Lifting low-dimensional local systems

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Abstract. Let k be a field of characteristic $p > 0$. Denote by $\mathbf{W}_r(k)$ the ring of truntacted Witt vectors of length $r \geq 2$, built out of k. In this text, we consider the following question, depending on a given profinite group G.

$Q(G)$: Does every (continuous) representation $G \longrightarrow \mathrm{GL}_d(k)$ lift to a representation $G \longrightarrow \mathrm{GL}_d(\mathbf{W}_r(k))$?

We work in the class of cyclotomic pairs (Definition 4.3), first introduced in [DCF] under the name "smooth profinite groups". Using Grothendieck-Hilbert' theorem 90, we show that the algebraic fundamental groups of the following schemes are cyclotomic: spectra of semilocal rings over $\mathbb{Z}[\frac{1}{p}]$, smooth curves over algebraically closed fields, and affine schemes over $\mathbb{F}_p$. In particular, absolute Galois groups of fields fit into this class. We then give a positive partial answer to $Q(G)$, for a cyclotomic profinite group G: the answer is positive, when $d = 2$ and $r = 2$. When $d = 2$ and $r = \infty$, we show that any 2-dimensional representation of G *stably* lifts to a representation over $\mathbf{W}(k)$: see Theorem 6.1.
When $p = 2$ and $k = \mathbb{F}_2$, we prove the same results, up to dimension $d = 4$.
We then give a concrete application to algebraic geometry: we prove that local systems of low dimension lift Zariski-locally (Corollary 6.3).

Contents

1. Introduction

Let k be a field of characteristic p and let G be a profinite group. This paper deals with the deformation theory (more accurately, the liftability) of continuous representations

$$\rho : G \longrightarrow \mathrm{GL}_d(k),$$

ultimately with coefficients in the ring of Witt vectors $\mathbf{W}(k)$. A fundamental instance is given by Galois representations.

Existence of such lifts has been extensively investigated, in the case of absolute Galois groups of local and global fields. In [K], Khare proves the existence of lifts to $\mathbf{W}(k)$, for 2-dimensional reducible representations, in the case where G is the absolute Galois group of a number field F, and when k is a finite field. As noticed by Serre, the proof actually works for any field F. If G is the absolute Galois group of $\mathbb{Q}$, and under mild assumptions, such lifts exist more generally, by the work of Ramakrishna [R]. Some time after the present text was released, Khare and Larsen ([KL]) proved that the answer to $Q(G)$ is positive, when G is the absolute Galois group of a non-archimedean local field, or a global field, when $k = \mathbb{F}_p$ for odd p, $d \leq 3$ and $r = 2$.

A class of profinite groups, whose mod p representations are likely to lift mod p^2, was first introduced in [DCF] under the name *smooth profinite groups*. Due to the recent progress made in the series of papers [DCF1], [F2] and [DCF3], it is now clear that one should distinguish between the notions of *smooth profinite groups* and of *cyclotomic pairs*. In the present paper, we focus on cyclotomic pairs. Loosely speaking, a cyclotomic pair consists of a profinite group, equipped with a so-called cyclotomic module, which will play the role of the cyclotomic character in Galois cohomology- see Definition 4.3 . We say that a profinite group is cyclotomic, if it fits into a cyclotomic pair.

The main contribution of this paper, is to give important examples of cyclotomic profinite groups among algebraic fundamental groups, using Kummer and Artin-Schreier theory.

More precisely, in Propositions 4.9, 4.10, 4.11 and 4.12, we show that the fundamental group $\pi_1(S, \overline{s})$, of a given scheme S at a geometric point $\overline{s}$, is cyclotomic in each of the following cases.

a) S is a semilocal $\mathbb{Z}[\frac{1}{p}]$-scheme;
b) S is an affine $\mathbb{F}_p$-scheme;
c) S is a smooth curve, over an algebraically closed field.
d) More generally, S is a smooth projective variety over an algebraically closed field, such that for every finite étale cover U/S, the Néron-Severi group of U is torsion-free.

Note that, in all four cases, the cyclotomic module is taken to be the Tate module of roots of unity $\mathbb{Z}_p(1)$ when p is invertible on S, or the trivial module $\mathbb{Z}_p$ when p vanishes on S. In particular, absolute Galois groups of fields are cyclotomic profinite groups.

Our next result is Theorem 6.1: continuous representations of dimension 2 of a cyclotomic profinite group (e.g. of type a), b), c) or d) above), with values in an *arbitrary* field k of characteristic $p > 0$ (possibly infinite), lift to p^2-torsion coefficients. They also *stably* lift to arbitrary torsion (see Definition 3.1 for the notion of stable lifting).

For $p = 2$ and $k = \mathbb{F}_2$, we prove the same results, for representations of dimension *up to* 4. After this paper was first drafted, the recent text [F2] was released, in which it is proved that mod p representations of a cyclotomic profinite group lift mod p^2- in all dimensions d. The proof involves a delicate new technology. Our theorem 6.1 here is the particular case $d = 2$; it is much easier to read.

The paper is structured as follows. In section 2, we recall the machinery of Witt vectors and of Yoneda extensions, which is a convenient computational tool in our proofs. We give precise definitions of what is meant by "lifting" in section 3. In section 4, we recall the notion of a cyclotomic pair. In the remaining sections, we prove the lifting theorem and deal with its applications.

2. Modules over Witt vectors and Yoneda extensions

Fixing a field k of characteristic $p > 0$, we consider the ring $\mathbf{W}(k)$ of Witt vectors built out of k. Recall that if k is perfect, $\mathbf{W}(k)$ is the unique complete discrete valuation ring of characteristic 0 whose uniformizer is p and residue field is k. We shall also consider the truncated Witt vectors of size $r \geq 1$, defined by the quotient

$$\mathbf{W}_r(k) := \mathbf{W}(k)/\mathrm{Ver}^r(\mathbf{W}(k)),$$

where $\mathrm{Ver} : \mathbf{W}(k) \longrightarrow \mathbf{W}(k)$ denotes the Verschiebung endomorphism. We set $\mathbf{W}_\infty(k) := \mathbf{W}(k)$. Note that if k is perfect, we have $\mathbf{W}_r(k) = \mathbf{W}(k)/p^r\mathbf{W}(k)$.

