Rational approximations to linear subspaces

Nicolas de Saxcé

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Abstract

We study intrinsic diophantine approximation on Grassmannian varieties. Using a new correspondence between the diophantine properties of a linear subspace x in $\mathbb{R}^d$ and certain diagonal orbits in the space of lattices, we are able to solve some problems suggested by Schmidt in 1967. In particular we obtain a version of Dirichlet's principle in this setting with an optimal exponent.

Contents

Introduction

Since the discovery of irrational numbers by the ancient Greeks, mathematicians have tried to approximate real numbers by rationals. This study of rational approximations $\frac{p}{q}$ to a given real number θ is the starting point of diophantine approximation. The problem can be reformulated in the projective space $\mathbb{P}^1(\mathbb{R})$: one seeks to approach an irrational line x with homogeneous coordinates $[1 : \theta]$ by a rational line $v = [p : q]$, where $(p, q) \in \mathbb{Z} \times \mathbb{N}$. For instance, writing $d(x, v)$ to denote the usual distance on $\mathbb{P}^1(\mathbb{R})$ between x and v, and $H(v) = \max(|p|, |q|)$ the height of the rational point v, Dirichlet's celebrated theorem [8] states that for every irrational point $x \in \mathbb{P}^1(\mathbb{R})$, the inequality

$$d(x, v) \leq H(v)^{-2}$$

has infinitely many solutions $v \in \mathbb{P}^1(\mathbb{Q})$. On the other hand, it is not difficult to check that the line $x = [1 : \sqrt{2}]$ satisfies, for some $c > 0$, for every $v \in \mathbb{P}^1(\mathbb{Q})$,

$d(x,v) \geq cH(v)^{-2}$, which shows that 2 is the optimal exponent in Dirichlet's theorem.

In higher dimension, one can generalize the problem in several ways. For example, given a real line $x \in \mathbb{P}^n(\mathbb{R})$, one may study approximations to x by rational lines (one then speaks of *simultaneous approximation*) or try to understand rational hyperplanes that are close to x (one then speaks of *approximation by linear forms*). In a foundational paper [29], Schmidt suggested the following general problem, which places those different questions in one simple natural geometric setting:

Fix integers d, k and ℓ such that $d \geq 2$ et $0 < k, \ell < d$. Given an ℓ-dimensional subspace x in $\mathbb{R}^d$, study k-dimensional rational subspaces v that come close to x.

Momentarily admitting that distance and height are properly defined on the Grassmann variety of subspaces of $\mathbb{R}^d$, the problems we wish to study here can be summarized as follows:

(A) *Dirichlet's principle:* Determine the supremum of all $\beta > 0$ such that for every $x \in X_\ell(\mathbb{R})$, the inequality $d(x,v) \leq H(v)^{-\beta}$ has solutions $v \in X_k(\mathbb{Q})$ arbitrarily close to x.

(B) *Roth-type theorem:* Compute the diophantine exponents $\beta_k(x)$ (defined below) of any subspace x defined over $\overline{\mathbb{Q}}$.

(C) *Metric theory:* Study diophantine properties of a point x chosen randomly in $X_\ell(\mathbb{R})$.

These problems fall into the realm of *intrinsic diophantine approximation*: having fixed an ambient algebraic variety X with dense set of rational points, one studies approximations to a real point $x \in X(\mathbb{R})$ by rational points $v \in X(\mathbb{Q})$. The classical theory of diophantine approximation corresponds to the case where $X = \mathbb{P}^n$ is the full projective space. Recently, the work of Kleinbock, Merrill, Fishman and Simmons [21, 11] and its continuations [20, 24] showed that homogeneous dynamics could be used to study the case of quadric hypersurfaces. We follow a similar approach here, based on the correspondence between the diophantine properties of a point $x \in X_\ell(\mathbb{R})$ and the asymptotic behavior of a well-chosen orbit in the space of lattices. But first, we need to define the distance and the height on the Grassmann variety.

In the sequel, the integers d, k and ℓ are fixed, and satisfy $0 < k, \ell < d$. *We write* $X_\ell = \mathrm{Grass}(\ell, d)$ *to denote the Grassmann variety of ℓ-dimensional linear subspaces of a d-dimensional ambient space; the set of real points of X_ℓ, naturally identified with real ℓ-dimensional subspaces of $\mathbb{R}^d$, is denoted* $X_\ell(\mathbb{R})$. *Similarly, we let* $X_k = \mathrm{Grass}(k,d)$ *and write* $X_k(\mathbb{Q})$ *to denote the set of* rational *points of X_k, i.e. k-dimensional rational linear subspaces of* $\mathbb{R}^d$.

Diophantine exponents of a linear subspace

In order to evaluate the quality of a rational approximation to a real subspace, we must first define a *distance* on the Grassmann variety. Just for this definition, we write the dimension of a linear subspace as an exponent: x^1 denotes a line, y^2 a plane, etc. This slightly cumbersome notation should help the reader follow the argument. When $x = x^1$ and $y = y^1$ are lines, the distance is the usual distance on the projective space, given by

$$d(x^1, y^1) = \sin \measuredangle(x^1, y^1) = \frac{\|u_x \wedge u_y\|}{\|u_x\|\|u_y\|}$$

where u_x and u_y are non-zero vectors on x and y, respectively. Then, if $x = x^1$ is a line and $y = y^k$ a k-dimensional subspace, we set

$$d(x^1, y^k) = \min\{d(x^1, y^1) \ ; \ y^1 \subset y^k\}$$

Finally, in general, if $x = x^\ell$ is ℓ-dimensional and $y = y^k$ is k-dimensional,

$$d(x^\ell, y^k) = \begin{cases} \max_{x^1 \in x^\ell} d(x^1, y^k) & \text{if } \ell \le k \\ \max_{y^1 \in y^k} d(y^1, x^\ell) & \text{if } k \ge \ell \end{cases}$$

Remark. If $\dim x = \dim y = \ell$, this indeed defines a distance on $X_\ell(\mathbb{R})$, and this distance is equivalent to any Riemannian distance on $X_\ell(\mathbb{R})$. In general, the equality $d(x, y) = 0$ only means that x and y are comparable, i.e. $x \subset y$ or $y \subset x$.

We shall also need a *height* on rational subspaces. For $v \in X_k(\mathbb{Q})$, choose a basis $(v_i)_{1 \le i \le k}$ for v consisting of vectors in $\mathbb{Z}^d$ with no common divisor and set

$$H(v) = \|v_1 \wedge \cdots \wedge v_k\|$$

where the norm on $\wedge^k \mathbb{R}^d$ is the usual Euclidean norm. Actually, any other norm could be taken for that definition: the associated height would then be comparable to H within some multiplicative constant, and the properties we study in the sequel would not be affected by such a change.

The above-defined distance and height allow one to define a family of diophantine exponents for each linear subspace x in $\mathbb{R}^d$ in the following way.

Definition (Diophantine exponents)**.** For $x \in X_\ell(\mathbb{R})$ and $k = 1, \ldots, d-1$, define the *diophantine exponent of* x for approximation by k-dimensional rational subspaces by

$$\beta_k(x) = \inf\{\beta > 0 \mid \exists c > 0 : \forall v \in X_k(\mathbb{Q}),\, d(v, x) \ge cH(v)^{-\beta}\}.$$

In order to compute $\beta_k(x)$, one may always reduce to the case $k \le \ell$, since for every $x \in X_\ell(\mathbb{R})$ and $v \in X_k(\mathbb{R})$, one has $d(x, v) = d(x^\perp, v^\perp)$ and so

$$\beta_k(x) = \beta_{d-k}(x^\perp). \tag{1}$$

Therefore, we shall always assume $k \le \ell$ in the sequel, which will simplify some of our formulas[1].

[1] This convention differs from that of Schmidt, who restrict attention to the case $k + \ell \le d$, but one can easily go from one assumption to the other using (1).

Main results

Schmidt [29, Theorem 3] showed that the number $N_{X_k}(H)$ of points v in $X_k(\mathbb{Q})$ of height at most H satisfies

$$N_{X_k}(H) \asymp_{H\to\infty} H^d. \tag{2}$$

Now, for each $v \in X_k$, the inequality $d(x,v) \leq r$ defines an r-neighborhood of a submanifold of codimension $k(d-\ell)$ in $X_\ell(\mathbb{R})$. The volume of such a neighborhood is essentially equal to $r^{k(d-\ell)}$, so one may expect the union of such neighborhoods for v varying among all rational points of height at most H to have total measure comparable to $H^d r^{k(d-\ell)}$, at least if $r \ll H^{-\frac{d}{k(d-\ell)}}$. Taking $r = H^{-\beta}$, this heuristic argument suggests that $\beta = \frac{d}{k(d-\ell)}$ is a critical value.

The main result of this article is that this critical value is indeed a uniform lower bound for $\beta_k(x)$ for *every* point $x \in X_\ell(\mathbb{R})$. It is not hard to check that it is optimal, either by considering a random point x in $X_\ell(\mathbb{R})$ for the Lebesgue measure, or by constructing an explicit example with coordinates in a number field, as in [29, Theorem 15]. When k or ℓ equals 1 or $d-1$, that result is nothing but Dirichlet's famous principle [8]. But in general, the known lower bounds, obtained by Schmidt [29, Theorem 12], are only optimal if the integers d, k and ℓ satisfy the very restrictive inequality $d \geq \min(k,\ell)(d-\max(k,\ell))$. We note that Elio Joseph [14] has recently improved some of these bounds, and obtained an optimal lower bound for a slightly different problem, also suggested by Schmidt; this will be explained in more detail at the end of the paper, with the concluding remarks.

Theorem 1 (Dirichlet's principle in Grassmann varieties)**.** *For all x in $X_\ell(\mathbb{R})$ and all $k \in [\![1,\ell]\!]$, $\beta_k(x) \geq \frac{d}{k(d-\ell)}$. That lower bound is optimal, since equality holds almost surely if x is a random point distributed according to the Lebesgue measure on $X_\ell(\mathbb{R})$.*

Let $\overline{\mathbb{Q}}$ denote the field of real algebraic numbers. When the linear subspace x is defined over $\overline{\mathbb{Q}}$, we are even able to give an explicit formula for each $\beta_k(x)$, in terms of dimensions of intersections of x with rational subspaces. The reader is referred to Theorem 8 for the precise statement, which implies the following analog of Roth's theorem on rational approximations to algebraic numbers. Recall that a *pencil* in X_ℓ is a subvariety of the form

$$\mathcal{P}_{W,r} = \{x \in X_\ell \mid \dim x \cap W \geq r\}$$

where W is a linear subspace in $\mathbb{R}^d$ and $r \in [\![0,d]\!]$. The pencil is said to be *rational* if W is defined over $\mathbb{Q}$, and *constraining* if $\frac{r}{\dim W} > \frac{\ell}{d}$.

Theorem 2 (Roth's theorem for Grassmann varieties)**.** *Let $x \in X_\ell(\overline{\mathbb{Q}})$. If x is not included in any rational constraining pencil, then for every $k \in [\![1,d-1]\!]$, $\beta_k(x) = \frac{d}{k(d-\ell)}$. Conversely, if x belongs to a constraining rational pencil, then for every $k = 1,\dots,d-1$, $\beta_k(x) > \frac{d}{k(d-\ell)}$.*

In metric diophantine approximation, we shall first prove a version of Khintchine's theorem.

Theorem 3 (Khintchine's theorem for Grassmann varieties)**.** *Let* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a non-increasing function. For* $x \in X_\ell(\mathbb{R})$ *and* $v \in X_k(\mathbb{Q})$*, one considers the inequality*

$$d(x,v) \leq H(v)^{-\frac{d}{k(d-\ell)}} \psi(H(v)). \tag{3}$$

(i) If $\int_1^{+\infty} \psi(u)^{k(d-\ell)} \frac{\mathrm{d}u}{u} < +\infty$*, then, for almost every* $x \in X_\ell(\mathbb{R})$*, the inequality* (3) *has only finitely many solutions* $v \in X_k(\mathbb{Q})$*.*

(ii) If $\int_1^{+\infty} \psi(u)^{k(d-\ell)} \frac{\mathrm{d}u}{u} = +\infty$*, then, for almost every* $x \in X_\ell(\mathbb{R})$*, the inequality* (3) *has infinitely many solutions* $v \in X_k(\mathbb{Q})$*.*

Then, we obtain a formula, analogous to Jarník's theorem, for the Hausdorff dimension of the set $W_{k,\ell}(\tau)$ of subspaces $x \in X_\ell(\mathbb{R})$ satisfying $\beta_k(x) \geq \tau$.

Theorem 4 (Jarník's theorem for Grassmann varieties)**.** *Fix integers* $1 \leq k \leq \ell < d$*. For every* $\tau \geq \frac{d}{k(d-\ell)}$*,*

$$\dim_H W_{k,\ell}(\tau) = (\ell - k)(d-\ell) + \frac{d}{\tau}$$

Finally, we shall study the diophantine exponents of a point x chosen randomly on an analytic submaniold in $X_\ell(\mathbb{R})$, and show the following theorem, inspired by the works of Kleinbock and Margulis [17] and of Kleinbock [19] on Sprindzuk's conjecture.

Theorem 5 (Diophantine approximation on analytic submanifolds)**.** *Let* $M \subset X_\ell(\mathbb{R})$ *be a connected analytic submanifold.*

1. *For each* $k = 1, \ldots, \ell$*, there exists* $\beta_k(M)$ *such that for almost every* x *in* M*,* $\beta_k(x) = \beta_k(M)$*.*

2. *The exponent* $\beta_k(M)$ *is determined by the Zariski closure of* M *in* X_ℓ*.*

3. *If* M *is not included in any constraining pencil, then, for* $k = 1, \ldots, \ell$*,* $\beta_k(M) = \frac{d}{k(d-\ell)}$*.*

When the Zariski closure of M is defined over $\overline{\mathbb{Q}}$, one can also give an explicit formula for the exponents $\beta_k(M)$; we refer the reader to Section 6 for its precise statement.

Plan of the paper

All the above theorems are proved using a correspondence between the diophantine properties of a point $x \in X_\ell(\mathbb{R})$ and the asymptotic behavior of an associated diagonal orbit in the space of lattices $\Omega = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{R})$. The statement of that correspondence is given as Proposition 1, and its proof is the goal of Section 1.

In Section 2, we combine this correspondence with the results of [5], based of Schmidt's subspace theorem, to compute all diophantine exponents $\beta_k(x)$, $k = 1, \ldots, d-1$ of an arbitrary point $x \in X_\ell(\overline{\mathbb{Q}})$, and then deduce Theorem 2.

Section 3 is devoted to Theorem 1. The proof is once more based on the correspondence with diagonal orbits in Ω, but this time, one cannot use Schmidt's subspace theorem, and the behavior of diagonal orbits cannot be so easily described. This makes our argument much more involved. Nonetheless, using tools inspired from the *parametric geometry of numbers* recently introduced by Schmidt and Summerer [31], and developed in particular by Roy [25] and by Das, Fishman, Simmons and Urbański [7], we are able to obtain enough information on the orbit to conclude our proof.

