



## Surface BVP Addon <sup>1</sup>

for the FC-VFEM<sub>P1</sub> Matlab toolbox

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### **Abstract**

The **Surface BVP Addon** for the FC-VFEM $\mathbb{P}_1$  Matlab toolbox contains codes which allow to compute numerically solutions of *scalar* Boundary Value Problems on regular surfaces. These codes use the FC-VFEM $\mathbb{P}_1$  Matlab toolbox and thus a good knowledge of the use of the latter is a prerequisite with the reading of this report.

# Contents

<b>1</b>	<b>Generic Boundary Value Problems</b>	<b>3</b>
1.1	Scalar boundary value problem . . . . .	3
1.2	Vector boundary value problem . . . . .	5
<b>2</b>	<b>Notations and results on regular surfaces</b>	<b>8</b>
2.1	Elementary geometric analysis . . . . .	8
2.2	Sobolev space on surface . . . . .	10
2.3	Variational formulation of a surface BVP . . . . .	11
<b>3</b>	<b>scalar surface BVP</b>	<b>13</b>
3.1	Analytical solutions of surface BVPs . . . . .	13
3.1.1	Sage computation of Laplace-Beltrami operator on hypersurfaces . . . . .	13
3.2	Numerical samples on closed hypersurfaces . . . . .	17
3.2.1	$-\Delta_{\Gamma}u + a_0u = f$ with exact solution on $S^2$ . . . . .	17
3.2.2	$-\Delta_{\Gamma}u + a_0u = f$ on $F(S^2)$ with exact solution . . . . .	18
3.2.3	$-\Delta_{\Gamma}u + a_0u = f$ on $F(S^2)$ without explicit solution . . . . .	20
3.2.4	$-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$ on $F(S^2)$ with exact solution . . . . .	20
3.2.5	$-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$ on $F(S^2)$ without exact solution . . . . .	22
3.2.6	3D surface Laplace-Beltrami BVP on closed hypersurface . . . . .	23
3.3	Numerical samples on non-closed hypersurfaces . . . . .	30
3.3.1	$-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$ with Dirichlet boundary conditions on an half-sphere . . . . .	30
3.3.2	$-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$ with mixed boundary conditions on an half-sphere . . . . .	32
3.3.3	Mixed boundary conditions on an half-sphere mapping by a function . . . . .	34

Let  $\Gamma \subset \mathbb{R}^{n+1}$  be a regular surface. For all  $x \in \Gamma$ ,  $\mathbb{A}(x) = (A_{i,j}(x))_{i,j=1,\dots,n+1}$  be a matrix which maps the tangent space  $T_x\Gamma$  into itself (i.e.  $\mathbb{A}(x) : T_x\Gamma \rightarrow T_x\Gamma$ ) and let  $\mathbf{b}(x) \in T_x\Gamma$ ,  $\mathbf{c} : \Gamma \rightarrow \mathbb{R}^{n+1}$  and  $a_0 : \Gamma \rightarrow \mathbb{R}$ . We denote by  $\mathcal{L}$  the second order linear differential operator acting on scalar fields defined,  $\forall u \in H^2(\Gamma)$ , by

$$\mathcal{L}(u) \stackrel{\text{def}}{=} -\text{div}_\Gamma(\mathbb{A} \nabla_\Gamma u) + \text{div}_\Gamma(\mathbf{b}u) + \langle \nabla_\Gamma u, \mathbf{c} \rangle + a_0 u. \quad (1)$$

The main of the **Surface BVP Addon** is to solve generic *scalar* surface BVP given by

### **Scalar surface BVP 1 : generic problem**

Find  $u \in H^2(\Gamma)$  such that

$$\mathcal{L}(u) = f \quad \text{in } \Gamma, \quad (2)$$

$$u = g^D \quad \text{on } \partial\Gamma^D, \quad (3)$$

$$\frac{\partial u}{\partial n_\mathcal{L}} + a^R u = g^R \quad \text{on } \partial\Gamma^R. \quad (4)$$

The **conormal derivative** of  $u$  is defined by

$$\frac{\partial u}{\partial n_\mathcal{L}} \stackrel{\text{def}}{=} \langle \mathbb{A} \nabla_\Gamma u, \boldsymbol{\mu} \rangle - \langle \mathbf{b}u, \boldsymbol{\mu} \rangle \quad (5)$$

Before getting to the heart of the matter, the first chapter rapidly presents the FC-VFEM $\mathbb{P}_1$  toolbox. For a more complete description one can refer to [1]. Thereafter, notations and results on regular surfaces given in [2] are recalled. Finally some surface BVPs are presented and numerically solved by using the **Surface BVP Addon** for the FC-VFEM $\mathbb{P}_1$  Matlab toolbox.

## Generic Boundary Value Problems

The notations of [3] are employed in this section and extended to the vector case.

### 1.1 Scalar boundary value problem

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^d$ ,  $d \geq 1$ . The boundary of  $\Omega$  is denoted by  $\Gamma$ .

We denote by  $\mathcal{L}_{\mathbb{A}, \mathbf{b}, \mathbf{c}, a_0} = \mathcal{L} : \mathbf{H}^2(\Omega) \rightarrow L^2(\Omega)$  the second order linear differential operator acting on *scalar fields* defined,  $\forall u \in \mathbf{H}^2(\Omega)$ , by

$$\mathcal{L}_{\mathbb{A}, \mathbf{b}, \mathbf{c}, a_0}(u) \stackrel{\text{def}}{=} -\operatorname{div}(\mathbb{A} \nabla u) + \operatorname{div}(\mathbf{b}u) + \langle \nabla u, \mathbf{c} \rangle + a_0 u \quad (1.1)$$

where  $\mathbb{A} \in (L^\infty(\Omega))^{d \times d}$ ,  $\mathbf{b} \in (L^\infty(\Omega))^d$ ,  $\mathbf{c} \in (L^\infty(\Omega))^d$  and  $a_0 \in L^\infty(\Omega)$  are given functions and  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $\mathbb{R}^d$ . We use the same notations as in the chapter 6 of [3] and we note that we can omit either  $\operatorname{div}(\mathbf{b}u)$  or  $\langle \nabla u, \mathbf{c} \rangle$  if  $\mathbf{b}$  and  $\mathbf{c}$  are sufficiently regular functions. We keep both terms with  $\mathbf{b}$  and  $\mathbf{c}$  to deal with more boundary conditions. It should be also noted that it is important to preserve the two terms  $\mathbf{b}$  and  $\mathbf{c}$  in the generic formulation to enable a greater flexibility in the choice of the boundary conditions.

Let  $\Gamma^D, \Gamma^R$  be open subsets of  $\Gamma$ , possibly empty and  $f \in L^2(\Omega)$ ,  $g^D \in \mathbf{H}^{1/2}(\Gamma^D)$ ,  $g^R \in L^2(\Gamma^R)$ ,  $a^R \in L^\infty(\Gamma^R)$  be given data.

A *scalar* boundary value problem is given by

#### **Scalar BVP 1 : generic problem**

Find  $u \in \mathbf{H}^2(\Omega)$  such that

$$\mathcal{L}(u) = f \quad \text{in } \Omega, \quad (1.2)$$

$$u = g^D \quad \text{on } \Gamma^D, \quad (1.3)$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = g^R \quad \text{on } \Gamma^R. \quad (1.4)$$

The **conormal derivative** of  $u$  is defined by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} \stackrel{\text{def}}{=} \langle \mathbb{A} \nabla u, \mathbf{n} \rangle - \langle \mathbf{b}u, \mathbf{n} \rangle \quad (1.5)$$

The boundary conditions (1.3) and (1.4) are respectively **Dirichlet** and **Robin** boundary conditions.

**Neumann** boundary conditions are particular Robin boundary conditions with  $a^R \equiv 0$ .

To have an outline of the FC-VFEM $\mathbb{P}_1$  toolbox, a first and simple problem is quickly present. Explanations will be given in next sections.

The problem to solve is the Laplace problem for a condenser.

### Usual BVP 1 : 2D condenser problem

Find  $u \in H^2(\Omega)$  such that

$$-\Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^2, \quad (1.6)$$

$$u = 0 \text{ on } \Gamma_1, \quad (1.7)$$

$$u = -12 \text{ on } \Gamma_{98}, \quad (1.8)$$

$$u = 12 \text{ on } \Gamma_{99}, \quad (1.9)$$

where  $\Omega$  and its boundaries are given in Figure 1.1.

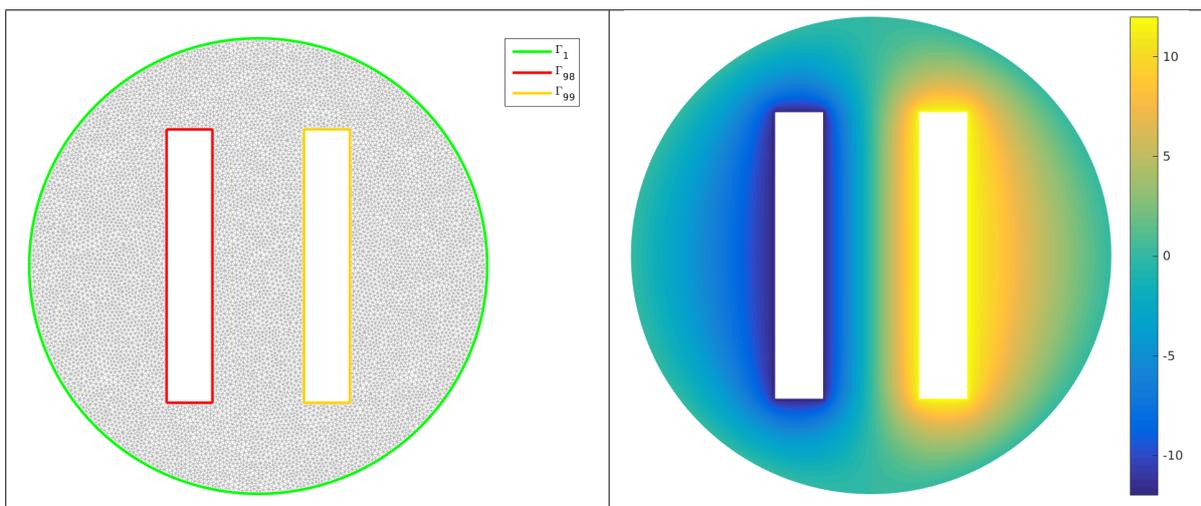


Figure 1.1: 2D condenser mesh and boundaries (left) and numerical solution (right)

The problem (1.6)-(1.9) can be equivalently expressed as the scalar BVP (1.2)-(1.4) :

### Scalar BVP 2 : 2D condenser problem

Find  $u \in H^2(\Omega)$  such that

$$\mathcal{L}(u) = f \quad \text{in } \Omega,$$

$$u = g^D \quad \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99}.$$

where  $\mathcal{L} := \mathcal{L}_{\mathbf{1}, \mathbf{0}, \mathbf{0}, 0}$ ,  $f \equiv 0$ , and

$$g^D := 0 \text{ on } \Gamma_1, \quad g^D := -12 \text{ on } \Gamma_{98}, \quad g^D := +12 \text{ on } \Gamma_{99}$$

In Listing 19 a complete code is given to solve this problem.

```

1 meshfile=gmsb.buildmesh2d('condenser',10); % generate mesh
2 Th=siMesh(meshfile); % read mesh
3 Lop=Loperator(2,2,{1,0;0,1},[],[],[]);
4 pde=PDEelt(Lop);
5 bvp=BVP(Th,pde);
6 bvp.setDirichlet( 1, 0.);
7 bvp.setDirichlet( 98, 12.);
8 bvp.setDirichlet( 99, +12.);
9 U=bvp.solve();
10 % Graphic parts
11 figure(1)
12 Th.plotmesh('color',0.7*[1,1,1])
13 hold on

```

```

14 Th.plotmesh('d',1,'Linewidth',2,'legend',true)
15 axis off,axis image
16 figure(2)
17 Th.plot(U,'edgecolor','none','facecolor','interp')
18 axis off,axis image;colorbar

```

Listing 1.1: Complete Matlab code to solve the 2D condenser problem with graphical representations

## 1.2 Vector boundary value problem

Let  $m \geq 1$  and  $\mathcal{H}$  be the  $m$ -by- $m$  matrix of second order linear differential operators defined by

$$\begin{cases} \mathcal{H} : (\mathbb{H}^2(\Omega))^m & \longrightarrow & (L^2(\Omega))^m \\ \mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m) & \longmapsto & \mathbf{f} = (\mathbf{f}_1, \dots, \mathbf{f}_m) \stackrel{\text{def}}{=} \mathcal{H}(\mathbf{u}) \end{cases} \quad (1.10)$$

where

$$\mathbf{f}_\alpha = \sum_{\beta=1}^m \mathcal{H}_{\alpha,\beta}(\mathbf{u}_\beta), \quad \forall \alpha \in \llbracket 1, m \rrbracket, \quad (1.11)$$

with, for all  $(\alpha, \beta) \in \llbracket 1, m \rrbracket^2$ ,

$$\mathcal{H}_{\alpha,\beta} \stackrel{\text{def}}{=} \mathcal{L}_{\mathbb{A}^{\alpha,\beta}, \mathbf{b}^{\alpha,\beta}, \mathbf{c}^{\alpha,\beta}, a_0^{\alpha,\beta}} \quad (1.12)$$

and  $\mathbb{A}^{\alpha,\beta} \in (L^\infty(\Omega))^{d \times d}$ ,  $\mathbf{b}^{\alpha,\beta} \in (L^\infty(\Omega))^d$ ,  $\mathbf{c}^{\alpha,\beta} \in (L^\infty(\Omega))^d$  and  $a_0^{\alpha,\beta} \in L^\infty(\Omega)$  are given functions. We can also write in matrix form

$$\mathcal{H}(\mathbf{u}) = \begin{pmatrix} \mathcal{L}_{\mathbb{A}^{1,1}, \mathbf{b}^{1,1}, \mathbf{c}^{1,1}, a_0^{1,1}} & \cdots & \mathcal{L}_{\mathbb{A}^{1,m}, \mathbf{b}^{1,m}, \mathbf{c}^{1,m}, a_0^{1,m}} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{\mathbb{A}^{m,1}, \mathbf{b}^{m,1}, \mathbf{c}^{m,1}, a_0^{m,1}} & \cdots & \mathcal{L}_{\mathbb{A}^{m,m}, \mathbf{b}^{m,m}, \mathbf{c}^{m,m}, a_0^{m,m}} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_m \end{pmatrix}. \quad (1.13)$$

We remark that the  $\mathcal{H}$  operator for  $m = 1$  is equivalent to the  $\mathcal{L}$  operator.

For  $\alpha \in \llbracket 1, m \rrbracket$ , we define  $\Gamma_\alpha^D$  and  $\Gamma_\alpha^R$  as open subsets of  $\Gamma$ , possibly empty, such that  $\Gamma_\alpha^D \cap \Gamma_\alpha^R = \emptyset$ . Let  $\mathbf{f} \in (L^2(\Omega))^m$ ,  $g_\alpha^D \in H^{1/2}(\Gamma_\alpha^D)$ ,  $g_\alpha^R \in L^2(\Gamma_\alpha^R)$ ,  $a_\alpha^R \in L^\infty(\Gamma_\alpha^R)$  be given data.

A *vector boundary value problem* is given by

### Vector BVP 1 : generic problem

Find  $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m) \in (\mathbb{H}^2(\Omega))^m$  such that

$$\mathcal{H}(\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega, \quad (1.14)$$

$$\mathbf{u}_\alpha = g_\alpha^D \quad \text{on } \Gamma_\alpha^D, \quad \forall \alpha \in \llbracket 1, m \rrbracket, \quad (1.15)$$

$$\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \mathbf{u}_\alpha = g_\alpha^R \quad \text{on } \Gamma_\alpha^R, \quad \forall \alpha \in \llbracket 1, m \rrbracket, \quad (1.16)$$

where the  $\alpha$ -th component of the **conormal derivative** of  $\mathbf{u}$  is defined by

$$\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_\alpha}} \stackrel{\text{def}}{=} \sum_{\beta=1}^m \frac{\partial \mathbf{u}_\beta}{\partial n_{\mathcal{H}_{\alpha,\beta}}} = \sum_{\beta=1}^m \left( \langle \mathbb{A}^{\alpha,\beta} \nabla \mathbf{u}_\beta, \mathbf{n} \rangle - \langle \mathbf{b}^{\alpha,\beta} \mathbf{u}_\beta, \mathbf{n} \rangle \right). \quad (1.17)$$

The boundary conditions (1.16) are the **Robin** boundary conditions and (1.15) is the **Dirichlet** boundary condition. The **Neumann** boundary conditions are particular Robin boundary conditions with  $a_\alpha^R \equiv 0$ .