DEFINITION 2.1. *Let $r \in \mathbb{N}^* \cup \{\infty\}$. Let M be a $\mathbf{W}_r(k)$-module of finite type. We endow it with the topology having the submodules $\mathrm{Ver}^i(\mathbf{W}(k))M$, $i \in \mathbb{N}$, as a basis for open neighborhoods of 0. This is simply the discrete topology if $r < \infty$. Note also that, if k is perfect, this defines the p-adic topology on M.*

DEFINITION 2.2. *Let G be a profinite group, and let $r \in \mathbb{N}^* \cup \{\infty\}$.*

A $(\mathbf{W}_r(k), G)$-module is a $\mathbf{W}_r(k)$-module of finite type, endowed with a continuous $\mathbf{W}_r(k)$-linear action of G.

In particular, for $r = 1$, a (k, G)-module is a finite-dimensional k-vector space endowed with an action of G, that factors through an open subgroup of G.

If V is a $(\mathbf{W}_r(k), G)$-module, we set

$$V^\vee := \mathrm{Hom}_{\mathbf{W}_r(k)}(V, \mathbf{W}_r(k)).$$

We refer to $V^\vee$ as the Pontryagin dual of V.

If k is perfect, then Pontryagin duality is perfect, in the sense that the natural arrow

$$V \longrightarrow V^{\vee\vee}$$

is an isomorphism.

If G is a profinite group and V is a G-module, we will denote by $H^n(G, V)$ the n-th cohomology group of G, with values in V. If V is equipped with the discrete topology, it is taken in the sense of [Se]. Otherwise (e.g. when $V = \mathbb{Z}_p(n)$), it is in the sense of Tate's continuous cohomology. In the context of the present paper, laying too much stress on continuity issues would, we believe, be smoke and mirrors.

The abelian categories $\mathcal{M}(\mathbf{W}_r(k), G)$ (resp. $\mathcal{M}(k, G)$) of $(\mathbf{W}_r(k), G)$-modules (resp. (k, G)-modules) are monoidal through the tensor product. For any positive integer n and $A, B \in \mathcal{M}(\mathbf{W}_r(k), G)$, one can then define the notion of

Yoneda n-extensions of B by A, as follows. First, define $\text{YExt}^0_{(\mathbf{W}_r(k),G)}(B,A)$ as $\text{Hom}_{(\mathbf{W}_r(k),G)}(B,A)$.

A n-extension of B by A is an exact sequence of $(\mathbf{W}_r(k),G)$-modules

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow A_1 \longrightarrow \ldots \longrightarrow A_n \longrightarrow B \longrightarrow 0.$$

Setting morphisms $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$ between two n-extensions of B by A to be morphisms of complexes for which the induced morphisms between A and B are the identity maps, we get the category $\mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A)$ of Yoneda extensions, of B by A and of size n. It is additive through the Baer sum.

Any morphism of $(\mathbf{W}_r(k),G)$-modules $f : A \longrightarrow A'$ (resp. $g : B' \longrightarrow B$) induces a pushforward functor

$$f_* : \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A'),$$

resp. a pullback functor

$$g^* : \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B',A).$$

Those functors commute, in the sense that $f_* g^*$ and $g^* f_*$ are canonically isomorphic.

Let us say that two Yoneda extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ in $\mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A)$ are linked if there exists a third extension $\mathcal{E} \in \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A)$ and morphisms of n-extensions

$$\mathcal{E}_1 \longleftarrow \mathcal{E} \longrightarrow \mathcal{E}_2.$$

In our setting, this indeed defines an equivalence relation between elements of $\mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A)$, compatible with Baer sum.

DEFINITION 2.3. *We denote by* $\text{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A)$ *the Abelian group of equivalence classes of Yoneda n-extensions, in the category* $\mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A)$.

PROPOSITION 2.4. *Let* $r \in \mathbb{N}^* \cup \{\infty\}$, *and let* V *be a* $(\mathbf{W}_r(k),G)$-*module. Then, for any* $n \geq 0$, *there is a canonical isomorphism*

$$\text{YExt}^n_{(\mathbf{W}_r(k),G)}(\mathbf{W}_r(k),V) \simeq H^n(G,V).$$

Proof. Let us first deal with the case where G and r are finite.
The group $H^n(G,V)$ is the n-th derived functor of the functor

$$V \mapsto V^G = \text{Hom}_{(\mathbf{W}_r(k),G)}(\mathbf{W}_r(k),V).$$

Thus, it is nothing but the usual Ext group $\text{Ext}^n_{(\mathbf{W}_r(k),G)}(\mathbf{W}_r(k),V)$, computed using injective resolutions. But, for any Abelian category with enough injectives, the derived Ext's coincide with the Yoneda YExt's ([Ve], Ch. III, Par. 3).
The general case follows from a classical limit argument, over the finite quotients of G. $\qquad\square$

LEMMA 2.5. *Let* $r \in \mathbb{N}^* \cup \{\infty\}$ *and let* A, B *be two* $(\mathbf{W}_r(k),G)$-*modules,* B *assumed to be free as a* $\mathbf{W}_r(k)$-*module. Then, for any* $n \geq 0$, *there is a canonical isomorphism*

$$\text{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A) \xrightarrow{\sim} \text{YExt}^n_{(\mathbf{W}_r(k),G)}(\mathbf{W}_r(k),\text{Hom}_{\mathbf{W}_r(k)}(B,A)).$$

