \cup \Gamma_{31}$ where $\Gamma_{10} \cup \Gamma_{11}$ is the cylinder part and Γ_{31} the plane part.

The 3D problem to solve is the following

Usual BVP 9 : 3D stationary heat with potential flow

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$
\begin{aligned}
-\operatorname{div}(\alpha \boldsymbol{\nabla} u) + \langle \boldsymbol{V}, \boldsymbol{\nabla} u \rangle + \beta u &= 0 \text{ in } \Omega \subset \mathbb{R}^3, && (4.30)\\
u &= 30 \text{ on } \Gamma_{1020} \cup \Gamma_{2020}, && (4.31)\\
u &= 10\delta_{|z-1|>0.5} \text{ on } \Gamma_{10}, && (4.32)\\
\frac{\partial u}{\partial n} &= 0 \text{ otherwise} && (4.33)
\end{aligned}
$$

where Ω and its boundaries are given in Figure 4.6. This problem is well posed if $\alpha(\boldsymbol{x}) > 0$ and $\beta(\boldsymbol{x}) \geqslant 0$.

We choose α and β in Ω as :

$$
\begin{aligned}
\alpha(\boldsymbol{x}) &= 1 + (x_3 - 1)^2; ,\\
\beta(\boldsymbol{x}) &= 0.01
\end{aligned}
$$

The potential flow is the velocity field $\boldsymbol{V} = \boldsymbol{\nabla} \phi$ where the scalar function ϕ is the velocity potential solution of the 3D BVP (4.34)-(4.37)

Usual BVP 10 : 3D velocity potential

Find $\phi \in \mathrm{H}^1(\Omega)$ such that

$$
\begin{aligned}
-\Delta\phi &= 0 \text{ in } \Omega, && (4.34)\\
\phi &= 1 \text{ on } \Gamma_{1021} \cup \Gamma_{2021}, && (4.35)\\
\phi &= -1 \text{ on } \Gamma_{1020} \cup \Gamma_{2020}, && (4.36)\\
\frac{\partial \phi}{\partial n} &= 0 \text{ otherwise} && (4.37)
\end{aligned}
$$

Then the potential flow $\boldsymbol{V}$ is *solution* of (4.38)

***Usual* vector BVP 5 : 3D potential flow**

Find $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ such that

$$\boldsymbol{V} = \boldsymbol{\nabla}\phi \quad \text{in } \Omega, \tag{4.38}$$

For a given mesh, the numerical result for heat u is represented in Figure 4.7, velocity potential ϕ and potential flow $\boldsymbol{V}$ are shown in Figure 4.8.

