Exercice 1: a finite volume scheme for an elliptic equation with discontinuous coefficients

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The following questions have priority:

- Section 1.1: question 1.
- Section 1.2.
- Section 1.3: question 2.
- Section 1.4.
- Section 2.

Let Ω be a bounded domain in $\mathbb{R}^d$ (with $d = 1$ or $d = 2$) and $f \in L^2(\Omega)$. We consider the following problem: find $u \in H_0^1(\Omega)$ such that

$$-\text{div}(k(x)\nabla u) = f \quad \text{in} \quad \mathcal{D}'(\Omega), \tag{1}$$

where $k \in L^\infty(\Omega)$ is a positive function. More precisely, we assume that there exist two constants $C_+ > 0$ and $C_- > 0$, such that, for almost every $x \in \Omega$,

$$0 \leq C_- \leq k(x) \leq C_+. \tag{2}$$

The objective of this exercice is to prove that Problem (1) is well-posed and to construct a finite volume approximation of its solution in the 1d case.

1 Theoretical part

1.1 Two general results

1. **Proof of the Poincare inequality in a square:** Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2, 0 < x_1 < L_1, \quad 0 < x_2 < L_2\}$, $L_1 > 0$, $L_2 > 0$.

   a- Let $u \in \mathcal{D}(\Omega)$. Prove the following equality

   $$u(x_1, x_2) = \int_0^{x_1} \frac{\partial u(t, x_2)}{\partial t} dt.$$

   b- Using the previous equality and the Cauchy-Schwarz inequality, show that for any $u \in \mathcal{D}(\Omega)$,

   $$\|u\|_{L^2(\Omega)} \leq L_1 \|\nabla u\|_{L^2(\Omega)}.$$

   c- Deduce the Poincare inequality: there is a constant $\mathcal{P} > 0$ such that,

   $$\forall\, u \in H_0^1(\Omega), \quad \|u\|_{L^2(\Omega)} \leq \mathcal{P} \|\nabla u\|_{L^2(\Omega)}$$

2. **Proof of the Lax-Milgram Lemma in the symmetric case:**

Let V be a Hilbert space (associated norm $\|\cdot\|_V$). Let $a : V \times V \to \mathbb{R}$ be a **symmetric** bilinear form such that

- a is continuous: $\exists M > 0$ such that $\forall\,(u, v) \in V^2, a(u, v) \leq M\|u\|_V\|v\|_V$.
- a is coercive: $\exists \alpha > 0$ such that $\forall\, u \in V, a(u, u) \geq \alpha\|u\|_V^2$.

We denote by V' the dual space of V. Its associated norm is given by

$$\forall \ell \in V', \|\ell\|_{V'} = \sup_{v \in V, v \neq 0} \frac{f(v)}{\|v\|_V}.$$

For a given $\ell \in V'$, we consider the following problem: find $u \in V$ such that

$$\forall v \in V, \quad a(u, v) = \ell(v). \tag{3}$$

The objective of the next four questions is to prove that Problem (3) is well-posed:

a- Show that Problem (3) has at most one solution.

b- Prove that the bilinear form $(u, v) \to a(u, v)$ defines a scalar product on V.

c- Using the Riesz representation theorem, deduce that Problem (3) has at least one solution.

d- Prove that Problem (3) has a unique solution that satisfies the following stability estimate:

$$\|u\|_V \leq \frac{1}{\alpha}\|\ell\|_{V'}.$$

1.2 Well-posedness of Problem (1)

1. Show that Problem (1) is equivalent to the following variational problem: find $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \quad \int_\Omega k(x)\nabla u(x) \cdot \nabla v(x) dx = \int_\Omega f(x)v(x)dx. \tag{4}$$

2. Deduce from the previous question that Problem (1) has a unique solution and that there is a constant $C > 0$ such that

$$\|\nabla u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

1.3 Investigation of Problem (1) in the one dimensional case

In this part, we consider $\Omega = (0, L)$, $L > 0$.

1. **Strong maximum principle**. Assume that $k \in C^\infty(\overline{\Omega})$ and $f \in C^0(\Omega)$ is such that

$$\exists\, \eta > 0, \quad f(x) \geq \eta > 0.$$

a- Prove that if $u \in C^2(\Omega)$ has a local minimum at a point $x_0 \in \Omega$, then $k(x_0)u''(x_0) \geq 0$.

b- Using a contradiction argument, prove that

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u = 0,$$

and conclude.

