ON THE FOURIER DECAY OF MULTIPLICATIVE CONVOLUTIONS

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ABSTRACT. We prove the following. Let $\mu_1, \ldots, \mu_n$ be Borel probability measures on $[-1, 1]$ such that μ_j has finite s_j-energy for certain indices $s_j \in (0, 1]$ with $s_1 + \ldots + s_n > 1$. Then, the multiplicative convolution of the measures $\mu_1, \ldots, \mu_n$ has power Fourier decay: there exists a constant $\tau = \tau(s_1, \ldots, s_n) > 0$ such that

$$\left| \int e^{-2\pi i \xi \cdot x_1 \cdots x_n} \, d\mu_1(x_1) \cdots d\mu_n(x_n) \right| \leqslant |\xi|^{-\tau}$$

for sufficiently large $|\xi|$. This verifies a suggestion of Bourgain from 2010.

1. INTRODUCTION

In 2010, Bourgain [2, Theorem 6] proved the following remarkable Fourier decay property for multiplicative convolutions of Frostman measures on the real line.

Theorem 1.1 (Fourier decay for multiplicative convolutions)**.** *For all $s > 0$, there exists $\epsilon > 0$ and $n \in \mathbb{Z}_+$ such that the following holds for every $\delta > 0$ sufficiently small.*
If μ is a probability measure on $[-1, 1]$ satisfying

$$\forall r \in [\delta, \delta^\epsilon], \qquad \sup_{a \in [-1,1]} \mu(B(a, r)) < r^s$$

then for all $\xi \in \mathbb{R}$ with $\delta^{-1} \leqslant |\xi| \leqslant 2\delta^{-1}$,

$$\int e^{2\pi i \xi x_1 \ldots x_n} d\mu(x_1) \ldots d\mu(x_n) \leqslant \delta^{-\epsilon}. \tag{1.2}$$

This result found striking applications in the Fourier decay of fractal measures and resulting spectral gaps for hyperbolic surfaces [3, 9]. It was recently generalised to higher dimensions by Li [5].

At the end of the introduction of [2], Bourgain proposes to study the optimal relation between s and n. Our goal here is to show that, as suggested by Bourgain, Theorem 1.1 holds under the condition $n > 1/s$, which is optimal up to the endpoint, as we shall see in Example 1.10 below.

The statement we obtain applies more generally to multiplicative convolutions of different measures, and our proof also allows us to replace the Frostman condition by a

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slightly weaker condition. Precisely, for a finite Borel measure μ on $\mathbb{R}$, given $s \in (0,1]$ and $\delta > 0$, the *s-energy* of μ is defined as

$$I_s(\mu) := \iint |x-y|^{-s}\, d\mu(x)\, d\mu(y).$$

We refer the reader to [6] for the basic properties of the energy of a measure. As in Bourgain's theorem, we shall be mostly interested in the properties of measures up to some fixed small scale δ; for that reason, we also define the s-energy of μ at scale δ by

$$I_s^\delta(\mu) = I_s(\mu_\delta),$$

where $\mu_\delta = \mu * P_\delta$ is the regularisation of μ at scale δ, and P_δ a smooth approximate unit of size δ. The main result of the present article is the following.

Theorem 1.3 (Fourier decay under optimal entropy condition)**.** *Let $n \geqslant 2$, and $\{s_j\}_{j=1}^n \subset (0,1]$ such that $\sum s_j > 1$. Then, there exist $\delta_0, \epsilon, \tau \in (0,1]$, depending only on the parameters above, such that the following holds for $\delta \in (0,\delta_0]$. Let $\mu_1,\dots,\mu_n$ be Borel probability measures on $[-1,1]$ satisfying the energy conditions*

$$I_{s_j}^\delta(\mu_j) \leqslant \delta^{-\epsilon}, \qquad 1 \leqslant j \leqslant n. \tag{1.4}$$

Then, for all ξ satisfying $\delta^{-1} \leqslant |\xi| \leqslant 2\delta^{-1}$,

$$\left| \int e^{-2\pi i \xi x_1 \dots x_n}\, d\mu_1(x_1) \dots d\mu_n(x_n) \right| \leqslant |\xi|^{-\tau}. \tag{1.5}$$

Remark 1.6. It is not difficult to check that the Frostman condition $\mu(B(a,r)) \leqslant r^s$ from Bourgain's Theorem 1.1 is stronger that the assumption on the s-energy at scale δ used above. The reader is referred to Lemma 3.8 for a detailed argument.

Remark 1.7. The values of the parameters $\delta_0, \epsilon > 0$ stay bounded away from 0 as long as $\min\{s_1,\dots,s_n\} > 0$ stays bounded away from 0, and $\sum_j s_j > 1$ stays bounded away from 1, and n ranges in a bounded subset of $\mathbb{N}$.

The following corollary is immediate:

Corollary 1.8. *Let $n \geqslant 2$, and $\{s_j\}_{j=1}^n \subset (0,1]$ such that $\sum s_j > 1$. There exists $\tau = \tau(n,\{s_j\}) > 0$ such that the following holds. Let $\mu_1,\dots,\mu_n$ be Borel probability measures on $\mathbb{R}$ such that $I_{s_j}(\mu_j) < +\infty$. Then there is $C = C(\{\mu_j\}) > 0$ such that*

$$\left| \int e^{-2\pi i \xi x_1 \dots x_n}\, d\mu_1(x_1) \dots d\mu_n(x_n) \right| \leqslant C \cdot |\xi|^{-\tau}, \quad \xi \in \mathbb{R}. \tag{1.9}$$

Writing $\mu_1 \boxtimes \cdots \boxtimes \mu_n$ for the image of the measure $\mu_1 \times \cdots \times \mu_n$ under the product map $(x_1,\dots,x_n) \mapsto x_1 \dots x_n$, the Fourier decay condition (1.9) implies that additive convolution powers of $\mu_1 \boxtimes \cdots \boxtimes \mu_n$ become absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with arbitrarily smooth densities. In particular, if A_i denotes the support of the measure μ_i, for $i = 1,\dots,n$, a sumset of the product set

$$A_1 A_2 \dots A_n = \{a_1 a_2 \dots a_n \;:\; a_i \in A_i\}$$

must contain a non-empty interval. This observation, together with the example below, shows that the condition $\sum s_j > 1$ used in Theorem 1.3 is essentially optimal.

Example 1.10. Given $s \in (0,1)$ and an increasing sequence of integers $(n_k)_{k\geqslant 1}$, define a subset H_s in $\mathbb{R}$ by

$$H_s = \{x \in [0,1] \ : \ \forall k \geqslant 1,\ d(x, n_k^{-s}\mathbb{Z}) \leqslant n_k^{-1}\}.$$

If (n_k) grows fast enough, then both H_s and the additive subgroup it generates will have Hausdorff dimension s.