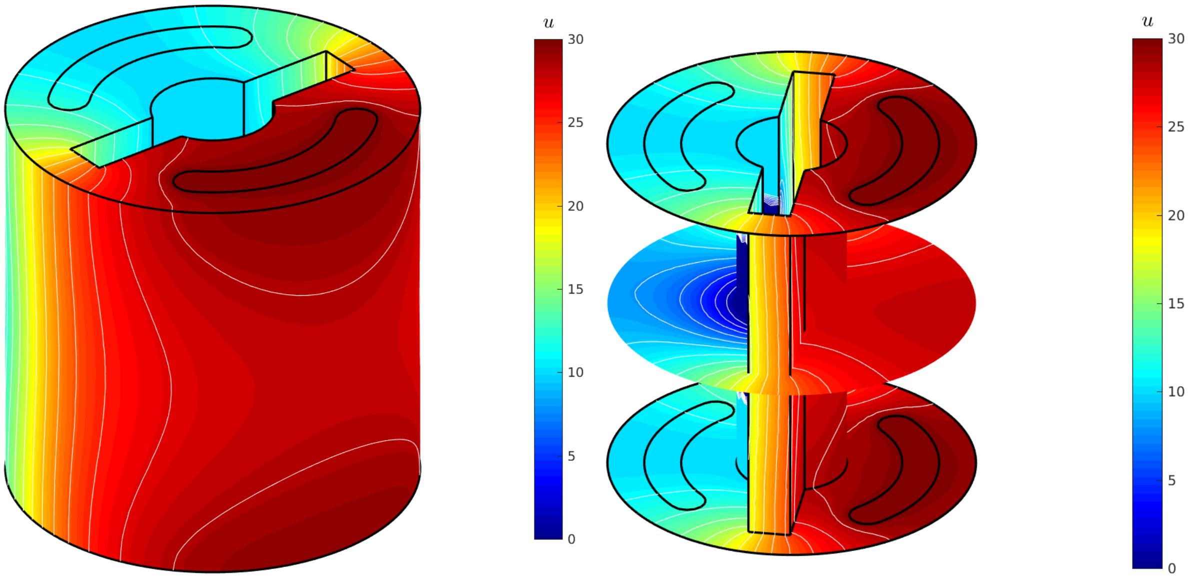


Figure 4.7: Heat solution u

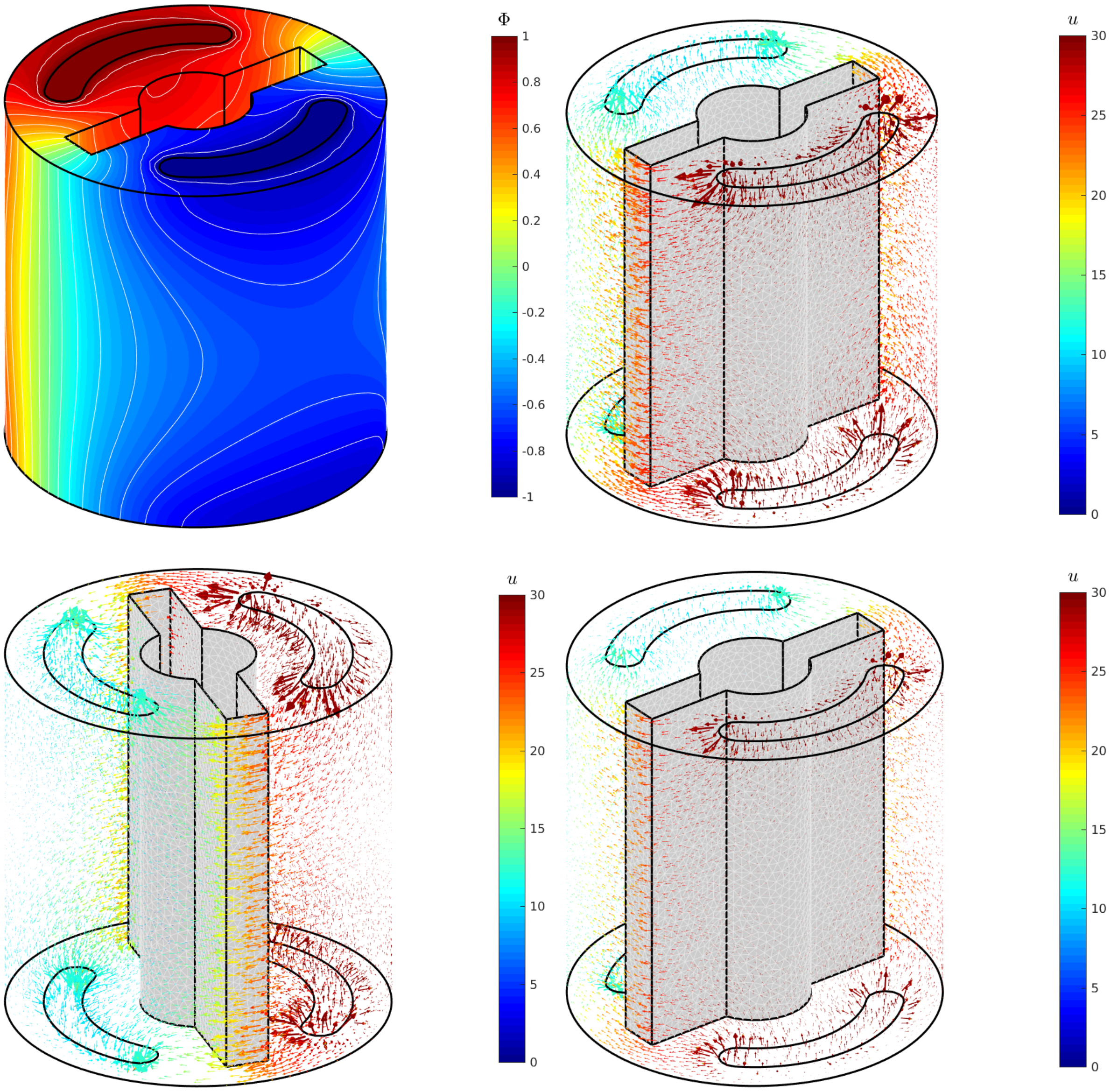


Figure 4.8: Velocity potential Φ (bottom) and velocity field $\boldsymbol{V} = \boldsymbol{\nabla}\,\Phi$ (upper)

Now we will present two manners of solving these problems using FC-VFEM$\mathbb{P}_1$ codes.

4.3.1 Method 1 : split in three parts

The 3D potential velocity problem (4.34)-(4.37) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

Scalar BVP 10 : 3D potential velocity

Find $\phi \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} \mathcal{L}(\phi) &= f && \text{in } \Omega, \\ \phi &= g^D && \text{on } \Gamma^D, \\ \frac{\partial \phi}{\partial n_{\mathcal{L}}} + a^R \phi &= g^R && \text{on } \Gamma^R. \end{aligned}$$

where

- $\mathcal{L} := \mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}$, and then the conormal derivative of ϕ is given by
$$\frac{\partial \phi}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \boldsymbol{\nabla} \phi, \boldsymbol{n} \rangle - \langle \boldsymbol{b}\phi, \boldsymbol{n} \rangle = \frac{\partial \phi}{\partial n}.$$
- $f(\boldsymbol{x}) := 0$
- $\Gamma^D = \Gamma_{1020} \cup \Gamma_{1021} \cup \Gamma_{2020} \cup \Gamma_{2021}$
- $\Gamma^R = \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{11} \cup \Gamma_{31} \cup \Gamma_{1000} \cup \Gamma_{2000}$
