Lifting vector bundles to Witt vector bundles

Charles De Clercq[1], Mathieu Florence[2] and Giancarlo Lucchini Arteche [3]

Abstract.

Let p be a prime, and let S be a scheme of characteristic p. Let $n \geq 2$ be an integer. Denote by $\mathbf{W}_n(S)$ the scheme of Witt vectors of length n, built out of S. The main objective of this paper concerns the question of extending (=lifting) vector bundles on S to vector bundles on $\mathbf{W}_n(S)$. After introducing the formalism of Witt-Frobenius Modules and Witt vector bundles, we study two significant particular cases, for which the answer is positive: that of line bundles, and that of the tautological vector bundle of a projective space. We give several applications of our point of view to classical questions in deformation theory—see the Introduction for details. In particular, we show that the tautological vector bundle of the Grassmannian $\mathrm{Gr}_{\mathbb{F}_p}(m,n)$ does not extend to $\mathbf{W}_2(\mathrm{Gr}_{\mathbb{F}_p}(m,n))$, if $2 \leq m \leq n-2$. In the Appendix, we give algebraic details on our (new) approach to Witt vectors, using polynomial laws and divided powers. It is, we believe, very convenient to tackle lifting questions.

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1. Introduction.

The main objective of this paper concerns the question of lifting vector bundles to Witt vector bundles. More precisely, let p be a prime number, and let S be a scheme of characteristic p. For any $n \geq 2$, denote by $\mathbf{W}_n(S)$ the scheme of Witt vectors of length n, built out of S. The closed immersion $S \hookrightarrow \mathbf{W}_n(S)$ can be thought of as a universal thickening of S, of characteristic p^n. Let V be a vector bundle over S.

Question: is V the restriction to S of a vector bundle defined over $\mathbf{W}_n(S)$?

Note that extending V to $\mathbf{W}_n(S)$ is, in a sense made precise in Section 4.6, the "universal" deformation problem for V.
In Section 3, we give a positive answer for line bundles: every line bundle L admits a (canonical and elementary) lift to a $\mathbf{W}_n$-bundle: its n-th Witt lift. In Section 4, we consider the case of the tautological vector bundle on the base $S = \mathbb{P}_X(V)$, the projective space of a vector bundle V, defined over an affine base X. The answer is positive again—see Theorem 4.2.
Global sections of Witt lifts of line bundles can be naturally described as noteworthy algebraic objects: Pontryagin duals of divided powers of modules over Witt vectors (cf. Proposition 3.4). We use this point of view to give, in Section 5, a unified functorial formulation of three questions in deformation theory. These are:
(I) Let S be a scheme over a perfect field k, of characteristic p. Lift S to a flat $\mathbf{W}_2(k)$-scheme S_2 (a problem made famous by the article [DI]).
(I') Same as (I), with the—major—extra requirement that the Frobenius of S lifts to S_2 (see the nice paper [MS]).
(II) Lift a vector bundle over S to a vector bundle over $\mathbf{W}_2(S)$—the main topic of this paper.
We describe very explicit equivalences of categories, allowing us to interpret these questions in the common framework of Witt-Frobenius Modules: Theorem 5.16, 5.20 and 5.29. As a first application, we prove that (I) has an affirmative and functorial answer, for any Frobenius-split k-scheme: this is Corollary 5.17 (for an introduction to Frobenius splitting, see [MR]). We also give explicit 2-extensions realizing the classical cohomological obstructions in deformation theory (Corollary 5.18 and Remark 5.30). As a second application, we show (Theorem 5.32) that the tautological vector bundle on the Grassmannian $\mathrm{Gr}(m,n)$ never lifts to a $\mathbf{W}_2$-bundle, if $2 \leq m \leq n-2$.
In the Appendix, we venture to explore algebraic connections between Witt vectors and divided powers. In particular, we give a new description of Witt vectors "by generators and relations": Proposition 6.9. This appendix, for which only the first and second authors endorse responsibility, in written in the spirit of the preprint [DCF].

2. Witt-Frobenius modules and $\mathbf{W}_n$-bundles.

In this section, unless otherwise stated, all sheafs (and local properties) are considered with respect to the Zariski topology.
In this text, all schemes are assumed to be quasi-compact.
We fix a scheme S, of characteristic p. For each integer $n \geq 1$, and for each commutative ring A of characteristic p, we denote by $\mathbf{W}_n(A)$ the ring of Witt vectors of A, of length n. Note that $\mathbf{W}_n$ itself might be considered as a ring scheme defined over $\mathbb{Z}$, isomorphic, as a scheme, to the n-dimensional affine space $\mathbb{A}^n$. For details on the construction of Witt vectors (adopting different viewpoints) we refer to [BO], [Ha], [Le], and [Se1]. For a simple recursive definition of Witt vectors, using divided powers of $\mathbb{Z}$-modules, see also the Appendix of the present paper, or [DCF] (for the case of a perfect field).

The following Definition is classical. It originates from Serre's paper [Se2].

Definition 2.1. *Let $n \geq 1$ be an integer. The association*

$$U \mapsto \mathbf{W}_n(\mathcal{O}_S(U))$$

defines a sheaf of commutative rings on S. We denote it by $\mathbf{W}_n(\mathcal{O}_S)$. As a sheaf of sets, it is represented by the n-dimensional affine space $\mathbb{A}^n_S$; it is thus also a sheaf for the fppf topology. If $m \leq n$ is a positive integer, we denote by

$$\pi_{n,m} : \mathbf{W}_n(\mathcal{O}_S) \longrightarrow \mathbf{W}_m(\mathcal{O}_S)$$

the quotient morphism (of sheaves of commutative rings on S), and by

$$i_{n-m,m} : \mathbf{W}_{n-m}(\mathcal{O}_S) \longrightarrow \mathbf{W}_n(\mathcal{O}_S)$$

the natural injection, whose image equals $\mathrm{Ker}(\pi_{n,m})$. *We thus have an exact sequence*

$$0 \longrightarrow \mathbf{W}_{n-m}(\mathcal{O}_S) \xrightarrow{i_{n-m,n}} \mathbf{W}_n(\mathcal{O}_S) \xrightarrow{\pi_{n,m}} \mathbf{W}_m(\mathcal{O}_S) \longrightarrow 0.$$

Definition 2.2. *Let $n \geq 1$ be an integer. The association*

$$U \mapsto \mathbf{W}_n(\mathcal{O}_S(U))^\times$$

defines a sheaf of Abelian groups on S. It is an affine and smooth S-group scheme, which we denote by $\mathbf{W}_{n,S}^\times$, or simply by $\mathbf{W}_n^\times$ if the dependence in S is clear. Note that

$$\mathbf{W}_1^\times = \mathbb{G}_m.$$

The Teichmüller representative is given by the formula

$$\tau = \tau_n : \mathbb{G}_m = \mathbf{W}_1^\times \longrightarrow \mathbf{W}_n^\times$$

$$x \mapsto (x, 0, 0, \ldots, 0).$$

It is a homomorphism of S-groups schemes; a canonical splitting of the natural quotient map

$$\mathbf{W}_n^\times \longrightarrow \mathbb{G}_m.$$

Remark 2.3. On the level of functors of points, we have the equality

$$i_{n,n+1}(x) i_{n,n+1}(y) = p i_{n,n+1}(xy),$$

for all $x, y \in \mathbf{W}_n$. In this formula, quantities on the left hand side (resp. on the right hand side) are multiplied in the ring $\mathbf{W}_{n+1}$ (resp. $\mathbf{W}_n$).
We also have

$$p\tau_{n+1}(x) = i_{n,n+1}(\tau_n(x^p)) \in \mathbf{W}_{n+1},$$

for all $x \in \mathbf{W}_1$.

Remark 2.4. Let $n \geq 1$ be an integer. The following is well-known.
The Teichmüller representative yields an isomorphism of linear algebraic groups (over $\mathbb{F}_p$)

$$\mathbb{G}_m \times (1 + \mathbf{W}_n)^\times \xrightarrow{\sim} \mathbf{W}_{n+1}^\times,$$
$$(x, a) \mapsto \tau_{n+1}(x)a.$$

The logarithm

$$\log : (1 + \mathbf{W}_n)^\times \longrightarrow (\mathbf{W}_n, +),$$

$$1 - x \mapsto x + \frac{p}{2}x^2 + \frac{p^2}{3}x^3 + \frac{p^3}{4}x^4 + \ldots$$

is well-defined. If p is odd, it is an isomorphism of linear algebraic groups, with inverse

$$\exp : (\mathbf{W}_n, +) \longrightarrow (1 + \mathbf{W}_n)^\times,$$

$$t \mapsto 1 + t + \frac{p}{2!}t^2 + \frac{p^3}{3!}t^3 + \ldots$$

Note that the sums occuring above are actually finite, and that the multiplication used is that of $\mathbf{W}_n$. Note also that, strictly speaking, we should write $(1 + i_{n,n+1}(\mathbf{W}_n))^\times$ instead of $(1 + \mathbf{W}_n)^\times$.
If $p = 2$, the logarithm is an isogeny of degree two, with kernel $\{1, -1\}$. In that case, the algebraic group $(1 + \mathbf{W}_n)^\times$ is isomorphic to the middle term of the pullback of the exact sequence

$$0 \longrightarrow \mathbf{W}_{n-1} \longrightarrow \mathbf{W}_n \longrightarrow \mathbb{G}_a \longrightarrow 0$$

by the Lang isogeny (which is the reduction of log mod 2)

$$\mathrm{Frob} - \mathrm{Id} : \mathbb{G}_a \longrightarrow \mathbb{G}_a.$$

Thus, from the point of view of linear algebraic groups, the multiplicative group scheme $\mathbf{W}_{n+1}^\times$, over $\mathbb{F}_p$, is "nothing new". See, however, the next exercise.

Exercise 2.5. Put

$$D_n := \mathrm{Ker}(\mathbf{W}_{n+2}^\times \longrightarrow \mathbf{W}_2^\times);$$

it is a smooth affine group scheme over $\mathbb{Z}$. Show that its generic fiber $D_n \times_{\mathbb{Z}} \mathbb{Q}$ is a split algebraic torus of dimension n, whereas its special fiber $D_n \times_{\mathbb{Z}} \mathbb{F}_p$ is isomorphic to $(\mathbf{W}_n, +)$ (also for $p = 2$). Thus, D_n is a simple example of a deformation of (the additive group of) Witt vectors to an algebraic torus, in the spirit of [TO].

Definition 2.6. *Assume that* $S = \mathrm{Spec}(A)$ *is affine. Let* $n \geq 1$ *be a positive integer. Let* M *be a* $\mathbf{W}_n(A)$*-module. The formula*

$$U \mapsto M \otimes_{\mathbf{W}_n(A)} \mathbf{W}_n(\mathcal{O}_S(U))$$

defines a presheaf (for the Zariski topology) on S*. We denote by* $\tilde{M}$ *the associated sheaf. It is a sheaf of* $\mathbf{W}_n(\mathcal{O}_S)$*-modules.*

Definition 2.7. *(Witt-Frobenius Modules).*
Let $n \geq 1$ be two positive integers. A Witt-Frobenius Module of height n, over S is a sheaf of $\mathbf{W}_n(\mathcal{O}_S)$-modules, which is locally isomorphic to a sheaf of the shape $\tilde{M}$ (cf. Definition 2.6).
When no reference to its height is necessary, a Witt-Frobenius Module will simply be referred to as a WtF-Module.

Definition 2.8. *($\mathbf{W}_n$-bundles).*
Let $r, n \geq 1$ be two positive integers. A $\mathbf{W}_n$-bundle over S, of rank r, is a Witt-Frobenius Module of height n, which is locally free of rank r; in other words, which is locally isomorphic to $\mathbf{W}_n(\mathcal{O}_S)^r$.

Remark 2.9. A Witt-Frobenius Module of height 1 (resp. a $\mathbf{W}_1$-bundle) over S, is nothing but a (quasi-coherent) $\mathcal{O}_S$-Module (resp. vector bundle). The notion of WtF-Modules and of $\mathbf{W}_n$-bundles can also be rephrased using the Greenberg functor, the classical notion allowing to descend schemes of $\mathbf{W}_n(S)$ to schemes over S. Indeed, a WtF-Module (resp. a $\mathbf{W}_n$-bundle) is exactly a Module (resp. vector bundle) over the scheme $\mathbf{W}_n(S)$, descended down to S this way.

Definition 2.10. *Let n be a positive integer. Let V/S be a $\mathbf{W}_n$-bundle over S. We denote by $V^\vee$ the sheaf*

$$\underline{\mathrm{Hom}}_{\mathbf{W}_n(\mathcal{O}_S)-\mathrm{Mod}}(V, \mathbf{W}_n(\mathcal{O}_S));$$

it is a $\mathbf{W}_n$-bundle over S.
Let $m \leq n$ be a positive integer. We denote by

$$V\{m\} := (\pi_{n,m})_*(V)$$

the sheaf associated to the presheaf

$$U \mapsto V(U) \otimes_{\mathbf{W}_n(\mathcal{O}_S(U))} \mathbf{W}_m(\mathcal{O}_S(U));$$

it is a $\mathbf{W}_m$-bundle over S.

We conclude this section by instructive exercises on Witt vectors and their topological behaviour. For the authors of the present text, question 4) of the first exercise below is the hardest, and we refer to [BLM, 5.4.1] and [BMS, Theorem 10.4] or [Il, Proposition 1.5.8] for a proof.
A good starting point for the reader interested in studying Frobenius base change phenomena is [Sz].

Exercise 2.11. Assume that A is a ring of characteristic p.

1) Show that $\mathbf{W}_2(A)$ is Noetherian if, and only if, A itself is Noetherian, and the Frobenius morphism $\mathrm{Frob} : A \longrightarrow A$ is finite.
2) Assuming that the equivalent conditions of 1) hold, show that $\mathbf{W}_n(A)$ is Noetherian for all $n \geq 2$.
3) Let f be an element of A. Show that the natural map

$$\mathbf{W}_n(A_f) \longrightarrow \mathbf{W}_n(A)_{\tau(f)}$$

is an isomorphism.
4) Let

$$f : A \longrightarrow B$$

be an étale morphism (of commutative rings of characteristic p). Show that

$$\mathbf{W}_n(f) : \mathbf{W}_n(A) \longrightarrow \mathbf{W}_n(B)$$

is étale, for every $n \geq 2$.
5) Give an example of a finite flat homomorphism

$$f : A \longrightarrow B,$$

such that

$$\mathbf{W}_2(f) : \mathbf{W}_2(A) \longrightarrow \mathbf{W}_2(B)$$

is not flat.