Now assume that the parameters $s_1, \dots, s_n$ satisfy $\sum s_i < 1$. Fixing $s_i' > s_i$ such that one still has $\sum s_i' < 1$, Frostman's lemma yields probability measures μ_i supported on $H_{s_i'}$ and satisfying $\mu_i(B(a,r)) < r^{s_i}$ for all $r > 0$ sufficiently small. However, since the support A_i of μ_i satisfies $A_i \subset H_{s_i'}$, one has

$$A_1 \dots A_n \subset \left\{ x \in [0,1] \ : \forall k \geqslant 1,\ d\left(x, n_k^{-\sum s_i'}\mathbb{Z}\right) \leqslant n \cdot n_k^{-1} \right\}.$$

This shows that the subgroup generated by $A_1 \dots A_n$ has dimension bounded above by $\sum s_i' < 1$ and so is not equal to $\mathbb{R}$. So the measure $\mu_1 \boxtimes \dots \boxtimes \mu_n$ cannot have polynomial Fourier decay.

An analogue of Theorem 1.3 in the prime field setting was obtained by Bourgain in [1], and our proof follows a similar general strategy, based on sum-product estimates and flattening for additive-multiplicative convolutions of measures. Example 1.10 above shows that there exist compact sets A and B in $\mathbb{R}$ such that the additive subgroup $\langle AB \rangle$ generated by the product set AB satisfies $\dim_{\mathrm{H}} \langle AB \rangle \leqslant \dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B$. Conversely, it was shown in [7] as a consequence of the discretised radial projection theorem [8] that for Borel sets $A, B \subset \mathbb{R}$, one has

$$\dim_{\mathrm{H}}(AB + AB - AB - AB) \geqslant \min\{\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B, 1\}. \tag{1.11}$$

The main ingredient in the proof of Theorem 1.3 is a discretised version of this inequality; the precise statement is given below as Proposition 3.6 and is taken from [7, Proposition 3.7]. It can be understood as a precise version of the discretised sum-product theorem, which allows us to improve on the strategy used by Bourgain in [2] and obtain Fourier decay of multiplicative convolutions under optimal entropy conditions. Before turning to the detailed proof, let us give a general idea of the argument.

Notation. We fix for the rest of the article a standard, L^1-normalized approximate identity $\{P_\delta\}_{\delta>0} = \{\delta^{-1}P(\cdot/\delta)\}_{\delta>0}$. Given a measure μ on $\mathbb{R}$, recall that we write μ_δ for the density of μ at scale δ, or equivalently, $\mu_\delta = \mu * P_\delta$.

Below, we shall use both additive and multiplicative convolution of measures. To avoid any confusion, we write $\mu \boxplus \nu$, $\mu \boxminus \nu$ and $\mu \boxtimes \nu$ to denote the image of $\mu \times \nu$ under the maps $(x,y) \mapsto x+y$, $(x,y) \mapsto x-y$, and $(x,y) \mapsto xy$, respectively. Similarly, we denote additive and multiplicative k-convolution powers of measures by $\mu^{\boxplus k}$ and $\mu^{\boxtimes k}$, respectively.

The push-forward of a Borel measure μ on the real line under a Borel map $g : \mathbb{R} \to \mathbb{R}$ is denoted $g_\sharp \mu$, that is,

$$\int f\, d(g_\sharp \mu) = \int f \circ g\, d\mu.$$

Sketch of proof of Theorem 1.3. The $n = 2$ case of Theorem 1.3 is classical and already appears in Bourgain's paper [2, Theorem 7]: If μ and ν are two probability measures on $[-1, 1]$ such that $\|\mu_\delta\|_2^2 \leqslant \delta^{-1+s}$ and $\|\nu_\delta\|_2^2 \leqslant \delta^{-1+t}$, then the multiplicative convolution $\mu \boxtimes \nu$ satisfies

$$|\widehat{\mu \boxtimes \nu}(\xi)| \lesssim \delta^{\frac{s+t-1}{2}}, \quad \delta^{-1} \leqslant |\xi| \leqslant 2\delta^{-1}.$$

For the reader's convenience we record the detailed argument below, see Section 2.

We want to use induction to reduce to this base case. To explain the induction step, we focus on the case $n = 3$. The main point is to translate equation (1.11) into a flattening statement for additive-multiplicative convolutions of measures. For simplicity, assume we knew that if μ and ν are probability measures on $[-1, 1]$, then the measure

$$\eta := (\mu \boxtimes \nu) \boxplus (\mu \boxtimes \nu) \boxminus (\mu \boxtimes \nu) \boxminus (\mu \boxtimes \nu)$$

satisfies, for $\epsilon > 0$ arbitrarily small,

$$\|\eta_\delta\|_2^2 \leqslant \delta^{1-\epsilon}\|\mu_\delta\|_2^2\|\nu_\delta\|_2^2. \tag{1.12}$$

(Note that this is the exact analogue of (1.11) for L^2-dimensions of measures at scale δ.) If μ_1, μ_2 and μ_3 satisfy $\|(\mu_i)_\delta\|_2^2 \leqslant \delta^{-1+s_i}$ for some parameters s_i with $s_1 + s_2 + s_3 > 1$, we apply the above inequality to μ_1 and μ_2 to obtain

$$\|\eta_\delta\|_2^2 \leqslant \delta^{-1-\epsilon+s_1+s_2},$$

where $\eta = (\mu_1 \boxtimes \mu_2) \boxplus (\mu_1 \boxtimes \mu_2) \boxminus (\mu_1 \boxtimes \mu_2) \boxminus (\mu_1 \boxtimes \mu_2)$. If ϵ is chosen small enough, we have $(s_1 + s_2 - \epsilon) + s_3 > 1$, and so we may apply the $n = 2$ case to the measures η and μ_3 to get, for $\delta^{-1} < |\xi| < 2\delta^{-1}$,

$$|\widehat{\eta \boxtimes \mu_3}(\xi)| < \delta^{\frac{s_1+s_2+s_3-\epsilon-1}{2}}.$$

To conclude, one observes from the Cauchy-Schwarz inequality that for any two probability measures μ and ν, one always has $|\widehat{\mu \boxtimes \nu}(\xi)|^2 \leqslant \widehat{(\mu \boxminus \mu) \boxtimes \nu}(\xi)$. This elementary observation applied twice yields

$$|\widehat{\mu_1 \boxtimes \mu_2 \boxtimes \mu_3}(\xi)|^4 \leqslant \widehat{\eta \boxtimes \mu_3}(\xi)^4 < \delta^{\frac{1}{2}(s_1+s_2+s_3-\epsilon-1)}$$

which is the desired Fourier decay, with parameter $\tau = \frac{1}{8}(s_1 + s_2 + s_3 - \epsilon - 1)$.

Unfortunately, the assumptions on the L^2-norms of μ and ν are not sufficient to ensure inequality (1.12) in general. One also needs some kind of non-concentration condition on μ and ν, and for that purpose we use the notion of energy of the measure at scale δ, which gives information on the behaviour of the measure at all scales between δ and 1. The precise statement we use for the induction is given as Lemma 3.1 below. It is also worth noting that to obtain the correct bound on the energy, we need to use a large number k of additive convolutions, whereas $k = 4$ was sufficient in the analogous statement (1.11) for Hausdorff dimension of sum-product sets. We do not know whether Lemma 3.1 holds for $k = 4$, or even for k bounded by some absolute constant.