Sections 4, 5 and 6 are dedicated to metric diophantine approximation; they contain the proofs of Theorems 3, 4 and 5, respectively.

1 Diagonal orbits and diophantine exponents

sec:correspondance

We now present the relation between diophantine approximation on the Grassmann variety X_ℓ and diagonal orbits in the space of lattices in $\mathbb{R}^d$.

The algebraic group $G = \mathrm{SL}_d$ acts transitively on the Grassmann variety X_ℓ. Let P denote the stabilizer in G of the base point $x_0 = \mathrm{Span}(e_1, \dots, e_\ell)$, i.e. the set of elements in G whose matrix in the standard representation is of the form $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, with $A \in M_\ell(\mathbb{R})$, $B \in M_{\ell,d-\ell}(\mathbb{R})$ and $C \in M_{d-\ell}(\mathbb{R})$. This action of G gives an identification of X_ℓ with the quotient variety $P\backslash G$:

$$\begin{array}{ccc} P\backslash G & \mapsto & X_\ell \\ Pg & \mapsto & g^{-1}x_0 \end{array}$$

The main result of this section is a formula relating the diophantine exponents of a point $x = Ps_x$ in $X_\ell(\mathbb{R})$ and the asymptotic behavior of the orbit of the lattice

$$\Delta_x = s_x\mathbb{Z}^d$$

under the diagonal subgroup

$$a_t = \mathrm{diag}(\underbrace{e^{-\frac{t}{\ell}}, \dots, e^{-\frac{t}{\ell}}}_{\ell \text{ times}}, \underbrace{e^{\frac{t}{d-\ell}}, \dots, e^{\frac{t}{d-\ell}}}_{d-\ell \text{ times}}).$$

As before, we assume for clarity that $k \leq \ell$. The eigenvalues of a_t in $\wedge^k\mathbb{R}^d$ are then equal to

$$e^{-t\frac{k}{\ell}},\ e^{-t[\frac{k}{\ell}-(\frac{1}{\ell}+\frac{1}{d-\ell})]},\ e^{-t[\frac{k}{\ell}-2(\frac{1}{\ell}+\frac{1}{d-\ell})]},\ \dots,\ e^{-t[\frac{k}{\ell}-\min(k,d-\ell)(\frac{1}{\ell}+\frac{1}{d-\ell})]}.$$

Let $\pi^+ : \wedge^k\mathbb{R}^d \to \wedge^k\mathbb{R}^d$ be the projection to the eigenspace of a_t associated to the eigenvalue $e^{-t\frac{k}{\ell}}$, parallel to all other eigenspaces of a_t. Denote also

$$C_+ = \{\mathbf{v} \in \wedge^k\mathbb{R}^d \mid \|\pi_+(\mathbf{v})\| \geq \frac{1}{2}\|\mathbf{v}\|\}$$

and set

$$\gamma_k(x) = \inf\{\gamma \in \mathbb{R} \mid \exists c > 0 : \forall t > 0, \forall \mathbf{v} \in C_+ \cap \wedge^k a_t\Delta_x,\ \|\mathbf{v}\| \geq ce^{-\gamma t}\}.$$

Theorem 6 below relates the escape rate $\gamma_k(x)$ of the diagonal orbit in the space of lattices to the diophantine exponent $\beta_k(x)$ for approximation of x by rational subspaces of dimension k. In the particular case where $k = 1$, this correspondence was put into light by Dani [6], and has been a central tool since then, in particular in the works of Kleinbock and Margulis [17, 18, 16]. The novelty here is that we study lattices in the exterior power representation $\wedge^k \mathbb{R}^d$ to obtain information about approximation by k-dimensional rational subspaces; this requires us to introduce the cone C_+ in order to control the direction of short vectors.

tfed **Theorem 6** (Rate of escape and diophantine exponent)**.** *For every x in $X_\ell(\mathbb{R})$,*

$$\beta_k(x) = \frac{d}{(k - \ell\gamma_k(x))(d-\ell)}.$$

That theorem will follow from Proposition 1 below, which gives a more precise correspondence between k-dimensional rational subspaces near x and short vectors along the orbit $\wedge^k a_t \Delta_x$.

corr **Proposition 1** (Dani's correspondence for Grassmann varieties)**.** *Let x in $X_\ell(\mathbb{R})$ and $s_x \in \mathrm{SL}_d(\mathbb{R})$ such that $x = P s_x$.*

1. *Let $v \in X_k(\mathbb{Q})$ be close to x, and $t > 0$ such that $e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})} = d(v,x)$. Within multiplicative constants depending on the choice of s_x, the pure tensor $\mathbf{v} \in \wedge^k \mathbb{Z}^d$ associated to v satisfies*
$$\|\pi_+(a_t s_x \mathbf{v})\| \gg \|a_t s_x \mathbf{v}\| \quad \text{and} \quad \|a_t s_x \mathbf{v}\| \ll e^{-t\frac{k}{\ell}} H(v).$$

2. *Let $t > 0$ and $\mathbf{v} \in \wedge^k \mathbb{Z}^d$ a pure tensor such that $\|\pi_+(a_t s_x \mathbf{v})\| \geq c \cdot \|a_t s_x \mathbf{v}\|$, where $c > 0$ is some fixed parameter. Within multiplicative constants depending on the choice of s_x and on c, the rational subspace $v \in X_k(\mathbb{Q})$ associated to $\mathbf{v}$ satisfies*
$$H(v) \ll e^{t\frac{k}{\ell}} \|a_t s_x \mathbf{v}\| \quad \text{and} \quad d(v,x) \ll e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})}.$$

The proof of the above proposition is based on two lemmas. The first expresses the distance between to linear subspaces in terms of vectors in some exterior power of $\mathbb{R}^d$. Given $\mathbf{v} \in \wedge^k \mathbb{R}^d$, we write

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 + \dots$$

its decomposition according to the eigenspaces of a_t, where $\mathbf{v}_i$ is an eigenvector associated to the eigenvalue $e^{-t(\frac{k}{\ell} - i(\frac{1}{\ell}+\frac{1}{d-\ell}))}$, for $i = 0, \dots, \min(k, d-\ell)$. We shall use the standard basis $(e_i)_{1 \leq i \leq d}$ for $\mathbb{R}^d$, and the associated basis

$$(\mathbf{e}_I)_{I \subset [\![1,d]\!],\ |I|=k} \quad \text{for } \wedge^k \mathbb{R}^d,$$

where

$$\mathbf{e}_I = e_{i_1} \wedge \dots \wedge e_{i_k} \quad \text{if} \quad I = \{i_1 < i_2 < \dots < i_k\}.$$

dist **Lemma 1.** *Let $x \in X_\ell(\mathbb{R})$ and $s_x \in \mathrm{SL}_d(\mathbb{R})$ so that $x = P s_x$. Within multiplicative constants depending on the choice of s_x, for every $v \in X_k(\mathbb{R})$ close to x, if $\mathbf{v} \in \wedge^k \mathbb{R}^d$ represents v and $\tilde{\mathbf{v}} = s_x \mathbf{v}$, one has*

$$d(v,x) \asymp \frac{\|\tilde{\mathbf{v}}_1\|}{\|\tilde{\mathbf{v}}_0\|}.$$

Proof. It is enough to prove this approximate equality when $d(v,x) \leq c$, where $c > 0$ is a fixed constant. Applying an isometry of $\mathbb{R}^d$ if necessary, we may assume that $x = \mathrm{Span}(e_1, \dots, e_\ell)$ and that the orthogonal projection of $\mathbf{v}$ to $\wedge^k x$ is $\pi_x(\mathbf{v}) = \mathbf{e}_{[\![1,k]\!]}$. We may then choose a basis $(u_i)_{1\leq i\leq k}$ for v so that for $i = 1, \dots, k$,

$$u_i = e_i + \sum_{s>\ell} u_{i,s} e_s,$$

which implies

$$\mathbf{v}_0 = \mathbf{e}_{[\![1,k]\!]} \quad \text{and} \quad \mathbf{v}_1 = \sum_{i,s} u_{i,s} \mathbf{e}_{[\![1,k]\!]\cup\{s\}\setminus\{i\}}.$$

Since s_x stabilizes the subspace $x = \mathrm{Span}(e_1, \dots, e_\ell)$, it preserves the eigenspaces of a_t and we must have, for every i, $\tilde{\mathbf{v}}_i = s_x \mathbf{v}_i$. Therefore,

$$\frac{\|\tilde{\mathbf{v}}_1\|}{\|\tilde{\mathbf{v}}_0\|} \asymp \frac{\|\mathbf{v}_1\|}{\|\mathbf{v}_0\|} \asymp \max_{i,s} |u_{i,s}|.$$

On the other hand,

$$d(v,x) = \max_{u\in v;\ \|u\|=1} \frac{\|u \wedge x\|}{\|u\|\|x\|} \asymp \max_{1\leq i\leq k} \frac{\|u_i \wedge x\|}{\|u_i\|\|x\|} \asymp \max_{i,s} |u_{i,s}|.$$

where the second step comes from the fact that the basis (u_i) for v is almost orthonormal (the coefficients $u_{i,s}$ are arbitrarily small if one restricts v to a small neighborhood of x). □

The statement of the second lemma is slightly more technical; it will allow us to control the components $\mathbf{v}_i$, $i \geq 2$ of a pure tensor $\mathbf{v}$ in terms of the first two components $\mathbf{v}_0$ and $\mathbf{v}_1$.

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Lemma 2. *If $\mathbf{v} \in \wedge^k \mathbb{R}^d$ is a pure tensor, then, for every $r \geq 1$,*

$$\|\mathbf{v}_0\|^{r-1} \|\mathbf{v}_r\| \ll \|\mathbf{v}_1\|^r.$$

Proof. We proceed by induction on r, using the Grassmann (or Plücker) relations [4, chapitre III, §13]. For $r = 1$ the result is clear. Now assume it has been proven for some $r \geq 1$. Write

$$\mathbf{v} = \sum_{I\subset[\![1,d]\!],\ |I|=k} v_I \mathbf{e}_I,$$

so that for each $r \geq 0$,

$$\mathbf{v}_r = \sum_{\substack{I\subset[\![1,d]\!],\ |I|=k \\ |I\setminus[\![1,\ell]\!]|=r}} v_I \mathbf{e}_i.$$

We want to show that if I is a subset of $[\![1,d]\!]$ with k elements and such that $|I \setminus [\![1,\ell]\!]| = r+1$, then $\|\mathbf{v}_0\|^r |v_I| \ll \|\mathbf{v}_1\|^{r+1}$.

Let $J_0 \subset [\![1,\ell]\!]$ be a subset of cardinality k such that $|v_{J_0}| \asymp \|\mathbf{v}_0\|$ and $i_0 \in I \setminus [\![1,\ell]\!]$. The Grassmann relation [4, (84-(J,H)), page 172, chapitre III] for the sets $H = I \setminus \{i_0\}$ and $J = J_0 \cup \{i_0\}$ writes

$$v_{J_0} v_I = \sum_{j\in J_0\setminus I} \pm v_{J\setminus\{j\}} v_{H\cup\{j\}}.$$

For each $j \in J_0 \setminus I$, one has $|(J \setminus \{j\}) \setminus [\![1, \ell]\!]| = 1$, and therefore $|v_{J \setminus \{j\}}| \leq \|\mathbf{v}_1\|$. Similarly, $|H \cup \{j\} \setminus [\![1, \ell]\!]| = r$, and so $|v_{H \cup \{j\}}| \leq \|\mathbf{v}_r\|$. This implies $|v_{J_0} v_I| \ll \|\mathbf{v}_1\| \|\mathbf{v}_r\|$, whence $\|\mathbf{v}_0\| \|\mathbf{v}_{r+1}\| \ll \|\mathbf{v}_1\| \|\mathbf{v}_r\|$. One concludes from the induction hypothesis. $\square$

Using the above two lemmas, we can now derive Proposition 1.

Proof of Proposition 1. Let $v \in X_k(\mathbb{Q})$ be close to x, and $t > 0$ such that

$$e^{-t(\frac{1}{\ell} + \frac{1}{d-\ell})} = d(v, x).$$

Let $\mathbf{v} \in \wedge^k \mathbb{Z}^d$ be a primitive pure tensor representing v and $\tilde{\mathbf{v}} = s_x \mathbf{v} \in \wedge^k \Delta_x$. The vector $a_t \tilde{\mathbf{v}}$ can be decomposed according to the eigenspaces of a_t:

$$\begin{aligned} a_t \tilde{\mathbf{v}} &= a_t \tilde{\mathbf{v}}_0 + a_t \tilde{\mathbf{v}}_1 + \dots \\ &= e^{-t\frac{k}{\ell}} \tilde{\mathbf{v}}_0 + e^{-t[\frac{k}{\ell} - (\frac{1}{\ell} + \frac{1}{d-\ell})]} \tilde{\mathbf{v}}_1 + \dots \end{aligned}$$

Then, from Lemma 1, $\frac{\|\tilde{\mathbf{v}}_1\|}{\|\tilde{\mathbf{v}}_0\|} \asymp d(v, x)$ and therefore, by choice of the parameter $t > 0$,

$$e^{-t[\frac{k}{\ell} - (\frac{1}{\ell} + \frac{1}{d-\ell})]} \|\tilde{\mathbf{v}}_1\| \ll e^{-t[\frac{k}{\ell} - (\frac{1}{\ell} + \frac{1}{d-\ell})]} \|\tilde{\mathbf{v}}_0\| d(v, x) \ll e^{-t\frac{k}{\ell}} \|\tilde{\mathbf{v}}_0\|.$$

Consequently, $\frac{\|a_t \tilde{\mathbf{v}}_1\|}{\|a_t \tilde{\mathbf{v}}_0\|} \ll 1$, and by Lemma 2, for all $i \geq 1$, $\frac{\|a_t \tilde{\mathbf{v}}_i\|}{\|a_t \tilde{\mathbf{v}}_0\|} \ll 1$. Thus,

$$\frac{\|\pi_+(a_t \tilde{\mathbf{v}})\|}{\|a_t \tilde{\mathbf{v}}\|} = \frac{\|a_t \tilde{\mathbf{v}}_0\|}{\|a_t \tilde{\mathbf{v}}\|} \gg 1 \quad \text{and} \quad \|a_t \tilde{\mathbf{v}}\| \ll e^{-t\frac{k}{\ell}} H(v),$$

which proves the first part of the proposition.

