Eigenvalues Addon [1]

for the FC-VFEM$\mathbb{P}_1$ Matlab toolbox

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Abstract

The **Eigenvalues Addon** for the FC-VFEM$\mathbb{P}_1$ Matlab toolbox contains codes which allow to compute numerically the eigenvalues and eigenfunctions of *scalar* or *vector* Boundary Value Problems. These codes use the FC-VFEM$\mathbb{P}_1$ Matlab toolbox and thus a good knowledge of the use of the latter is a prerequisite with the reading of this report.

Contents

Chapter 1

Generic Boundary Value Problems

The notations of [5] are employed in this section and extended to the vector case.

1.1 Scalar boundary value problem

Let Ω be a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$. The boundary of Ω is denoted by Γ.

We denote by $\mathcal{L}_{\mathbb{A},\boldsymbol{b},\boldsymbol{c},a_0} = \mathcal{L} : \mathrm{H}^2(\Omega) \longrightarrow L^2(\Omega)$ the second order linear differential operator acting on *scalar fields* defined, $\forall u \in \mathrm{H}^2(\Omega)$, by

$$\mathcal{L}_{\mathbb{A},\boldsymbol{b},\boldsymbol{c},a_0}(u) \stackrel{\mathsf{def}}{=} -\operatorname{div}(\mathbb{A}\boldsymbol{\nabla} u) + \operatorname{div}(\boldsymbol{b}u) + \langle \boldsymbol{\nabla} u, \boldsymbol{c}\rangle + a_0 u \tag{1.1}$$

where $\mathbb{A} \in (L^\infty(\Omega))^{d\times d}$, $\boldsymbol{b} \in (L^\infty(\Omega))^d$, $\boldsymbol{c} \in (L^\infty(\Omega))^d$ and $a_0 \in L^\infty(\Omega)$ are given functions and $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^d$. We use the same notations as in the chapter 6 of [5] and we note that we can omit either $\operatorname{div}(\boldsymbol{b}u)$ or $\langle \boldsymbol{\nabla} u, \boldsymbol{c}\rangle$ if $\boldsymbol{b}$ and $\boldsymbol{c}$ are sufficiently regular functions. We keep both terms with $\boldsymbol{b}$ and $\boldsymbol{c}$ to deal with more boundary conditions. It should be also noted that it is important to preserve the two terms $\boldsymbol{b}$ and $\boldsymbol{c}$ in the generic formulation to enable a greater flexibility in the choice of the boundary conditions.

Let Γ^D, Γ^R be open subsets of Γ, possibly empty and $f \in L^2(\Omega)$, $g^D \in \mathrm{H}^{1/2}(\Gamma^D)$, $g^R \in L^2(\Gamma^R)$, $a^R \in L^\infty(\Gamma^R)$ be given data.

A *scalar* boundary value problem is given by

Scalar BVP 1 : generic problem

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = f \quad \text{in } \Omega, \tag{1.2}$$

$$u = g^D \quad \text{on } \Gamma^D, \tag{1.3}$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = g^R \quad \text{on } \Gamma^R. \tag{1.4}$$

The **conormal derivative** of u is defined by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} \stackrel{\mathsf{def}}{=} \langle \mathbb{A}\boldsymbol{\nabla} u, \boldsymbol{n}\rangle - \langle \boldsymbol{b}u, \boldsymbol{n}\rangle \tag{1.5}$$

The boundary conditions (1.3) and (1.4) are respectively **Dirichlet** and **Robin** boundary conditions. **Neumann** boundary conditions are particular Robin boundary conditions with $a^R \equiv 0$.

To have an outline of the FC-VFEM$\mathbb{P}_1$ toolbox, a first and simple problem is quickly present. Explanations will be given in next sections.

The problem to solve is the Laplace problem for a condenser.

Usual BVP 1 : 2D condenser problem

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$-\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.6}$$
$$u = 0 \quad \text{on } \Gamma_1, \tag{1.7}$$
$$u = -12 \quad \text{on } \Gamma_{98}, \tag{1.8}$$
$$u = 12 \quad \text{on } \Gamma_{99}, \tag{1.9}$$

where Ω and its boundaries are given in Figure 1.1.

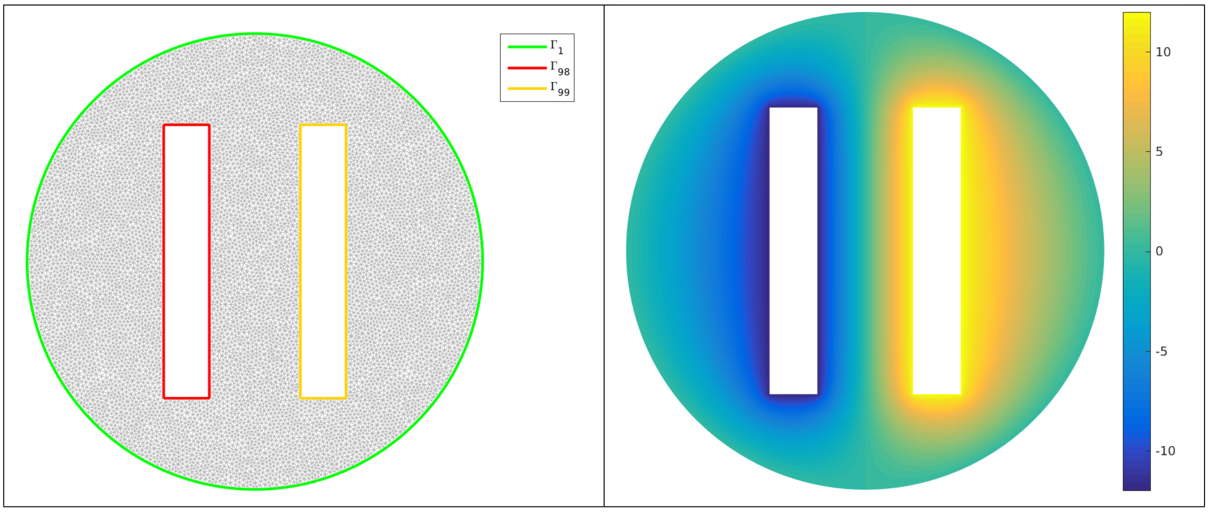

Figure 1.1: 2D condenser mesh and boundaries (left) and numerical solution (right)

The problem (1.6)-(1.9) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

Scalar BVP 2 : 2D condenser problem

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = f \quad \text{in } \Omega,$$
$$u = g^D \quad \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99}.$$

where $\mathcal{L} := \mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}$, $f \equiv 0$, and

$$g^D := 0 \text{ on } \Gamma_1,\ g^D := -12 \text{ on } \Gamma_{98},\ g^D := +12 \text{ on } \Gamma_{99}$$

In Listing 19 a complete code is given to solve this problem.

```
meshfile=gmsh.buildmesh2d('condenser',10); % generate mesh
Th=siMesh(meshfile); % read mesh
Lop=Loperator(2,2,{1,0;0,1},[],[],[]);
pde=PDEelt(Lop);
bvp=BVP(Th,pde);
bvp.setDirichlet( 1, 0.);
bvp.setDirichlet( 98, -12.);
bvp.setDirichlet( 99, +12.);
U=bvp.solve();
% Graphic parts
figure(1)
Th.plotmesh('color',0.7*[1,1,1])
hold on
Th.plotmesh('d',1,'Linewidth',2,'legend',true)
```

```
axis off,axis image
figure(2)
Th.plot(U,'edgecolor','none','facecolor','interp')
axis off,axis image;colorbar
```

Listing 1.1: Complete Matlab code to solve the 2D condenser problem with graphical representations

1.2 Vector boundary value problem

Let $m \geqslant 1$ and $\mathcal{H}$ be the m-by-m matrix of second order linear differential operators defined by

$$\left\{ \begin{array}{lllll} \mathcal{H} & : & \left(\mathrm{H}^2(\Omega)\right)^m & \longrightarrow & \left(L^2(\Omega)\right)^m \\ & & \boldsymbol{u} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) & \longmapsto & \boldsymbol{f} = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m) \stackrel{\text{def}}{=} \mathcal{H}(\boldsymbol{u}) \end{array} \right. \tag{1.10}$$

where

$$\boldsymbol{f}_\alpha = \sum_{\beta=1}^{m} \mathcal{H}_{\alpha,\beta}(\boldsymbol{u}_\beta), \quad \forall \alpha \in [\![1, m]\!], \tag{1.11}$$

with, for all $(\alpha, \beta) \in [\![1, m]\!]^2$,

$$\mathcal{H}_{\alpha,\beta} \stackrel{\text{def}}{=} \mathcal{L}_{\mathbb{A}^{\alpha,\beta}, \boldsymbol{b}^{\alpha,\beta}, \boldsymbol{c}^{\alpha,\beta}, a_0^{\alpha,\beta}} \tag{1.12}$$

and $\mathbb{A}^{\alpha,\beta} \in (L^\infty(\Omega))^{d \times d}$, $\boldsymbol{b}^{\alpha,\beta} \in (L^\infty(\Omega))^d$, $\boldsymbol{c}^{\alpha,\beta} \in (L^\infty(\Omega))^d$ and $a_0^{\alpha,\beta} \in L^\infty(\Omega)$ are given functions. We can also write in matrix form

$$\mathcal{H}(\boldsymbol{u}) = \begin{pmatrix} \mathcal{L}_{\mathbb{A}^{1,1}, \boldsymbol{b}^{1,1}, \boldsymbol{c}^{1,1}, a_0^{1,1}} & \dots & \mathcal{L}_{\mathbb{A}^{1,m}, \boldsymbol{b}^{1,m}, \boldsymbol{c}^{1,m}, a_0^{1,m}} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{\mathbb{A}^{m,1}, \boldsymbol{b}^{m,1}, \boldsymbol{c}^{m,1}, a_0^{m,1}} & \dots & \mathcal{L}_{\mathbb{A}^{m,m}, \boldsymbol{b}^{m,m}, \boldsymbol{c}^{m,m}, a_0^{m,m}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_m \end{pmatrix}. \tag{1.13}$$

We remark that the $\mathcal{H}$ operator for $m = 1$ is equivalent to the $\mathcal{L}$ operator.

For $\alpha \in [\![1, m]\!]$, we define Γ_α^D and Γ_α^R as open subsets of Γ, possibly empty, such that $\Gamma_\alpha^D \cap \Gamma_\alpha^R = \emptyset$. Let $\boldsymbol{f} \in (L^2(\Omega))^m$, $g_\alpha^D \in \mathrm{H}^{1/2}(\Gamma_\alpha^D)$, $g_\alpha^R \in L^2(\Gamma_\alpha^R)$, $a_\alpha^R \in L^\infty(\Gamma_\alpha^R)$ be given data.