2.1. Lifting towers.

Definition 2.12. *Let V/S be a vector bundle, of constant rank d. A (complete) lifting tower for V is the data, for every $r \geq 1$, of a $\mathbf{W}_r$-bundle V_r on S, together with isomorphisms (of $\mathbf{W}_r$-bundles)*

$$f_r : V_{r+1}\{r\} \longrightarrow V_r.$$

Remark 2.13. The notion of a lifting tower for V is the same as that of a lift of V to a $\mathbf{W}_\infty$-bundle, where $\mathbf{W}_\infty = \varprojlim \mathbf{W}_r$. For instance, if $S = \mathrm{Spec}(\mathbb{F}_p)$, a lifting tower for V is simply the data of a free $\mathbb{Z}_p$-module V_∞, together with an isomorphism of $\mathbb{F}_p$-vector spaces $V_\infty/p \xrightarrow{\sim} V$.
Our definition is here to emphasize the following (classical) philosophical statement. To lift a vector bundle to a $\mathbf{W}_\infty$-bundle is a highly nonabelian problem, whereas lifting a $\mathbf{W}_r$-bundle to a $\mathbf{W}_{r+1}$-bundle is an Abelian problem—equivalent to the vanishing of some class in a second cohomology group. To lift mod p algebro-geometric structures to their mod p^n counterparts, it is often advisable to proceed step by step, as slowly as possible. We hope that this loose remark will acquire a meaning in the sequel.

3. The Teichmüller representative lifts line bundles.

In this section, S is a scheme of characteristic p. Hypothesis will be made on S, when necessary.

Definition 3.1. *Let L be a line bundle over S. Denote by P the corresponding $\mathbb{G}_m$-torsor over S. Let $n \geq 1$ be an integer. Then*

$$P_n := (\tau_n)_*(P)$$

is a $\mathbf{W}_n^\times$-torsor over S. Twisting the trivial invertible $\mathbf{W}_n$ bundle $\mathbf{W}_n(\mathcal{O}_S)$ by P_n yields an invertible $\mathbf{W}_n$-bundle, which we denote by $\mathbf{W}_n(L)$. It is the n-th Witt lift of L.
The sequence $(\mathbf{W}_n(L))_{n\geq 1}$, together with the canonical isomorphisms $\mathbf{W}_{n+1}(L)\{n\} \simeq \mathbf{W}_n(L)$, is a canonical lifting tower for L.

Remark 3.2. Note that, for a group scheme G, twisting G-schemes by G-torsors usually requires extra hypothesis—e.g. quasiprojectivity. Here however, no assumption on S is needed, since line bundles are locally trivial for the Zariski topology.

Now, consider the quotient map

$$\pi = \pi_{n+1,n} : \mathbf{W}_{n+1}(\mathcal{O}_S) \longrightarrow \mathbf{W}_n(\mathcal{O}_S),$$

and the natural exact sequence (of sheaves of Abelian groups on S)

$$0 \longrightarrow \mathcal{O}_S \xrightarrow{i=i_{1,n+1}} \mathbf{W}_{n+1}(\mathcal{O}_S) \xrightarrow{\pi} \mathbf{W}_n(\mathcal{O}_S) \longrightarrow 0.$$

A meditation on the formula

$$xi(y) = i(\pi(x)^{p^n} y) = i(\mathrm{Frob}^n(\pi(x))y),$$

valid for all $x \in \mathbf{W}_{n+1}(\mathcal{O}_S)$ and all $y \in \mathcal{O}_S$, reveals that the arrows of this sequence are $\mathbb{G}_m$-equivariant, for the following actions:

i) the group $\mathbb{G}_m$ acts on $\mathbf{W}_n(\mathcal{O}_S)$ (resp. $\mathbf{W}_{n+1}(\mathcal{O}_S)$) via τ_n (resp. τ_{n+1}),
ii) the group $\mathbb{G}_m$ acts on $\mathcal{O}_S$ via Frob^n (formula: $x.s := x^{p^n} s$).

Twisting by the $\mathbb{G}_m$-torsor P corresponding to the line bundle L, we then get an exact sequence

$$0 \longrightarrow (\mathrm{Frob}_*)^n(\mathrm{Frob}^*)^n(L) \longrightarrow \mathbf{W}_{n+1}(L) \xrightarrow{\pi_L = \pi_{L,n+1,n}} \mathbf{W}_n(L) \longrightarrow 0,$$

where

$$\mathrm{Frob}^n : S \longrightarrow S$$

is the n-th power of the (absolute) Frobenius of S (recall that $\mathrm{Frob}^*(L)$ is canonically isomorphic to $L^{\otimes p}$). Note that, written as such, it is more than an exact sequence of sheaves of Abelian groups on S: it is an exact sequence of WtF-Modules on S.
Similarly, twisting the exact sequence

$$0 \longrightarrow \mathbf{W}_n(\mathcal{O}_S) \longrightarrow \mathbf{W}_{n+1}(\mathcal{O}_S) \longrightarrow \mathcal{O}_S \longrightarrow 0$$

yields the (less useful) exact sequence

$$0 \longrightarrow \mathrm{Frob}_*\mathrm{Frob}^*(\mathbf{W}_n(L)) \longrightarrow \mathbf{W}_{n+1}(L) \longrightarrow L \longrightarrow 0.$$

3.1. The Teichmüller section, for Witt lifts of line bundles. We keep the notation of the preceding paragraph.
The surjection $\pi : \mathbf{W}_{n+1} \longrightarrow \mathbf{W}_n$ has a canonical (scheme-theoretic) section

$$s : \mathbf{W}_n \longrightarrow \mathbf{W}_{n+1},$$

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0).$$

This section is $\mathbb{G}_m$-equivariant, for the $\mathbb{G}_m$-action on $\mathbf{W}_n$ (resp. on $\mathbf{W}_{n+1}$) given by τ_n (resp. τ_{n+1}). Twisting by P then yields a canonical (scheme-theoretic) section of π_L—its Teichmüller section, denoted by

$$s_L : \mathbf{W}_n(L) \longrightarrow \mathbf{W}_{n+1}(L).$$

Note that s_L is, of course, not additive. In a sense clarified later in the Appendix, it is in fact polynomial, of degree p. Taking global sections, its existence shows that the sequence

$$0 \longrightarrow H^0(S, L^{\otimes p^n}) \longrightarrow H^0(S, \mathbf{W}_{n+1}(L)) \longrightarrow H^0(S, \mathbf{W}_n(L)) \longrightarrow 0$$

is always exact.
Furthermore, assume that G is a group (assumed to be finite for simplicity) that acts on S by scheme automorphisms, and on L accordingly. In other words, assume that L is a G-linearized line bundle. Then, the sequence of G-invariant global sections

$$0 \longrightarrow H^0(S, L^{\otimes p^n})^G \longrightarrow H^0(S, \mathbf{W}_{n+1}(L))^G \longrightarrow H^0(S, \mathbf{W}_n(L))^G \longrightarrow 0$$

is exact as well. Indeed, the Teichmüller section s_L is then G-equivariant.

3.2. A link with divided powers of modules over Witt vectors. Let V be a finite-dimensional vector space over a *perfect* field k, of characteristic p. Denote by $S := \mathbb{P}_k(V)$ the projective space parametrizing *quotient* line bundles of V.

We begin this section with a very instructive exercise.

Exercise 3.3. We have the natural exact sequence

$$0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}(p)) \longrightarrow \mathbf{W}_2(\mathcal{O}(1)) \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Show that the exact sequence of $\mathbf{W}_2(k)$-modules obtained by taking its global sections is canonically isomorphic to the exact sequence

$$0 \longrightarrow \mathrm{Sym}^p_k(W) \longrightarrow \Gamma^p_{\mathbf{W}_2(k)}(W) \xrightarrow{\mathrm{Frob}_W} W^{(1)} = V \longrightarrow 0,$$

where $W := V^{(-1)} = \mathrm{Frob}_*(V)$, and where $\Gamma^p_{\mathbf{W}_2(k)}(W)$ denotes the p-th divided power of the k-vector space W, seen as a $\mathbf{W}_2(k)$-module.
For the definition of the arrow Frob_W, see the preprint [DCF]. Our notation $V^{(-1)} = \mathrm{Frob}_*(V)$ is coherent with their notation $V^{(1)} = \mathrm{Frob}^*(V)$.
What happens if the field k is not perfect?

Now, let $r \geq 2$ be any integer. A general description of $H^0(S, \mathbf{W}_r(\mathcal{O}(1)))$ can be obtained as follows.
Let A/k be an arbitrary commutative k-algebra. Let $\phi \in V^\vee \otimes_k A$ be an A-linear form on $V \otimes_k A$. Assume first that $\phi : V \otimes_k A \longrightarrow A$ is surjective. It then gives an A-point $s \in S(A)$, as well as a canonical isomorphism

$$H^0(s, \mathbf{W}_r(\mathcal{O}(1))_{|s}) \xrightarrow{\sim} \mathbf{W}_r(A).$$

Evaluating at s then yields a canonical $\mathbf{W}_r(k)$-linear homomorphism

$$\mathrm{ev}_s : H^0(S, \mathbf{W}_r(\mathcal{O}(1))) \longrightarrow \mathbf{W}_r(A).$$

Now, let $A = \mathrm{Sym}(V)$ denote the symmetric algebra of V, and let ϕ stand for the canonical (tautological) A-linear form on $V \otimes_k A$. Let $(e_1, \ldots, e_d)$ be a k-basis of V. The preceding discussion applies to the important particular case of $A_i := A_{e_i}$, the k-algebra obtained by inverting e_i in A, and where ϕ_i is the linear form induced by ϕ. Note that $\mathbf{W}_r(A_i) = \mathbf{W}_r(\mathrm{Sym}(V))_{\tau(e_i)}$, by Question 3) of Exercise 2.11. For $i \neq j$, the homomorphism obtained by composing

$$\mathrm{ev}_i : H^0(S, \mathbf{W}_r(\mathcal{O}(1))) \longrightarrow \mathbf{W}_r(A_i),$$

with the canonical map $\mathbf{W}_r(A_i) \longrightarrow \mathbf{W}_r(A_{e_i e_j}) = \mathbf{W}_r(A)_{\tau(e_i e_j)}$ is easily seen to be independent of the order of i, j. Hence, ev_i actually takes values in $\mathbf{W}_r(A)$, thanks to the exactness of the sequence

$$0 \longrightarrow \mathbf{W}_r(A) \xrightarrow{x \mapsto (x, \ldots, x)} \prod_{1 \leq i \leq d} \mathbf{W}_r(A)_{\tau(e_i)} \xrightarrow{(x_i) \mapsto (x_j - x_i)} \prod_{1 \leq i < j \leq d} \mathbf{W}_r(A)_{\tau(e_i e_j)}.$$

We have thus shown that there exists a canonical $\mathbf{W}_r(k)$-linear homomorphism

$$\mathrm{Ev} = \mathrm{Ev}_{S,r} : H^0(S, \mathbf{W}_r(\mathcal{O}(1))) \longrightarrow \mathbf{W}_r(\mathrm{Sym}(V)).$$

Put

$$W := \mathrm{Frob}_*^{r-1}(V) = V^{(-r+1)},$$

as in Exercise 3.3. Recall that W, as an Abelian group, is equal to V, and that its scalar multiplication . is given by

$$\lambda.v := \lambda^{p^{r-1}} v.$$

Pick a global section $s \in H^0(S, \mathbf{W}_r(\mathcal{O}(1)))$. We now define a *polynomial law* of $\mathbf{W}_r(k)$-modules

$$F_s : W^\vee \longrightarrow \mathbf{W}_r(k),$$

homogeneous of degree p^{r-1}, as follows (for an introduction to polynomial laws, see [DCF], section 5).

Let $\mathcal{A}$ be a $\mathbf{W}_r(k)$-algebra. Put $A := \mathcal{A} \otimes_{\mathbf{W}_r(k)} k$. Let

$$\phi \in W^\vee \otimes_{\mathbf{W}_r(k)} \mathcal{A} = \mathrm{Hom}_k(W, A)$$

be a linear form. By adjunction, we get

$$\mathrm{Hom}_k(W, A) = \mathrm{Hom}_k(\mathrm{Frob}_*^{r-1}(V), A) = \mathrm{Hom}_k(V, (\mathrm{Frob}^{r-1})^*(A)),$$

where $(\mathrm{Frob}^{r-1})^*(A)$ is naturally identified to A, viewed as a k-algebra via

$$x \in k \mapsto x^{p^{-r+1}} \in A$$

(remember that k is assumed to be perfect). By functoriality, this data induces, successively, a homomorphism of k-algebras

$$\mathrm{Sym}(V) \longrightarrow (\mathrm{Frob}^{r-1})^*(A)$$

and a homomorphism of $\mathbf{W}_r(k)$-algebras

$$\Phi : \mathbf{W}_r(\mathrm{Sym}(V)) \longrightarrow \mathbf{W}_r((\mathrm{Frob}^{r-1})^*(A)).$$

By Corollary 6.15 in the appendix, we have a canonical homomorphism of $\mathbf{W}_r(k)$-algebras

$$\begin{aligned} f_r : \mathbf{W}_r((\mathrm{Frob}^{r-1})^*(A)) &\longrightarrow \mathcal{A}, \\ \tau(\overline{a}) &\longmapsto a^{p^{r-1}}. \end{aligned}$$

Here a denotes any element of $\mathcal{A}$, and $\overline{a}$ stands for its reduction to an element of A. We then set

$$F_s(\phi) := (f_r(\Phi(\mathrm{Ev}(s))) \in \mathcal{A}.$$

The association

$$\phi \mapsto F_s(\phi)$$

is then a *polynomial law* between $\mathbf{W}_r(k)$-modules, homogeneous of degree p^{r-1} (see [DCF, Proposition 6.8]). As such, by the universal property of divided powers, it induces a $\mathbf{W}_r(k)$-linear homomorphism

$$\Gamma^{p^{r-1}}_{\mathbf{W}_r(k)}(W^*) \longrightarrow \mathrm{Hom}_{\mathbf{W}_r(k)}(H^0(S, \mathbf{W}_r(\mathcal{O}(1))), \mathbf{W}_r(k)).$$

In other words, we get a $\mathbf{W}_r(k)$-linear pairing

$$\langle \cdot, \cdot \rangle : \Gamma^{p^{r-1}}_{\mathbf{W}_r(k)}(W^*) \times H^0(S, \mathbf{W}_r(\mathcal{O}(1))) \longrightarrow \mathbf{W}_r(k).$$

Proposition 3.4. *The pairing $\langle \cdot, \cdot \rangle$ is perfect. It thus yields a canonical isomorphism*

$$h_{S,r} : H^0(S, \mathbf{W}_r(\mathcal{O}(1))) \xrightarrow{\sim} \Gamma^{p^{r-1}}_{\mathbf{W}_r(k)}(W^*)^\vee.$$

