Lifting low-dimensional local systems

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Abstract. Let k be a field of characteristic $p > 0$. Denote by $\mathbf{W}_d(k)$ the ring of truntacted Witt vectors of length $d \geq 2$, built out of k. In this text, we consider the following problem.

Let G be a profinite group. Find conditions on G, ensuring that every (continuous) representation $G \longrightarrow \mathrm{GL}_n(k)$ lifts to a representation $G \longrightarrow \mathrm{GL}_n(\mathbf{W}_d(k))$.

Following [DCF], we introduce the class of smooth profinite groups (Definition 3.4). Using Grothendieck-Hilbert' theorem 90, we show that the algebraic fundamental groups of the following schemes are smooth profinite groups: spectra of semilocal rings over $\mathbb{Z}[\frac{1}{p}]$, smooth curves over algebraically closed fields, and affine schemes over $\mathbb{F}_p$. In particular, absolute Galois groups of fields are smooth. We then give a positive partial answer to the question above, for smooth profinite groups. Concretely, we show that any 2-dimensional representation of a smooth profinite group lifts to p^2-torsion, and stably lifts to arbitrary torsion (see Theorem 5.1). When $p = 2$ and $k = \mathbb{F}_2$, we prove the same result up to dimension $n = 4$.
We then give a concrete application to algebraic geometry: we prove that local systems of low dimension lift Zariski-locally (see Corollary 5.3).

Contents

1. Introduction

Let k be a field of characteristic p and let G be a profinite group. The present paper deals with the deformation theory of continuous representations

$$\rho : G \longrightarrow \mathrm{GL}_n(k),$$

more precisely with the existence of liftings of such representations to higher torsion, and ultimately with coefficients in the ring of Witt vectors built out of k. A fundamental case of this study is given by Galois representations. Existence of such lifts has been extensively investigated, in the case of absolute Galois groups of local and global fields. In [K], Khare thus proves the existence of lifts to $\mathbf{W}(k)$,

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for 2-dimensional reducible representations, in the case where G is the absolute Galois group of a number field, and where k is a finite field. If G is the absolute Galois groups of $\mathbb{Q}$, and under mild assumptions, such lifts also exist by Ramakrishna [R1,"R2] and Hamblen [H]. The existence of p^2-torsion lifts also holds for absolute Galois groups of function fields of curves over number fields, by [BK]. In a private communication to the second author of this paper, Serre proves that any Galois representation $G \longrightarrow \mathrm{GL}_2(\mathbb{F}_p)$ lifts to $G \longrightarrow \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$, when G is the absolute Galois group of an *arbitrary* field F.

In this paper, we study the deformation (or more accurately, the liftability) of continuous representations of *smooth profinite groups*. First introduced in [DCF], smooth profinite groups come equipped with a so-called cyclotomic module (see Definition 3.3), which will play the role of the cyclotomic character in Galois cohomology.
We first give important examples of smooth profinite groups, among algebraic fundamental groups, using Kummer and Artin-Schreier theory.
More precisely, in Propositions 3.9, 3.10 and 3.11, we show that the following groups are $(1,\infty)$-smooth, relatively to any field k of characteristic p :

a) The fundamental group of a semilocal $\mathbb{Z}[\frac{1}{p}]$-scheme;
b) The fundamental group of an affine $\mathbb{F}_p$-scheme;
c) The fundamental group of a smooth curve, over an algebraically closed field.

In particular, absolute Galois groups of fields are smooth profinite groups.
Note that, in all cases, the cyclotomic module is taken to be the Tate module of roots of unity $\mathbb{Z}_p(1)$ when p is invertible, or the trivial module $\mathbb{Z}_p$ when $p=0$.

The main (perhaps misleadingly formal) result of this paper is then the following.

Theorem 1.1. *Let k be a field of characteristic p, and $d \in \mathbb{N}^* \cup \{\infty\}$. Let G be a $(1,d)$-smooth profinite group relatively to k, and let V be a (k,G)-module.*
Assume that there is an open subgroup $G_0 \subset G$, of prime-to-p index, two permutation (k,G_0)-modules A and B, and a short exact sequence of (k,G_0)-modules

$$0 \longrightarrow A \longrightarrow V \longrightarrow B \longrightarrow 0.$$

Then, there exists a (k,G)-module W, such that $V \oplus W$ lifts to p^{d+1}-torsion.

Furthermore, V itself lifts to p^2-torsion.

Note that, by "a (k,G)-module Z lifts to p^{d+1}-torsion", we mean here that "Z is the reduction of a $(\mathbf{W}_{d+1}(k),G)$-module, free as a $\mathbf{W}_{d+1}(k)$-module"- see section 2 for details.