In this problem, we may consider on a given boundary some conditions which can vary depending on the component. For example we may have a Robin boundary condition satisfying  $\frac{\partial \mathbf{u}}{\partial n_{\mathcal{H}_1}} + a_1^R \mathbf{u}_1 = g_1^R$  and a Dirichlet one with  $\mathbf{u}_2 = g_2^D$ .

To have an outline of the FC-VFEMP<sub>1</sub> toolbox, a second and simple problem is quickly present.

### 💡 Usual vector BVP 1 : 2D simple vector problem

Find  $\mathbf{u} = (u_1, u_2) \in (H^2(\Omega))^2$  such that

$$-\Delta u_1 + u_2 = 0 \text{ in } \Omega \subset \mathbb{R}^2, \quad (1.18)$$

$$-\Delta u_2 + u_1 = 0 \text{ in } \Omega \subset \mathbb{R}^2, \quad (1.19)$$

$$(u_1, u_2) = (0, 0) \text{ on } \Gamma_1, \quad (1.20)$$

$$(u_1, u_2) = (-12., +12.) \text{ on } \Gamma_{98}, \quad (1.21)$$

$$(u_1, u_2) = (+12., -12.) \text{ on } \Gamma_{99}, \quad (1.22)$$

where  $\Omega$  and its boundaries are given in Figure 1.1.

The problem (1.18)-(1.22) can be equivalently expressed as the vector BVP (1.2)-(1.4) :

### 🔗 Vector BVP 2 : 2D simple vector problem

Find  $\mathbf{u} = (u_1, u_2) \in (H^2(\Omega))^2$  such that

$$\mathcal{H}(\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega,$$

$$u_1 = g_1^D \quad \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99},$$

$$u_2 = g_2^D \quad \text{on } \Gamma^D = \Gamma_1 \cup \Gamma_{98} \cup \Gamma_{99},$$

where

$$\mathcal{H} := \begin{pmatrix} \mathcal{L}_{1,0,0,0} & \mathcal{L}_{0,0,0,1} \\ \mathcal{L}_{0,0,0,1} & \mathcal{L}_{1,0,0,0} \end{pmatrix}, \text{ as } \mathcal{H} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -\Delta & 1 \\ 1 & -\Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$f \equiv 0,$$

and

$$g_1^D = g_2^D := 0 \text{ on } \Gamma_1, \quad g_1^D := -12, \quad g_2^D := +12 \text{ on } \Gamma_{98}, \quad g_1^D := +12, \quad g_2^D := -12 \text{ on } \Gamma_{99}$$

In Listing 21 a complete code is given to solve this problem. Numerical solutions are given in Figure 1.2.

```

1 meshfile=gsmsh.buildmesh2d('condenser',10); % generate mesh
2 Th=siMesh(meshfile); % read mesh
3 Hop=Hoperator(2,2,2);
4 Hop.set([1,2],[1,2],Loperator(2,2,{1,[],[],1},[],[],[]));
5 Hop.set([1,2],[2,1],Loperator(2,2,[],[],[],1));
6 pde=PDEelt(Hop);
7 bvp=BVP(Th,pde);
8 bvp.setDirichlet( 1, 0.,1:2);
9 bvp.setDirichlet( 98, {12,+12},1:2);
10 bvp.setDirichlet( 99, {+12,12},1:2);
11 U=bvp.solve('split',true);
12 % Graphic parts
13 figure(1)
14 Th.plot(U{1})
15 axis image;axis off;shading interp
16 colorbar
17 figure(2);
18 Th.plot(U{2})
19 axis image;axis off;shading interp
20 colorbar

```

Listing 1.2: Complete Matlab code to solve the funny 2D vector problem with graphical representations

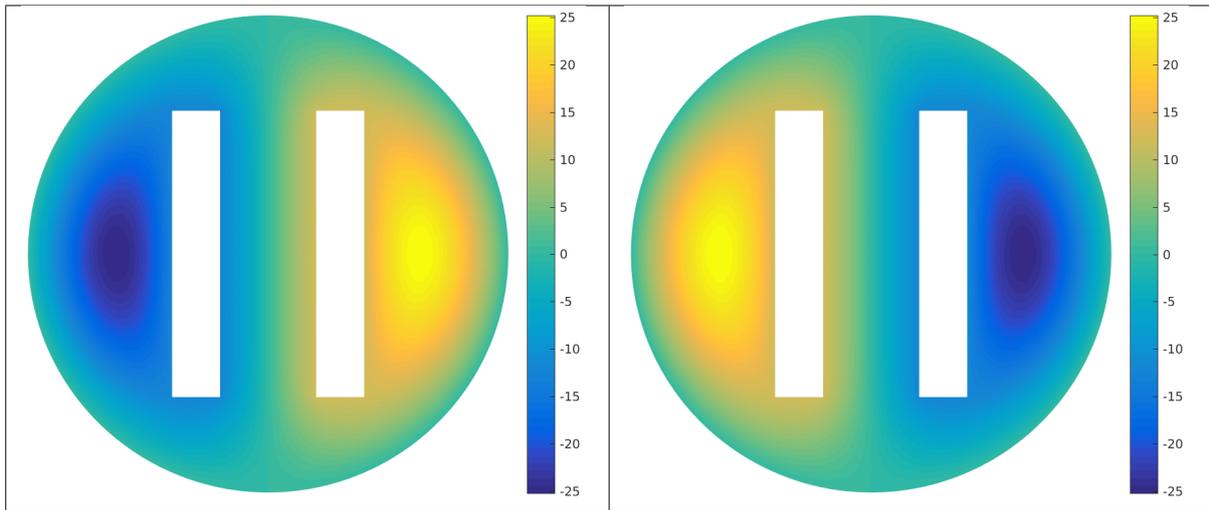


Figure 1.2: Funny vector BVP,  $u_1$  numerical solution (left) and  $u_2$  numerical solution (right)

# Chapter 2

## Notations and results on regular surfaces

All the notations and results of this chapter are directly obtain from [2].

### 2.1 Elementary geometric analysis

#### ♥ Definition 2.1: $C^k$ -hypersurface

Let  $k \in \mathbb{N} \cup \infty$ .  $\Gamma \subset \mathbb{R}^{n+1}$  is called a  $C^k$ -hypersurface if, for each point  $x_0 \in \Gamma$ , there exists an open set  $U \subset \mathbb{R}^{n+1}$  containing  $x_0$  and a function  $\Phi \in C^k(U; \mathbb{R})$  with the property that  $\nabla \Phi \neq 0$  on  $\Gamma \cap U$  and such that

$$\Gamma \cap U = \{x \in U \mid \Phi(x) = 0\}. \quad (2.1)$$

The linear space

$$T_x \Gamma = [\nabla \Phi(x)]^\perp \quad (2.2)$$

is called the *tangent space* to  $\Gamma$  at  $x \in \Gamma$ .

A vector  $\nu(x) \in \mathbb{R}^{n+1}$  is called a *unit normal vector* at  $x \in \Gamma$  if  $\nu(x) \perp T_x \Gamma$  and  $\|\nu(x)\| = 1$ . We have

$$\nu(x) = \frac{\nabla \Phi(x)}{\|\nabla \Phi(x)\|} \quad \text{or} \quad \nu(x) = -\frac{\nabla \Phi(x)}{\|\nabla \Phi(x)\|} \quad (2.3)$$

A  $C^1$ -hypersurface is called *orientable* if there exists a continuous vector field  $\nu : \Gamma \rightarrow \mathbb{R}^{n+1}$  such that  $\nu(x)$  is a unit normal vector to  $\Gamma$  for all  $x \in \Gamma$ .

#### ♥ Definition 2.2: Tangential gradient or surface gradient

Let  $\Gamma \subset \mathbb{R}^{n+1}$  be a  $C^1$ -hypersurface and let  $f : \Gamma \rightarrow \mathbb{R}$  be differentiable at  $x \in \Gamma$ . we define the tangential gradient or surface gradient of  $f$  at  $x \in \Gamma$  by

$$\nabla_\Gamma f(x) = \nabla \bar{f}(x) - \langle \nabla \bar{f}(x), \nu(x) \rangle \nu(x) = \mathbb{P}(x) \nabla \bar{f}(x), \quad (2.4)$$

where  $(\mathbb{P}(x))_{i,j} = \delta_{i,j} - \nu_i(x)\nu_j(x)$ ,  $\forall (i,j) \in \llbracket 1, n+1 \rrbracket^2$ . Here  $\bar{f}$  is a smooth extension of  $f$  to a neighbourhood  $U \subset \mathbb{R}^{n+1}$  of the surface  $\Gamma$ , so that  $\bar{f}|_\Gamma = f$ .

We shall use the notation

$$\nabla_{\Gamma} f(x) = (\underline{D}_1 f(x), \dots, \underline{D}_{n+1} f(x))^t \quad (2.5)$$

for the  $n+1$  components of the surface gradient. Note that  $\langle \nabla_{\Gamma} f(x), \nu(x) \rangle = 0$  and hence  $\nabla_{\Gamma} f(x) \in T_x \Gamma$ . We denote by :

- $\mathcal{C}^0(\Gamma; \mathbb{R})$  the set of functions  $f : \Gamma \rightarrow \mathbb{R}$  which are continuous.
- $\mathcal{C}^1(\Gamma; \mathbb{R})$  the set of functions in  $\mathcal{C}^0(\Gamma; \mathbb{R})$  which are differentiable at every point  $x \in \Gamma$  and for which  $\underline{D}_i \in \mathcal{C}^0(\Gamma; \mathbb{R})$ , for all  $i \in \llbracket 1, n+1 \rrbracket$ .
- $\mathcal{C}^l(\Gamma; \mathbb{R})$ ,  $l \in \mathbb{N}$  provide that  $\Gamma$  is a  $\mathcal{C}^k$ -hypersurface with  $k \geq l$ .

Let  $\mathbf{g} : \Gamma \rightarrow \mathbb{R}^{n+1}$  be differentiable at  $x \in \Gamma$  ( $\mathbf{g}$  is a vector field) and  $f : \Gamma \rightarrow \mathbb{R}$  be twice differentiable at  $x \in \Gamma$ . The surface divergence is given by

$$\operatorname{div}_{\Gamma} \mathbf{g}(x) = \sum_{i=1}^{n+1} \underline{D}_i \mathbf{g}_i(x) \quad (2.6)$$

and, the *Laplace-Beltrami operator* is given by

$$\Delta_{\Gamma} f(x) = \operatorname{div}_{\Gamma} \nabla_{\Gamma} f(x) = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i f(x) \quad (2.7)$$

### ♥ Definition 2.3: Extend Weingarten map

Let  $\Gamma$  be a  $\mathcal{C}^2$ -hypersurface. The extend Weingarten map is the matrix  $\mathbb{H} \in \mathcal{M}_{n+1}(\mathcal{C}^0(\Gamma; \mathbb{R}))$  define by

$$\mathbb{H}_{i,j} = \underline{D}_i \nu_j, \quad \forall (i,j) \in \llbracket 1, n+1 \rrbracket^2. \quad (2.8)$$

The matrix  $\mathbb{H}$  is symmetric and  $\mathbb{H}\nu = 0$ . The restriction of  $\mathbb{H}$  to the tangent space is called the *Weingarten map*.

The *mean curvature* of  $\Gamma$  at point  $x \in \Gamma$  is the quantity

$$H(x) = \operatorname{trace} \mathbb{H}(x) = \sum_{i=1}^{n+1} \mathbb{H}_{i,i}(x). \quad (2.9)$$

It differs from the common definition by a factor of  $n$ .

### 📖 Lemme 2.4

Let  $\Gamma$  be a  $\mathcal{C}^2$ -hypersurface and  $u \in \mathcal{C}^2(\Gamma; \mathbb{R})$ . Then, we have

$$\underline{D}_i \underline{D}_j u - \underline{D}_j \underline{D}_i u = (\mathbb{H} \nabla_{\Gamma} u)_j \nu_i - (\mathbb{H} \nabla_{\Gamma} u)_i \nu_j, \quad \forall (i,j) \in \llbracket 1, n+1 \rrbracket^2. \quad (2.10)$$

### 📖 Theorem 2.5: integration by parts on surfaces

Assume that  $\Gamma$  is a hypersurface in  $\mathbb{R}^{n+1}$  with smooth boundary  $\partial\Gamma$  and that  $f \in \mathcal{C}^1(\bar{\Gamma}; \mathbb{R})$ . Then

$$\int_{\Gamma} \underline{D}_i f dA = \int_{\Gamma} f H \nu_i dA + \int_{\partial\Gamma} f \mu_i dA, \quad \forall i \in \llbracket 1, n+1 \rrbracket \quad (2.11)$$

and so

$$\int_{\Gamma} \nabla_{\Gamma} f dA = \int_{\Gamma} f H \nu dA + \int_{\partial\Gamma} f \mu dA. \quad (2.12)$$

Here,  $\mu$  denotes the co-normal vector which is normal to  $\partial\Gamma$  and tangent to  $\Gamma$ .

A compact hypersurface  $\Gamma$  does not have boundary,  $\partial\Gamma = \emptyset$ , and the last term on the right-hand side vanishes.

Note that  $dA$  in connection with an integral over  $\Gamma$  denotes the  $n$ -dimensional surface measure, while  $dA$  in connection with an integral over  $\partial\Gamma$  is the  $n - 1$ -dimensional surface measure.

## 2.2 Sobolev space on surface

Let  $\Gamma$  be a  $\mathcal{C}^2$ -hypersurface for the following.

For  $p \in \llbracket 1, \infty \rrbracket$  we let  $L^p(\Gamma)$  denote the space of functions  $f : \Gamma \rightarrow \mathbb{R}$  which are measurable with respect to the surface measure  $dA$  and have a finite norm, where the norm is defined by

$$\|f\|_{L^p(\Gamma)} = \left( \int_{\Gamma} |f|^p dA \right)^{\frac{1}{p}}$$

for  $p < \infty$ , and for  $p = \infty$  we mean the essential supremum norm.

$L^p(\Gamma)$  is a Banach space and for  $p = 2$  a Hilbert space. For  $1 \leq p < \infty$  the spaces  $\mathcal{C}^0(\Gamma; \mathbb{R})$  and  $\mathcal{C}^1(\Gamma; \mathbb{R})$  are dense in  $L^p(\Gamma)$ .

### ♥ Definition 2.6

A function  $f \in L^p(\Gamma)$  has the weak derivative  $v_i = \underline{D}_i f \in L^p(\Gamma)$  if, for every function  $\varphi \in \mathcal{C}^1(\Gamma; \mathbb{R})$  with compact support  $\overline{\{x \in \Gamma \mid \varphi(x) \neq 0\}} \subset \Gamma$ , we have the relation

$$\int_{\Gamma} f \underline{D}_i \varphi dA = - \int_{\Gamma} f v_i dA + \int_{\Gamma} f \varphi H \boldsymbol{\nu}_i dA.$$

The Sobolev space  $H^{1,p}(\Gamma)$  is defined by

$$H^{1,p}(\Gamma) = \{f \in L^p(\Gamma) \mid \underline{D}_i f \in L^p(\Gamma), i \in \llbracket 1, n+1 \rrbracket\} \quad (2.13)$$

with norm

$$\|f\|_{H^{1,p}(\Gamma)} = \left( \|f\|_{L^p(\Gamma)}^p + \|\nabla_{\Gamma} f\|_{L^p(\Gamma)}^p \right)^{\frac{1}{p}} \quad (2.14)$$

### 📖 Theorem 2.7: Poincaré's inequality

Assume that  $\Gamma$  is a  $\mathcal{C}^3$ -hypersurface and  $1 \leq p < \infty$ . Then there is a constant  $C > 0$  such that, for every function  $f \in H^{1,p}(\Gamma)$  with  $\int_{\Gamma} f dA = 0$ , we have the inequality

$$\|f\|_{L^p(\Gamma)} \leq C \|\nabla_{\Gamma} f\|_{L^p(\Gamma)}. \quad (2.15)$$

### 📖 Theorem 2.8: Green's formula

Let  $\Gamma$  be a  $\mathcal{C}^2$ -hypersurface. Then for all  $f \in H^{1,p}(\Gamma)$  and  $g \in H^{2,p}(\Gamma)$  we have

$$\int_{\Gamma} \langle \nabla_{\Gamma} f, \nabla_{\Gamma} g \rangle dA = - \int_{\Gamma} f \Delta_{\Gamma} g dA + \int_{\partial\Gamma} f \langle \nabla_{\Gamma} g, \boldsymbol{\mu} \rangle dA. \quad (2.16)$$

A compact hypersurface  $\Gamma$  does not have boundary,  $\partial\Gamma = \emptyset$ , and the last term on the right-hand side vanishes.