Proof. By [DCF], Proposition 7.4, we have an exact sequence

$$0 \longrightarrow \Gamma^{p^{r-1}}_{\mathbf{W}_r(k)}(W^*) \xrightarrow{\mathrm{Ver}} \Gamma^{p^r}_{\mathbf{W}_{r+1}(k)}(W^*) \xrightarrow{can} \Gamma^{p^r}_k(W^*) \longrightarrow 0.$$

Its Pontryagin dual (obtained by applying $\mathrm{Hom}(\,\cdot\,, \mathbf{W}_{r+1}(k))$) reads as

$$0 \longrightarrow \mathrm{Sym}^{p^r}_k(W) \xrightarrow{\mathrm{Ver}} \Gamma^{p^r}_{\mathbf{W}_{r+1}(k)}(W^*)^\vee \xrightarrow{can} \Gamma^{p^{r-1}}_{\mathbf{W}_r(k)}(W^*)^\vee \longrightarrow 0.$$

Now, compare it to the exact sequence

$$0 \to \mathrm{Frob}^r_*(H^0(\mathbb{P}(V), \mathcal{O}(p^r))) \to H^0(S, \mathbf{W}_{r+1}(\mathcal{O}(1))) \to H^0(S, \mathbf{W}_r(\mathcal{O}(1))) \to 0$$

of Section 3.1 (where $L = \mathcal{O}(1)$). We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Frob}^r_*(H^0(\mathbb{P}(V), \mathcal{O}(p^r))) & \to & H^0(S, \mathbf{W}_{r+1}(\mathcal{O}(1))) & \to & H^0(S, \mathbf{W}_r(\mathcal{O}(1))) & \to & 0 \\ & & \downarrow \wr & & \downarrow h_{S,r+1} & & \downarrow h_{S,r} & & \\ 0 & \longrightarrow & \mathrm{Sym}^{p^r}_k(W) & \longrightarrow & \Gamma^{p^r}_{\mathbf{W}_{r+1}(k)}(W^*)^\vee & \longrightarrow & \Gamma^{p^{r-1}}_{\mathbf{W}_r(k)}(W^*)^\vee & \longrightarrow & 0. \end{array}$$

Noting that the vertical arrow on the left is an isomorphism, we see, by induction on r, that $h_{S,r}$ is an isomorphism, for all $r \geq 0$.
Alternatively, one can use that the finite torsion $\mathbf{W}(k)$-modules $H^0(S, \mathbf{W}_r(\mathcal{O}(1)))$ and $\Gamma^{p^{r-1}}_{\mathbf{W}_r(k)}(W^*)^\vee$ have the same length, and that the arrow $h_{S,r}$ is injective (this follows from the fact that $\mathrm{Ev}_{S,r}$ is injective). □ This isomorphism

will not be used in this paper. We mention it here to suggest that the algebraic constructions introduced in [DCF] (involving divided powers of modules over Witt vectors and Pontryagin duality) are very natural, from a geometric point of view. Indeed, they correspond to global sections of the Witt lifts of Serre's line bundle $\mathcal{O}(1)$, on a projective space.

4. A lifting tower, for the tautological vector bundle of a projective space.

Let S be a scheme of characteristic $p > 0$, and let V/S be a vector bundle, of constant rank $n \geq 1$. Denote by

$$f : \mathbb{P}(V) \longrightarrow S$$

the projective space of V. We adopt here Grothendieck's point of view: $\mathbb{P}(V)$ parametrizes *quotient* line bundles (of V).

Definition 4.1. *We denote by $\mathcal{H}_V$, or simply by $\mathcal{H}$ if the dependence in V is clear, the tautological vector bundle on $\mathbb{P}(V)$.*
We have a (tautological) exact sequence of vector bundles on $\mathbb{P}(V)$

$$\mathcal{T}(= \mathcal{T}_V) : 0 \longrightarrow \mathcal{H} \longrightarrow f^*(V) \longrightarrow \mathcal{O}(1) \longrightarrow 0.$$

Theorem 4.2. *Assume that S is affine. Then, V admits a lifting tower.*
For each choice of a lifting tower $(V_r)_{r\geq 1}$ of V (over S), there exists a lifting tower $(\mathcal{H}_r)_{r\geq 1}$ of $\mathcal{H}$ (over $\mathbb{P}(V)$), such that each $\mathcal{H}_r$ fits into an exact sequence (of $\mathbf{W}_r$-bundles over $\mathbb{P}(V)$)

$$\mathcal{T}_r : 0 \longrightarrow \mathcal{H}_r \longrightarrow f^*(V_r) \longrightarrow \mathbf{W}_r(\mathcal{O}(1)) \longrightarrow 0.$$

Proof. Assume that V_r is a given lift of V, to a $\mathbf{W}_r$-bundle. Then, the obstruction to lifting V_r to a $\mathbf{W}_{r+1}$-bundle V_{r+1} lies in

$$\mathrm{Ext}^2_{\mathcal{O}_S-\mathrm{Mod}}(V, (\mathrm{Frob}^r)_*(\mathrm{Frob}^r)^*(V)) = H^2_{Zar}(S, (\mathrm{Frob}^r)^*(\mathrm{End}_S(V))).$$

This cohomology group vanishes since S is affine. The first claim follows.
We now prove the second claim.
Assume that $\mathcal{H}_r$, together with the extension

$$\mathcal{T}_r : 0 \longrightarrow \mathcal{H}_r \longrightarrow f^*(V_r) \xrightarrow{\rho_r} \mathbf{W}_r(\mathcal{O}(1)) \longrightarrow 0$$

has been constructed. Since $f^*(V_r)$ is a $\mathbf{W}_r$-bundle, we have a canonical duality isomorphism

$$\underline{\mathrm{Hom}}_{WtF-\mathrm{Mod}}(f^*(V_r), \mathbf{W}_r(\mathcal{O}(1))) = f^*(V_r)^\vee(1) \xrightarrow{\sim} f^*(V_r^\vee)(1).$$

The surjection ρ_r thus corresponds to a global section

$$s_r \in H^0(\mathbb{P}(V), f^*(V_r^\vee)(1)).$$

We would like to lift it through the epimorphism of the exact sequence (on $\mathbb{P}(V)$)

$$0 \longrightarrow (\mathrm{Frob}^r)_*(\mathrm{Frob}^r)^*(f_*(V^\vee)(1)) \longrightarrow f^*(V_{r+1}^\vee)(1) \longrightarrow f^*(V_r^\vee)(1) \longrightarrow 0.$$

The obstruction to do so is a class

$$\begin{aligned} c &\in H^1_{Zar}(\mathbb{P}(V), (\mathrm{Frob}^r)_*(\mathrm{Frob}^r)^*(f_*(V^\vee)(1))) \\ &= H^1_{Zar}(\mathbb{P}(V), (\mathrm{Frob}^r)^*(f_*(V^\vee))(p^r)) = 0. \end{aligned}$$

[To get this vanishing, use $H^1_{Zar}(\mathbb{P}(V), \mathcal{O}(p^r)) = 0$, together with the projection formula.]
Hence, s_r can be (non-canonically in general) lifted to a global section

$$s_{r+1} \in H^0(\mathbb{P}(V), f^*(V_{r+1}^\vee)(1)).$$

Dualizing, it corresponds to a homomorphism

$$\rho_{r+1} : f^*(V_{r+1}) \longrightarrow \mathbf{W}_{r+1}(\mathcal{O}(1)),$$

lifting ρ_r. Since ρ_r is surjective, ρ_{r+1} is surjective as well. Define the $\mathbf{W}_{r+1}$-bundle $\mathcal{H}_{r+1}$ to be its kernel. We then have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_{r+1} & \longrightarrow & f^*(V_{r+1}) & \xrightarrow{\rho_{r+1}} & \mathbf{W}_{r+1}(\mathcal{O}(1)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_{r+1,r}} & & \\ 0 & \longrightarrow & \mathcal{H}_r & \longrightarrow & f^*(V_r) & \xrightarrow{\rho_r} & \mathbf{W}_r(\mathcal{O}(1)) & \longrightarrow & 0. \end{array}$$

The Theorem is proved. □

4.1. Two general facts about liftings of vector bundles. The following easy Lemma asserts that being liftable is invariant under twists by line bundles.

Lemma 4.3. *Let S be a scheme of characteristic p. Let V be a vector bundle on S, and let L be a line bundle on S. Then, V admits a lifting tower if and only if $V \otimes L$ admits a lifting tower.*

Proof. Let $(V_r)_{r\geq 1}$ be a lifting tower for V. By Section 3, the line bundle L admits a (canonical) lifting tower $(\mathbf{W}_r(L))_{r\geq 1}$. Then, $(V_r \otimes \mathbf{W}_r(L))_{r\geq 1}$ is a lifting tower for $V \otimes L$. □

Definition 4.4. *(geometric formulation of Definition 6.6). Let S be any scheme. Let $i : S \hookrightarrow T$ be a closed immersion. We say that i is a p-elementary thickening of S, if the sheaf of Ideals $I \subset \mathcal{O}_T$ defining S satisfies $I^p = pI = 0$.*
Let $n \geq 1$ be an integer. We say that i is a p-thickening of depth n of S, if it can be written as a composite

$$s : S = S_0 \hookrightarrow S_1 \hookrightarrow \ldots \hookrightarrow S_{n-1} \hookrightarrow S_n = T$$

of n p-elementary thickenings.

Examples 4.5. Let S be a scheme of characteristic p. Then, the closed immersion $\mathbf{W}_n(S) \hookrightarrow \mathbf{W}_{n+1}(S)$ is p-elementary.
If S is a smooth variety over a perfect field k of characteristic p, and if S_{n+1} is a lift of S to a scheme S_{n+1}, flat over $\mathbf{W}_{n+1}(k)$, then the closed immersion $S \hookrightarrow S_{n+1}$ is a p-thickening of depth n.
More generally, let $i : S \hookrightarrow T$ be a closed immersion of schemes, defined by an Ideal $I \subset \mathcal{O}_T$. Assume that $I^m = 0$, and that $p^n\mathcal{O}_T = 0$, for some positive integers m and n. Then, one can show that i is a p-thickening, of depth $n(m-1)$.

Extending to Witt vector bundles is (up to Frobenius twist) the "most difficult" problem, in the deformation theory of vector bundles over a scheme of characteristic p. We make this statement precise in the next Lemma.

Lemma 4.6. *Let S be a scheme of characteristic p. Let $n \geq 1$ be an integer, and let $i : S \hookrightarrow T$ be a p-thickening of S, of depth n. Let V be a vector bundle on S. Assume that V admits a lift to a $\mathbf{W}_{n+1}$-bundle on S.*
Then, $(\mathrm{Frob}^n)^*(V)$ *extends (via i) to a vector bundle on T.*

Proof. Denote by

$$f : S \longrightarrow \mathbf{W}_{n+1}(S)$$

the natural thickening. Using (the geometric formulation of) Lemma 6.11 of the Appendix, we get a canonical morphism

$$F : T \longrightarrow \mathbf{W}_{n+1}(S),$$

yielding a factorization

$$F \circ i = f \circ \mathrm{Frob}^n.$$

Thus, the existence of an extension of V to (a vector bundle over) $\mathbf{W}_{n+1}(S)$ implies that of an extension of $(\mathrm{Frob}^n)^*(V)$ to T. □

Exercise 4.7. Let $n \geq 2$ be an integer.
Let S_n be a scheme of characteristic p^n. Denote by S_1 its reduction mod p. Put $A := H^0(S_1, \mathcal{O}_{S_1})$.
Let V/S_1 be a vector bundle, of rank m. Assume that there a line bundle L on S_1, such that $V \otimes L$ is generated by $m+1$ global sections. Show that $(\mathrm{Frob}^{n-1})^*(V)$ admits a lift to a vector bundle V_n on S_n.

5. Some equivalences of categories in deformation theory.

Let k be a perfect field of characteristic p.
Let S be a k-scheme. We denote by Frob the (absolute) Frobenius of S.

We would like to present a functorial description of three problems in deformation theory (the first two of which are classical), using the point of view of WtF Modules.

(I): Lift S to a scheme S_2, flat over $\mathbf{W}_2(k)$.
(I'): Lift S to a scheme S_2, flat over $\mathbf{W}_2(k)$, *together with its Frobenius morphism.*
(II): Lift a given vector bundle V/S to a $\mathbf{W}_2$-bundle V_2.

In (I'), we require the existence of an endomorphism F_S of S_2, whose mod p reduction equals the (absolute) Frobenius of S. Since k is perfect, such an F_2 is then automatically compatible with the Frobenius of $\mathbf{W}_2(k)$.

Problems (I) and (I') have been the subject of sustained investigation from many authors—see, for instance, the seminal papers [DI] and [MS]. Note that (I') is a—much—stronger requirement than (I). Problem (II), to the knowledge of the authors of the present text, is new.

We begin with a bunch of elementary technical remarks. The first concern pullbacks and pushforwards of extensions, that we shall constantly use later on. We present them an exercise in homological algebra.

Exercise 5.1. Let $\mathcal{A}$ be an Abelian category. Let

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

be an exact sequence in $\mathcal{A}$, thought of as an extension of C by A.

1) Let $f : A \longrightarrow A'$ be a morphism in $\mathcal{A}$. Show that there exists a unique extension $f_*(\mathcal{E})$, fitting in a commutative diagram

$$\begin{array}{rcccccccc} \mathcal{E} : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \Vert & & \\ f_*(\mathcal{E}) : 0 & \longrightarrow & A' & \longrightarrow & D & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

We refer to $f_*(\mathcal{E})$ as the pushforward of $\mathcal{E}$ by f.
2) Let $g : C' \longrightarrow C$ be a morphism in $\mathcal{A}$. Show that there exists a unique extension $g^*(\mathcal{E})$, fitting in a commutative diagram

$$\begin{array}{rcccccccc} g^*(\mathcal{E}) : 0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & C' & \longrightarrow & 0 \\ & & \Vert & & \downarrow & & \downarrow{\scriptstyle g} & & \\ \mathcal{E} : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0. \end{array}$$

We refer to $g^*(\mathcal{E})$ as the pullback of $\mathcal{E}$ by g.
3) Let

$$\mathcal{E}' : 0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0,$$

be another extension in $\mathcal{A}$. Show that the data of a commutative diagram

$$\begin{array}{rcccccccc} \mathcal{E} : 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle g} & & \\ \mathcal{E}' : 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

is equivalent to an isomorphism (of extensions of C by A') $f_*(\mathcal{E}) \xrightarrow{\sim} g^*(\mathcal{E}')$.

