A generic way to solve partial differential equations with P1-finite elements in vector languages : User guide

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Contents

1 Description of the generic problems

The notations of [2] are employed in this section and extended to the vector case.

1.1 Scalar boundary value problem

Let Ω be a bounded open subset of $\mathbb{R}^d$, $d \geqslant 1$. The boundary of Ω is denoted by Γ.

We denote by $\mathcal{L}_{\mathbb{A},\boldsymbol{b},\boldsymbol{c},a_0} = \mathcal{L} : \mathrm{H}^2(\Omega) \longrightarrow L^2(\Omega)$ the second order linear differential operator acting on *scalar fields* defined, $\forall u \in \mathrm{H}^2(\Omega)$, by

$$\mathcal{L}_{\mathbb{A},\boldsymbol{b},\boldsymbol{c},a_0}(u) \stackrel{\text{def}}{=} -\operatorname{div}(\mathbb{A}\nabla u) + \operatorname{div}(\boldsymbol{b}u) + \langle \nabla u, \boldsymbol{c}\rangle + a_0 u \tag{1.1}$$

where we suppose that $\mathbb{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and a_0 are sufficiently regular functions. $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^d$. We use the same notations as in the chapter 6 of [2] . It should be also noted that it is important to preserve the two terms $\boldsymbol{b}$ and $\boldsymbol{c}$ in the generic formulation to allow a greater flexibility in the choice of the boundary conditions.

Let Γ^D, Γ^R be open subsets of Γ, possibly empty and $f \in L^2(\Omega)$, $g^D \in \mathrm{H}^{1/2}(\Gamma^D)$, $g^R \in L^2(\Gamma^R)$, $a^R \in L^\infty(\Gamma^R)$ be given data.

A *scalar* boundary value problem is given by

***Scalar* BVP**

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = f \qquad \text{in } \Omega, \tag{1.2}$$

$$u = g^D \qquad \text{on } \Gamma^D, \tag{1.3}$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = g^R \qquad \text{on } \Gamma^R. \tag{1.4}$$

The **conormal derivative** of u is defined by

$$\frac{\partial u}{\partial n_{\mathcal{L}}} = \langle \mathbb{A} \nabla u, \boldsymbol{n} \rangle - \langle \boldsymbol{b} u, \boldsymbol{n} \rangle \tag{1.5}$$

The boundary conditions (1.3) and (1.4) are respectively **Dirichlet** and **Robin** boundary conditions. **Neumann** boundary conditions are particular Robin boundary conditions with $a^R \equiv 0$.

1.2 Vector boundary value problem

Let $m \geqslant 1$ and $\mathcal{H}$ be the m-by-m matrix of second order linear differential operators defined by

$$\begin{cases} \mathcal{H} : & \left(\mathrm{H}^2(\Omega)\right)^m & \longrightarrow & \left(L^2(\Omega)\right)^m \\ & \boldsymbol{u} = (\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) & \longmapsto & \boldsymbol{f} = (\boldsymbol{f}_1, \dots, \boldsymbol{f}_m) \stackrel{\text{def}}{=} \mathcal{H}(\boldsymbol{u}) \end{cases} \tag{1.6}$$

where

$$\boldsymbol{f}_\alpha = \sum_{\beta=1}^{m} \mathcal{H}_{\alpha,\beta}(\boldsymbol{u}_\beta), \quad \forall \alpha \in [\![1, m]\!], \tag{1.7}$$

with, for all $(\alpha, \beta) \in [\![1, m]\!]^2$,

$$\mathcal{H}_{\alpha,\beta} \stackrel{\text{def}}{=} \mathcal{L}_{\mathbb{A}^{\alpha,\beta}, \boldsymbol{b}^{\alpha,\beta}, \boldsymbol{c}^{\alpha,\beta}, a_0^{\alpha,\beta}} \tag{1.8}$$

and $\mathbb{A}^{\alpha,\beta} \in (L^\infty(\Omega))^{d\times d}$, $\boldsymbol{b}^{\alpha,\beta} \in (L^\infty(\Omega))^d$, $\boldsymbol{c}^{\alpha,\beta} \in (L^\infty(\Omega))^d$ and $a_0^{\alpha,\beta} \in L^\infty(\Omega)$ are given functions. We can also write in matrix form

$$\mathcal{H}(\boldsymbol{u}) = \begin{pmatrix} \mathcal{L}_{\mathbb{A}^{1,1}, \boldsymbol{b}^{1,1}, \boldsymbol{c}^{1,1}, a_0^{1,1}} & \dots & \mathcal{L}_{\mathbb{A}^{1,m}, \boldsymbol{b}^{1,m}, \boldsymbol{c}^{1,m}, a_0^{1,m}} \\ \vdots & \ddots & \vdots \\ \mathcal{L}_{\mathbb{A}^{m,1}, \boldsymbol{b}^{m,1}, \boldsymbol{c}^{m,1}, a_0^{m,1}} & \dots & \mathcal{L}_{\mathbb{A}^{m,m}, \boldsymbol{b}^{m,m}, \boldsymbol{c}^{m,m}, a_0^{m,m}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_m \end{pmatrix}. \tag{1.9}$$

We remark that the $\mathcal{H}$ operator for $m = 1$ is equivalent to the $\mathcal{L}$ operator.

For $\alpha \in [\![1, m]\!]$, we define Γ_α^D and Γ_α^R as open subsets of Γ, possibly empty. Let $\boldsymbol{f} \in (L^2(\Omega))^m$, $g_\alpha^D \in \mathrm{H}^{1/2}(\Gamma_\alpha^D)$, $g_\alpha^R \in L^2(\Gamma_\alpha^R)$, $a_\alpha^R \in L^\infty(\Gamma_\alpha^R)$ be given data.

A *vector* boundary value problem is given by

***Vector* BVP**

Find $\boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) \in (\mathrm{H}^2(\Omega))^m$ such that

$$\mathcal{H}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.10}$$

$$\boldsymbol{u}_\alpha = g_\alpha^D \quad \text{on } \Gamma_\alpha^D,\ \forall \alpha \in [\![1, m]\!], \tag{1.11}$$

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \boldsymbol{u}_\alpha = g_\alpha^R \quad \text{on } \Gamma_\alpha^R,\ \forall \alpha \in [\![1, m]\!], \tag{1.12}$$

where the α-th component of the **conormal derivative** of $\boldsymbol{u}$ is defined by

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} \stackrel{\text{def}}{=} \sum_{\beta=1}^{m} \frac{\partial \boldsymbol{u}_\beta}{\partial n_{\mathcal{H}_{\alpha,\beta}}} = \sum_{\beta=1}^{m} \left(\left\langle \mathbb{A}^{\alpha,\beta} \nabla \boldsymbol{u}_\beta, \boldsymbol{n} \right\rangle - \left\langle \boldsymbol{b}^{\alpha,\beta} \boldsymbol{u}_\beta, \boldsymbol{n} \right\rangle \right). \tag{1.13}$$

The boundary conditions (1.12) are the **Robin** boundary conditions and (1.11) is the **Dirichlet** boundary condition. The **Neumann** boundary conditions are particular Robin boundary conditions with $a_\alpha^R \equiv 0$.

In the following of the report we will solve by a P_1-Lagrange finite element method *scalar* BVP (1.2) to (1.4) and *vector* BVP (1.10) to (1.12) without additional restrictive assumption.

2 Examples

2.1 First level functions or commonly used functions

We briefly describe the main functions that will be used in the sequel.

$\mathcal{T}_h \leftarrow$ getMesh(`FileName`) : to define the mesh $\mathcal{T}_h$ by reading a 2d or 3d mesh from the file `FileName`.

$\mathcal{T}_h \leftarrow$ HyperCube($d, N, < trans = \Phi >$) : to define the mesh $\mathcal{T}_h$ as the unit hypercube $[0,1]^d$. There are $N(i)$ (or N if N is a scalar) points in each direction and the mesh of the hypercube contains $\prod_{i=1}^{d} N(i)$ points. Optionnal parameter `trans` set the displacement vector of mesh transformation $\Phi(\mathrm{q}) = [\Phi_1(\mathrm{q}), \ldots, \Phi_d(\mathrm{q})]$.

`Lop` $\leftarrow$ Loperator($d, \mathbb{A}, \boldsymbol{b}, \boldsymbol{c}, a_0$) : to initialize the operator $\mathcal{L}$ in dimension d given by (1.1) : `Lop` $\leftarrow \mathcal{L}_{\mathbb{A},\boldsymbol{b},\boldsymbol{c},a_0}$.