For the second assertion, let $t > 0$ and $\mathbf{v} \in \wedge^k \mathbb{Z}^d$ be a pure tensor such that $\|\pi_+(a_t s_x \mathbf{v})\| \geq c \cdot \|a_t s_x \mathbf{v}\|$. As above, set $\tilde{\mathbf{v}} = s_x \mathbf{v}$. Note that $\pi_+(a_t \tilde{\mathbf{v}}) = a_t \tilde{\mathbf{v}}_0 = e^{-t\frac{k}{\ell}} \tilde{\mathbf{v}}_0$ so that, for each $i \geq 0$,

$$\|\tilde{\mathbf{v}}_i\| \leq e^{t[\frac{k}{\ell} - i(\frac{1}{\ell} + \frac{1}{d-\ell})]} \|a_t \tilde{\mathbf{v}}\| \ll e^{-ti(\frac{1}{\ell} + \frac{1}{d-\ell})} \|\tilde{\mathbf{v}}_0\|.$$

If $v \in X_k(\mathbb{Q})$ is the linear subspace associated to $\mathbf{v}$, one indeed finds

$$H(v) = \|\mathbf{v}\| \asymp \|\tilde{\mathbf{v}}\| \ll \|\tilde{\mathbf{v}}_0\| \ll e^{t\frac{k}{\ell}} \|a_t \tilde{\mathbf{v}}\|$$

and with Lemma 1,

$$d(v, x) \asymp \frac{\|\tilde{\mathbf{v}}_1\|}{\|\tilde{\mathbf{v}}_0\|} \ll e^{-t(\frac{1}{\ell} + \frac{1}{d-\ell})}.$$

$\square$

Proof of Theorem 6. Let $\beta < \beta_k(x)$. One may find $v \in X_k(\mathbb{Q})$ arbitrarily close to x such that $d(v, x) \leq H(v)^{-\beta}$. The first part of Proposition 1 shows that if $t > 0$ satisfies $e^{-t(\frac{1}{\ell} + \frac{1}{d-\ell})} = d(v, x)$, then $\|\pi_+(a_t s_x \mathbf{v}\| \gg \|a_t s_x \mathbf{v}\|$ and

$$\|a_t s_x \mathbf{v}\| \ll e^{-t\frac{k}{\ell}} H(v) \leq e^{-t\frac{k}{\ell}} d(v, x)^{-\frac{1}{\beta}} \leq e^{-t[\frac{k}{\ell} - \frac{1}{\beta}(\frac{1}{\ell} + \frac{1}{d-\ell})]}.$$

Replacing t by $t - C$, for some constant C depending only on d, one may ensure that $\|\pi_+(a_t s_x \mathbf{v})\| \geq \frac{1}{2} \|a_t s_x \mathbf{v}\|$, and the above inequality is essentially preserved.

This shows that $\gamma_k(x) \geq \frac{k}{\ell} - \frac{1}{\beta}(\frac{1}{\ell} + \frac{1}{d-\ell})$, and letting β go to $\beta_k(x)$, we obtain $\beta_k(x) \leq \frac{d}{(k-\ell\gamma_k(x))(d-\ell)}$.

To show the converse inequality, fix $\gamma > \gamma_k(x)$, so that for $t > 0$ arbitrarily large, there exists a non-zero vector $\mathbf{v} \in \wedge^k\mathbb{Z}^d$ satisfying $\|\pi_+(a_t s_x \mathbf{v})\| \geq \frac{1}{2}\|a_t s_x \mathbf{v}\|$ et $\|a_t s_x \mathbf{v}\| \leq e^{-\gamma t}$. A priori, it does not follow from the definition of $\gamma_k(x)$ that $\mathbf{v}$ is a pure tensor. However, Siegel's reduction theory shows that there exists an essentially orthogonal basis $(\mathbf{u}_1, \dots, \mathbf{u}_D)$, where $D = \binom{d}{k}$, of the lattice $\wedge^k a_t s_x \mathbb{Z}^d$ consisting only of pure tensors. Writing $\mathbf{v} = \sum_i \lambda_i \mathbf{u}_i$, $\lambda_i \in \mathbb{Z}$, we find

$$\|\mathbf{v}\| \asymp \max_i |\lambda_i| \|\mathbf{u}_i\|. \tag{4}$$

Besides, $\pi_+(\mathbf{v}) = \sum_i \lambda_i \pi_+(\mathbf{u}_i)$ and $\sum_i |\lambda_i| \|\pi_+(\mathbf{u}_i)\| \geq \|\pi_+(\mathbf{v})\| \gg \|\mathbf{v}\|$, so there must exist i such that $|\lambda_i| \|\pi_+(\mathbf{u}_i)\| \gg \|\mathbf{v}\|$. Given (4), this implies that $\|\pi_+(\mathbf{u}_i)\| \gg \|\mathbf{u}_i\|$, and since $|\lambda_i| \geq 1$, $\|\mathbf{u}_i\| \ll \|\mathbf{v}\|$. In other words, replacing $\mathbf{v}$ by $\mathbf{u}_i$ if necessary, we may assume that $\mathbf{v}$ is a pure tensor in $\wedge^k \mathbb{R}^d$. It follows from the second part of Proposition 1 that the point $v \in X_k(\mathbb{Q})$ associated to $\mathbf{v}$ satisfies $H(v) \ll e^{t(\frac{k}{\ell} - \gamma)}$ and $d(v, x) \leq e^{-t(\frac{1}{\ell} + \frac{1}{d-\ell})}$, which yields $d(v,x) \ll H(v)^{-\frac{d}{(k-\ell\gamma)(d-\ell)}}$ whence $\beta_k(x) \geq \frac{d}{(k-\ell\gamma)(d-\ell)}$. □

The ergodicity of the flow $(a_t)_{t>0}$ on the space of lattices implies that the escape rate of the orbit $(a_t \Delta_x)_{t>0}$ is equal to zero for almost every $x \in X_\ell(\mathbb{R})$. This yields a simple proof that the diophantine exponent $\beta_k(x)$ is almost everywhere constant, equal to $\frac{d}{k(d-\ell)}$. Recall that the first minimum of a lattice Δ in $\mathbb{R}^d$ is defined by

$$\lambda_1(\Delta) = \min\{\|v\| \ ; \ v \in \Delta \setminus \{0\}\}.$$

Corollary 1 (Zero escape rate and generic diophantine exponent)**.** *Fix integers $1 \leq k \leq \ell \leq d$. For each $x \in X_\ell(\mathbb{R})$ such that $\lim_{t\to+\infty} \frac{1}{t}\log \lambda_1(a_t s_x \mathbb{Z}^d) = 0$, one has $\beta_k(x) = \frac{d}{k(d-\ell)}$. This holds in particular for almosts every $x \in X_\ell(\mathbb{R})$.*

Proof. For this proof, we use the concept of successive minima of a lattice Δ in $\mathbb{R}^d$: for $i = 1, \dots, d$, the i-th successive minimum is defined by

$$\lambda_i(\Delta) = \min\{\lambda > 0 \mid \Delta \cap B(0,\lambda) \text{ contains } i \text{ linearly independent vectors}\}$$

so that

$$\lambda_1(\Delta) \leq \lambda_2(\Delta) \leq \dots \leq \lambda_d(\Delta).$$

Minkowski's second theorem [30, Theorem 1A*], states that within constants depending only on d, the covolume of Δ in $\mathbb{R}^d$ is comparable to the product $\lambda_1(\Delta) \dots \lambda_d(\Delta)$.

Let $x \in X_\ell(\mathbb{R})$ be such that $\lim_{t\to\infty} \frac{1}{t}\log \lambda_1(a_t s_x \mathbb{Z}^d) = 0$. For every $\varepsilon > 0$, one has for every large enough $t > 0$, $\lambda_1(a_t s_x \mathbb{Z}^d) \geq e^{-\varepsilon t}$. By Minkowski's second theorem, this implies

$$e^{-\varepsilon t} \leq \lambda_1(a_t s_x \mathbb{Z}^d) \leq \dots \leq \lambda_d(a_t s_x \mathbb{Z}^d) \leq e^{d\varepsilon t}.$$

Let $u_1, \dots, u_d$ be vectors in $a_t s_x \mathbb{Z}^d$ achieving those successive minima. By a lemma of Mahler [22, Theorem 3], the vectors

$$\mathbf{u}_\tau = u_{\tau_1} \wedge \dots \wedge u_{\tau_k}, \qquad \tau = \{\tau_1 < \dots < \tau_k\} \subset [\![1, d]\!],$$

achieve the successive minima of $\wedge^k a_t s_x \mathbb{Z}^d$, within a multiplicative constant depending only on d. In particular, letting $\varepsilon > 0$ go to 0, we find

$$\lim_{t\to\infty} \frac{1}{t} \log \lambda_1(\wedge^k a_t s_x \mathbb{Z}^d) = 0$$

whence $\gamma_k(x) \leq 0$ i.e. $\beta_k(x) \leq \frac{d}{k(d-\ell)}$.

For the converse inequality, note that there exists $\tau > 0$ such that the vector $\mathbf{u} = \mathbf{u}_\tau$ satisfies $\|\pi_+(\mathbf{u})\| \gg \|\mathbf{u}\|$. Indeed, the vectors $\mathbf{u}_\tau$ generate a sublattice of bounded index in $\wedge^k a_t s_x \mathbb{Z}^d$ and are almost orthogonal, i.e. $\|\wedge_\tau \mathbf{u}_\tau\| \gg \prod_\tau \|\mathbf{u}_\tau\|$. As one also has $\|\mathbf{u}_\tau\| \ll e^{kd\varepsilon}$, the second point in Proposition 1 implies that the subspace $v \in X_k(\mathbb{Q})$ associated to $\mathbf{v} = s_x^{-1} a_{-t} \mathbf{u}$ satisfies $H(v) \ll e^{t(\frac{k}{\ell}+kd\varepsilon)}$, whence

$$d(v,x) \ll e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})} \ll H(v)^{-\frac{1}{\frac{k}{\ell}+kd\varepsilon}(\frac{1}{\ell}+\frac{1}{d-\ell}))} = H(v)^{-\frac{d}{k(d-\ell)}+O(\varepsilon)}.$$

Letting ε tend to zero, one indeed finds $\beta_k(x) = \frac{d}{k(d-\ell)}$.

For the last assertion, it is enough to note that the ergodicity of the flow (a_t) on the space of unimodular lattices implies that for Lebesgue almost every x in $X_\ell(\mathbb{R})$, one has $\lim_{t\to\infty} \frac{1}{t} \log \lambda_1(a_t s_x \mathbb{Z}^d) = 0$. To see this, recall that if f is an integrable function on an ergodic dynamical system (Ω, T), then $\lim \frac{1}{n} f(T^n x) = 0$ for almost every x. This applies to the function on Ω defined by $f(x) = \log \frac{1}{\lambda_1(x)}$, which is readily seen to be integrable: $m(\{\log \frac{1}{\lambda_1} \geq t\}) = m(\{\lambda_1 \leq e^{-t}\}) = c \cdot e^{-2t}$ (Siegel's formula) so $\mathbb{E}[\log \frac{1}{\lambda_1}] = \int_0^{+\infty} m(\{\log \frac{1}{\lambda_1} \geq t\})\mathrm{d}t = c\int_0^{+\infty} e^{-2t}\mathrm{d}t < +\infty$. $\square$

Definition. A point x in $X_\ell(\mathbb{R})$ is *very well approximable by rational k-planes* if $\beta_k(x) > \frac{d}{k(d-\ell)}$. We denote by $\mathrm{VWA}_k(X_\ell)$ the set of points in $X_\ell(\mathbb{R})$ that are very well approximable by rational k-planes.

The above corollary shows that the set $\mathrm{VWA}_k(X_\ell)$ has zero Lebesgue measure. Proposition 1 can also be used to derive some inequalities between the different exponents $\beta_k(x)$, $k = 1, \dots, \ell$.

compexp **Proposition 2** (Inequalities among the β_k, $k \geq 1$)**.** *For every $x \in X_\ell(\mathbb{R})$ and every $k \in [\![1,\ell]\!]$, $\beta_1(x) \geq k\beta_k(x)$. In particular,* $\mathrm{VWA}_k(X_\ell) \subset \mathrm{VWA}_1(X_\ell)$.

Proof. Note that by definition, $\gamma_1(x) \leq \limsup_{t\to\infty} \frac{-1}{t} \log \lambda_1(a_t s_x \mathbb{Z}^d)$. Let us show that equality holds. This follows from the fact that a_t has only one contracting eigenvalue on $\mathbb{R}^d$. Indeed, for $\mathbf{v} \in \mathbb{Z}^d$, one may write $\tilde{\mathbf{v}} = s_x \mathbf{v} = \tilde{\mathbf{v}}_0 + \tilde{\mathbf{v}}_1$ whence $a_t \tilde{\mathbf{v}} = e^{-\frac{t}{\ell}} \tilde{\mathbf{v}}_0 + e^{\frac{t}{d-\ell}} \tilde{\mathbf{v}}_1$ and

$$\|a_t \tilde{\mathbf{v}}\| \asymp \max(e^{-\frac{t}{\ell}} \|\tilde{\mathbf{v}}_0\|, e^{\frac{t}{d-\ell}} \|\tilde{\mathbf{v}}_1\|).$$

Consequently, if for some $t > 0$ and $\gamma \in]0, \frac{1}{\ell}[$, one has $\|a_t s_x \mathbf{v}\| \leq e^{-\gamma t}$, the function $f \colon \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(t) = \max(e^{-t(\frac{1}{\ell}-\gamma)} \|\tilde{\mathbf{v}}_0\|, e^{t(\frac{1}{d-\ell}+\gamma)} \|\tilde{\mathbf{v}}_1\|)$$

satisfies $f(t) \ll 1$ for some $t > 0$. But this function achieves its minimum at $t = t_m$ satisfying $e^{t_m(\frac{1}{\ell}+\frac{1}{d-\ell})} = \frac{\|\tilde{\mathbf{v}}_0\|}{\|\tilde{\mathbf{v}}_1\|}$. So we find $\|a_{t_m} \tilde{\mathbf{v}}\| \ll e^{-\gamma t_m}$ whereas by definition of t_m, $\|\pi_+(a_{t_m} \tilde{\mathbf{v}})\| \gg \|a_{t_m} \tilde{\mathbf{v}}\|$. This proves the desired inequality.