Remark 5.2. Let

$$\mathcal{E} : 0 \longrightarrow \mathcal{O}_S \longrightarrow B \longrightarrow C \longrightarrow 0,$$

be an extension of (quasi-coherent) Modules over a scheme S, of characteristic p. The Frobenius

$$\mathrm{Frob} : \mathcal{O}_S \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S),$$
$$x \longmapsto x^p$$

is $\mathcal{O}_S$-linear. The notation $\mathrm{Frob}_*(\mathcal{E})$ may then refer either to the pushforward

$$\mathrm{Frob}_*(\mathcal{E}) : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow B \longrightarrow C \longrightarrow 0,$$

in the sense of the exercise above, or to the extension

$$\mathrm{Frob}_*(\mathcal{E}) : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathrm{Frob}_*(B) \longrightarrow \mathrm{Frob}_*(C) \longrightarrow 0,$$

obtained by applying the exact functor (on $\mathcal{O}_S$-Modules) Frob_*.
We believe that this ambiguity is nothing serious, and that what is meant by $\mathrm{Frob}_*(\mathcal{E})$ is always clear from the context.
Note that a similar issue may arise, for the notation $\mathrm{Frob}^*(\mathcal{E})$.

Now, assume given an extension of Abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

where A and C are killed by p. Then, B is killed by p^2, and there exists a canonical homomorphism

$$\kappa : C \longrightarrow A,$$

which vanishes if, and only if, $pB = 0$. In the opposite situation, κ is an isomorphism if, and only if, B is a lift of the $\mathbb{F}_p$-vector space C to a free $(\mathbb{Z}/p^2\mathbb{Z})$-module. The construction of κ can be done in a much wider context; this is the object of the Exercise below.

Exercise 5.3. Let $\mathcal{S}$ be a site, and let

$$\mathcal{E} : 0 \longrightarrow A \longrightarrow B \xrightarrow{\pi} C \longrightarrow 0,$$

be an exact sequence of sheaves of Abelian groups on $\mathcal{S}$.
Assume given an endomorphism $f \in \underline{\mathrm{Hom}}_{\mathcal{S}}(B, B)$, leaving A stable. It thus induces an endomorphism of C, still denoted by f. Assume that $f(A) = f(C) = 0$.

1) Explain how one defines a homomorphism

$$\kappa = \kappa_{E,f} \in \underline{\mathrm{Hom}}_{\mathcal{S}}(C, A),$$

through the following process. Let X be an object of $\mathcal{S}$. For any section $s \in C(X)$, pick a covering (X_i) of X such that, for each i, the restriction $s_i = s_{|X_i}$ lifts to $t_i \in B(X_i)$. Then $u_i := f(t_i)$ belongs to $A(X_i)$. The u_i glue to give an element in $A(X)$, depending only on s. We denote it by $\kappa(s)$.
2) (trivial) Show that $\kappa = 0$ if and only if $F = 0$.
3) Here $\mathcal{S}$ is (at the reader's convenience) either the Zariski, étale or fppf site, big or small, of a scheme S of characteristic p. Consider the exact sequence

$$\mathcal{E} : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathbf{W}_2(\mathcal{O}_S) \xrightarrow{\pi} \mathcal{O}_S \longrightarrow 0.$$

Take f to be multiplication by p. Show that

$$\kappa_{\mathcal{E},p} = \mathrm{Frob} : \mathcal{O}_S \xrightarrow{x \mapsto x^p} \mathrm{Frob}_*(\mathcal{O}_S).$$

5.1. An equivalence of categories for Problem (I). To begin with, we assume that S/k is smooth.
One can then freely replace "flat" by "smooth" in (I).
By deformation theory, we know that there exists a canonical class

$$\mathrm{Obs}(S) \in \mathrm{Ext}^2_{\mathcal{O}_S-\mathrm{Mod}}(\Omega^1_{S/k}, \mathcal{O}_S),$$

which vanishes if, and only if, problem (I) has a positive answer.
In what follows, we give a simple functorial interpretation of this class. Note that, in Proposition 2.2 of the recent preprint [Yo], a similar goal is achieved. The approach chosen there is quite different from ours: a main input in its proof is Proposition 1 of the Appendix of [MS], which is concerned with Problem (I').

Let us first recall some well-known concepts and facts, from deformation theory. For details, we refer to [Il].

Definition 5.4. *Let B be a base scheme. Let S be a scheme over B, and let $\mathcal{M}$ be a coherent $\mathcal{O}_S$-module. A square-zero extension of S by $\mathcal{M}$ (in the category of B-schemes) is the data of a B-scheme T, together with a closed embedding*

$$i : S \longrightarrow T,$$

defined by an Ideal $\mathcal{I} \subset \mathcal{O}_T$ of square zero, equipped with an isomorphism of $\mathcal{O}_S$-Modules

$$f : \mathcal{I} \xrightarrow{\sim} M.$$

Square-zero extensions of S by M (in the category of B-schemes) form a category (where morphisms are isomorphisms), which we denote by $\mathbf{Exal_B}(S, M)$.

Remark 5.5. The previous Definition implicitly uses the fact that the $\mathcal{O}_T$-module $\mathcal{I}$ is actually an $\mathcal{O}_S$-module.
In short, a square zero extension can be thought of as an extension

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{O}_T \xrightarrow{\pi} \mathcal{O}_S \longrightarrow 0,$$

for which $\mathcal{M}$ is an ideal of square zero, and such that π is a homomorphism of $\mathcal{O}_B$-algebras.

Definition 5.6. *Let S be a scheme, and let M and N be $\mathcal{O}_S$-modules. We denote by $\mathbf{Ext}^1_{\mathcal{O}_S}(N, M)$ the category whose objects are extensions of $\mathcal{O}_S$-modules*

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0,$$

with morphisms being morphisms of exact sequences, which are the identity on M on N.

Remark 5.7. Morphisms in $\mathbf{Ext}^1_{\mathcal{O}_S}(N, M)$ are isomorphisms, and the group of automorphisms of any object is canonically isomorphic to $\mathrm{Hom}_{\mathcal{O}_S}(N, M)$.

The following fundamental Proposition is well-known. We refer to [Il], III, Proposition 1.1.9, for a proof (and also for understanding how the hypothesis that S/k is smooth can be removed, using the cotangent complex).

Proposition 5.8. *Let S be a smooth k-variety. Let M be an $\mathcal{O}_S$-Module.
There is an equivalence of categories*

$$\mathbf{Exal}_k(S, M) \xrightarrow{\sim} \mathbf{Ext}^1_{\mathcal{O}_S}(\Omega^1_{S/k}, M).$$

This equivalence is compatible with Baer sum.

From now on, unless specified otherwise, we do not assumme S/k to be smooth anymore.

Lemma 5.9. *Let S be a k-scheme.
The data of a lifting of S/k to a scheme S_2, flat over $\mathbf{W}_2(k)$, is equivalent to that of a square-zero extension—of schemes over $\mathbf{W}_2(k)$—*

$$(\mathcal{E} : 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_S \longrightarrow 0) \in \mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathcal{O}_S),$$

such that $\kappa_{\mathcal{E},p} = \mathrm{Id}$ (cf. Exercise 5.3 for the definition of κ).

Proof. Clear, from the following observation. Let M be a $\mathbf{W}_2(k)$-module. Then M is flat if and only if it is free. Equivalently, for the exact sequence

$$\mathcal{M} : 0 \longrightarrow pM \longrightarrow M \longrightarrow M/pM \longrightarrow 0,$$

the homomorphism $\kappa_{\mathcal{M},p} : M/pM \longrightarrow pM$ is an isomorphism. ∎

Remark 5.10. In Lemma 5.9, it is of course crucial to work in the category of $\mathbf{W}_2(k)$-schemes, and hence consider square-zero extensions in $\mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathcal{O}_S)$.

DEFINITION 5.11. *Using the description of the previous Lemma, the liftings of S/k to a scheme flat over $\mathbf{W}_2(k)$ form a category (with morphisms being isomorphisms). It is the full subcategory of $\mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathcal{O}_S)$ consisting of extensions having $\kappa = \mathrm{Id}$. We denote it by $\mathcal{L}_{\mathbf{W}_2(k)}(S)$, or simply by $\mathcal{L}_2(S)$—which is unambiguous thanks to the next elementary Lemma.*

LEMMA 5.12. *Let S be a k-scheme. Then, the forgetful functor*

$$\mathcal{L}_{\mathbf{W}_2(k)}(S) \longrightarrow \mathcal{L}_{\mathbb{Z}/p^2\mathbb{Z}}(S)$$

is an equivalence of categories.

Proof. It is enough to deal with the affine case $S = \mathrm{Spec}(A)$. Since k is perfect, a lift of A to a flat $(\mathbb{Z}/p^2\mathbb{Z})$-algebra can be given the structure of a (flat) $\mathbf{W}_2(k)$-algebra in a unique way (standard verification, left to the reader). Hence also, a lift of a homomorphism of k-algebras $A \longrightarrow A'$ to a homomorphism of flat $(\mathbb{Z}/p^2\mathbb{Z})$-algebras $A_2 \longrightarrow A'_2$ is automatically $\mathbf{W}_2(k)$-linear. This proves the Lemma. $\square$

From now on, the reader may thus do one of the following:

a) Bluntly assume that $k = \mathbb{F}_p$ everywhere.
b) Go on with an arbitrary perfect field k, but stop (or do not start) worrying about checking $\mathbf{W}_2(k)$-linearity of homomorphisms.

From now on, we denote by S a *reduced* k-scheme.

The next Definition is the key prerequisite for formulating our equivalence of categories.

DEFINITION 5.13. *Denote by*

$$\mathcal{E}F(S) : 0 \longrightarrow \mathcal{O}_S \xrightarrow{\mathrm{Frob}} \mathrm{Frob}_*(\mathcal{O}_S) \xrightarrow{d} \mathrm{Frob}_*(B^1_S) \longrightarrow 0$$

the natural sequence of $\mathcal{O}_S$-Modules, in which $\mathrm{Frob}_*(B^1_S)$ is the cokernel of Frobenius (this notation is in accordance with the usual one).
Recall that, for the natural exact sequence

$$\mathcal{E}\mathbf{W}_2(S) : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathbf{W}_2(\mathcal{O}_S) \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

one has

$$\kappa_{\mathcal{E}\mathbf{W}_2(S),p} = \mathrm{Frob} : \mathcal{O}_S \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S),$$

by Exercise 5.3.
The pushforward of $\mathcal{E}\mathbf{W}_2(S)$ by d is a square-zero extension of S by $\mathrm{Frob}_*(B^1_S)$, denoted by

$$\mathrm{C}\mathbf{W}_2(S) : 0 \longrightarrow \mathrm{Frob}_*(B^1_S) \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Clearly, it has $\kappa = 0$. Hence, we have $p\mathcal{O}_T = 0$. In other words,

$$\mathrm{C}\mathbf{W}_2(S) \in \mathbf{Exal}_k(S, \mathrm{Frob}_*(B^1_S))$$

is a square-zero extension of S in the category of k-schemes.
If S/k is smooth, we get using Proposition 5.8 an extension of $\mathcal{O}_S$-modules

$$C\Omega(S) : 0 \longrightarrow \mathrm{Frob}_*(B^1_S) \longrightarrow E \longrightarrow \Omega^1_{S/k} \longrightarrow 0,$$

corresponding to $\mathrm{C}\mathbf{W}_2(S)$.

The extensions $\mathcal{E}F(S)$, $\mathrm{C}\mathbf{W}_2(S)$ and $C\Omega(S)$ canonically depend on the k-scheme S.

Remark 5.14. If S/k is smooth, the interested reader can check that $C\Omega(S)$ is given by the Cartier operator. This explains our choice of terminology: C stands for Cartier.

We can now state and prove the equivalence of categories announced in the beginning of this Section. Recall that S is any reduced k-scheme.

Definition 5.15. *Denote by*

$$d : \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathrm{Frob}_*(B^1_S)$$

the natural $\mathcal{O}_S$-linear quotient. Denote by $\tilde{\mathcal{L}}_2(S)$ the category whose objects are pairs $(\mathcal{E}, f)$, consisting of a square-zero extension

$$(\mathcal{F} : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_S \longrightarrow 0) \in \mathbf{Exal}_k(S, \mathrm{Frob}_*(\mathcal{O}_S)),$$

together with an isomorphism (in $\mathbf{Exal}_k(S, \mathrm{Frob}_*(B^1_S))$)

$$f : d_*(\mathcal{F}) \xrightarrow{\sim} \mathrm{C}\mathbf{W}_2(S).$$

Morphisms in $\tilde{\mathcal{L}}_2(S)$ are (iso)morphisms commuting to the given isomorphisms.

Theorem 5.16. *There is an equivalence of categories*

$$\Phi : \mathcal{L}_2(S) \xrightarrow{\sim} \tilde{\mathcal{L}}_2(S),$$

given with a quasi-inverse in the proof below.

Proof.

Let

$$(\mathcal{E} : 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{S_2} \longrightarrow \mathcal{O}_S \longrightarrow 0) \in \mathcal{L}_2(S)$$

be a lift of S to a scheme flat over $\mathbf{W}_2(k)$. One has $\kappa_{\mathcal{E},p} = \mathrm{Id}$ (cf. Exercise 5.3 for the definition of κ). The pushforward

$$\mathrm{Frob}_*(\mathcal{E}) : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathcal{O}_{S_2} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

and

$$\mathcal{E}\mathbf{W}_2(S) : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathbf{W}_2(\mathcal{O}_S) \longrightarrow \mathcal{O}_S \longrightarrow 0$$

are both square-zero extensions of S by $\mathrm{Frob}_*(\mathcal{O}_S)$, in the category of schemes over $\mathbf{W}_2(k)$. They both have $\kappa = \mathrm{Frob}$, so that their difference $\mathcal{E}\mathbf{W}_2(S) - \mathrm{Frob}_*(\mathcal{E})$ (using Baer sum) is a square-zero extension

$$\mathcal{F} : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

in the category of k-schemes. Indeed, the computation of κ commutes to (Baer) sum, so that $\mathcal{F}$ indeed has $\kappa = 0$. Because $d \circ \mathrm{Frob} = 0$, the extension $d_*(\mathrm{Frob}_*(\mathcal{E}))$ has a canonical trivialization, so that the square-zero extension $d_*(\mathcal{F})$ is canonically isomorphic to $d_*(\mathcal{E}\mathbf{W}_2(S)) = \mathrm{C}\mathbf{W}_2(S)$. Denoting the canonical isomorphism by f, we see that the assignment

$$\mathcal{E} \longrightarrow (\mathcal{F}, f)$$

is a functor Φ, from $\mathcal{L}_2(S)$ to $\tilde{\mathcal{L}}_2(S)$.
A quasi-inverse to Φ is obtained as follows. Pick an object $(\mathcal{F}, f)$ in $\tilde{\mathcal{L}}_2(S)$.
We view

$$\mathcal{F} : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_S \longrightarrow 0$$

as a square-zero extension in $\mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathrm{Frob}_*(\mathcal{O}_S))$, having $\kappa = 0$. Then, the (Baer) difference

$$\tilde{\mathcal{E}} := (\mathcal{E}\mathbf{W}_2(S) - \mathcal{F}) \in \mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathrm{Frob}_*(\mathcal{O}_S))$$

has $\kappa = \mathrm{Frob}$, and $d_*(\tilde{\mathcal{E}})$ is equipped with the canonical trivialization induced by f. Therefore, it naturally yields a square-zero extension

$$\mathcal{E} : 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{S_2} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

in $\mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathcal{O}_S)$, together with a canonical isomorphism $\mathrm{Frob}_*(\mathcal{E}) \xrightarrow{\sim} \tilde{\mathcal{E}}$. The extension $\mathcal{E}$ automatically has $\kappa = \mathrm{Id}$, hence belongs to $\mathcal{L}_2(S)$. The assignment $(\mathcal{F}, f) \mapsto \mathcal{E}$ defines a functor Ψ in the reverse direction.
Checking that Φ and Ψ are mutually inverse is left to the reader. $\square$

We present two consequences of the preceding Proposition. The first one is the existence of liftings of Frobenius-split k-schemes. This question has been studied by several authors. See for instance the recent preprint [Yo], Theorem 4.4 (where S/k is assumed to be smooth, but where Frobenius-splitting is replaced by a weaker notion).