We conclude this introduction by an example showing that for $n \geqslant 3$, the assumption $I_{s_j}^\delta(\mu_j) \leqslant \delta^{-\epsilon}$ in Theorem 1.3 cannot be replaced by the "single-scale" L^2-bound $\|\mu_\delta\|_2^2 \leqslant \delta^{s_j-1-\epsilon}$. This is mildly surprising, because the situation is opposite in the case $n = 2$, as shown by Proposition 2.1 below. We only write down the details of the example in the case $n = 3$, but it is straightforward to generalise to $n \geqslant 3$.

Example 1.13. For every $s \in (0, \frac{1}{2})$ and $\delta_0 > 0$ there exists a scale $\delta \in (0, \delta_0]$ and a Borel probability measure $\mu = \mu_{\delta,s}$ on $[1,2]$ with the following properties:

- $\|\mu\|_2^2 \sim \delta^{s-1}$.
- $|\widehat{\mu \boxtimes \mu \boxtimes \mu}(\delta^{-1})| \sim 1$.

The building block for the construction is the following. For $r \in 2^{-\mathbb{N}}$, and a suitable absolute constant $c > 0$, let $\mathcal{I} = \mathcal{I}_r$ be a family of r^{-1} intervals of length cr, centred around the points $r\mathbb{Z} \cap [0,1]$. Then, if $c > 0$ is small enough, we have

$$\cos(2\pi x/r) \geqslant \tfrac{1}{2}, \qquad \forall x \in \cup\mathcal{I}.$$

Consequently, if $\boldsymbol{\rho}$ is any probability measure supported on $\cup\mathcal{I}$, then $|\widehat{\boldsymbol{\rho}}(r^{-1})| \geqslant \frac{1}{2}$.

Fix $s \in (0, \frac{1}{2})$, $\delta > 0$, and let $\boldsymbol{\rho}$ be the uniform probability measure on the intervals $\mathcal{I}_{\delta^s}$. As we just discussed, $|\widehat{\boldsymbol{\rho}}(\delta^{-s})| \sim 1$. Next, let $\mu = \mu_{\delta,s}$ be a rescaled copy of $\boldsymbol{\rho}$ inside the interval $[1, 1+\delta^{1-s}] \subset [1,2]$. More precisely, $\mu = \tau_\sharp \lambda_\sharp \boldsymbol{\rho}$, where $\tau(x) = x + 1$ and $\lambda(x) = \delta^{1-s}x$. Now μ is a uniform probability measure on a collection of δ^{-s} intervals of length δ, and consequently $\|\mu\|_2^2 \sim \delta^{s-1}$.

We next investigate the Fourier transform of $\mu \boxtimes \mu \boxtimes \mu$. Writing $\mu' := \lambda_\sharp \boldsymbol{\rho}$, we have

$$\widehat{\mu \boxtimes \mu \boxtimes \mu}(\delta^{-1}) = \iiint e^{-2\pi i \delta^{-1}(x+1)(y+1)(z+1)}\, d\mu'(x)\, d\mu'(y)\, d\mu'(z).$$

We expand

$$\delta^{-1}(x+1)(y+1)(z+1) = \delta^{-1}xyz + \delta^{-1}(xy + xz + yz) + \delta^{-1}(x+y+z) + \delta^{-1}.$$

Now the key point: since $x, y, z \in \operatorname{spt} \mu' \subset [0, \delta^{1-s}]$, we have both $|\delta^{-1}xyz| \leqslant \delta^{2-3s}$ and $|\delta^{-1}(xy + xz + yz)| \lesssim \delta^{1-2s}$. Since $s < \frac{1}{2}$, both exponents $2 - 3s$ and $1 - 2s$ are strictly positive and consequently,

$$e^{-2\pi i \delta^{-1}(x+1)(y+1)(z+1)} = e^{-2\pi i \delta^{-1}(x+y+z+1)} + o_{\delta\to 0}(1).$$

Using this, and also that $\hat{\mu}'(\xi) = \widehat{\boldsymbol{\rho}}(\delta^{1-s}\xi)$, we find

$$\begin{aligned}\widehat{\mu \boxtimes \mu \boxtimes \mu}(\delta^{-1}) &= e^{-2\pi i \delta^{-1}} \iiint e^{-2\pi i \delta^{-1}(x+y+z)}\, d\mu'(x)\, d\mu'(y)\, d\mu'(z) + o_{\delta\to 0}(1) \\ &= e^{-2\pi i \delta^{-1}} (\widehat{\boldsymbol{\rho}}(\delta^{-s}))^3 + o_{\delta\to 0}(1).\end{aligned}$$

In particular, $|\widehat{\mu \boxtimes \mu \boxtimes \mu}(\delta^{-1})| \sim 1$ for $\delta > 0$ sufficiently small.

If we allow μ to be supported on $[-1,1]$, as in Theorem 1.3, an even simpler example is available, namely $\mu = \delta^{-1+s}\mathbf{1}_{[0,c\delta^{1-s}]}$, where $s < 2/3$. Then $\mu \boxtimes \mu \boxtimes \mu$ is supported on $[0, c^3\delta^{3-3s}] \subset [0, c\delta]$, so it satisfies $|\widehat{\mu \boxtimes \mu \boxtimes \mu}(\delta^{-1})| \gtrsim 1$ if $c > 0$ is chosen small enough.

2. The base case $n = 2$

In the $n = 2$ case, Theorem 1.3 is proved by a direct elementary computation. In fact, to obtain the desired Fourier decay, one only needs an assumption on the L^2-norms of the measures at scale δ.

Proposition 2.1 (Base case $n = 2$). *Let $\delta \in (0,1]$ and let μ, ν be Borel probability measures on $[-1,1]$ and $s, t \in [0,1]$ such that*

$$\|\mu_\delta\|_2^2 \leqslant \delta^{-1+s} \quad \textit{and} \quad \|\nu_\delta\|_2^2 \leqslant \delta^{-1+t}.$$

Then, for all ξ with $\delta^{-1} \leqslant |\xi| \leqslant 2\delta^{-1}$,

$$\left| \iint e^{-2\pi i \xi \cdot xy} \, d\mu(x) \, d\nu(y) \right| \lesssim \delta^{\frac{s+t-1}{2}}.$$

Remark 2.2. If μ and ν are equal to the normalized Lebesgue measure on balls of size δ^{1-s} and δ^{1-t}, respectively, the assumptions of the proposition are satisfied. In that case, the multiplicative convolution $\mu \boxtimes \nu$ is supported on a ball of size $\delta^{1-\max(s,t)}$, so the Fourier decay cannot hold for $|\xi| \leqslant \delta^{-1+\max(s,t)}$.

The above proposition is an easy consequence of the lemma below, which is essentially [2, Theorem 7], except that we keep slightly more careful track of the constants. We include the proof for completeness.