*Proof.* Let  $i \in \llbracket 1, n \rrbracket$ , we have

$$\underline{D}_i (f \underline{D}_i (g)) = \underline{D}_i (f) \underline{D}_i (g) + f \underline{D}_i \underline{D}_i (g)$$

and then

$$\int_{\Gamma} \underline{D}_i (f) \underline{D}_i (g) dA = \int_{\Gamma} \underline{D}_i (f \underline{D}_i (g)) dA - \int_{\Gamma} f \underline{D}_i \underline{D}_i (g) dA$$

Using integration by parts formula (2.11) with  $f$  identify to  $f\underline{D}_i(g)$  we obtain

$$\int_{\Gamma} \underline{D}_i(f)\underline{D}_i(g)dA = \int_{\Gamma} f\underline{D}_i(g)H\underline{\nu}_i dA + \int_{\partial\Gamma} f\underline{D}_i(g)\underline{\mu}_i dA - \int_{\Gamma} f\underline{D}_i\underline{D}_i(g)dA$$

By summing in  $i$ , we obtain

$$\int_{\Gamma} \langle \nabla_{\Gamma} f, \nabla_{\Gamma} g \rangle dA = \int_{\Gamma} f \langle \nabla_{\Gamma} g, \underline{\nu} \rangle H dA + \int_{\partial\Gamma} f \langle \nabla_{\Gamma} g, \underline{\mu} \rangle dA - \int_{\Gamma} f \Delta_{\Gamma} g dA$$

Since  $\langle \nabla_{\Gamma} g, \underline{\nu} \rangle = 0$ , the formula is proved.  $\square$

## 2.3 Variational formulation of a surface BVP

In this section, the variational formulation of the generic *scalar* surface BVP (1.2)-(1.4) is established by using previous results.

### Lemme 2.9

Let  $\Gamma$  be a  $\mathcal{C}^2$ -hypersurface. Then, for all  $f, g \in H^1(\Gamma)$  and for all  $i \in \llbracket 1, n+1 \rrbracket$ , we have

$$\int_{\Gamma} \underline{D}_i(f)g dA = \int_{\Gamma} fgH\underline{\nu}_i dA - \int_{\Gamma} f\underline{D}_i(g) dA + \int_{\partial\Gamma} fg\underline{\mu}_i dA. \quad (2.17)$$

*Proof.* We have  $\underline{D}_i(f)g = \underline{D}_i(fg) - f\underline{D}_i(g)$  and by integration we obtain using ...

$$\begin{aligned} \int_{\Gamma} \underline{D}_i(f)g dA &= \int_{\Gamma} \underline{D}_i(fg) dA - \int_{\Gamma} f\underline{D}_i(g) dA \\ &= \int_{\Gamma} fgH\underline{\nu}_i dA + \int_{\partial\Gamma} fg\underline{\mu}_i dA - \int_{\Gamma} f\underline{D}_i(g) dA \end{aligned}$$

$\square$

### Lemme 2.10

Let  $\Gamma$  be a  $\mathcal{C}^2$ -hypersurface. Then, for all  $u \in H^2(\Gamma)$  and  $v \in H^1(\Gamma)$ , we have

$$\int_{\Gamma} \operatorname{div}_{\Gamma}(\mathbb{A} \nabla_{\Gamma} u) v dA = - \int_{\Gamma} \langle \mathbb{A} \nabla_{\Gamma} u, \nabla_{\Gamma} v \rangle dA + \int_{\partial\Gamma} \langle \mathbb{A} \nabla_{\Gamma} u, \underline{\mu} \rangle dA. \quad (2.18)$$

*Proof.* Using (2.17) with  $f = A_{i,j}\underline{D}_j(u)$  and  $g = v$  we obtain

$$\begin{aligned} \int_{\Gamma} \underline{D}_i(A_{i,j}\underline{D}_j(u))v dA &= \int_{\Gamma} A_{i,j}\underline{D}_j(u)vH\underline{\nu}_i dA - \int_{\Gamma} A_{i,j}\underline{D}_j(u)\underline{D}_i(v) dA \\ &\quad + \int_{\partial\Gamma} A_{i,j}\underline{D}_j(u)v\underline{\mu}_i dA. \end{aligned}$$

Summation in  $i$  and  $j$  give

$$\begin{aligned} \int_{\Gamma} \operatorname{div}_{\Gamma}(\mathbb{A} \nabla_{\Gamma} u) v dA &= \int_{\Gamma} \langle \mathbb{A} \nabla_{\Gamma} u, \underline{\nu} \rangle H v dA - \int_{\Gamma} \langle \mathbb{A} \nabla_{\Gamma} u, \nabla_{\Gamma} v \rangle dA \\ &\quad + \int_{\partial\Gamma} \langle \mathbb{A} \nabla_{\Gamma} u, \underline{\mu} \rangle dA. \end{aligned}$$

By hypothesis on  $\mathbb{A}$ ,  $\mathbb{A}(x) \nabla_{\Gamma} u(x) \in T_x\Gamma$  and so  $\langle \mathbb{A}(x) \nabla_{\Gamma} u(x), \underline{\nu}(x) \rangle = 0$ .  $\square$

 **Lemme 2.11**

Let  $\Gamma$  be a  $\mathcal{C}^2$ -hypersurface. Then, for all  $u, v \in H^1(\Gamma)$ , we have

$$\int_{\Gamma} \operatorname{div}_{\Gamma}(\mathbf{b}u)v \, dA = - \int_{\Gamma} \langle \mathbf{b}u, \nabla_{\Gamma} v \rangle \, dA + \int_{\partial\Gamma} \langle \mathbf{b}u, \boldsymbol{\mu} \rangle \, v \, dA. \quad (2.19)$$

*Proof.* Using (2.17) with  $f = b_i u$  and  $g = v$  we obtain

$$\int_{\Gamma} \underline{D}_i(b_i u)v \, dA = \int_{\Gamma} b_i u v H \boldsymbol{\nu}_i \, dA - \int_{\Gamma} b_i u \underline{D}_i(v) \, dA + \int_{\partial\Gamma} b_i u v \boldsymbol{\mu}_i \, dA.$$

Summation in  $i$  give

$$\begin{aligned} \int_{\Gamma} \operatorname{div}_{\Gamma}(\mathbf{b}u)v \, dA &= \int_{\Gamma} \langle \mathbf{b}u, \boldsymbol{\nu} \rangle H v \, dA - \int_{\Gamma} \langle \mathbf{b}u, \nabla_{\Gamma} v \rangle \, dA \\ &\quad + \int_{\partial\Gamma} \langle \mathbf{b}u, \boldsymbol{\mu} \rangle \, v \, dA. \end{aligned}$$

By hypothesis, for all  $x \in \Gamma$ ,  $\mathbf{b}(x) \in T_x \Gamma$  and so  $\langle \mathbf{b}(x)u(x), \boldsymbol{\nu}(x) \rangle = u(x) \langle \mathbf{b}(x), \boldsymbol{\nu}(x) \rangle = 0$ .  $\square$

So we have proved the following theorem

 **Theorem 2.12**

Let  $\Gamma \subset \mathbb{R}^{n+1}$  be a  $\mathcal{C}^2$ -hypersurface. Let  $a_0, b_i, c_i, A_{i,j} \in L^{\infty}(\Gamma)$ ,  $\forall (i, j) = \llbracket 1, n+1 \rrbracket^2$ . We also assume that for all  $x \in \Gamma$ ,  $\mathbf{b}(x) \in T_x \Gamma$  and the matrix  $\mathbb{A}(x) = (A_{i,j}(x))_{i,j=1,\dots,n+1}$  maps the tangent space  $T_x \Gamma$  into itself (i.e.  $\mathbb{A}(x) : T_x \Gamma \rightarrow T_x \Gamma$ ). We denote by  $\mathcal{L}$  the second order linear differential operator acting on scalar fields defined by

$$\mathcal{L}(u) \stackrel{\text{def}}{=} - \operatorname{div}_{\Gamma}(\mathbb{A} \nabla_{\Gamma} u) + \operatorname{div}_{\Gamma}(\mathbf{b}u) + \langle \nabla_{\Gamma} u, \mathbf{c} \rangle + a_0 u, \quad \forall u \in H^2(\Gamma).$$

Then, for all  $u \in H^2(\Gamma)$  and  $v \in H^1(\Gamma)$ , we have

$$\begin{aligned} \int_{\Gamma} \mathcal{L}(u)v \, dA &= \int_{\Gamma} \langle \mathbb{A} \nabla_{\Gamma} u, \nabla_{\Gamma} v \rangle \, dA + \int_{\Gamma} a_0 u v \, dA \\ &\quad - \left( \int_{\Gamma} u \langle \nabla_{\Gamma} v, \mathbf{b} \rangle \, dA - \int_{\Gamma} \langle \nabla_{\Gamma} u, \mathbf{c} \rangle \, v \, dA \right) \\ &\quad - \int_{\partial\Gamma} \langle \mathbb{A} \nabla_{\Gamma} u, \boldsymbol{\mu} \rangle \, v \, dA + \int_{\partial\Gamma} \langle \mathbf{b}u, \boldsymbol{\mu} \rangle \, v \, dA. \end{aligned}$$

Let  $\mathcal{D}_{\mathcal{L}} = \mathcal{D}_{\mathbb{A}, \mathbf{b}, \mathbf{c}, a_0}$  be the first order bilinear differential operator acting on *scalar fields* associated to the  $\mathcal{L}$  operator defined  $\forall (u, v) \in (H^1(\Gamma))^2$  by

$$\mathcal{D}_{\mathcal{L}}(u, v) = \langle \mathbb{A} \nabla_{\Gamma} u, \nabla_{\Gamma} v \rangle - (\langle \nabla_{\Gamma} v, \mathbf{b} \rangle u - \langle \nabla_{\Gamma} u, \mathbf{c} \rangle v) + a_0 u v. \quad (2.20)$$

The variational formulation associated to the scalar surface BVP (1.2)-(1.4) is given by


**Variational formulation of the scalar surface BVP (1.2)-(1.4)**

Find  $u \in H_{g^D, \partial\Gamma^D}^1(\Gamma)$  such that

$$\mathcal{A}_{\mathcal{L}}(u, v) = \mathcal{F}(v) \quad \forall v \in H_{0, \partial\Gamma^D}^1(\Gamma) \quad (2.21)$$

where

$$\mathcal{A}_{\mathcal{L}}(u, v) = \int_{\Gamma} \mathcal{D}_{\mathcal{L}}(u, v) \, dA + \int_{\partial\Gamma^R} a^R u v \, dA \quad (2.22)$$

$$\mathcal{F}(v) = \int_{\Gamma} f v \, dA + \int_{\partial\Gamma^R} g^R v \, dA \quad (2.23)$$

# Chapter 3

## scalar surface BVP

This next section looks first at the construction of surface BVPs where analytic solutions are known by using Sage (a formal calculation software). Thereafter, some surface BVPs are numerically solved by using the FC-VFEM $P_1$  Matlab toolbox and solutions compared to exact solutions when we know them.

### 3.1 Analytical solutions of surface BVPs

#### 3.1.1 Sage computation of Laplace-Beltrami operator on hypersurfaces

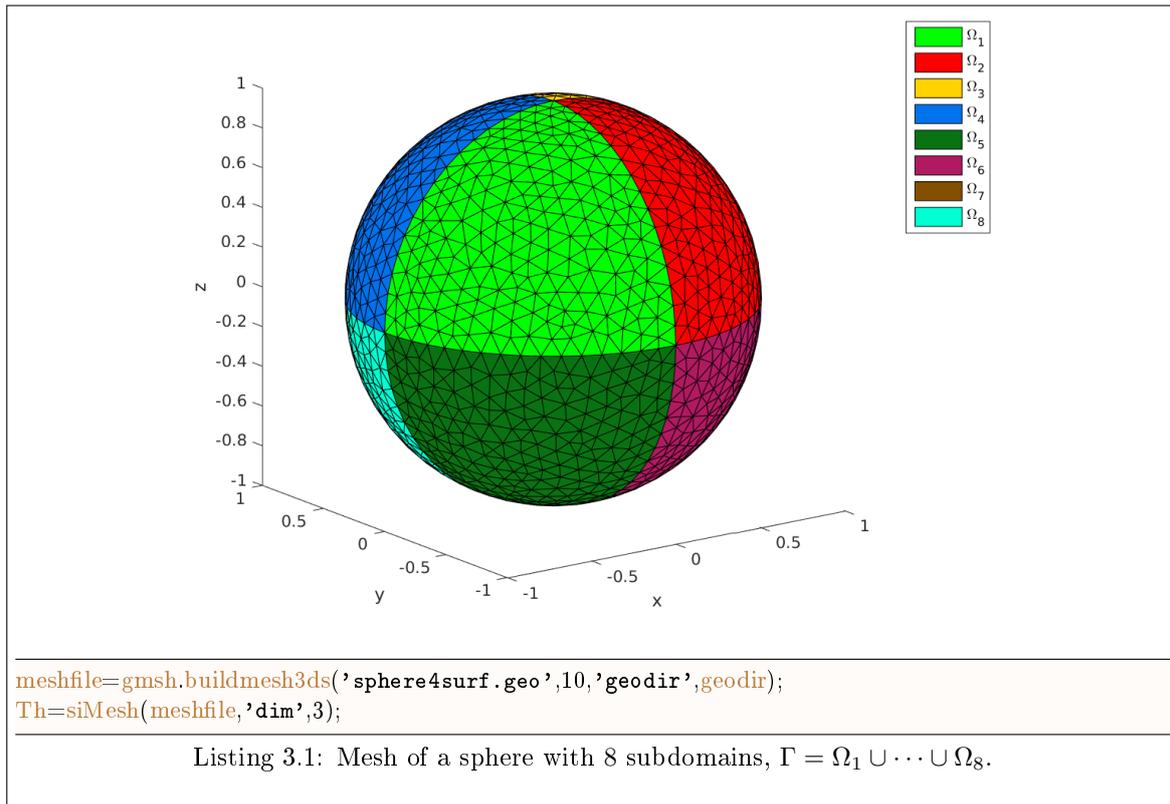
Let the hypersurface  $\Gamma$  given by a level set function  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}$  :

$$\Gamma = \{\mathbf{x} \in \mathbb{R}^3 \mid \Phi(\mathbf{x}) = 0.\}$$

For the unit sphere, one can use

$$\Phi : \mathbf{x} \mapsto x_1^2 + x_2^2 + x_3^2 - 1.$$

With FC-VFEM $P_1$ , one can easily generated this surface from a sphere mesh made by `gmsh` and `sphere4surf.geo` :



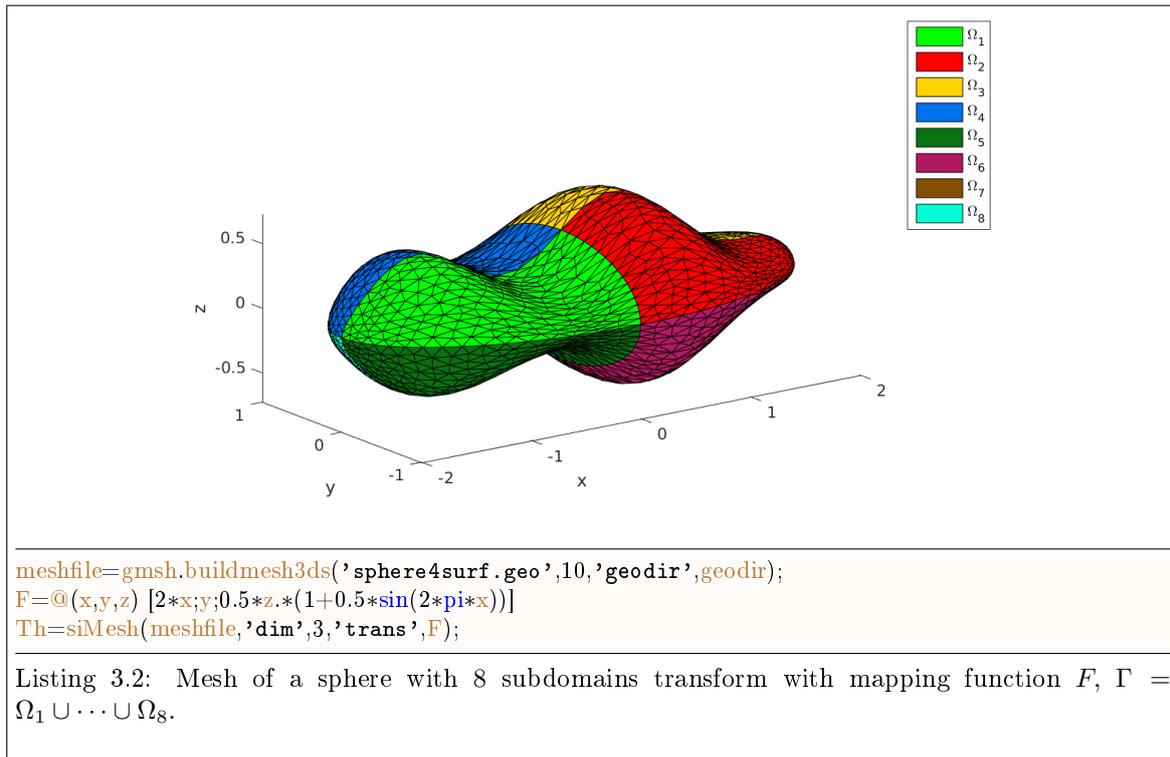
More complicated surfaces can be obtained by mapping the unit sphere  $S^2$  onto  $\Gamma$ . we note  $F : S^2 \rightarrow \Gamma$  the map function. From example 4.8 in [2], we take

$$F(y_1, y_2, y_3) = \left( 2y_1, y_2, \frac{1}{2}y_3 \left( 1 + \frac{1}{2} \sin(2\pi y_1) \right) \right) \in \Gamma, \quad \forall \mathbf{y} = (y_1, y_2, y_3) \in S^2 \quad (3.1)$$

The representation of  $\Gamma = \mathbf{F}(S^2)$  as a hypersurface  $\{\mathbf{x} \in \mathbb{R}^3 \mid \phi(\mathbf{x}) = 0\}$  follows from  $y_1^2 + y_2^2 + y_3^2 = 1$  with the level set function

$$\phi(\mathbf{x}) = \frac{1}{4}x_1^2 + x_2^2 + \frac{4x_3^2}{\left(1 + \frac{1}{2} \sin(\pi x_1)\right)^2} - 1. \quad (3.2)$$

With  $\text{FC-VFEM}\mathbb{P}_1$ , one can easily generated this surface from a mesh of the sphere made by `gmsht` and `sphere4surf.geo` :



To compute the Laplace-Beltrami operator apply on a given function over a given surface we use the Sage software (see [1]) by implementing :

- the surface gradient operator  $\nabla_{\Gamma}$ , formula (2.4),
- the surface divergence operator  $\text{div}_{\Gamma}$ , formula (2.6),
- the Laplace-Beltrami operator  $\Delta_{\Gamma}$ , formula (2.7).