A *vector* boundary value problem is given by

Vector BVP 1 : generic problem

Find $\boldsymbol{u} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) \in (\mathrm{H}^2(\Omega))^m$ such that

$$\mathcal{H}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.14}$$

$$\boldsymbol{u}_\alpha = g_\alpha^D \quad \text{on } \Gamma_\alpha^D, \ \forall \alpha \in [\![1, m]\!], \tag{1.15}$$

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \boldsymbol{u}_\alpha = g_\alpha^R \quad \text{on } \Gamma_\alpha^R, \ \forall \alpha \in [\![1, m]\!], \tag{1.16}$$

where the α-th component of the **conormal derivative** of $\boldsymbol{u}$ is defined by

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} \stackrel{\text{def}}{=} \sum_{\beta=1}^{m} \frac{\partial \boldsymbol{u}_\beta}{\partial n_{\mathcal{H}_{\alpha,\beta}}} = \sum_{\beta=1}^{m} \left(\left\langle \mathbb{A}^{\alpha,\beta} \boldsymbol{\nabla} \boldsymbol{u}_\beta, \boldsymbol{n} \right\rangle - \left\langle \boldsymbol{b}^{\alpha,\beta} \boldsymbol{u}_\beta, \boldsymbol{n} \right\rangle \right). \tag{1.17}$$

The boundary conditions (1.16) are the **Robin** boundary conditions and (1.15) is the **Dirichlet** boundary condition. The **Neumann** boundary conditions are particular Robin boundary conditions with $a_\alpha^R \equiv 0$.

In this problem, we may consider on a given boundary some conditions which can vary depending on the component. For example we may have a Robin boundary condition satisfying $\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_1}} + a_1^R \boldsymbol{u}_1 = g_1^R$ and a Dirichlet one with $\boldsymbol{u}_2 = g_2^D$.

To have an outline of the FC-VFEM$\mathbb{P}_1$ toolbox, a second and simple problem is quickly present.

Usual vector BVP 1 : 2D simple vector problem

Find $\boldsymbol{u} = (u_1, u_2) \in (\mathrm{H}^2(\Omega))^2$ such that

$$-\Delta u_1 + u_2 = 0 \;\text{ in } \Omega \subset \mathbb{R}^2, \tag{1.18}$$

$$-\Delta u_2 + u_1 = 0 \;\text{ in } \Omega \subset \mathbb{R}^2, \tag{1.19}$$

$$(u_1, u_2) = (0, 0) \;\text{ on } \Gamma_1, \tag{1.20}$$

$$(u_1, u_2) = (-12., +12.) \;\text{ on } \Gamma_{98}, \tag{1.21}$$

$$(u_1, u_2) = (+12., -12.) \;\text{ on } \Gamma_{99}, \tag{1.22}$$

where Ω and its boundaries are given in Figure 1.1.
The problem (1.18)-(1.22) can be equivalently expressed as the vector BVP (1.2)-(1.4) :

Vector BVP 2 : 2D simple vector problem

Find $\boldsymbol{u} = (u_1, u_2) \in (\mathrm{H}^2(\Omega))^2$ such that

$$\begin{aligned} \mathcal{H}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in } \Omega, \\ u_1 &= g_1^D && \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99}, \\ u_2 &= g_2^D && \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99}, \end{aligned}$$

where

$$\mathcal{H} := \begin{pmatrix} \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},0} & \mathcal{L}_{\mathbb{O},\boldsymbol{O},\boldsymbol{O},1} \\ \mathcal{L}_{\mathbb{O},\boldsymbol{O},\boldsymbol{O},1} & \mathcal{L}_{\mathbb{I},\boldsymbol{O},\boldsymbol{O},0} \end{pmatrix}, \text{ as } \mathcal{H}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\Delta & 1 \\ 1 & -\Delta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$f \equiv 0,$$

and

$$g_1^D = g_2^D := 0 \text{ on } \Gamma_1,\; g_1^D := -12,\; g_2^D := +12 \text{ on } \Gamma_{98},\; g_1^D := +12,\; g_2^D := -12 \text{ on } \Gamma_{99}$$

In Listing 21 a complete code is given to solve this problem. Numerical solutions are given in Figure 1.2.

```
meshfile=gmsh.buildmesh2d('condenser',10); % generate mesh
Th=siMesh(meshfile); % read mesh
Hop=Hoperator(2,2,2);
Hop.set([1,2],[1,2],Loperator(2,2,{1,[];[],1},[],[],[]));
Hop.set([1,2],[2,1],Loperator(2,2,[],[],[],1));
pde=PDEelt(Hop);
bvp=BVP(Th,pde);
bvp.setDirichlet( 1, 0.,1:2);
bvp.setDirichlet( 98, {-12,+12},1:2);
bvp.setDirichlet( 99, {+12,-12},1:2);
U=bvp.solve('split',true);
% Graphic parts
figure(1)
Th.plot(U{1})
axis image;axis off;shading interp
colorbar
figure(2);
Th.plot(U{2})
axis image;axis off;shading interp
colorbar
```

Listing 1.2: Complete Matlab code to solve the funny 2D vector problem with graphical representations

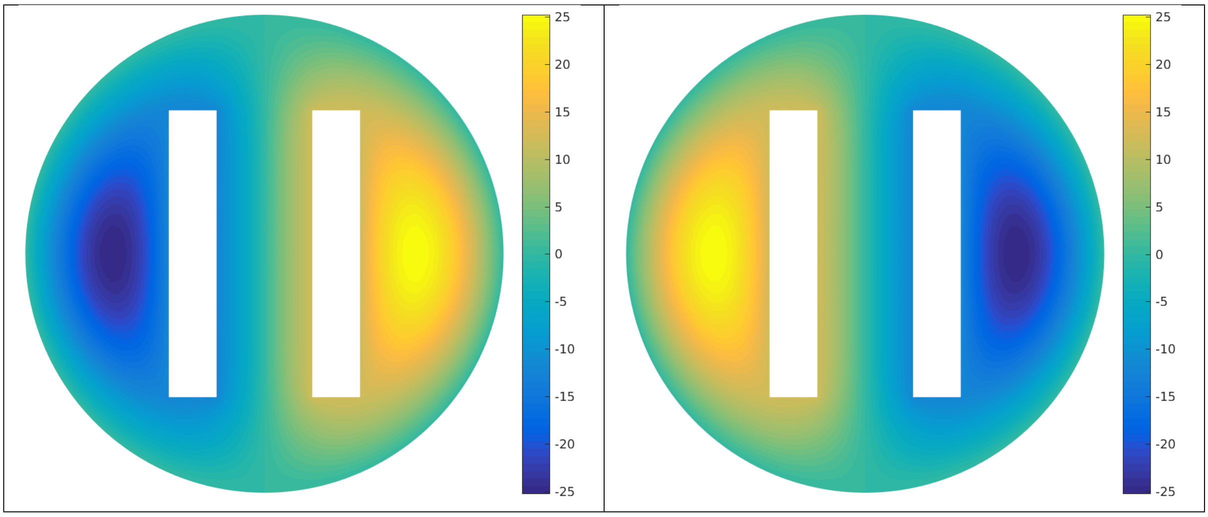

Figure 1.2: Funny vector BVP, u_1 numerical solution (left) and u_2 numerical solution (right)

Chapter 2

Generalized Eigenvalue scalar BVP

We want to solve generalized eigenvalue problems coming from scalar BVP's.

The **generalized eigenvalue problem** associated with *scalar* BVP (1.2)-(1.4) can be written as

***Scalar* EBVP 1 : generic problem**

Find $(\lambda, u) \in \mathbb{K} \times H^2(\Omega)$ such that

$$\mathcal{L}(u) = \lambda \mathcal{B}(u) \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on } \Gamma^D, \tag{2.2}$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = 0 \quad \text{on } \Gamma^R. \tag{2.3}$$

where $\mathcal{B} = \mathcal{L}_{\mathbb{O}_{d\times d}, \mathbf{0}_d, \tilde{\mathbf{c}}, \tilde{a_0}}$.

We briefly describe the main function that will be used to solve eigenvalues problems. Let bvp be a BVP object.

2.1 fc_vfemp1.addon.eigs.solve function

The function fc_vfemp1.addon.eigs.solve returns a few eigenvalues and eigenvectors obtained by solving a generalized eigenvalue scalar BVP with $\mathbb{P}_1$-Lagrange finite elements.

```
[U,lambda]=fc_vfemp1.addon.eigs.solve(bvp)
[U,lambda]=fc_vfemp1.addon.eigs.solve(bvp,options{:})
[U,lambda]=fc_vfemp1.addon.eigs.solve(bvp,Bop,options{:})
```

Description The inputs are :

- bvp a BVP object which described the *scalar* BVP (1.2)-(1.4) with all right-hand sides equal to zeros, i.e. $f := 0$, $g^D := 0$ and $g^R := 0$.
- Bop a Loperator object corresponding to operator $\mathcal{B}$. By default Bop is the operator $\mathcal{L}_{\mathbb{O}_{d\times d}, \mathbf{0}_d, \mathbf{0}_d, 1}$ for scalar BVP.
- options{:} are the parameters of the eigs Matlab function:

- options={k}, return the k *largest magnitude* eigenvalues. By default k value is 6.
- options={k,sigma}, return k eigenvalues based on sigma. For example, if sigma is `'sm'`, return the k *smallest magnitude* eigenvalues. By default sigma value is `'lm'` correponding to *largest magnitude*.

The outputs are those given by the eigs Matlab function:

- U a BVP object which described the *scalar* BVP (1.2)-(1.4) with all right-hand sides equal to zeros, i.e. $f := 0$, $g^D := 0$ and $g^R := 0$.
- lambda a Loperator object corresponding to operator $\mathcal{B}$.
- options{:} are the parameters of the eigs Matlab function:
  - options={k}, return the k *largest magnitude* eigenvalues. By default k value is 6.
  - options={k,sigma}, return k eigenvalues based on sigma. For example, if sigma is `'sm'`, return the k *smallest magnitude* eigenvalues. By default sigma value is `'lm'` correponding to *largest magnitude*.

2.2 2D samples

2.2.1 2D Laplace eigenvalues problem with Dirichlet boundary condition

We want to solve the eigenvalue problem given by (2.4)-(2.5)).

Usual EBVP 1 : 2D Laplace with Dirichlet boundary condition

Find $(\lambda, u) \in \mathbb{K} \times \mathrm{H}^2(\Omega)$ such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{2.4}$$

$$u = 0 \quad \text{on } \Gamma, \tag{2.5}$$

The problem (2.4)-(2.5) can be equivalently write as the *Scalar* EBVP 1:

Scalar EBVP 2 : 2D Laplace with Dirichlet boundary condition

Find $(\lambda, u) \in \mathbb{K} \times \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = \lambda \mathcal{B}(u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \Gamma^D = \Gamma,$$

where $\mathcal{L} = \mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}$, $\mathcal{B} = \mathcal{L}_{\mathbb{O},\mathbf{0},\mathbf{0},1}$.

Application on the rectangle $\Omega = [0, L] \times [0, H]$.