That observation, combined with Minkowski's first theorem applied in a k-dimensional subspace almost achieving the limit value $\gamma_k(x)$, shows that $\gamma_1(x) \geq \frac{\gamma_k(x)}{k}$, and therefore

$$\beta_1(x) = \frac{d}{(1-\gamma_1)(d-\ell)} \geq \frac{d}{(1-\frac{\gamma_k}{k})(d-\ell)} = k\beta_k(x).$$

$\square$

Remark. The argument we used to derive the inequality

$$\gamma_1(x) = \limsup \frac{-1}{t} \log \lambda_1(a_t s_x \mathbb{Z}^d)$$

can be generalized in the following way: if the flow (a_t) has only one contracting eigenvalue en $\wedge^j \mathbb{R}^d$, then

$$\gamma_j(x) = \limsup \frac{-1}{t} \log \lambda_j(a_t s_x \mathbb{Z}^d).$$

Assume for clarity that $j \leq \ell$. Since the second smallest eigenvalue of a_t on $\wedge^j \mathbb{R}^d$ is equal to $\frac{j}{\ell} - \frac{d}{\ell(d-\ell)}$, this condition can be rewritten $\frac{j}{\ell} \leq \frac{d}{\ell(d-\ell)}$, i.e.

$$j \leq \frac{d}{d-\ell}.$$

(For example, if $j = \ell$ this is only possible if $\frac{1}{d-\ell} + \frac{1}{\ell} \geq 1$, i.e. $\ell = 1$, or $\ell = d-1$ or $(d,\ell) = (4,2)$.) Under that condition, one always has $\gamma_j(x) = \limsup \frac{-1}{t} \sum_{i=1}^{j} \log \lambda_i(a_t s_x \mathbb{Z}^d) \geq 0$, and therefore

$$\beta_j(x) \geq \frac{d}{j(d-\ell)}.$$

This optimal lower bound in the case $j \leq \frac{d}{d-\ell}$ was already given by Schmidt [29, Theorem 15, page 462], with a different proof. Under that condition, one can also adapt the proof of Proposition 2 and show that for every $k \in [\![j,\ell]\!]$, $\beta_j(x) \geq \frac{j}{k}\beta_k(x)$.

Remark. One may formulate Theorem 6 and Proposition 1 so as to cover also the case $k > \ell$. For that, let $e^{-t\omega_{k,\ell}}$ denote the smallest eigenvalue of a_t in the exterior power $\wedge^k \mathbb{R}^d$, i.e.

$$\omega_{k,\ell} = \begin{cases} \frac{k}{\ell} & \text{if } k \leq \ell \\ \frac{d-k}{d-\ell} & \text{if } k \geq \ell \end{cases}$$

One then obtains $\beta_k(x) = \frac{1}{\omega_{k,\ell}-\gamma_k(x)}(\frac{1}{\ell} + \frac{1}{d-\ell})$, and the inequality between $\beta_1(x)$ and $\beta_k(x)$ becomes

$$\beta_1(x) \geq \begin{cases} k\beta_k(x) & \text{if } k \leq \ell \\ \frac{k\beta_k(x)}{(k-\ell)\beta_k+1} & \text{if } k > \ell. \end{cases}$$

The inclusion $\mathrm{VWA}_k(X_\ell) \subset \mathrm{VWA}_1(X_\ell)$ is still valid.

Definition (Badly approximable points)**.** One may also define the set of points in $X_\ell(\mathbb{R})$ that are *badly approximable by rational k-planes*:

$$\mathrm{BA}_k(X_\ell) = \{x \in X_\ell(\mathbb{R}) \mid \exists c > 0 : \forall v \in X_k(\mathbb{Q}),\ d(v,x) \geq cH(v)^{-\frac{d}{k(d-\ell)}}\}.$$

It follows from Proposition 1 that any point x in $X_\ell(\mathbb{R})$ such that the orbit $a_t s_x \mathbb{Z}^d$ is bounded in the space of lattices, belongs to $\mathrm{BA}_k(X_\ell)$, for each $k = 1,\dots,d$. By a result of Schmidt [28], the set of such points x has full Hausdorff dimension in $X_\ell(\mathbb{R})$. So we immediately deduce the same property for $\mathrm{BA}_k(X_\ell)$.

Proposition 3 (Winning property for $\mathrm{BA}_k(X_\ell)$)**.** *The set $\mathrm{BA}_k(X_\ell)$ is winning in the sense of Schmidt. In particular, it has full Hausdorff dimension:*

$$\dim_H \mathrm{BA}_k(X_\ell) = \ell(d-\ell).$$

2 Rational approximations to algebraic points

In this section, as first application of Theorem 6, we derive a formula for the diophantine exponents of an arbitrary point $x \in X_\ell(\overline{\mathbb{Q}})$. The proof is based on the interpretation of Schmidt's subspace theorem in terms of diagonal orbits in the space of lattices, as presented in [5]. We shall in particular use the following result [5, Theorem 3].

Theorem 7. *Let $(a_t)_{t>0}$ be a one-parameter diagonal subgroup in $\mathrm{GL}_d(\mathbb{R})$ and L an element of $\mathrm{GL}_d(\overline{\mathbb{Q}})$. For each $i \in [\![1,d]\!]$, the limit $\Lambda_i = \lim_{t\to\infty} \frac{1}{t}\log\lambda_i(a_t L\mathbb{Z}^d)$ exists. Moreover, if the indices $i_1 < \dots < i_r$ are chosen so that*

$$\Lambda_1 = \dots = \Lambda_{i_1} < \Lambda_{i_1+1} = \dots = \Lambda_{i_2} < \dots < \Lambda_{i_r+1} = \dots = \Lambda_d,$$

then there exists a partial flag

$$0 = T_0 < T_1 < \dots < T_{r+1} = \mathbb{Z}^d$$

of rational subspaces such that for each $s = 1,\dots,r$,

- $\dim T_s = i_s$*;*
- *for every large enough $t > 0$, $a_t L T_s$ contains the first i_s successive minima of $a_t L\mathbb{Z}^d$.*

When one considers the subgroup

$$a_t = \mathrm{diag}(e^{-\frac{t}{\ell}},\dots,e^{-\frac{t}{\ell}},e^{\frac{t}{d-\ell}},\dots,e^{\frac{t}{d-\ell}})$$

and a representative $L = s_x$ of a point $x = Ps_x$ in the Grassmann variety $X_\ell(\overline{\mathbb{Q}})$, the asymptotic behavior of the orbit $(a_t s_x\mathbb{Z}^d)_{t>0}$ in the space of lattices can be interpreted geometrically. Indeed, T_1 is the unique rational subspace of maximal dimension maximizing the ratio $\frac{\dim x\cap T_1}{\dim T_1}$, and by induction, T_s is the unique subspace containing T_{s-1}, of maximal dimension, and maximizing $\frac{\dim x\cap T_s - \dim x\cap T_{s-1}}{\dim T_s - \dim T_{s-1}}$. Moreover,

$$\Lambda_i = \frac{1}{d-\ell} - \left(\frac{1}{\ell} + \frac{1}{d-\ell}\right)\frac{\dim x\cap T_s - \dim x\cap T_{s-1}}{\dim T_s - \dim T_{s-1}} \quad \text{if } i_s < i \leq i_{s+1}.$$

With the correspondence obtained in the preceding section, those observations will allow us to give a general formula for the diophantine exponents $\beta_k(x)$ of a subspace x of $\mathbb{R}^d$ defined over $\overline{\mathbb{Q}}$. By Theorem 6, this is equivalent to a formula for $\gamma_k(x)$, and this is how we now state the result.

Theorem 8 (Diophantine exponents of an algebraic subspace)**.** *Let $x \in X_\ell(\overline{\mathbb{Q}})$ and $\{0\} = T_0 < T_1 < \cdots < T_r < T_{r+1} = \mathbb{Z}^d$ the partial flag defined above. For $s = 1, \dots, r$, set*

$$i_s = \dim T_s \quad \textit{and} \quad j_s = \dim x \cap T_s.$$

Then, for $k = 1, \dots, \ell$, denoting $k_s = \min(j_s, k)$,

$$\gamma_k(x) = -\sum_{s=0}^{r}(k_{s+1}-k_s)\Lambda_{i_s+1} = \frac{-k}{d-\ell} + \left(\frac{1}{\ell} + \frac{1}{d-\ell}\right)\sum_{s=0}^{r} \frac{(k_{s+1}-k_s)(j_{s+1}-j_s)}{i_{s+1}-i_s}.$$

Proof. Consider the sublattice in $\wedge^k\mathbb{Z}^d$ defined by

$$S = \wedge^{k_1}T_1 \wedge (\wedge^{k_2-k_1}T_2) \wedge \cdots \wedge (\wedge^{k_{r+1}-k_r}T_{r+1}).$$

Theorem 7 implies that for every $\varepsilon > 0$, for all large enough $t > 0$, the sublattice $a_t s_x S$ has a basis consisting of elements of norm at most

$$\exp[t(\varepsilon + \sum_{s=0}^{r}(k_{s+1}-k_s)\Lambda_{i_s+1})].$$

Moreover, by definition of the integers k_s, this subspace contains a non-zero pure tensor $\mathbf{v}_x \in \wedge^k x$. One may always choose s_x so that $s_x\mathbf{v}_x = e_1 \wedge \cdots \wedge e_k$, and then, there exists in $a_t s_x S$ a vector $\mathbf{v}$ such that $\|\pi_+(\mathbf{v})\| \geq \frac{1}{2}\|\mathbf{v}\|$ and $\|\mathbf{v}\| \leq e^{t(\sum_{s=0}^{r}(k_{s+1}-k_s)\Lambda_{i_s+1}+O(\varepsilon))}$. Since $\varepsilon > 0$ can be arbitrarily small, this already shows that

$$\gamma_k(x) \geq -\sum_{s=0}^{r}(k_{s+1}-k_s)\Lambda_{i_s+1}.$$

Conversely, Theorem 7 also shows that for $\varepsilon > 0$, for all large enough $t > 0$, any vector $\mathbf{v}$ in $\wedge^k\mathbb{Z}^d$ satisfying

$$\|a_t s_x \mathbf{v}\| \leq e^{t(-\varepsilon+\sum_{s=0}^{r}(k_{s+1}-k_s)\Lambda_{i_s+1})}$$

must belong to a subspace

$$S' = \wedge^{k_1'}T_1 \wedge (\wedge^{k_2'-k_1'}T_2) \wedge \cdots \wedge (\wedge^{k_{r+1}'-k_r'}T_{r+1})$$

where $0 \leq k_1' \leq k_2' \leq \cdots \leq k_{r+1}' = k$ are integers with for some u, $k_u < k_u'$. By definition of the integers k_s, $s = 1, \dots, r$, the subspace S' contains no pure tensor in $\wedge^k x$, so by a compactness argument, there exists $c > 0$ such that for every pure tensor $\mathbf{v} \in S'$, $\|s_x\mathbf{v} - \pi_+(s_x\mathbf{v})\| \geq c\|s_x\mathbf{v}\|$. This implies

$$\begin{aligned}\|a_t s_x \mathbf{v}\| &\geq c e^{-t(\frac{k}{\ell}-\frac{1}{\ell}-\frac{1}{d-\ell})}\|s_x\mathbf{v}\| \\ &\gg e^{t(\frac{1}{\ell}+\frac{1}{d-\ell})}\|\pi_+(a_t s_x \mathbf{v})\|\end{aligned}$$

and shows that $a_t s_x \mathbf{v}$ cannot lie inside C^+. Thus, $\gamma_k(x) \leq \varepsilon - \sum_{s=0}^{r}(k_{s+1} - k_s)\Lambda_{i_s+1}$. Letting ε go to zero, we obtain the desired result. $\square$

This formula implies that the diophantine exponent $\beta_k(x)$ of a subspace defined over $\overline{\mathbb{Q}}$ is always greater than or equal to the generic exponent $\frac{d}{k(d-\ell)}$, and it also gives a necessary and sufficient condition for equality to hold.

Corollary 2. *For every $x \in X_\ell(\overline{\mathbb{Q}})$ and every $k \leq \ell$, one has $\beta_k(x) \geq \frac{d}{k(d-\ell)}$, with equality if and only if x is not included in any rational constraining pencil.*

Proof. From Proposition 1, we know that $\beta_k(x)$ is given by $\beta_k(x) = \frac{d}{(k-\ell\gamma_k(x))(d-\ell)}$ so it suffices to show that $\gamma_k(x) \geq 0$, with equality if and only if x is not included in any rational constraining pencil.

Let us start with the case $k = \ell$, where one has $k_s = j_s$ for each s, and so

$$\gamma_\ell(x) = \frac{-\ell}{d-\ell} + \frac{d}{\ell(d-\ell)} \sum_{s=0}^{r} \frac{(j_{s+1}-j_s)^2}{i_{s+1}-i_s}.$$

One can then write

$$\begin{aligned} d\sum_{s=0}^{r} \frac{(j_{s+1}-j_s)^2}{i_{s+1}-i_s} &= (\sum_{s=0}^{r} \frac{(i_{s+1}-i_s)^2}{i_{s+1}-i_s})(\sum_{s=0}^{r} \frac{(j_{s+1}-j_s)^2}{i_{s+1}-i_s}) \\ &\geq (\sum_{s=0}^{r} j_{s+1}-j_s)^2 = \ell^2 \end{aligned}$$

which shows that $\gamma_\ell(x) \geq 0$. The inequality $\gamma_k(x) \geq 0$ for $k \leq \ell$ follows from the particular case $k = \ell$. Indeed, using the fact that the map $s \mapsto \frac{j_{s+1}-j_s}{i_{s+1}-i_s}$ is non-increasing, one has

$$\sum_{s=0}^{r} \frac{(k_{s+1}-k_s)(j_{s+1}-j_s)}{i_{s+1}-i_s} \geq \frac{k}{\ell} \sum_{s=0}^{r} \frac{(j_{s+1}-j_s)^2}{i_{s+1}-i_s}.$$

To see this, note that the non-negative function on $[0,d]$ defined by $f\colon x \mapsto \frac{j_{s+1}-j_s}{i_{s+1}-i_s}$ if $j_s < x \leq j_{s+1}$ is non-increasing, and that the above inequality can be written $\int_0^k f(x)\mathrm{d}x \geq \frac{k}{\ell}\int_0^\ell f(x)\mathrm{d}x$.

If $\gamma_k(x) = 0$, the above computation shows that $\gamma_1(x) = 0$, and this implies $\frac{j_1}{i_1} = \frac{\ell}{d}$, which by definition of T_1 is only possible if $T_1 = \mathbb{Q}^d$, i.e. if x does not belong to any rational constraining pencil. Conversely, if x does not belong to any rational constraining pencil, then one must have $T_1 = \mathbb{Q}^d$ and therefore $\gamma_k(x) = 0$ for $k = 1, \ldots, \ell$. □

3 Dirichlet's principle

We saw in the previous section that every point x in $X_\ell(\overline{\mathbb{Q}})$ satisfies $\beta_k(x) \geq \frac{d}{k(d-\ell)}$ for each $k \in [\![1,\ell]\!]$. In other words, the almost sure diophantine exponent is a uniform lower bound for the diophantine exponent of any point in $X_\ell(\overline{\mathbb{Q}})$. From the remark following the proof of Proposition 2, we also know that this lower bound is still valid for any x in $X_\ell(\mathbb{R})$ provided $k \leq \frac{d}{d-\ell}$. The goal of the present section is to remove this restrictive condition by showing that this optimal lower bound holds for all values $1 \leq k \leq d < d$. For that, we shall use the correspondence from Section 1 and a general description of diagonal orbits in the space of lattices, in the spirit of recents papers of Schmidt and Summerer [31], Roy [25], and Das, Fishman, Simmons and Urbański [7].

We now derive Theorem 1 announced in the introduction. It will follow from the slightly more precise statement below.

