Exercice 3: a finite volume scheme for the heat equation

Let $\Omega = (0,1)$ and $\nu > 0$. We consider the one dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} - \nu \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad x \in \Omega, t > 0, \tag{1}$$

together with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0 \ , \ \forall t > 0, \tag{2}$$

and the initial condition

$$u(x, t=0) = u_0(x), \tag{3}$$

where u_0 is a given function belonging to $L^2(\Omega)$.

Equation (1) is called the heat equation because it models the temperature distribution u in the domain Ω at the time t. The heat equation and its variants occur in many diffusion phenomena and ν is called the diffusion parameter. It is the simplest example of a parabolic equation. Equation (2) means that the heat flux across the boundary vanishes.

Problem (1-2-3) has a unique solution in $C((0,+\infty); H^2(\Omega))$. Moreover, for any $t > 0$, $u(\cdot,t) \in C^\infty(\overline{\Omega})$ (cf. Brezis).

1 Analysis of the solution of the continuous problem

1. Prove that, for $v(x,t)$ is smooth enough

$$2\int_{[0,1]} \frac{\partial v(x,t)}{\partial t} v(x,t)dx = \int_{[0,1]} \frac{\partial}{\partial t}\left(v^2(x,t)\right)dx = \frac{d}{dt}\left(\int_0^1 v^2(x,t)dx\right),$$

Deduce that $t \longrightarrow \int_0^1 u^2(x,t)dx$ is a decreasing function.

2. Show that u satisfies

$$\int_{[0,1]} u(x,t)dx = \int_{[0,1]} u_0(x)dx \ \forall t \geq 0.$$

3. We admit that, as t tends to $+\infty$, $u(\cdot,t)$ tends to a limit function denoted by $\bar{u} \in H^2(\overline{\Omega})$ and that $\frac{\partial u}{\partial t}$ tends to 0. Give the equation verified by the function $\bar{u}$. What are the associated boundary conditions ? Using the previous question, compute the function $\bar{u}$. Give the explicite value of $\bar{u}$ in the particular case

$$u_0(x) = \begin{cases} 1 & \text{if} \quad x < 1/2 \\ 0 & \text{if} \quad x > 1/2 \end{cases}. \tag{4}$$

4. Using the Poincare-Wirtinger inequality, prove $\|u - \bar{u}\|_{L^2(\Omega)}$ tends exponentially fast to 0 as t tends to $+\infty$.

2 Finite volume approximation

Let $T > 0$. We shall approach numerically the equation (1) on $(0,T) \times \Omega$. For the space discretization, we use a regular mesh obtained by splitting the segment $[0,1]$ into N cells of length $\Delta x := \frac{1}{N}$. For $i \in [: N]$, we denote by x_i the midpoint of T_i and we set $x_0 = 0$ and $x_{N+1} = 1$. For the time discretization, we split the time interval $(0,T)$ of the simulation is also split into time steps of equal size $\Delta t > 0$. With each point x_i and each time step $t^n := n\Delta t$, we associate a discrete unknown u_i^n, which we expect to be an approximation of the value $u(x_i, t^n)$.

We then consider the following scheme : for any $i \in [1 : N]$,

$$u_i^{n+1} = u_i^n + \frac{\nu \Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n). \tag{5}$$

where, for any $n \in \mathbb{N}$, and in oder to approximate the Neumann boundary condition, we set

$$u_0^n = u_1^n \quad \text{and} \quad u_{N+1}^n = u_N^n \tag{6}$$

The scheme 5 is said to be explicit, because the computation of u_i^{n+1} can be done directly (using the values of u_{i-1}^n, u_i^n and u_{i+1}^n) without solving any linear system.

1. Denoting by $\mathbf{u}^n = (u_i^n)_{i\in[1:N]} \in \mathbb{R}^N$, show that the previous scheme may be written the following matricial form: $\forall n \in \mathbb{N}$.

$$\mathbf{u}^{n+1} = \mathbf{M}\mathbf{u}^n \quad \mathbf{M} \in \mathcal{M}_N(\mathbb{R}). \tag{7}$$

Make the matrix $\mathbf{M}$ explicite.

In order to study the stability of the scheme, we introduce the notion of L^∞-stability. For any $\mathbf{u} = (u_i)_{i\in[1:N]} \in \mathbb{R}^N$ we denote by

$$\|\mathbf{u}\|_\infty := \sup_{i\in[1,N]} |u_i|. \tag{8}$$

A scheme is said to be L^∞ stable if its solution $(\mathbf{u}^n)_{n\in\mathbb{N}}$ satisfies

$$\forall n \in \mathbb{N} \quad ||\mathbf{u}^{n+1}||_\infty \leq ||\mathbf{u}^n||_\infty,$$

which means that the L^∞ norm of its solution $\mathbf{u}^n$ does not grow with respect to n. In other words, the matrix $\mathbf{M}$ of equation (7) satisfies

$$\|\mathbf{M}\|_\infty = \sup_{\mathbf{v}\in\mathbb{R}^N, \mathbf{v}\neq 0} \|\mathbf{M}\mathbf{v}\|_\infty \leq 1.$$

