# An introduction to finite volume schemes for elliptic equations

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# The class of problems under consideration

## Equation under consideration :

$\Omega$ bounded domain of $\mathbb{R}^N$

$$-\mathrm{div}\,(\mathbf{A}\nabla u) + \mathbf{b}\cdot\nabla u + cu = f$$ + boundary conditions

$\mathbf{A}$ square matrix of size $N \times N$ symmetric (or hermitian) positive and definite.

$$\mathbf{A} = \mathbf{A}^T \qquad x^T\mathbf{A}x > 0 \quad \forall x \neq 0$$

$\mathbf{b}$ vector of $\mathbb{R}^N$ (or $\mathbb{C}^N$)

$c$ constant

$\longrightarrow$ model 1d problem:

$$\begin{cases} -u'' = f \text{ in } \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

# Famous numerical methods

- ✓ finite difference methods

  advantage: easy implementation

  drawback: regular mesh

- ✓ finite element methods (based on a variational formulation)

  advantage: very powerful theory for variational problem

  drawbacks:

  - application to non variational problem (hyperbolic)
  - lack of conservativity

- ✓ spectral methods

  advantage: super-algebraically convergence

  drawback: the domain has to be regular

# Outlook of the lecture

✓ Chapter 1: the continuous model problem

✓ Chapter 2: Construction and analysis of finite difference and finite volume methods for the numerical resolution of the one dimensional model problem

✓ Chapter 3: a finite volume method for the resolution of the 2d model problem

+ 3 tutorial classes (theoretical and pratical parts)

# Introduction

## Bibliography

✓ teaching materials (including lecture notes of P. Omnes)

http://www.math.univ-paris13.fr/~delourme/TeachingFV2015.html

✓ books

*Finite Volume Methods*, R. Eymard, T. Gallouet, R. Herbin.

*Scientific Computing with Matlab and Octave, 3rd edition*, A. Quarteroni, F. Saleri, P. Gervasio.

*Functional Analysis, Sobolev spaces and Partial Differential Equations*, H. Brezis.

# Evaluation

- ✓ Exam in november
- ✓ 3 tutorials (individual reports and codes to send before November 15)

# The continuous model problem

## 1 Basic concepts of functional analysis

- 1.1 the space $L^p(\Omega)$
- 1.2 the space $\mathcal{D}(\Omega)$, $\mathcal{D}(\overline{\Omega})$ and the space of distributions $\mathcal{D}'(\Omega)$
- 1.3 the Sobolev space $H^m(\Omega)$
- 1.4 Lax-Milgram Lemma

## 2 Application to our model problem

## 3 Properties of the solution

- 2.1 Elliptic regularity
- 2.2 Maximum principle

# I-Outlook of the second chapter

Construction and analysis of Finite difference and finite volume methods for the numerical resolution of the one dimensional model problem

# 1-Outlook of the second chapter

Construction and analysis of Finite difference and finite volume methods for the numerical resolution of the one dimensional model problem