Lemma 2.3. *Let $\delta \in (0,1]$, and let μ, ν be Borel probability measures on $[-1,1]$ with*

$$A := \int_{|\xi| \leqslant 2\delta^{-1}} |\hat{\mu}(\xi)|^2 \, d\xi \quad \text{and} \quad B := \int_{|\xi| \leqslant 2\delta^{-1}} |\hat{\nu}(\xi)|^2 \, d\xi.$$

Then, for all ξ with $1 \leqslant |\xi| \leqslant \delta^{-1}$,

$$\left| \iint e^{-2\pi i \xi \cdot xy} \, d\mu(x) \, d\nu(y) \right| \lesssim \sqrt{AB/|\xi|} + \delta. \tag{2.4}$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R})$ be an auxiliary function with the properties $\mathbf{1}_{[-1,1]} \leqslant \varphi \leqslant \mathbf{1}_{[-2,2]}$ (thus $\varphi \equiv 1$ on $\operatorname{spt} \mu$) and $\widehat{\varphi} \geqslant 0$. Fixing $1 \leqslant |\xi| \leqslant \delta^{-1}$, the left-hand side of (2.4) can be estimated by

$$\begin{aligned}\left| \iint e^{-2\pi i \xi \cdot xy} \, d\mu(x) \, d\nu(y) \right| &= \left| \int \widehat{\varphi\mu}(y\xi) \, d\nu(y) \right| \leqslant \iint \widehat{\varphi}(x - y\xi) |\hat{\mu}(x)| \, dx \, d\nu(y) \\ &= \int |\hat{\mu}(x)| \left(\int \widehat{\varphi}(x - y\xi) \, d\nu(y) \right) dx.\end{aligned}$$

We split the right-hand side as the sum

$$\int_{|x| \leqslant 2\delta^{-1}} |\hat{\mu}(x)| \left(\int \widehat{\varphi}(x - y\xi) \, d\nu(y) \right) dx + \iint_{|x| \geqslant 2\delta^{-1}} |\hat{\mu}(x)| \widehat{\varphi}(x - y\xi) \, dx \, d\nu(y) =: I_1 + I_2.$$

For the term I_2, we use that $\widehat{\varphi}(x - y\xi) \lesssim |x - y\xi|^{-2}$, $|\hat{\mu}(x)| \leqslant 1$, and $\nu(\mathbb{R}) = 1$:

$$I_2 \lesssim \max_{y \in [-1,1]} \int_{|x| \geqslant 2\delta^{-1}} \frac{dx}{|x - y\xi|^2} \lesssim \int_{|x| \geqslant \delta^{-1}} \frac{dx}{|x|^2} \lesssim \delta.$$

For the term I_1, we first use the Cauchy-Schwarz inequality and the definition of A to deduce

$$I_1 \leqslant \sqrt{A} \left(\int \left[\int \widehat{\varphi}(x - y\xi) \, d\nu(y) \right]^2 dx \right)^{1/2}.$$

Finally, for the remaining factor, assume $\xi > 0$ without loss of generality, and write $\widehat{\varphi}(x - y\xi) = \widehat{\varphi_\xi}(x/\xi - y)$, where $\varphi_\xi = \xi^{-1}\varphi(\cdot/\xi)$. With this notation, and by Plancherel's formula,

$$\int \left[\int \widehat{\varphi}(x-y\xi)\,d\nu(y)\right]^2 dx = \int (\widehat{\varphi_\xi} * \nu)(x/\xi)^2\,dx = \xi \int (\widehat{\varphi_\xi} * \nu)(z)^2\,dz$$
$$= \xi \int \varphi_\xi(u)^2 |\hat{\nu}(u)|^2\,du \lesssim \xi^{-1} \int_{\operatorname{spt}\varphi_\xi} |\hat{\nu}(u)|^2\,du.$$

Finally, recall that $\operatorname{spt}\varphi \subset [-2,2]$, so $\operatorname{spt}\varphi_\xi \subset [-2\xi, 2\xi] \subset [-2\delta^{-1}, 2\delta^{-1}]$. This shows that $I_1 \lesssim \sqrt{AB/\xi}$, and the proof of (2.4) is complete. □

Proof of Proposition 2.1. Observe that by Plancherel's formula

$$A = \int_{|\xi|\leqslant 4\delta^{-1}} |\hat{\mu}(\xi)|^2\,d\xi \leqslant \|\mu_{\frac{\delta}{10}}\|_2^2 \lesssim \|\mu_\delta\|_2^2 \leqslant \delta^{-1+s}$$

and similarly

$$B = \int_{|\xi|\leqslant 4\delta^{-1}} |\hat{\mu}(\xi)|^2\,d\xi \lesssim \delta^{-1+t}.$$

So Lemma 3.1 applied at scale $\delta/2$ implies that for $\delta^{-1} \leqslant |\xi| \leqslant 2\delta^{-1}$,

$$\left|\iint e^{-2\pi i\xi\cdot xy}\,d\mu(x)\,d\nu(y)\right| \lesssim \sqrt{AB/|\xi|} + \delta$$
$$\lesssim \delta^{\frac{s+t-1}{2}}.$$

□

3. Dimension and energy of additive-multiplicative convolutions

This section is the central part of the proof of Theorem 1.3. Its goal is to derive Lemma 3.1 below, whose statement can be qualitatively understood in the following way: If μ and ν are two Borel probability measures on $\mathbb{R}$ with respective dimensions s and t, then there exists some additive convolution of $\mu \boxtimes \nu$ with dimension at least $s+t-\epsilon$, where $\epsilon > 0$ can be arbitrarily small. The precise formulation in terms of the energies of the measures at scale δ will be essential in our proof of Fourier decay for multiplicative convolutions.

Lemma 3.1. *For all $s, t \in (0,1]$ with $s+t \leqslant 1$, and for all $\kappa > 0$, there exist $\epsilon = \epsilon(s,t,\kappa) > 0$, $\delta_0 = \delta_0(s,t,\kappa,\epsilon) > 0$, and $k_0 = k_0(s,t,\kappa) \in \mathbb{N}$ such that the following holds for all $\delta \in (0,\delta_0]$ and $k \geqslant k_0$. Let μ, ν be Borel probability measures on $[-1,1]$ satisfying*

$$I_s^\delta(\mu) \leqslant \delta^{-\epsilon} \quad \textit{and} \quad I_t^\delta(\nu) \leqslant \delta^{-\epsilon}. \tag{3.2}$$

Then, with $\Pi := (\mu \boxminus \mu) \boxtimes (\nu \boxminus \nu)$, we have

$$I_{s+t}^\delta(\Pi^{\boxplus k}) \leqslant \delta^{-\kappa}.$$

Moreover, the value of k_0 stays bounded as long as $\min\{s,t\} > 0$ stays bounded away from zero.

The main component of the proof of Lemma 3.1 will be a combinatorial result from [7] which we will apply in the following form. Let $N(E,\delta)$ denote the smallest number of δ-balls required to cover E.

Lemma 3.3. *For all $s, t \in (0,1]$ with $s + t \leqslant 1$, and for all $\kappa > 0$, there exist $\epsilon = \epsilon(s,t,\kappa) > 0$ and $\delta_0 = \delta_0(s,t,\kappa,\epsilon) > 0$ such that the following holds for all $\delta \in (0, \delta_0]$ and $k \geqslant 2$. Let μ, ν be Borel probability measures on $[-1,1]$ satisfying*

$$I_s^\delta(\mu) \leqslant \delta^{-\epsilon} \quad \text{and} \quad I_t^\delta(\nu) \leqslant \delta^{-\epsilon}. \tag{3.4}$$

Let $\Pi := (\mu \boxminus \mu) \boxtimes (\nu \boxminus \nu)$ and assume that $E \subset \mathbb{R}$ is a set with $\Pi^{\boxplus k}(E) \geqslant \delta^\epsilon$. Then,

$$N(E, \delta) \geqslant \delta^{-s-t+\kappa}. \tag{3.5}$$

Since this lemma does not explicitly appear in [7], we now briefly explain how to derive it from the results of that paper. Recall that a Borel measure μ on $\mathbb{R}$ is said to be (s, C)*-Frostman* if it satisfies

$$\mu(B(x,r)) \leqslant Cr^s \quad \text{for all } x \in \mathbb{R}, r > 0.$$

The precise statement we shall need is [7, Proposition 3.7], see also [7, Remark 3.11], which reads as follows.