Proof. Considering the Pontryagin dual $B^\vee = \text{Hom}_{\mathbf{W}_r(k)}(B,\mathbf{W}_r(k))$, we have a canonical isomorphism

$$B^\vee \otimes A \xrightarrow{\sim} \text{Hom}_{\mathbf{W}_r(k)}(B,A).$$

The exact functor $B^\vee \otimes .$ yields a functor

$$T : \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B^\vee \otimes B, B^\vee \otimes A)$$

which maps a Yoneda n-extension

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow A_1 \longrightarrow \ldots \longrightarrow A_n \longrightarrow B \longrightarrow 0$$

to the Yoneda n-extension

$$B^\vee \otimes \mathcal{E} : 0 \longrightarrow B^\vee \otimes A \longrightarrow B^\vee \otimes A_1 \longrightarrow \ldots \longrightarrow B^\vee \otimes A_n \longrightarrow B^\vee \otimes B \longrightarrow 0.$$

But the G-equivariant map

$$I : \mathbf{W}_r(k) \longrightarrow B^\vee \otimes B = \mathrm{End}_{\mathbf{W}_r(k)}(B),$$

$$\lambda \mapsto \lambda \mathrm{Id}$$

gives a pullback functor

$$I^* : \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B^\vee \otimes B, B^\vee \otimes A) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(\mathbf{W}_r(k), \mathrm{Hom}_{\mathbf{W}_r(k)}(B,A)),$$

and the composite

$$I^* \circ T : \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_r(k),G)}(\mathbf{W}_r(k), \mathrm{Hom}_{\mathbf{W}_r(k)}(B,A))$$

gives, by passing to isomorphism classes of objects, a group homomorphism

$$\Phi : \mathrm{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A) \longrightarrow \mathrm{YExt}^n_{(\mathbf{W}_r(k),G)}(\mathbf{W}_r(k), \mathrm{Hom}_{\mathbf{W}_r(k)}(B,A)),$$

which is the desired isomorphism. Its inverse can be constructed in a similar fashion, as follows. Given a Yoneda n-extension of $(\mathbf{W}_r(k),G)$-modules

$$\mathcal{F} : 0 \longrightarrow B^\vee \otimes A \longrightarrow F_1 \longrightarrow \ldots \longrightarrow F_n \longrightarrow \mathbf{W}_r(k) \longrightarrow 0,$$

form the tensor product

$$\mathcal{F} \otimes B : 0 \longrightarrow B^\vee \otimes B \otimes A \longrightarrow F_1 \otimes B \longrightarrow \ldots \longrightarrow F_n \otimes B \longrightarrow B \longrightarrow 0.$$

Applying pushforward w.r.t. the trace

$$B^\vee \otimes B \otimes A \longrightarrow A,$$

$$\phi \otimes b \otimes a \mapsto \phi(b)a,$$

we get an n-extension

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow A_1 \longrightarrow \ldots \longrightarrow A_n \longrightarrow B \longrightarrow 0.$$

Passing to isomorphism classes of extensions, we get an arrow

$$\Psi : \mathrm{YExt}^n_{(\mathbf{W}_r(k),G)}(\mathbf{W}_r(k), \mathrm{Hom}_{\mathbf{W}_r(k)}(B,A)) \longrightarrow \mathrm{YExt}^n_{(\mathbf{W}_r(k),G)}(B,A).$$

Checking that Φ and Ψ are mutual inverses is left as an exercise for the reader- in the spirit of Morita equivalence. $\qquad\square$

3. Lifting and stable lifting

The purpose of this section is to give a precise meaning to "lifting representations". Here G is a profinite group, and k is any field of characteristic p.

Definition 3.1. *(Lifting, stable lifting).*
Let $1 \le r \le s$ be integers.
Let V_r be a $(\mathbf{W}_r(k), G)$-module, free as a $\mathbf{W}_r(k)$-module. We say that V lifts to p^s-torsion coefficients, if there exists a $(\mathbf{W}_s(k), G)$-module V_s, free as a $\mathbf{W}_s(k)$-module, such that the $(\mathbf{W}_r(k), G)$-modules V_r and $V_s \otimes_{\mathbf{W}_s(k)} \mathbf{W}_r(k)$ are isomorphic. We say that V_r stably lifts to p^s-torsion coefficients, if there exists an open subgroup $G_0 \subset G$, of prime-to-p index, such that, as a $(\mathbf{W}_r(k), G_0)$-module, V_r lifts to p^s-torsion coefficients.

The terminology "stable" is motivated by the following Lemma.

Lemma 3.2. *Let $1 \le r \le s$ be integers. Let V_r be a $(\mathbf{W}_r(k), G)$-module, free as a $\mathbf{W}_r(k)$-module. Assume that V_r stably lifts to p^s-torsion coefficients. Then, there exists a $(\mathbf{W}_r(k), G)$-module W_r, free as a $\mathbf{W}_r(k)$-module, such that $V_r \oplus W_r$ lifts to p^s-torsion coefficients.*

Proof. Let $G_0 \subset G$ be an open subgroup, of prime-to-p index, such that the $(\mathbf{W}_r(k), G_0)$-module V_r lifts to p^s-torsion coefficients. Let V_s be a $(\mathbf{W}_s(k), G_0)$-module, free as a $\mathbf{W}_s(k)$-module, lifting V_r.
Denote by $V_r^{(G/G_0)}$ the product of copies of V_r, indexed by the finite set G/G_0. It is a $(\mathbf{W}_r(k), G)$-module in a natural way, canonically isomorphic to $\mathrm{Ind}_{G_0}^G(\mathrm{Res}_{G_0}^G(V_r))$.
Consider the morphisms of $(\mathbf{W}_r(k), G)$-modules

$$V_r \xrightarrow{i} V_r^{(G/G_0)} \xrightarrow{N} V_r,$$

where i is the diagonal embedding, and N is the norm

$$N : \quad \begin{array}{ccc} V_r^{(G/G_0)} & \longrightarrow & V_r \\ (v_c)_{c \in G/G_0} & \longmapsto & \sum_{c \in G/G_0} v_c. \end{array}$$

The composite $N \circ i$ is multiplication by the index of G_0 in G, which is prime to p. The $(\mathbf{W}_r(k), G)$-module V_r is thus a direct summand of $V_r^{(G/G_0)}$, with complement $W_r := \mathrm{Ker}(N)$. But the $(\mathbf{W}_r(k), G)$-module module $V_r^{(G/G_0)}$ admits the induced module

$$\mathrm{Ind}_{G_0}^G V_s = V_s \otimes_{\mathbf{W}_s(k)[G_0]} \mathbf{W}_s(k)[G]$$

as a lift to p^s-torsion coefficients. The claim follows. $\qquad\square$

Remark 3.3. In the previous Lemma, once G_0 is known, W_r is pretty much explicit.