Theorem 9 (Dirichlet's principle in $X_\ell(\mathbb{R})$)**.** *Given integers $1 \leq k \leq \ell < d$, there exists a constant $C = C_{d,\ell,k}$ such that for every x in $X_\ell(\mathbb{R})$, there exists $v \in X_\ell(\mathbb{Q})$ arbitrarily close to x such that*

$$d(v,x) \leq CH(v)^{-\frac{d}{k(d-\ell)}}.$$

First, to each lattice Δ in the Euclidean space $\mathbb{R}^d$ we associate a convex function c_Δ on the set of integers $[\![0,d]\!]$. This function encodes the data of all successive minima of Δ; it is a variant of the Grayson polygon described in [12], for which one replaces the Euclidean norm by the norm given by the maximal absolute value of a coordinate in the canonical basis. The behavior of that norm with respect to differentiation is particularly simple, and this will be convenient for the rest of our argument.

Some Grayson polygon

Recall that $(e_i)_{1\leq i\leq d}$ denotes the standard basis for $\mathbb{R}^d$, and that for $I = \{i_1 < \cdots < i_k\} \subset [\![1,d]\!]$, we write $\mathbf{e}_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. For each $k \in [\![1,d]\!]$, we endow $\wedge^k\mathbb{R}^d$ with the norm

$$\|x\| = \max_{|I|=k}|x_I|, \quad \text{where} \quad x = \sum_I x_I \mathbf{e}_I.$$

Then, if $W = \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_k$ is a discrete subgroup of rank k in $\mathbb{R}^d$, we write $\|W\| = \|\mathbf{w}\|$, where $\mathbf{w} = w_1 \wedge \cdots \wedge w_k$ represents W in $\wedge^k\mathbb{R}^d$. This definition does not depend of the choice of the basis $(w_i)_{1\leq i\leq k}$.

Lemma 3. *There exists a constant $A > 0$ depending only on d such that the function ϕ defined on discrete subgroups of $\mathbb{R}^d$ by*

$$\phi(W) = \log\|W\| + A \cdot (\dim W)(d - \dim W)$$

is submodular, i.e. satisfies

$$\forall V, W, \quad \phi(V) + \phi(W) \geq \phi(V \cap W) + \phi(V + W).$$

Proof. For the Euclidean norm $\|\cdot\|_2$ on $\mathbb{R}^d$ and its exterior powers, one has, for every V, W, $\|V\|_2\|W\|_2 \geq \|V \cap W\|_2\|V + W\|_2$. Since the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on $\wedge^*\mathbb{R}^d$, there exists a constant $A > 0$ such that for every V, $|\log\|V\| - \log\|V\|_2| \leq \frac{A}{2}$. Therefore, for every V, W,

$$\log\|V\| + \log\|W\| \geq \log\|V \cap W\| + \log\|V + W\| - 2A.$$

Set $k = \dim V$, $\ell = \dim W$, $m = \dim V \cap W$ and $n = \dim V + W$, so that $k + \ell = m + m$. If V, W are not comparable for inclusion, then $m < \min(k, \ell)$, $n > \max(k, \ell)$, and an elementary computation shows that

$$k(d-k) + \ell(d-\ell) \geq 2 + m(d-m) + n(d-n).$$

The submodularity property of ϕ follows from these observations. $\square$

Definition (Grayson polygon)**.** Given a lattice Δ, the *Grayson polygon*

$$c_\Delta \colon [0,d] \to \mathbb{R}$$

associated to the function ϕ on the set of discrete subgroups of Δ is the largest convex function whose graph lies below all points $(\dim W, \phi(W))$, $W \leq \Delta$.

Recall from [5, §1.3] that if $J_\Delta = \{j_1 < \cdots < j_r\}$ denotes the set of angular points of c_Δ, there exists a unique partial flag

$$\{0\} < V_{j_1} < \cdots < V_{j_r} < \Delta$$

of discrete subgroups in Δ such that for $s = 1, \ldots, r$, $\dim V_{j_s} = j_s$ and $\phi(V_{j_s}) = c_\Delta(j_s)$. This partial flag is called the *Harder-Narasimhan filtration* of Δ. Any primitive subgroup $W \leq \Delta$ such that $\phi(W) = c_\Delta(\dim W)$ is compatible with this filtration. We shall need to relate the Grayson polygon and the Harder-Narasimhan filtration to the successive minima of Δ; this is the content of the following proposition, which can be seen as a reformulation of Minkowski's second theorem.

Proposition 4 (Harder-Narasimhan filtration and successive minima)**.** *Let Δ be a lattice in $\mathbb{R}^d$ with Harder-Narasimhan filtration $\{0\} < V_{j_1} < \cdots < V_{j_r} < \Delta$ and Grayson polygon c_Δ. Then,*

1. *There exists a family of linearly independent vectors $(v_1, \ldots, v_d)$ in Δ such that for each i, $\log\|v_i\| = c_\Delta(i) - c_\Delta(i-1) + O(1)$ and $v_i \in V_{j_r}$ if $i \leq j_r$.*
2. $\forall i = 1, \ldots, d, \quad \log \lambda_i(\Delta) = c_\Delta(i) - c_\Delta(i-1) + O(1)$*;*

Proof. We construct by induction on $s = 1, \ldots, r$ a basis $v_1, \ldots, v_{j_s}$ for V_{j_s} such that $\log\|v_i\| = c_\Delta(i) - c_\Delta(i-1) + O(1)$ and $v_i \in V_{j_s}$ if $i \leq j_s$. For $s = 0$, the empty family has the required property, so we assume that for some $s \geq 0$ the vectors $v_1, \ldots, v_{j_s}$ have been constructed. By definition of the Grayson polygon, the covolume of the quotient lattice $W_{s+1} = V_{j_{s+1}}/V_{j_s}$ is comparable to $e^{c_\Delta(j_{s+1}) - c_\Delta(j_s)}$ and contains no vector of norm less than $e^{\frac{c_\Delta(j_{s+1}) - c_\Delta(j_s)}{j_{s+1} - j_s}}$. (The quotient lattice is identified with the orthogonal projection of $V_{j_{s+1}}$ on the orthogonal complement of V_{j_s}.) If $w_{j_s+1}, \ldots, w_{j_{s+1}}$ is a family of vectors realizing the successive minima of W_{s+1}, Minkowski's second theorem shows that for $j = j_s + 1, \ldots, j_{s+1}$,

$$\log\|w_j\| = \frac{c_\Delta(j_{s+1}) - c_\Delta(j_s)}{j_{s+1} - j_s} + O(1) = c_\Delta(j) - c_\Delta(j-1) + O(1).$$

In order to lift those vectors to elements in $V_{j_{s+1}}$, we use the induction hypothesis: since the function c_Δ is convex, for every $i \leq j_s$, we have the upper bound, $\|v_i\| \ll e^{c_\Delta(j_s) - c_\Delta(j_s - 1)}$, and therefore, every element of the vector space spanned by V_{j_s} is at distance at most $O(e^{c_\Delta(j_s) - c_\Delta(j_s-1)})$ from an element of V_{j_s}. So we may lift w_j, $j = j_s + 1, \ldots, j_{s+1}$ to a vector $v_j \in V_{j_{s+1}}$ such that

$$\|v_j\| = \|w_j\| + O(e^{c_\Delta(j_s) - c_\Delta(j_s-1)}) \ll \|w_j\|,$$

which implies

$$\log\|v_j\| = \log\|w_j\| + O(1) = c_\Delta(j) - c_\Delta(j-1) + O(1)$$

and finishes the proof of the first part of the proposition.

For the second item, the above already shows that for each i, $\log \lambda_i(\Delta) \leq c_\Delta(i) - c_\Delta(i-1) + O(1)$. But Δ is unimodular, so one also has $\sum \log \lambda_i(\Delta) \geq 0$, and since $\sum_i c_\Delta(i) - c_\Delta(i-1) = 0$, this implies that for each i, $\log \lambda_i(\Delta) = c_\Delta(i) - c_\Delta(i-1) + O(1)$. $\square$

Trajectory and derivative along a diagonal orbit

In order to bound from below the diophantine exponents of a point $x \in X_\ell(\mathbb{R})$, we shall study the Grayson polygon along the orbit $(a_t s_x \mathbb{Z}^d)_{t>0}$, where

$$a_t = \mathrm{diag}(e^{-\frac{t}{\ell}}, \dots, e^{-\frac{t}{\ell}}, e^{\frac{t}{d-\ell}}, \dots, e^{\frac{t}{d-\ell}})$$

and $s_x \in G$ is such that $x = P s_x$. In the sequel, the point x is fixed, as well as its representative s_x in G. To make notation less cumbersome, we shall write

$$c_t = c_{a_t s_x \mathbb{Z}^d} \quad \text{and} \quad \dot{c}_t = \frac{dc_t}{dt}.$$

The space $\mathfrak{a}$ of real-valued functions on $[\![0, d]\!]$ that vanish at 0 and d is endowed with the Euclidean norm defined by

$$\|f\|^2 = \sum_{i=0}^{d-1} (f(i+1) - f(i))^2$$

and with the associated scalar product. The heart of the proof of Theorem 9 (dirichlet) is the following proposition.

bonnebase **Proposition 5.** *For every $t > 0$, there exist vectors $v_1, \dots, v_\ell$ in $a_t s_x \mathbb{Z}^d$ such that*

$$\|\pi_+(v_1 \wedge \dots \wedge v_\ell)\| \gg \|v_1 \wedge \dots \wedge v_\ell\|$$

and

$$\log\|v_1 \wedge \dots \wedge v_\ell\| \leq \frac{-\ell(d-\ell)}{d} \langle \dot{c}_t, c_t \rangle + O(1).$$

We let $J_t = \{j_1(t) < \dots < j_r(t)\}$ denote the set of angular points of c_t, and $\{0\} < V_{j_1} < \dots < V_{j_r} < \mathbb{Z}^d$ the Harder-Narasimhan filtration at time t. The slopes of the Grayson polygon c_t are denoted by

$$\Lambda_1 = \dots = \Lambda_{j_1} < \Lambda_{j_1+1} = \dots = \Lambda_{j_2} < \dots < \Lambda_{j_r+1} = \dots = \Lambda_d.$$

For $t > 0$, we let $\ell_1, \dots, \ell_r$ be the integers defined for $s = 1, \dots, r$ by

$$\frac{d}{dt} \log\|a_t s_x V_{j_s}\| = \frac{-\ell_s}{\ell} + \frac{j_s - \ell_s}{d - \ell}.$$

Note that the subspaces V_{j_s}, the real numbers Λ_i and the integers ℓ_s depend on t, although our notation does not make this explicit.

deriv **Lemma 4.** *With the above notation,*

$$\langle \dot{c}_t, c_t \rangle = -(\frac{1}{\ell} + \frac{1}{d-\ell}) \sum_{s=0}^{r} (\ell_{s+1} - \ell_s) \Lambda_{j_s+1}.$$

Proof. For each i, $c_t(i+1) - c_t(i) = \Lambda_i$, and by definition of ℓ_s, for $s = 1, \dots, r$,

$$\dot{c}_t(j_{s+1}) - \dot{c}_t(j_s) = \frac{j_{s+1} - j_s}{d - \ell} - (\ell_{s+1} - \ell_s)(\frac{1}{\ell} + \frac{1}{d - \ell}).$$

Therefore,

$$\begin{aligned} \langle \dot{c}_t, c_t \rangle &= \sum_{i=0}^{d-1} (\dot{c}_t(i+1) - \dot{c}_t(i))\Lambda_i \\ &= \sum_{s=0}^{r} (\dot{c}_t(j_{s+1}) - \dot{c}_t(j_s))\Lambda_{j_s+1} \\ &= \frac{1}{d-\ell} \sum_{s=0}^{r} (j_{s+1} - j_s)\Lambda_{j_s+1} - (\frac{1}{\ell} + \frac{1}{d-\ell}) \sum_{s=0}^{r} (\ell_{s+1} - \ell_s)\Lambda_{j_s+1}. \end{aligned}$$

This yields the desired result, because $\sum_{s=0}^{r} (j_{s+1} - j_s)\Lambda_{j_s+1} = \sum_{i=1}^{d} \Lambda_i = 0$. $\square$

One should expect the sequence $\ell_1, \dots, \ell_r$ to be non-decreasing; this however might not always be the case. For that reason, we define a sequence $\ell'_1 \leq \dots \leq \ell'_r$ by

$$\ell'_s = \max_{i \leq s} \ell_i.$$

To construct the vectors $v_1, \dots, v_\ell$, we proceed by induction, starting with $v_1, \dots, v_{\ell'_1}$ in V_{j_1}. At each step, the vectors $v_1, \dots, v_{\ell'_s}$ will belong to the subspace V_{j_s} from the Harder-Narasimhan filtration. For the lower bound on the projection $\pi_+(v_1 \wedge \dots \wedge v_{\ell'_s})$, we shall use an elementary lemma. Let $L = \mathrm{Span}(e_1, \dots, e_\ell)$. For $j \in [\![1, d]\!]$, consider the direct sum decomposition

$$\wedge^j \mathbb{R}^d = \wedge^j L \oplus (\wedge^{j-1} L \otimes L^\perp) \oplus \dots \oplus \wedge^j L^\perp.$$

For $i = 0, \dots, j$, we let $\pi_i \colon \wedge^j \mathbb{R}^d \to \wedge^{j-i} L \otimes \wedge^i L^\perp$ denote the projection whose kernel is equal to the sum of all other subspaces occurring in the above decomposition.

proj **Lemma 5.** *Let $i \leq j$ be integers in $[\![1, d]\!]$ and let $V \subset \mathbb{R}^d$ be a j-dimensional subspace represented by a vector $\mathbf{v}$ in $\wedge^j \mathbb{R}^d$ satisfying*

$$\|\pi_{j-i}(\mathbf{v})\| \gg \|\mathbf{v}\|.$$

There exists an orthonormal family of vectors $u_1, \dots, u_i$ inside V such that

$$\|\pi_+(u_1 \wedge \dots \wedge u_i)\| \gg 1.$$

More generally, for each $s \leq i$, if $v_1, \dots, v_s$ is a family of elements in V such that $\|\pi_+(v_1 \wedge \dots \wedge v_s)\| \gg \|v_1 \wedge \dots \wedge v_s\|$, then one may find an orthonormal family of vectors $v_{s+1}, \dots, v_i$ in V such that $\|\pi_+(v_1 \wedge \dots \wedge v_i)\| \gg \|v_1 \wedge \dots \wedge v_i\|$.