Proposition 3.6. *Given $s, t \in (0,1]$ and $\sigma \in [0, \min\{s+t, 1\})$, there exist $\epsilon = \epsilon(s,t,\sigma) > 0$ and $\delta_0 = \delta_0(s,t,\sigma,\epsilon) > 0$ such that the following holds for all $\delta \in (0, \delta_0]$.*
Let μ_1, μ_2 be $(s, \delta^{-\epsilon})$-Frostman probability measures, let ν_1, ν_2 be $(t, \delta^{-\epsilon})$-Frostman probability measures, all four measures supported on $[-1,1]$, and let $\boldsymbol{\rho}$ be an $(s+t, \delta^{-\epsilon})$-Frostman probability measure supported on $[-1,1]^2$. Then there is a set $\mathbf{Bad} \subset \mathbb{R}^4$ with

$$(\mu_1 \times \mu_2 \times \nu_1 \times \nu_2)(\mathbf{Bad}) \leqslant \delta^\epsilon,$$

such that for every $(a_1, a_2, b_1, b_2) \in \mathbb{R}^4 \setminus \mathbf{Bad}$ and every subset $G \subset \mathbb{R}^2$ satisfying $\boldsymbol{\rho}(G) \geqslant \delta^\epsilon$, one has

$$N(\{(b_1 - b_2)a + (a_1 - a_2)b : (a,b) \in G\}, \delta) \geqslant \delta^{-\sigma}. \tag{3.7}$$

The derivation of Lemma 3.3 from Proposition 3.6 is relatively formal; it mostly uses the link between the Frostman condition and the energy at scale δ, and the pigeonhole principle to construct large fibres in product sets. Let us first record an elementary statement about the energy at scale δ of a Frostman measure.

Lemma 3.8 (Frostman condition and s-energy)**.** *Fix $C \geqslant 1$, $s \in (0,d)$, and $\epsilon \in (0, \frac{1}{2}]$. Then the following holds for all $\delta > 0$ small enough. Let μ be a probability measure on $B(1) \subset \mathbb{R}^d$.*

1. *If μ satisfies $\mu(B(x,r)) \leqslant Cr^s$ for all $x \in \mathbb{R}^d$ and all $r \in [\delta, \delta^\epsilon]$, then $I_s^\delta(\mu) \leqslant C\delta^{-d\epsilon}$.*
2. *Conversely, if $I_s^\delta(\mu) \leqslant \delta^{-\epsilon}$, there exists a set A such that $\mu(A) \geqslant 1 - (\log 1/\delta)\delta^\epsilon$ and for every $r \in [\delta, 1]$, $\mu|_A(B(x,r)) \leqslant \delta^{-2\epsilon} r^s$.*

Proof. Assume first that μ satisfies the Frostman condition $\mu(B(x,r)) \leqslant Cr^s$ for $r \in [\delta, \delta^\epsilon]$. It is not difficult to check that the measure μ_δ with density $\mu * P_\delta$ satisfies

$$\mu_\delta(B(x,r)) \lesssim \begin{cases} C\delta^{s-d} r^d, & 0 < r \leqslant \delta, \\ Cr^s, & \delta \leqslant r \leqslant \delta^\epsilon, \\ 1, & r \geqslant \delta^\epsilon. \end{cases}$$

Consequently, for $x \in \mathbb{R}^d$ fixed,

$$\int |x-y|^{-s} \, d\mu_\delta(y) \lesssim \sum_{2^k \leqslant \delta} C\delta^{s-d} 2^{(d-s)k} + \sum_{\delta \leqslant 2^k \leqslant \delta^\epsilon} C + \sum_{2^k \geqslant \delta^\epsilon} 2^{-ks} \lesssim_s C\delta^{-s\epsilon}.$$

Since μ_δ is a probability measure, and $s < d$, this implies $I_s^\delta(\mu) \leqslant C\delta^{-d\epsilon}$ for $\delta > 0$ small enough.

For the converse, we observe that

$$\int |x-y|^{-s}\, d\mu_\delta(y) = s \int \mu_\delta(B(x,r)) r^{-s-1}\, dr,$$

and so, with a change of variables,

$$I_s^\delta(\mu) = s \iint \mu_\delta(B(x,r)) r^{-s} \frac{dr}{r}\, d\mu(x) = (\log 2) \cdot s \iint \mu_\delta(B(x,2^{-u})) 2^{su}\, du\, d\mu(x).$$

If $I_s^\delta(\mu) \leqslant \delta^{-\epsilon}$, letting

$$E_u = \{x \;:\; 2^{su}\mu_\delta(B(x,2^{-u})) > \delta^{-2\epsilon}\},$$

one gets $\mu(E_u) \leqslant \delta^\epsilon$. So, for $E = \bigcup E_u$, where $u = 0, 1, \ldots, \lfloor \log 1/\delta \rfloor$, we find $\mu(E) \leqslant (\log 1/\delta)\delta^\epsilon$. Thus, letting $A = \mathbb{R}^d \setminus E$, one indeed has

$$\mu(A) \geqslant 1 - (\log 1/\delta)\delta^\epsilon$$

and for all x in A, for all $r \in [\delta, 1]$, $\mu(B(x,r)) \lesssim \delta^{-2\epsilon} r^s$. □

Proof of Lemma 3.3. First of all, we may assume that $k = 2$, since if $k > 2$, we may write

$$\Pi^{\boxplus k}(E) = \int \Pi^{\boxplus 2}(E - x_3 - \ldots - x_k)\, d\Pi(x_3) \cdots d\Pi(x_k),$$

and in particular there exists a vector $(x_3, \ldots, x_k)$ such that $\Pi^{\boxplus 2}(E - x_3 - \ldots - x_k) \geqslant \delta^\epsilon$. After this, it suffices to prove (3.5) with $E - x_3 - \ldots - x_k$ in place of E.