`Hop` $\leftarrow$ Hoperator(d, m) : to initialize the operator $\mathcal{H}$ given by (1.6) verifying $\mathcal{H}_{\alpha,\beta} = 0$, $\forall (\alpha, \beta) \in [\![1, m]\!]^2$. Each operator $\mathcal{H}_{\alpha,\beta}$ corresponds to `Hop.H`(α, β) and can be initialized by the function Loperator

PDE $\leftarrow$ initPDE(`Op`, $\mathcal{T}_h$) : to initialize a PDE structure fron an operator (either $\mathcal{L}$-operator or $\mathcal{H}$-operator) and a mesh. Default boundary conditions are homogeneous generalized Neumann.

PDE $\leftarrow$ **setBC_PDE**(PDE, `label`, `comps`, `type`, `g`, `ar`) : to define or modify the boundary conditions on the boundary $\Gamma_{\texttt{label}}$ on the mesh PDE.$\mathcal{T}_h$ for components of index `comps` (in the scalar case `comps` $\equiv 1$). For a scalar PDE, we have for example

- Dirichlet condition : $\boldsymbol{u}_2 = g$ *on* Γ_{11}, then
  PDE $\leftarrow$ setBC_PDE(PDE, 11, 2, `'Dirichlet'`, g, $\emptyset$)
- generalized Neumann condition : $\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_3}} = g$ *on* Γ_{12}, then
  PDE $\leftarrow$ setBC_PDE(PDE, 12, 3, `'Neumann'`, g, $\emptyset$)
- generalized Robin condition : $\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_2}} + a_2^R \boldsymbol{u}_2 = g$ *on* Γ_{13}, then
  PDE $\leftarrow$ setBC_PDE(PDE, 13, 2, `'Robin'`, g, a_2^R)

$\boldsymbol{x} \leftarrow$ **SolvePDE**(PDE) : to solve by P_1 Lagrange finite elements the partial differential equation defined by the structure PDE. This function returns the solution $\boldsymbol{x}$ ()

2.2 Scalar case

2.2.1 2D condenser problem

The problem to solve is the Laplace problem for a condenser.

2D condenser problem

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$-\Delta u = 0 \text{ in } \Omega \subset \mathbb{R}^2, \tag{2.1}$$

$$u = 0 \text{ on } \Gamma_1, \tag{2.2}$$

$$u = -1 \text{ on } \Gamma_{98}, \tag{2.3}$$

$$u = 1 \text{ on } \Gamma_{99}, \tag{2.4}$$

where Ω and its boundaries are given in Figure 1a.

The operator in (2.1) is the *Stiffness* operator given by $\mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}$ and the conormal derivative $\frac{\partial u}{\partial n_{\mathcal{L}}}$ is

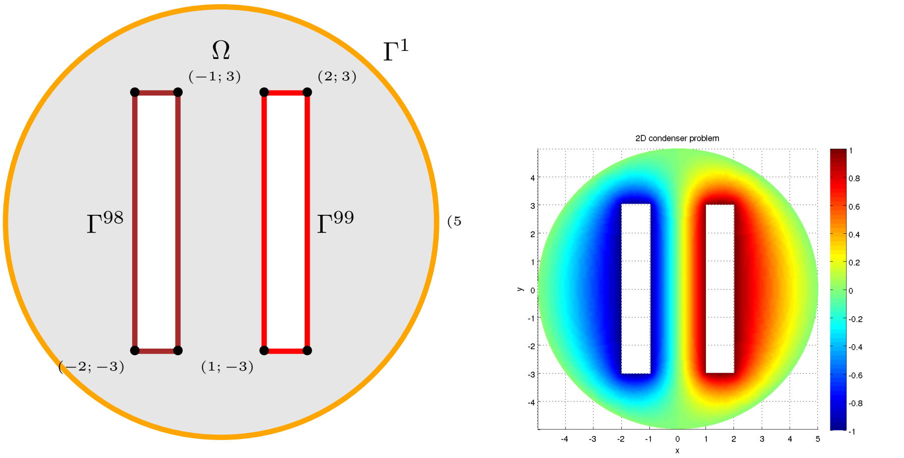
$$\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \nabla u, \boldsymbol{n} \rangle - \langle \boldsymbol{b} u, \boldsymbol{n} \rangle = \frac{\partial u}{\partial n}.$$

The algorithm using the toolbox for solving (2.1)-(2.4) is the following:

Algorithm 2.1 2D condenser

```
Th ← getMesh(...)                                  ▷ Load FreeFEM++ mesh
Dop ← Loperator(I, 0, 0, 0)                        ▷ Stiffness operator
PDE ← initPDE(Dop, Th)
PDE ← setBCLabel(PDE, 'Dirichlet', 1, 1, 0.)       ▷ u = 0 on Γ1
PDE ← setBCLabel(PDE, 'Dirichlet', 99, 1, 1.)      ▷ u = 1 on Γ99
PDE ← setBCLabel(PDE, 'Dirichlet', 98, 1, −1.)     ▷ u = −1 on Γ98
x ← SolvePDE(PDE)
```

The solution for a given mesh is shown on Figure 1b



(a) Domain for the condenser problem (b) Result for the 2D condenser problem

Figure 1: Condenser problem

2.2.2 Poisson PDE with mixed boundary conditions in 2D (2)

We first consider the classical 2D Poisson problem with various boundary conditions. The problem to solve is the following

2D Poisson problem

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega \subset \mathbb{R}^2, && (2.5)\\
u &= 0 \quad \text{on } \Gamma_1, && (2.6)\\
u &= 1 \quad \text{on } \Gamma_2, && (2.7)\\
\frac{\partial u}{\partial n} + a_R\, u &= -0.5 \quad \text{on } \Gamma_3, && (2.8)\\
\frac{\partial u}{\partial n} &= 0.5 \quad \text{on } \Gamma_4 && (2.9)
\end{aligned}
$$

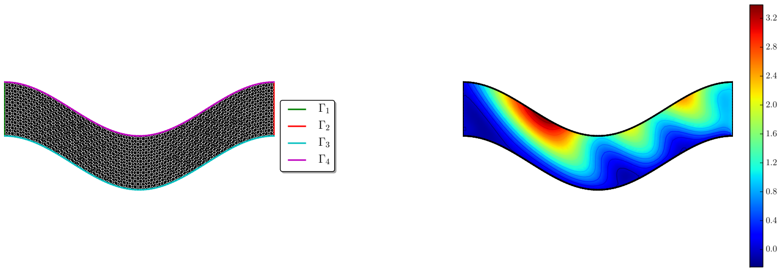
where Ω is the unit hypercube transformed by function $\boldsymbol{\Phi}(x, y) = (20x,\ 2(2y-1+\cos(2\pi x))$ and its boundaries are given in Figure 2a.
f and a_R satisfy:

$$
\begin{aligned}
f(\boldsymbol{x}) &= \cos(\boldsymbol{x}_1 + \boldsymbol{x}_2) \quad \forall \boldsymbol{x} \in \Omega\\
a_R(\boldsymbol{x}) &= 1 + \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 \quad \forall \boldsymbol{x} \in \Omega
\end{aligned}
$$

The operator in (2.5) is the *Stiffness* operator : $\mathcal{L}_{\mathbb{I},\mathbf{0},\mathbf{0},0}$.
The conormal derivative $\frac{\partial u}{\partial n_{\mathcal{L}}}$ is

$$
\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A}\, \nabla\, u, \boldsymbol{n}\rangle - \langle \boldsymbol{b}u, \boldsymbol{n}\rangle = \frac{\partial u}{\partial n}.
$$

The algorithm using the toolbox for solving (2.5)-(2.9) is given in Algorithm 2.2. The corresponding Matlab/Octave and Python codes are given in Listing 1.