2. We remind that a number p is a convex combination of the numbers $(q_j)_{j\in[1:k]}$ if there exists a set of k real numbers $(\theta_j)_{j\in[1:k]}$ such that

$$p = \sum_{j=1}^{k} \theta_j q_j \quad \text{and} \quad \forall j \in [1:k], \theta_j \in [0,1].$$

- Show that if, for all $i \in [1:N]$, u_i^{n+1} is a convex combination of u_{i-1}^n, u_i^n and u_{i+1}^n, then the scheme is L^∞ stable.
- For $i \in [2:N-1]$, find a condition linking ν, Δt and Δx such that u_i^{n+1} given by (5) is a convex combination of $(u_{i-1}^n, u_i^n, u_{i+1}^n)$.
- Similarly, find a condition ν, Δt and Δx such that u_1^{n+1} is a convex combination of (u_1^n, u_2^n) and u_N^{n+1} is a convex combination of (u_{N-1}^n, u_N^n) ?
- Deduce that the scheme (5-6) is L^∞ stable under a "CFL" condition. Explain why is this type of condition is a severe drawback of this explicit scheme.

3. Compute the truncation (or consistancy) error $\mathbf{r}^n = (r_i^n)_{i\in[1:N]} \in \mathbb{R}^N$

$$r_i^n = \begin{cases} \dfrac{u(x_1, t^{n+1}) - u(x_1, t^n)}{\Delta t} - \dfrac{\nu}{\Delta x^2}\left(u(x_2, t^n) - u(x_1, t^n)\right) & i = 1, \\ \dfrac{u(x_i, t^{n+1}) - u(x_i, t^n)}{\Delta t} - \dfrac{\nu}{\Delta x^2}\left(u(x_{i+1}, t^n) - 2u(x_1, t^n) + u(x_{i-1}, t^n)\right) & i \in [2:N-1], \\ \dfrac{u(x_N, t^{n+1}) - u(x_N, t^n)}{\Delta t} - \dfrac{\nu}{\Delta x^2}\left(u(x_{N-1}, t^n) - u(x_N, t^n)\right) & i = N. \end{cases}$$

and show that, for any $n \leq \frac{T}{\Delta t}$ (T is the final time of the simulation),

$$\|\mathbf{r}^n\|_\infty \leq C(T)(\Delta t + \Delta x)$$

Prove that the error $\mathbf{e}^n = (e_i^n)_{i\in[1:N]} \in \mathbb{R}^N$ defined by

$$e_i^n = u_i^n - u(x_i, t^n). \tag{9}$$

satisfies

$$\mathbf{e}^{n+1} = \mathbf{M}\mathbf{e}^n \ + \ \Delta t\, \mathbf{r}^n.$$

4. Under the CFL condition of question 2, show the following inequality:

$$||\mathbf{e}^{n+1}||_\infty \leq ||\mathbf{e}^n||_\infty + \Delta t\, ||\mathbf{r}^n||_\infty.$$

Deduce that

$$||\mathbf{e}^n||_\infty \leq ||\mathbf{e}^0||_\infty + \Delta t \sum_{m=0}^{n-1} ||\mathbf{r}^n||_\infty.$$

5. Assuming that $u_i^0 = u_0(x_i)$, prove that, under the CFL condition of question 2, the scheme (5-6) converges and show that for $n \leq \frac{T}{\Delta t}$, there is a constant $C(T)$ (which depends on T) such that

$$||\mathbf{e}^n||_\infty \leq C(T)\Delta x$$

3 Practical part

1. Write the program associated with the scheme (5-6). You may take N, ν, T, and $\lambda = \frac{\nu \Delta t}{\Delta x^2}$ as the input parameters of your program.

2. Compute the exact solution associated with the initial data $u_0(x) = \cos(\pi x)$ and $\nu = 1$. To find the exact solution, you can use the technic of separations of variables, i.e. you may look for a solution of the form

$$u(x,t) = \cos(\pi x) g(t).$$

3. Take $N = 10$, $\nu = 1$, $T = 1$ and choose $\lambda = 0.25$, $\lambda = 0.49$ and $\lambda = 0.51$. Visualize the evolution of $\mathbf{u}$ with respect to t. What do you observe ?

4. For $\lambda = 0.49$ ($N = 10$m, $T = 1$, $\nu = 1$), visualize the time evolution of the L^∞ norm of the error (9). Visualize the time evolution of the discrete L_2 energy

$$E^n = \left(\sum_{i=1}^{N} \Delta x \, (u_i^n)^2 \right)^{1/2}.$$

5. Let $\lambda = 0.49$, $T = 0.25$ and $\nu = 1$. Plot the L^∞ norm of the error (9) with respect to N.

6. We consider the initial data u_0 given by (4). We fix $\lambda = 0.4$, $N = 20$ and $T = 1$. Run the simulations for different values of the diffusion coefficient $\nu = 1$, $\nu = 0.5$, $\nu = 0.25$ and $\nu = 0.125$.Visualize and comment the time evolution of the energy deviation:

$$\tilde{e}^n = \left(\sum_{i=1}^{N} \Delta x \, (u_i^n - 1/2)^2 \right)^{1/2}.$$

Representing the values of u at different time steps, to which function seems u to converge ? How does this convergence depend on the value of ν ? Is it coherent with the theory?