The source code is given in Listing 3.3.

Listing 3.3: Implementation of Laplace-Beltrami operator, Sage code

```

# gradient of function Phi
def Gradient(Phi):
    return Phi.diff()
# normal to the surface function Phi
def normal(Phi):
    G=Gradient(Phi)
    return G/G.norm()
# Projection matrix on tangent space
def Pmat(normal):
    P=[];dim=len(normal)
    for i in range(dim):
        Pr=[];
        for j in range(dim):
            if i==j:
                Pr.append(1*normal[i]*normal[j])
            else:
                Pr.append(normal[i]*normal[j])
        P.append(Pr)
    return matrix(P)
# Surface gradient of u over surface function Phi
def SurfGrad(u,Phi):
    P=Pmat(normal(Phi))
    gradu=Gradient(u)
    return P*gradu
# Surface divergence of u over surface function Phi
def SurfDiv(V,Phi):
    P=Pmat(normal(Phi))
    f=0
    for i in range(len(V)):
        g=P*Gradient(V[i])
        f=f+g[i]
    return f.simplify_full()
# LaplaceBeltrami operator on u over surface function Phi
def LaplaceBeltrami(u,Phi):
    return SurfDiv(SurfGrad(u,Phi),Phi).simplify_full()

```

Under Sage, one can now compute some exact solutions on hypersurfaces :

- sample 1 :  $\Gamma = S^2$

```

load('../sage/LaplaceBeltrami.sage')
var('x,y,z',domain=RR)
Phi(x,y,z)=x**2+y**2+z**2 -1
u(x,y,z)=x*y
f(x,y,z)=-LaplaceBeltrami(u,Phi)

```

So with  $\Phi : (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$  and  $u : (x, y, z) \mapsto xy$  we have

$$f = -\Delta_{\Gamma} u : (x, y, z) \mapsto \frac{6xy}{x^2 + y^2 + z^2}$$

- sample 2 :  $\Gamma = S^2$

```

load('../sage/LaplaceBeltrami.sage')
var('x,y,z',domain=RR)
Phi(x,y,z)=x**2+y**2+z**2 -1
u(x,y,z)=x**2*y**2
f(x,y,z)=-LaplaceBeltrami(u,Phi)

```

So with  $\Phi : (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$  and  $u : (x, y, z) \mapsto x^2 y^2$  we have

$$f = -\Delta_{\Gamma} u : (x, y, z) \mapsto -\frac{2(x^4 - 8x^2 y^2 + y^4 + (x^2 + y^2)z^2)}{x^2 + y^2 + z^2}$$

- sample 3 :  $\Gamma = F(S^2)$  with  $F$  given by (3.1) and the level set function  $\Phi$  defined in (3.2)

```
load('../sage/LaplaceBeltrami.sage')
var('x,y,z',domain=RR)
u(x,y,z)=x*y
Phi(x,y,z)=x**2/4+y**2+(4*z**2)/(1+sin(pi*x)/2)**2 - 1
f(x,y,z)=-LaplaceBeltrami(u,Phi)
```

In this case, the function  $f = -\Delta_\Gamma u$  obtained is too long to be written.

## 3.2 Numerical samples on closed hypersurfaces

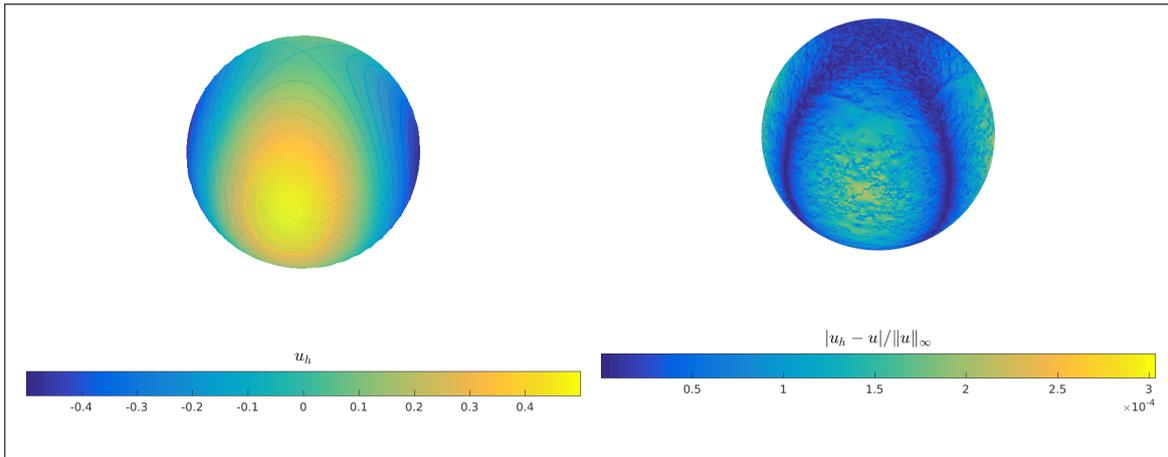
### 3.2.1 $-\Delta_\Gamma u + a_0 u = f$ with exact solution on $S^2$

We want to solve the surface PDE on a closed hypersurface  $\Gamma$  :

$$-\Delta_\Gamma u + a_0 u = f, \quad \text{on } \Gamma \quad (3.3)$$

where  $f \in L^2(\Gamma)$  and  $a_0 \in L^\infty(\Gamma)$  are given with  $a_0 > 0$ . With this last assumption we have existence and unicity of a  $u \in H^2(\Gamma)$ .

To solve and compare to exact solution, we choose  $u(\mathbf{x}) = x_1 x_2$ ,  $a_0(\mathbf{x}) = 1 + x_1^2$  and we calculate the right-hand side  $f$  as  $f = -\Delta_\Gamma u + a_0 u$  using sage with  $\Gamma = S^2$ . The complete code using the FC-VFEM $\mathbb{P}_1$  Matlab toolbox is given in Listing 3.4. In figure 3.1



```
1 fprintf('***_Building_reading_mesh\n')
2 N=50;geofile='sphere4surf';
3 meshfile=gmesh.buildmesh3ds(geofile,N,'geodir','geodir','meshdir','geodir');
4 Th=siMesh(meshfile,'dim',3);
5 Th.info()
6 fprintf('***_Setting_surface_BVP\n')
7 u=@(x,y,z) x.*y;
8 LPu=@(x,y,z) 6.*x.*y./(x.^2 + y.^2 + z.^2);
9 a0=@(x,y,z) 1+x.^2;
10 f=@(x,y,z) LPu(x,y,z)+a0(x,y,z).*u(x,y,z);
11 Lop=Loperator(3,2,{1,[],[];1,[],[];1,[],[]},[],[],a0);
12 pde=PDEelt(Lop,f);
13 bvp=BVP(Th,pde);
14 fprintf('***_Solving_surface_BVP\n')
15 U=bvp.solve();
```

Listing 3.4: Solving  $-\Delta_\Gamma u + a_0 u = f$  on unit sphere with  $a_0(x, y, z) = 1 + x^2$ , solution (left) and relative error (right)

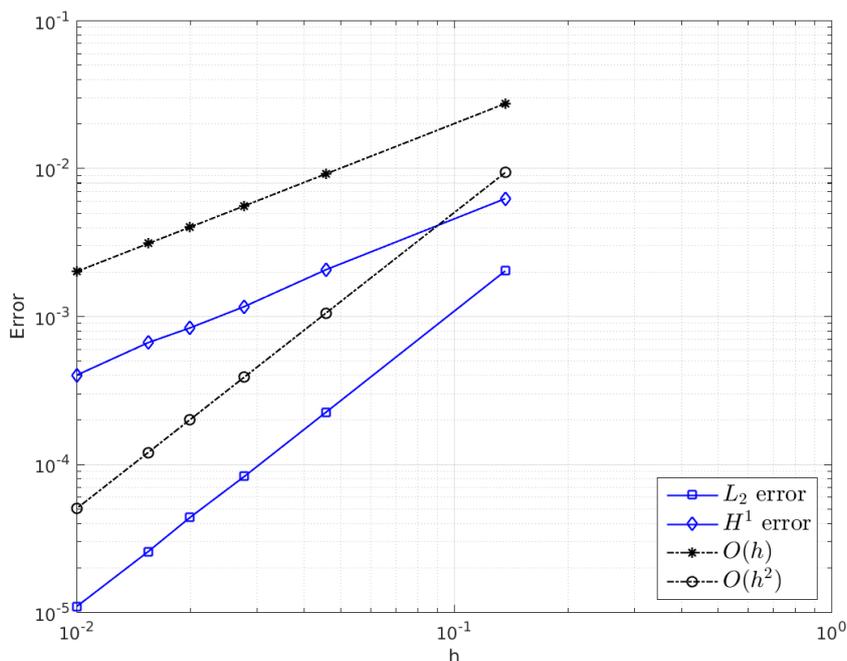


Figure 3.1: Order for surface BVP  $-\Delta_{\Gamma}u + a_0u = f$  on unit sphere with  $a_0(x, y, z) = 1 + x^2$  and exact solution  $u(x, y, z) = xy$ .

**3.2.2**  $-\Delta_{\Gamma}u + a_0u = f$  on  $F(S^2)$  with exact solution

To solve and compare to exact solution, we choose  $u(\mathbf{x}) = x_1x_2$ ,  $a_0(\mathbf{x}) = 1 + x_1^2$  and we calculate with Sage the right-hand side  $f$  as  $f = -\Delta_{\Gamma}u + a_0u$  where  $\Gamma = F(S^2)$  and  $F$  given by (3.1).

The complete code using the FC-VFEM $\mathbb{P}_1$  Matlab toolbox is given in Listing 3.5. In figure ?? the orders for the  $L^2$  and  $H^1$  norms are represented.

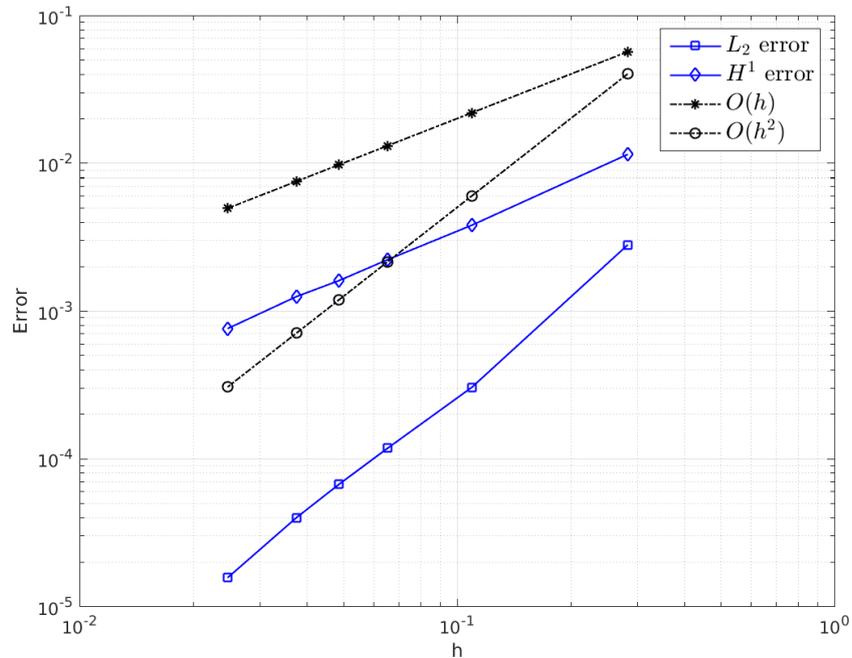
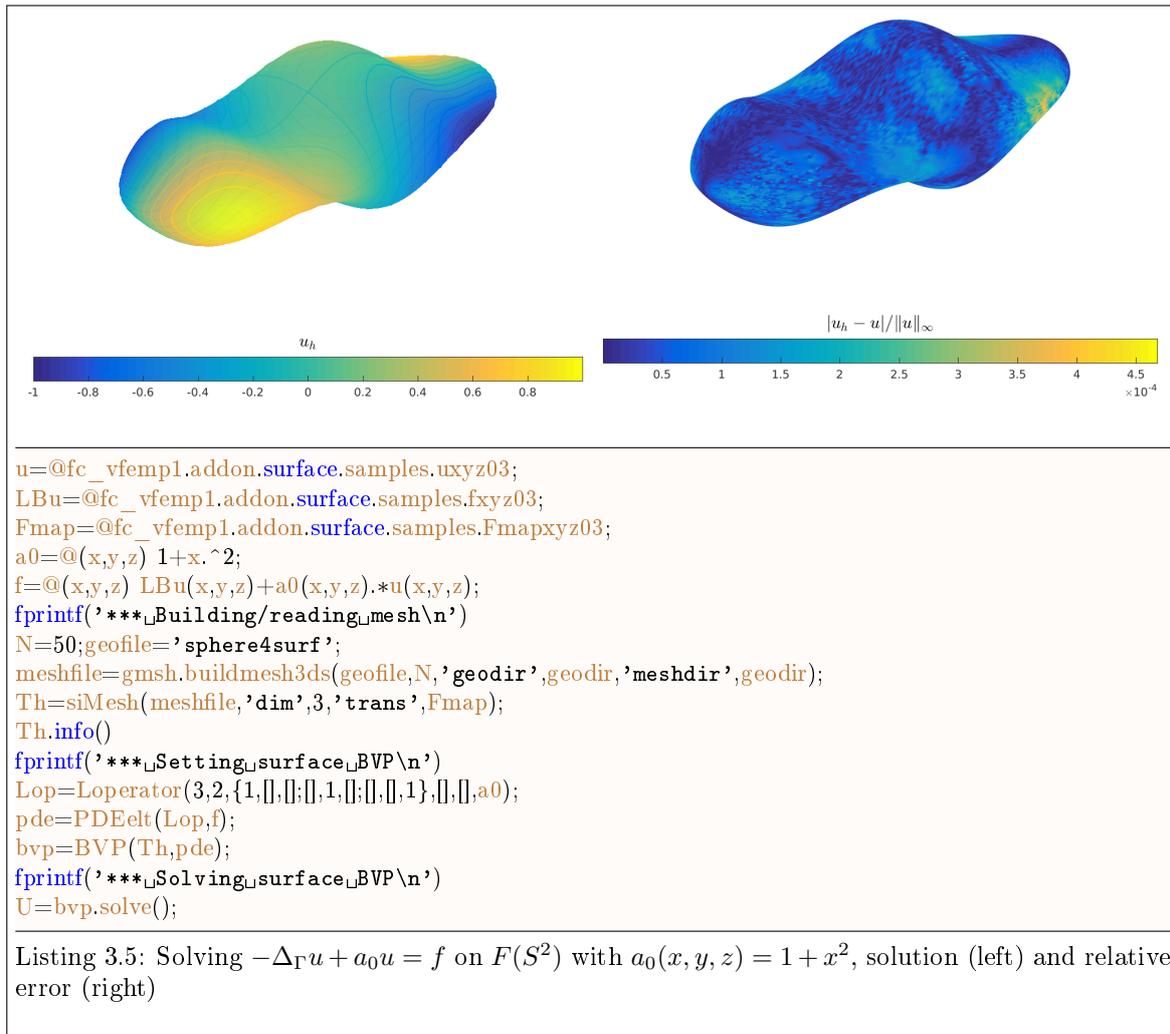