Corollary 5.17. *(Lifting Frobenius-split k-schemes) Let S be a reduced k-scheme. Assume that S is Frobenius-split, i.e. that the exact sequence of coherent $\mathcal{O}_S$-modules*

$$\mathcal{E}F(S) : 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow \mathrm{Frob}_*(B^1_S) \longrightarrow 0$$

splits. Then, S admits a lift to a scheme S_2, flat over $\mathbf{W}_2(k)$. More precisely, every splitting of $\mathcal{E}F(S)$ canonically determines such an S_2.

Proof. Denote by $s : \mathrm{Frob}_*(B^1_S) \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S)$ the splitting of $\mathcal{E}F(S)$. Put $\mathcal{F} := s_*(\mathrm{C}\mathbf{W}_2(S))$. Then it is clear that $\mathcal{F}$ is a square-zero extension and that $d_*(\mathcal{F})$ is isomorphic to $\mathrm{C}\mathbf{W}_2(S)$. We conclude by Theorem 5.16. $\square$

Corollary 5.18. *Assume that S/k is smooth.*
To give a lift of S to a scheme S_2, smooth over $\mathbf{W}_2(k)$, is then equivalent to giving an extension of $\mathcal{O}_S$-modules

$$\mathcal{E} : 0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_S) \longrightarrow E \longrightarrow \Omega^1_{S/k} \longrightarrow 0,$$

together with an isomorphism (in $\mathbf{Ext}^1_{\mathcal{O}_S}(\Omega^1_{S/k}, \mathrm{Frob}_*(B^1_S))$)

$$d_*(\mathcal{E}) \xrightarrow{\sim} C\Omega(S),$$

where $C\Omega(S) \in \mathbf{Ext}^1_{\mathcal{O}_S}(\Omega^1_{S/k}, \mathrm{Frob}_*(B^1_S))$ is the extension given in 5.14.
In particular, such a lift S_2 exists if and only if the cup-product

$$\mathcal{E}F(S) \cup C\Omega(S) \in \mathbf{Ext}^2_{\mathcal{O}_S}(\Omega^1_{S/k}, \mathcal{O}_S)$$

vanishes.
This cup-product is the obstruction $\mathrm{Obs}(S)$*, given by classical deformation theory.*

Proof. The first part of the Corollary is simply a translation of the equivalence of categories given in Theorem 5.16, using that of Theorem 5.8. The second part follows from the first, applying standard considerations in homological algebra.
That the cup-product $\mathcal{E}F(S) \cup C\Omega(S)$ is the usual obstruction is left as an exercise for the reader. $\square$

5.2. An equivalence of categories for Problem (I').

Lemma 5.19. *Let S be a reduced k-scheme. Let $S_2 \in \mathcal{L}_2(S)$ be a lift of S, viewed as a square-zero extension*

$$\mathcal{E} : 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{S_2} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

in $\mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathcal{O}_S)$, having $\kappa = \mathrm{Id}$. The following is true.

1) Form the pullback $\mathrm{Frob}^*(\mathcal{E}) \in \mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathrm{Frob}_*(\mathcal{O}_S))$. It is the extension whose middle term is the sheaf of $\mathbf{W}_2(k)$-algebras $\mathcal{O}_T$, defined as the fibered product

$$\begin{array}{ccc} \mathcal{O}_T & \longrightarrow & \mathcal{O}_S \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Frob}} \\ \mathcal{O}_{S_2} & \longrightarrow & \mathcal{O}_S, \end{array}$$

using the (absolute) Frobenius of S. It is then canonically isomorphic to $\mathcal{E}\mathbf{W}_2(S)$.

2) The data of a lift of the Frobenius of S to S_2 is equivalent to the data of an isomorphism of square-zero extensions

$$\mathrm{Frob}_*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}\mathbf{W}_2(S)$$

in $\mathbf{Exal}_{\mathbf{W}_2(k)}(S, \mathrm{Frob}_*(\mathcal{O}_S))$.

Proof. By glueing, it is enough to deal with the case where $S = \mathrm{Spec}(A)$ is affine. Then $S_2 = \mathrm{Spec}(A_2)$, for some free $\mathbf{W}_2(k)$-algebra A_2.
We have a commutative diagram

$$\begin{array}{rcccccccc} \mathcal{E}\mathbf{W}_2(A) : 0 & \longrightarrow & \mathrm{Frob}_*(A) & \longrightarrow & \mathbf{W}_2(A) & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \mathrm{Frob}} & & \\ \mathcal{E} : 0 & \longrightarrow & A & \longrightarrow & A_2 & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

where f_2 is the ring homomorphism defined in Corollary 6.15. The slightly abusive notation

$$\mathrm{Id} : \mathrm{Frob}_*(A) \longrightarrow A$$

makes sense, remembering that $\mathrm{Frob}_*(A) = A$, as Abelian groups. The existence of this diagram proves point 1). For point 2), note that a lift F_2 of the Frobenius of A to an endomorphism of the ring A_2 is equivalent to that of a commutative diagram

$$\begin{array}{rcccccccc} \mathcal{E} : 0 & \longrightarrow & A & \longrightarrow & A_2 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Frob}} & & \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle \mathrm{Frob}} & & \\ \mathcal{E} : 0 & \longrightarrow & A & \longrightarrow & A_2 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

In other words, it is equivalent to giving an isomorphism (in $\mathbf{Exal}_{\mathbf{W}_2(k)}(A, \mathrm{Frob}_*(A))$)

$$\mathrm{Frob}_*(\mathcal{E}) \xrightarrow{\sim} \mathrm{Frob}^*(\mathcal{E}).$$

The proof is finished using point 1). ∎

Proposition 5.20. *Let S/k be a reduced scheme. Then, the data of a lift of S to a scheme S_2, flat over $\mathbf{W}_2(k)$, together with a lift $\mathrm{Frob}_2 : S_2 \longrightarrow S_2$ of the Frobenius of S, is equivalent to that of a splitting of the square-zero extension*

$$C\mathbf{W}_2(S) \in \mathbf{Exal}_k^1(S, \mathrm{Frob}_*(B_S^1)).$$

If S/k is smooth, this is equivalent to the data of a splitting of the extension

$$C\Omega(S) \in \mathbf{Ext}^1_{\mathcal{O}_S}(\Omega^1_{S/k}, \mathrm{Frob}_*(B_S^1)).$$

Proof. We use the equivalence of categories provided in Theorem 5.16. Keeping the notation of its proof, we see by Lemma 5.19 that the data of a lift of S, flat over $\mathbb{Z}/p^2\mathbb{Z}$, together with its Frobenius, amounts to specifying an isomorphism $\mathrm{Frob}_*(\mathcal{E}) \xrightarrow{\sim} \mathcal{E}\mathbf{W}_2(S)$; that is, a splitting of $C\mathbf{W}_2(S)$. To prove the last assertion (when S/k is smooth), just remember that $C\Omega(S)$ then corresponds to $C\mathbf{W}_2(S)$, through the equivalence of Proposition 5.8.

□

5.3. An equivalence of categories for Problem (II). The approach taken here is, *mutatis mutandis*, the same as that used to tackle Problem (I) in the previous paragraph. Some proofs are thus left to the reader. We shall use freely symmetric and divided powers of Modules. These are polynomial functors, characterized by a universal property (see the Appendix, [Fe], or [DCF] for details).

Let S be a scheme over k. Let V be a vector bundle over S. Denote by

$$f : P := \mathbb{P}_S(V) \longrightarrow S$$

the projective bundle of V. Denote by

$$\mathrm{ad} : V \longrightarrow \mathrm{Frob}_*(\mathrm{Frob}^*(V))$$

the (first) adjunction morphism. Recall that, if M a quasi-coherent $\mathcal{O}_S$-Module, we have an adjunction isomorphism

$$\mathbf{Ext}^1_{\mathcal{O}_S}(\mathrm{Frob}^*(V), M) \xrightarrow{\sim} \mathbf{Ext}^1_{\mathcal{O}_S}(V, \mathrm{Frob}_*(M)),$$

given by applying the exact functor Frob_*, followed by pulling back by ad. To get its inverse, apply Frob^*, and push forward by the (second) adjunction $\mathrm{Frob}^*(\mathrm{Frob}_*(V)) \longrightarrow V$.

Definition 5.21. *We have morphisms (between vector bundles over S)*

$$\mathrm{Ver}_V : \mathrm{Frob}^*(V) \longrightarrow \mathrm{Sym}^p_{\mathcal{O}_S}(V), \quad v \otimes 1 \longmapsto v^p;$$

and

$$\mathrm{Frob}_V : \Gamma^p_{\mathcal{O}_S}(V) \longrightarrow \mathrm{Frob}^*(V), \quad [v]_p \longmapsto v \otimes 1;$$

the Verschiebung and the Frobenius of V. They fit into exact sequences

$$\mathcal{E}\mathrm{Ver}(V) : 0 \longrightarrow \mathrm{Frob}^*(V) \xrightarrow{\mathrm{Ver}_V} \mathrm{Sym}^p_{\mathcal{O}_S}(V) \xrightarrow{q_V} \overline{\mathrm{Sym}}^p_{\mathcal{O}_S}(V) \longrightarrow 0$$

and

$$\mathcal{E}\mathrm{Frob}(V) : 0 \longrightarrow \overline{\Gamma}^p_{\mathcal{O}_S}(V) \longrightarrow \Gamma^p_{\mathcal{O}_S}(V) \xrightarrow{\mathrm{Frob}_V} \mathrm{Frob}^*(V) \longrightarrow 0,$$

where $\overline{\mathrm{Sym}}^p_{\mathcal{O}_S}(V) := \mathrm{Coker}(\mathrm{Ver}_V)$ and $\overline{\Gamma}^p_{\mathcal{O}_S}(V) := \mathrm{Ker}(\mathrm{Frob}_V)$.

Lemma 5.22. *We have a canonical isomorphism (of vector bundles over S)*

$$\Phi_V : \overline{\mathrm{Sym}}^p_{\mathcal{O}_S}(V) \xrightarrow{\sim} \overline{\Gamma}^p_{\mathcal{O}_S}(V),$$

given in the proof below.
In what follows, we may tacitly use it to identify these vector bundles.

Proof. There is a natural homomorphism

$$\alpha_V : \mathrm{Sym}^p_{\mathcal{O}_S}(V) \longrightarrow \Gamma^p_{\mathcal{O}_S}(V),$$

defined on sections by the formula

$$v_1 v_2 \dots v_p \mapsto [v_1]_1 \dots [v_p]_1.$$

It takes values in $\overline{\Gamma}^p_{\mathcal{O}_S}(V)$, and vanishes on $\mathrm{Im}(\mathrm{Ver}_V)$—as follows from the identity

$$[v]_1^p = p![v]_p = 0.$$

The resulting homomorphism

$$\overline{\mathrm{Sym}}^p_{\mathcal{O}_S}(V) \longrightarrow \overline{\Gamma}^p_{\mathcal{O}_S}(V)$$

is an isomorphism. To check this, one can assume that $V = \mathcal{O}_S^d$ is the trivial rank d vector bundle, and that $S = \mathrm{Spec}(A)$ is affine. The rest of the verification is left to the reader. □

Exercise 5.23. Show that the dual of the exact sequence (of vector bundles over S) $\mathcal{E}\mathrm{Ver}(V^\vee)$ is canonically isomorphic to $\mathcal{E}\mathrm{Frob}(V)$.

Remark 5.24. By adjunction, we have a canonical isomorphism

$$\mathbf{Ext}^1_{\mathcal{O}_S}(\mathrm{Frob}^*(V), \overline{\Gamma}^p_{\mathcal{O}_S}(V)) \xrightarrow{\sim} \mathbf{Ext}^1_{\mathcal{O}_S}(V, \mathrm{Frob}_*(\overline{\Gamma}^p_{\mathcal{O}_S}(V))),$$

through which $\mathcal{E}\mathrm{Frob}(V)$ corresponds to an extension

$$\overline{\mathcal{E}}\mathrm{Frob}(V) : 0 \longrightarrow \mathrm{Frob}_*(\overline{\Gamma}^p_{\mathcal{O}_S}(V)) \longrightarrow \Phi(V) \longrightarrow V \longrightarrow 0.$$

Now, it is tempting to consider the $\mathcal{O}_S$-Module V as a $\mathbf{W}_2(\mathcal{O}_S)$-Module, to get a "mod p^2 avatar" of $\mathcal{E}\mathrm{Frob}(V)$, as follows:

$$0 \longrightarrow \mathrm{Sym}^p_{\mathcal{O}_S}(V) \longrightarrow \Gamma^p_{\mathbf{W}_2(\mathcal{O}_S)}(V) \xrightarrow{\mathrm{Frob}_V} \mathrm{Frob}^*(V) \longrightarrow 0.$$

This idea is in the spirit of [DCF], Proposition 7.4. This indeed works, *if S is perfect*—which is assumed in *loc. cit.* However, in the case of an arbitrary S, the functor $\Gamma^p_{\mathbf{W}_2(\mathcal{O}_S)}(V)$ is rather nasty, and the previous sequence is not exact in the middle. As an alternate approach, we would now like to promote the use of Witt lifts of line bundles.