The eigenvalues and the associated eigen functions are given by

$$\lambda_{k,l} = \left(\frac{k\pi}{L}\right)^2 + \left(\frac{l\pi}{H}\right)^2, \quad u_{k,l}(x, y) = \sin(\frac{k\pi}{L}x)\sin(\frac{l\pi}{H}y), \quad \forall (k, l) \in \mathbb{N}^* \times \mathbb{N}^*.$$

In Table 2.1, the first eigenvalues are given for $(k, l) \in [\![1, 5]\!]$.

l \ k	1	2	3	4	5
1	$\frac{13}{36}\pi^2 \approx 3.56402$	$\frac{25}{36}\pi^2 \approx 6.85389$	$\frac{5}{4}\pi^2 \approx 12.3370$	$\frac{73}{36}\pi^2 \approx 20.0134$	$\frac{109}{36}\pi^2 \approx 29.8830$
2	$\frac{10}{9}\pi^2 \approx 10.9662$	$\frac{13}{9}\pi^2 \approx 14.2561$	$2\pi^2 \approx 19.7392$	$\frac{25}{9}\pi^2 \approx 27.4156$	$\frac{34}{9}\pi^2 \approx 37.2852$
3	$\frac{85}{36}\pi^2 \approx 23.3032$	$\frac{97}{36}\pi^2 \approx 26.5931$	$\frac{13}{4}\pi^2 \approx 32.0762$	$\frac{145}{36}\pi^2 \approx 39.7526$	$\frac{181}{36}\pi^2 \approx 49.6222$
4	$\frac{37}{9}\pi^2 \approx 40.5750$	$\frac{40}{9}\pi^2 \approx 43.8649$	$5\pi^2 \approx 49.3480$	$\frac{52}{9}\pi^2 \approx 57.0244$	$\frac{61}{9}\pi^2 \approx 66.8940$
5	$\frac{229}{36}\pi^2 \approx 62.7817$	$\frac{241}{36}\pi^2 \approx 66.0715$	$\frac{29}{4}\pi^2 \approx 71.5546$	$\frac{289}{36}\pi^2 \approx 79.2310$	$\frac{325}{36}\pi^2 \approx 89.1006$

Table 2.1: Eingenvalues $\lambda_{k,l}$ for $(k, l) \in [\![1, 5]\!]$ with $L = 2$, $H = 3$

In Listing ?? is given the complete code computing the first smallest magnitude eight eigenvalues and, in Figure 2.1, eigenvectors associated to the first eight smallest magnitude eigenvalues are plotted.

```
L=2;H=3;N=150;
Th=fc_simesh.HyperCube(2,[L*N,H*N],'trans',@(q) [L*q(1,:);H*q(2,:)]);
Lop=Loperator(2,2,{1,0;0,1},[],[],[]);
pde=PDEelt(Lop);
bvp=BVP(Th,pde);
for lab=1:4, bvp.setDirichlet( lab, 0);end
[eVec,lambda]=fc_vfemp1.addon.eigs.solve(bvp,8,'sm');
```

Listing 2.1: 2D Laplace eigenvalues problem with Dirichlet boundary condition $\Omega = [0, L] \times [0, H]$.

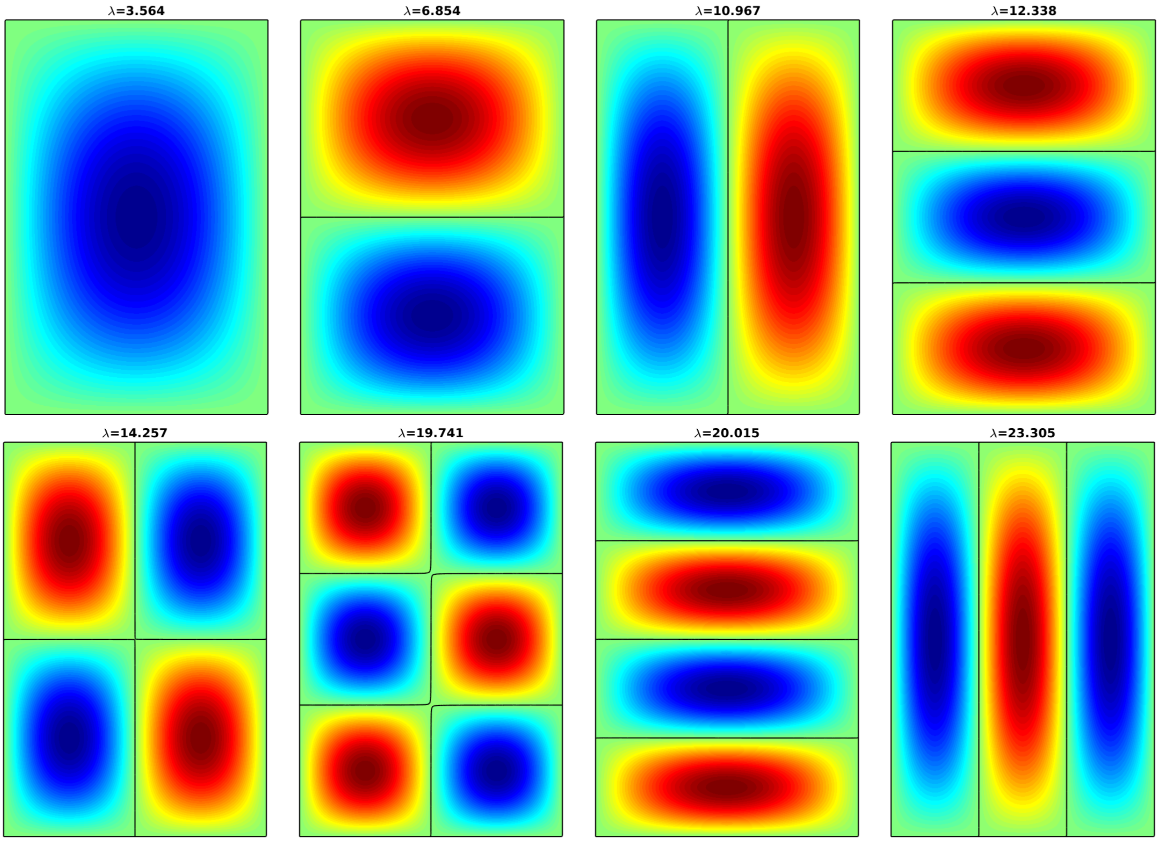

Figure 2.1: 2D Laplace in rectangle $[0, 2] \times [0, 3]$ with Dirichlet boundary conditions : eigenvectors of the smallest magnitude eigenvalues

We represent in Figure 2.2 the order of convergence of the first ten eigenvalues.

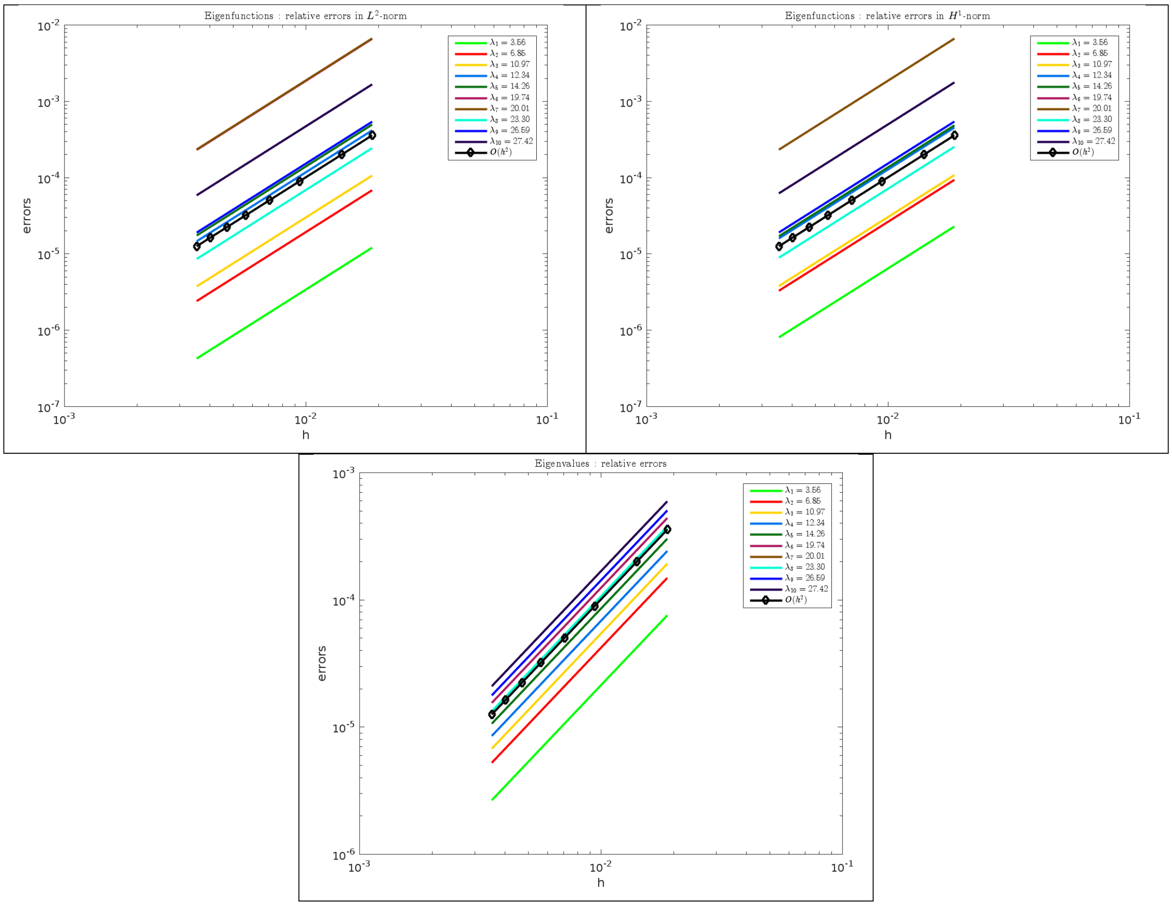

Figure 2.2: eigenvalues and eigenfunctions : order computation for 2D Laplace with Dirichlet boundary condition on rectangle (regular meshes) . Relative errors of eigenfunctions in L^2-norm (upper left) and H^1-norm (upper right). Relative errors of eigenvalues (bottom).

One can see that a superconvergence phenomena occurs due to regularity of the hypercube mesh. Indeed, for the H^1-norm an order 1 is expected. To hightlight it, gmsh is now used to generate all the meshes of Ω : results are given in Figure 2.3.