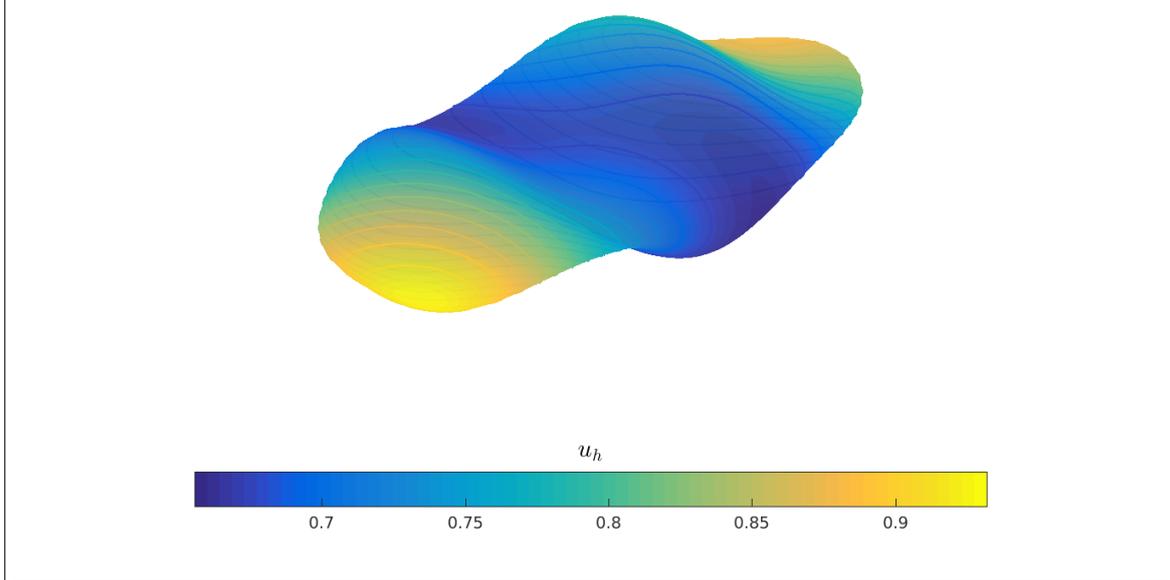
Figure 3.2: Order for surface BVP  $-\Delta_{\Gamma}u + a_0u = f$  on unit sphere mapping by function (3.1) :  $u(x, y, z) = C + xy$ .

3.2.3  $-\Delta_{\Gamma}u + a_0u = f$  on  $F(S^2)$  without explicit solution

We want to solve the surface PDE

$$-\Delta_{\Gamma}u + a_0u = f, \quad \text{on } \Gamma = F(S^2) \quad (3.4)$$

with  $a_0(x, y, z) = 1 + 0.9 \cos(x + y + z)$ ,  $f = 1$



```

1 N=30;
2 f=1;
3 a0=@(x,y,z) 1+0.9*cos(x+y+z);
4 Fmap=@(x,y,z) [ 2.*x; y;(1/2)*z.*(1/2)*sin(2.*pi.*x) + 1];
5 meshfile=gmsl.buildmesh3ds('sphere4surf.geo',N,'geodir','geodir','meshdir','geodir');
6 Th=siMesh(meshfile,'dim',3,'trans',Fmap);
7 Lop=Loperator(3,2,{1,[],[],1,[],[],1},[],[],a0);
8 pde=PDEelt(Lop,f);
9 bvp=BVP(Th,pde);
10 u=bvp.solve();
    
```

Listing 3.6: Solution of  $-\Delta_{\Gamma}u + a_0u = f$  PDE on unit sphere mapping by function (3.1) with  $a_0(x, y, z) = 1 + 0.9 \cos(x + y + z)$  and  $f = 1$ . Matlab code

3.2.4  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$  on  $F(S^2)$  with exact solution

We want to solve the surface PDE on a closed hypersurface  $\Gamma$  :

$$-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f, \quad \text{on } \Gamma \quad (3.5)$$

where  $f \in L^2(\Gamma)$ ,  $\mathbf{c} \in (L^\infty(\Gamma))^3$  and  $a_0 \in L^\infty(\Gamma)$  are given with  $a_0 > 0$ .

For this example we choose  $\Gamma$  as the surface obtained by mapping the unit sphere by the function defined in (3.1)

To solve and compare to exact solution, we choose

$$u(\mathbf{x}) = x_1x_2, \quad a_0(\mathbf{x}) = 1 + x_1^2 \quad \text{and} \quad \mathbf{c}(\mathbf{x}) = (\cos(x_1), \sin(x_2), 2 + x_1x_2x_3)^t$$

With Sage, we compute  $-\Delta_{\Gamma}u$  and  $\nabla_{\Gamma}u$ . The results are saved in FC-VFEM $\mathbb{P}_1$  toolbox respectively in

```

+fc_vfemp1/+addon/+surface/+samples/fxyz04.m
+fc_vfemp1/+addon/+surface/+samples/gradSuxyz04.m
    
```

as well as corresponding  $u$  and map function (3.1) respectively in

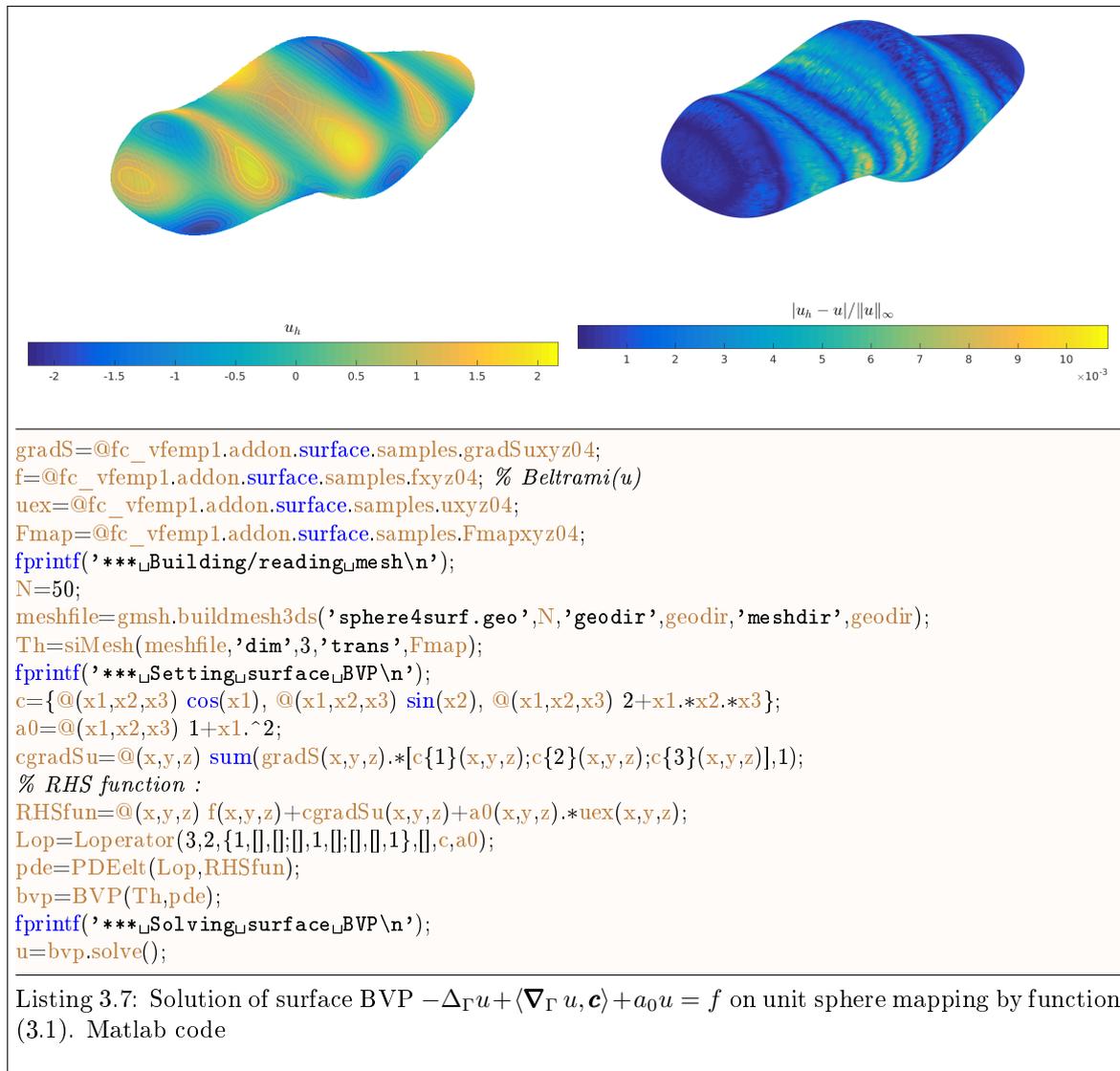
```

+fc_vfemp1/+addon/+surface/+samples/uxyz04.m
+fc_vfemp1/+addon/+surface/+samples/Fmapxyz04.m
    
```

We give in Listing 3.7 the complete Matlab code to solve 3.5 with

$$a_0 : \mathbf{x} \mapsto 1 + x_1^2 \quad \text{and} \quad \mathbf{c} : \mathbf{x} \mapsto \begin{pmatrix} \cos(x_1) \\ \sin(x_2) \\ 2 + x_1 x_2 x_3 \end{pmatrix}.$$

The order of the  $P_1$ -lagrange finite element method used in the FC-VFEM $P_1$  toolbox is given in Figure 3.3.



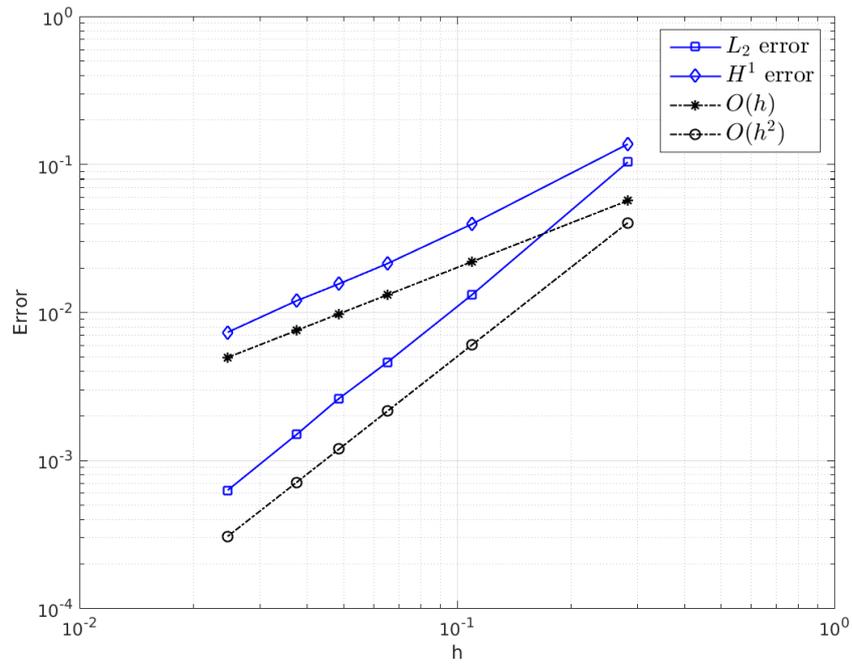
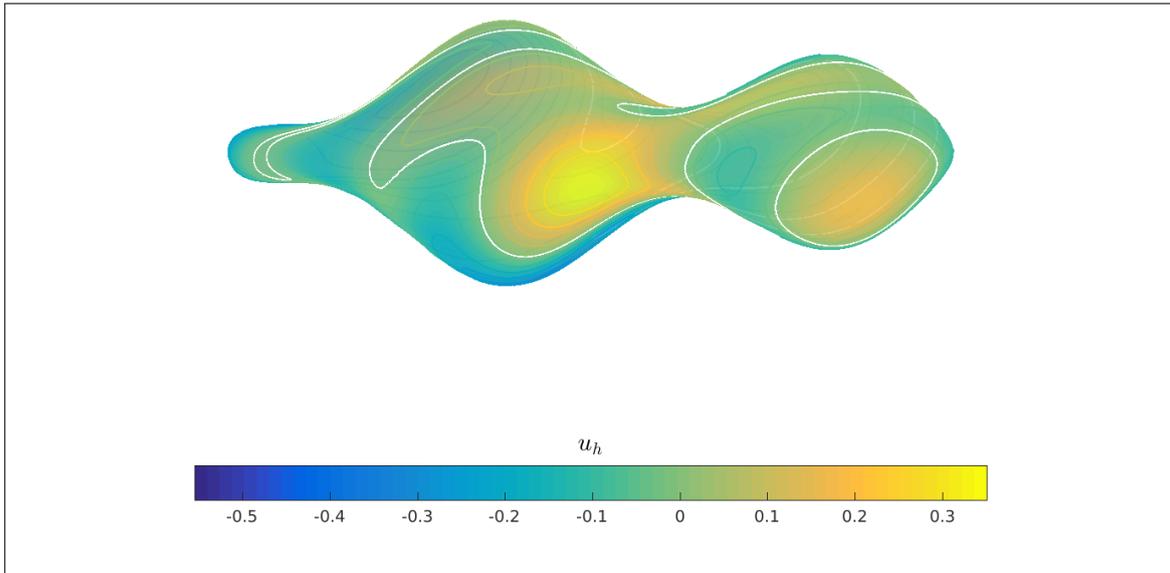


Figure 3.3: Order for the surface BVP  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$  on unit sphere mapping by function (3.1).

**3.2.5**  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$  on  $F(S^2)$  without exact solution

From previous sample, just differs the  $f$  function given by

$$f(\mathbf{x}) = 5 \cos(5x_1 - 5x_2 + 7x_3)$$



```

1 N=50;
2 Fmap=@(x,y,z) [ 2.*x; y;(1/2)*z.*((1/2)*sin(2.*pi.*x) + 1)];
3 c=@(x1,x2,x3) cos(x1), @(x1,x2,x3) sin(x2), @(x1,x2,x3) 2+x1.*x2.*x3};
4 a0=@(x1,x2,x3) 1+x1.^2;
5 f=@(x1,x2,x3) 5*cos(5*x15*x2+7*x3);
6 meshfile=gmsl.buildmesh3ds('sphere4surf.geo',N,'geodir','geodir','meshdir','geodir');
7 Th=siMesh(meshfile,'dim',3,'trans',Fmap);
8 Lop=Loperator(3,2,{1,[],[];[],1,[],[];[],[],1},[],c,a0);
9 pde=PDEelt(Lop,f);
10 bvp=BVP(Th,pde);
11 u=bvp.solve();
    
```

Listing 3.8: Solution of surface PDE  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma} u, \mathbf{c} \rangle + a_0u = f$  on unit sphere mapping by function (3.1) without exact solution. Matlab code

### 3.2.6 3D surface Laplace-Beltrami BVP on closed hypersurface

On closed and regular hypersurface there is no boundary. The surface Laplace-Beltrami BVP is the following

#### Scalar surface BVP 2 : Laplace-Beltrami

Let  $\Gamma$  be a regular and closed hypersurface. Find  $u \in H^1(\Gamma)$  such that

$$-\Delta_{\Gamma}u = f \text{ in } \Gamma \tag{3.6}$$

$$\int_{\Gamma} u dq = 0. \tag{3.7}$$

From [2], (Theorem 3.3, page 302 and Theorem 4.9, page 319) we obtain

#### Theorem 3.1

Suppose that  $f \in L^2(\Gamma)$  with  $\int_{\Gamma} f dq = 0$ . There exists a unique solution  $u \in H^2(\Gamma)$  of the Laplace-Beltrami surface BVP (3.6)-(3.7).