Recall (Section 3) that we have a natural exact sequence of WtF-Modules on P

$$0 \longrightarrow \mathrm{Frob}_*(\mathcal{O}_P(p)) \longrightarrow \mathbf{W}_2(\mathcal{O}_P(1)) \longrightarrow \mathcal{O}_P(1) \longrightarrow 0.$$

Applying f_*, we get an exact sequence of WtF Modules on S

$$0 \longrightarrow \mathrm{Frob}_*(\mathrm{Sym}^p_{\mathcal{O}_S}(V)) \longrightarrow f_*(\mathbf{W}_2(\mathcal{O}_P(1))) \xrightarrow{\Phi_V} V \longrightarrow 0.$$

The surjectivity of the last arrow follows from the (effect on global sections of the) Teichmüller section $s_{\mathcal{O}_P(1)}$; see 3.1.

Definition 5.25. *The exact sequence (of $\mathbf{W}_2(\mathcal{O}_S)$-Modules on S)*

$$0 \longrightarrow \mathrm{Frob}_*(\mathrm{Sym}^p_{\mathcal{O}_S}(V)) \xrightarrow{i_V} f_*(\mathbf{W}_2(\mathcal{O}_P(1))) \xrightarrow{\Phi_V} V \longrightarrow 0$$

will be denoted by $\overline{\mathcal{E}}\mathbf{W}_2(V)$.
We denote by s_V the canonical (sheaf-theoretic) section s_V of Φ_V induced by the Teichmüller section $s_{\mathcal{O}_P(1)}$.

Remark 5.26. Assume that $S = \mathrm{Spec}(A)$ is affine. Denote by B the symmetric algebra $\mathrm{Sym}_A(V) := \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i_A(V)$. One can also obtain $\overline{\mathcal{E}}\mathbf{W}_2(V)$ from the exact sequence

$$0 \longrightarrow \mathrm{Frob}_*(B) \longrightarrow \mathbf{W}_2(B) \longrightarrow B \longrightarrow 0,$$

by pulling it back by the inclusion $V \longrightarrow B$, and pushing it forward by the projection $\mathrm{Frob}_*(B) \longrightarrow \mathrm{Frob}_*(\mathrm{Sym}^p_A(V))$.

Lemma 5.27. *One has the formula*

$$s_V(x+y) = s_V(x) + s_V(y) + i_V\left(\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}\right).$$

Proof. This is the well-known formula sometimes used to define addition on $\mathbf{W}_2$. □

Lemma 5.28. *The extension $\overline{\mathcal{E}}\mathbf{W}_2(V)$ has $\kappa = \kappa_{\overline{\mathcal{E}}\mathbf{W}_2(V),p}$ given by the map adjoint to Ver_V. Concretely, it is given by*

$$\begin{aligned} V &\longrightarrow \mathrm{Frob}_*(\mathrm{Sym}^p_{\mathcal{O}_S}(V)), \\ x &\longmapsto x^p. \end{aligned}$$

We have a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Frob}_*(\mathrm{Sym}^p_{\mathcal{O}_S}(V)) & \longrightarrow & f_*(\mathbf{W}_2(\mathcal{O}_P(1))) & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \frac{1}{(p-1)!}\mathrm{Frob}_*(q_V)} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \mathrm{ad}} & & \\ 0 & \longrightarrow & \mathrm{Frob}_*(\overline{\mathrm{Sym}}^p_{\mathcal{O}_S}(V)) & \longrightarrow & \mathrm{Frob}_*(\Gamma^p_{\mathcal{O}_S}(V)) & \longrightarrow & \mathrm{Frob}_*(\mathrm{Frob}^*(V)) & \longrightarrow & 0, \end{array}$$

where the upper line is $\overline{\mathcal{E}}\mathbf{W}_2(V)$, the lower line is $\mathrm{Frob}_*(\mathcal{E}\mathrm{Frob}(V))$, and where F is defined (on sections) by the formula

$$F(s_V(x)) = [x]_p,$$

for $x \in V$.
Consequently, the pushforward $(\mathrm{Frob}_*(q_V))_*(\overline{\mathcal{E}}\mathbf{W}_2(V))$ is canonically isomorphic to $-\overline{\mathcal{E}}\mathrm{Frob}(V)$.

Proof. We can assume that $S = \mathrm{Spec}(A)$ is affine. The key point here is to check that the formula giving F makes sense, and indeed defines a homomorphism of WtF-Modules. This follows from the previous Lemma, combined to the similar formula

$$[x+y]_p = [x]_p + [y]_p + \sum_{i=1}^{p-1} [x]_i [y]_{p-i},$$

in $\Gamma^p_A(V)$. Details are left to the reader. □

Proposition 5.29. *(An equivalence of categories for Problem (II)) To give a lift of V to a $\mathbf{W}_2$-bundle V_2 on S is equivalent to give an extension*

$$\mathcal{F} \in \mathbf{Ext}^1_{\mathcal{O}_S}(\mathrm{Frob}^*(V), \mathrm{Sym}^p_{\mathcal{O}_S}(V)),$$

together with an isomorphism

$$(q_V)_*(\mathcal{F}) \xrightarrow{\sim} \mathcal{E}\mathrm{Frob}(V),$$

in $\mathbf{Ext}^1_{\mathcal{O}_S}(\mathrm{Frob}^*(V), \overline{\mathrm{Sym}}^p_{\mathcal{O}_S}(V))$.

Proof. Similar to that of Theorem 5.16. □

Remark 5.30. In purely cohomological terms, the previous Proposition implies that the obstruction $\mathrm{Obs}(V)$ to lifting V to a $\mathbf{W}_2$-bundle on S is the element of $\mathrm{Ext}^2_{\mathcal{O}_S}(\mathrm{Frob}^*(V), \mathrm{Frob}^*(V))$ represented by the 2-extension

$$0 \longrightarrow \mathrm{Frob}^*(V) \longrightarrow \mathrm{Sym}^p_{\mathcal{O}_S}(V) \longrightarrow \Gamma^p_{\mathcal{O}_S}(V) \longrightarrow \mathrm{Frob}^*(V) \longrightarrow 0,$$

defined as the cup-product of $\mathcal{E}\mathrm{Frob}(V)$ and $\mathcal{E}\mathrm{Ver}(V)$. A more elementary instance of this extension was first introduced in an earlier version of [DCF], where it was named "the fundamental 2-extension of V".

We finish this Section with an exercise, connecting Problems (I) and (II).

Exercise 5.31. Let S be a smooth variety, over a perfect field k of characteristic p. Denote by V the tangent bundle of S/k.
Consider the following assertions.

a) Problem (I') for S: the variety S can be lifted to a smooth scheme over $\mathbf{W}_2(k)$, together with its Frobenius morphism.
b) The variety S can be lifted to a smooth scheme $S_2/\mathbf{W}_2(k)$, in such a way that $\mathrm{Frob}^*(V)$ extends to a vector bundle on S_2.
c) Problem (II) for V/S: the vector bundle V/S admits a lift to a $\mathbf{W}_2$-bundle on S.

Show that $a) \Longrightarrow b) \Longrightarrow c)$.

5.4. An application: Grassmannians whose tautological bundle does not lift. In a previous version of this paper, available at arXiv:1807.04859v1, we claimed that all vector bundles admit lifting towers. This turned out to be false: the proof of Theorem 3.5 in *loc. cit.* is incorrect. We are grateful to Bhargav Bhatt and to Alexander Petrov for (independently) providing a counterexample. Bhatt's counterexample is available on his homepage, in the Notes section. A counterexample close to his is also available, in a previous version of this paper (arXiv:1807.04859v2), or on our webpages.
We now give a counterexample of another kind, for tautological bundles of Grassmannians. Our purpose here is to suggest that the equivalences of categories offered in the previous section allow very concrete computations of obstructions in deformation theory.

Theorem 5.32. *Let m and n be two integers, with $2 \leq m \leq n-2$.*
Then, the tautological vector bundle of the Grassmannian $\mathrm{Gr}(m,n)$ *(over any base of characteristic p) does not lift to a $\mathbf{W}_2$-bundle.*

The rest of this section is devoted to the proof of this Proposition.
Let k be a (perfect) field of characteristic p. Let E be a k-vector space, of dimension n.

Denote by

$$f : S = \mathrm{Gr}(m,E) \longrightarrow \mathrm{Spec}(k)$$

the structure morphism of the Grassmannian, parametrizing m-dimensional subspaces of E. We denote by V the tautological bundle of S. We thus have an exact sequence

$$0 \longrightarrow V \longrightarrow f^*(E) \longrightarrow W \longrightarrow 0,$$

whose cokernel is a vector bundle W on S. We have seen that V admits a lifting tower, in the (dual) cases $m=1$ or $m=n-1$.
We now assume that $2 \leq m \leq n-2$.
We are going to show that V does not lift to a $\mathbf{W}_2$-bundle on S.
Consider the extension (of vector bundles on S)

$$\mathcal{E}\mathrm{Frob}(V) : 0 \longrightarrow \overline{\mathrm{Sym}}^p_{\mathcal{O}_S}(V) \longrightarrow \Gamma^p_{\mathcal{O}_S}(V) \xrightarrow{\mathrm{Frob}_V} \mathrm{Frob}^*(V) \longrightarrow 0.$$

Assume that V can be lifted to a $\mathbf{W}_2$-bundle. By Proposition 5.29, this means that $\mathcal{E}\mathrm{Frob}(V)$ admits a lift to an extension

$$\mathcal{F} : 0 \longrightarrow \mathrm{Sym}^p_{\mathcal{O}_S}(V) \longrightarrow F \longrightarrow \mathrm{Frob}^*(V) \longrightarrow 0.$$

By point 1) of Lemma 5.33, $\mathcal{F}$ would be split, hence so would $\mathcal{E}\mathrm{Frob}(V)$—contradicting point 3) of the same Lemma.

LEMMA 5.33. *The following statements are true:*

1) We have $\mathrm{Ext}^1_{\mathcal{O}_S}(\mathrm{Frob}^*(V), \mathrm{Sym}^p_{\mathcal{O}_S}(V)) = 0$.
2) For all integers $i \geq 0$, we have $\mathrm{End}_{\mathcal{O}_S}(\Gamma^i_{\mathcal{O}_S}(V)) = k\mathrm{Id}$.
3) We have $\mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Frob}^*(V), \Gamma^p_{\mathcal{O}_S}(V)) = 0$.

Proof.

Put

$$T := \mathrm{Gr}(m-1,m,E);$$

it is the flag variety parametrizing partial flags

$$0 \subset V_{m-1} \subset V_m \subset E,$$

with $\dim(V_{m-1}) = m-1$ and $\dim(V_m) = m$. Consider the natural projections

$$g : T \longrightarrow S = \mathrm{Gr}(m,E).$$

and

$$g' : T \longrightarrow S' := \mathrm{Gr}(m-1,E).$$

Denote by $f : S' \longrightarrow \mathrm{Spec}(k)$ the structure morphism, and by V' the tautological bundle on S'. The morphism g (resp. g') is the Grassmann bundle $\mathbb{P}_S(V) = \mathrm{Gr}_S(m-1,V)$ (resp. $\mathrm{Gr}_{S'}(1, f'^*(E)/V')$). We have $V_{m-1} = g'^*(V')$ and $V_m = g^*(V)$.
We now prove 1). Consider the exact sequence (on T)

$$0 \longrightarrow g'^*(V') \longrightarrow g^*(V) \longrightarrow L \longrightarrow 0.$$

The line bundle L can be identified either to $\mathcal{O}_T(1)$ (Serre's twisting sheaf with respect to the projective bundle g), or to $\mathcal{O}'_T(-1)$ (the tautological bundle of g'). Adopting the first point of view, we have

$$\mathrm{Ext}^1_{\mathcal{O}_S}(\mathrm{Frob}^*(V), \mathrm{Sym}^p_{\mathcal{O}_S}(V)) = \mathrm{Ext}^1_{\mathcal{O}_T}(\mathrm{Frob}^*(g^*(V)), \mathcal{O}_T(p)),$$

using Leray's spectral sequence for $f \circ g$, combined to the projection formula. Taking the long exact sequence for $\mathrm{Ext}^{-}_{\mathcal{O}_T}(., \mathcal{O}_T(p))$'s associated to

$$0 \longrightarrow \mathrm{Frob}^*({g'}^*(V')) \longrightarrow \mathrm{Frob}^*(g^*(V)) \longrightarrow \mathcal{O}_T(p) \longrightarrow 0,$$

we get

$$0 = H^1_{Zar}(T, \mathcal{O}_T) = \mathrm{Ext}^1_{\mathcal{O}_T}(\mathcal{O}_T(p), \mathcal{O}_T(p))$$
$$\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_T}(\mathrm{Frob}^*(g^*(V)), \mathcal{O}_T(p)) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_T}(\mathrm{Frob}^*({g'}^*(V')), \mathcal{O}_T(p)).$$

To conclude, it thus suffices to show that $\mathrm{Ext}^1_{\mathcal{O}_T}(\mathrm{Frob}^*({g'}^*(V')), \mathcal{O}_T(p))$ vanishes.

To see this, we change our point of view, and identify L to $\mathcal{O}'_T(-1)$, using g'. We know that

$$R^i {g'}_*(\mathcal{O}'_T(-p)) = 0,$$

for $i = 0, 1$. Indeed, since $\dim(E) - \dim(V') = n - m + 1 \geq 3$, the projective bundle g' has relative dimension at least two, and the result follows from the usual computation of the cohomology of projective bundles. Using Leray's spectral sequence for $f' \circ g'$, together with the projection formula, we thus finally get

$$\mathrm{Ext}^1_{\mathcal{O}_T}(\mathrm{Frob}^*({g'}^*(V')), \mathcal{O}_T(p)) = \mathrm{Ext}^1_{\mathcal{O}_T}(\mathrm{Frob}^*({g'}^*(V')), \mathcal{O}'_T(-p)) = 0,$$

qed.

We prove point 2) by induction on $1 \leq m \leq n-1$, and on $i \geq 0$. Note that the result holds for $m = 1$ (and i arbitrary), or for $i = 0$ (and m arbitrary), because then $\mathrm{End}_{\mathcal{O}_S}(\Gamma^i_{\mathcal{O}_S}(V)) \simeq H^0(S, \mathcal{O}_S) = k$.