Second, we may assume that the measures μ, ν satisfy the Frostman conditions

$$\mu(B(x,r)) \leqslant \delta^{-6\epsilon} r^s \quad \text{and} \quad \nu(B(x,r)) \leqslant \delta^{-6\epsilon} r^t$$

for $\delta \leqslant r \leqslant 1$ and all $x \in \mathbb{R}$. Indeed, since $I_s^\delta(\mu) \leqslant \delta^{-3\epsilon}$, Lemma 3.8 shows that there exists a Borel set $A \subset \mathbb{R}$ of measure $\mu(A) \geqslant 1 - \delta^{2\epsilon}$ with the property $(\mu|_A)(B(x,r)) \leqslant \delta^{-6\epsilon} r^s$ for all $x \in \mathbb{R}$ and all $r \in [\delta, 1]$. Similarly, we may find a Borel set $B \subset \mathbb{R}$ of measure $\nu(B) \geqslant 1 - \delta^{2\epsilon}$ with the property $(\nu|_B)(B(x,r)) \leqslant \delta^{-6\epsilon} r^t$. Now, we still have $\overline{\Pi}^{\boxplus 2}(E) \geqslant \frac{1}{2}\delta^\epsilon$, where

$$\overline{\Pi} := (\mu|_A \boxminus \mu|_A) \boxtimes (\nu|_B \boxminus \nu|_B).$$

Therefore, we may proceed with the argument, with μ, ν replaced by $\mu|_A, \nu|_B$.

Let us rewrite the condition $\Pi^{\boxplus 2}(E) \geqslant \delta^\epsilon$ as $(\mu \times \nu)^4(G_8) \geqslant \delta^\epsilon$, where

$$G_8 := \{(a_1, b_1, \ldots, a_4, b_4) \in \mathbb{R}^8 : (a_1 - a_2)(b_3 - b_4) + (b_1 - b_2)(a_3 - a_4) \in E\}.$$

In particular, there exists a subset $G_6 \subset \mathbb{R}^6$ of measure $(\mu \times \nu)^3(G_6) \geqslant \delta^{2\epsilon}$ such that for every $(a_1, b_1, a_2, b_2, a_3, b_3)$ in G_6, one has $(\mu \times \nu)(G_2) \geqslant \delta^{2\epsilon}$, where

$$G_2 := \{(a_4, b_4) \in \mathbb{R}^2 : (a_1, b_1, \ldots, a_4, b_4) \in G_8\}. \tag{3.9}$$

Next, we plan to apply Proposition 3.6. To make this formally correct, let us "freeze" two of the variables, say (a_3, b_3): more precisely, fix (a_3, b_3) in such a way that $(\mu \times \nu)^2(G_4) \geqslant \delta^{2\epsilon}$, where

$$G_4 := \{(a_1, b_1, a_2, b_2) \in \mathbb{R}^4 : (a_1, b_1, a_2, b_2, a_3, b_3) \in G_6\}.$$

If ϵ is chosen small enough in terms of s, t and $\sigma := s + t - \kappa$, we may apply Proposition 3.6 with $\mu_1 = \mu_2 = \mu$ and $\nu_1 = \nu_2 = \nu$, and $\boldsymbol{\rho} = \mu \times \nu$, using 6ϵ instead of ϵ. Then, one has

$$(\mu \times \nu)^2(\mathbf{Bad}) < \delta^{6\epsilon} \leqslant (\mu \times \nu)^2(G_4),$$

and (3.7) holds for all $(a_1, b_1, a_2, b_2) \in \mathbb{R}^4 \setminus \mathbf{Bad}$. Consequently, we may find a 4-tuple $(a_1, b_1, a_2, b_2) \in G_4 \setminus \mathbf{Bad}$, and eventually a 6-tuple

$$(a_1, b_1, a_2, b_2, a_3, b_3) \in G_6$$

such that whenever $G \subset \mathbb{R}^2$ is a Borel set with $(\mu \times \nu)(G) = \boldsymbol{\rho}(G) \geqslant \delta^{6\epsilon}$, then

$$\begin{aligned} &N(\{(a_1 - a_2)(b_3 - b_4) + (b_1 - b_2)(a_3 - a_4) : (a_4, b_4) \in G\}, \delta) \\ &= N(\{(a_1 - a_2)b_4 + (b_1 - b_2)a_4 : (a_4, b_4) \in G\}, \delta) \geqslant \delta^{-\sigma} = \delta^{-s-t+\kappa}. \end{aligned}$$

In particular, by (3.9), this can be applied to the set $G := G_2$, and the conclusion is that

$$N(\{(a_1 - a_2)(b_3 - b_4) + (b_1 - b_2)(a_3 - a_4) : (a_4, b_4) \in G_2\}, \delta) \geqslant \delta^{-s-t+\kappa}. \tag{3.10}$$

However, since $(a_1, b_1, a_2, b_2, a_3, b_3) \in G_6$, we have $(a_1, b_1, \ldots, a_4, b_4) \in G_8$ for all $(a_4, b_4) \in G_2$, and consequently

$$\forall (a_4, b_4) \in G_2, \qquad (a_1 - a_2)(b_3 - b_4) + (b_1 - b_2)(a_3 - a_4) \in E.$$

Therefore, (3.10) implies (3.5). □

We now want to go from the combinatorial conclusion of Lemma 3.3 to the more measure theoretic statement of Lemma 3.1 involving energies at scale δ. For that, our strategy is similar in flavour to the one used by Bourgain and Gamburd [4] to derive their flattening lemma, decomposing the measures into dyadic level sets.

Proof of Lemma 3.1. Let $\Pi_r := \Pi * P_r$, where we recall that $\{P_r\}_{r>0} = \{r^{-1}P(\cdot/r)\}_{r>0}$ is a standard approximate identity. The goal will be to show that if $k \geqslant 1$ is sufficiently large (depending on s, t, κ), then, for all $r \in [\delta, 1]$,

$$J_r(k) := \|\Pi_r^{\boxplus 2^k}\|_2 \leqslant \delta^{-\kappa/2} r^{(s+t-1)/2}. \tag{3.11}$$

This implies in a standard manner (using for example [6, Lemma 12.12], Plancherel and a dyadic frequency decomposition) that

$$I_{s+t}^{\delta}(\Pi^{\boxplus 2^k}) \lesssim \delta^{-2\kappa}.$$

Note that the sequence $\{J_r(k)\}_{k \in \mathbb{N}}$ is decreasing in k, since by Young's inequality

$$J_r(k+1) = \|\Pi_r^{\boxplus 2^k} \boxplus \Pi_r^{\boxplus 2^k}\|_2 \leqslant \|\Pi_r^{\boxplus 2^k}\|_1 \|\Pi_r^{\boxplus 2^k}\|_2 = \|\Pi_r^{\boxplus 2^k}\|_2 = J_r(k).$$

Therefore, in order to prove (3.11), the value of k may depend on r, as long as it is uniformly bounded in terms of s, t, κ. Eventually, the maximum of all possible values for k will work for all $\delta \leqslant r \leqslant 1$.

Let us start by disposing of large r, i.e. $r \geqslant \delta^{\kappa/2}$. For that, we have the trivial bound (recalling also that we assumed $s + t \leqslant 1$)

$$J_r(k) \leqslant J_r(0) \lesssim r^{-1} \leqslant \delta^{-\kappa/2} r^{s+t-1}.$$

So, it remains to treat the case $r \in [\delta, \delta^{\kappa/2}]$. We now fix such a scale r. By the pigeonhole principle, given a small parameter $\epsilon \in (0, \frac{\kappa}{4})$ to be fixed later (the choice will roughly be determined by applying Lemma 3.3 to the parameters s, t, κ), there exists $k \lesssim 1/\epsilon$, depending on r, such that

$$\|\Pi_r^{\boxplus 2^{k+1}}\|_2 = J_r(k+1) \geqslant r^{\epsilon} J_r(k) = r^{\epsilon}\|\Pi_r^{\boxplus 2^k}\|_2. \tag{3.12}$$

This index k is fixed for the rest of the argument, so we will not display it in (all) subsequent notation. We may assume that $\|\Pi_r^{\boxplus 2^k}\|_2 \geqslant 1$, otherwise (3.11) is clear.