In Theorem 5.1, we deduce from Theorem 1.1 the following result. Continuous representations of dimension 2 of a smooth profinite group (e.g. of type a), b) or c) above), with values in a field k of characteristic $p>0$, lift to p^2-torsion. Up to direct sum, they actually lift to arbitrary torsion.
Note that, for $p=2$ and $k=\mathbb{F}_2$, we prove that this result holds for representations of dimension *up to* 4.

The paper is structured as follows. In section 2, we recall the technical machinery of smooth profinite groups introduced in [DCF], and that of Yoneda extensions- which is a convenient computational tool in our proofs. We then prove the main

theorem in section 3 and deal with its consequences in section 4.
Note that this text, though adressing the same topics, does not rely on [DCF].

2. Modules over Witt vectors and Yoneda extensions

Fixing a field k of characteristic $p > 0$, we consider the ring $\mathbf{W}(k)$ of Witt vectors built out of k. Recall that if k is perfect, $\mathbf{W}(k)$ is the unique complete discrete valuation ring of characteristic 0 whose uniformizer is p and residue field is k. We shall also consider the truncated Witt vectors of size $d \geq 1$, defined by the quotient

$$\mathbf{W}_d(k) := \mathbf{W}(k)/\mathrm{Ver}^d(\mathbf{W}(k)),$$

where $\mathrm{Ver} : \mathbf{W}(k) \longrightarrow \mathbf{W}(k)$ denotes the Verschiebung endomorphism. We set $\mathbf{W}_\infty(k) := \mathbf{W}(k)$. Note that if k is perfect, we have $\mathbf{W}_d(k) = \mathbf{W}(k)/p^d\mathbf{W}(k)$.

Definition 2.1. *Let $d \in \mathbb{N}^* \cup \{\infty\}$. Let M be a $\mathbf{W}_d(k)$-module of finite type. We endow it with the topology having the submodules $\mathrm{Ver}^i(\mathbf{W}(k))M$, $i \in \mathbb{N}$, as a basis for open neighborhoods of 0. This is simply the discrete topology if $d < \infty$. Note also that, if k is perfect, this defines the p-adic topology on M.*

Definition 2.2. *Let G be a profinite group, and let $d \in \mathbb{N}^* \cup \{\infty\}$.*

A $(\mathbf{W}_d(k), G)$-module is a $\mathbf{W}_d(k)$-module of finite type, endowed with a continuous $\mathbf{W}_d(k)$-linear action of G.
In particular, for $d = 1$, a (k, G)-module is a finite-dimensional k-vector space endowed with an action of G, that factors through an open subgroup of G.

In the sequel, a $(\mathbf{W}_d(k), G)$-module which is free as a $\mathbf{W}_d(k)$-module will be called a free $(\mathbf{W}_d(k), G)$-module.

If G is a profinite group and V is a G-module, we will denote by $H^n(G, V)$ the n-th cohomology group of G, with values in V. If V is equipped with the discrete topology, it is taken in the sense of [Se]. Otherwise (e.g. when $V = \mathbb{Z}_p(n)$), it is in the sense of Tate's continuous cohomology. In the context of the present paper, laying too much stress on continuity issues would, we believe, be smoke and mirrors.
The abelian categories $\mathcal{M}(\mathbf{W}_d(k), G)$ (resp. $\mathcal{M}(k, G)$) of $(\mathbf{W}_d(k), G)$-modules (resp. (k, G)-modules) are monoidal through the tensor product. For any positive integer n and $A, B \in \mathcal{M}(\mathbf{W}_d(k), G)$, one can then define the notion of Yoneda n-extensions of A by B, as follows. First, define $\mathrm{YExt}^0_{(\mathbf{W}_d(k),G)}(A, B)$ as $\mathrm{Hom}_{(\mathbf{W}_d(k),G)}(A, B)$.

A n-extension of A by B is an exact sequence of $(\mathbf{W}_d(k), G)$-modules

$$\mathcal{E} : 0 \longrightarrow B \longrightarrow A_1 \longrightarrow \ldots \longrightarrow A_n \longrightarrow A \longrightarrow 0.$$

Setting morphisms $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$ between two n-extensions of A by B to be morphisms of complexes for which the induced morphisms between A and B are the identity maps, we get the category $\mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A, B)$ of Yoneda extensions between A and B of size n, which is additive through the Baer sum.

Any morphism of $(\mathbf{W}_d(k), G)$-modules $f : B \longrightarrow B'$ (resp. $g : A' \longrightarrow A$) induces a pushforward functor

$$f_* : \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A, B) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A, B'),$$

resp. a pullback functor

$$g^* : \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A, B) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A', B).$$

Those functors commute, in the sense that f_*g^* and g^*f_* are canonically isomorphic.