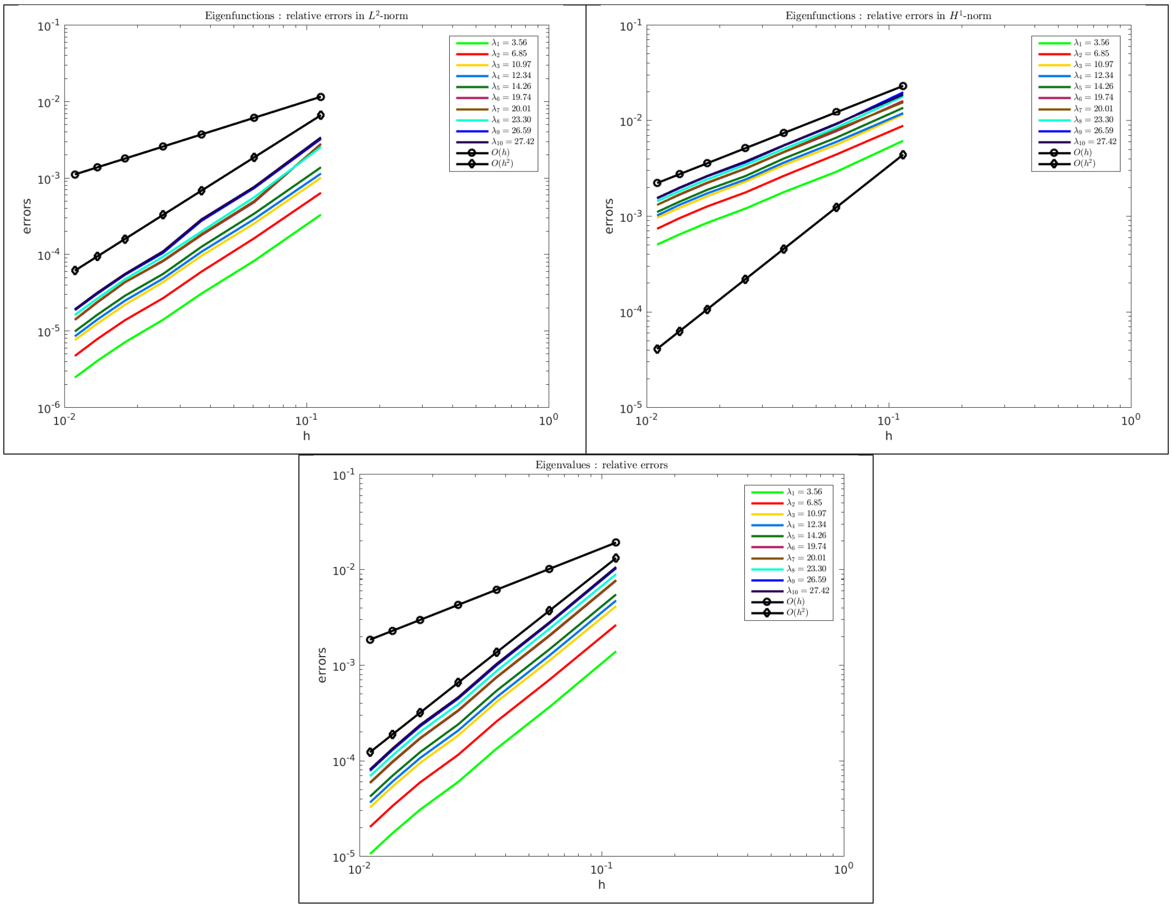

Figure 2.3: eigenvalues and eigenfunctions : order computation for 2D Laplace with Dirichlet boundary condition on rectangle (gmsh meshes). Relative errors of eigenfunctions in L^2-norm (upper left) and H^1-norm (upper right). Relative errors of eigenvalues (bottom).

Application on the unit disk.

Let $\Omega \subset \mathbb{R}^2$ be the unit disk meshed by gmsh and given in Figure 2.4.

Let α_{nl} bet the l-th zero of the Bessel function of the first kind J_n. The eigenvalues are given by

$$\lambda_{n,l} = \alpha_{nl}^2 \quad \forall (n,l) \in \mathbb{N} \times \mathbb{N}^*$$

In Table 2.1, the values of α_{nl} are given for $(n,l) \in [\![0,4]\!][\![1,5]\!]$.

The eigenvalues are simple for $n = 0$ and twice degenerate for $n > 0$.

l	J_0	J_1	J_2	J_3	J_4	J_5	J_6
1	2.4048256	3.8317060	5.1356223	6.3801619	7.5883424	8.7714838	9.9361095
2	5.5200781	7.0155867	8.4172441	9.7610231	11.064709	12.338604	13.589290
3	8.6537279	10.173468	11.619841	13.015201	14.372537	15.700174	17.003820
4	11.791534	13.323692	14.795952	16.223466	17.615966	18.980134	20.320789
5	14.930918	16.470630	17.959819	19.409415	20.826933	22.217800	23.586084
6	18.071064	19.615859	21.116997	22.582730	24.019020	25.430341	26.820152

Table 2.2: Zeros of the Bessel function of the first kind J_n

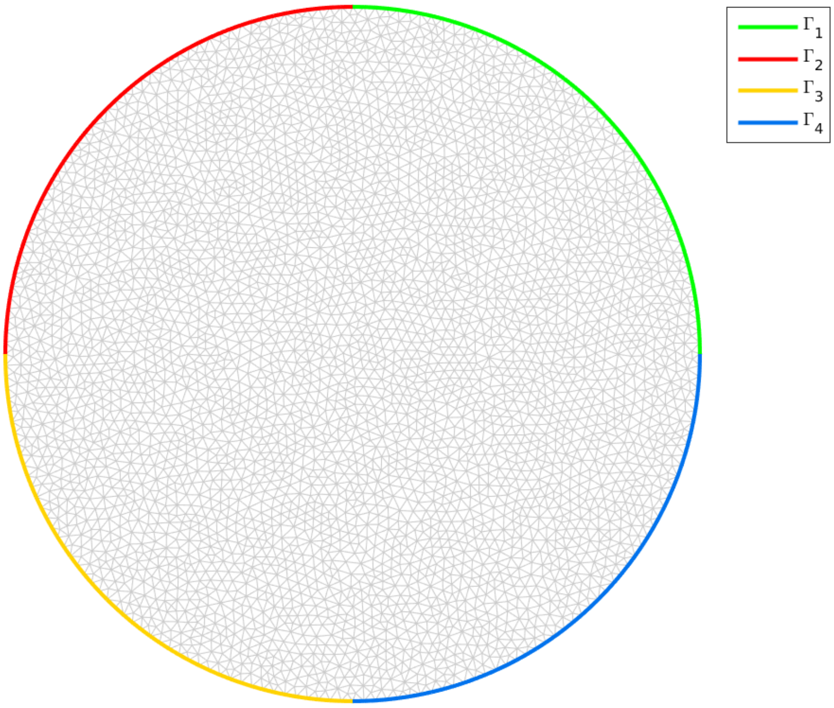

Figure 2.4: Unit disk with four boundaries

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8
5.7831860	14.681971	14.681971	26.374616	26.374616	30.471262	40.706466	40.706466
λ_9	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}	λ_{15}	λ_{16}
49.218456	49.218456	57.582941	57.582941	70.849999	70.849999	74.887007	76.938928
λ_{17}	λ_{18}	λ_{19}	λ_{20}	λ_{21}	λ_{22}	λ_{23}	λ_{24}
76.938928	95.277573	95.277573	98.726272	98.726272	103.49945	103.49945	122.42780

Table 2.3: twenty four first eigenvalues

We represent in Figure 2.5 eigenvectors associated to the first twenty-four smallest magnitude eigenvalues.

Application on the L-shape domain. The *Lshape* domain Ω meshed by gmsh is given in Figure 2.6.

Part of the source code To compute the eigenvalues and the eigenfunctions of the Laplacian with Dirichlet boundary condition using $\mathbb{P}_1$-Lagrange finite elements one can used the Matlabcommand fc_vfemp1.addon.eigs.sam Part of the source code (file `+fc_vfemp1/+addon/+eigs/+samples/+2d/Laplacian_Dir_Lshape_01.m`) is given in Listing 2.2

```
meshfile=gmsh.buildmesh2d(geofile,N,'geodir',[curpath,filesep,'geodir'],'meshdir',[curpath,filesep,'meshdir']);
Th=siMesh(meshfile,'dim',2,'format','gmsh');
dim=2;d=2;
Lop=Loperator(dim,d,{1,0;0,1},[],[],[]);
pde=PDEelt(Lop);
bvp=BVP(Th,pde);
BDLabels=Th.sThlab(Th.find(1));
for lab=BDLabels, bvp.setDirichlet( lab, 0);end
[eVec,lambda,flag]=fc_vfemp1.addon.eigs.solve(bvp,NumEigs,sigma);
```

Listing 2.2: 2D Laplace eigenvalues problem with Dirichlet boundary condition on *Lshape* domain

Results can be found in [6], Figure 1 page 4 and [3]. From [4] section 6.52 page 122 or [7] Table 1 page 1088, we have the bounds to the first ten eigenvalues of the L-shaped Laplacian problem is given Table 2.4.

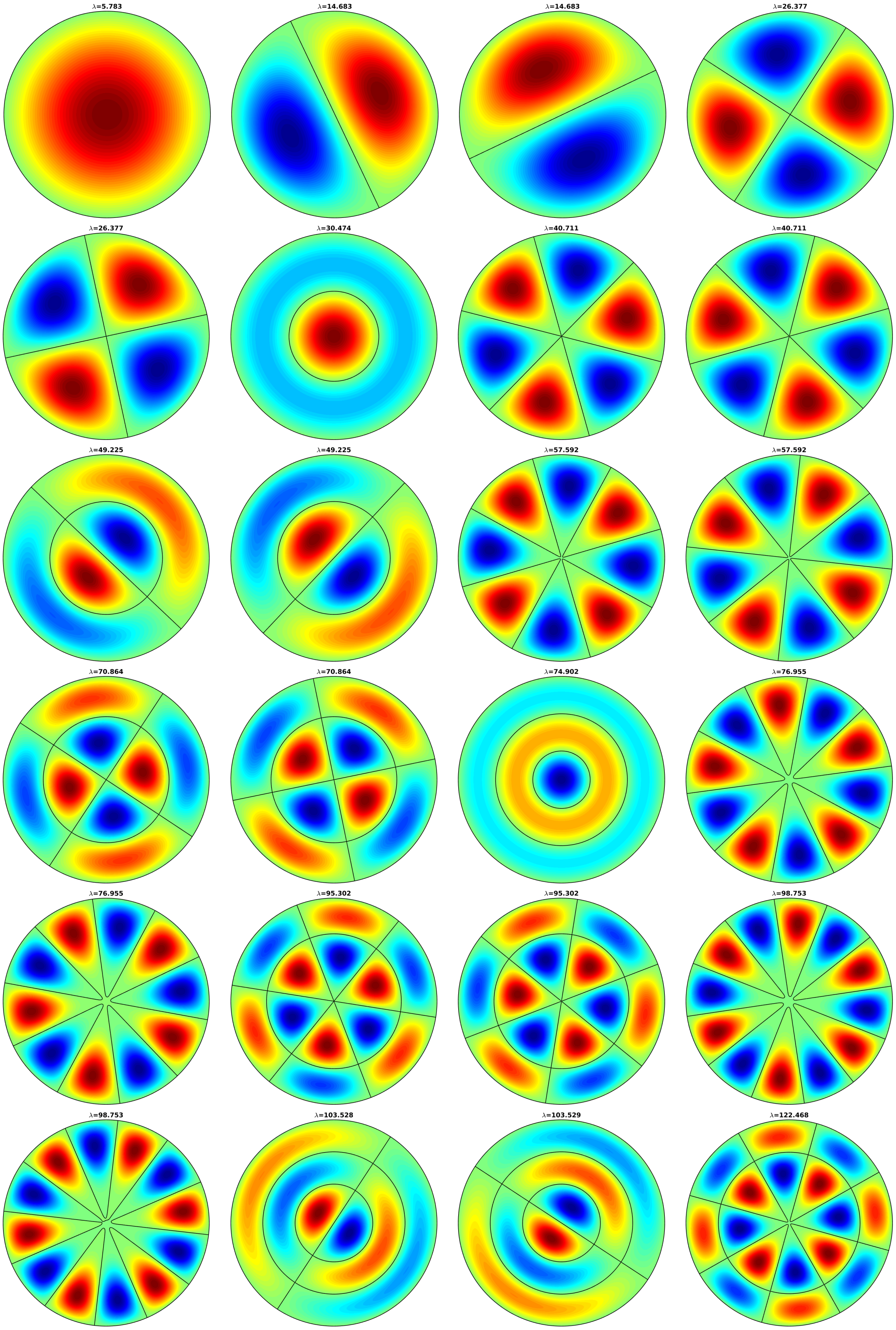

Figure 2.5: 2D Laplace in unit disk with Dirichlet boundary conditions : eigenvectors of the smallest magnitude eigenvalues