- $g^D := 1$ on $\Gamma_{1021} \cup \Gamma_{2021}$, and $g^D := -1$ on $\Gamma_{1020} \cup \Gamma_{2020}$
- $g^R = a^R := 0$ on Γ^R. (Neumann boundary condition)

The code using the toolbox for solving (4.34)-(4.37) is given in Listing 4.7

Listing 4.7: **Stationary heat with potential flow in 3D, Matlab code (method 1)**

```
d=3;dim=3;
Lop=Loperator(dim,d, {1,[],[];[],1,[];[],[],1},[],[],[]);
bvpFlow=BVP(Th,PDEelt(Lop));
bvpFlow.setDirichlet(1021,1.);
bvpFlow.setDirichlet(2021,1.);
bvpFlow.setDirichlet(1020,-1.);
bvpFlow.setDirichlet(2020,-1.);
Phi=bvpFlow.solve();
```

Now to compute $\boldsymbol{V}$, we can write the potential flow problem (4.38)

- with $\mathcal{H}$-operators as
$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \\ \boldsymbol{V}_2 \end{pmatrix} = \mathcal{B} \begin{pmatrix} \phi \\ \phi \\ \phi \end{pmatrix}$$
where
$$\mathcal{B} = \begin{pmatrix} \mathcal{L}_{\mathbb{O}_3,\mathbf{0}_3,(1,0,0)^t,1} & 0 & 0 \\ 0 & \mathcal{L}_{\mathbb{O}_3,\mathbf{0}_3,(0,1,0)^t,0} & 0 \\ 0 & 0 & \mathcal{L}_{\mathbb{O}_3,\mathbf{0}_3,(0,0,1)^t,0} \end{pmatrix}$$
- with $\mathcal{L}$-operators as
$$\boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \\ \boldsymbol{V}_2 \end{pmatrix} = \boldsymbol{\nabla} \phi = \begin{pmatrix} \mathcal{L}_{\mathbb{O}_3,\mathbf{0}_3,(1,0,0)^t,0}(\phi) \\ \mathcal{L}_{\mathbb{O}_3,\mathbf{0}_3,(0,1,0)^t,0}(\phi) \\ \mathcal{L}_{\mathbb{O}_3,\mathbf{0}_3,(0,0,1)^t,0}(\phi) \end{pmatrix}$$