(a) Domain and boundaries (b) Result

Figure 2: Laplace2d01 problem

Algorithm 2.2 2D Poisson problem

$\mathcal{T}_h \leftarrow \textsc{HyperCube}(2, 50)$ ▷ Build unit square mesh
$\texttt{Dop} \leftarrow \textsc{Loperator}(2, \mathbb{I}, \mathbf{0}, \mathbf{0}, 0)$ ▷ Stiffness operator
$\mathrm{PDE} \leftarrow \textsc{initPDE}(\texttt{Dop}, \mathcal{T}_h)$
$\mathrm{PDE} \leftarrow \textsc{setBC\_PDE}(\mathrm{PDE}, 1, 1, \texttt{'Dirichlet'}, 0., \emptyset)$ ▷ $u = 0$ on Γ_1
$\mathrm{PDE} \leftarrow \textsc{setBC\_PDE}(\mathrm{PDE}, 2, 1, \texttt{'Dirichlet'}, 1., \emptyset)$ ▷ $u = 1$ on Γ_2
$\mathrm{PDE} \leftarrow \textsc{setBC\_PDE}(\mathrm{PDE}, 3, 1, \texttt{'Robin'}, -0.5, \boldsymbol{x} \to 1 + \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2)$ ▷ $\frac{\partial u}{\partial n} + a_R\, u = -0.5$ on Γ_3
$\mathrm{PDE} \leftarrow \textsc{setBC\_PDE}(\mathrm{PDE}, 4, 1, \texttt{'Neumann'}, 0.5, \emptyset)$ ▷ $\frac{\partial u}{\partial n} = 0.5$ on Γ_4
$\mathrm{PDE}.f \leftarrow (\boldsymbol{x}_1, \boldsymbol{x}_2) \mapsto \cos(\boldsymbol{x}_1 + \boldsymbol{x}_2)$
$\boldsymbol{u}_h \leftarrow \textsc{SolvePDE}(\mathrm{PDE})$

A numerical solution for a given mesh is shown on Figure 2b

```
Th=HyperCube(2,50);
LOp=Loperator(2,{1,0;0,1},[],[],[]);
PDE=initPDE(LOp,Th);
PDE=setBC_PDE(PDE,1,1,'Dirichlet',0);
PDE=setBC_PDE(PDE,2,1,'Dirichlet',1);
PDE=setBC_PDE(PDE,3,1,'Robin',-0.5 , ...
                   @(x1,x2) 1+x1.^2+x2.^2 );
PDE=setBC_PDE(PDE,4,1,'Neumann',0.5);
PDE.f=@(x,y) cos(x+y) ;
uh=solvePDE(PDE);
```

(a) Matlab/Octave

```
from pyVecFEMP1Light.mesh import *
from pyVecFEMP1Light.pde import *
from pyVecFEMP1Light.operators import *
Th=HyperCube(2,50)
LOp=Loperator(2,[[1,0],[0,1]],None,None,None)
pde=initPDE(LOp,Th)
pde=setBC_PDE(pde,1,0,'Dirichlet',0.,None)
pde=setBC_PDE(pde,2,0,'Dirichlet',1.,None)
pde=setBC_PDE(pde,3,0,'Robin',-0.5,
                   lambda x,y: 1+x**2+y**2)
pde=setBC_PDE(pde,4,0,'Neumann',0.5,None)
pde.f=lambda x,y: cos(x+y)
uh=solvePDE(pde)
```

(b) Python

Listing 1: Poisson ...

2.3 Vector case

2.3.1 Elasticity problem

General case ($d = 2, 3$)

We consider here Hooke's law in linear elasticity, under small strain hypothesis (see for example [1]).

For a sufficiently regular vector field $\boldsymbol{u} = (u_1, \ldots, u_d) : \Omega \to \mathbb{R}^d$, we define the linearized strain tensor $\underline{\boldsymbol{\epsilon}}$ by

$$\underline{\boldsymbol{\epsilon}}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}(\boldsymbol{u}) + \boldsymbol{\nabla}^t(\boldsymbol{u})\right).$$

We set $\underline{\boldsymbol{\epsilon}} = (\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})^t$ in 2d and $\underline{\boldsymbol{\epsilon}} = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{13})^t$ in 3d, with $\epsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. Then the Hooke's law writes

$$\underline{\boldsymbol{\sigma}} = \mathbb{C}\underline{\boldsymbol{\epsilon}},$$

where $\underline{\boldsymbol{\sigma}}$ is the elastic stress tensor and $\mathbb{C}$ the elasticity tensor.

The material is supposed to be isotropic. Thus the elasticity tensor $\mathbb{C}$ is only defined by the Lamé parameters λ and μ, which satisfy $\lambda + \mu > 0$. We also set $\gamma = 2\,\mu + \lambda$. For $d = 2$ or $d = 3$, $\mathbb{C}$ is given by

$$\mathbb{C} = \begin{pmatrix} \lambda\mathbb{1}_2 + 2\mu\mathbb{I}_2 & 0 \\ 0 & \mu \end{pmatrix}_{3\times 3} \quad \text{or} \quad \mathbb{C} = \begin{pmatrix} \lambda\mathbb{1}_3 + 2\mu\mathbb{I}_3 & 0 \\ 0 & \mu\mathbb{I}_3 \end{pmatrix}_{6\times 6},$$

respectively, where $\mathbb{1}_d$ is a d-by-d matrix of ones, and $\mathbb{I}_d$ the d-by-d identity matrix.

For dimension $d = 2$ or $d = 3$, we have:

$$\boldsymbol{\sigma}_{\alpha\beta}(\boldsymbol{u}) = 2\,\mu\,\boldsymbol{\epsilon}_{\alpha\beta}(\boldsymbol{u}) + \lambda\,\mathrm{tr}(\boldsymbol{\epsilon}(\boldsymbol{u}))\delta_{\alpha\beta} \quad \forall \alpha, \beta \in [\![1, d]\!]$$

The problem to solve is the following

Elasticity problem

Find $\boldsymbol{u} = \mathrm{H}^2(\Omega)^d$ such that

$$-\,\mathrm{div}(\boldsymbol{\sigma}(\boldsymbol{u})) = \boldsymbol{f}, \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{2.10}$$
$$\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma^N, \tag{2.11}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma^D. \tag{2.12}$$

Now, with the following lemma, we obtain that this problem can be rewritten as the vector BVP defined by (1.10) to (1.12).

Lemma 1. *Let $\mathcal{H}$ be the d-by-d matrix of the second order linear differential operators defined in* (1.6) *where $\mathcal{H}_{\alpha,\beta} = \mathcal{L}_{\mathbb{A}^{\alpha,\beta},\boldsymbol{0},\boldsymbol{0},0}$, $\forall(\alpha,\beta) \in [\![1, d]\!]^2$, with*

$$(\mathbb{A}^{\alpha,\beta})_{k,l} = \mu\delta_{\alpha\beta}\delta_{kl} + \mu\delta_{k\beta}\delta_{l\alpha} + \lambda\delta_{k\alpha}\delta_{l\beta}, \ \forall(k,l) \in [\![1, d]\!]^2. \tag{2.13}$$

then

$$\mathcal{H}(\boldsymbol{u}) = -\,\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) \tag{2.14}$$

and, $\forall \alpha \in [\![1, d]\!]$,

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} = (\boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{n})_\alpha. \tag{2.15}$$

The proof is given in appendix **??**. So we obtain

Elasticity problem with $\mathcal{H}$ operator in dimension $d = 2$ or $d = 3$

Let $\mathcal{H}$ be the d-by-d matrix of the second order linear differential operators defined in (1.6) where $\forall(\alpha, \beta) \in [\![1, d]\!]^2$, $\mathcal{H}_{\alpha,\beta} = \mathcal{L}_{\mathbb{A}^{\alpha,\beta},\mathbf{0},\mathbf{0},0}$, with

- for $d = 2$,
$$\mathbb{A}^{1,1} = \begin{pmatrix} \gamma & 0 \\ 0 & \mu \end{pmatrix}, \ \mathbb{A}^{1,2} = \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}, \ \mathbb{A}^{2,1} = \begin{pmatrix} 0 & \mu \\ \lambda & 0 \end{pmatrix}, \ \mathbb{A}^{2,2} = \begin{pmatrix} \mu & 0 \\ 0 & \gamma \end{pmatrix}$$

- for $d = 3$,
$$\mathbb{A}^{1,1} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \mathbb{A}^{1,2} = \begin{pmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{A}^{1,3} = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}$$
$$\mathbb{A}^{2,1} = \begin{pmatrix} 0 & \mu & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{A}^{2,2} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \mu \end{pmatrix}, \quad \mathbb{A}^{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \mu & 0 \end{pmatrix},$$
$$\mathbb{A}^{3,1} = \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}, \quad \mathbb{A}^{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & \lambda & 0 \end{pmatrix}, \quad \mathbb{A}^{3,3} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

The elasticity problem (2.10) to (2.12) can be rewritten as :
Find $\boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d) \in (\mathrm{H}^2(\Omega))^d$ such that

$$\mathcal{H}(\boldsymbol{u}) = \boldsymbol{f}, \quad \text{in } \Omega, \tag{2.16}$$
$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} = 0, \quad \text{on } \Gamma^N_\alpha = \Gamma^N, \ \forall \alpha \in [\![1, d]\!] \tag{2.17}$$
$$\boldsymbol{u}_\alpha = 0, \quad \text{on } \Gamma^D_\alpha = \Gamma^D, \ \forall \alpha \in [\![1, d]\!]. \tag{2.18}$$

2D example

For example, in 2d, we want to solve the elasticity problem (2.10) to (2.12) where Ω and its boundaries are given in Figure 3. We have $\Gamma^N = \Gamma^1 \cup \Gamma^2 \cup \Gamma^3$, $\Gamma^D = \Gamma^4$.