Lifting from mod p^r to mod p^{r+1} is an "abelian" question: there is no difference between lifting and stable lifting, as illustrated by the following Lemma.

Lemma 3.4. *Let $1 \le r$ be an integer. Let V_r be a $(\mathbf{W}_r(k), G)$-module, free as a $\mathbf{W}_r(k)$-module. Assume that there is another $(\mathbf{W}_r(k), G)$-module W_r, such that $V_r \oplus W_r$ lifts to p^{r+1}-torsion coefficients.*
Then, V_r itself lifts to p^{r+1}-torsion coefficients.

Proof. We give a constructive proof, avoiding the use of cohomological obstructions.

Denote by V_1 (resp. W_1) the reduction of V_r (resp. W_r) to a (k, G)-module. If M is a $\mathbf{W}(k)$-module, denote by

$$M^{(i)} := M \otimes_{\mathrm{Frob}^i} \mathbf{W}(k)$$

its i-th Frobenius twist. By assumption, there is a free $(\mathbf{W}_{r+1}(k), G)$-module Z_{r+1} and a short exact sequence of $(\mathbf{W}_{r+1}(k), G)$-modules

$$\mathcal{E} : 0 \longrightarrow V_1^{(r)} \oplus W_1^{(r)} = \mathrm{Ver}^r(\mathbf{W}_{r+1}(k)) Z_{r+1} \longrightarrow Z_{r+1} \longrightarrow V_r \oplus W_r \longrightarrow 0.$$

Denote by $i : V_r \longrightarrow V_r \oplus W_r$ and $\pi : W_1^{(r)} \oplus V_1^{(r)} \longrightarrow V_1^{(r)}$ the natural inclusion and projection. Form the extension of $(\mathbf{W}_{r+1}(k), G)$-modules

$$\pi^* i_*(\mathcal{E}) : 0 \longrightarrow V_1^{(r)} \longrightarrow V_{r+1} \longrightarrow V_r \longrightarrow 0,$$

which serves as the definition of V_{r+1}. Recall that the extensions $\pi^* i_*(\mathcal{E})$ and $i_* \pi^*(\mathcal{E})$ are canonically isomorphic, so that this construction does not depend on the order in which the pullback and the pushforward are applied. We claim that V_{r+1} is a lift of V_r to p^{r+1}-torsion coefficients. To see why, it suffices to justify that V_{r+1} is free, as a $\mathbf{W}_{r+1}(k)$-module. We may thus dismiss the action of G. Picking bases, we then get that V_r (resp. W_r, Z_{r+1}) is isomorphic to $\mathbf{W}_r(k)^m$ (resp. $\mathbf{W}_r(k)^n$, $\mathbf{W}_{r+1}(k)^m \oplus \mathbf{W}_{r+1}(k)^n$), and that $\mathcal{E}$ is isomorphic to

$$0 \longrightarrow k^m \oplus k^n \longrightarrow \mathbf{W}_{r+1}(k)^m \oplus \mathbf{W}_{r+1}(k)^n \longrightarrow \mathbf{W}_r(k)^m \oplus \mathbf{W}_r(k)^n \longrightarrow 0,$$

which is the direct sum of the extensions

$$\mathcal{F}_i := 0 \longrightarrow k^i \longrightarrow \mathbf{W}_{r+1}(k)^i \longrightarrow \mathbf{W}_r(k)^i \longrightarrow 0,$$

for $i = m, n$. Applying $i_* \pi^*$ to $\mathcal{E}$ yields $\mathcal{F}_m$ as a result. The claim is proved. $\square$

4. Cyclotomic modules and cyclotomic profinite groups

From now on, we fix a field k of characteristic p and a profinite group G.

In this section, we recall the notion of cyclotomic pair from [DCF1], and provide important examples.

Notation 4.1. Given two positive integers $s \leq r$ in $\mathbb{N}^* \cup \{\infty\}$ and a $(\mathbf{W}_r(k), G)$-module M_r, we put

$$M_s := M_r \otimes_{\mathbf{W}_r(k)} \mathbf{W}_s(k).$$

DEFINITION 4.2. *Let $r \in \mathbb{N}^* \cup \{\infty\}$ and $n \geq 1$ be an integer. Let*

$$f : M_r \longrightarrow N_r$$

be a morphism of $(\mathbf{W}_r(k), G)$-modules. We say that f is n-surjective if, for every open subgroup $H \subset G$, the map

$$f_* : H^n(H, M_r) \longrightarrow H^n(H, N_r)$$

is surjective.

DEFINITION 4.3. *Let $n \geq 1$ be an integer and $e \in \mathbb{N}^* \cup \{\infty\}$. Let $\mathcal{T}_{e+1}$ be a $(\mathbf{W}_{e+1}(k), G)$-module, free of rank 1 as a $\mathbf{W}_{e+1}(k)$-module. Assume that the quotient*

$$\mathcal{T}_{e+1}^{\otimes^n_{\mathbf{W}_{e+1}(k)}} \to \mathcal{T}_1^{\otimes^n_k}$$

is n-surjective.

We then say that the pair $(G, \mathcal{T})$ is (n, e)-cyclotomic, and that the profinite group G is (n, e)-cyclotomic (relatively to k).