Let us say that two Yoneda extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ in $\mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B)$ are linked if there exists a third extension $\mathcal{E} \in \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B)$ and morphisms of n-extensions

$$\mathcal{E}_1 \longleftarrow \mathcal{E} \longrightarrow \mathcal{E}_2.$$

In our setting, this indeed defines an equivalence relation between elements of $\mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B)$, compatible with Baer sum.

Definition 2.3. *We denote by* $\mathrm{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B)$ *the Abelian group of equivalence classes of Yoneda n-extensions, in the category* $\mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B)$.

Proposition 2.4. *Let $d \in \mathbb{N}^* \cup \{\infty\}$, and let V be a $(\mathbf{W}_d(k),G)$-module. Then, for any $n \geq 0$, there is a canonical isomorphism*

$$\mathrm{YExt}^n_{(\mathbf{W}_d(k),G)}(\mathbf{W}_d(k),V) \simeq H^n(G,V).$$

Proof. Let us first deal with the case where G and d are finite.
The group $H^n(G,V)$ is the n-th derived functor of the functor

$$V \mapsto V^G = \mathrm{Hom}_{(\mathbf{W}_d(k),G)}(\mathbf{W}_d(k),V).$$

Thus, it is nothing but the usual Ext group $\mathrm{Ext}^n_{(\mathbf{W}_d(k),G)}(\mathbf{W}_d(k),V)$, computed using injective resolutions. But, for any Abelian category with enough injectives, the derived Ext's coincide with the Yoneda YExt's ([Ve], Ch. III, Par. 3).
The general case follows from a classical limit argument, over the finite quotients of G. □

Lemma 2.5. *Let $d \in \mathbb{N}^* \cup \{\infty\}$ and let A, B be two $(\mathbf{W}_d(k),G)$-modules, A assumed to be free. Then, for any $n \geq 0$, there is a canonical isomorphism*

$$\mathrm{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B) \xrightarrow{\sim} \mathrm{YExt}^n_{(\mathbf{W}_d(k),G)}(\mathbf{W}_d(k),\mathrm{Hom}_{\mathbf{W}_d(k)}(A,B)).$$

Proof. Considering the Pontryagin dual $A^\vee = \mathrm{Hom}_{\mathbf{W}(k)}(A,\mathbf{W}_d(k))$, we have a canonical isomorphism

$$A^\vee \otimes B \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{W}_d(k)}(A,B).$$

The functor $A^\vee \otimes .$ yields a functor

$$\Phi : \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A^\vee \otimes A, A^\vee \otimes B)$$

which maps a Yoneda n-extension

$$(\mathcal{E} : 0 \longrightarrow B \longrightarrow A_1 \longrightarrow \ldots \longrightarrow A_n \longrightarrow A \longrightarrow 0)$$

to the Yoneda n-extension

$$(A^\vee \otimes \mathcal{E} : 0 \longrightarrow A^\vee \otimes B \longrightarrow A^\vee \otimes A_1 \longrightarrow \ldots \longrightarrow A^\vee \otimes A_n \longrightarrow A^\vee \otimes A \longrightarrow 0).$$

But the G-equivariant map

$$\Psi : \mathbf{W}_d(k) \longrightarrow A^\vee \otimes A = \mathrm{End}_{\mathbf{W}_d(k)}(A),$$
$$\lambda \mapsto \lambda \mathrm{Id}$$

gives a pullback functor

$$\Psi^* : \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A^\vee \otimes A, A^\vee \otimes B) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(\mathbf{W}_d(k),\mathrm{Hom}_{\mathbf{W}_d(k)}(A,B)),$$

and the composite

$$\Psi^* \circ \Phi : \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B) \longrightarrow \mathbf{YExt}^n_{(\mathbf{W}_d(k),G)}(\mathbf{W}_d(k),\mathrm{Hom}_{\mathbf{W}_d(k)}(A,B))$$

gives, by passing to isomorphism classes of objects, a group homomorphism

$$\mathrm{YExt}^n_{(\mathbf{W}_d(k),G)}(A,B) \longrightarrow \mathrm{YExt}^n_{(\mathbf{W}_d(k),G)}(\mathbf{W}_d(k), \mathrm{Hom}_{\mathbf{W}_d(k)}(A,B)),$$

which is the desired isomorphism. □

3. Cyclotomic modules and smooth profinite groups

From now on, we fix a field k of characteristic p and a profinite group G.

In this section, we recall the notion of cyclotomic modules and of smooth profinite groups from [DCF], and provide important examples.