The material's properties are given by Young's modulus E and Poisson's coefficient ν. As we use plane strain hypothesis, Lame's coefficients verify

$$\mu = \frac{E}{2\,(1+\nu)}, \quad \lambda = \frac{E\,\nu}{(1+\nu)\,(1-2\,\nu)}, \quad \gamma = 2\mu + \lambda$$

The material is rubber so that $E = 21.10^5$Pa and $\nu = 0.45$. We also have $\boldsymbol{f} = \boldsymbol{x} \mapsto (0, -1)^t$.

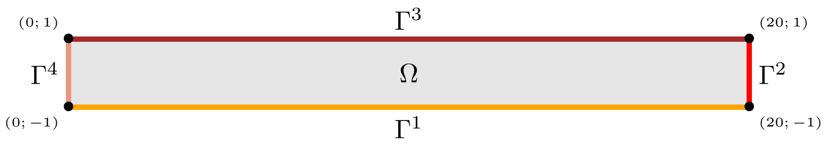


Figure 3: Domain for the 2D elasticity problem

Using **??** the operator in (2.10) is the *Elastic Stiffness* operator. Its conormal derivative corresponds to the stress vector.

The algorithm using the toolbox for solving (2.10)-(2.12) is the following:

Algorithm 2.3 2D elasticity

$\mathcal{T}_h \leftarrow$ GETMESH(...) ▷ Load FreeFEM++ mesh
$\lambda \leftarrow \frac{E\nu}{(1+\nu)(1-2\nu)}$
$\mu \leftarrow \frac{E}{2(1+\nu)}$
`Hop` $\leftarrow$ INITHOPERATOR$(2,2)$
`Hop`$(1,1) \leftarrow$ LOPERATOR$(2,[2\mu+\lambda,0;0,\mu],\mathbf{0},\mathbf{0},0)$
`Hop`$(2,1) \leftarrow$ LOPERATOR$(2,[0,\lambda;\mu,0],\mathbf{0},\mathbf{0},0)$
`Hop`$(1,2) \leftarrow$ LOPERATOR$(2,[0,\mu;\lambda,0],\mathbf{0},\mathbf{0},0)$
`Hop`$(2,2) \leftarrow$ LOPERATOR$(2,[\mu,0;0,2\mu+\lambda],\mathbf{0},\mathbf{0},0)$
PDE $\leftarrow$ INITPDE(`Hop`, $\mathcal{T}_h$)
PDE $\leftarrow$ SETBC_PDE(PDE$, 4, 1:2,$ `'Dirichlet'`, $\boldsymbol{x} \rightarrow \mathbf{0}$)
PDE.$f \leftarrow \boldsymbol{x} \rightarrow [0,-1]$
$\boldsymbol{x} \leftarrow$ SOLVEPDE(PDE)

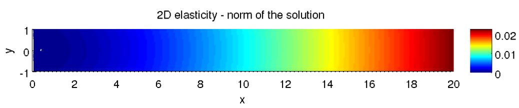


Figure 4: Result for the 2D elasticity problem

The solution for a given mesh is shown on Figure 4

2.3.2 Stationary heat with potential flow in 2D

Let Γ_1 be the unit circle, Γ_{10} be the circle with center point $(0,0)$ and radius 0.3. Let Γ_{20}, Γ_{21}, Γ_{22} and Γ_{23} be the circles with radius 0.1 and respectively with center point $(0,-0.7)$, $(0,0.7)$, $(-0.7,0)$ and $(0.7,0)$. The domain $\Omega \subset \mathbb{R}^2$ is defined as the inner of Γ_1 and the outer of all other circles (see Figure 5).

The 2D problem to solve is the following

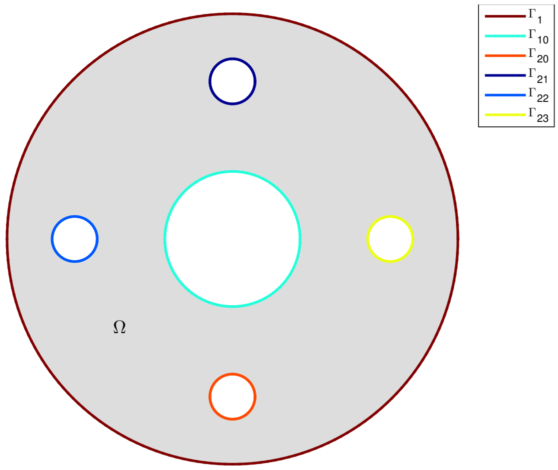


Figure 5: Domain and boundaries

2D problem : stationary heat with potential flow

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$-\operatorname{div}(\alpha \nabla u) + \langle \boldsymbol{V}, \nabla u \rangle + \beta u = 0 \text{ in } \Omega \subset \mathbb{R}^2, \tag{2.19}$$
$$u = 20 * y \text{ on } \Gamma_{21}, \tag{2.20}$$
$$u = 0 \text{ on } \Gamma_{22} \cup \Gamma_{23}, \tag{2.21}$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{21} \tag{2.22}$$

where Ω and its boundaries are given in Figure 5. This problem is well posed if $\alpha(\boldsymbol{x}) > 0$ and $\beta(\boldsymbol{x}) \geqslant 0$.
We choose α and β in Ω as :

$$\alpha(\boldsymbol{x}) = 0.1 + x_2^2,$$
$$\beta(\boldsymbol{x}) = 0.01$$

The potential flow is the velocity field $\boldsymbol{V} = \nabla \phi$ where the scalar function ϕ is the velocity potential solution of the PDE

Velocity potential in 2d

Find $\phi \in \mathrm{H}^2(\Omega)$ such that

$$-\Delta\phi = 0 \text{ in } \Omega, \tag{2.23}$$
$$\phi = -20 \text{ on } \Gamma_{21}, \tag{2.24}$$
$$\phi = 20 \text{ on } \Gamma_{20}, \tag{2.25}$$
$$\frac{\partial \phi}{\partial n} = 0 \text{ on } \Gamma_1 \cup \Gamma_{23} \cup \Gamma_{22} \tag{2.26}$$

To solve problem (2.19)-(2.22), we need to compute the velocity field $\boldsymbol{V}$.

Method 1 : have fun with $\mathcal{H}$-operators

To compute the velocity field $\boldsymbol{V}$ we can rewrite the potential flow problem (2.23)-(2.26), by introducing $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2)$ as unknowns :

Velocity potential and velocity field in 2d

Find $\phi \in \mathrm{H}^2(\Omega)$ and $\boldsymbol{V} \in \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega)$ such that

$$-\left(\frac{\partial \boldsymbol{V}_1}{\partial x} + \frac{\partial \boldsymbol{V}_2}{\partial y}\right) = 0 \quad \text{in } \Omega, \tag{2.27}$$

$$\boldsymbol{V}_1 - \frac{\partial \phi}{\partial x} = 0 \quad \text{in } \Omega, \tag{2.28}$$

$$\boldsymbol{V}_2 - \frac{\partial \phi}{\partial y} = 0 \quad \text{in } \Omega, \tag{2.29}$$

with boundary conditions (2.24) to (2.26).

We can also replace (2.27) by $-\Delta\phi = 0$.

Let $\boldsymbol{w} = \begin{pmatrix} \phi \\ \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{pmatrix}$, the previous PDE can be written as a vector boundary value problem (see Section 1.2) where the $\mathcal{H}$-operator is given by

$$\mathcal{H}(\boldsymbol{w}) = 0 \tag{2.30}$$

with

$$\mathcal{H}_{1,1} = 0, \qquad \mathcal{H}_{1,2} = \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_1,\mathbf{0},0}, \qquad \mathcal{H}_{1,3} = \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_2,\mathbf{0},0} \tag{2.31}$$

$$\mathcal{H}_{2,1} = \mathcal{L}_{\mathbb{O},\mathbf{0},-\boldsymbol{e}_1,0}, \qquad \mathcal{H}_{2,2} = \mathcal{L}_{\mathbb{O},\mathbf{0},\mathbf{0},1}, \qquad \mathcal{H}_{2,3} = 0, \tag{2.32}$$

$$\mathcal{H}_{3,1} = \mathcal{L}_{\mathbb{O},\mathbf{0},-\boldsymbol{e}_2,0}, \qquad \mathcal{H}_{3,2} = 0, \qquad \mathcal{H}_{3,3} = \mathcal{L}_{\mathbb{O},\mathbf{0},\mathbf{0},1}, \tag{2.33}$$

and $\boldsymbol{e}_1 = (1,0)^t$, $\boldsymbol{e}_2 = (0,1)^t$.