Here again, it suffices to show that

$$\mathrm{End}_{\mathcal{O}_T}(\Gamma^i_{\mathcal{O}_T}(g^*(V))) = k\mathrm{Id}.$$

From the exact sequence (relative to the projective bundle g)

$$\mathcal{T} : 0 \longrightarrow {g'}^*(V') \longrightarrow g^*(V) \longrightarrow \mathcal{O}_T(1) \longrightarrow 0,$$

we deduce an exact sequence

$$\Gamma^i(\mathcal{T}) : 0 \longrightarrow \Gamma^i_{\mathcal{O}_T}({g'}^*(V')) \longrightarrow \Gamma^i_{\mathcal{O}_T}(g^*(V)) \longrightarrow \Gamma^{i-1}_{\mathcal{O}_T}(g^*(V))(1) \longrightarrow 0.$$

Applying $\mathrm{Hom}_{\mathcal{O}_T}(., \Gamma^i_{\mathcal{O}_T}(g^*(V)))$, we get an exact sequence

$$0 = \mathrm{Hom}_{\mathcal{O}_T}(\Gamma^{i-1}_{\mathcal{O}_T}(g^*(V))(1), \Gamma^i_{\mathcal{O}_T}(g^*(V))) \longrightarrow \mathrm{End}_{\mathcal{O}_T}(\Gamma^i_{\mathcal{O}_T}(g^*(V)))$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{O}_T}(\Gamma^i_{\mathcal{O}_T}({g'}^*(V')), \Gamma^i_{\mathcal{O}_T}(g^*(V))).$$

To see that its first term indeed vanishes, use the projection formula (for g) to identify it to

$$f_*(g_*(\mathcal{O}_T(-1)) \otimes_{\mathcal{O}_S} \mathrm{Hom}_{\mathcal{O}_S}(\Gamma^{i-1}_{\mathcal{O}_S}(V), \Gamma^i_{\mathcal{O}_S}(V))) = 0,$$

because $g_*(\mathcal{O}_T(-1)) = 0$. To compute its last term, insert it in the (middle term of the) exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_T}(\Gamma^i_{\mathcal{O}_T}({g'}^*(V')), \Gamma^i_{\mathcal{O}_T}({g'}^*(V'))) \xrightarrow{\phi} \mathrm{Hom}_{\mathcal{O}_T}(\Gamma^i_{\mathcal{O}_T}({g'}^*(V')), \Gamma^i_{\mathcal{O}_T}(g^*(V)))$$
$$\longrightarrow \mathrm{Hom}_{\mathcal{O}_T}(\Gamma^i_{\mathcal{O}_T}({g'}^*(V')), \Gamma^{i-1}_{\mathcal{O}_T}(g^*(V))(1)).$$

Note then that $\Gamma^{i-1}_{\mathcal{O}_T}(g^*(V))(1)$ has a filtration by subbundles, with graded pieces reading as

$$\Gamma^{i-j}_{\mathcal{O}_T}(g'^*(V'))(j),$$

for $j = 1 \dots i$. Thanks to the projection formula again, but using g' instead of g, we see that (for $j = 1 \dots i-1$)

$$\mathrm{Hom}_{\mathcal{O}_T}(\Gamma^{i}_{\mathcal{O}_T}(g'^*(V')), \Gamma^{i-j}_{\mathcal{O}_T}(g'^*(V'))(j))$$

$$= \mathrm{Hom}_{\mathcal{O}_T}(\Gamma^{i}_{\mathcal{O}_T}(g'^*(V')), \Gamma^{i-j}_{\mathcal{O}_T}(g'^*(V')) \otimes_{\mathcal{O}_T} \mathcal{O}'_T(-j)) = 0.$$

Thus, we get $\mathrm{Hom}_{\mathcal{O}_T}(\Gamma^{i}_{\mathcal{O}_T}(g'^*(V')), \Gamma^{i-1}_{\mathcal{O}_T}(g^*(V))(1)) = 0$, and ϕ is an isomorphism. Now, pick an element $f \in \mathrm{End}_{\mathcal{O}_T}(\Gamma^{i}_{\mathcal{O}_T}(g^*(V)))$. By what we just saw, f restricts to an endomorphism f' of $\Gamma^{i}_{\mathcal{O}_T}(g'^*(V'))$. By the induction hypothesis, $f' = \lambda \mathrm{Id}$, for some $\lambda \in k$. Replacing f by $f - \lambda\mathrm{Id}$, we reduce to the case $f' = 0$, implying $f = 0$, qed.
Finally, we prove 3). Pick an element $g \in \mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Frob}^*(V), \Gamma^{p}_{\mathcal{O}_S}(V))$. Then, $g \circ \mathrm{Frob}_V$ belongs to $\mathrm{Hom}_{\mathcal{O}_S}(\Gamma^{p}_{\mathcal{O}_S}(V), \Gamma^{p}_{\mathcal{O}_S}(V))$, hence is equal to $\lambda\mathrm{Id}$ for some $\lambda \in k$, by point 2) (for $i = p$). If λ were invertible, g would be surjective, which is clearly impossible since $\dim(V) \geq 2$. Thus, $\lambda = 0$, implying $g = 0$ by the surjectivity of Frob_V. □

6. Appendix: lifting $\mathbf{W}_n$-schemes, and a universal property of Witt vectors.

Let S be a scheme of characteristic p.
For any integer $n \geq 2$, the thickening $\mathbf{W}_n(S)$ of S is then universal up to Frobenius, among all lifts of S to a scheme of characteristic p^n.
The goal of this Appendix is to make this statement precise, in a slightly more general context. Along the way, we give a simple recursive definition of Witt vectors, using the (very convenient) divided power functor $\Gamma^p_{\mathbb{Z}}$. Basic properties of divided powers (of modules over commutative rings) and of polynomial laws can be found in [Fe], or in [DCF].
Before proceeding further, we would like to point out that we made the choice to state and prove our constructions in the language of commutative algebra (i.e. of affine schemes). They are entirely canonical. By gluing, the statements that follow can thus be extended to arbitrary schemes, in an obvious fashion.

We begin with an algebraic warm up (see also [DCF], Lemma 6.13).

Lemma 6.1. *Let $n \geq 1$ be an integer.*
Let R be a commutative ring, in which $(n-1)!$ is invertible.
Let M be an R-module.
Then, the R-module $\Gamma^n_R(M)$ is generated by the pure symbols $[x]_n$, for $x \in M$.
Furthermore, let $N \subset M$ be a submodule. Then, the kernel of the natural reduction

$$\rho : \Gamma^n_R(M) \longrightarrow \Gamma^n_R(M/N)$$

is generated by elements of the shape

$$[x+y]_n - [x]_n,$$

for $x \in M$ and $y \in N$.

Proof. The first statement is a particular case of the second one (take $N = M$), which we now prove. Denote by L the submodule of $\Gamma^n_R(M)$ spanned by all differences $[x+y]_n - [x]_n$. We have to show that the inclusion $L \subset \mathrm{Ker}(\rho)$ is an equality. For any $\alpha \in R$, we have

$$[\alpha x + y]_n - [\alpha x]_n = \sum_{i=0}^{n-1} \alpha^i [x]_i [y]_{n-i} \in L.$$

Now, note that the invertibility of $(n-1)!$ in R is equivalent to that of the $n \times n$ Vandermonde matrix

$$((j^i))_{0 \leq i,j \leq n-1}.$$

Thus, for all $x \in M$, $y \in N$ and $i = 0 \ldots n-1$, the element

$$[x]_i [y]_{n-i}$$

belongs to L.

Put $Q := \Gamma^p_R(M)/L$. The association

$$f : x \mapsto [x]_p$$

naturally defines a polynomial law between the R-modules M and Q, homogeneous of degree p. We claim that the law f factors through the quotient $M \longrightarrow M/N$. To see this, it is sufficient to check that, for any commutative R-algebra R', for every $x' = \sum x_i \otimes \alpha'_i \in M' := M \otimes_R R'$, and for very $y' = \sum y_j \otimes \beta'_j \in N' := N \otimes_R R'$, we have

$$[x' + y']_n = [x']_n,$$

in $Q \otimes_R R'$ (recall that forming divided powers commutes to base change). Developping symbols, this directly follows from what we have proved above. By the universal property of divided powers, we get an R-linear arrow

$$\Gamma^n_R(M/N) \longrightarrow Q,$$

clearly seen to be the inverse of

$$\overline{\rho} : Q \longrightarrow \Gamma^n_R(M/N).$$

The proof is over.

□

From now on, A is a *perfect* commutative ring of characteristic p.
Let $n \geq 1$ be an integer, and let R be an arbitrary commutative $\mathbf{W}_n(A)$-algebra. We would like to "lift" R to a commutative $\mathbf{W}_{n+1}(A)$-algebra. What follows is an attempt to do so.
Consider R as a $\mathbf{W}_{n+1}(A)$-module, through the quotient ring homomorphism

$$\pi_{n+1,n} : \mathbf{W}_{n+1}(A) \longrightarrow \mathbf{W}_n(A).$$

The p-th divided power module

$$\Gamma^p_{\mathbf{W}_{n+1}(A)}(R)$$

carries a natural structure of a $\mathbf{W}_{n+1}(A)$-algebra, given on pure symbols by $[x]_p[y]_p = [xy]_p$, for all $x, y \in R$. The symmetric power $\mathrm{Sym}^p_{\mathbf{W}_{n+1}(A)}(R)$ can be naturally turned into a $\Gamma^p_{\mathbf{W}_{n+1}(A)}(R)$-module, through the formula

$$[a]_p.(x_1 \otimes \ldots \otimes x_p) := (ax_1) \otimes \ldots \otimes (ax_p),$$

for all $a, x_1, \ldots, x_p \in R$.

Remark 6.2. It is crucial here to consider R as a $\mathbf{W}_{n+1}(A)$-module, in order to build its p-th divided power. For symmetric powers, this is not an issue: the canonical map $\mathrm{Sym}^p_{\mathbf{W}_{n+1}(A)}(R) \longrightarrow \mathrm{Sym}^p_{\mathbf{W}_n(A)}(R)$ is an isomorphism.

We now have a canonical $\mathbf{W}_{n+1}(R)$-linear homomorphism

$$j_p : \mathrm{Sym}^p_{\mathbf{W}_{n+1}(A)}(R) \longrightarrow \Gamma^p_{\mathbf{W}_{n+1}(A)}(R),$$
$$x_1 \otimes \ldots \otimes x_p \mapsto \frac{1}{(p-1)!}[x_1]_1 \ldots [x_p]_1.$$

Note that it needs not be injective, and that it is not a ring homomorphism: it sends 1 to p.
Denote by J_p the kernel of the ring homomorphism

$$\mathrm{Sym}^p_{\mathbf{W}_{n+1}(A)}(R) \longrightarrow R,$$
$$x_1 \otimes \ldots \otimes x_p \mapsto x_1 \ldots x_p;$$

it is a submodule of the $\mathbf{W}_{n+1}(R)$-module $\mathrm{Sym}^p_{\mathbf{W}_{n+1}(A)}(R)$. Denote by $\tilde{J}_p$ its homomorphic image in $\Gamma^p_{\mathbf{W}_{n+1}(A)}(R)$; it is an ideal in this ring.

Definition 6.3. *Using the notation above, we set*

$$\Phi_{n+1}(R) := \Gamma^p_{\mathbf{W}_{n+1}(A)}(R)/\tilde{J}_p.$$

The assignment

$$R \mapsto \Phi_{n+1}(R)$$

is a functor, from the category of $\mathbf{W}_n(A)$*-algebras to that of* $\mathbf{W}_{n+1}(A)$*-algebras.*

Remark 6.4. The functor Φ_{n+1} can be extended to modules. More precisely, for any R-module M, the formula

$$\Phi_{n+1}(M) := \Gamma^p_{\mathbf{W}_{n+1}(A)}(M) \otimes_{\Gamma^p_{\mathbf{W}_{n+1}(A)}(R)} \Phi_{n+1}(R)$$

defines a $\Phi_{n+1}(R)$-module. This construction shall not be needed in the sequel.

Definition 6.5. *The canonical homomorphism*

$$j_p : \mathrm{Sym}^p_{\mathbf{W}_{n+1}(A)}(R) \longrightarrow \Gamma^p_{\mathbf{W}_{n+1}(A)}(R)$$

yields, by passing to the quotient by $\tilde{J}_p$*, a homomorphism of* $\Phi_{n+1}(R)$*-modules*

$$R \longrightarrow \Phi_{n+1}(R),$$

which we denote by ϕ_{n+1}*. Its image is an ideal of* $\Phi_{n+1}(R)$*.*

Definition 6.6. *A homomorphism of commutative rings*

$$f : R \longrightarrow S$$

is said to be p*-elementary if it is surjective, and if its kernel* I *satisfies*

$$pI = I^p = 0.$$

If this is the case, for every $x \in S$*, we set*

$$V(x) := pX \in R,$$

where X *is any preimage of* x *by* f*.*

Example 6.7. For any $n \geq 1$, the homomorphism

$$\pi_{n+1,n} : \mathbf{W}_{n+1}(B) \longrightarrow \mathbf{W}_n(B)$$

is p-elementary.

Now, let $\mathcal{R}$ be a $\mathbf{W}_{n+1}(A)$-algebra. Assume given a p-elementary homomorphism of $\mathbf{W}_{n+1}(A)$-algebras

$$f : \mathcal{R} \longrightarrow R.$$

The formula

$$x \mapsto x^p$$

defines a multiplicative polynomial law, homogeneous of degree p, from the $\mathbf{W}_{n+1}(A)$-algebra $\mathcal{R}$ to itself. From the fact that f is p-elementary, it is readily checked that this polynomial law factors through f. Indeed, we have

$$(x+i)^p = x^p + \sum_{j=1}^{p-1} \binom{p}{j} x^j i^{p-j} + i^p = x^p + 0 + 0,$$

for all $x \in \mathcal{R}$ and all $i \in \mathrm{Ker}(f)$- and the same formula holds after a base-change to an arbitrary $\mathbf{W}_{n+1}(A)$-algebra. By the universal property of divided powers, it is thus given by a homomorphism of $\mathbf{W}_{n+1}(A)$-algebras

$$F : \Gamma^p_{\mathbf{W}_{n+1}(A)}(R) \longrightarrow \mathcal{R},$$

$$[f(x)]_p \mapsto x^p,$$

for any $x \in \mathcal{R}$. We have

$$F([f(x_1)]_1 \ldots [f(x_p)]_1) = p! x_1 \ldots x_p.$$

Hence, F vanishes on $\tilde{J}_p$, yielding a homomorphism of $\mathbf{W}_{n+1}(A)$-algebras

$$\Phi_{n+1}(R) \longrightarrow \mathcal{R},$$

$$[f(x)]_p \mapsto x^p.$$

Definition 6.8. *The morphism*

$$\Phi_{n+1}(R) \longrightarrow \mathcal{R},$$

$$[f(x)]_p \mapsto x^p$$

just constructed will be denoted by $L_{n+1}(f)$.
It canonically depends on the p-elementary arrow $f : \mathcal{R} \longrightarrow R$.