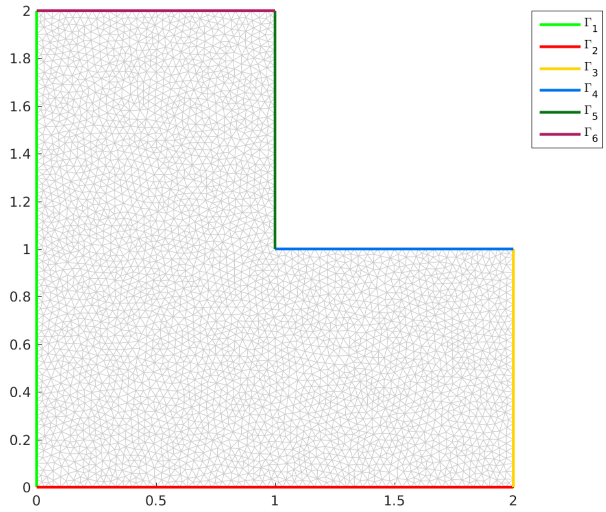

Figure 2.6: *Lshape* domain with four boundaries

We also give the computed values from a L-shaped mesh with $\mathrm{n_q} = 357991$, $\mathrm{n_{me}} = 713580$ and $h \approx 0.0052$.

n	bounds of λ_n from [7]	computed here
1	9.63972384^{44}_{04}	9.6400753491384
2	15.19725192^{66}_{59}	15.1964331319672
3	$2\pi^2 = 19.739208802178$	19.7380016650384
4	29.52148111^{42}_{38}	29.5191207646982
5	31.9126359^{59}_{37}	31.9101116684313
6	41.4745098^{92}_{66}	41.4681961047908
7	44.9484877^{82}_{77}	44.9402450596564
8	$5\pi^2 = 49.34802200544$	49.3399944563230
9	$5\pi^2 = 49.34802200544$	49.3400650877821
10	56.7096098^{90}_{18}	56.6971710161666

Table 2.4: Bounds to the first ten eigenvalues of the L-shaped Laplacian problem

We represent in Figure 2.7 eigenvectors associated to the first twenty-four smallest magnitude eigenvalues.

In Figure 2.8 the eigenvectors associated with the four eigenvalues nearest 250 (multiplicity 1) and 493 (multiplicity 3) are represented. This is done by setting *sigma* option to 250 for the first case and to 493 for the second one.

2.2.2 2D Laplace eigenvalues problem with mixed boundary conditions

We want to solve the eigenvalue problem given by (2.6)-(2.9)).

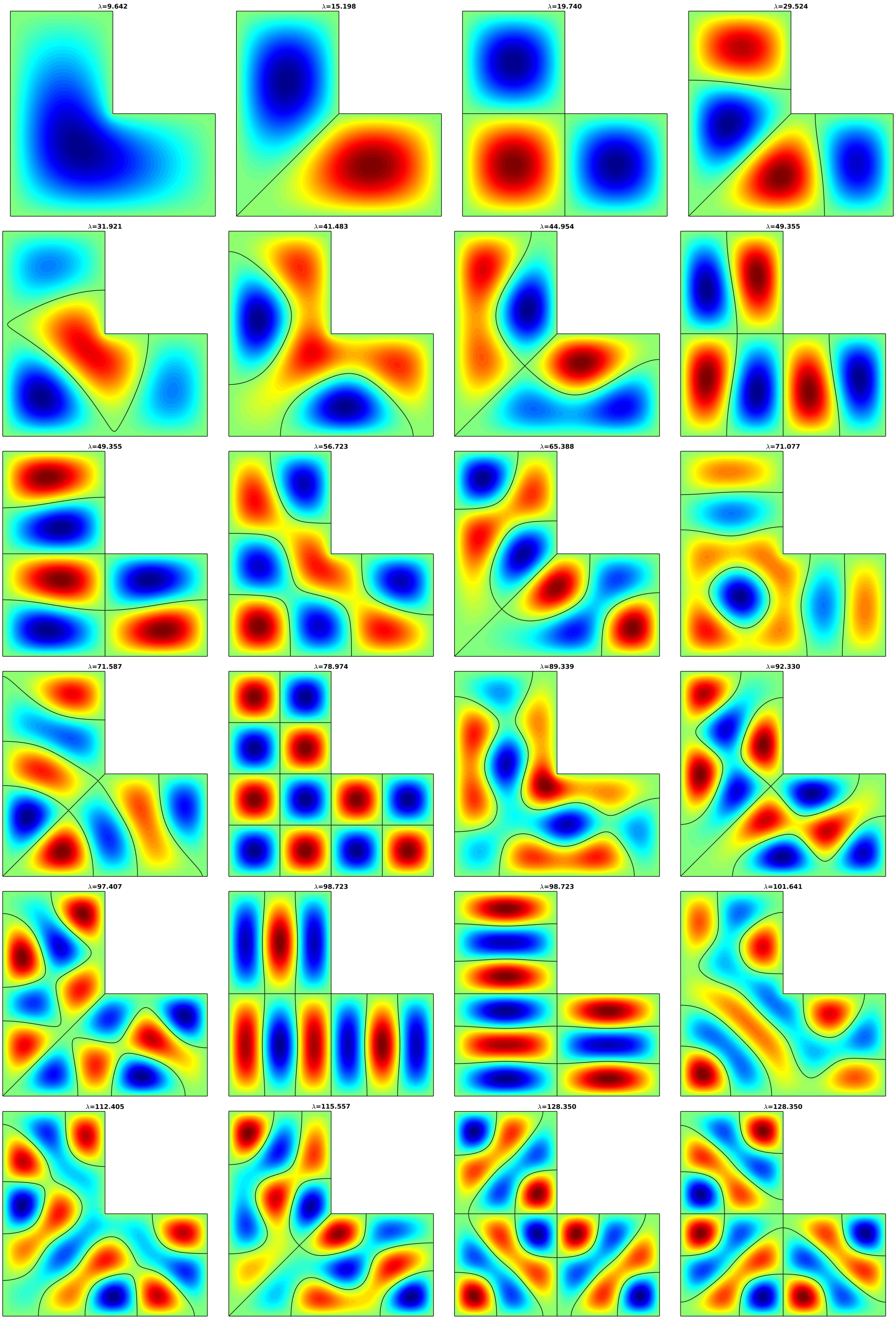

Figure 2.7: 2D Laplace in L-shaped domain with Dirichlet boundary conditions : eigenvectors of the smallest magnitude eigenvalues

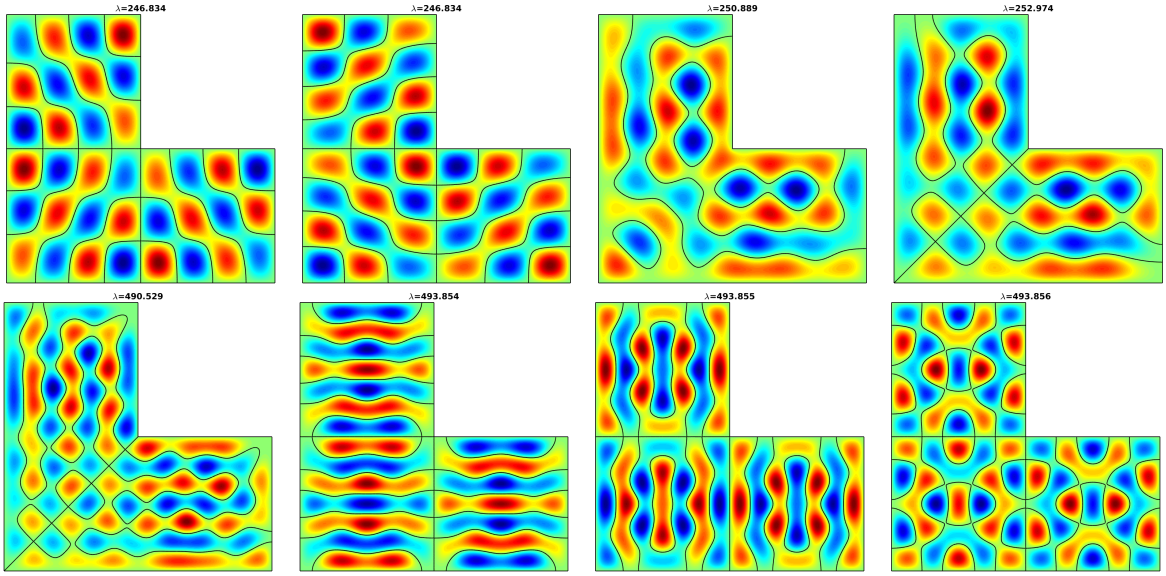

Figure 2.8: 2D Laplace with Dirichlet boundary conditions : eigenvectors of the eigenvalues near $\lambda_{50} = 250.78548$ (multiplicity 1) and $\lambda_{104} = 493.48022$ (multiplicity 3)

Usual EBVP 2 : 2D Laplace with mixed boundary condition

Find $(\lambda, u) \in \mathbb{K} \times \mathrm{H}^2(\Omega)$ such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{2.6}$$

$$\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \Gamma^a, \tag{2.7}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma^b, \tag{2.8}$$

$$u = 0 \quad \text{on } \Gamma^c, \tag{2.9}$$

The problem (2.6)-(2.9) can be equivalently written as the *Scalar* EBVP 1:

Scalar EBVP 3 : 2D Laplace with mixed boundary condition

Find $(\lambda, u) \in \mathbb{K} \times \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = \lambda \mathcal{B}(u) \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = 0 \quad \text{on } \Gamma^R = \Gamma^a \cup \Gamma^b,$$

$$u = 0 \quad \text{on } \Gamma^D = \Gamma^c,$$

where $\mathcal{L} = \mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}$ (and then $\frac{\partial u}{\partial n_{\mathcal{L}}} = \frac{\partial u}{\partial n}$), $\mathcal{B} = \mathcal{L}_{\mathbb{O},\mathbf{0},\mathbf{0},1}$, $a^R = \alpha \delta_{\Gamma^b}$

Application on the disk with 5 holes domain. Let Γ_1 be the unit disk, Γ_{10} be the disk with center point $(0, 0)$ and radius 0.3. Let Γ_{20}, Γ_{21}, Γ_{22} and Γ_{23} be the disks with radius 0.1 and respectively with center point $(0, -0.7)$, $(0, 0.7)$, $(-0.7, 0)$ and $(0.7, 0)$. The domain $\Omega \subset \mathbb{R}^2$ is defined as the inner of Γ_1 and the outer of all other disks (see Figure 2.9).

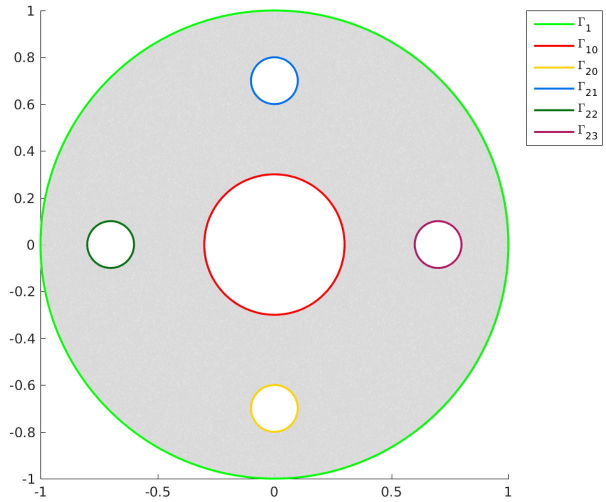

Figure 2.9: Domain and boundaries

We want to solve the eigenvalue problem given by (2.10)-(2.13)).