Let $\mathcal{D}_r$ be the dyadic intervals of $\mathbb{R}$ of length r. For each $I \in \mathcal{D}_r$, we set

$$a_I := \sup_{x \in I} \Pi_r^{\boxplus 2^k}(x).$$

Next, we fix an absolute constant $C_0 \geqslant 1$ to be specified momentarily, and we define the collections

$$\mathcal{A}_j := \begin{cases} \{I \in \mathcal{D}_r : a_I \leqslant C_0\}, & j = 0, \\ \{I \in \mathcal{D}_r : C_0 2^{j-1} < a_I \leqslant C_0 2^j\}, & j \geqslant 1. \end{cases}$$

We also define the sets $A_j := \cup \mathcal{A}_j$; note that the sets A_j are disjoint for distinct j indices. Since Π is a probability measure, $\Pi_r^{\boxplus 2^k} \lesssim 1/r$ for all $k \geqslant 1$. Therefore $A_j = \emptyset$ for $j \geqslant C \log(1/r)$, and evidently

$$\Pi_r^{\boxplus 2^k} \lesssim \sum_{j=0}^{C\log(1/r)} 2^j \cdot \mathbf{1}_{A_j}. \tag{3.13}$$

Here the implicit constants may depend on C_0. Conversely, we claim that

$$\sum_{j=1}^{C\log(1/r)} 2^j \cdot \mathbf{1}_{A_j} \lesssim \Pi_r^{\boxplus 2^k}. \tag{3.14}$$

To see this, fix $x \in A_j$ with $j \geqslant 1$, and let $I = I(x) \in \mathcal{D}_r$ be the dyadic r-interval containing x. Then $a_I \geqslant C_0 2^{j-1}$, which means that there exists another point $x' \in I$ with $\Pi_r^{\boxplus 2^k}(x') \geqslant C_0 2^{j-1}$. Now the key point: the function $\Pi_r^{\boxplus 2^k}$ is C_1/r-Lipschitz for some absolute constant $C_1 > 0$. Therefore, $\Pi_r^{\boxplus 2^k}(x) \geqslant C_0 2^{j-1} - C_1|x - x'|/r \sim 2^j$, provided that $C_0 \geqslant 2C_1$. This proves (3.14).

Based on (3.14) (and our hypothesis $\|\Pi_r^{\boxplus 2^k}\|_2 \geqslant 1$ to treat the case $j = 0$) we may deduce, in particular, that

$$2^j \|\mathbf{1}_{A_j}\|_2 \lesssim \|\Pi_r^{\boxplus 2^k}\|_2, \qquad j \geqslant 0. \tag{3.15}$$

Next, using (3.13), we may pigeonhole an index $j \geqslant 0$ and a set $A := A_j$ with the property

$$\|\Pi_r^{\boxplus 2^{k+1}}\|_2 \leqslant \|\Pi_r^{\boxplus 2^k} \boxminus \Pi_r^{\boxplus 2^k}\|_2 \lesssim (\log 1/r) \cdot 2^j \cdot \|\mathbf{1}_A \boxminus \Pi_r^{\boxplus 2^k}\|_2.$$

Since further, by Plancherel and Cauchy-Schwarz,

$$\|\mathbf{1}_A \boxminus \Pi_r^{\boxplus 2^k}\|_2 \leqslant \|\mathbf{1}_A \boxminus \mathbf{1}_A\|_2^{1/2} \|\Pi_r^{\boxplus 2^{k+1}}\|_2^{1/2}$$

we deduce that

$$\begin{aligned} r^{\epsilon}\|\Pi_r^{\boxplus 2^k}\|_2 &\overset{(3.12)}{\leqslant} \|\Pi_r^{\boxplus 2^{k+1}}\|_2 \lesssim (\log 1/r)^2 \cdot 2^{2j} \cdot \|\mathbf{1}_A \boxminus \mathbf{1}_A\|_2 \\ &\lesssim r^{-\epsilon} \cdot 2^{2j} \cdot \|\mathbf{1}_A\|_1 \|\mathbf{1}_A\|_2 \overset{(3.15)}{\lesssim} r^{-\epsilon} \cdot 2^j \cdot \|\mathbf{1}_A\|_1 \|\Pi_r^{\boxplus 2^k}\|_2. \end{aligned} \tag{3.16}$$

At this point we note that if $j = 0$, then the preceding inequality shows that $\|\Pi_r^{\boxplus 2^k}\|_2 \lesssim r^{-2\epsilon}$, which is better than (3.11), since we declared that $\epsilon \leqslant \kappa/4$. So, we may and will assume that $j \geqslant 1$ in the sequel.

In (i) below we combine (3.16) and (3.14), whereas in (ii) below we combine (3.16) with $2^j\|\mathbf{1}_A\|_1 \lesssim \|\Pi_r^{\boxplus 2^k}\|_1 = 1$:

(i) $r^{2\epsilon} \lesssim 2^j\|\mathbf{1}_A\|_1 \lesssim \Pi_r^{\boxplus 2^k}(A)$,
(ii) $r^{2\epsilon}\|\Pi_r^{\boxplus 2^k}\|_2 \lesssim 2^j\|\mathbf{1}_A\|_2 \lesssim \|\mathbf{1}_A\|_1^{-1}\|\mathbf{1}_A\|_2$.

Since A is a union of intervals in $\mathcal{D}_r$, one has

$$\|\mathbf{1}_A\|_1 \sim rN(A,r) \quad \text{and} \quad \|\mathbf{1}_A\|_2 \sim r^{1/2}N(A,r)^{1/2},$$

so item (ii) yields

$$\|\Pi_r^{\boxplus 2^k}\|_2 \lesssim r^{-\frac{1}{2}-2\epsilon}N(A,r)^{-1/2}.$$

On the other hand, since

$$I_s^r(\mu) \leqslant I_s^\delta(\mu) \leqslant \delta^{-\epsilon} \leqslant r^{-\epsilon/\kappa} \quad \text{and} \quad I_t^r(\nu) \leqslant r^{-\epsilon/\kappa},$$

Lemma 3.3 applied at scale r (and recalling (i) above) shows that if ϵ is chosen small enough in terms of s, t, κ, then

$$N(A,r) \geqslant r^{-s-t+\kappa}.$$

We thus obtain what we claimed in (3.11):

$$\|\Pi_r^{\boxplus 2^k}\|_2 \lesssim \delta^{-\kappa/2}r^{(s+t-1)/2}.$$

This completes the proof of Lemma 3.1. □

4. The induction step

The proof of Theorem 1.3 is by induction on n, starting from the $n = 2$ case, already studied in Section 2. The induction step is based on the flattening results for additive-multiplicative convolutions developed in the previous section. It will be essential in the argument to be able to switch the order of addition and multiplication. For that we record the following lemma, which is a simple application of the Cauchy-Schwarz inequality.