The code using FC-VFEM$\mathbb{P}_1$ toolbox for solving this problem with $\mathcal{L}$-operators is given in Listing 4.8.

Listing 4.8: **Stationary heat with potential flow in 3D, Matlab code (method 1)**

```
Lop=Loperator(dim,d,[],[],{1,0,0},[]);
V{1}=Lop.apply(Th,Phi);
Lop=Loperator(dim,d,[],[],{0,1,0},[]);
V{2}=Lop.apply(Th,Phi);
Lop=Loperator(dim,d,[],[],{0,0,1},[]);
V{3}=Lop.apply(Th,Phi);
```

Finally, the stationary heat BVP (4.30)-(**??**) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

Scalar BVP 11 : 3D stationary heat

Find $u \in \mathrm{H}^1(\Omega)$ such that

$$\begin{aligned} \mathcal{L}(u) &= f && \text{in } \Omega, \\ u &= g^D && \text{on } \Gamma^D, \\ \frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u &= g^R && \text{on } \Gamma^R. \end{aligned}$$

where

- $\mathcal{L} := \mathcal{L}_{\begin{pmatrix}\alpha & 0 & 0\\ 0 & \alpha & 0\\ 0 & 0 & \alpha\end{pmatrix},\mathbf{0},\boldsymbol{V},\beta}$, and then the conormal derivative of u is given by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A}\,\boldsymbol{\nabla}\, u, \boldsymbol{n}\rangle - \langle \boldsymbol{b}u, \boldsymbol{n}\rangle = \alpha\frac{\partial u}{\partial n}.$$

- $f := 0$
- $\Gamma^D = \Gamma_{1020} \cup \Gamma_{2020} \cup \Gamma_{10}$
- $\Gamma^R = \Gamma_1 \cup \Gamma_{11} \cup \Gamma_{31} \cup \Gamma_{1000} \cup \Gamma_{1021} \cup \Gamma_{2000} \cup \Gamma_{2021}$
- $g^D(x,y,z) := 30$ on $\Gamma_{1020} \cup \Gamma_{2020}$, and $g^D(x,y,z) := 10(|z-1| > 0.5)$ on Γ_{10}
- $g^R := 0$ and $a^R := 0$ on Γ^R

The code using the toolbox for solving (4.30)-(**??**) is given in Figure 4.9.

Listing 4.9: **Stationary heat with potential flow in 3D, Matlab code (method 1)**

```
af=@(x,y,z) 1+(z-1).^2;
Lop=Loperator(dim,d, {af,[],[];[],af,[];[],[],af},[], {V{1},V{2},V{3}},0.01);
bvpHeat=BVP(Th,PDEelt(Lop));
bvpHeat.setDirichlet(1020,30.);
bvpHeat.setDirichlet(2020,30.);
bvpHeat.setDirichlet(10, @(x,y,z) 10*(abs(z-1)>0.5));
U=bvpHeat.solve();
```

4.3.2 Method 2 : have fun with $\mathcal{H}$-operators

To solve problem (4.30)-(4.33), we need to compute the velocity field $\boldsymbol{V}$. For that we can rewrite the potential flow problem (4.34)-(4.37), by introducing $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3)$ as unknowns :