Notation 3.1. Given two positive integers $m \leq n$ in $\mathbb{N}^* \cup \{\infty\}$ and a $(\mathbf{W}_n(k), G)$-module $\mathcal{M}$, we put

$$\mathcal{M}\{m\} := \mathcal{M} \otimes_{\mathbf{W}_n(k)} \mathbf{W}_m(k).$$

Definition 3.2. *Let $d \in \mathbb{N}^* \cup \{\infty\}$ and $n \geq 1$ be an integer. Let*

$$f : M \longrightarrow N$$

be a morphism of $(\mathbf{W}_d(k), G)$-modules. We say that f is n-surjective if, for every open subgroup $H \subset G$, the map

$$f_* : H^n(H, M) \longrightarrow H^n(H, N)$$

is surjective.

Definition 3.3. *Let $n \geq 1$ be an integer and $d \in \mathbb{N}^* \cup \{\infty\}$. An n-cyclotomic module $\mathcal{T}$ of depth d (relatively to G and k) is a free $(\mathbf{W}_{d+1}(k), G)$-module of rank 1 (as a $\mathbf{W}_{d+1}(k)$-module) such that the quotient*

$$\otimes^n_{\mathbf{W}_{d+1}(k)} \mathcal{T} \to (\otimes^n_{\mathbf{W}_{d+1}(k)} \mathcal{T})\{1\}$$

is n-surjective.

Let $\mathcal{T}$ be a n-cyclotomic G-module, of depth d. For i a non negative integer, we put

$$\mathbf{W}_{d+1}(k)(i) = \otimes^i_{\mathbf{W}_{d+1}(k)} \mathcal{T},$$

and for any $\mathbf{W}_{d+1}(k)$-module M, we put

$$M(i) = \mathbf{W}_{d+1}(k)(i) \otimes_{\mathbf{W}_{d+1}(k)} M.$$

A cyclotomic module of depth d is given by a continuous character

$$\chi : G \longrightarrow \mathbf{W}_{d+1}(k)^\times,$$

and provides an analogue of the cyclotomic character in Galois theory. This allows to freely to freely mimick Kummer theory, in the framework of smooth profinite groups.

Definition 3.4. *Let n be a positive integer, and let $d \in \mathbb{N}^* \cup \{\infty\}$.*
A profinite group G is (n,d)-smooth, relatively to the field k, if there is an n-cyclotomic module of depth d (relatively to G and k).

Remark 3.5. A closed subgroup of a smooth profinite group is smooth as well.
For p odd, there is no nontrivial $(1,1)$-smooth finite group. For $p = 2$, the only nontrivial $(1,1)$-smooth finite group is $\mathbb{Z}/2\mathbb{Z}$ (see [DCF], Exercise 14.27).

The following provides a first class of smooth profinite groups.

Lemma 3.6. *Let $n \geq 1$ be an integer. Assume that the profinite group G is of cohomological p-dimension at most 1. Then, any free $(\mathbf{W}(k), G)$-module of rank 1 is n-cyclotomic. In particular, G is (n, ∞)-smooth (relatively to any k).*

Proof. See [Se], paragraph 3.4. □

Proposition 3.7. *Assume that G is (n, d)-smooth relatively to k. Fix an n-cyclotomic module $\mathbf{W}_{d+1}(k)(1)$ of depth d, relatively to G and k.*
Then for any surjection $\pi : M \longrightarrow N$ of $\mathbf{W}_{d+1}(k)$-modules (with trivial G-action), the induced morphism

$$\pi(n) : M(n) \longrightarrow N(n)$$

is n-surjective.

Proof. By a limit argument, we can assume that d is finite. We then proceed by induction, on the lowest integer $m \geq 1$ such that N is a $\mathbf{W}_m(k)$-module. If $m = 1$, then N is a k-vector space. Pick a k-basis $\mathcal{B}$ for N. Consider the natural surjection

$$F : \mathbf{W}_{d+1}(k)^{(\mathcal{B})} \longrightarrow k^{(\mathcal{B})} \simeq N.$$

There exists a $\mathbf{W}_{d+1}(k)$-linear map $\rho : \mathbf{W}_{d+1}(k)^{(\mathcal{B})} \longrightarrow M$, such that $\pi \circ \rho = F$. Since $F(n)$ is n-surjective by definition of a cyclotomic module (combined to the fact that taking cohomology commutes with direct sums), we indeed conclude that $\pi(n)$ is n-surjective as well.
In general, denote by $\mathcal{M} := \mathrm{Ver}(\mathbf{W}_{d+1}(k))$ the maximal ideal of $\mathbf{W}_{d+1}(k)$. Consider the composite

$$M \xrightarrow{\pi} N \xrightarrow{q} N/\mathcal{M}N,$$

where q is the natural quotient. By what precedes, $(q \circ \pi)(n)$ is n-surjective. By a diagram chase left to the reader, it thus suffices to prove that $\pi'(n)$ is n-surjective, where

$$\pi' : \mathcal{M}M \longrightarrow \mathcal{M}N$$

denotes the map induced by π. But $\mathcal{M}N$ is a $\mathbf{W}_{m-1}(k)$-module, so that induction applies. □

Corollary 3.8. *Let l/k be a field extension. Let $n \geq 0$ be an integer, and let $d \in \mathbb{N}^* \cup \{\infty\}$. Let $\mathcal{T}$ be a n-cyclotomic module of depth d, relatively to G and k. Then $\mathcal{T}' := \mathcal{T} \otimes_{\mathbf{W}_{d+1}(k)} \mathbf{W}_{d+1}(l)$ is a n-cyclotomic module of depth d, relatively to G and l.*
In particular, (n, d)-smoothness of G is preserved under field extensions of k.