Lemma 4.1. *Given two Borel probability measures μ and ν on $\mathbb{R}$, one has, for all ξ in $\mathbb{R}$,*

$$|\widehat{\mu \boxtimes \nu}(\xi)|^2 \leqslant (\widehat{\mu \boxminus \mu) \boxtimes \nu}(\xi).$$

Proof. Writing the Fourier transforms explicitly, and applying the Cauchy-Schwarz inequality, one gets

$$\left|\iint e^{-2\pi i\xi xy}\,dx\,dy\right|^2 \leqslant \int \left|\int e^{-2\pi i\xi xy}\,dx\right|^2 dy = \iiint e^{-2\pi i\xi(x_1-x_2)y}\,dx_1\,dx_2\,dy.$$

□

Proof of Theorem 1.3. For the base case $n = 2$, we may apply Proposition 2.1. Indeed, assuming $I_{s_i}^\delta(\mu_i) \leqslant \delta^{-\epsilon}$ for $i = 1, 2$, one has

$$\|\mu_{i,\delta}\|_2^2 = \int |\widehat{\mu_{i,\delta}}(\xi)|^2\,d\xi \lesssim \delta^{-1+s_i+\epsilon}\int |\xi|^{1-s_i}|\widehat{\mu_{i,\delta}}(\xi)|^2\,d\xi \lesssim \delta^{-1+s_i+2\epsilon}$$

so if $s_1 + s_2 > 1$, taking ϵ small enough to ensure $s_1 + s_2 - 4\epsilon > 1$, one finds, for every $\delta^{-1} \leqslant |\xi| \leqslant 2\delta^{-1}$,

$$|\widehat{\mu_1 \boxtimes \mu_2}(\xi)| \leqslant \delta^{\frac{s_1+s_2-4\epsilon-1}{2}},$$

which is the desired Fourier decay.

Now let $n \geqslant 3$, and assume that we have already established the case $n-1$ with the collection of parameters $\mathcal{S}_{n-1} := \{s_1 + s_2, s_3, \dots, s_n\}$, and some constants

$$\epsilon_{n-1}(\mathcal{S}_{n-1}) > 0 \quad \text{and} \quad \delta_0 := \delta_0(\mathcal{S}_{n-1}) > 0. \tag{4.2}$$

It is easy to reduce to the case $s_1 + s_2 \leqslant 1$, so we assume this in the sequel.

Given ξ with $\delta^{-1} \leqslant \xi \leqslant 2\delta^{-1}$, our goal is to bound

$$\mathcal{F}(\xi) := (\mu_1 \boxtimes \dots \boxtimes \mu_n)^\wedge(\xi).$$

Applying Lemma 4.1 twice, first with $\mu = \mu_1$ and $\nu = \mu_2 \boxtimes \dots \boxtimes \mu_n$ and then with $\mu = \mu_2$ and $\nu = (\mu_1 \boxminus \mu_1) \boxtimes \mu_3 \boxtimes \dots \boxtimes \mu_n$, yields

$$|\mathcal{F}(\xi)|^4 \leqslant (\Pi \boxtimes \mu_3 \boxtimes \dots \boxtimes \mu_n)^\wedge(\xi),$$

where $\Pi = (\mu_1 \boxminus \mu_1) \boxtimes (\mu_2 \boxminus \mu_2)$. Using the same lemma again k times, we further get

$$|\mathcal{F}(\xi)|^{2^{k+2}} \leqslant (\Pi^{\boxplus 2^k} \boxtimes \mu_3 \boxtimes \dots \boxtimes \mu_n)^\wedge(\xi).$$

Lemma 3.1 applied with constants $s := s_1$ and $t := s_2$, and $\kappa := \epsilon_{n-1} := \epsilon_{n-1}(\mathcal{S}_{n-1})$, shows that if $\epsilon = \epsilon(s_1, s_2, \epsilon_{n-1}) > 0$ is sufficiently small, $k = k(s_1, s_2, \epsilon_{n-1})$ is sufficiently large, and μ_1, μ_2 satisfy $I^\delta_{s_j}(\mu_j) \leqslant \delta^{-\epsilon}$ for $j = 1, 2$, then

$$I^\delta_{s_1+s_2}(\Pi^{\boxplus 2^k}) \leqslant \delta^{-\kappa} = \delta^{-\epsilon_{n-1}}.$$

We apply our induction hypothesis to the collection of $n-1$ probability measures

$$\{\bar{\mu}_1, \dots, \bar{\mu}_{n-1}\} = \{\Pi^{\boxplus 2^k}, \mu_3, \dots, \mu_n\}$$

with exponents $\{s_1 + s_2, s_3, \dots, s_n\}$ to get

$$|\mathcal{F}(\xi)|^{2^{k+2}} \leqslant |\xi|^{-\epsilon_{n-1}}.$$

(To be precise, since the measure $\Pi^{\boxplus 2^k}$ is not supported on $[-1, 1]$ but on $[-2^{k+2}, 2^{k+2}]$, so one rather needs to consider the rescaled measure $\bar{\mu}_1 = (2^{-k-2})_* \Pi^{\boxplus 2^k}$, which satisfies $I^\delta_{s_1+s_2}(\bar{\mu}_1) \sim_k I^\delta_{s_1+s_2}(\Pi^{\boxplus 2^k})$ but the involved constant depending on k is harmless.) This shows that the Fourier decay property holds for n, with constants $\epsilon_n := \min\{\epsilon, \epsilon_{n-1}\}$ and $\tau_n = \frac{\tau_{n-1}}{2^k}$. The necessary size of k is determined by the application of Lemma 3.1, so it depends only on $\min\{s_1, s_2\} > 0$. The proof of Theorem 1.3 is complete. □

References

[1] Jean Bourgain. Multilinear exponential sums in prime fields under optimal entropy condition on the sources. *Geom. Funct. Anal.*, 18(5):1477–1502, 2009.

[2] Jean Bourgain. The discretized sum-product and projection theorems. *J. Anal. Math.*, 112:193–236, 2010.

[3] Jean Bourgain and Semyon Dyatlov. Fourier dimension and spectral gaps for hyperbolic surfaces. *Geom. Funct. Anal.*, 27(4):744–771, 2017.

[4] Jean Bourgain and Alex Gamburd. On the spectral gap for finitely generated subgroups of $SU(2)$. *Inventiones Mathematicae*, 171:83–121, 2008.

[5] Jialun Li. Discretized sum-product and Fourier decay in $\mathbb{R}^n$. *J. Anal. Math.*, 143(2):763–800, 2021.

[6] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability. 1st paperback ed.* Cambridge University Press, 1999.

[7] Tuomas Orponen and Pablo Shmerkin. Projections, Furstenberg sets, and the ABC sum-product problem. Preprint, arXiv:2301.10199, January 2023.

[8] Tuomas Orponen, Pablo Shmerkin, and Hong Wang. Kaufman and Falconer estimates for radial projections and a continuum version of Beck's theorem. Preprint arXiv:2209.00348v1.pdf, September 2022.

[9] Tuomas Sahlsten and Connor Stevens. Fourier transform and expanding maps on Cantor sets. *Amer. J. Math.*, Accepted for publication, 2022.

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