Let  $u_h$  be the  $P_1$ -Lagrange FEM solution of the discrete Laplace-Beltrami surface BVP then

$$\|u - u_h\|_{H^1(\Gamma_h)} \leq C_1 h \text{ and } \|u - u_h\|_{L^2(\Gamma_h)} \leq C_0 h^2. \tag{3.8}$$

To have uniqueness of the solution we fix the free constant with (3.7). This can be done by replacing

one line of the linear system  $\mathbb{A}\mathbf{u}_h = \mathbf{b}$  obtain from 3.6 by

$$\sum_{i=1}^n (\mathbb{M}\mathbf{u}_h)_i = 0 \tag{3.9}$$

This can be done using FC-VFEM $\mathbb{P}_1$  by

```

1 ...
2 % Set surface BVP problem
3 Lop=Loperator(3,2,{1,[],[];[],1,[],[];[],[],1},[],[],0);
4 pde=PDEelt(Lop,f);
5 bvp=BVP(Th,pde);
6 % Assembly of the linear system
7 [A,b]=bvp.Assembly('local',true);
8 % Compute Surface Mass matrix : M
9 bvpMass=BVP(Th,PDEelt(Loperator(3,2,[],[],1))),
10 [M,~]=bvpMass.Assembly('local',true);
11 % Change last line
12 A(end,:)=sum(M,1);
13 b(end)=0;
14 % Solve linear system
15 u=A\b;
    
```

The constant  $C$  is computed so that  $\int_{\Gamma} u dA = 0$ .  
 By construction, we necessarily have  $\int_{\Gamma} f dA = 0$ . Indeed for all  $\varphi \in H^1(\Gamma)$  we have

$$-\int_{\Gamma} \Delta_{\Gamma} u \varphi dA = \int_{\Gamma} f \varphi dA$$

By using Green formula, we obtain

$$\int_{\Gamma} \langle \nabla_{\Gamma} u, \nabla_{\Gamma} \varphi \rangle dA = \int_{\Gamma} f \varphi dA$$

Taking  $\varphi \equiv 1$  gives  $\int_{\Gamma} f dA = 0$ .

**On the unit sphere  $\Gamma = S^2$ , exact solution  $u_{\text{ex}}(x, y, z) = C + xy$**

With Sage, one can compute  $f = -\Delta_{\Gamma} u : (x, y, z) \mapsto \frac{6xy}{x^2+y^2+z^2}$  where  $u : (x, y, z) \mapsto xy$ . Furthermore, we have  $\int_{S^2} u dq = 0$  and so the constant  $C$  is equal to zero.

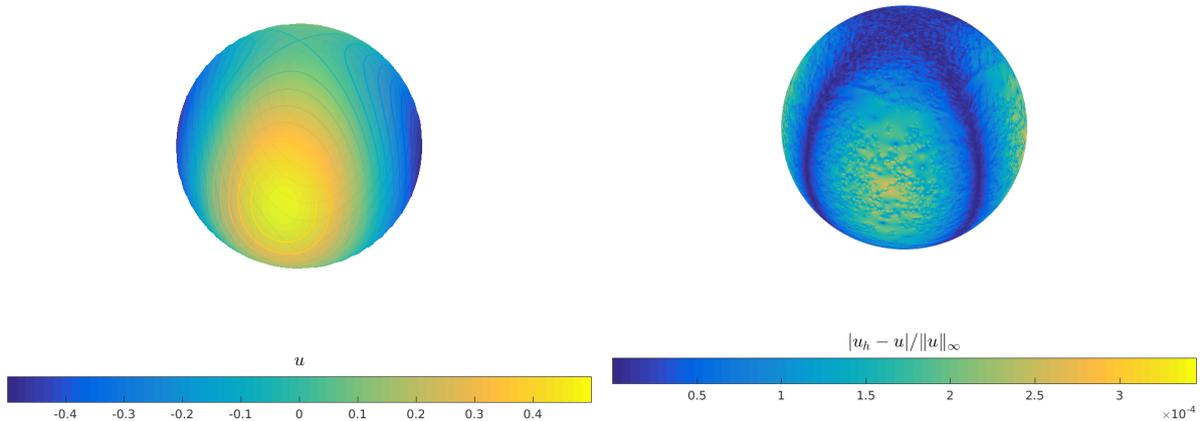


Figure 3.4: Numerical solution for 3D surface Laplace-Beltrami BVP on unit sphere (left) and error where exact solution is  $u(x, y, z) = xy$  (right)

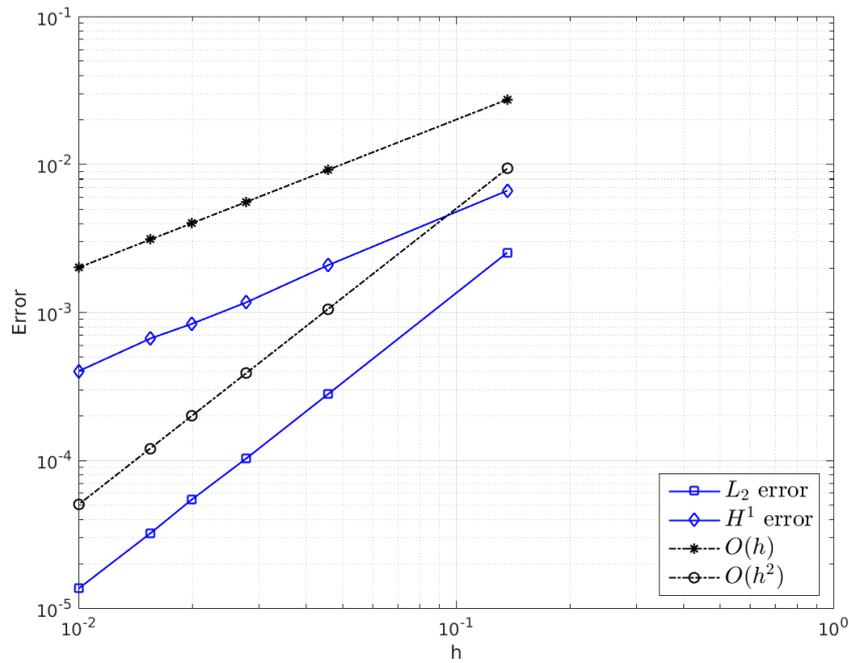


Figure 3.5: Order for 3D surface Laplace-Beltrami BVP on unit sphere :  $u(x, y, z) = xy$ .

**On the unit sphere  $\Gamma = S^2$ , exact solution  $u_{\text{ex}}(x, y, z) = C + x^2y^2$**

With  $u : (x, y, z) \mapsto x^2y^2$  we obtain from Sage

$$f = -\Delta_{\Gamma} u : (x, y, z) \mapsto -\frac{2(x^4 - 8x^2y^2 + y^4 + (x^2 + y^2)z^2)}{x^2 + y^2 + z^2}$$

Furthermore, we have  $\int_{S^2} u_{\text{ex}} d\mathbf{q} \neq 0$  and so the constant  $C$  is not equal to zero.

To compute the constant  $C$  we use (??) and

$$\int_{S^2} u_{\text{ex}} d\mathbf{q} = \int_{S^2} C d\mathbf{q} + \int_{S^2} u d\mathbf{q} = 4\pi C + \frac{4}{15} \pi = 0.$$

So we obtain

$$C = -\frac{1}{15}.$$

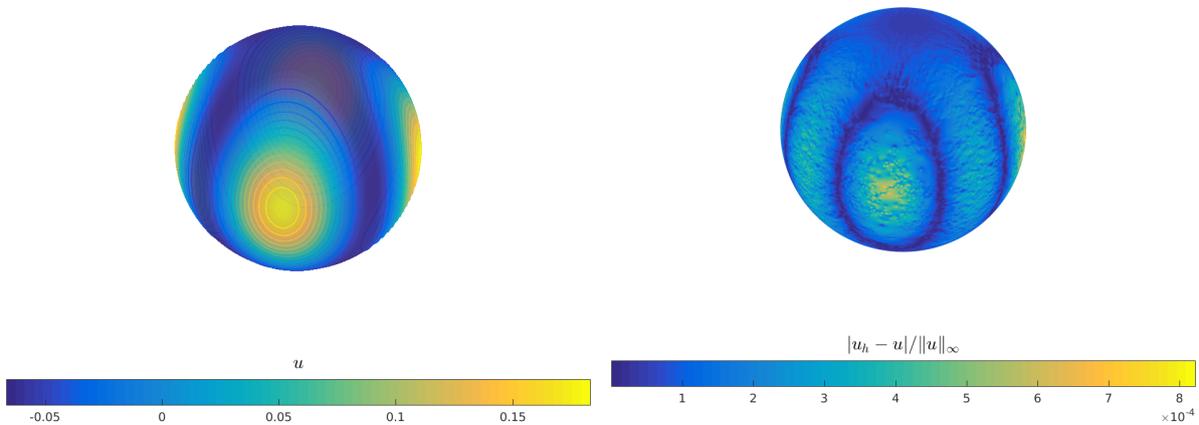


Figure 3.6: Numerical solution for 3D surface Laplace-Beltrami BVP on unit sphere (left) and error where exact solution is  $u(x, y, z) = x^2y^2 + C$  (right)

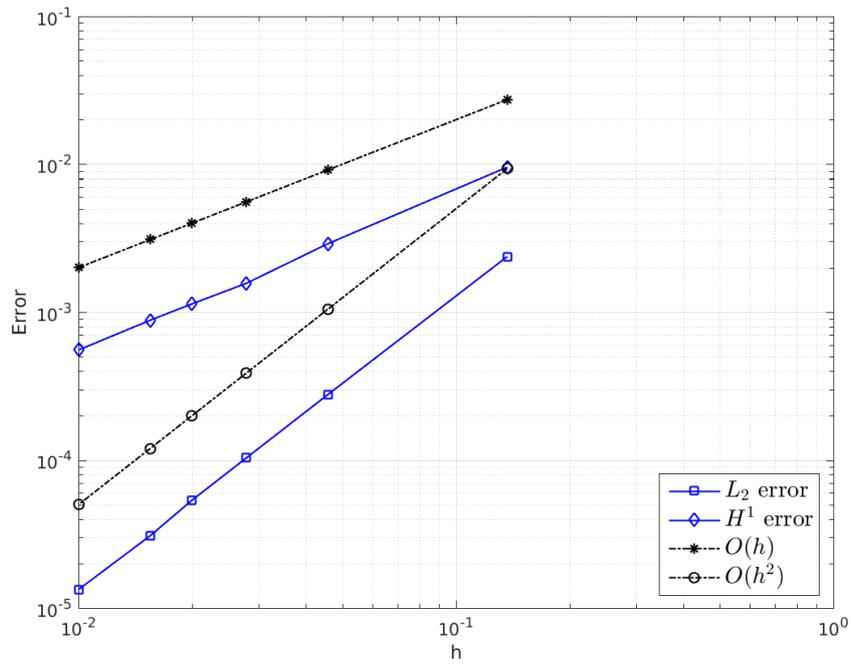


Figure 3.7: Order for 3D surface Laplace-Beltrami BVP on unit sphere :  $u(x, y, z) = x^2y^2 + C$ .

On the unit sphere, exact solution  $u(x, y, z) = C + \cos(2\pi xy) \sin(2\pi z)$

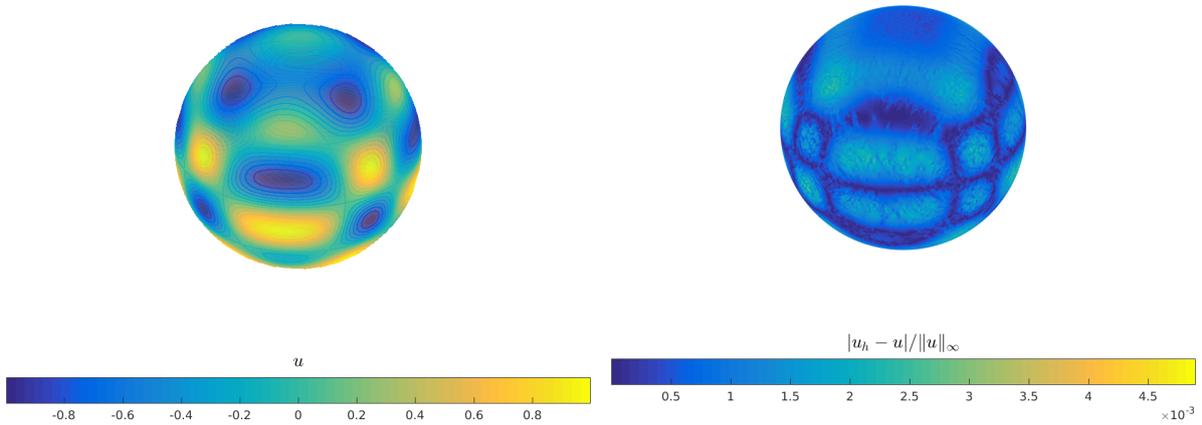


Figure 3.8: Numerical solution for 3D surface Laplace-Beltrami BVP on unit sphere (left) and error where exact solution is  $u(x, y, z) = C + \cos(2\pi xy) \sin(2\pi z)$  (right)

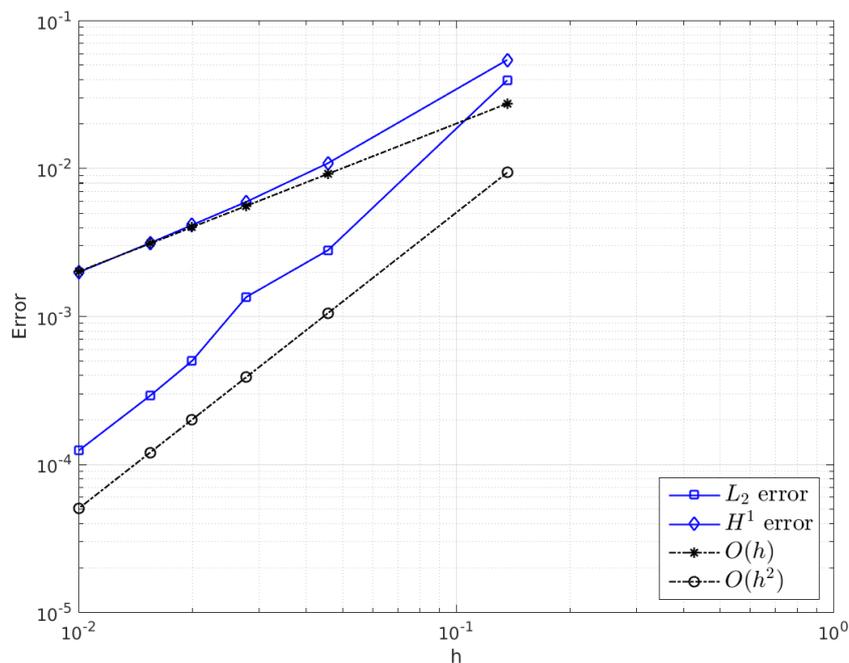


Figure 3.9: Order for 3D surface Laplace-Beltrami BVP on unit sphere :  $u(x, y, z) = C + \cos(2\pi xy) \sin(2\pi z)$ .

**On the unit sphere mapping by a function  $F$ , exact solution  $u(x, y, z) = C + xy$**

The mapping function is given in (3.1).