Proof. Starting from an orthonormal basis $u_1, \dots, u_j$ for V, write for each s, $u_s = u_s^0 + u_s^1$, où $u_s^0 \in L$ et $u_s^1 \in L^\perp$. Then,

$$\pi_{j-i}(\mathbf{v}) = \sum_{\substack{(\varepsilon_s) \in \{0,1\}^j \\ \varepsilon_1 + \dots + \varepsilon_j = j-i}} u_1^{\varepsilon_1} \wedge \dots \wedge u_j^{\varepsilon_j},$$

so that the inequality $\|\pi_{j-i}(\mathbf{v})\| \gg \|\mathbf{v}\| = 1$ implies that there exists $(\varepsilon_s)_s$ such that $\|u_1^{\varepsilon_1} \wedge \cdots \wedge u_j^{\varepsilon_j}\| \gg 1$. Reordoring the vectors u_s if necessary, we may assume that $\varepsilon_s = 0$ if $s \leq i$ and $\varepsilon_s = 1$ otherwise. Then,

$$\begin{aligned} 1 &\ll \|u_1^0 \wedge \cdots \wedge u_i^0 \wedge u_{i+1}^1 \wedge \cdots \wedge u_j^1\| \\ &\leq \|u_1^0 \wedge \cdots \wedge u_i^0\| \|u_{i+1}^1 \wedge \cdots \wedge u_j^1\| \\ &\leq \|u_1^0 \wedge \cdots \wedge u_i^0\|. \end{aligned}$$

This shows the first assertion of the lemma, because $\pi_+(u_1 \wedge \cdots \wedge u_i) = u_1^0 \wedge \cdots \wedge u_i^0$.

For the second assertion, write $v_1 \wedge \cdots \wedge v_s = \mathbf{u}^0 + \mathbf{u}^1$, with $\mathbf{u}^0 \in \wedge^s L$ and $\mathbf{u}^1 \in (\wedge^s L)^\perp$. If $s = i$, there is nothing to prove. Otherwise, there must exist an element u_k in the above family such that $\|\mathbf{u}^0 \wedge u_k^0\| \gg 1$. Indeed, if this were not the case, all vectors u_k^0 would lie close to the subspace U^0 associated to $\mathbf{u}^0$, but since $\dim U^0 = s < i$, this would contradict $\|u_1^0 \wedge \cdots \wedge u_i^0\| \gg 1$. So we set $v_{s+1} = u_k$ to get $\|\pi_+(v_1 \wedge \cdots \wedge v_{s+1})\| \gg \|v_1 \wedge \cdots \wedge v_{s+1}\|$. The lemma follows by induction. $\square$

Proof of Proposition 5. We construct the vectors $v_1, \ldots, v_\ell$ by induction, so that at each step s, one has:

1. if $\ell'_{s-1} < i \leq \ell'_s$, then $v_i \in V_{j_s}(\mathbb{Z})$ and $\|v_i\| \ll \lambda_{j_s}(a_t s_x \mathbb{Z}^d)$;
2. $\|\pi_+(v_1 \wedge \cdots \wedge v_{\ell'_s})\| \gg \|v_1 \wedge \cdots \wedge v_{\ell'_s}\|$.

For $s = 1$, Lemma 5 applied to $V = a_t s_x V_{j_1}(\mathbb{R})$ shows that there exists an orthonormal family $u_1, \ldots, u_{\ell_1}$ inside $a_t s_x V_{j_1}(\mathbb{R})$ so that $\|\pi_+(u_1 \wedge \cdots \wedge u_{\ell_1})\| \gg 1$. Since all successive minima of $a_t s_x V_{j_1}(\mathbb{Z})$ are comparable to $\lambda_1(a_t s_x \mathbb{Z}^d)$, there exists, for its action on its linear span, a fundamental domain included in a ball of radius $C\lambda_1(a_t s_x \mathbb{Z}^d)$, for some constant C depending only on d. Therefore, one can find vectors $v_1, \ldots, v_{\ell_1}$ in $a_t s_x V_{j_1}(\mathbb{Z})$ with norm in the interval $[10C\lambda_1(a_t s_x \mathbb{Z}^d), 12C\lambda_1(a_t s_x \mathbb{Z}^d)]$ and such that $d(v_i, \mathbb{R}u_i) \leq C\lambda_1(a_t s_x \mathbb{Z}^d)$. This implies

$$\begin{aligned} \|\pi_+(v_1 \wedge \cdots \wedge v_{\ell_1})\| &= \Big(\prod_{1 \leq i \leq \ell_1} \|v_i\|\Big) \left\| \pi_+\Big(\frac{v_1}{\|v_1\|} \wedge \cdots \wedge \frac{v_{\ell_1}}{\|v_{\ell_1}\|}\Big) \right\| \\ &\gg \prod_{1 \leq i \leq \ell_1} \|v_i\| \\ &\gg \|v_1 \wedge \cdots \wedge v_{\ell_1}\|. \end{aligned}$$

Suppose now that $v_1, \ldots, v_{\ell'_{s-1}}$ have been defined, and satisfy the required properties. By Lemma 5, there exists an orthonormal family of vectors $u_{\ell'_{s-1}+1}, \ldots, u_{\ell'_s}$ in $V_{j_s}(\mathbb{R})$ such that

$$\|\pi_+(v_1 \wedge \cdots \wedge v_{\ell'_{s-1}} \wedge u_{\ell'_{s-1}+1} \wedge \cdots \wedge u_{\ell'_s})\| \gg \|v_1 \wedge \cdots \wedge v_{\ell'_{s-1}} \wedge u_{\ell'_{s-1}+1} \wedge \cdots \wedge u_{\ell'_s}\|.$$

Indeed, to an orthonormal basis for $v_1 \wedge \cdots \wedge v_{\ell'_{s-1}}$, we may add a vector so that the new family is orthonormal and satisfies the desired inequality. The successive minima of the sublattice $a_t s_x V_{j_s}(\mathbb{Z})$ are all bounded by $\lambda_{j_s}(a_t s_x \mathbb{Z}^d)$, and this allows us to approach u_i, $i > \ell'_{s-1}$ with vectors in $a_t s_x V_{j_s}(\mathbb{Z})$ of controlled size: we obtain $v_{\ell'_{s-1}+1}, \ldots, v_{\ell'_s}$ in $a_t s_x V_{j_s}(\mathbb{Z})$ of norm in the interval

$[10C\lambda_{j_s}(a_t s_x \mathbb{Z}^d), 12C\lambda_{j_s}(a_t s_x \mathbb{Z}^d)]$ and such that $d(v_i, \mathbb{R}u_i) \leq \lambda_{j_s}(a_t s_x \mathbb{Z}^d)$. This gives the desired result.

We are left to show that for $\ell'_s = \ell$, one has

$$\log\|v_1 \wedge \cdots \wedge v_\ell\| \leq \frac{-\ell(d-\ell)}{d}\langle \dot{c}_t, c_t\rangle + O(1).$$

For that, write

$$\begin{aligned}
\log\|v_1 \wedge \cdots \wedge v_\ell\| &\leq \sum_{i=1}^{\ell} \log\|v_i\| \\
&\leq \ell'_1\Lambda_1 + (\ell'_2 - \ell'_1)\Lambda_{j_1+1} + \cdots + (\ell'_{s-1} - \ell'_s)\Lambda_{j_{s-1}+1} + O(1) \\
&= \ell'_1(\Lambda_1 - \Lambda_{j_1+1}) + \cdots + \ell'_{s-1}(\Lambda_{j_{s-1}+1} - \Lambda_{j_{s-2}+1}) - \ell\Lambda_{j_{s-1}+1} + O(1) \\
&\leq \ell_1(\Lambda_1 - \Lambda_{j_1+1}) + \cdots + \ell_{s-1}(\Lambda_{j_{s-1}+1} - \Lambda_{j_{s-2}+1}) - \ell\Lambda_{j_{s-1}+1} + O(1) \\
&= \ell_1\Lambda_1 + (\ell_2 - \ell_1)\Lambda_{j_1+1} + \cdots + (\ell_{s-1} - \ell_s)\Lambda_{j_s+1} + O(1).
\end{aligned}$$

So we can conclude using Lemma 4. □

End of proof for Dirichlet's principle

We now derive Theorem 9, according to which the almost sure value of the diophantine exponent $\beta_k(x)$ on $X_\ell(\mathbb{R})$ is also its minimal value. In short, given a point $x \in X_\ell(\mathbb{R})$, we shall apply the second point in Proposition 1 to the vectors $v_1, \ldots, v_\ell$ given by Proposition 5, in order to construct good rational approximations to x.

Proof of Theorem 9. Notation is as in the above paragraph. In particular, $x = Ps_x$ is a point in $X_\ell(\mathbb{R})$, and c_t denotes the Grayson polygon of the lattice $a_t s_x \mathbb{Z}^d$, where $a_t = \mathrm{diag}(e^{-\frac{t}{\ell}}, \ldots, e^{-\frac{t}{\ell}}, e^{\frac{t}{d-\ell}}, \ldots, e^{\frac{t}{d-\ell}})$. By Proposition 5, for every $t > 0$, there exist vectors $v_1, \ldots, v_\ell$ in $a_t s_x \mathbb{Z}^d$ such that

$$\|\pi_+(v_1 \wedge \cdots \wedge v_\ell)\| \gg \|v_1 \wedge \cdots \wedge v_\ell\|$$

and

$$\log\|v_1 \wedge \cdots \wedge v_\ell\| \leq \frac{-\ell(d-\ell)}{d}\langle \dot{c}_t, c_t\rangle + O(1).$$

Fix $k \leq \ell$. Without loss of generality, we may assume that the vectors $v_1, \ldots, v_\ell$ achieve the successive minima of the sublattice they generate, and then, one also has

$$\|\pi_+(v_1 \wedge \cdots \wedge v_k)\| \gg \|v_1 \wedge \cdots \wedge v_k\|$$

and

$$\log\|v_1 \wedge \cdots \wedge v_k\| \leq \frac{-k(d-\ell)}{d}\langle \dot{c}_t, c_t\rangle + O(1).$$

Note that $\langle \dot{c}_t, c_t\rangle = \frac{d}{dt}\frac{\|c_t\|^2}{2}$ and that since $\|c_t\|^2 \geq 0$, there must exist arbitrary large values of $t > 0$ for which $\langle \dot{c}_t, c_t\rangle \geq -1$, and then

$$\log\|v_1 \wedge \cdots \wedge v_k\| \leq O(1).$$

The second point in Proposition 1 applied to the vector $\mathbf{v} = v_1 \wedge \cdots \wedge v_k$ shows that the rational point $v \in X_k(\mathbb{Q})$ associated to $\mathbf{v}$ satisfies

$$H(v) \ll e^{t\frac{k}{\ell}} \quad \text{and} \quad d(x, v) \ll e^{-t(\frac{1}{\ell} + \frac{1}{d-\ell})}.$$

Thus, $d(x,v) \ll H(v)^{-\frac{d}{k(d-\ell)}}$. $\square$

The ideas used in the above proof can also be used to show that every point lying on a rational constraining pencil is very well approximable by rational k-planes.

cpvwa **Proposition 6** (Constraining rational pencils are included in $\mathrm{VWA}_k(X_\ell)$)**.** *Let $\mathcal{P}_{W,r} \subset X_\ell(\mathbb{R})$ be a constraining rational pencil. For every $x \in \mathcal{P}_{W,r}$ and every $k \leq \ell$,*

$$\gamma_k(x) \geq \frac{2k(rd-\ell\dim W)^2}{\ell^2(d-\ell)(\dim W)(d-\dim W)} > 0.$$

In particular, every point x in $\mathcal{P}_{W,r}$ is very well approximable by rational k-planes.

Proof. Note that the vector $\mathbf{w} \in \wedge^{\dim W}\mathbb{Z}^d$ associated to W satisfies

$$\begin{aligned}\log\|a_t s_x \mathbf{w}\| &= t \cdot \Big(-\frac{r}{\ell} + \frac{\dim W - r}{d-\ell}\Big) + O(1)\\ &= -t \cdot \frac{d}{\ell(d-\ell)}\Big(r - \frac{\ell\dim W}{d}\Big) + O(1).\end{aligned}$$

This readily implies a lower bound

$$\|c_t\|^2 \geq t^2 \cdot \left(\frac{d}{\ell(d-\ell)}\right)^2 \Big(r - \frac{\ell\dim W}{d}\Big)^2\Big(\frac{1}{\dim W} + \frac{1}{d-\dim W}\Big) - O(t)$$

and therefore, there exists $t > 0$ arbitrarily large such that

$$\langle \dot{c}_t, c_t\rangle \geq 2t\left(\frac{d}{\ell(d-\ell)}\right)^2 \Big(r - \frac{\ell\dim W}{d}\Big)^2\Big(\frac{1}{\dim W} + \frac{1}{d-\dim W}\Big) - O(1).$$

Using Proposition bonnebase 5 as in the proof of Theorem dirichlet 9, one infers that for $t > 0$ arbitrarily large, there exist vectors $v_1, \dots, v_k$ in $a_t s_x \mathbb{Z}^d$ such that

$$\|\pi_+(v_1 \wedge \cdots \wedge v_k)\| \gg \|v_1 \wedge \cdots \wedge v_k\|$$

and

$$\begin{aligned}\log\|v_1 \wedge \cdots \wedge v_k\| &\leq \frac{-k(d-\ell)}{d}\langle \dot{c}_t, c_t\rangle + O(1)\\ &\leq 2t \cdot \frac{dk}{\ell^2(d-\ell)}\Big(r - \frac{\ell\dim W}{d}\Big)^2\Big(\frac{1}{\dim W} + \frac{1}{d-\dim W}\Big) + O(1).\end{aligned}$$

This shows that

$$\gamma_k(x) \geq \frac{2k(rd-\ell\dim W)^2}{\ell^2(d-\ell)(\dim W)(d-\dim W)} > 0.$$

$\square$

4 Khintchine's theorem

In this section, we prove Theorem 3 announced in the introduction. This result generalizes Khintchine's famous theorem [15], which corresponds to the particular case $k = \ell = 1$. Just as in the classical case of Khintchine's theorem, the case where the sum is convergent is easier, and follows from a simple application of the Borel-Cantelli lemma.