Proof. The $(\mathbf{W}_{d+1}(l), G)$-module $\mathcal{T}'$ is free of rank 1. As a morphism of $\mathbf{W}_{d+1}(k)$-modules, the map $\mathcal{T}' \longrightarrow \mathcal{T}'\{1\}$ is surjective. It remains to apply Proposition 3.7. □

Hilbert 90 theorem implies that absolute Galois groups are $(1, \infty)$-smooth. This elementary fact was discussed in [DCF, Proposition 14.19], which also includes other examples of Galois groups satisfying the smoothness condition.
We now provide more geometric examples of smooth profinite groups.

Proposition 3.9. *Let A be a semilocal $\mathbb{Z}[\frac{1}{p}]$-algebra. Then, "the" étale fundamental group of $\mathrm{Spec}(A)$ is $(1, \infty)$-smooth, relatively to any field k of characteristic p.*

Proof. May assume that the coefficient field k is $\mathbb{F}_p$ by lemma 3.8, and that the semilocal ring A is connected. We work on the small étale site over $S = \mathrm{Spec}(A)$. Fixing a geometric point $\bar{x}$ of S, denote by G the fundamental group $\pi(S;\bar{x})$ and by $\mathbb{Z}_p(1)$ the Tate module (of roots of unity).

An open subgroup of $\pi(S;\bar{x})$ corresponds to the fundamental group G_U of a finite étale cover $U \longrightarrow S$. Consider for $s \geq 1$ the diagram of étale sheaves

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_{p,U} & \longrightarrow & \mu_{p^{s+1},U} & \xrightarrow{\phi} & \mu_{p^s,U} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_{p,U} & \longrightarrow & \mathbb{G}_{m,U} & \xrightarrow{\phi'} & \mathbb{G}_{m,U} & \longrightarrow & 0
\end{array}$$

where ϕ and ϕ' denote the p-power maps. As U is the spectrum of a semilocal ring, its Picard group is trivial by Grothendieck-Hilbert's theorem 90, and ϕ' certainly(!) induces a surjection

$$H^1_{\acute{e}t}(U, \mathbb{G}_m) \longrightarrow H^1_{\acute{e}t}(U, \mathbb{G}_m).$$

A simple diagram chase then implies that ϕ also induces a surjection

$$H^1(G_U, \mathbb{Z}_p(1)\{s+1\}) \simeq H^1_{\acute{e}t}(U, \mu_{p^{s+1},U}) \longrightarrow H^1_{\acute{e}t}(U, \mu_{p^s,U}) \simeq H^1(G_U, \mathbb{Z}_p(1)\{s\}).$$

□

PROPOSITION 3.10. *(* [Gi, Proposition 1.6]*) Let A be a commutative ring of characteristic p. Denote by G "the" étale fundamental group of A. Then G is of p-cohomological dimension ≤ 1. In particular, it is $(1,\infty)$-smooth (relatively to any k).*

Proof. (sketch; see [Gi, Proposition 1.6] for details)
As before, we can assume that k is $\mathbb{F}_p$, we denote by S the spectrum of A and we work in the small étale site over S. Let G be "the" étale fundamental group of A. Consider the Artin-Schreier sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \overset{\mathrm{Frob}-\mathrm{Id}}{\longrightarrow} \mathbb{G}_a \longrightarrow 0.$$

By Grothendieck-Hilbert 90 for $\mathbb{G}_a$, combined with the vanishing of coherent cohomology over an affine base, we know that $H^1_{\acute{e}t}(S, \mathbb{G}_a) = H^2_{\acute{e}t}(S, \mathbb{G}_a) = 0$. Considering the associated long sequence in étale cohomology, we get $H^2_{\acute{e}t}(S, \mathbb{Z}/p\mathbb{Z}) = 0$. Using Leray's spectral sequence, we conclude that $H^2(G, \mathbb{Z}/p\mathbb{Z}) = 0$. Similarly, $H^2(H, \mathbb{Z}/p\mathbb{Z}) = 0$ for any open subgroup H of G. The group G is therefore of cohomological p-dimension ≤ 1, and it remains to apply lemma 3.6. □