The conormal derivatives are given by

$$\begin{array}{lll}
\dfrac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{1,1}}} = 0, & \dfrac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{2,1}}} = 0, & \dfrac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{3,1}}} = 0, \\
\dfrac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{1,2}}} = \boldsymbol{V}_1\boldsymbol{n}_1, & \dfrac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{2,2}}} = 0, & \dfrac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{3,2}}} = 0, \\
\dfrac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{1,3}}} = \boldsymbol{V}_2\boldsymbol{n}_2, & \dfrac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{2,3}}} = 0, & \dfrac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{3,3}}} = 0.
\end{array}$$

So we obtain

$$\sum_{\alpha=1}^{3} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{1,\alpha}}} = \langle \boldsymbol{V}, \boldsymbol{n} \rangle = \langle \nabla \phi, \boldsymbol{n} \rangle, \tag{2.34}$$

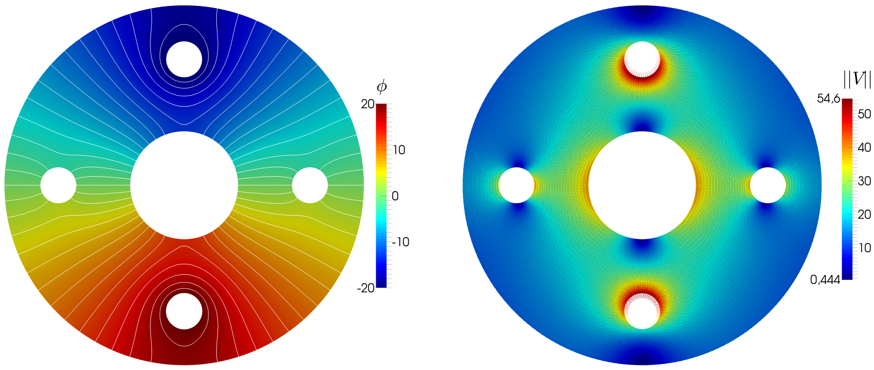
and

$$\sum_{\alpha=1}^{3} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{2,\alpha}}} = \sum_{\alpha=1}^{3} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{3,\alpha}}} = 0. \tag{2.35}$$

From (2.35), we cannot impose boundary conditions on components 2 and 3. Thus, with notations of Section 1.2, we have $\Gamma_2^N = \Gamma_3^N = \Gamma$ with $g_2^N = g_3^N = 0$.

To take into account boundary conditions (2.24) to (2.26), we set $\Gamma_1^D = \Gamma_{20} \cup \Gamma_{21}$, $\Gamma_1^N = \Gamma_1 \cup \Gamma_{10} \cup \Gamma_{22} \cup \Gamma_{23}$ and $g_1^D = 20\delta_{\Gamma_{20}} - 20\delta_{\Gamma_{21}}$, $g_1^N = 0$.

The resolution of this vector BVP is given on lines 3 to 13 of Algorithm 2.4. A representation of the velocity potential ϕ and potential flow $\boldsymbol{V}$ is given in Figure 6.



(a) Velocity potential (b) Potential flow

Figure 6: Stationary heat with potential flow in 2D

The operator in (2.19) is given by $\mathcal{L}_{\alpha\mathbb{I},\mathbf{0},\boldsymbol{V},\beta}$. The conormal derivative $\frac{\partial u}{\partial n_{\mathcal{L}}}$ is

$$\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \nabla u, \boldsymbol{n} \rangle - \langle \boldsymbol{b} u, \boldsymbol{n} \rangle = \alpha \frac{\partial u}{\partial n}.$$

The algorithm using the toolbox for solving (2.23)-(2.26) is the following:

Algorithm 2.4 Stationary heat with potential velocity problem

$\mathcal{T}_h \leftarrow \textsc{getMesh}(\ldots)$ ▷ Load FreeFEM++ mesh
$\boldsymbol{e}_1 \leftarrow \begin{pmatrix}1\\0\end{pmatrix}, \quad \boldsymbol{e}_2 \leftarrow \begin{pmatrix}0\\1\end{pmatrix}$
$\texttt{Hop} \leftarrow \textsc{Hoperator}(2,3)$
$\texttt{Hop.H}(1,2) \leftarrow \textsc{Loperator}(\mathbb{O}_2, -\boldsymbol{e}_1, \boldsymbol{0}, 0)$
$\texttt{Hop.H}(1,3) \leftarrow \textsc{Loperator}(\mathbb{O}_2, -\boldsymbol{e}_2, \boldsymbol{0}, 0)$
$\texttt{Hop.H}(2,1) \leftarrow \textsc{Loperator}(\mathbb{O}_2, \boldsymbol{0}, -\boldsymbol{e}_1, 0)$
$\texttt{Hop.H}(2,2) \leftarrow \textsc{Loperator}(\mathbb{O}_2, \boldsymbol{0}, \boldsymbol{0}, 1)$
$\texttt{Hop.H}(3,1) \leftarrow \textsc{Loperator}(\mathbb{O}_2, \boldsymbol{0}, -\boldsymbol{e}_2, 0)$
$\texttt{Hop.H}(3,3) \leftarrow \textsc{Loperator}(\mathbb{O}_2, \boldsymbol{0}, \boldsymbol{0}, 1)$
$\text{PDEflow} \leftarrow \textsc{init}\text{PDE}(\texttt{Hop}, \mathcal{T}_h)$
$\text{PDEflow} \leftarrow \textsc{setBC\_PDE}(\text{PDEflow}, 20, 1, \texttt{'Dirichlet'}, 20., \varnothing)$
$\text{PDEflow} \leftarrow \textsc{setBC\_PDE}(\text{PDEflow}, 21, 1, \texttt{'Dirichlet'}, -20., \varnothing)$
$[\boldsymbol{\phi}, \boldsymbol{V}_1, \boldsymbol{V}_2] \leftarrow \textsc{Solve}\text{PDE}(\text{PDEflux})$
$\alpha \leftarrow (x,y) \longmapsto 0.1 + y^2$
$g_{21} \leftarrow (x,y) \longmapsto 20y$
$\beta \leftarrow 0.01$
$\texttt{Dop} \leftarrow \textsc{Loperator}(\begin{pmatrix}\alpha & 0\\0 & \alpha\end{pmatrix}, \boldsymbol{0}, \begin{pmatrix}\boldsymbol{V}_1\\\boldsymbol{V}_2\end{pmatrix}, \beta)$
$\text{PDE} \leftarrow \textsc{init}\text{PDE}(\texttt{Dop}, \mathcal{T}_h)$ ▷ Set homogeneous `'Neumann'` condition on all boundaries
$\text{PDE} \leftarrow \textsc{setBC\_PDE}(\text{PDE}, 21, 1, \texttt{'Dirichlet'}, g_{21}, \varnothing)$ ▷ $u = 4$ on Γ_2
$\text{PDE} \leftarrow \textsc{setBC\_PDE}(\text{PDE}, 22, 1, \texttt{'Dirichlet'}, 0, \varnothing)$ ▷ $u = -4$ on Γ_4
$\text{PDE} \leftarrow \textsc{setBC\_PDE}(\text{PDE}, 23, 1, \texttt{'Dirichlet'}, 0, \varnothing$ ▷ $u = 0$ on Γ_{20}
$\boldsymbol{u} \leftarrow \textsc{Solve}\text{PDE}(\text{PDE})$

The numerical solution for a given mesh is shown on Figure 7

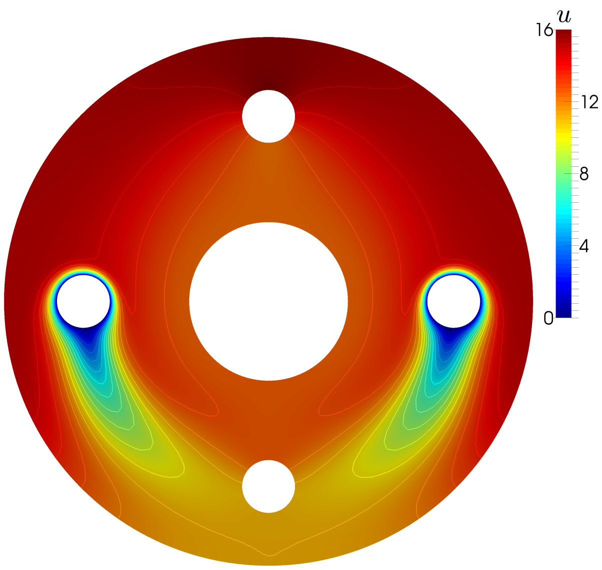
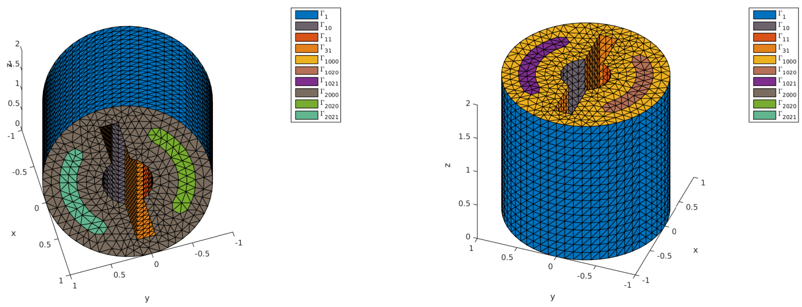


Figure 7: u

Method 2 :

2.3.3 Stationary heat with potential flow in 3D

Let $\Omega \subset \mathbb{R}^3$ be the cylinder given in Figure 8.