Usual vector BVP 6 : Velocity potential and velocity field in 3D

Find $\phi \in \mathrm{H}^2(\Omega)$ and $\boldsymbol{V} \in \mathrm{H}^1(\Omega)^3$ such that

$$-\left(\frac{\partial \boldsymbol{V}_1}{\partial x} + \frac{\partial \boldsymbol{V}_2}{\partial y} + \frac{\partial \boldsymbol{V}_3}{\partial z}\right) = 0 \text{ in } \Omega, \tag{4.39}$$

$$\boldsymbol{V}_1 - \frac{\partial \phi}{\partial x} = 0 \text{ in } \Omega, \tag{4.40}$$

$$\boldsymbol{V}_2 - \frac{\partial \phi}{\partial y} = 0 \text{ in } \Omega, \tag{4.41}$$

$$\boldsymbol{V}_3 - \frac{\partial \phi}{\partial z} = 0 \text{ in } \Omega, \tag{4.42}$$

with boundary conditions (4.35) to (4.37).

We can also replace (4.39) by $-\Delta\phi = 0$.

Let $\boldsymbol{w} = \begin{pmatrix}\phi\\ \boldsymbol{V}_1\\ \boldsymbol{V}_2\\ \boldsymbol{V}_3\end{pmatrix}$, the previous PDE can be written as a vector boundary value problem (see section 1.2) where the $\mathcal{H}$-operator is given by

$$\mathcal{H}(\boldsymbol{w}) = 0 \tag{4.43}$$

with

$$\mathcal{H}_{1,1} = 0, \quad \mathcal{H}_{1,2} = \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_1,\mathbf{0},0}, \quad \mathcal{H}_{1,3} = \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_2,\mathbf{0},0}, \quad \mathcal{H}_{1,4} = \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_3,\mathbf{0},0}, \tag{4.44}$$

$$\mathcal{H}_{2,1} = \mathcal{L}_{\mathbb{O},\mathbf{0},-\boldsymbol{e}_1,0}, \quad \mathcal{H}_{2,2} = \mathcal{L}_{\mathbb{O},\mathbf{0},\mathbf{0},1}, \quad \mathcal{H}_{2,3} = 0, \quad \mathcal{H}_{2,4} = 0, \tag{4.45}$$

$$\mathcal{H}_{3,1} = \mathcal{L}_{\mathbb{O},\mathbf{0},-\boldsymbol{e}_2,0}, \quad \mathcal{H}_{3,2} = 0, \quad \mathcal{H}_{3,3} = \mathcal{L}_{\mathbb{O},\mathbf{0},\mathbf{0},1}, \quad \mathcal{H}_{3,4} = 0, \tag{4.46}$$

$$\mathcal{H}_{4,1} = \mathcal{L}_{\mathbb{O},\mathbf{0},-\boldsymbol{e}_3,0}, \quad \mathcal{H}_{4,2} = 0, \quad \mathcal{H}_{4,3} = 0, \quad \mathcal{H}_{4,4} = \mathcal{L}_{\mathbb{O},\mathbf{0},\mathbf{0},1}, \tag{4.47}$$

and $\boldsymbol{e}_1 = (1,0,0)^t$, $\boldsymbol{e}_2 = (0,1,0)^t$, $\boldsymbol{e}_3 = (0,0,1)^t$.

The conormal derivatives are given by

$$\begin{array}{llll}
\frac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{1,1}}} = 0, & \frac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{2,1}}} = 0, & \frac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{3,1}}} = 0, & \frac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{4,1}}} = 0, \\
\frac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{1,2}}} = \boldsymbol{V}_1 \boldsymbol{n}_1, & \frac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{2,2}}} = 0, & \frac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{3,2}}} = 0, & \frac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{4,2}}} = 0, \\
\frac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{1,3}}} = \boldsymbol{V}_2 \boldsymbol{n}_2, & \frac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{2,3}}} = 0, & \frac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{3,3}}} = 0, & \frac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{4,3}}} = 0, \\
\frac{\partial \boldsymbol{w}_4}{\partial n_{\mathcal{H}_{1,4}}} = \boldsymbol{V}_3 \boldsymbol{n}_3, & \frac{\partial \boldsymbol{w}_4}{\partial n_{\mathcal{H}_{2,4}}} = 0, & \frac{\partial \boldsymbol{w}_4}{\partial n_{\mathcal{H}_{3,4}}} = 0, & \frac{\partial \boldsymbol{w}_4}{\partial n_{\mathcal{H}_{4,4}}} = 0,
\end{array}$$