PROPOSITION 3.11. *Let C be a smooth curve over an algebraically closed field F. Denote by G "its" fundamental group. Then G is $(1,\infty)$-smooth, relatively to any field of characteristic p.*

Proof. Again, we may assume $k = \mathbb{F}_p$, and C connected. If F has characteristic p, one can adapt the proof of Proposition 3.10. How to do it is obvious if C is affine. If C is proper, note that one still has $H^2_{\acute{e}t}(C, \mathbb{G}_a) = 0$, and that $(\mathrm{Frob} - \mathrm{Id})$ induces a surjection on the finite-dimensional F-vector space $H^1_{\acute{e}t}(C, \mathbb{G}_a)$ (see [Bh], Lemma 0.5). A similar proof then goes through.
We thus assume that F has characteristic $\ell \neq p$, and work over the small étale site over C. First assume that C is proper and consider a connected finite étale cover

$U \longrightarrow C$, given by an open subgroup H of G. The curve U is then smooth and proper, as well. Write the short exact sequence

$$0 \longrightarrow \mathrm{Pic}_0(U) \longrightarrow \mathrm{Pic}(U) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

The abelian group $\mathrm{Pic}_0(U)$ consists of the F-rational points of the Jacobian $\mathrm{Jac}_F(U)$. It is hence divisible, since F is algebraically closed.

Note that for any $s \geq 1$, the natural map $H^1_{ét}(U, \mu_{p^s,U}) \longrightarrow H^1_{ét}(U, \mathbb{G}_{m,U})$ lands in $\mathrm{Pic}_0(U)$. Considering the same diagram as in the proof of Proposition 3.9, we see that the image of any class of $H^1_{ét}(U, \mu_{p^s,U})$ in $H^1_{ét}(U, \mathbb{G}_{m,U})$ lies in the image of the endomorphism of $H^1_{ét}(U, \mathbb{G}_{m,U})$ induced by ϕ' (which is multiplication by p). The map

$$H^1(H, \mathbb{Z}/p^{s+1}\mathbb{Z}(1)) = H^1_{ét}(U, \mu_{p^{s+1},U}) \longrightarrow H^1_{ét}(U, \mu_{p^s,U}) = H^1(H, \mathbb{Z}/p^s\mathbb{Z}(1))$$

induced by ϕ is thus also surjective. Therefore, the group G is $(1,\infty)$-smooth, with cyclotomic character the Tate module $\mathbb{Z}_p(1)$.
We now deal with the case of a non-proper (i.e. affine) smooth connected curve C over F. Denote by $\tilde{C}$ the smooth proper curve containing C, and by x a closed point in $\tilde{C} \setminus C$. Adjusting by multiples of $[x] \in \mathrm{Pic}(\tilde{C})$, one easily sees that the restriction morphism $\mathrm{Pic}_0(\tilde{C}) \longrightarrow \mathrm{Pic}(C)$ is surjective, hence that the abelian group $\mathrm{Pic}(C)$ is divisible. The same holds for any étale cover of C, and we can conclude as before. □

Remark 3.12. Let X be a smooth projective variety, over an algebraically closed field F, of characteristic $\neq p$. Assume that, for every finite étale cover $Y \longrightarrow X$, the Néron-Severi group of Y has no p-torsion. Then, the fundamental group of X is $(1,\infty)$-smooth. The proof is the same as that of Proposition 3.11.

4. Lifting representations of smooth profinite groups

The following is a reformulation of Definition 2.2, for free $(\mathbf{W}_d(k), G)$-modules.

Definition 4.1. *Let G be a profinite group, p be a prime, k be a field of characteristic p and $d \geq 1$ be an integer. A representation*

$$G \longrightarrow \mathrm{GL}_n(\mathbf{W}_d(k))$$

is continuous if its kernel is open in G.
A continuous representation

$$\rho : G \longrightarrow \mathrm{GL}_n(\mathbf{W}(k))$$

is a compatible data, for all $d \in \mathbb{N}^*$, of continuous representations

$$\rho_d : G \longrightarrow \mathrm{GL}_n(\mathbf{W}_d(k)).$$

The compatiblity condition simply means that ρ_{d+1} reduces to ρ_d, for all d.

We now prove Theorem 1.1, whose first statement provides sufficient conditions for existence of mod p^2 liftings for continuous representations of smooth profinite groups. The second statement of the result asserts that under the same assumptions, continuous representation of smooth profinite groups lift, up to direct sum, to arbitrary torsion- bounded by the depth of the underlying cyclotomic module.