Scalar EBVP 4 : 2D Laplace eigenvalues problem with mixed boundary conditions

Find $(\lambda, u) \in \mathbb{K} \times \mathrm{H}^2(\Omega)$ such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \tag{2.10}$$

$$\frac{\partial u}{\partial n} + 10u = 0 \quad \text{on } \Gamma_{22} \cup \Gamma_{23}. \tag{2.11}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_{20} \cup \Gamma_{21}, \tag{2.12}$$

$$u = 0 \quad \text{on } \Gamma_1 \cup \Gamma_{10}. \tag{2.13}$$

So we have, $\Gamma^D = \Gamma_1 \cup \Gamma_{10}$, $\Gamma^R = \bigcup_{i=20}^{23} \Gamma_i$, and $a^R = 10\delta_{\Gamma_{22} \cup \Gamma_{23}}$.
We give in Listing 2.3 the corresponding Matlab code.

Listing 2.3: 2D Laplacian eigenvalues problem with mixed boundary conditions on a domain with 5 holes

```
geofile='disk5holes';
meshfile=gmsh.buildmesh2d(geofile,N,'geodir',[curpath,filesep,'geodir'],'meshdir',[curpath,filesep,'meshdir']);
dim=2;d=2;
Th=siMesh(meshfile,'dim',dim,'format','gmsh');
Lop=Loperator(dim,d,{1,0;0,1},[],[],[]);
pde=PDEelt(Lop);
bvp=BVP(Th,pde);
bvp.setDirichlet( 1, 0.);
bvp.setDirichlet( 10, 0.);
bvp.setRobin( 20, 0.,[]);bvp.setRobin( 21, 0.,[]);
bvp.setRobin( 22, 0.,10.);bvp.setRobin( 23, 0.,10.);
[eVec,eVal,flag]=fc_vfemp1.addon.eigs.solve(bvp,NumEigs,sigma);
```

We represent in Figure 2.10 the twenty four first eigenvectors.

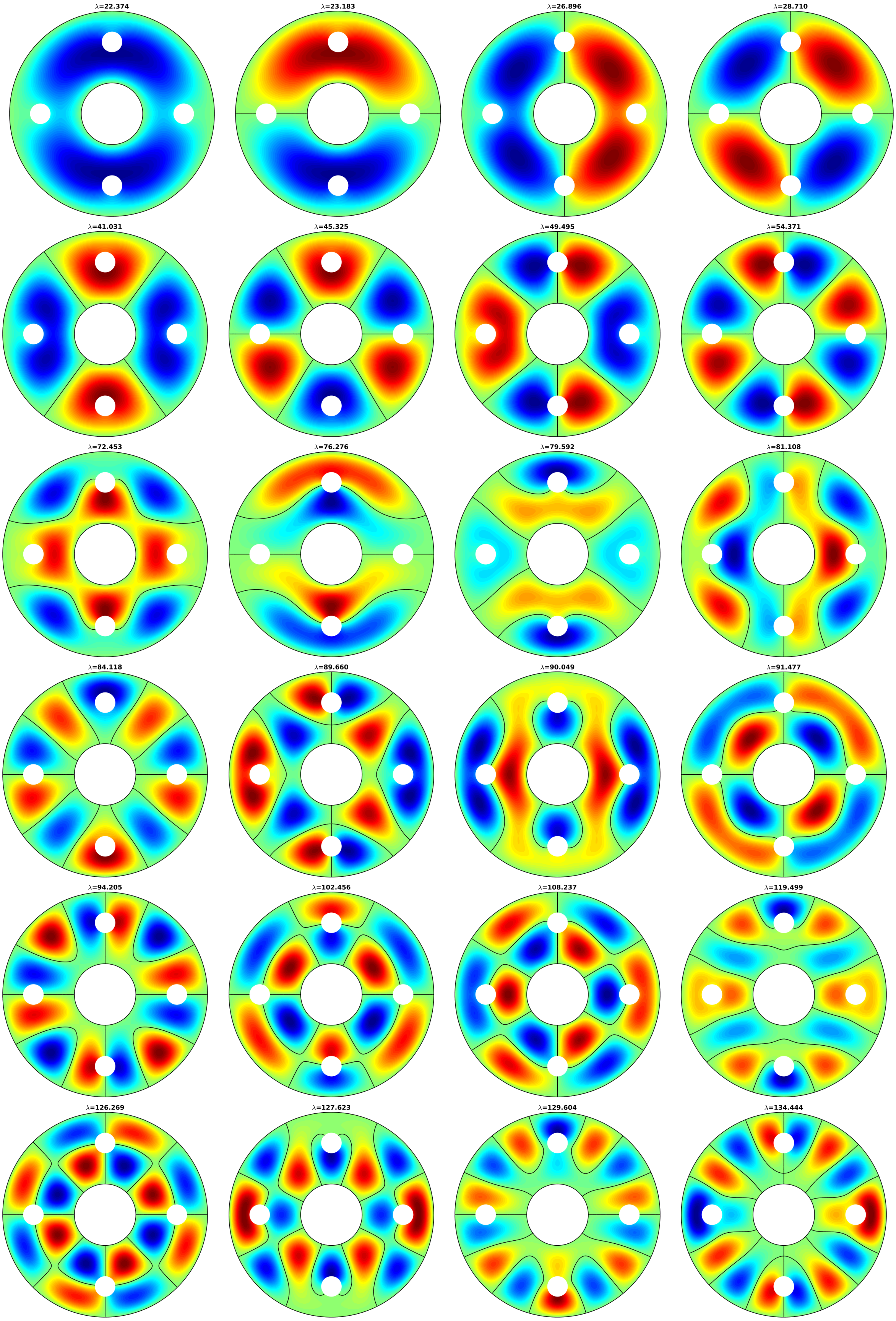

Figure 2.10: 2D Laplace with mixed boundary conditions : eigenvectors of the smallest magnitude eigenvalues

2.2.3 Other 2D eigenvalues problems with Dirichlet boundary condition

Convection-Diffusion on the L-shaped domain.

We want to solve the eigenvalue problem given by

***Usual* EBVP 3 : 2D Convection-Diffusion eigenvalues problem with Dirichlet boundary condition**

Find $(\lambda, u) \in \mathbb{K} \times \mathrm{H}^2(\Omega)$ such that

$$\begin{aligned} -\Delta u + \boldsymbol{\beta}.\boldsymbol{\nabla}\, u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

with constant convection parameter $\boldsymbol{\beta} \in \mathbb{R}^2$.

From [4] section 6.52 page 122 the eigenvalues of *Usual* EBVP 3 are $\lambda_i^{\boldsymbol{\beta}} = |\boldsymbol{\beta}|/4 + \lambda_i$ where λ_i are the eigenvalues of the L-shaped Laplacian problem with Dirichlet boundary condition (the ten first are given in Table 2.4). We have for example

$$\begin{aligned} \lambda_1^{\boldsymbol{\beta}} &\approx |\boldsymbol{\beta}|/4 + 9.63972, & \lambda_3^{\boldsymbol{\beta}} &= |\boldsymbol{\beta}|/4 + 2\pi^2 \\ \lambda_5^{\boldsymbol{\beta}} &\approx |\boldsymbol{\beta}|/4 + 31.912636, & \lambda_8^{\boldsymbol{\beta}} = \lambda_9^{\boldsymbol{\beta}} &= |\boldsymbol{\beta}|/4 + 5\pi^2 \\ \lambda_{20}^{\boldsymbol{\beta}} &\approx |\boldsymbol{\beta}|/4 + 101.60529, & \lambda_{50}^{\boldsymbol{\beta}} &\approx |\boldsymbol{\beta}|/4 + 250.78548. \end{aligned}$$

We give in Listing **??** the corresponding Matlab code.

Listing 2.4: 2D L-shaped Convection-Diffusion problem with $\boldsymbol{\beta} = (3, 0)$: Matlab code

```
geofile='Lshape';
meshfile=gmsh.buildmesh2d(geofile,N,'geodir',[curpath,filesep,'geodir'],'meshdir',[curpath,filesep,'meshdir']);
Th=siMesh(meshfile,'dim',2,'format','gmsh');
dim=2;d=2;
beta={3,0};
Lop=Loperator(dim,d,{1,0;0,1},[],beta,[]);
pde=PDEelt(Lop);
bvp=BVP(Th,pde);
for lab=Th.sThlab(Th.find(1)), bvp.setDirichlet( lab, 0);end
[eVec,eVal,flag]=fc_vfemp1.addon.eigs.solve(bvp,NumEigs,sigma);
```

We give the computed values from a L-shaped mesh with $\mathrm{n_q} = 89780$, $\mathrm{n_{me}} = 178358$ and $h \approx 0.0105$. We represent in Figure 2.10 the twenty four first eigenvectors with $\boldsymbol{\beta} = (3, 0)$.