Proof of Theorem 3 (i): convergent sum. For each $v \in X_k(\mathbb{Q})$, the set of points $x \in X_\ell(\mathbb{R})$ such that $d(x,v) \leq H(v)^{-\frac{d}{k(d-\ell)}}\psi(H(v))$ is a neighborhood of the submanifold $\{x \mid x \supset v\}$. That submanifold has codimension $k(d-\ell)$ in $X_\ell(\mathbb{R})$, and therefore, within bounded multiplicative constants, the considered neighborhood has measure

$$H(v)^{-d}\psi(H(v))^{k(d-\ell)}.$$

Since the number of rational points in $X_\ell(\mathbb{Q})$ of height at most H is $O(H^d)$, we may bound from above the sum of all measures of those neighborhoods by grouping together all points v such that $2^p \leq H(v) < 2^{p+1}$:

$$\begin{aligned}\sum_{v \in X_k(\mathbb{Q})} &|\{x \in X_\ell(\mathbb{R}) \mid d(v,x) \leq H(v)^{-\frac{d}{k(d-\ell)}}\psi(H(v))\}| \\ &\ll \sum_{v \in X_k(\mathbb{Q})} H(v)^{-d}\psi(H(v))^{k(d-\ell)} \\ &\ll \sum_{p \geq 1} 2^{-dp}\psi(2^p)^{k(d-\ell)}2^{dp} = \sum_{p \geq 1}\psi(2^p)^{k(d-\ell)}.\end{aligned}$$

Then,

$$\begin{aligned}\sum_{p\geq 1}\psi(2^p)^{k(d-\ell)} &\leq \int_{t>0}\psi(2^t)^{k(d-\ell)}\mathrm{d}t \\ &= (\log 2)\int_u \psi(u)^{k(d-\ell)}\frac{\mathrm{d}u}{u} < +\infty\end{aligned}$$

and therefore,

$$\sum_{v \in X_k(\mathbb{Q})} |\{x \in X_\ell(\mathbb{R}) \mid d(v,x) \leq H(v)^{-\frac{d}{k(d-\ell)}}\psi(H(v))\}| < +\infty.$$

Thus, by the Borel-Cantelli Lemma, for almost every $x \in X_\ell(\mathbb{R})$, the inequality (3) has only finitely many solutions. □

In order to prove the second assertion of the theorem, where the sum diverges, we follow the strategy of Kleinbock and Margulis [18], based on the correspondence from Section 1 and on exponential mixing for the action of the subgroup $(a_t)_{t>0}$ on the space of lattices.

Proof of Theorem 3(ii): divergent sum. Let

$$\Psi(u) = u^{-\frac{d}{k(d-\ell)}}\psi(u).$$

By the second item in Proposition 1, it is enough to show that when the sum diverges, for almost every x in $X_\ell(\mathbb{R})$, for arbitrarily large $t>0$, there exists $\mathbf{v}\in\wedge^k\mathbb{Z}^d$ such that

$$\|\pi_+(a_t s_x\mathbf{v})\| \geq \frac{\|a_t s_x \mathbf{v}\|}{2} \quad \text{and} \quad \|a_t s_x\mathbf{v}\| \leq e^{-t\frac{k}{\ell}}\Psi^{-1}(e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})}).$$

Indeed, if $v\in X_k(\mathbb{Q})$ denotes the subspace associated to $\mathbf{v}$, then one has $H(v)\ll \Psi^{-1}(e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})})$ and $d(x,v)\ll e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})}\leq \Psi(H(v))$. Since one may always replace $\psi(u)$ by $\frac{1}{C}\psi(u)$, for some large constant $C>0$, without changing the nature of the sum, this gives the desired statement. For $r>0$, let Ω'_r denote the set of unimodular lattices Δ in $\mathbb{R}^d$ for which there exists a pure tensor $\mathbf{v}\in\wedge^k\Delta$ satisfying $\|\pi_+(\mathbf{v})\|\geq\frac{\|\mathbf{v}\|}{2}$ and $\|\mathbf{v}\|\leq r$. Given $t>0$, set

$$r_t = e^{-t\frac{k}{\ell}}\Psi^{-1}(e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})}).$$

We want to show that for almost every $x\in X_\ell(\mathbb{R})$, the orbit $(a_t\Delta_x)_{t>0}$ meets Ω'_{r_t} for arbitrarily large $t>0$. This follows from Proposition 7 below, and from the fact that

$$\sum_{t\geq 1}|\Omega'_{r_t}| = +\infty.$$

Indeed, within multiplicative constants, reduction theory for $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$, as presented in [3, chapitre I, théorème 1.4 and §4], allows one to bound from below the Haar measure of Ω'_r by

$$|\Omega'_r|\gg r^d$$

so we only need to check that

$$\int\left(e^{-t\frac{k}{\ell}}\Psi^{-1}(e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})})\right)^d \mathrm{d}t = +\infty.$$

Write $\beta=\frac{d}{k(d-\ell)}$. Since ψ is non-increasing, we may bound, for each $u\geq 1$, $\Psi(u)\leq u^{-\beta}\psi(1)$ whence $\Psi^{-1}(s)\leq Cs^{-\frac{1}{\beta}}$ for $s\in(0,1)$, and then

$$\Psi^{-1}(s) = s^{-\frac{1}{\beta}}\psi(\Psi^{-1}(s))^{\frac{1}{\beta}} \geq s^{-\frac{1}{\beta}}\psi(Cs^{-\frac{1}{\beta}})^{\frac{1}{\beta}}.$$

So we get a lower bound

$$\int\left(e^{-t\frac{k}{\ell}}\Psi^{-1}(e^{-t(\frac{1}{\ell}+\frac{1}{d-\ell})})\right)^d\mathrm{d}t \geq \int \psi(Ce^{-\frac{t}{\beta}(\frac{1}{\ell}+\frac{1}{d-\ell})})^{k(d-\ell)}\mathrm{d}t$$

and with the change of variable $u=Ce^{-\frac{t}{\beta}(\frac{1}{\ell}+\frac{1}{d-\ell})}$, $\frac{\mathrm{d}u}{u}=\frac{1}{\beta}(\frac{1}{\ell}+\frac{1}{d-\ell})\mathrm{d}t$,

$$\gg \int \psi(u)^{k(d-\ell)}\frac{\mathrm{d}u}{u} = +\infty.$$

□

Exponential mixing for the action of (a_t) on the space of lattices is used in a essential way for the proof of the proposition below, following a strategy of Kleinbock and Margulis [18]. For the proof, we refer the reader to [26, Proposition 3.4.1] where the result is proved in a slightly different setting, but which can easily be adapted to our problem.

flkhin **Proposition 7.** *Let (r_t) be a sequence of positive real numbers such that*

$$\sum_{t\geq 1} m_\Omega(\Omega'_{r_t}) = +\infty.$$

For almost every $x = Ps_x \in X_\ell(\mathbb{R})$, there exists arbitrarily large $t \in \mathbb{N}$ such that

$$a_t s_x \mathbb{Z}^d \in \Omega'_{r_t}.$$

5 Jarník's theorem

sec:jarnik

Given $\tau \geq \frac{d}{k(d-\ell)}$, we now consider the set $W_{k,\ell}(\tau)$ of points in $X_\ell(\mathbb{R})$ whose diophantine exponent for approximation by rational subspaces of dimension k is at least τ:

$$W_{k,\ell}(\tau) = \{x \in X_\ell \mid \beta_k(x) \geq \tau\}.$$

When $k = \ell = 1$, Jarník's celebrated theorem [**jarnik2**, 13] gives a formula for the Hausdorff dimension of $W_{1,1}(\tau)$, and Dodson [9] generalized that result to approximation of a subspace by rational lines, which corresponds to the case where $k = 1$ and ℓ is arbitrary. Theorem 4 from the introduction, which we now want to prove, gives an analogous formula for arbitrary values $1 \leq k \leq \ell < d$:

$$\dim_H W_{k,\ell}(\tau) = \begin{cases} (\ell-k)(d-\ell) + \frac{d}{\tau} & \text{if } \tau > \frac{d}{k(d-\ell)} \\ \ell(d-\ell) & \text{if } \tau \leq \frac{d}{k(d-\ell)}. \end{cases}$$

Proof of Theorem 4. If $\tau \leq \frac{d}{k(d-\ell)}$, then Theorem 1 shows that $W_{k,\ell}(\tau) = X_\ell(\mathbb{R})$, so the result is clear. From now on, we therefore assume $\tau > \frac{d}{k(d-\ell)}$.

For $s > 0$, denote by $\mathcal{H}^{(s)}$ the s-dimensional Hausdorff measure on $X_\ell(\mathbb{R})$, and let us show that for $s > \frac{d}{\tau} + (\ell-k)(d-\ell)$, one has $\mathcal{H}^{(s)}(W_{k,\ell}(\tau)) < +\infty$. For each rational point $v \in X_k(\mathbb{Q})$ of height $2^p \leq H(v) < 2^{p+1}$, one has the inclusion

$$\{x \in X_\ell(\mathbb{R}) \mid d(x,v) \leq H(v)^{-\tau}\} \subset L_v^{(2^{-p\tau})}$$

where $L_v^{(\rho)}$ is the ρ-neighborhood of the submanifold $L_v = \{x \mid x \supset v\}$. The submanifold L_v has dimension $(\ell-k)(d-\ell)$, so we may cover $L_v^{(2^{-p\tau})}$ with $O(2^{p\tau(\ell-k)(d-\ell)})$ balls of radius $2^{-p\tau}$. Taking the union over $v \in X_k(\mathbb{Q})$ of all these covers, we get a cover of $W_{k,\ell}^{(\tau)}$ whose Hausdorff measure is bounded above by:

$$\begin{aligned}\mathcal{H}^{(s)}(W_{k,\ell}^{(\tau)}) &\leq \sum_p \mathrm{card}\{v \in X_k(\mathbb{Q}) \mid 2^p \leq H(v) < 2^{p+1}\} 2^{p\tau(\ell-k)(d-\ell)} 2^{-p\tau s} \\ &\ll \sum_p 2^{p(d+\tau(\ell-k)(d-\ell)-\tau s)}\end{aligned}$$

and since we assumed $s > \frac{d}{\tau} + (\ell-k)(d-\ell)$, this sum converges, and $\dim_H W_{k,\ell}(\tau) \leq \frac{d}{\tau} + (\ell-k)(d-\ell)$.

In order to prove the converse inequality, we shall use the notion of *ubiquity*, introduced by Dodson, Rynne and Vickers in [10]. Given a function $\rho\colon \mathbb{R}^+ \to$

$\mathbb{R}^+$, the family of sets L_v, $v \in X_k(\mathbb{Q})$ is *ubiquitous with respect to* ρ if

$$\lim_{H\to\infty} \left| \bigcup_{H(v)\leq H} L_v^{(\rho(H))} \right| = 1.$$

Let us show that if $C > 0$ is a large enough constant, then $(L_v)_{v\in X_k(\mathbb{Q})}$ is ubiquitous with respect to

$$\rho(H) = (\log H)^C \cdot H^{-\frac{d}{k(d-\ell)}}.$$

For that, note that for almost every x in $X_\ell(\mathbb{R})$, for all large enough $t > 0$, $\lambda_d(a_t s_x \mathbb{Z}^d) \leq t$. Indeed, the measure of the set $\{\lambda_d \leq \delta^{-1}\}$ is equal to that of $\{\lambda_1 \leq \delta\}$, which is of order δ^d, by Siegel's formula. So if $\alpha > \frac{1}{d}$, then $\sum_{t\in\mathbb{N}} |\{x \mid \lambda_d(a_t s_x \mathbb{Z}^d) \leq t^\alpha\}| \ll \sum_{t\in\mathbb{N}} t^{d\alpha} < +\infty$, and the Borel-Cantelli lemma shows that almost surely, for all large enough $t > 0$, $\lambda_d(a_t s_x \mathbb{Z}^d) \leq t^\alpha$. Consequently, we may find in $\mathbb{Z}^d$ a vector $\mathbf{v}$ such that $\|a_t s_x \mathbf{v}\| \leq t$ and $\|\pi_+(a_t s_x \mathbf{v})\| \gg \|a_t s_x \mathbf{v}\|$. By the second part of Proposition 1, and setting $H = te^{t^{\frac{k}{\ell}}}$, this implies that for all large enough $H \geq 1$, there exists $v \in X_k(\mathbb{Q})$ such that

$$\begin{cases} H(v) \leq H \\ d(x,v) \leq (\log H)^C \cdot H^{-\frac{d}{k(d-\ell)}}. \end{cases}$$

Since this holds for almost every x in $X_\ell(\mathbb{R})$, we find

$$\lim_{H'\to\infty} \left| \bigcap_{H\geq H'} \bigcup_{H(v)\leq H} L_v^{(\rho(H))} \right| = 1$$

whence

$$\left| \bigcup_{H(v)\leq H'} L_v^{(\rho(H'))} \right| \geq \left| \bigcap_{H\geq H'} \bigcup_{H(v)\leq H} L_v^{(\rho(H))} \right| \to_{H\to\infty} 1$$

which is the desired ubiquity property. By [10, Theorem 1], if $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a decreasing function, one can bound from below the dimension of the set

$$W(\psi) = \limsup_{H\to\infty} L_v^{(\psi(H(v))} = \bigcap_{H\geq 1} \bigcup_{H(v)\geq H} L_v^{(\psi(H(v))}$$

by

$$\dim_H W(\psi) \geq \dim \mathcal{L} + \gamma \operatorname{codim} \mathcal{L}$$

where $\dim \mathcal{L} = (\ell - k)(d-\ell)$ and $\operatorname{codim} \mathcal{L} = k(d-\ell)$ are the common dimension and codimension of all L_v, $v \in X_k$, and

$$\gamma = \min(1, \limsup_{H\to\infty} \frac{\log \rho(H)}{\log \psi(H)}).$$

For the function ψ defined by $\psi(H) = H^{-\tau}$, one has $W(\psi) = W_{k,\ell}^{(\tau)}$ and $\gamma = \frac{d}{\tau k(d-\ell)}$ whence

$$\dim_H W_{k,\ell}^{(\tau)} \geq (\ell - k)(d-\ell) + \frac{d}{\tau}.$$

□

Remark. The transference principle from Beresnevich and Velani [2, Theorem 3] does not apply when $k \neq \ell$, because then the resonant sets L_v are not points, but submanifolds of positive dimension. This is the reason why we went back to the coarser, but more robust, notion of ubiquity.

6 Approximation on submanifolds

In this section, we consider a connected analytic submanifold $M \subset X_\ell(\mathbb{R})$, and we study diophantine properties of a point x chosen randomly on M. Our first notable result is that the exponent of a point chosen randomly on an analytic submanifold is almost surely constant. When $k = 1$, this result is due to Kleinbock [19]. Just as in this particular case, our proof is based on Dani's correspondence and on the quantitative non-divergence estimates. But when $k > 1$, one has to control the direction of short vectors along the diagonal orbit, and this makes the argument more intricate.

Theorem 10 (Exponent of an analytic submanifold)**.** *For each $k = 1, \dots, \ell$, there exists $\beta_k(M)$ such that for almost every x in M, $\beta_k(x) = \beta_k(M)$. Moreover, $\beta_k(M)$ is entirely determined by the Zariski closure of M.*

The diophantine exponent $\beta_k(x)$ is determined by the escape rate $\gamma_k(x)$, so the theorem will follow from Lemma 6 below, applied to the diagonal flow $\wedge^k a_t$ on the space $\mathbb{R}^D = \wedge^k \mathbb{R}^d$. Recall that if G is a metric space, $C, \alpha > 0$ two constants, and μ a finite Borel measure on G, a function $f\colon X \to \mathbb{R}$ is (C,α)*-good* for μ if for every ball $B = B(x,r)$ centered at $x \in \operatorname{Supp}\mu$, for all $\varepsilon > 0$,

$$\mu(\{g \in B \mid |f(g)| \leq \varepsilon \|f\|_{B,\mu}\}) \leq C\varepsilon^\alpha \mu(B),$$

where $\|f\|_{B,\mu} = \sup_{y \in B \cap \operatorname{Supp}\mu} |f(y)|$. In the proof of the lemma below, we shall use several times the following important property of analytic functions, which results from the work of Kleinbock and Margulis [17, Proposition 2.3]: if $\mathcal{F}$ is a finite-dimensional linear space of analytic functions on an open set $O \subset \mathbb{R}^d$, then for every $x \in O$, there exists a neighborhood U of x and constants $C, \alpha > 0$ such that every function $f \in \mathcal{F}$ is (C,α)-good on U.