Proof of theorem 1.1. We show the first statement of the theorem. Let $\mathcal{T}$ be a 1-cyclotomic module of depth d, relatively to k and G. We may replace G_0 by its intersection with the kernel of the character $G \longrightarrow k^\times$ associated to $\mathcal{T}\{1\}$ (which has index prime-to-p as well). We can thus assume that $\mathcal{T}\{1\} \simeq k$ has the trivial G_0-action. The Yoneda extension

$$0 \longrightarrow A \longrightarrow V \longrightarrow B \longrightarrow 0,$$

corresponds to a cohomology class $\mathcal{E} \in H^1(G_0, \mathrm{Hom}(B, A))$. Fixing two respective G_0-basis Y and X of A and B, we have a G_0-equivariant isomorphism

$$\mathrm{Hom}(B, A) \simeq \mathrm{Hom}(k^X, k^Y) \simeq k^{X \times Y},$$

through which the class $\mathcal{E}$ is given by an element of $H^1(G_0, k^{X\times Y})$. The G_0-set $X \times Y$ decomposes as a disjoint union

$$X \times Y = \bigsqcup_{i \in \mathcal{I}} G_0/G_i$$

of G_0-orbits, where all G_i's are open in G_0. Shapiro's lemma yields an isomorphism

$$H^1(G_0, k^{X \times Y}) \simeq \bigoplus_{i \in \mathcal{I}} H^1(G_i, k)$$

Deciphering the definition of $(1, d)$-smoothness and the comparison lemma 2.4, we get a surjection

$$H^1(G_i, \mathcal{T}) = \mathrm{YExt}^1_{(\mathbf{W}_{d+1}(k), G_i)}(\mathbf{W}_{d+1}(k), \mathcal{T}) \longrightarrow \mathrm{YExt}^1_{(k, G_i)}(k, \mathcal{T}\{1\}) = H^1(G_i, k),$$

for all $i \in \mathcal{I}$. Hence, the natural map

$$\mathrm{YExt}^1_{(\mathbf{W}_{d+1}(k), G_0)}(\mathbf{W}_{d+1}(k)^X, \mathcal{T}^Y) \longrightarrow \mathrm{YExt}^1_{(k, G_0)}(k^X, \mathcal{T}\{1\}^Y)$$

is surjective. As a consequence, V fits into a commutative diagram of $(\mathbf{W}_{d+1}(k), G_0)$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T}^Y & \longrightarrow & \mathcal{V} & \longrightarrow & \mathbf{W}_{d+1}(k)^X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\ 0 & \longrightarrow & k^Y & \longrightarrow & V & \longrightarrow & k^X & \longrightarrow & 0. \end{array} \tag{1}$$

The cyclotomic module $\mathcal{T}$ is a free $\mathbf{W}_{d+1}(k)$-module, hence so is $\mathcal{V}$. This shows that the (k, G_0)-module $\mathrm{Res}^G_{G_0}(V)$ admits the $(\mathbf{W}_{d+1}(k), G_0)$-module $\mathcal{V}$ as a lift to p^{d+1} torsion. Consider the morphisms of (k, G)-modules

$$V \xrightarrow{i} V^{G/G_0} \xrightarrow{N} V,$$

where i is the diagonal embedding, and N is the norm

$$\begin{array}{rccc} N: & V^{G/G_0} & \longrightarrow & V \\ & (v_c)_{c \in G/G_0} & \longmapsto & \sum_{c \in G/G_0} v_c. \end{array}$$

The composite $N \circ i$ is multiplication by the index of G_0 in G, which is prime to p. The (k, G)-module V is thus a direct summand of $V^{G/G_0} = \mathrm{Ind}^G_{G_0}(\mathrm{Res}^G_{G_0}(V))$, with complement $W := \mathrm{Ker}(N)$. But the (k, G)-module module V^{G/G_0} admits the induced module $\mathrm{Ind}^G_{G_0}\mathcal{V} = \mathcal{V} \otimes_{\mathbf{W}_{d+1}[G_0]} \mathbf{W}_{d+1}[G]$ as a lift to p^{d+1}-torsion. The first statement is proved.

The second statement of Theorem 1.1 is deduced from the next lemma.

Lemma 4.2. *Let k be a field of characteristic p, G be a profinite group, and let V be a (k,G)-module. Assume that there is another (k,G)-module W such that $V \oplus W$ lifts to p^2-torsion.*
Then V itself lifts to p^2-torsion.

Proof. Denote by $V^{(1)} := V \otimes_{\mathrm{Frob}} k$ the Frobenius twist of V. By assumption, there is a free $(\mathbf{W}_2(k), G)$-module $\mathcal{V}$ and a short exact sequence of $(\mathbf{W}_2(k), G)$-modules

$$\mathcal{E} : 0 \longrightarrow V^{(1)} \oplus W^{(1)} = \mathrm{Ver}(\mathbf{W}_2(k))\mathcal{V} \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}\{1\} = V \oplus W \longrightarrow 0.$$

Denoting by $i : V \longrightarrow V \oplus W$ and $\pi : W^{(1)} \oplus V^{(1)} \longrightarrow V^{(1)}$ the natural inclusion and projection, we see that the short exact sequence

$$\pi^* i_*(\mathcal{E}) : 0 \longrightarrow V^{(1)} \longrightarrow \mathcal{V}' \longrightarrow V \longrightarrow 0$$

is a lift of V to p^2-torsion.