(a) Bottom face (b) Top face

Figure 8: Stationary heat with potential flow : 3d mesh

The bottom and top faces of the cylinder are respectively $\Gamma_{1000} \cup \Gamma_{1020} \cup \Gamma_{1021}$ and $\Gamma_{2000} \cup \Gamma_{2020} \cup \Gamma_{2021}$. The hole surface is $\Gamma_{10} \cup \Gamma_{11} \cup \Gamma_{31}$ where $\Gamma_{10} \cup \Gamma_{11}$ is the cylinder part and Γ_{31} the plane part.

The 3D problem to solve is the following

3D problem : stationary heat with potential flow

Find $u \in \mathrm{H}^2(\Omega)$ such that

$$
\begin{aligned}
-\operatorname{div}(\alpha \nabla u) + \langle \boldsymbol{V}, \nabla u \rangle + \beta u &= 0 \;\; \text{in } \Omega \subset \mathbb{R}^3, && (2.36)\\
u &= 30 \;\; \text{on } \Gamma_{1020} \cup \Gamma_{2020}, && (2.37)\\
u &= 10\delta_{|z-1|>0.5} \;\; \text{on } \Gamma_{10}, && (2.38)\\
\frac{\partial u}{\partial n} &= 0 \;\; \text{otherwise} && (2.39)
\end{aligned}
$$

where Ω and its boundaries are given in Figure 8. This problem is well posed if $\alpha(\boldsymbol{x}) > 0$ and $\beta(\boldsymbol{x}) \geqslant 0$.
We choose α and β in Ω as :

$$
\begin{aligned}
\alpha(\boldsymbol{x}) &= 1,\\
\beta(\boldsymbol{x}) &= 0.01
\end{aligned}
$$

The potential flow is the velocity field $\boldsymbol{V} = \nabla \phi$ where the scalar function ϕ is the velocity potential solution of the PDE :

Velocity potential in 3d

Find $\phi \in \mathrm{H}^2(\Omega)$ such that

$$
\begin{aligned}
-\Delta\phi &= 0 \ \text{ in } \Omega, &(2.40)\\
\phi &= 1 \ \text{ on } \Gamma_{1021} \cup \Gamma_{2021}, &(2.41)\\
\phi &= -1 \ \text{ on } \Gamma_{1020} \cup \Gamma_{2020}, &(2.42)\\
\frac{\partial \phi}{\partial n} &= 0 \ \text{ otherwise} &(2.43)
\end{aligned}
$$

To solve problem (2.36)-(2.39), we need to compute the velocity field $\boldsymbol{V}$. For that we can rewrite the potential flow problem (2.40)-(2.43), by introducing $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3)$ as unknowns :

Velocity potential and velocity field in 3d

Find $\phi \in \mathrm{H}^2(\Omega)$ and $\boldsymbol{V} \in \mathrm{H}^1(\Omega)^3$ such that

$$
\begin{aligned}
-\left(\frac{\partial \boldsymbol{V}_1}{\partial x} + \frac{\partial \boldsymbol{V}_2}{\partial y} + \frac{\partial \boldsymbol{V}_3}{\partial z}\right) &= 0 \ \text{ in } \Omega, &(2.44)\\
\boldsymbol{V}_1 - \frac{\partial \phi}{\partial x} &= 0 \ \text{ in } \Omega, &(2.45)\\
\boldsymbol{V}_2 - \frac{\partial \phi}{\partial y} &= 0 \ \text{ in } \Omega, &(2.46)\\
\boldsymbol{V}_3 - \frac{\partial \phi}{\partial z} &= 0 \ \text{ in } \Omega, &(2.47)
\end{aligned}
$$

with boundary conditions (2.41) to (2.43).

We can also replace (2.44) by $-\Delta\phi = 0$.

Let $\boldsymbol{w} = \begin{pmatrix} \phi \\ \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \\ \boldsymbol{V}_3 \end{pmatrix}$, the previous PDE can be written as a vector boundary value problem (see section 1.2) where the $\mathcal{H}$-operator is given by

$$\mathcal{H}(\boldsymbol{w}) = 0 \tag{2.48}$$

with

$$
\begin{aligned}
&\mathcal{H}_{1,1} = 0, && \mathcal{H}_{1,2} = \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_1,\boldsymbol{0},0}, && \mathcal{H}_{1,3} = \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_2,\boldsymbol{0},0}, && \mathcal{H}_{1,4} = \mathcal{L}_{\mathbb{O},-\boldsymbol{e}_3,\boldsymbol{0},0}, &(2.49)\\
&\mathcal{H}_{2,1} = \mathcal{L}_{\mathbb{O},\boldsymbol{0},-\boldsymbol{e}_1,0}, && \mathcal{H}_{2,2} = \mathcal{L}_{\mathbb{O},\boldsymbol{0},\boldsymbol{0},1}, && \mathcal{H}_{2,3} = 0, && \mathcal{H}_{2,4} = 0, &(2.50)\\
&\mathcal{H}_{3,1} = \mathcal{L}_{\mathbb{O},\boldsymbol{0},-\boldsymbol{e}_2,0}, && \mathcal{H}_{3,2} = 0, && \mathcal{H}_{3,3} = \mathcal{L}_{\mathbb{O},\boldsymbol{0},\boldsymbol{0},1}, && \mathcal{H}_{3,4} = 0, &(2.51)\\
&\mathcal{H}_{4,1} = \mathcal{L}_{\mathbb{O},\boldsymbol{0},-\boldsymbol{e}_3,0}, && \mathcal{H}_{4,2} = 0, && \mathcal{H}_{4,3} = 0, && \mathcal{H}_{4,4} = \mathcal{L}_{\mathbb{O},\boldsymbol{0},\boldsymbol{0},1}, &(2.52)
\end{aligned}
$$

and $\boldsymbol{e}_1 = (1,0,0)^t$, $\boldsymbol{e}_2 = (0,1,0)^t$, $\boldsymbol{e}_3 = (0,0,1)^t$.

The conormal derivatives are given by

$$\begin{array}{llll}
\dfrac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{1,1}}} = 0, & \dfrac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{2,1}}} = 0, & \dfrac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{3,1}}} = 0, & \dfrac{\partial \boldsymbol{w}_1}{\partial n_{\mathcal{H}_{4,1}}} = 0, \\
\dfrac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{1,2}}} = \boldsymbol{V}_1\boldsymbol{n}_1, & \dfrac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{2,2}}} = 0, & \dfrac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{3,2}}} = 0, & \dfrac{\partial \boldsymbol{w}_2}{\partial n_{\mathcal{H}_{4,2}}} = 0, \\
\dfrac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{1,3}}} = \boldsymbol{V}_2\boldsymbol{n}_2, & \dfrac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{2,3}}} = 0, & \dfrac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{3,3}}} = 0, & \dfrac{\partial \boldsymbol{w}_3}{\partial n_{\mathcal{H}_{4,3}}} = 0, \\
\dfrac{\partial \boldsymbol{w}_4}{\partial n_{\mathcal{H}_{1,4}}} = \boldsymbol{V}_3\boldsymbol{n}_3, & \dfrac{\partial \boldsymbol{w}_4}{\partial n_{\mathcal{H}_{2,4}}} = 0, & \dfrac{\partial \boldsymbol{w}_4}{\partial n_{\mathcal{H}_{3,4}}} = 0, & \dfrac{\partial \boldsymbol{w}_4}{\partial n_{\mathcal{H}_{4,4}}} = 0,
\end{array}$$

So we obtain

$$\sum_{\alpha=1}^{4} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{1,\alpha}}} = \langle \boldsymbol{V}, \boldsymbol{n} \rangle = \langle \nabla \phi, \boldsymbol{n} \rangle, \tag{2.53}$$