2. **Regularity of u**

a- Using the Sobolev embedding theorem, prove that the function ku (where $u \in H_0^1(\Omega)$ is the unique solution to Problem (1)) is continuous.

b- Assume that $\Omega = (0, 1)$ $(L = 1)$ and that there are two positive constant $k_1 > 0$ and $k_2 > 0$ such that

$$k(x) = \begin{cases} k_1 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ k_2 & \text{if } \frac{1}{2} < x \leq 1. \end{cases} \tag{5}$$

We define the following jump values,

$$[u]_{1/2} = \lim_{\varepsilon \to 0^+} \left(u(\frac{1}{2} + \varepsilon) - u(\frac{1}{2} - \varepsilon) \right)$$
$$[ku']_{1/2} = \lim_{\varepsilon \to 0^+} \left(k_2 u'(\frac{1}{2} + \varepsilon) - k_1 u'(\frac{1}{2} - \varepsilon) \right)$$

Prove that $[u]_{1/2}$ and $[ku']_{1/2}$ vanish. Is u' continuous at the point $x = \frac{1}{2}$?

1.4 Construction of the finite volume scheme

In this section, we assume that $\Omega = (0, 1)$ and that k is the piecewise constant function given by (5). We consider a regular mesh (see Figure 1) of $N = 2P$ ($P \in \mathbb{N}^*$) intervalls (or control volume) $T_i = (x_{i-1/2}, x_{i+1/2})$, $i \in [1 : N]$ ($i \in \mathbb{N}$, $1 \leq i \leq N$). Since the mesh is uniform (or regular), for $i \in [1 : N]$,

$$x_{i-1/2} = \frac{i-1}{h}, \quad \text{with} \quad h = \frac{1}{N}.$$

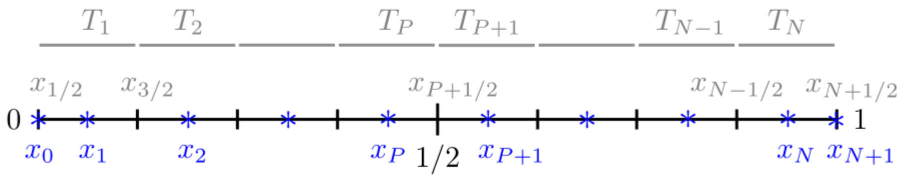

Figure 1: Representation of the mesh

For $i \in [1 : N]$, We denote by x_i the midpoint of T_i, and we denote by u_i the associated discrete unknown (note that u_i is expected to be a good approximation of $u(x_i)$ but u_i **is not equal to** $u(x_i)$). Moreover, we set $x_0 = x_{1/2} = 0$ and $x_{N+1} = x_{N+1/2} = 1$. We associate with x_0 and x_{N+1} the unknowns u_0 and u_{N+1}. In view of the regularity of the mesh, and since $N = 2P$, we remark that $x_{P+1/2} = \frac{1}{2}$.

1. For $i \in [1 : P-1]$ and $i \in [P+2, n]$, following the method of the lecture, propose a method to discretize Problem (1) on the control volume T_i.

2. What is the difficulty raised by the discretization of Problem (1) over T_P and T_{P+1}?

In order to overcome this difficulty, we introduce a temporary additional unknown $u_{P+1/2}$ associated with the point $x_{P+1/2}$ (here again, we expect $u_{P+1/2}$ to be an approximation of $u(x_{P+1/2})$).

3. In view of the regularity of the exact solution u, explain why there is a priori no need to introduce two new temporary additional unknowns $u^-_{P+1/2}$ and $u^+_{P+1/2}$, $u^-_{P+1/2}$ being an approximation of the value of u on the left of $\frac{1}{2}$ (i.e. $\lim_{x \to \frac{1}{2}^-} u(x)$) and $u^+_{P+1/2}$ being an approximation the value of u on the right of $\frac{1}{2}$ ($\lim_{x \to \frac{1}{2}^+} u(x)$).

4. Using $u_{P+1/2}$ and u_P, propose a method for the discretization of the flux $u'(\frac{1}{2})$ for the volume equation on T_p. Similarly, using $u_{P+1/2}$ and u_{P+1}, propose a discretization of the flux $u'(\frac{1}{2})$ for the volume equation on T_{p+1}.

5. In view of the jump condition $[ku']_{1/2} = 0$, propose a method to evaluate $u_{P+1/2}$ as a function of k_1, k_2, u_P and u_{P+1}. Then, eliminate the unknown $u_{P+1/2}$ in the discrete equations of the previous question.

6. Write the systems of equations of question 1 and question 5 as a system of N linear equations associated with N unknowns by eliminating u_0 and u_{N+1} using the Dirichlet boundary conditions. You may write the matricial form of this system.

2 Practical part

Write the Matlab code associated with the system of equations of the previous question (you may use the Matlab command \ to solve the linear system).

In order to validate your code, you may compute the exact solution for $k_1 = 1$, $k_2 = \alpha$ and

$$f(x) = \begin{cases} 0 & \text{if } 0 < x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x < 1, \end{cases}$$

and, for different values of α, plot the L^∞ error ($\max_{i \in [1:N]} |u_i - u(x_i)|$) as a function of h in logarithmic scale (Matlab command loglog). What is the observed convergence rate?