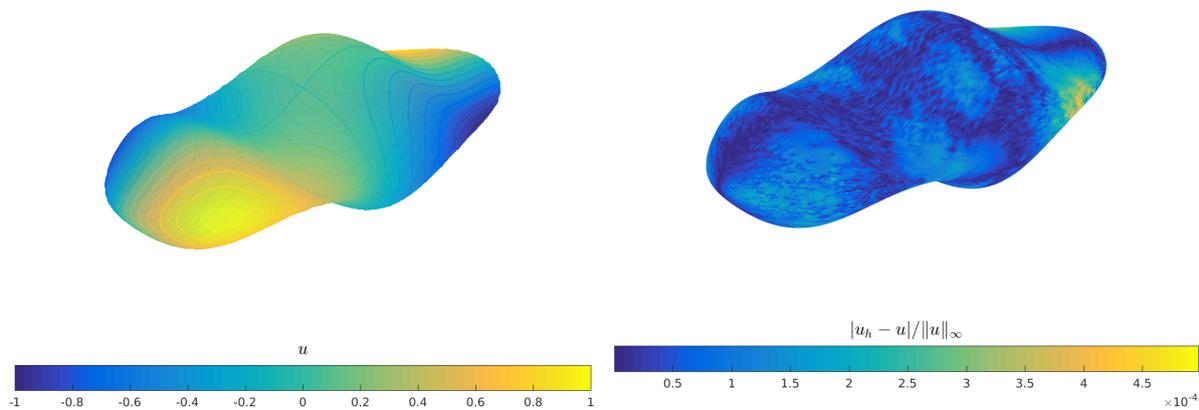


Figure 3.10: Numerical solution for 3D surface Laplace-Beltrami BVP on unit sphere mapping by function (3.1) (left) and error where exact solution is  $u(x, y, z) = C + xy$  (right)

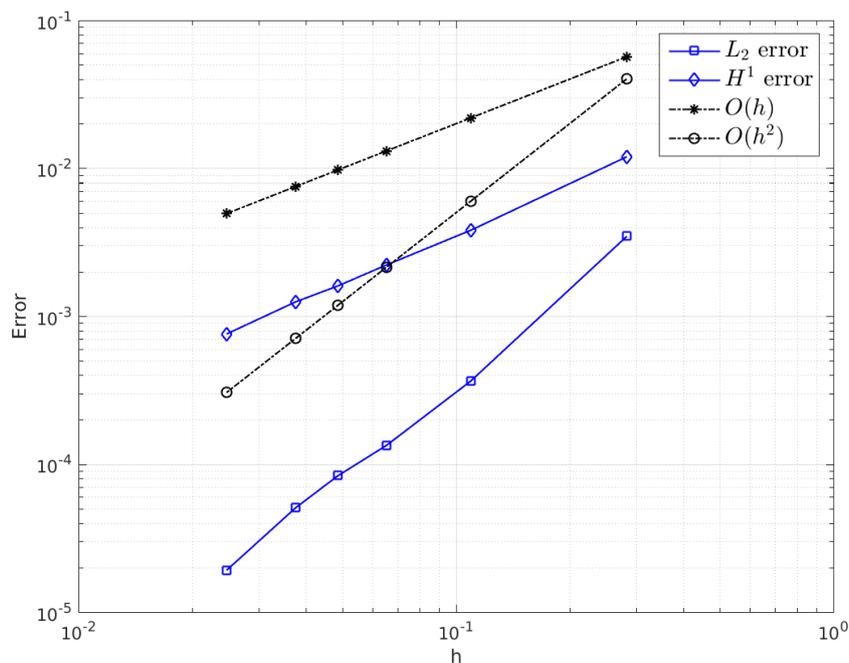


Figure 3.11: Order for 3D surface Laplace-Beltrami BVP on unit sphere mapping by function (3.1) :  $u(x, y, z) = C + xy$ .

On the unit sphere mapping by a function  $F$ , exact solution  $u(x, y, z) = C + \cos(2\pi x) + \sin(2\pi z) + xyz$

The mapping function is given in (3.1).

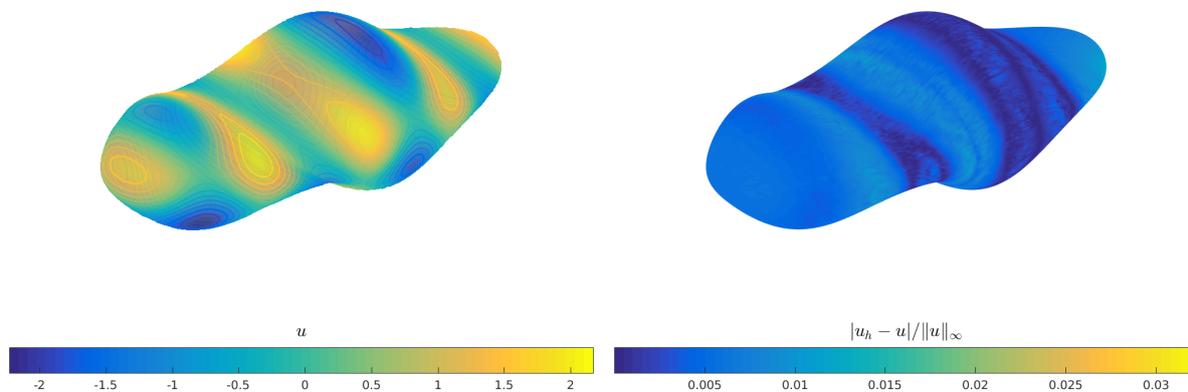


Figure 3.12: Numerical solution for 3D surface Laplace-Beltrami BVP on unit sphere mapping by function (3.1) (left) and error where exact solution is  $u(x, y, z) = C + \cos(2\pi x) + \sin(2\pi z) + xyz$  (right)

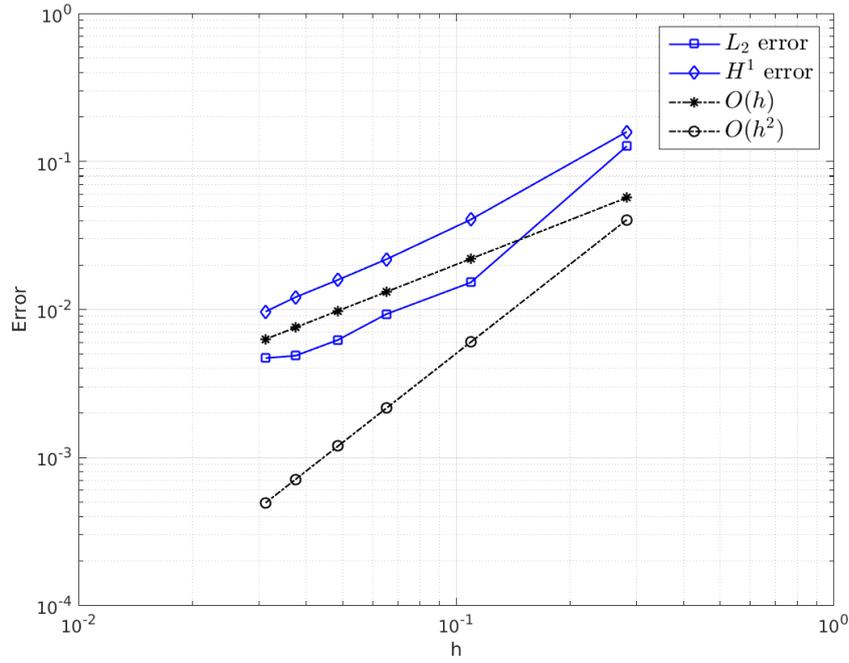


Figure 3.13: Order for 3D surface Laplace-Beltrami BVP on unit sphere mapping by function (3.1) :  $u(x, y, z) = C + \cos(2\pi x) + \sin(2\pi z) + xyz$ .

**On the unit sphere mapping by a function  $F$ , exact solution  $C + xy$**

The hypersurface was constructed by mapping a discretization of the unit sphere  $S^2$  onto the surface  $\Gamma$  by

$$F(\mathbf{y}) = \left( y_1, y_2, y_3 \sqrt{\frac{1}{20} + 2y_1^2 + \frac{1}{2}y_2^2} \right), \quad \mathbf{y} = (y_1, y_2, y_3) \in S^2. \tag{3.10}$$

The representation of  $\Gamma = F(S^2)$  as a hypersurface  $\{\mathbf{x} \in \mathbb{R}^3 \mid \phi(\mathbf{x}) = 0\}$  follows from  $y_1^2 + y_2^2 + y_3^2 = 1$  with the level set function

$$\phi(\mathbf{x}) = x_1^2 + x_2^2 + \frac{x_3^2}{\frac{1}{20} + 2y_1^2 + \frac{1}{2}y_2^2} - 1. \tag{3.11}$$

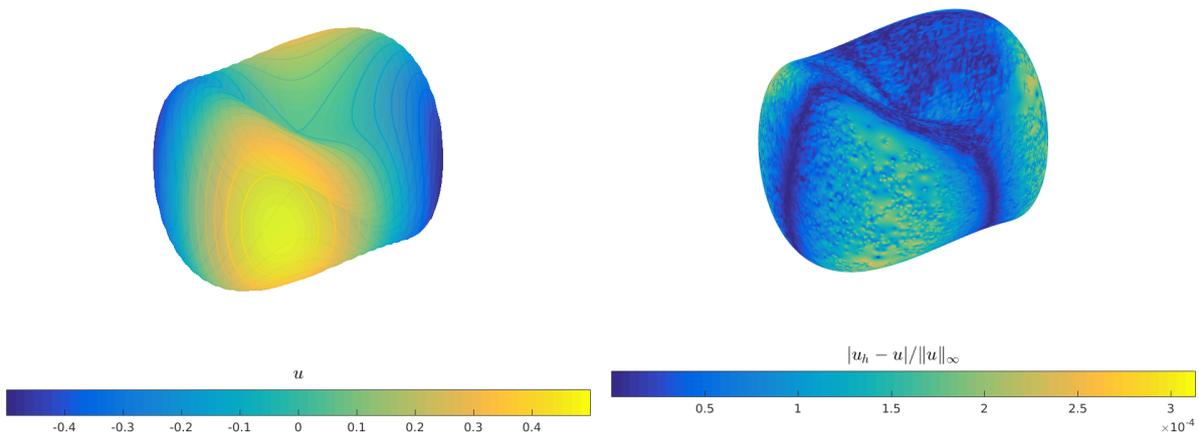


Figure 3.14: Numerical solution for 3D surface Laplace-Beltrami BVP on unit sphere mapping by function (3.10) (left) and error where exact solution is  $u(x, y, z) = C + xy$  (right)

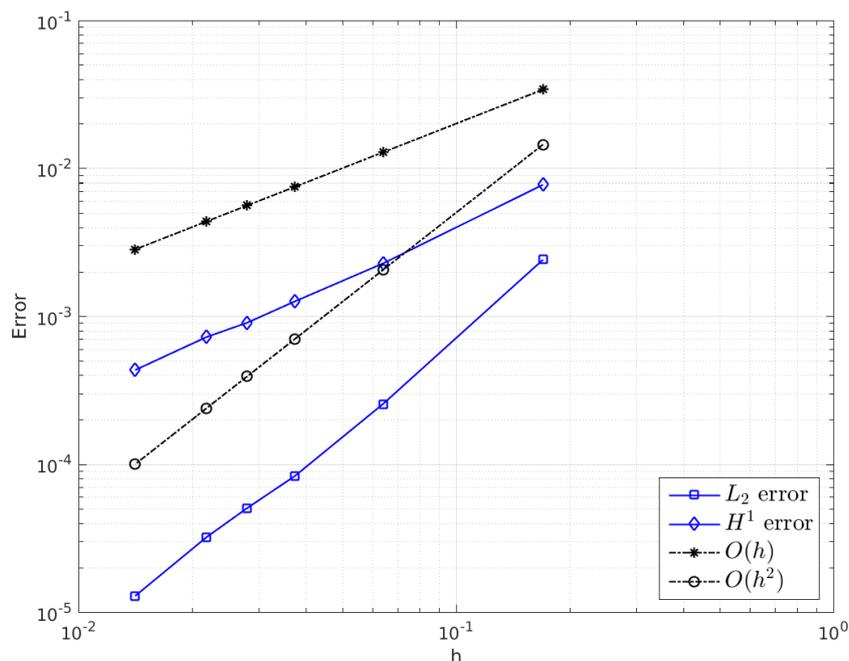


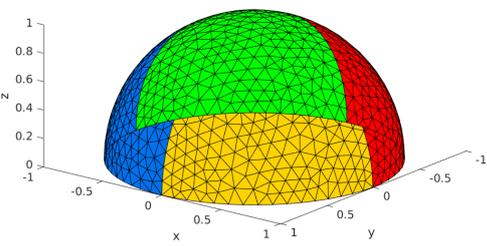
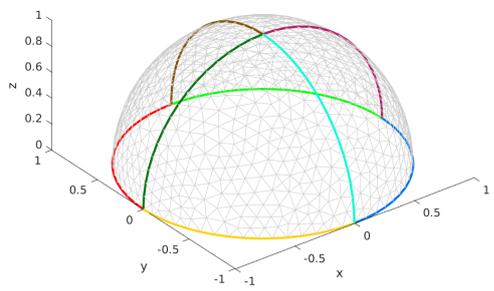
Figure 3.15: Order for 3D surface Laplace-Beltrami BVP on unit sphere mapping by function (3.10) :  $u(x, y, z) = C + xy$ .

### 3.3 Numerical samples on non-closed hypersurfaces

Let  $\Gamma$  be a non-closed hypersurfaces and  $\partial\Gamma$  be its none empty boundary. We want to solve the scalar BVP (1.2)-(1.4)

3.3.1  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$  with Dirichlet boundary conditions on an half-sphere

We represente in Listing 3.9, the unit demi-sphere  $z \geq 0$ . As we can remark, the *physical* boundaries are labeled from 1 to 4 : the label 5 to 8 only serves us for graphic representation purposes.

```

1 meshfile=gmesh.buildmesh3ds('halfsphere4surf.geo',10,'geodir',geodir,'meshdir',geodir);
2 Th=siMesh(meshfile,'dim',3);
    
```

Listing 3.9: Mesh of a demi-sphere with 4 subdomains,  $\Gamma = \Omega_1 \cup \dots \cup \Omega_4$  (left) and boundaries  $\partial\Gamma = \Gamma_1 \cup \dots \cup \Gamma_4$  (right)

We want to solve the

### Scalar surface BVP 3 : non-closed surface, sample 1

Find  $u \in H^1(\Gamma)$  such that

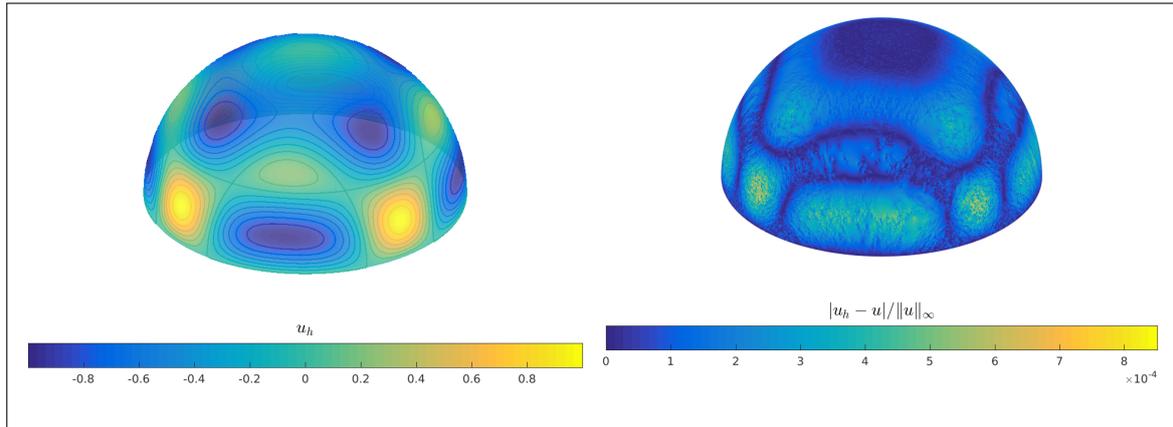
$$-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f, \text{ in } \Gamma \quad (3.12)$$

$$u = g, \text{ on } \partial\Gamma. \quad (3.13)$$

With exact solution  $u(x, y, z) = \cos(2\pi xy) \sin(2\pi z)$ , we compute  $-\Delta_{\Gamma}u$  and  $\nabla_{\Gamma}u$  with Sage and store the results as Matlab functions in FC-VFEM $\mathbb{P}_1$  toolbox respectively in file `+fc_vfemp1/+addon/+surface/+samples/fxyz02` and `+fc_vfemp1/+addon/+surface/+samples/gradSuxyz02.m`.