## 3 Properties of the approximate solution

### 3.1 The finite difference method

### 3.2 The finite volume method

1.2.1 Conservatively of the flux

1.2.2 Maximum principle

## 4 Convergence analysis

### 4-1 Finite difference method

4.1.1 Consistency error

4.1.2 A stability result

4.1.3 Convergence result

### 4-2 Finite volume method

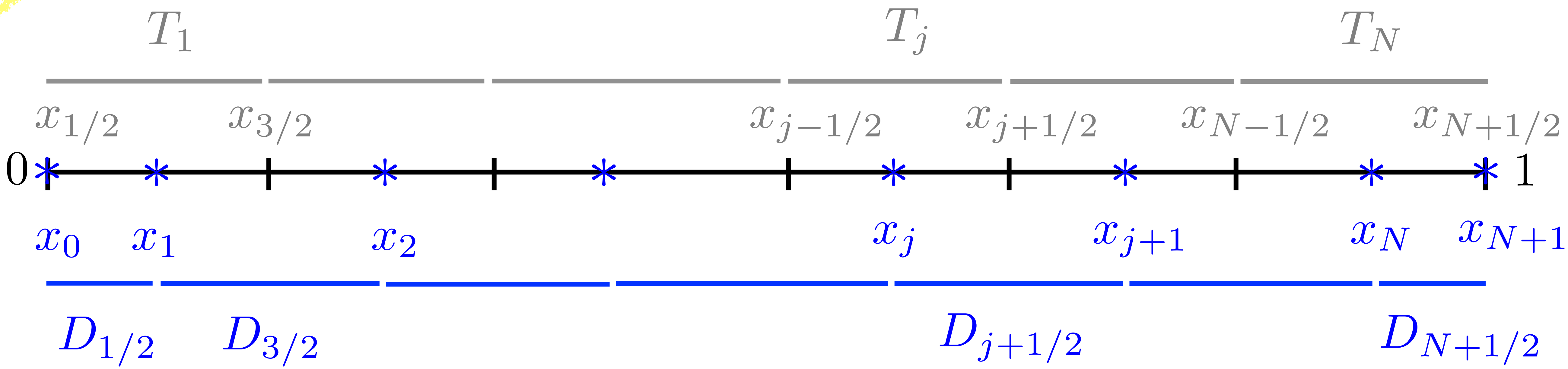
4.2.1 Two 'projection' operators

4.2.2 Consistency error for the flux

4.2.3 A stability result

4.2.4 Convergence result

# The finite volume mesh and dual mesh




$T_j = [x_{j-1/2} \quad x_{j+1/2}]$

$x_j \subset T_j \quad x_0 = x_{1/2} \quad x_{N+1} = x_{N+1/2}$

$D_{j+1/2} = [x_j \quad x_{j+1}]$

$x_{j+1/2} \in D_{j+1/2}$

$N$ segments $T_j$ $(1 \leq j \leq N)$

$N+1$ points $x_{j+1/2}$ $(0 \leq j \leq N)$

$N+1$ segments $D_{j+1/2}$ $(0 \leq j \leq N)$

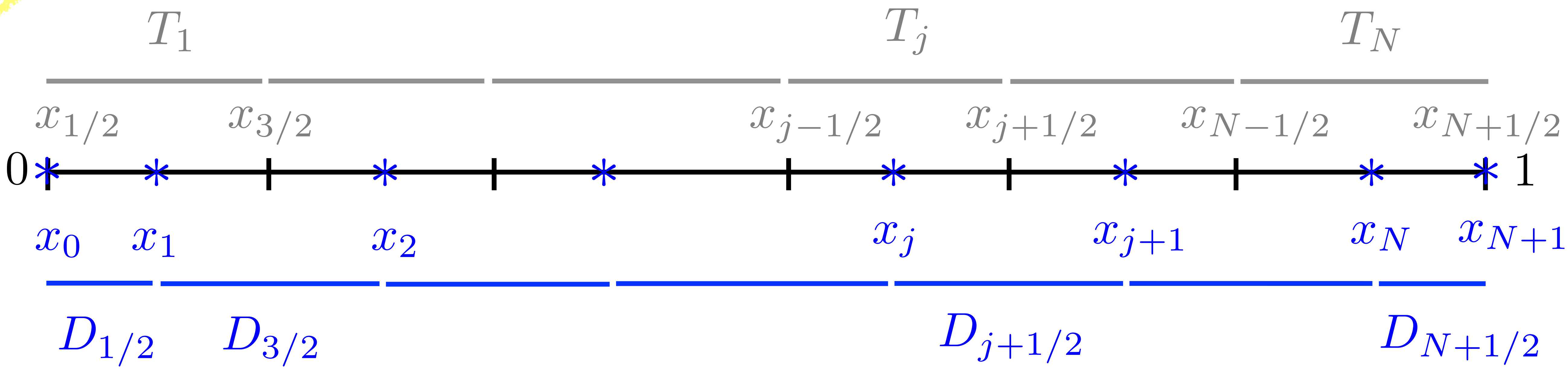
$N+2$ points $x_j$ $(0 \leq j \leq N+1)$

$\{T_j\}_{j=1}^{N}$ primal mesh

$\{D_{j+1/2}\}_{j=0}^{N}$ dual mesh

# The finite volume mesh and dual mesh

## 2-a The mesh and the dual mesh




- ✓ The segments $T_j$ do not have the same size (non uniform mesh)
- ✓ The point $x_j$ is not necessarily the middle of $T_j$