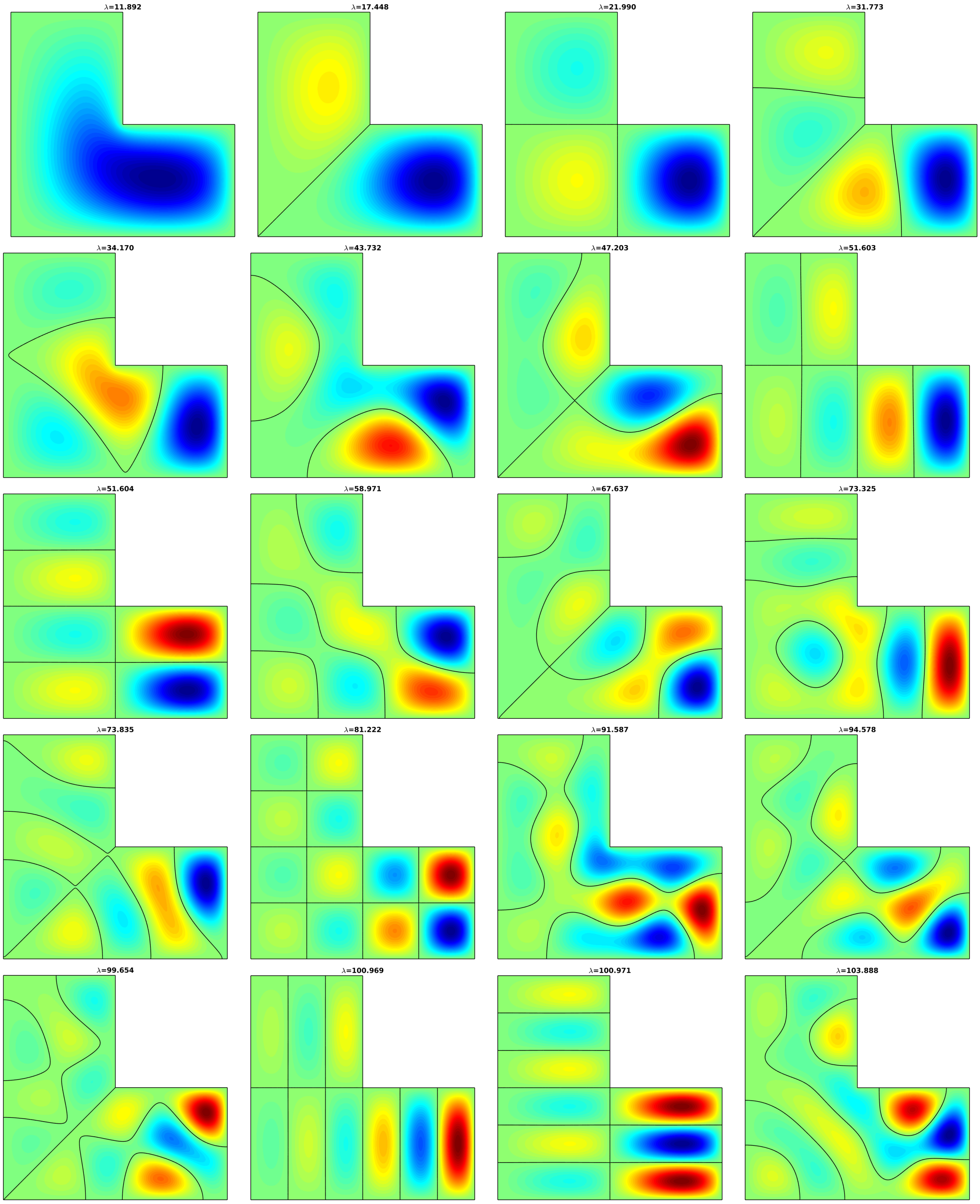

Figure 2.11: 2D L-shaped Convection-Diffusion problem with $\boldsymbol{\beta} = (3, 0)$. Eigenvectors of the smallest magnitude eigenvalues ($n_q = 89780$)

n	bounds of $\lambda_n^{\boldsymbol{\beta}}$ from [7]	computed here
1	$\lvert\boldsymbol{\beta}\rvert/4 + 9.63972384_{04}^{44} \approx 11.88972384$	11.891874
2	$\lvert\boldsymbol{\beta}\rvert/4 + 15.19725192_{59}^{66} \approx 17.44725192$	17.447521
3	$\lvert\boldsymbol{\beta}\rvert/4 + 2\pi^2 = 21.989208802178716$	21.989805
4	$\lvert\boldsymbol{\beta}\rvert/4 + 29.52148111_{38}^{42} \approx 31.77148111$	31.773153
5	$\lvert\boldsymbol{\beta}\rvert/4 + 31.9126359_{37}^{59} \approx 34.1626359$	34.169822
6	$\lvert\boldsymbol{\beta}\rvert/4 + 41.4745098_{66}^{92} \approx 43.7245099$	43.732106
7	$\lvert\boldsymbol{\beta}\rvert/4 + 44.9484877_{77}^{82} \approx 47.19848777$	47.202920
8	$\lvert\boldsymbol{\beta}\rvert/4 + 5\pi^2 = 51.598022005446794$	51.603105
9	$\lvert\boldsymbol{\beta}\rvert/4 + 5\pi^2 = 51.598022005446794$	51.603864
10	$\lvert\boldsymbol{\beta}\rvert/4 + 56.7096098_{18}^{90} \approx 58.9596098$	58.971362
20	$\lvert\boldsymbol{\beta}\rvert/4 + 101.60529 \approx 103.85529$	103.88828
50	$\lvert\boldsymbol{\beta}\rvert/4 + 250.78548 \approx 253.03548$	253.21186

Table 2.5: Eigenvalues of the L-shaped Convection-Diffusion problem with $\boldsymbol{\beta} = (3, 0)$.

Chapter 3

Generalized Eigenvalue vector BVP

The eigenvalue problems associated with *vector* BVP (1.14)-(1.16) can be written as

Vector EBVP 1 : generic problem

Find $\lambda \in \mathbb{K}$ and $\boldsymbol{u} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) \in (\mathrm{H}^2(\Omega))^m$ such that

$$\mathcal{H}(\boldsymbol{u}) = \lambda \mathcal{B}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{3.1}$$

$$\boldsymbol{u}_\alpha = 0 \quad \text{on } \Gamma_\alpha^D, \ \forall \alpha \in [\![1, m]\!], \tag{3.2}$$

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \boldsymbol{u}_\alpha = 0 \quad \text{on } \Gamma_\alpha^R, \ \forall \alpha \in [\![1, m]\!], \tag{3.3}$$

where $\mathcal{B}$ is a given $\mathcal{H}$-operator.

In most cases $\mathcal{B}$ is the identity operator ($\mathcal{B}$ is a diagonal $\mathcal{H}$-operator with $\mathcal{B}_{\alpha,\alpha} = \mathcal{L}_{\mathbb{O}_{d\times d}, \mathbf{0}_d, \mathbf{0}_d, 1}$, $\forall \alpha \in [\![1, m]\!]$).

3.0.4 Linear elasticity

Elasticity problem

Let $d = 2$ or $d = 3$. We consider here Hooke's law in linear elasticity, under small strain hypothesis (see for example [2]).

For a sufficiently regular vector field $\boldsymbol{u} = (u_1, \dots, u_d) : \Omega \to \mathbb{R}^d$, we define the linearized strain tensor $\underline{\boldsymbol{\epsilon}}$ by

$$\underline{\boldsymbol{\epsilon}}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}(\boldsymbol{u}) + \boldsymbol{\nabla}^t(\boldsymbol{u})\right).$$

We set $\underline{\boldsymbol{\epsilon}} = (\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})^t$ in 2d and $\underline{\boldsymbol{\epsilon}} = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{13})^t$ in 3d, with $\epsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. Then the Hooke's law writes

$$\underline{\boldsymbol{\sigma}} = \mathbb{C}\underline{\boldsymbol{\epsilon}},$$

where $\underline{\boldsymbol{\sigma}}$ is the elastic stress tensor and $\mathbb{C}$ the elasticity tensor.

The material is supposed to be isotropic. Thus the elasticity tensor $\mathbb{C}$ is only defined by the Lamé parameters λ and μ, which satisfy $\lambda + \mu > 0$. We also set $\gamma = 2\mu + \lambda$. For $d = 2$ or $d = 3$, $\mathbb{C}$ is given by

$$\mathbb{C} = \begin{pmatrix} \lambda \mathbb{1}_2 + 2\mu \mathbb{I}_2 & 0 \\ 0 & \mu \end{pmatrix}_{3\times 3} \quad \text{or} \quad \mathbb{C} = \begin{pmatrix} \lambda \mathbb{1}_3 + 2\mu \mathbb{I}_3 & 0 \\ 0 & \mu \mathbb{I}_3 \end{pmatrix}_{6\times 6},$$

respectively, where $\mathbb{1}_d$ is a d-by-d matrix of ones, and $\mathbb{I}_d$ the d-by-d identity matrix.

For dimension $d = 2$ or $d = 3$, we have:

$$\boldsymbol{\sigma}_{\alpha\beta}(\boldsymbol{u}) \quad = \quad 2\,\mu\,\boldsymbol{\epsilon}_{\alpha\beta}(\boldsymbol{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{u}))\delta_{\alpha\beta} \quad \forall \alpha, \beta \in [\![1, d]\!]$$

The problem to solve is the following

Usual EBVP 4 : Elasticity problem

Find $(k, \boldsymbol{u}) = \mathbb{K} \times \mathrm{H}^2(\Omega)^d$ such that

$$-\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u})) = k\boldsymbol{u}, \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{3.4}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma^R, \tag{3.5}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma^D. \tag{3.6}$$

We recall the following lemma (see [1])

Lemme 3.1

Let $\mathcal{H}^{\boldsymbol{\sigma}}$ be the $\mathcal{H}$-operator defined in (1.10) by

$$\mathcal{H}_{\alpha,\beta} = \mathcal{L}_{\mathbb{A}^{\alpha,\beta},\boldsymbol{0},\boldsymbol{0},0}, \quad \forall (\alpha, \beta) \in [\![1, d]\!]^2 \tag{3.7}$$

with

$$(\mathbb{A}^{\alpha,\beta})_{k,l} = \mu\delta_{\alpha\beta}\delta_{kl} + \mu\delta_{k\beta}\delta_{l\alpha} + \lambda\delta_{k\alpha}\delta_{l\beta}, \quad \forall (k, l) \in [\![1, d]\!]^2. \tag{3.8}$$

Then, we have

$$\mathcal{H}^{\boldsymbol{\sigma}}(\boldsymbol{u}) = -\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}) \tag{3.9}$$

and, $\forall \alpha \in [\![1, d]\!]$,

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}^{\boldsymbol{\sigma}}_\alpha}} = (\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{n})_\alpha. \tag{3.10}$$

The matrices $\mathbb{A}^{\alpha,\beta}$ of previous lemma are explicitely given by

- for $d = 2$,

$$\mathbb{A}^{1,1} = \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix}, \ \mathbb{A}^{1,2} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}, \ \mathbb{A}^{2,1} = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix}, \ \mathbb{A}^{2,2} = \begin{pmatrix} \mu & 0 \\ 0 & \gamma \end{pmatrix}$$

- for $d = 3$,

$$\mathbb{A}^{1,1} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \mathbb{A}^{1,2} = \begin{pmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{A}^{1,3} = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}$$

$$\mathbb{A}^{2,1} = \begin{pmatrix} 0 & \mu & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{A}^{2,2} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \mathbb{A}^{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \mu & 0 \end{pmatrix},$$

$$\mathbb{A}^{3,1} = \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, \quad \mathbb{A}^{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \lambda & 0 \end{pmatrix}, \quad \mathbb{A}^{3,3} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

So the elasticity problem (3.4) to (3.6) can be equivalently written as :

Vector EBVP 2 : Linear elasticity in dimension $d = 2$ or $d = 3$

Find $(k, \boldsymbol{u}) \in \mathbb{K} \times (\mathrm{H}^2(\Omega))^d$ such that

$$\mathcal{H}^{\boldsymbol{\sigma}}(\boldsymbol{u}) = k\mathcal{B}^{\boldsymbol{\sigma}}(\boldsymbol{u}), \quad \text{in } \Omega, \tag{3.11}$$

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}^{\boldsymbol{\sigma}}_\alpha}} = 0, \quad \text{on } \Gamma^R_\alpha = \Gamma^R, \ \forall \alpha \in [\![1, d]\!] \tag{3.12}$$

$$\boldsymbol{u}_\alpha = 0, \quad \text{on } \Gamma^D_\alpha = \Gamma^D, \ \forall \alpha \in [\![1, d]\!]. \tag{3.13}$$

with $\mathcal{B}^{\boldsymbol{\sigma}}_{\alpha,\beta} = \delta_{\alpha,\beta}\mathcal{L}_{\mathbb{O},\mathbf{0},\mathbf{0},1}$.