We give in Listing 3.7 the complet Matlab code to solve 3.5 with

$$a_0 : \mathbf{x} \mapsto 1 + x_1^2 \text{ and } \mathbf{c} : \mathbf{x} \mapsto \begin{pmatrix} \cos(x_1) \\ \sin(x_2) \\ 2 + x_1x_2x_3 \end{pmatrix}.$$



```

1 N=100;
2 gradSu=@fc_vfemp1.addon.surface.samples.gradSuxyz02;
3 mBu=@fc_vfemp1.addon.surface.samples.fxyz02;
4 uex=@fc_vfemp1.addon.surface.samples.uxyz02;
5 meshfile=gsmh.buildmesh3ds('halfsphere4surf.geo',N,'geodir','geodir','meshdir','geodir');
6 Th=siMesh(meshfile,'dim',3);
7 c=@(x1,x2,x3) cos(x1), @(x1,x2,x3) sin(x2), @(x1,x2,x3) 2+x1.*x2.*x3;
8 a0=@(x1,x2,x3) 1+x1.^2;
9 cgradSu=@(x,y,z) sum(gradSu(x,y,z).*[c{1}(x,y,z);c{2}(x,y,z);c{3}(x,y,z)],1);
10 % RHS function :
11 RHSfun=@(x,y,z) mBu(x,y,z)+cgradSu(x,y,z)+a0(x,y,z).*uex(x,y,z);
12 Lop=Loperator(3,2,{1,[],[];[],1,[];[],[],1},[],c,a0);
13 pde=PDEelt(Lop,RHSfun);
14 bvp=BVP(Th,pde);
15 for lab=1:4, bvp.setDirichlet(lab,uex);end
16 u=bvp.solve();

```

Listing 3.10: Solution of  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$  with Dirichlet boundary condition

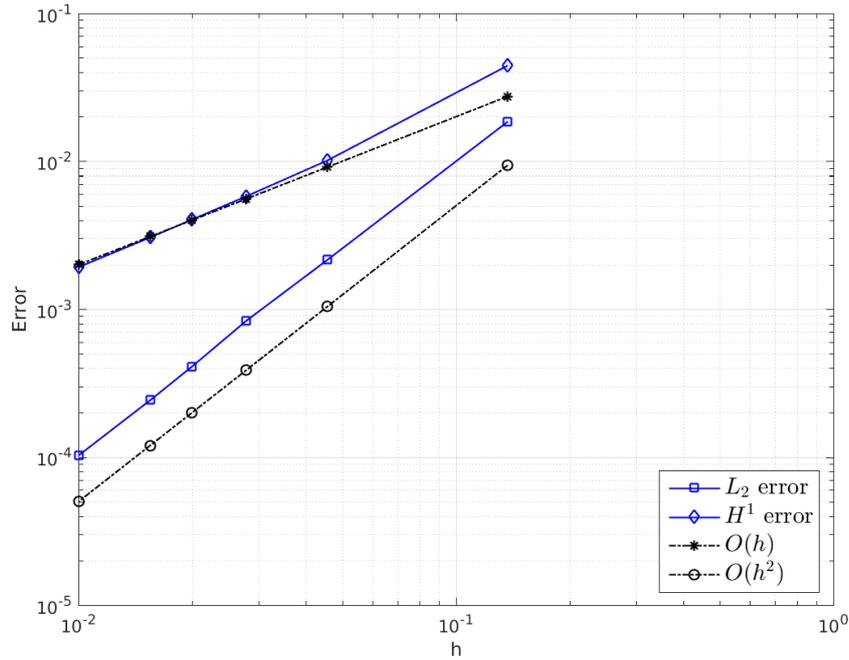


Figure 3.16: Order for 3Ds BVP  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$  with Dirichlet boundary condition on unit half-sphere

### 3.3.2 $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$ with mixed boundary conditions on an half-sphere

We want to solve the

#### **Scalar surface BVP 4 : non-closed surface, sample 2**

Find  $u \in H^1(\Gamma)$  such that

$$-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f, \text{ in } \Gamma \quad (3.14)$$

$$u = g_D, \text{ on } \partial\Gamma_D. \quad (3.15)$$

$$\langle \nabla_{\Gamma}g, \boldsymbol{\mu} \rangle = g_N, \text{ on } \partial\Gamma_N. \quad (3.16)$$

$$\langle \nabla_{\Gamma}g, \boldsymbol{\mu} \rangle + \alpha u = g_R, \text{ on } \partial\Gamma_R. \quad (3.17)$$

On the half-sphere ( $z \geq 0$ ), we have  $\boldsymbol{\mu} = (0, 0, -1)^{\top}$  on  $\partial\Gamma$ .

With exact solution  $u(x, y, z) = \cos(2\pi xy) \sin(2\pi z)$ , we compute  $-\Delta_{\Gamma}u$  and  $\nabla_{\Gamma}u$  with Sage and store the results as a Matlab functions in FC-VFEMP<sub>1</sub> toolbox respectively in files `+fc_vfemp1/+addon/+surface/+samples/fxy` and `+fc_vfemp1/+addon/+surface/+samples/gradSuxyz02.m`.

We also have to set  $g_N$  and  $g_R$  functions. We obtain them with the following Matlab code

```

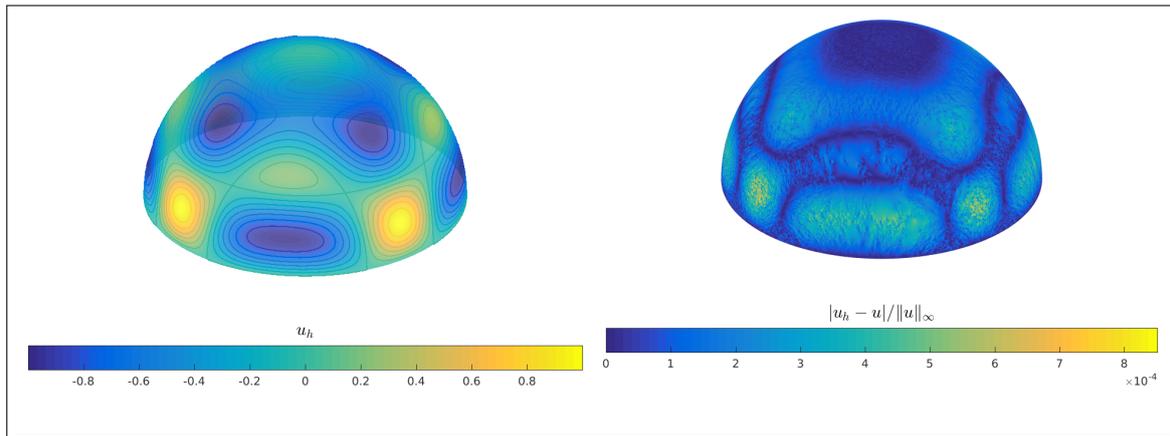
1 u=@fc_vfemp1.addon.surface.samples.uxyz02;
2 gradSu=@fc_vfemp1.addon.surface.samples.gradSuxyz02;
3 subindex = @(A,r) A(r,:);
4 gN=@(x,y,z) subindex(gradSu(x,y,z),3);
5 gR=@(x,y,z) subindex(gradSu(x,y,z),3)+alpha(x,y,z).*u(x,y,z);
    
```

We give in Listing 3.7 the complet Matlab code to solve (3.14)-(3.17) with

$$\partial\Gamma_D = \Gamma_1 \cup \Gamma_3, \quad \partial\Gamma_N = \Gamma_2, \quad \partial\Gamma_R = \Gamma_4$$

and

$$a_0 : \mathbf{x} \mapsto 1 + x_1^2, \quad \mathbf{c} : \mathbf{x} \mapsto \begin{pmatrix} \cos(x_1) \\ \sin(x_2) \\ 2 + x_1x_2x_3 \end{pmatrix} \quad \text{and} \quad \alpha : \mathbf{x} \mapsto 1 + x_3^2$$



```

1 N=100;
2 gradSu=@fc_vfemp1.addon.surface.samples.gradSuxyz02;
3 mBu=@fc_vfemp1.addon.surface.samples.fxyz02;
4 uex=@fc_vfemp1.addon.surface.samples.uxyz02;
5 meshfile=gmesh.buildmesh3ds('halfsphere4surf.geo',N,'geodir','geodir','meshdir','geodir');
6 Th=siMesh(meshfile,'dim',3);
7 c=@(x1,x2,x3) cos(x1), @(x1,x2,x3) sin(x2), @(x1,x2,x3) 2+x1.*x2.*x3};
8 a0=@(x1,x2,x3) 1+x1.^2;
9 cgradSu=@(x,y,z) sum(gradSu(x,y,z).*[c{1}(x,y,z);c{2}(x,y,z);c{3}(x,y,z)],1);
10 alpha=@(x,y,z) 1+z.^2;
11 % RHS function :
12 RHSfun=@(x,y,z) mBu(x,y,z)+cgradSu(x,y,z)+a0(x,y,z).*uex(x,y,z);
13 Lop=Loperator(3,2,{1,[],[];[],1,[],[],1},[],c,a0);
14 pde=PDEelt(Lop,RHSfun);
15 bvp=BVP(Th,pde);
16 for lab=[1,3], bvp.setDirichlet(lab,uex);end
17 subindex = @(A,r) A(r,:);
18 gN=@(x,y,z) subindex(gradSu(x,y,z),3);
19 gR=@(x,y,z) subindex(gradSu(x,y,z),3)+alpha(x,y,z).*uex(x,y,z);
20 bvp.setRobin(2,gN,[]);
21 bvp.setRobin(4,gR,alpha);
22 u=bvp.solve();

```

Listing 3.11: Solution of  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma} u, \mathbf{c} \rangle + a_0 u = f$  with mixed boundary condition

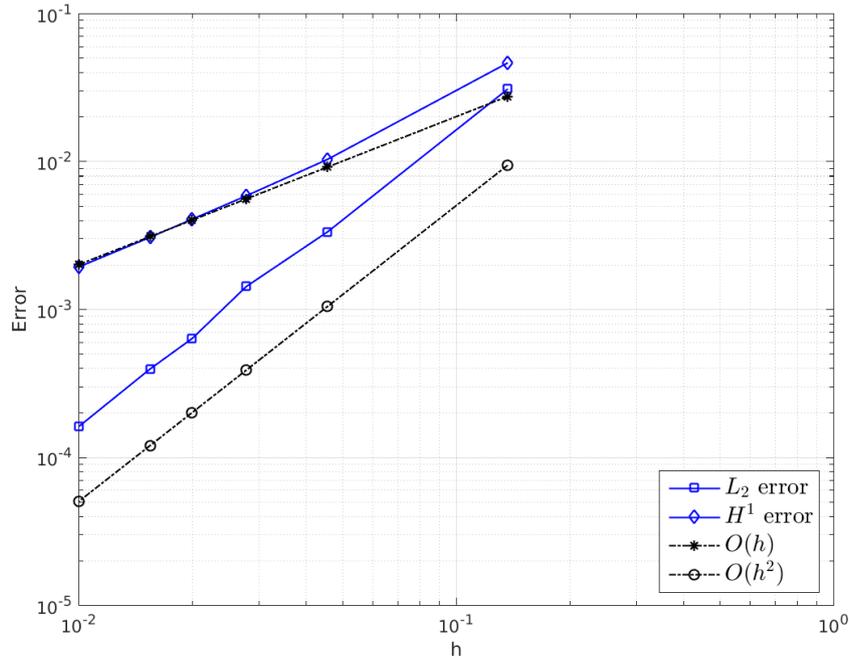


Figure 3.17: Order for 3Ds BVP  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma} u, \mathbf{c} \rangle + a_0u = f$  with mixed boundary condition on unit half-sphere

### 3.3.3 Mixed boundary conditions on an half-sphere mapping by a function

We represent in Listing 3.9, the unit demi-sphere  $z \geq 0$ . As we can remark, the *physical* boundaries are labeled from 1 to 4 : the label 5 to 8 only serves us for graphic representation purposes.

```

1 F=@(x,y,z) [2*x;y;0.5*z.*(1+0.5*sin(2*pi*x))];
2 meshfile=gmsb.buildmesh3ds('halfsphere4surf.geo',10,'geodir','geodir','meshdir','geodir');
3 Th=siMesh(meshfile,'dim',3,'trans',F);
    
```

Listing 3.12: Mesh of a demi-sphere with 4 subdomains,  $\Gamma = \Omega_1 \cup \dots \cup \Omega_4$  (left) and boundaries  $\partial\Gamma = \Gamma_1 \cup \dots \cup \Gamma_4$  (right)

On this mesh, we also have  $\boldsymbol{\mu} = (0, 0, -1)^t$  on  $\partial\Gamma$ .

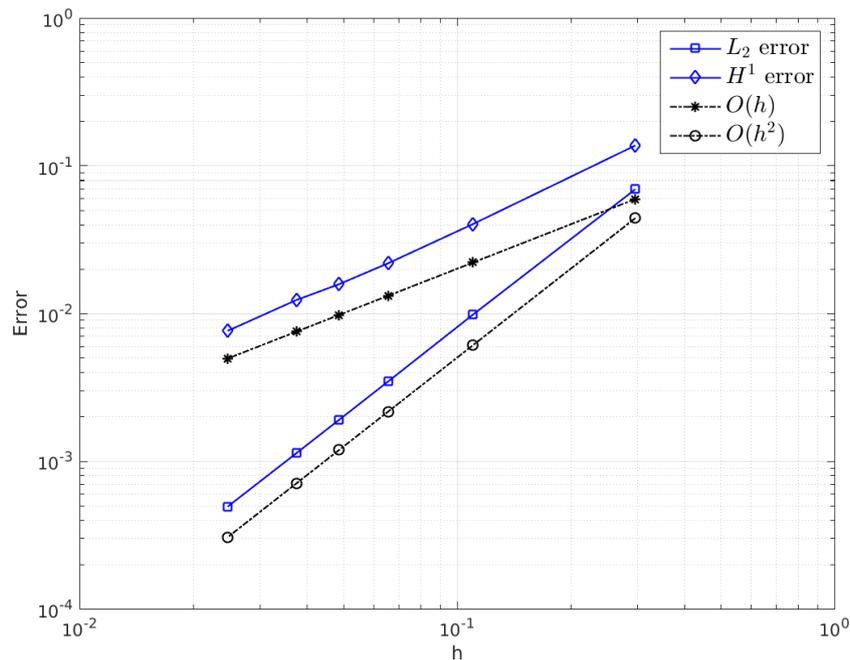
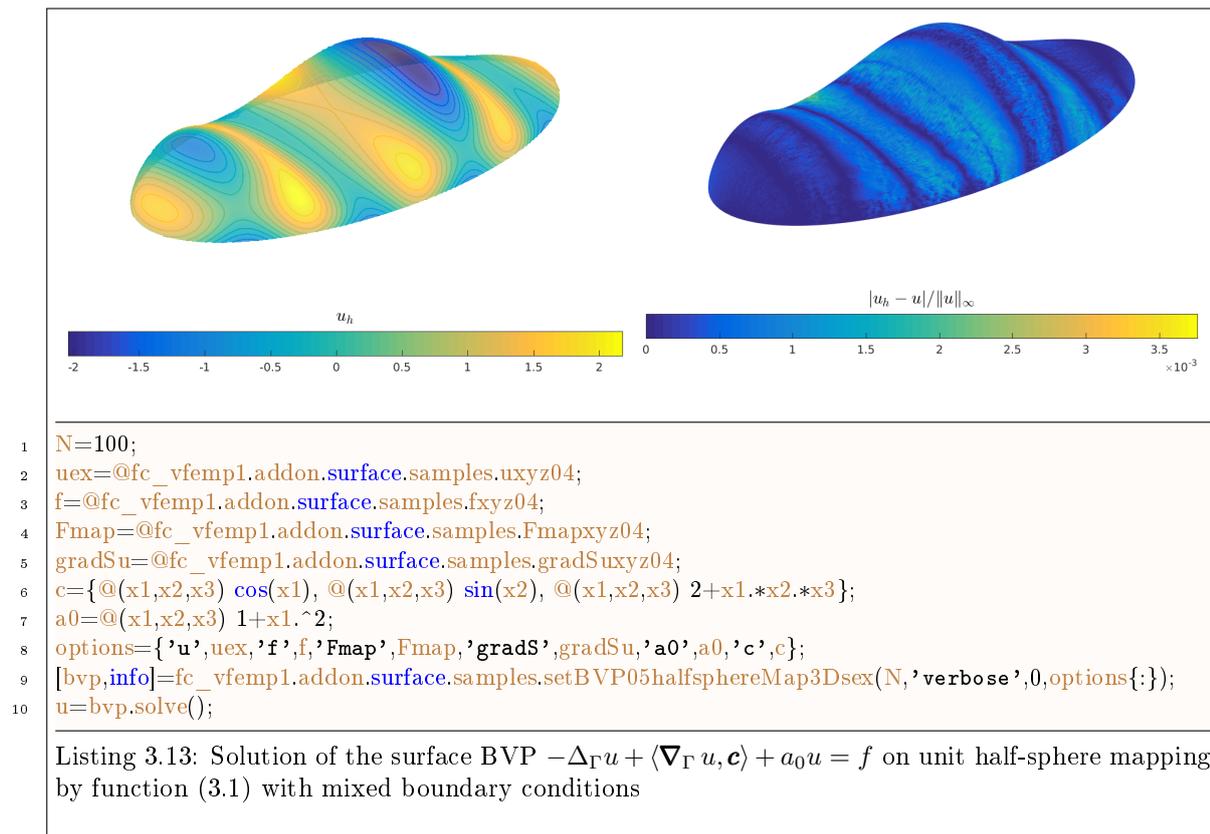


Figure 3.18: Order for the surface BVP  $-\Delta_{\Gamma}u + \langle \nabla_{\Gamma}u, \mathbf{c} \rangle + a_0u = f$  with mixed boundary condition on unit half-sphere mapping by function (3.1).

## Bibliography

- [1] The Sage Developers. *Sage Mathematics Software (Version 6.8)*, 2015. <http://www.sagemath.org>.
- [2] Gerhard Dziuk and Charles M. Elliott. Finite element methods for surface pdes. *Acta Numerica*, 22:289–396, 5 2013.
- [3] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*, volume 23 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1994.