So we obtain

$$\sum_{\alpha=1}^{4} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{1,\alpha}}} = \langle \boldsymbol{V}, \boldsymbol{n} \rangle = \langle \boldsymbol{\nabla} \phi, \boldsymbol{n} \rangle, \tag{4.48}$$

and

$$\sum_{\alpha=1}^{4} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{2,\alpha}}} = \sum_{\alpha=1}^{4} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{3,\alpha}}} = \sum_{\alpha=1}^{4} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{4,\alpha}}} = 0. \tag{4.49}$$

From (4.49), we cannot impose boundary conditions on components 2 to 4. Thus, with notation of section 1.2, we have $\Gamma_2^N = \Gamma_3^N = \Gamma_4^N = \Gamma$ with $g_2^N = g_3^N = g_4^N = 0$.

To take into account boundary conditions (4.35) to (4.37), we set $\Gamma_1^D = \Gamma_{1020} \cup \Gamma_{1021} \cup \Gamma_{2020} \cup \Gamma_{2021}$, $\Gamma_1^N = \Gamma \backslash \Gamma_1^D$ and $g_1^D = \delta_{\Gamma_{1020} \cup \Gamma_{2020}} - \delta_{\Gamma_{1021} \cup \Gamma_{2021}}$, $g_1^N = 0$.

The operator in (4.30) is given by $\mathcal{L}_{\alpha\mathbb{I},\mathbf{0},\boldsymbol{V},\beta}$. The conormal derivative $\frac{\partial u}{\partial n_{\mathcal{L}}}$ is

$$\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \boldsymbol{\nabla} u, \boldsymbol{n} \rangle - \langle \boldsymbol{b} u, \boldsymbol{n} \rangle = \alpha \frac{\partial u}{\partial n}.$$

The code using the toolbox for solving (4.39)-(4.42) is given in Listing 4.10

Listing 4.10: **Stationary heat with potential flow in 3D, Matlab code (method 2)**

```
d=3;dim=3;m=4;
Hop=Hoperator(dim,d,m);
Hop.set(1,2,Loperator(dim,d,[],{-1,0,0},[],[]));
Hop.set(1,3,Loperator(dim,d,[],{0,-1,0},[],[]));
Hop.set(1,4,Loperator(dim,d,[],{0,0,-1},[],[]));
Hop.set(2,1,Loperator(dim,d,[],[],{-1,0,0},[]));
Hop.set(2,2,Loperator(dim,d,[],[],[],1));
Hop.set(3,1,Loperator(dim,d,[],[],{0,-1,0},[]));
Hop.set(3,3,Loperator(dim,d,[],[],[],1));
Hop.set(4,1,Loperator(dim,d,[],[],{0,0,-1},[]));
Hop.set(4,4,Loperator(dim,d,[],[],[],1));
bvpFlow=BVP(Th,PDEelt(Hop));
bvpFlow.setDirichlet(1020,-1,1);
bvpFlow.setDirichlet(1021,1,1);
bvpFlow.setDirichlet(2020,-1,1);
bvpFlow.setDirichlet(2021,1,1);
W=bvpFlow.solve('split',true);
af=@(x,y,z) 1+(z-1).^2;
Lop=Loperator(dim,d,{af,[],[];[],af,[];[],[],af},[],{W{2},W{3},W{4}},0.01);
bvpHeat=BVP(Th,PDEelt(Lop));
bvpHeat.setDirichlet(1020,30.);
bvpHeat.setDirichlet(2020,30.);
bvpHeat.setDirichlet(10, @(x,y,z) 10*(abs(z-1)>0.5));
U=bvpHeat.solve();
```

Bibliography

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