Application on the unit square with Dirichlet boundary condition The phycical parameters are $E = 2100000.0$ and $\nu = 0.45$. We take $\Gamma^D = \Gamma$ and thus $\Gamma^R = \emptyset$. For each eigenfunction $\boldsymbol{u}$, we represent $\sqrt{\boldsymbol{u}_1^2 + \boldsymbol{u}_2^2}$ in Figure 3.1 for the first twelve smallest magnitude eigenvalues. The mesh parameters are $\mathrm{n_q} = 90601$, $\mathrm{n_{me}} = 180000$ and $h = 0.0047140$.

```
Th=fc_simesh.HyperCube(dim,N);
Hop=Hoperator();
Hop.opStiffElas(dim,lambda,mu);
pde=PDEelt(Hop);
bvp=BVP(Th,pde);
for lab=Th.sThlab(Th.find(1)), bvp.setDirichlet( lab, 0.,1:2);end
[eVec,eVal,flag]=fc_vfemp1.addon.eigs.solve(bvp,NumEigs,sigma);
```

Listing 3.1: 2D linear elasticity eigenvalues problem with Dirichlet condition on the unit square

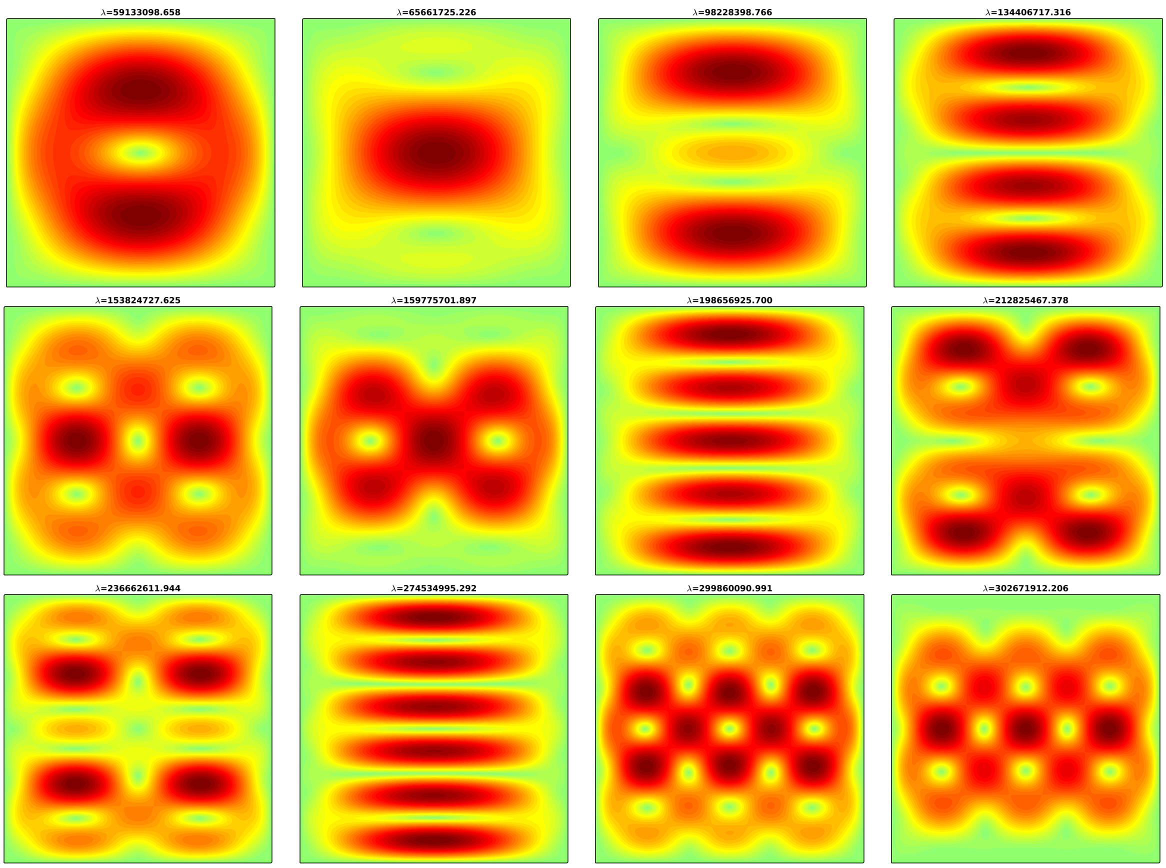

Figure 3.1: 2D linear elasticity on the unit square with Dirichlet boundary condition : euclidean norm of eigenfunctions associated with the smallest magnitude eigenvalues

Application on the bar $[0,7]\times[-1,1]$ with mixed boundary conditions The phycical parameters are $E = 0.45000$ and $\nu = 584010.$. We take $\Gamma^D = \Gamma^1 \cup \Gamma^2$ and $\Gamma^R = \Gamma^3 \cup \Gamma^4$. For each eigenfunction $\boldsymbol{u}$, we represent $\sqrt{\boldsymbol{u}_1^2 + \boldsymbol{u}_2^2}$ in Figure 3.2 for the first twelve smallest magnitude eigenvalues. The mesh parameters are $\mathrm{n_q} = 140901$, $\mathrm{n_{me}} = 280000$ and $h = 2.1000 \times 10^6$.

```
Th=fc_simesh.HyperCube(dim,N*[7,2],'trans',@(q) [7*q(1,:);2*q(2,:)-1]);
Hop=Hoperator();
Hop.opStiffElas(dim,lambda,mu);
pde=PDEelt(Hop);
bvp=BVP(Th,pde);
bvp.setDirichlet( 1, 0.,1:2);
bvp.setDirichlet( 2, 0.,1:2);
[eVec,eVal,flag]=fc_vfemp1.addon.eigs.solve(bvp,NumEigs,sigma);
```

Listing 3.2: 2D linear elasticity eigenvalues problem with mixed boundary conditions on $\Omega = [0, L] \times [0, H]$.

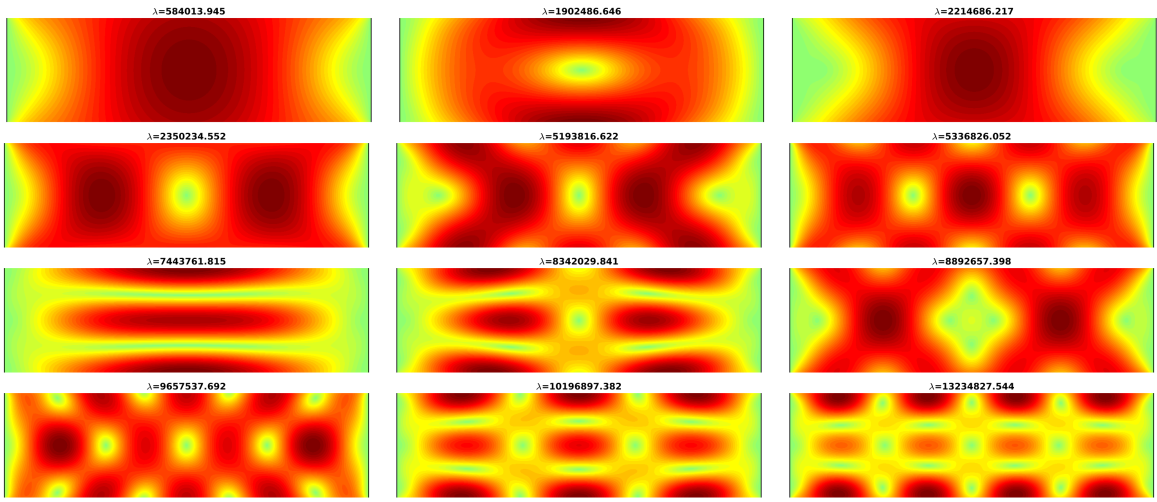

Figure 3.2: 2D linear elasticity on the bar $[0,7] \times [-1,1]$ with mixed boundary conditions : euclidean norm of eigenfunctions assocuated with the smallest magnitude eigenvalues

Application on a bar with 4 holes and with mixed boundary conditions The domain $\Omega \subset \mathbb{R}^2$ is given in Figure 3.3. The phycical parameters are $E = 0.45000$ and $\nu = 59604.$. We take $\Gamma^D = \Gamma^1 \cup \Gamma^2$ and $\Gamma^R = \Gamma^3 \cup \Gamma^4$. For each eigenfunction $\boldsymbol{u}$, we represent $\sqrt{\boldsymbol{u}_1^2 + \boldsymbol{u}_2^2}$ in Figure **??** for the first twelve smallest magnitude eigenvalues. The mesh parameters are $\mathrm{n_q} = 252489$, $\mathrm{n_{me}} = 501824$ and $h = 2.1000 \times 10^6$.

```
meshfile=gmsh.buildmesh2d(geofile,N,'geodir',[curpath,filesep,'geodir'],'meshdir',[curpath,filesep,'meshdir']);
Th=siMesh(meshfile);
Hop=Hoperator();
Hop.opStiffElas(dim,lambda,mu);
pde=PDEelt(Hop);
bvp=BVP(Th,pde);
bvp.setDirichlet( 1, 0.,1:2);
[eVec,eVal,flag]=fc_vfemp1.addon.eigs.solve(bvp,NumEigs,sigma);
```

Listing 3.3: 2D linear elasticity eigenvalues problem with mixed boundary conditions on a bar with 4 holes.

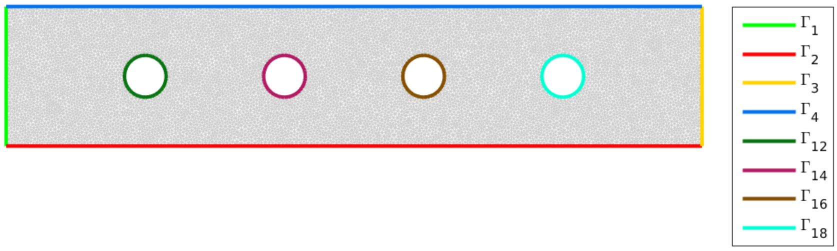

Figure 3.3: bar with 4 holes : domain and boundaries

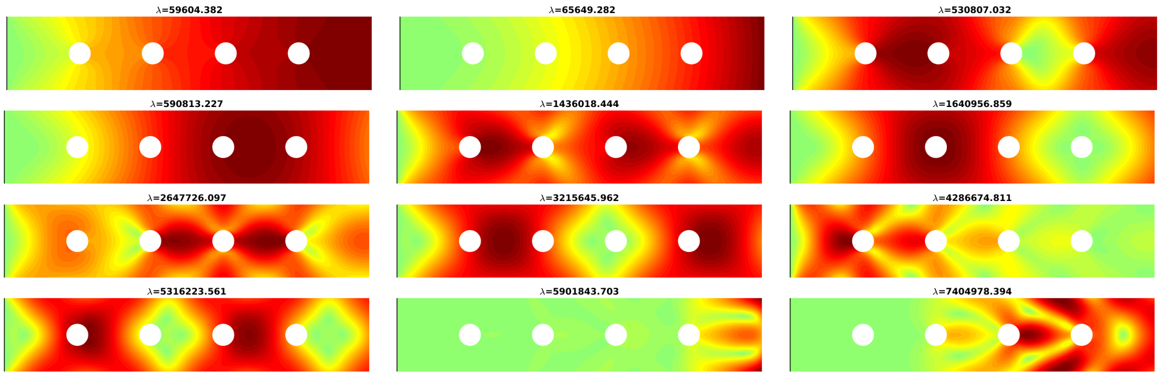

Figure 3.4: 2D linear elasticity on the bar with 4 holes and with mixed boundary conditions : euclidean norm of eigenfunctions assocuated with the smallest magnitude eigenvalues

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