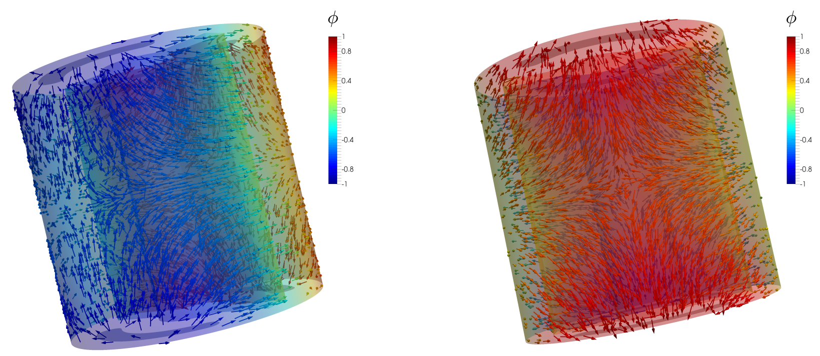
and

$$\sum_{\alpha=1}^{4} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{2,\alpha}}} = \sum_{\alpha=1}^{4} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{3,\alpha}}} = \sum_{\alpha=1}^{4} \frac{\partial \boldsymbol{w}_\alpha}{\partial n_{\mathcal{H}_{4,\alpha}}} = 0. \tag{2.54}$$

From (2.54), we cannot impose boundary conditions on components 2 to 4. Thus, with notation of section 1.2, we have $\Gamma_2^N = \Gamma_3^N = \Gamma_4^N = \Gamma$ with $g_2^N = g_3^N = g_4^N = 0$.

To take into account boundary conditions (2.41) to (2.43), we set $\Gamma_1^D = \Gamma_{1020} \cup \Gamma_{1021} \cup \Gamma_{2020} \cup \Gamma_{2021}$, $\Gamma_1^N = \Gamma \backslash \Gamma_1^D$ and $g_1^D = \delta_{\Gamma_{1020}\cup\Gamma_{2020}} - \delta_{\Gamma_{1021}\cup\Gamma_{2021}}$, $g_1^N = 0$.

The solution of this vector boundary value problem is given in lines 3 to 13 of Algorithm 2.5. A representation of velocity potential ϕ and potential flow $\boldsymbol{V}$ is given in Figure 9.



(a) ϕ and $\boldsymbol{V}$: first view

(b) ϕ and $\boldsymbol{V}$: second view

Figure 9: HeatAndFlowVelocity3d01 problem

The operator in (2.36) is given by $\mathcal{L}_{\alpha\mathbb{I},\mathbf{0},\boldsymbol{V},\beta}$. The conormal derivative $\frac{\partial u}{\partial n_{\mathcal{L}}}$ is

$$\frac{\partial u}{\partial n_{\mathcal{L}}} := \langle \mathbb{A} \nabla u, \boldsymbol{n} \rangle - \langle \boldsymbol{b} u, \boldsymbol{n} \rangle = \alpha \frac{\partial u}{\partial n}.$$

The algorithm using the toolbox for solving (2.44)-(2.47) is the following:

Algorithm 2.5 Stationary heat with potential velocity problem

$\mathcal{T}_h \leftarrow$ GETMESH(...) ▷ Load FreeFEM++ mesh
$\boldsymbol{e}_1 \leftarrow \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \boldsymbol{e}_2 \leftarrow \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \boldsymbol{e}_3 \leftarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$
$\texttt{Hop} \leftarrow$ HOPERATOR$(3,4)$
$\texttt{Hop.H}(1,2) \leftarrow$ LOPERATOR$(\mathbb{O}_3, -\boldsymbol{e}_1, \mathbf{0}, 0)$
$\texttt{Hop.H}(1,3) \leftarrow$ LOPERATOR$(\mathbb{O}_3, -\boldsymbol{e}_2, \mathbf{0}, 0)$
$\texttt{Hop.H}(1,4) \leftarrow$ LOPERATOR$(\mathbb{O}_3, -\boldsymbol{e}_3, \mathbf{0}, 0)$
$\texttt{Hop.H}(2,1) \leftarrow$ LOPERATOR$(\mathbb{O}_3, \mathbf{0}, -\boldsymbol{e}_1, 0)$, $\texttt{Hop.H}(2,2) \leftarrow$ LOPERATOR$(\mathbb{O}_3, \mathbf{0}, \mathbf{0}, 1)$
$\texttt{Hop.H}(3,1) \leftarrow$ LOPERATOR$(\mathbb{O}_3, \mathbf{0}, -\boldsymbol{e}_2, 0)$, $\texttt{Hop.H}(3,3) \leftarrow$ LOPERATOR$(\mathbb{O}_3, \mathbf{0}, \mathbf{0}, 1)$
$\texttt{Hop.H}(4,1) \leftarrow$ LOPERATOR$(\mathbb{O}_3, \mathbf{0}, -\boldsymbol{e}_3, 0)$, $\texttt{Hop.H}(4,4) \leftarrow$ LOPERATOR$(\mathbb{O}_3, \mathbf{0}, \mathbf{0}, 1)$
PDEflow $\leftarrow$ INITPDE$(\texttt{Hop}, \mathcal{T}_h)$
PDEflow $\leftarrow$ SETBC_PDE(PDEflow, $20, 1$, `'Dirichlet'`, $20., \varnothing$)
PDEflow $\leftarrow$ SETBC_PDE(PDEflow, $21, 1$, `'Dirichlet'`, $-20., \varnothing$)
$[\boldsymbol{\phi}, \boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3] \leftarrow$ SOLVEPDE(PDEflow)
$\alpha \leftarrow (x,y,z) \longmapsto 1$
$g_{20} \leftarrow (x,y,z) \longmapsto 30, \quad g_{10} \leftarrow (x,y,z) \longmapsto 10 * (|z-1| > 0.5)$
$\beta \leftarrow 0.01$
$\texttt{Dop} \leftarrow$ LOPERATOR$\left(\begin{pmatrix}\alpha & 0 & 0\\0 & \alpha & 0\\0 & 0 & \alpha\end{pmatrix}, \mathbf{0}, \begin{pmatrix}\boldsymbol{V}_1\\\boldsymbol{V}_2\\\boldsymbol{V}_3\end{pmatrix}, \beta\right)$
PDE $\leftarrow$ INITPDE$(\texttt{Dop}, \mathcal{T}_h)$ ▷ Set homogeneous `'Neumann'` condition on all boundaries
PDE $\leftarrow$ SETBC_PDE(PDE, $1020, 1$, `'Dirichlet'`, $g_{20}, \varnothing$)
PDE $\leftarrow$ SETBC_PDE(PDE, $2022, 1$, `'Dirichlet'`, $g_{20}, \varnothing$)
PDE $\leftarrow$ SETBC_PDE(PDE, $10, 1$, `'Dirichlet'`, $g_{10}, \varnothing$)
$\boldsymbol{u} \leftarrow$ SOLVEPDE(PDE)

The numerical solution for a given mesh is shown on Figure 10

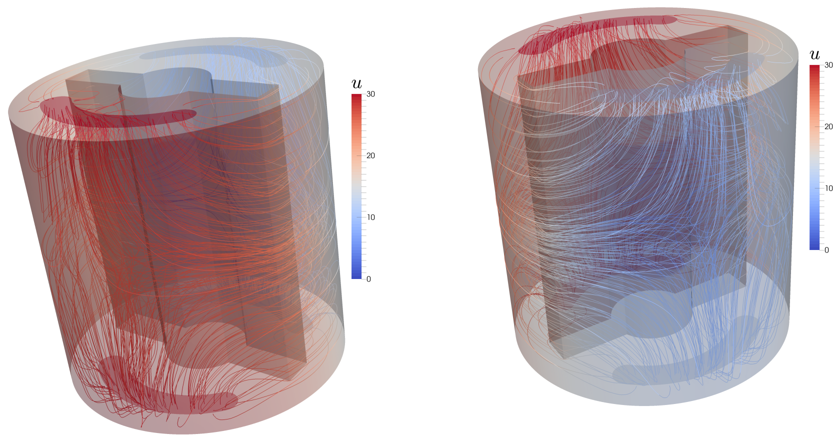
2.4 Eigenvalues

3 Eigenvalue problems

We want to solve eigenvalue problems coming from scalar or vector BVP's.