For a commutative A-algebra B, we "almost" have $\Phi_{n+1}(\mathbf{W}_n(B)) = \mathbf{W}_{n+1}(B)$. We now make this precise.

Proposition 6.9. *Let B/A be a commutative algebra. Then, there exists a canonical surjection of $\mathbf{W}_{n+1}(A)$-algebras*

$$s_{n+1} : \Phi_{n+1}(\mathbf{W}_n(B)) \longrightarrow \mathbf{W}_{n+1}(B),$$

mapping $[\tau_n(x)]_p$ to $\tau_{n+1}(x)$. We have

$$\mathrm{Frob} \circ s_{n+1} = L_{n+1}(\pi_{n+1,n}) : \Phi_{n+1}(\mathbf{W}_n(B)) \longrightarrow \mathbf{W}_{n+1}(B).$$

If B is perfect, s_{n+1} is an isomorphism.
For B arbitrary, its kernel is (additively) spanned by elements of the shape

$$[i(\tau_{n-1}(x))]_p - p^{p-1}\phi_{n+1}(\tau_n(x)),$$

where $i = i_{n-1,n}$ and where x runs through all elements of B.

Proof. The statement does not really depend on the perfect $\mathbb{F}_p$-algebra A: we can thus assume that $A = \mathbb{F}_p$.
Assume first that B is reduced.
Consider the morphism

$$L := L_{n+1}(\pi_{n+1,n}) : \Phi_{n+1}(\mathbf{W}_n(B)) \longrightarrow \mathbf{W}_{n+1}(B)$$

$$[\pi_{n+1,n}(x))]_p \mapsto x^p,$$

associated to $\pi_{n+1,n} : \mathbf{W}_{n+1}(B) \longrightarrow \mathbf{W}_n(B)$ (see Definition 6.8). The image of L is the subring of $\mathbf{W}_{n+1}(B)$ generated by p-th powers, which coincides with

$$\mathbf{W}_{n+1}(B^p) \subset \mathbf{W}_{n+1}(B)$$

(exercise for the reader, using the fact that $\mathbf{W}_{n+1}(B)$ is additively spanned by Teichmüller representatives, and that $p\mathbf{W}_{n+1}(B) \subset \mathbf{W}_{n+1}(B^p)$). Since B is reduced, we thus see that L naturally factors through the injective ring homomorphism

$$\mathrm{Frob} : \mathbf{W}_{n+1}(B) \longrightarrow \mathbf{W}_{n+1}(B),$$

$$\tau_{n+1}(x) \mapsto \tau_{n+1}(x^p),$$

whence the existence (and unicity) of the surjection s_{n+1}.
Denote by

$$M \subset \Phi_{n+1}(\mathbf{W}_n(B))$$

the *subgroup* spanned by elements of the shape

$$[i(\tau_{n-1}(x))]_p - p^{p-1}\phi_{n+1}(\tau_n(x)),$$

for $x \in B$.
We now compute, for $x \in \mathbf{W}_n(B)$,

$$L(i(\tau_{n-1}(x))) = i_{n,n+1}(\tau_n(x))^p = p^{p-1} i_{n,n+1}(\tau_n(x^p)) = p^p \tau_{n+1}(x) \in \mathbf{W}_{n+1}(B).$$

On the other hand, we have

$$L(\phi_{n+1}(\tau_n(x))) = L(j_p(\tau_n(x) \otimes 1 \ldots \otimes 1)) = L([\tau_n(x)]_1[1]_{p-1}) = p\tau_{n+1}(x),$$

so that M is indeed contained in $\mathrm{Ker}(s_{n+1})$. Put

$$\Psi_{n+1}(B) := \Phi_{n+1}(\mathbf{W}_n(B))/M;$$

it is (a priori) an Abelian group. Now, there exists a unique map

$$f = f_{n,B} : \mathbf{W}_{n+1}(B) \longrightarrow \Phi_{n+1}(\mathbf{W}_n(B))$$

such that

$$f(i_{n,n+1}(a) + \tau_{n+1}(x)) = \phi_{n+1}(a) + [\tau_n(x)]_p,$$

for all $a \in \mathbf{W}_n(B)$, and for all $x \in B$. It might not be surjective if B is not perfect. However, the composite

$$\rho : \mathbf{W}_{n+1}(B) \xrightarrow{f} \Phi_{n+1}(\mathbf{W}_n(B)) \xrightarrow{can} \Psi_{n+1}(B)$$

is always surjective. This follows from the fact that Teichmüller representatives additively span $\mathbf{W}_n(B)$, and that the association

$$x \in B \mapsto [\tau_n(x)]_p \in \Phi_{n+1}(\mathbf{W}_n(B))$$

is additive, modulo $\langle \mathrm{Im}(\phi_{n+1}), M \rangle$.
It is then easy to see that $s_{n+1} \circ \rho$ is the identity. In other words, we indeed have $M = \mathrm{Ker}(s_{n+1})$. If B is perfect, the ideal $\mathbf{W}_n(B) \subset \mathbf{W}_{n+1}(B)$ is generated by p, implying $M = 0$ (verification left to the reader).
We no longer assume that B is reduced. Denote by $\rho : C \longrightarrow B$ a surjective

homomorphism of $\mathbb{F}_p$-algebras, where $C := \mathbb{F}_p[X_i, i \in I]$ is a polynomial ring (say, with I infinite). We have a commutative diagram

$$\begin{array}{ccc} \mathbf{W}_{n+1}(C) & \xrightarrow{f_{n,C}} & \Psi_{n+1}(C) \\ \big\downarrow{\scriptstyle \mathbf{W}_{n+1}(\rho)} & & \big\downarrow{\scriptstyle \Psi_{n+1}(\rho)} \\ \mathbf{W}_{n+1}(B) & \xrightarrow{f_{n,B}} & \Psi_{n+1}(B). \end{array}$$

We know that $f_{n,C}$ is an isomorphism.
Put

$$X := (X_1, X_2, \dots, X_n) \in \mathbf{W}_n(C),$$

and

$$F = F(X_1, \dots, X_n) := f_{n,C}^{-1}([X]_p) \in \mathbf{W}_{n+1}(C).$$

By Lemma 6.1, we know that the kernel of $\Psi_{n+1}(\rho)$ is generated by (classes of) symbols of the shape $[Y]_p - [Y']_p$, for $Y = (Y_1, Y_2, \dots, Y_n)$ and $Y' = (Y_1', Y_2', \dots, Y_n')$ elements of $\mathbf{W}_n(C)$, such that $\rho(Y_i) = \rho(Y_i')$ for all i. Specializing, one has

$$f_{n,C}(Z) = [Y]_p,$$

where $Z := F(Y_1, \dots, Y_n)$, and similarly for Y' (with $Z' := F(Y_1', \dots, Y_n')$).
Thus, we get

$$f_{n,C}^{-1}([Y]_p - [Y']_p) = Z - Z'.$$

Since Z and Z' have the same image by $\mathbf{W}_{n+1}(\rho)$, we have shown that

$$f_{n,C}^{-1}(\mathrm{Ker}(\Psi_{n+1}(\rho))) = \mathrm{Ker}(\mathbf{W}_{n+1}(\rho)).$$

An easy diagram chase then shows that $f_{n,B}$ is an isomorphism. Setting $s_{n+1} := f_{n,B}^{-1}$ is the last step of the proof.

□

Remark 6.10. If $A = \mathbb{F}_p$, it is known (!) that $\mathbf{W}_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}$. Hence, the preceding lemma yields a simple recursive definition of the ring scheme $\mathbf{W}_n$, over $\mathbb{F}_p$. Note that it just uses the polynomial functor $\Gamma^p_{\mathbb{Z}}$. Indeed, for a $(\mathbb{Z}/p^n\mathbb{Z})$-module M, the canonical map $\Gamma^p_{\mathbb{Z}}(M) \longrightarrow \Gamma^p_{\mathbb{Z}/p^{n+1}\mathbb{Z}}(M)$ is an isomorphism.
In fact, our insight here is rather simple. If $\mathbf{W}_n$ has been defined, then $\mathbf{W}_{n+1}$ occurs as an explicit quotient of $\Gamma^p_{\mathbb{Z}}(\mathbf{W}_n)$, in which the pure symbol $[\tau_n(x)]_p$ becomes $\tau_{n+1}(x)$.

The following phenomenon is well-known. Let k be a perfect field of characteristic p. Let R be a complete local ring, with residue field k. Then, R is a $\mathbf{W}(k)$-algebra, in a natural way. We are now going to offer a "slow" generalization of this fact.
Until the end, we assume that $A = \mathbb{F}_p$.

Definition 6.11. *Let $n \geq 1$ be an integer.*
A p-elementary diagram is a diagram

$$\begin{array}{ccc} \mathbf{W}_n(B) & \xrightarrow{f_n} & R_n \\ \big\downarrow{\scriptstyle \pi_{n,n-1}} & & \big\downarrow{\scriptstyle F_n} \\ \mathbf{W}_{n-1}(B) & \xrightarrow{f_{n-1}} & R_{n-1}, \end{array}$$

in the category of commutative rings, with the following properties.
The letter B denotes a commutative $\mathbb{F}_p$-algebra.

The arrow F_n is assumed to be p-elementary.
The diagram "commutes up to Frob*", in the sense that*

$$f_{n-1} \circ \pi_{n,n-1} \circ \mathrm{Frob} = F_n \circ f_n : \mathbf{W}_n(B) \longrightarrow R_{n-1}.$$

We have

$$f_n(i_{n-1,n}(x)) = V(f_{n-1}(x)) \in R_n,$$

for all $x \in \mathbf{W}_{n-1}(B)$.

Note that, for $n = 1$, a p-elementary diagram is simply the data of a commutative B-algebra R_1.

Remark 6.12. If B is semi-perfect (i.e. if the Frobenius of B is surjective), the last condition of this Definition is a consequence of the preceding one (commutation up to Frob).

LEMMA 6.13. *(Lifting p-elementary diagrams)*
Let

$$\begin{array}{ccc} \mathbf{W}_n(B) & \xrightarrow{f_n} & R_n \\ \downarrow{\scriptstyle \pi_{n,n-1}} & & \downarrow{\scriptstyle F_n} \\ \mathbf{W}_{n-1}(B) & \xrightarrow{f_{n-1}} & R_{n-1} \end{array}$$

be a p-elementary diagram.
Let $F_{n+1} : R_{n+1} \longrightarrow R_n$ be any p-elementary homomorphism.
Then, there exists a canonical f_{n+1} such that

$$\begin{array}{ccc} \mathbf{W}_{n+1}(B) & \xrightarrow{f_{n+1}} & R_{n+1} \\ \downarrow{\scriptstyle \pi_{n+1,n}} & & \downarrow{\scriptstyle F_{n+1}} \\ \mathbf{W}_n(B) & \xrightarrow{f_n} & R_n \end{array}$$

is a p-elementary diagram.

Proof.

Applying Definition 6.8 to the p-elementary homomorphism F_{n+1}, we get a homomorphism

$$L_{n+1}(F_{n+1}) : \Phi_{n+1}(R_n) \longrightarrow R_{n+1}.$$

Composing it with $\Phi_{n+1}(f_n)$ yields

$$G : \Phi_{n+1}(\mathbf{W}_n(B)) \longrightarrow R_{n+1}.$$

Pick an element $x \in B$. We compute:

$$G([i_{n-1,n}(\tau_{n-1}(x))]_p) = L_{n+1}(F_n)([V(f_{n-1}(\tau_{n-1}(x)))]_p) = p^p X^p,$$

where $X \in R_{n+1}$ is any preimage of $f_{n-1}(\tau_{n-1}(x))$ by $F_n \circ F_{n+1}$. For such an X, we have

$$F_{n+1}(pX^p) = V(f_{n-1}(\tau_{n-1}(x^p))),$$

and it follows that

$$G([i_{n-1,n}(\tau_{n-1}(x))]_p) = p^{p-1} Y,$$

for any Y in the preimage of $V(f_{n-1}(\tau_{n-1}(x^p)))$. On the other hand, we have

$$G(\phi_{n+1}(\tau_n(x))) = G([\tau_n(x)]_1[1]_{p-1}) = pZ,$$

where $Z \in R_{n+1}$ is any preimage of $f_n(\tau_n(x))$ by F_{n+1}. But then, pZ is a preimage of

$$f_n(p\tau_n(x)) = f_n(i_{n-1,n}(\tau_{n-1}(x^p))) = V(f_{n-1}(\tau_{n-1}(x^p)))$$

by F_{n+1}, so that (choosing Y to be pZ) we get

$$G(p^{p-1}\phi_{n+1}(\tau_n(x)) - [i_{n-1,n}(\tau_{n-1}(x))]_p) = 0.$$

By Proposition 6.9, we infer that G passes to the quotient to a morphism

$$f_{n+1} : \mathbf{W}_{n+1}(B) \longrightarrow R_{n+1}.$$

Verifying that this f_{n+1} satisfies the needed assumptions is straightforward. □

Example 6.14. Consider the particular case $R_n = \mathbf{W}_n(B)$ and $F_{n+1} = \pi_{n+1,n}$ for all $n \geq 1$. Starting with $f_1 = \mathrm{Id}$, it is easy to check that the construction of Proposition 6.13 returns $f_n = \mathrm{Frob}^{n-1}$ for all n.

As a Corollary of Lemma 6.13, we recover the following well-known statement.

Corollary 6.15. *Let $n \geq 2$ be an integer. Let $\mathcal{B}$ be a $(\mathbb{Z}/p^n\mathbb{Z})$-algebra. Put $B := \mathcal{B}/p$. Then, there exists a unique ring homomorphism*

$$f_n : \mathbf{W}_n(B) \longrightarrow \mathcal{B}$$

mapping $\tau_n(\overline{x})$ to $x^{p^{n-1}}$, for all $x \in \mathcal{B}$.

Proof. For $i \geq 1$, put $R_i := \mathcal{B}/p^i$. The quotient maps $R_{i+1} \longrightarrow R_i$ are p-elementary. Induction, using the procedure of Proposition 6.13, shows that identity $f_1 : B \longrightarrow R_1$ can be lifted to an f_n with the required properties. □

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Charles De Clercq, Equipe Topologie Algébrique, Laboratoire Analyse, Géométrie et Applications, Université Paris 13, 93430 Villetaneuse.

Mathieu Florence, Equipe de Topologie et Géométrie Algébriques, Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4, place Jussieu, 75005 Paris.

Giancarlo Lucchini Arteche, Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Las Palmeras 3425, Ñuñoa, Santiago, Chile.