Let $\mathcal{T}_{e+1}$ be a (n, e)-cyclotomic module. For i a non negative integer, we put

$$\mathbf{W}_{e+1}(k)(i) := \mathcal{T}_{e+1}^{\otimes_{\mathbf{W}_{e+1}(k)}^{i}}$$

and

$$\mathbf{W}_{e+1}(k)(-i) := \mathbf{W}_{e+1}(k)(i)^{\vee}.$$

For a $\mathbf{W}_{e+1}(k)$-module M, we put

$$M(i) = M \otimes_{\mathbf{W}_{e+1}(k)} \mathbf{W}_{e+1}(k)(i).$$

A cyclotomic module of depth e is given by a continuous character

$$\chi : G \longrightarrow \mathbf{W}_{e+1}(k)^{\times},$$

and provides an analogue of the cyclotomic character in Galois theory. This allows to freely to freely mimic Kummer theory, in the framework of cyclotomic pairs.

Remark 4.4. Let $(G, \mathbf{W}_{e+1}(k)(1))$ be a cyclotomic pair. If $H \subset G$ is a closed subgroup, then $(H, \mathbf{W}_{e+1}(k)(1))$ is a cyclotomic pair as well. This follows, by a standard limit argument, from the (obvious) case where $H \subset G$ is open.

Remark 4.5. For p odd, there is no non-trivial finite $(1, 1)$-cyclotomic p-group. For $p = 2$, the only non-trivial $(1, 1)$-cyclotomic finite 2-group is $\mathbb{Z}/2\mathbb{Z}$. For each $e \in \mathbb{N}^* \cup \{\infty\}$, it fits into the unique $(1, e)$-cyclotomic pair $(\mathbb{Z}/2\mathbb{Z}, \mathbf{W}_{e+1}(k)(1))$, where the non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ acts by multiplication by -1. This result is a variation around Emil Artin's theorem: the only non-trivial finite group that occurs as an absolute Galois group is $\mathbb{Z}/2\mathbb{Z}$. See [DCF], Exercise 14.27, or [QW], Proposition 6.1.

We now provide a supply of cyclotomic pairs $(G, \mathcal{T})$ arising from geometry, i.e. where G is the algebraic fundamental group of a scheme. Note that, by [Se3, Proposition 15], it is known that every finite group occurs as the algebraic fundamental group of a smooth complex variety. Using Remark 4.5, we see that not every algebraic fundamental group is cyclotomic profinite.

LEMMA 4.6. *Let $n \geq 1$ be an integer. Assume that the profinite group G is of cohomological p-dimension at most 1. Let $\mathbf{W}_{e+1}(k)(1)$ be a $(\mathbf{W}_{e+1}(k), G)$-module, free of rank 1 as a $\mathbf{W}_{e+1}(k)$-module. Then, the pair $(G, \mathbf{W}_{e+1}(k)(1))$ is (n, e)-cyclotomic.*

Proof. Let $c_1 \in H^n(G, \mathbf{W}_1(k)(n))$ be a cohomology class. Using the exact sequence

$$0 \longrightarrow \mathbf{W}_e(k)(n) \longrightarrow \mathbf{W}_{e+1}(k)(n) \longrightarrow \mathbf{W}_1(k)(n) \longrightarrow 0,$$

we see that the obstruction to lifting c_1 to a class $c_{e+1} \in H^n(G, \mathbf{W}_{e+1}(k)(n))$ lies in $H^{n+1}(G, \mathbf{W}_e(k)(n))$. This group vanishes since G is of cohomological p-dimension at most 1. The claim is proved. $\qquad\qquad\square$

PROPOSITION 4.7. *Assume that $(G, \mathbf{W}_{e+1}(k)(1))$ is an (n,e)-cyclotomic pair. Then for any surjection $\pi : M \longrightarrow N$ of $\mathbf{W}_{e+1}(k)$-modules (with trivial G-action), the induced morphism*

$$\pi(n) : M(n) \longrightarrow N(n)$$

is n-surjective.

Proof. By a limit argument, we can assume that e is finite. We then proceed by induction, on the lowest integer $m \geq 1$ such that N is a $\mathbf{W}_m(k)$-module. If $m = 1$, then N is a k-vector space. Pick a k-basis $\mathcal{B}$ for N. Consider the natural surjection

$$F : \mathbf{W}_{e+1}(k)^{(\mathcal{B})} \longrightarrow k^{(\mathcal{B})} \simeq N.$$

There exists a $\mathbf{W}_{e+1}(k)$-linear map $\rho : \mathbf{W}_{e+1}(k)^{(\mathcal{B})} \longrightarrow M$, such that $\pi \circ \rho = F$. Since $F(n)$ is n-surjective by definition of a cyclotomic module (combined to the fact that taking cohomology commutes with direct sums), we indeed conclude that $\pi(n)$ is n-surjective as well.

In general, denote by $\mathcal{M} := \mathrm{Ver}(\mathbf{W}_{e+1}(k))$ the maximal ideal of $\mathbf{W}_{e+1}(k)$. Consider the composite

$$M \xrightarrow{\pi} N \xrightarrow{q} N/\mathcal{M}N,$$

where q is the natural quotient. By what precedes, $(q \circ \pi)(n)$ is n-surjective. It suffices to prove that $\pi'(n)$ is n-surjective, where

$$\pi' : \mathcal{M}M \longrightarrow \mathcal{M}N$$

denotes the map induced by π, by a diagram chase over

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}M & \longrightarrow & M & \longrightarrow & M/\mathcal{M}M & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}N & \longrightarrow & N & \xrightarrow{q} & N/\mathcal{M}N & \longrightarrow & 0
\end{array}
$$

But as $\mathcal{M}N$ is a $\mathbf{W}_{m-1}(k)$-module, induction applies. $\qquad \square$

COROLLARY 4.8. *Let l/k be a field extension. Let $n \geq 0$ be an integer, and let $e \in \mathbb{N}^* \cup \{\infty\}$. Let $(G, \mathbf{W}_{e+1}(k)(1))$ be an (n,e)-cyclotomic pair. Set*

$$\mathbf{W}_{e+1}(l)(1) := \mathbf{W}_{e+1}(k)(1) \otimes_{\mathbf{W}_{e+1}(k)} \mathbf{W}_{e+1}(l).$$

Then, the pair $(G, \mathbf{W}_{e+1}(l)(1))$ is (n,e)-cyclotomic, relatively to l.
In short: cyclotomic pairs are preserved under field extensions of k.