3.1 Scalar case

The eigenvalue problems associated with *scalar* BVP (1.2)-(1.4) can be written as



(a) us solution with streamline : first viewview (b) us solution with streamline : second

Figure 10: HeatAndFlowVelocity3d01 problem

Scalar eigenvalue problem

Find $(\lambda, u) \in \mathbb{K} \times \mathrm{H}^2(\Omega)$ such that

$$\mathcal{L}(u) = \lambda \mathcal{B}(u) \quad \text{in } \Omega, \tag{3.1}$$

$$u = 0 \quad \text{on } \Gamma^D, \tag{3.2}$$

$$\frac{\partial u}{\partial n_{\mathcal{L}}} + a^R u = 0 \quad \text{on } \Gamma^R. \tag{3.3}$$

where $\mathcal{B} = \mathcal{L}_{\mathbb{O}_{d\times d}, \mathbf{0}_d, \tilde{\boldsymbol{c}}, \tilde{a_0}}$.

A variational formulation of this problem is given by

Variational formulation

Find $\lambda \in \mathbb{K}$ and $u \in \mathrm{H}^1_{0,\Gamma^D}(\Omega)$ such that

$$\int_\Omega \mathcal{D}_{\mathcal{L}}(u, v) dq + \int_{\Gamma^R} a^R uv d\sigma = \lambda \int_\Omega \mathcal{D}_{\mathcal{B}}(u, v) dq \quad \forall v \in \mathrm{H}^1_{0,\Gamma^D}(\Omega) \tag{3.4}$$

As seen in section **??**, the discretization of this variational formulation by P_1-Lagrange finite element method leads to the linear system

Linear system from the discrete variational formulation (3.4)

Find $\boldsymbol{u} \in \mathbb{R}^{\mathrm{n_q}}$ such that

$$\sum_{j \in \mathcal{I}_D^c} \mathcal{A}^h_{\mathcal{L}}(\varphi_j, \varphi_i) \boldsymbol{u}_j = \sum_{j \in \mathcal{I}_D^c} \mathcal{A}^h_{\mathcal{B}}(\varphi_j, \varphi_i) \boldsymbol{u}_j \ \forall i \in \mathcal{I}_D^c, \tag{3.5}$$

$$\boldsymbol{u}_i = 0 \ \forall i \in \mathcal{I}_D. \tag{3.6}$$

and we have

$$u_h = \sum_{j \in \mathcal{I}_D^c} \boldsymbol{u}_j \varphi_j.$$

Let $\mathbb{A}^{\mathcal{B}}$ be the matrix associated with $\mathcal{B}$ operator. With notations of section **??** and

$$\mathbb{A}^{\mathcal{L}}(\mathcal{I}_D^c, \mathcal{I}_D^c)\boldsymbol{v} = \lambda \mathbb{A}^{\mathcal{B}}(\mathcal{I}_D^c, \mathcal{I}_D^c)\boldsymbol{v} \tag{3.7}$$

Algorithm 3.1 function EigsPDE : solve *scalar* or *vector* eigenvalue problems

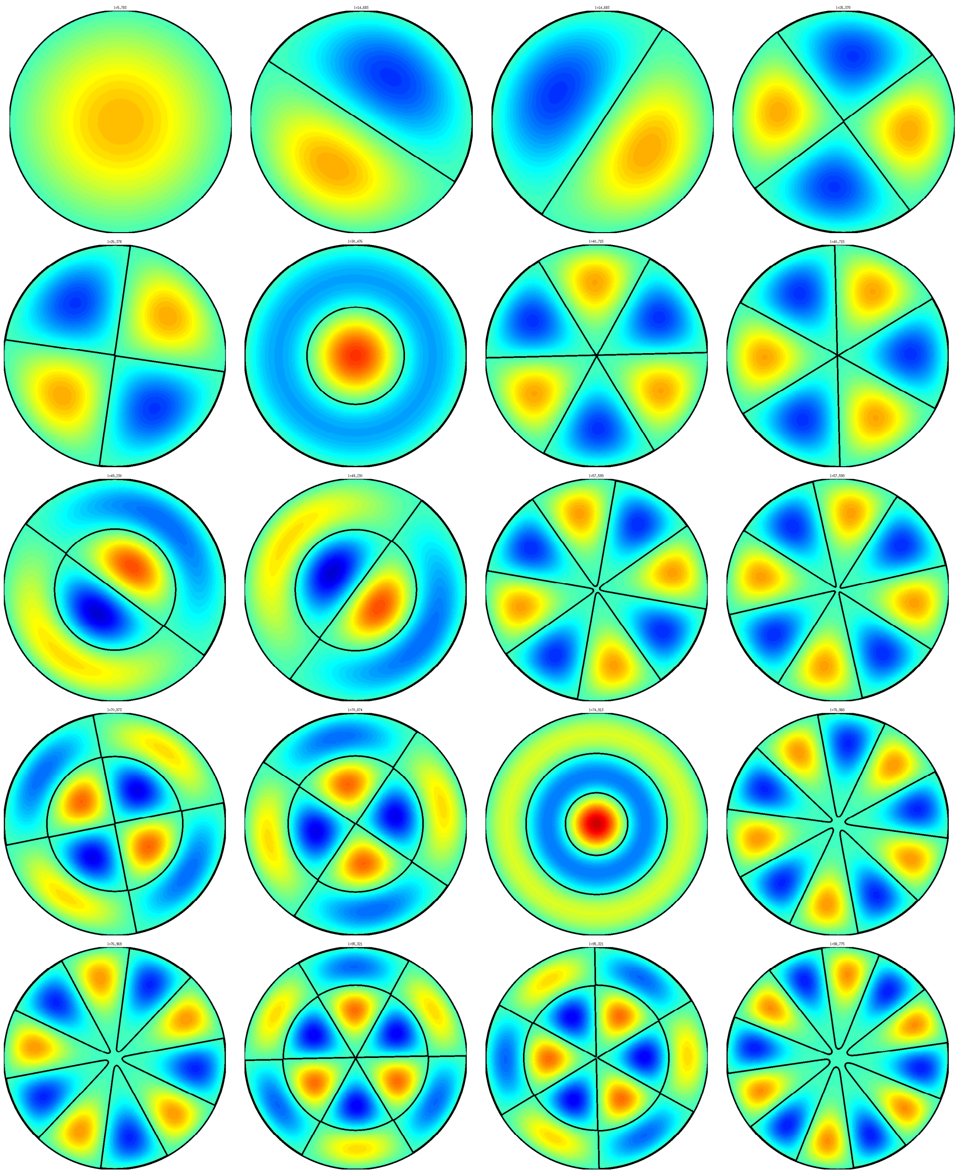
Input :

pde	:	a `PDE` structure
Bop	:	operator structure associated with $\mathcal{B}$
N_e	:	computes the N_e first eigenvalues and eigenvectors.
σ	:	start value for eigs function

Output :

$\boldsymbol{\lambda}$	:	1d-array of dimension N_e. Contains N_e-th first eigenvalues
$\boldsymbol{U}$	:	pde.$\mathcal{T}_\mathrm{h}$.$\mathrm{n_q}$-by-N_e array, $\boldsymbol{U}(:,i)$ is the i-th eigenvector associated with the $\boldsymbol{\lambda}(i)$ eigenvalue.

```
Function [U, λ] ← EigsPDE(pde, Bop, N_e, σ)
  n_dof ← pde.m × pde.T_h.n_q
  A ← AssemblyP1_OptV3(pde.T_h, pde.Op)
  [M^R, F^R] ← RobinBC(pde)
  A ← A + M^R
  B ← AssemblyP1_OptV3(pde.T_h, Bop)
  [R^D, I_D, I_D^c] ← DirichletBC(pde)
  U ← O_{n_dof, N_e}
  [U(I_D^c, :), λ] ← Eigs(A(I_D^c, I_D^c), B(I_D^c, I_D^c), N_e, σ)
end Function
```



3.2 Vector case

The eigenvalue problems associated with *vector* BVP (1.10)-(1.12) can be written as

Vector eigenvalue problems

Find $\lambda \in \mathbb{K}$ and $\boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) \in (\mathrm{H}^2(\Omega))^m$ such that

$$\mathcal{H}(\boldsymbol{u}) = \lambda \mathcal{B}(\boldsymbol{u}) \quad \text{in } \Omega, \tag{3.8}$$

$$\boldsymbol{u}_\alpha = 0 \quad \text{on } \Gamma_\alpha^D, \ \forall \alpha \in [\![1, m]\!], \tag{3.9}$$

$$\frac{\partial \boldsymbol{u}}{\partial n_{\mathcal{H}_\alpha}} + a_\alpha^R \boldsymbol{u}_\alpha = 0 \quad \text{on } \Gamma_\alpha^R, \ \forall \alpha \in [\![1, m]\!], \tag{3.10}$$

where $\mathcal{B}$ is a given $\mathcal{H}$-operator.

[]

References

[1] G. Dhatt, E. Lefrançois, and G. Touzot. *Finite Element Method*. Wiley, 2012.

[2] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer, 2008.