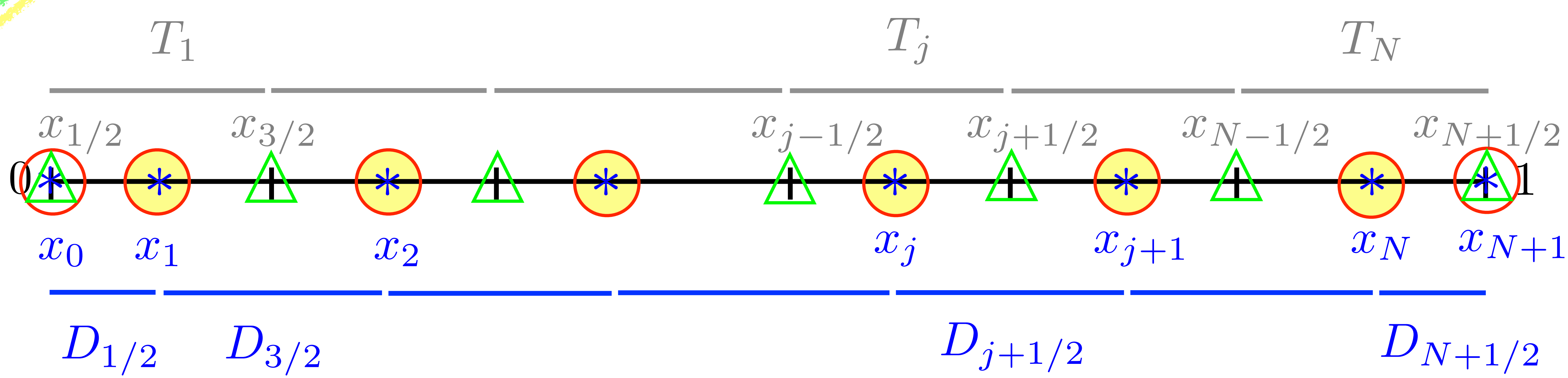
Notation

$|T_j| = x_{j+1/2} - x_{j-1/2}$ size of $T_j$

$|D_{j+1/2}| = x_{j+1} - x_j$ size of $D_{j+1/2}$

$$h := \max_{i \in [1:n]} |T_i|$$

# Discrete gradient and discrete divergence operators



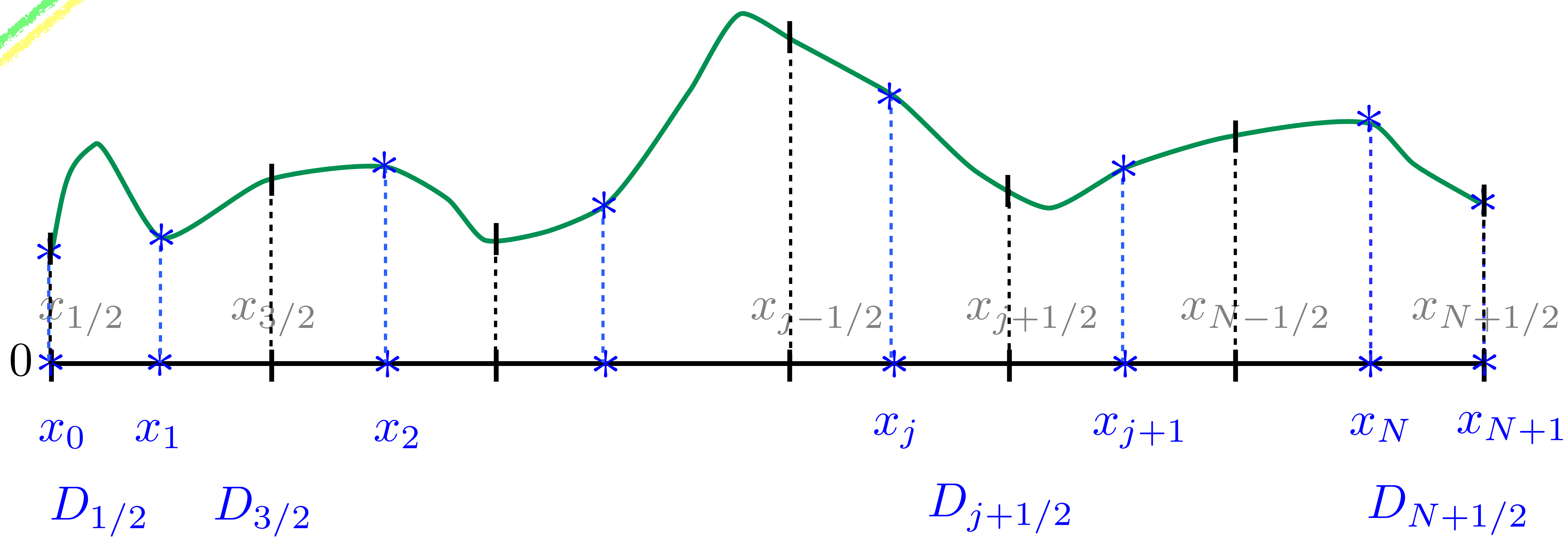

- ✓ The discret gradient operator:

$$g : \begin{cases} \mathbb{R}^{N+2} \to \mathbb{R}^{N+1} \\ \mathbf{v} = (v_j)_{j\in[0:N+1]} \mapsto (g\mathbf{v})_{j+1/2} := \dfrac{v_{j+1} - v_j}{|D_{j+1/2}|} \quad j \in [0:N] \end{cases}$$

- ✓ The discret divergence operator:

$$d : \begin{cases} \mathbb{R}^{N+1} \to \mathbb{R}^{N} \\ \mathbf{v} = (v_{j-1/2})_{j\in[0:N]} \mapsto (d\mathbf{v})_j := \dfrac{v_{j+1/2} - v_{j-1/2}}{|T_j|} \quad j \in [1:N] \end{cases}$$

# Projections operators




$$\Pi : \begin{cases} \mathcal{C}^0(\overline{\Omega}) \to \mathbb{R}^{N+2} \\ v \mapsto (\Pi v)_i = v(x_i) \qquad \forall i \in [0 : N+1] \end{cases}$$

$$\mathbf{P} : \begin{cases} \mathcal{C}^0(\overline{\Omega}) \to \mathbb{R}^{N+1} \\ v \mapsto (\mathbf{P} v)_{i+1/2} = v(x_{i+1/2}) \qquad \forall i \in [0 : N] \end{cases}$$