□

5. Applications to Galois representations and local systems

In this section we provide applications to Theorem 1.1 to liftings of Galois representations and local systems.

Theorem 5.1. *Let k be a field of characteristic p, $d \in \mathbb{N}^* \cup \{\infty\}$ and G be a $(1,d)$-smooth profinite group. Let*

$$\rho : G \to \mathrm{GL}_2(k)$$

be a continuous representation. Then, ρ lifts to p^2-torsion.

Furthermore, there is another continuous representation

$$\rho' : G \to \mathrm{GL}_n(k)$$

such that $\rho \oplus \rho'$ lifts to p^{d+1}-torsion.

If $k = \mathbb{F}_2$, these results also hold for representations of G of dimension up to 4.

Proof. Let V be a 2-dimensional (k,G)-module. There is a line $L \subset V$ fixed by a pro-p-Sylow G_p of G [Se2, 8.3]. The stabilizer $H = \mathrm{Stab}_{G_p}(L)$ is thus of prime-to-p index in G, and we get a short exact sequence of (k,H)-modules

$$0 \longrightarrow L \longrightarrow V \longrightarrow V/L \longrightarrow 0.$$

As in the previous proofs, we can consider an open subgroup G_0 of G, of prime-to-p index, for which the two characters giving the action on L and on V/L are trivial. We can now apply Theorem 1.1.

We now assume that $k = \mathbb{F}_2$ and that V is a 4-dimensional $(\mathbb{F}_2, G)$-module. Again, there is a plane $P \subset V$ stabilized by an open subgroup G_0 of G, of prime-to-2 index. The continuous representation V fits into a short exact sequence of $(\mathbb{F}_2, G_0)$-modules

$$0 \longrightarrow P \longrightarrow V \longrightarrow V/P \longrightarrow 0.$$

Replacing G_0 by an open subgroup of odd index, we can moreover assume that the 2-dimensional $(\mathbb{F}_2, G_0)$-modules P and V/P both admit an $\mathbb{F}_2$-basis permuted by G_0 (to check this is a good exercise, left to the reader). It remains, once more, to cast Theorem 1.1. □

Remark 5.2. Theorem 5.1 applies, in particular, to profinite groups of the types a), b) and c) given in the Introduction.
In particular, it applies to absolute Galois groups.
To conclude, we offer another application below.

Corollary 5.3. *(Zariski-local lifting of local systems of low dimension).*
Let k be a finite field of characteristic p.
Let S be a scheme, defined either over $\mathbb{F}_p$, or over $\mathbb{Z}[\frac{1}{p}]$.
Let $\mathcal{L}$ be a local system over S, with coefficients in k, of dimension 2.
(Equivalently, $\mathcal{L}$ is given by a representation $\pi_1(S) \longrightarrow \mathrm{GL}_2(k)$.)

Then, there exists another local system $\mathcal{L}'$ over S, with coefficients in k, and an open cover $(U_i)_{i\in I}$ of S, with the following property.

For every $i \in I$, the restriction of $\mathcal{L}$ to U_i lifts to a local system over U_i, with coefficients in $\mathbf{W}_2(k)$. Furthermore, the restriction of $\mathcal{L} \bigoplus \mathcal{L}'$ to U_i lifts to a local system over U_i, with coefficients in $\mathbf{W}(k)$.

If $k = \mathbb{F}_2$, the same result holds, for local systems of dimension up to 4.

Proof. By Proposition 3.9, combined with Theorem 5.1, we know that, for each point $s \in S$, the stalk of $\mathcal{L}$ at s (which is a local system over $\mathrm{Spec}(\mathcal{O}_{S,s})$) lifts as stated. It is perhaps worth noting that the representation ρ' of Theorem 5.1 (i.e. the module W of Theorem 1.1) is natural enough, so that the local system $\mathcal{L}'$ is actually defined globally on S. To conclude, use the fact that any finite étale cover of $\mathrm{Spec}(\mathcal{O}_{S,s})$ extends to an open $U \subset S$ containing s. □

Remark 5.4. Corollary 5.3 in fact holds for any field k of characteristic p.
We did not attempt to make the assumptions on S optimal. For instance, the result clearly extends to schemes S where p is nilpotent.

We conjecture that Theorem 5.1 (and hence Corollary 5.3) holds with no restriction on the dimension of V (or of $\mathcal{L}$). We shall investigate this subject in a forthcoming paper.

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