Proof. The $(\mathbf{W}_{e+1}(l), G)$-module $\mathbf{W}_{e+1}(l)(1)$ is free of rank 1. As a morphism of $\mathbf{W}_{e+1}(k)$-modules, the map $\mathbf{W}_{e+1}(l) \longrightarrow l$ is surjective. It remains to apply Proposition 4.7. $\qquad \square$

Hilbert 90 theorem implies that the absolute Galois group G of a field F of characteristic $\neq p$, together with its Tate module $\mathbb{Z}_p(1)$, form a $(1, \infty)$-cyclotomic pair. This elementary fact was discussed in [DCF, Proposition 14.19], which also includes other examples of (not necessarily absolute) Galois groups.
We now provide more geometric examples of cyclotomic pairs.

PROPOSITION 4.9. *Let A be a semilocal $\mathbb{Z}[\frac{1}{p}]$-algebra. Denote by G the étale fundamental group of $S := \mathrm{Spec}(A)$ (at a given geometric point) and by $\mathbb{Z}_p(1)$ its usual Tate module. Then, the pair $(G, \mathbb{Z}_p(1))$ is $(1, \infty)$-cyclotomic (over $k = \mathbb{F}_p$).*

Proof. May assume that the semilocal ring A is connected. We work on the small étale site over S.

An open subgroup of G corresponds to the fundamental group G_U of a finite étale cover $U \longrightarrow S$. Consider for $s \geq 1$ the diagram of étale sheaves

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_{p,U} & \longrightarrow & \mu_{p^{s+1},U} & \xrightarrow{\phi} & \mu_{p^s,U} & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_{p,U} & \longrightarrow & \mathbb{G}_{m,U} & \xrightarrow{\phi'} & \mathbb{G}_{m,U} & \longrightarrow & 0
\end{array}
$$

where ϕ and ϕ' denote the p-power maps. As U is the spectrum of a semilocal ring, its Picard group is trivial by Grothendieck-Hilbert's theorem 90, and ϕ' certainly(!) induces a surjection

$$H^1_{\acute{e}t}(U, \mathbb{G}_m) \longrightarrow H^1_{\acute{e}t}(U, \mathbb{G}_m).$$

A simple diagram chase then implies that ϕ also induces a surjection

$$H^1(G_U, \mathbb{Z}/p^{s+1}(1)) \simeq H^1_{\acute{e}t}(U, \mu_{p^{s+1},U}) \longrightarrow H^1_{\acute{e}t}(U, \mu_{p^s,U}) \simeq H^1(G_U, \mathbb{Z}/p^s(1)).$$

$\square$

PROPOSITION 4.10 ([Gi, Proposition 1.6]). *Let A be a commutative ring of characteristic p. Denote by G the étale fundamental group of $S := \mathrm{Spec}(A)$ (at a given geometric point). Then G is of p-cohomological dimension ≤ 1. Thus Lemma 4.6 applies: for any $n \geq 1$ and for any $\mathbf{W}(k)(1)$, the pair $(G, \mathbf{W}(k)(1))$ is (n, ∞)-cyclotomic.*

Proof. (sketch; see [Gi, Proposition 1.6] for details)
As before, we can assume that k is $\mathbb{F}_p$, and we work in the small étale site over S. Consider the Artin-Schreier sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{\mathrm{Frob-Id}} \mathbb{G}_a \longrightarrow 0.$$

By Grothendieck-Hilbert 90 for $\mathbb{G}_a$, combined with the vanishing of coherent cohomology over an affine base, we know that $H^1_{\acute{e}t}(S, \mathbb{G}_a) = H^2_{\acute{e}t}(S, \mathbb{G}_a) = 0$. Considering the associated long sequence in étale cohomology, we get $H^2_{\acute{e}t}(S, \mathbb{Z}/p\mathbb{Z}) = 0$. Using Leray's spectral sequence, we conclude that $H^2(G, \mathbb{Z}/p\mathbb{Z}) = 0$. Similarly, $H^2(H, \mathbb{Z}/p\mathbb{Z}) = 0$ for any open subgroup H of G. The group G is therefore of cohomological p-dimension ≤ 1, and it remains to apply lemma 4.6. $\square$

PROPOSITION 4.11. *Let $S = C$ be a smooth curve over an algebraically closed field F. Denote by G its fundamental group (at a given geometric point). Set $\mathcal{T} := \mathbb{Z}_p(1)$, the usual Tate module if $p \neq 0 \in F$, or $\mathcal{T} := \mathbb{Z}_p$, the trivial module if $p = 0 \in F$. Then, the pair $(G, \mathcal{T})$ is $(1, \infty)$-cyclotomic.*

Proof. We may assume that C is connected. If F has characteristic p, one can adapt the proof of Proposition 4.10. How to do it is obvious if C is affine. If C is proper, note that one still has $H^2_{\acute{e}t}(C, \mathbb{G}_a) = 0$, and that $(\mathrm{Frob} - \mathrm{Id})$ induces a surjection on the finite-dimensional F-vector space $H^1_{\acute{e}t}(C, \mathbb{G}_a)$ (see [Bh, Lemma 0.5]). A similar proof then goes through.
We thus assume that F has characteristic not p, and work over the small étale site over C. First assume that C is proper and consider a connected finite étale cover

$U \longrightarrow C$, given by an open subgroup H of G. The curve U is then smooth and proper, as well. Write the short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(U) \longrightarrow \mathrm{Pic}(U) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

The abelian group $\mathrm{Pic}^0(U)$ consists of the F-rational points of the Jacobian $\mathrm{Jac}_F(U)$. It is hence divisible, since F is algebraically closed.

Note that for any $s \geq 1$, the natural map $H^1_{\acute{e}t}(U, \mu_{p^s,U}) \longrightarrow H^1_{\acute{e}t}(U, \mathbb{G}_{m,U})$ lands in $\mathrm{Pic}^0(U)$. Considering the same diagram as in the proof of Proposition 4.9, we see that the image of any class of $H^1_{\acute{e}t}(U, \mu_{p^s,U})$ in $H^1_{\acute{e}t}(U, \mathbb{G}_{m,U})$ lies in the image of the endomorphism of $H^1_{\acute{e}t}(U, \mathbb{G}_{m,U})$ induced by ϕ' (which is multiplication by p). The map

$$H^1(H, \mathbb{Z}/p^{s+1}\mathbb{Z}(1)) = H^1_{\acute{e}t}(U, \mu_{p^{s+1},U}) \longrightarrow H^1_{\acute{e}t}(U, \mu_{p^s,U}) = H^1(H, \mathbb{Z}/p^s\mathbb{Z}(1))$$

induced by ϕ is thus also surjective. Therefore, the group G is $(1, \infty)$-cyclotomic, with cyclotomic character the Tate module $\mathbb{Z}_p(1)$.

We now deal with the case of a non-proper (i.e. affine) smooth connected curve C over F. Denote by $\tilde{C}$ the smooth proper curve containing C, and by x a closed point in $\tilde{C} \setminus C$. Adjusting by multiples of $[x] \in \mathrm{Pic}(\tilde{C})$, one easily sees that the restriction morphism $\mathrm{Pic}^0(\tilde{C}) \longrightarrow \mathrm{Pic}(C)$ is surjective, hence that the abelian group $\mathrm{Pic}(C)$ is divisible. The same holds for any étale cover of C, and we can conclude as before. $\qquad\square$

The previous result actually extends to higher dimensional varieties, as follows.

PROPOSITION 4.12. *Let S be a smooth projective variety, over an algebraically closed field F, of characteristic $\neq p$. Denote by G the fundamental group of S, at a given geometric point.*
Assume that, for every finite étale cover $U \longrightarrow S$, the Néron-Severi group of U has no p-torsion. Then, the pair $(G, \mathbb{Z}_p(1))$ is $(1, \infty)$-cyclotomic.
More generally, let $e \geq 1$ be an integer. Assume that, for every finite étale cover $U \longrightarrow S$, the p-primary part of the Néron-Severi group of U is isomorphic to a product $\prod_{i=1}^n \mathbb{Z}/p^{r_i}\mathbb{Z}$, with $r_i > e$ for $i = 1, \ldots, n$.
Then, the pair $(G, \mathbb{Z}/p^{1+e}(1))$ is $(1, e)$-cyclotomic.

Proof. It suffices to prove the second statement. Consider the exact sequence (of Abelian groups)

$$0 \longrightarrow \mathrm{Pic}^0(U) \longrightarrow \mathrm{Pic}(U) \longrightarrow \mathrm{NS}(U) \longrightarrow 0.$$

The group $\mathrm{Pic}^0(U)$ is divisible (as the group of F-points of an Abelian variety). From the hypothesis made on $\mathrm{NS}(U)$, we deduce that the map

$$\begin{array}{ccc} \mathrm{Pic}(U)[p^{r+1}] & \longrightarrow & \mathrm{Pic}(U)[p] \\ x & \mapsto & p^r x \end{array}$$

is onto, using an elementary diagram chase left to the reader. This holds for every finite étale cover U/S. The rest of the proof is the same as in Proposition 4.11. $\square$

Remark 4.13. Construction of new examples of surfaces satisfying hypothesis of Proposition 4.12 is an interesting problem. A good starting point may be Kodaira fibered surfaces.

5. Lifting representations of cyclotomic profinite groups

The following is a reformulation of Definition 2.2, for free $(\mathbf{W}_r(k), G)$-modules.

DEFINITION 5.1. *Let G be a profinite group, p be a prime, k be a field of characteristic p and $r \geq 1$ be an integer. A representation*

$$G \longrightarrow \mathrm{GL}_d(\mathbf{W}_r(k))$$

is continuous if its kernel is open in G.
A continuous representation

$$\rho_\infty : G \longrightarrow \mathrm{GL}_\infty(\mathbf{W}(k))$$

is a compatible data, for all $r \in \mathbb{N}^*$, of continuous representations

$$\rho_r : G \longrightarrow \mathrm{GL}_r(\mathbf{W}_d(k)).$$

The compatiblity condition simply means that ρ_{r+1} reduces to ρ_r, for all r.

We now prove Theorem 5.2, providing a sufficient condition for lifting continuous representations of cyclotomic profinite groups.

THEOREM 5.2. *Let k be a field of characteristic p, and $e \in \mathbb{N}^* \cup \{\infty\}$. Let G be a $(1, e)$-cyclotomic profinite group relatively to k, and let V_1 be a (k, G)-module. Assume that there is an open subgroup $G_0 \subset G$, of prime-to-p index, two permutation (k, G_0)-modules A and B, and a short exact sequence of (k, G_0)-modules*

$$0 \longrightarrow A \longrightarrow V_1 \longrightarrow B \longrightarrow 0.$$

Then, V_1 stably lifts to p^{e+1}-torsion coefficients.

Furthermore, V_1 itself lifts to p^2-torsion coefficients.

Proof. We show the first statement of the theorem. The second then follows from Lemma 3.4. Let $\mathbf{W}_{1+e}(1)$ be a $(1, e)$-cyclotomic module, relatively to k and G. We may replace G_0 by its intersection with the kernel of the character $G \longrightarrow k^\times$ giving the action of G on $\mathbf{W}_1(k)(1)$, which has index prime-to-p as well. We can thus assume that $\mathbf{W}_1(k)(1) \simeq k$ has the trivial G_0-action. The Yoneda extension

$$0 \longrightarrow A \longrightarrow V_1 \longrightarrow B \longrightarrow 0,$$

corresponds to a cohomology class

$$\mathcal{E} \in H^1(G_0, \mathrm{Hom}(B, A)).$$

Fixing two respective G_0-bases Y and X of A and B, we have a G_0-equivariant isomorphism

$$\mathrm{Hom}(B, A) \simeq \mathrm{Hom}(k^X, k^Y) \simeq k^{X \times Y},$$

through which the class $\mathcal{E}$ is given by an element of $H^1(G_0, k^{X \times Y})$. The G_0-set $X \times Y$ decomposes as a disjoint union

$$X \times Y = \bigsqcup_{i \in \mathcal{I}} G_0/G_i$$

of G_0-orbits, where all G_i's are open in G_0. Shapiro's lemma yields an isomorphism

$$H^1(G_0, k^{X \times Y}) \simeq \bigoplus_{i \in \mathcal{I}} H^1(G_i, k)$$

Deciphering the definition of a $(1, e)$-cyclotomic pair and the comparison lemma 2.4, we get a surjection

$$H^1(G_i, \mathbf{W}_{1+e}(1)) \longrightarrow H^1(G_i, k);$$

that is to say,

$$\mathrm{YExt}^1_{(\mathbf{W}_{e+1}(k),G_i)}(\mathbf{W}_{e+1}(k),\mathbf{W}_{1+e}(1)) \longrightarrow \mathrm{YExt}^1_{(k,G_i)}(k,k),$$

for all $i \in \mathcal{I}$. Hence, the natural map

$$\mathrm{YExt}^1_{(\mathbf{W}_{e+1}(k),G_0)}(\mathbf{W}_{e+1}(k)^X,\mathbf{W}_{1+e}(1)^Y) \longrightarrow \mathrm{YExt}^1_{(k,G_0)}(k^X,k^Y)$$

is surjective. As a consequence, V_1 fits into a commutative diagram of $(\mathbf{W}_{e+1}(k),G_0)$-modules

(1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{W}_{1+e}(1)^Y & \longrightarrow & V_{e+1} & \longrightarrow & \mathbf{W}_{e+1}(k)^X & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k^Y & \longrightarrow & V_1 & \longrightarrow & k^X & \longrightarrow & 0,
\end{array}$$

which serves as the definition of V_{e+1}. Note that the vertical arrows are the natural reductions.

The cyclotomic module $\mathbf{W}_{1+e}(k)(1)$ is a free $\mathbf{W}_{e+1}(k)$-module. Hence so is V_{e+1}-yielding a stable lifting of V_1, to p^{e+1}-torsion coefficients. □

6. Applications to Galois representations and local systems

In this section we provide applications to Theorem 5.2 to lifting Galois representations and local systems.

THEOREM 6.1. *Let k be a field of characteristic p, $e \in \mathbb{N}^* \cup \{\infty\}$. Let G be a $(1,e)$-cyclotomic profinite group. Let*

$$\rho : G \to \mathrm{GL}_2(k)$$

be a continuous representation. Then, ρ lifts to p^2-torsion coefficients.

Furthermore, ρ stably lifts to p^{e+1}-torsion coefficients.

If $k = \mathbb{F}_2$, these results also hold for representations of G of dimension up to 4.

Proof. Let V_1 be a 2-dimensional (k,G)-module. There is a line $L_1 \subset V_1$ fixed by a pro-p-Sylow G_p of G [Se2, 8.3]. The stabilizer $H = \mathrm{Stab}_{G_p}(L_1)$ is thus of prime-to-p index in G, and we get a short exact sequence of (k,H)-modules

$$0 \longrightarrow L_1 \longrightarrow V_1 \longrightarrow V_1/L_1 \longrightarrow 0.$$

As in the previous proofs, we can consider an open subgroup G_0 of G, of prime-to-p index, for which the two characters giving the action on L_1 and on V_1/L_1 are trivial. We can now apply Theorem 5.2.

We now assume that $k = \mathbb{F}_2$ and that V_1 is a 4-dimensional $(\mathbb{F}_2,G)$-module. Again, there is a plane $P_1 \subset V_1$ stabilized by an open subgroup G_0 of G, of odd index. The continuous representation V_1 fits into a short exact sequence of $(\mathbb{F}_2,G_0)$-modules

$$0 \longrightarrow P_1 \longrightarrow V_1 \longrightarrow V_1/P_1 \longrightarrow 0.$$

Replacing G_0 by an open subgroup of odd index, we can moreover assume that the 2-dimensional $(\mathbb{F}_2,G_0)$-modules P_1 and V_1/P_1 both admit an $\mathbb{F}_2$-basis permuted by G_0. It remains, once more, to invoke Theorem 5.2. □

Remark 6.2. Theorem 6.1 applies, in particular, to profinite groups of the types a), b), c) and d) given in the Introduction. In particular, it applies to Galois representations. To conclude, we offer another application below.

COROLLARY 6.3 (Zariski-local lifting of local systems of low dimension). .
Let k be a field of characteristic p and let S be a scheme, defined either over $\mathbb{F}_p$
or $\mathbb{Z}[\frac{1}{p}]$. Let $\mathcal{L}$ be a local system over S with coefficients in k, of dimension 2.
(Equivalently, $\mathcal{L}$ is given by a representation $\pi_1(S) \longrightarrow \mathrm{GL}_2(k)$)
Then, Zariski-locally on S, $\mathcal{L}$ lifts to a local system with coefficients in $\mathbf{W}_2(k)$.
Furthermore, Zariski-locally on S, $\mathcal{L}$ stably lifts to a local system with coefficients
in $\mathbf{W}(k)$.

If $k = \mathbb{F}_2$, the same result holds, for local systems of dimension up to 4.

Proof. By Proposition 4.9, combined with Theorem 6.1, we know that, for each
point $s \in S$, the stalk of $\mathcal{L}$ at s (which is a local system over $\mathrm{Spec}(\mathcal{O}_{S,s})$) lifts as
stated. To conclude, use the fact that any finite étale cover of $\mathrm{Spec}(\mathcal{O}_{S,s})$ extends
to an open $U \subset S$ containing s. $\qquad\square$

Remark 6.4. We did not attempt to make the assumptions on S optimal. For
instance, the result clearly extends to schemes S where p is nilpotent.

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