Lemma 6. *Let $(a_t)_{t>0}$ be a diagonal one-parameter subgroup in $\mathrm{GL}_D(\mathbb{R})$. Let $\pi_+\colon \mathbb{R}^D \to \mathbb{R}^D$ denote the projection to the eigenspace of (a_t) associated to the most contracting eigenvalue parallel to the sum of all other eigenspaces, and given a lattice Δ in $\mathbb{R}^D$, let*

$$\gamma(\Delta) = \limsup_{t\to\infty} \frac{-1}{t} \min\{\log\|v\| \ ;\ v \in \Delta \setminus \{0\} \text{ such that } \|\pi_+(v)\| \geq \frac{1}{2}\|v\|\}.$$

If $M \subset \mathrm{GL}_D(\mathbb{R})$ is a connected analytic submanifold, then there exists $\gamma_M \in \mathbb{R}$ such that for almost every $g \in M$,

$$\gamma(g\mathbb{Z}^d) = \gamma_M.$$

Moreover, γ_M only depends on the Zariski closure of M.

Proof. For g in $\mathrm{GL}_D(\mathbb{R})$ and $t > 0$, let c_t^g denote the Grayson polygon associated to the lattice $a_t g\mathbb{Z}^D$. It is an element in the cone $\mathfrak{a}^+$ of convex functions

$c\colon [\![0,D]\!] \to \mathbb{R}$ such that $c(0)=c(D)=0$. Fix $g_0 \in M$. The quantitative non-divergence estimate [27, Theorem 1] and the Borel-Cantelli lemma show that there exists a map $c_t^M : t \to \mathfrak{a}^+$ such that for almost every g in a neighborhood of g_0, for all $\varepsilon > 0$ and all large enough $t > 0$,

$$\|c_t^g - c_t^M\| \leq \varepsilon t.$$

Up to a $o(t)$ term, the map $t \mapsto c_t^M$ is uniquely defined, and it is entirely determined by the Zariski closure of M. (We refer the reader to [26, corollaire 7.2.2] for a detailed proof of that point, in a slightly more general context.) Then, it follows from [27, Corollary 1] that if

$$I_t = \{i_1 < \cdots < i_r\} \subset [\![1,D]\!]$$

denotes the set of points where the derivative of the map $i \mapsto c_t^M(i)$ has a discontinuity of size at least $t\varepsilon$ i.e. $c_t^M(i) - c_t^M(i-1) + t\varepsilon \leq c_t^M(i+1) - c_t^M(i)$, there exists a partial flag

$$\{0\} < V_{i_1} < \cdots < V_{i_r} < \mathbb{Z}^d$$

such that for every g in a neighborhood of g_0, with probability at least $1 - Ce^{-\alpha\varepsilon t}$, for $s = 1, \dots, r$, the i_s first successive minima of $a_t g\mathbb{Z}^d$ are achieved in $a_t g V_{i_s}$. Here again, the map $t \mapsto (V_{i_s})_{1\leq s\leq r}$ is entirely determined by the Zariski closure of M.

Let E^+ be the eigenspace of a_t associated to the most contracting eigenvalue, and E^- the sum of all other eigenspaces, so that $\mathbb{R}^D = E^+ \oplus E^-$. Given a linear subspace V in $\mathbb{R}^D$, denote the distance from V to E^- by

$$d(V, E^-) = \max_{v \in V\setminus\{0\}} \frac{d(v, E^-)}{\|v\|}.$$

The equality $d(V, E^-) = 0$ corresponds to the inclusion $V \subset E^-$, which is equivalent to V intersecting non-trivially every linear subspace of codimension $\dim V - 1$ in E^-. Therefore, if $\mathbf{v}$ is a representative of V in $\wedge^{\dim V}\mathbb{R}^D$ and $(\mathbf{u}_j)_{j\in J}$ an orthonormal basis for $\wedge^{\dim E^- - \dim V + 1} E^-$, then

$$d(V, E^-) \asymp \max_{j \in J} \frac{\|\mathbf{u}_j \wedge \mathbf{v}\|}{\|\mathbf{v}\|}.$$

Since they generate a finite-dimensional space of analytic maps, all maps $g \mapsto \|a_t g\mathbf{w}\|^2$, where $t > 0$ and $\mathbf{w} \in \wedge^*\mathbb{R}^d$, are (C,α)-good in a neighborhood of g_0 for the Lebesgue measure on M; so the above equation can be used to show that for each $i \in I_t$, there exists $f_i = f_i(t,\varepsilon) \geq 0$ such that

$$\lambda_M(\{g \in U \mid e^{-t(f_i+\varepsilon)} \leq d(a_t g V_i, E^-) \leq e^{-t(f_i - \varepsilon)}\}) \geq (1 - Ce^{-\alpha\varepsilon t})\lambda_M(U).$$

Indeed, if $\mathbf{v}_i$ represents V_i in $\wedge^i\mathbb{R}^D$ and if $(\mathbf{u}_j)_{j\in J}$ is an orthonormal basis for $\wedge^{\dim E^- - i + 1}E^-$, then

$$d(a_t g V_i, E^-) \asymp \max_{j\in J} \frac{\|\mathbf{u}_j \wedge a_t g \mathbf{v}_i\|}{\|a_t g \mathbf{v}_i\|}.$$

But, by the (C,α)-good property for the functions $g \mapsto \|\mathbf{u}_j \wedge a_t g\mathbf{v}_i\|$ and $x \mapsto \|a_t g\mathbf{v}_i\|$ for the measure λ_M in a neighborhood of g_0, one has, with λ_M-probability at least $1 - Ce^{-\alpha\varepsilon t}$,

$$\|\mathbf{u}_j \wedge a_t g\mathbf{v}_i\| \geq e^{-\varepsilon t} \sup_{h\in U\cap M} \|\mathbf{u}_j \wedge a_t h\mathbf{v}_i\|$$

and

$$\|a_t g\mathbf{v}_i\| \geq e^{-\varepsilon t} \sup_{h\in U\cap M} \|a_t h\mathbf{v}_i\|.$$

Therefore, if f_i is chosen so that $e^{-f_i t} = \max_j \frac{\sup_{h\in U\cap M}\|\mathbf{u}_j\wedge a_t h\mathbf{v}_i\|}{\sup_{h\in U\cap M}\|a_t h\mathbf{v}_i\|}$, one has, with probability at least $1 - Ce^{-\alpha\varepsilon t}$, for $g \in U \cap M$,

$$d(a_t gV_i, E^-) \leq \max_j \frac{\sup_{h\in U\cap M}\|\mathbf{u}_j \wedge a_t h\mathbf{v}_i\|}{\|a_t g\mathbf{v}_i\|} \leq e^{-t(f_i-\varepsilon)}$$

and

$$d(a_t gV_i, E^-) \geq \max_j \frac{\|\mathbf{u}_j \wedge a_t g\mathbf{v}_i\|}{\sup_h\|a_t h\mathbf{v}_i\|} \geq e^{-t(f_i+\varepsilon)}.$$

Here again, the maps $t \mapsto f_i$ are determined by the Zariski closure of M. This follows from the fact that the maps $h \mapsto \mathbf{u}_i a_t h\mathbf{v}_i$ are linear and therefore satisfy, within constants independent of $\mathbf{u}_i$, a_t and $\mathbf{v}_i$, $\sup_{h\in B\cap M}\|\mathbf{u}_i \wedge a_t h\mathbf{v}_i\| \asymp \sup_{h\in B\cap\mathcal{L}(M)}\|\mathbf{u}_i \wedge a_t h\mathbf{v}_i\|$, where $\mathcal{L}(M)$ is the linear span of the image of M in $\mathrm{End}(\wedge^{\dim V_i}\mathbb{R}^d)$. The same holds for $h \mapsto a_t h\mathbf{v}_i$. With the Borel-Cantelli lemma, those inequalities show that for almost every g in $U\cap M$, for every large enough $t > 0$,

$$\begin{cases} \|c_t^g - c_t^M\| \leq t\varepsilon \\ \forall s = 1,\dots,r, \quad c_t^g(i_s) = \log\|a_t gV_{i_s}\| \\ d(a_t gV_{i_s}, E^-) \in [e^{-t(f_i+\varepsilon)}, e^{-t(f_i-\varepsilon)}] \end{cases}$$

Note that $f_{i_1} \geq f_{i_2} \geq \cdots \geq f_{i_k}$. Let $j_t = j_t(\varepsilon) \in I_t$ be minimal such that $f_{j_t} \leq 2\varepsilon$. Let us show that for almost every g in $U \cap M$,

$$\gamma(g) = \limsup_{t\to\infty} \frac{-1}{t}(c_t^M(j_t) - c_t^M(j_t - 1)) + O(\varepsilon).$$

First, the inequality $f_{j_t} \leq 2\varepsilon$ shows that $d(a_t gV_{j_t}, E^-) \geq e^{-3t\varepsilon}$, and since $a_t gV_{j_t}(\mathbb{Z})$ admits a basis consisting of vectors of norm at most $e^{c_t^g(j_t)-c_t^g(j_t-1)} = e^{c_t^M(j_t)-c_t^M(j_t-1)+tO(\varepsilon)}$, there exists a vector $v \in a_t gV_{j_t}(\mathbb{Z})$ such that $\|\pi^+(v)\| \geq e^{-tO(\varepsilon)}\|v\|$ and $\|v\| = e^{c_t^M(j_t)-c_t^M(j_t-1)+tO(\varepsilon)}$. Since π^+ is the projection to the most contracted eigenspace, one may replace t by $t(1 - O(\varepsilon))$ to ensure that $\|\pi^+(v)\| \geq \frac{\|v\|}{2}$, and this can only change the norm of v by a factor $e^{O(\varepsilon)t}$. Therefore,

$$\gamma(g\mathbb{Z}^D) \geq \limsup_{t\to\infty} \frac{-1}{t}(c_t^M(j_t) - c_t^M(j_t - 1)) - O(\varepsilon).$$

Conversely, let $V_{j'_t}$ be the subspace preceding V_{j_t} in the partial flag $V_{i_1} < \cdots < V_{i_r}$. Since, for almost every $g \in U \cap M$, for all large enough t

$$d(a_t gV_{t,\varepsilon}^{j'_t}, E^-) \leq e^{-(f_{j-1}-\varepsilon)t} \leq e^{-\varepsilon t}$$

no vector $v \in a_t g V_{j'_t}$ can satisfy $\|\pi^+(v)\| \geq \frac{\|v\|}{2}$. But any vector $v \in a_t g\mathbb{Z}^D$ outside $a_t g V_{j'_t}$ satisfies

$$\|v\| \gg e^{c_t^M(j'_t+1)-c_t^M(j'_t)} \geq e^{c_t^M(j_t)-c_t^M(j_t-1)-(j_t-j'_t)t\varepsilon},$$

where the second step follows from the fact that all changes of slopes of $i \mapsto c_t^M(i)$ are bounded above by $t\varepsilon$ on the whole interval (j'_t, j_t). Therefore,

$$\gamma(g\mathbb{Z}^D) \leq \limsup_{t\to\infty} \frac{-1}{t}(c_t^M(j_t) - c_t^M(j_t-1)) - O(\varepsilon).$$

Thus, there exists a neighborhood U of g_0 such that for every $\varepsilon > 0$, there exists a real number $\gamma_\varepsilon = \limsup_{t\to\infty} \frac{-1}{t}(c_t^M(j_t) - c_t^M(j_t-1)$ entirely determined by the Zariski closure of M and such that for almost every g in $U \cap M$, $\gamma(g\mathbb{Z}^D) \in [\gamma_\varepsilon - \varepsilon, \gamma_\varepsilon + \varepsilon]$. Letting ε go to 0, we obtain that $\gamma(g\mathbb{Z}^D)$ is almost everywhere constant in a neighborhood of g_0 in M, and since M is connected, that $\gamma(g\mathbb{Z}^D)$ is almost everywhere constant on M. □

In view of Theorem 1, it is natural to wonder what submanifolds $M \subset X_\ell(\mathbb{R})$ satisfy $\beta_k(M) = \frac{d}{k(d-\ell)}$. The sufficient criterion obtained in [1] for the case $k = 1$ is easily extended to apply to all exponents β_k, $k = 1, \ldots, \ell$.

Theorem 11 (Sufficient criterion for extremality)**.** *Let $M \subset X_\ell(\mathbb{R})$ be a connected analytic submanifold. If M is not included in any constraining pencil, then, for $k = 1, \ldots, \ell$, for almost every $x \in M$,*

$$\beta_k(x) = \frac{d}{k(d-\ell)}.$$

Proof. By Proposition 2, the inequality $\frac{1}{k}\beta_1(x) \geq \beta_k(x) \geq \frac{d}{k(d-\ell)}$ always holds, so we only need prove $\beta_1(x) = \frac{d}{d-\ell}$ for almost every x in M. This is exactly the content of [1, Corollary 1.6]. □

This sufficient criterion is essentially optimal, since any submanifold M included in a *rational* constraining pencil, is not extremal. The converse statement is false in general, but it holds if one assumes that the Zariski closure of M is defined over the field $\overline{\mathbb{Q}}$ of real algebraic numbers. In that case, Theorem 12 below even gives a formula for all exponents $\beta_k(M)$, analogous to that of Theorem 8.

Given a connected analytic submanifold $M \subset X_\ell(\mathbb{R})$, the function

$$\begin{array}{rccc} \phi_M\colon & \mathrm{Grass}(\mathbb{Q}^d) & \to & \mathbb{R} \\ & V & \mapsto & \min_{x\in M} \dim x \cap V \end{array}$$

is submodular, and following [5, §1.3] one can associate to it a partial flag

$$0 = T_0 < T_1 < \cdots < T_{r+1} = \mathbb{Q}^d$$

such that for each $s = 1, \ldots, r$, T_s is the unique rational subspace containing T_{s-1} of maximal dimension maximizing the ratio $\frac{\phi_M(T_s)-\phi_M(T_{s-1})}{\dim T_s - \dim T_{s-1}}$. Moreover, by [5, Theorem 3] this flag determines the asymptotic behavior of the orbit $a_t s_x \mathbb{Z}^d$ when the subspace x is chosen randomly on M:

- for all large enough $t > 0$, the i_s first successive minima of $a_t s_x \mathbb{Z}^d$ are achieved in $a_t s_x T_s$;
- for $i \in [\![i_s, i_{s+1} - 1]\!]$, $\lim_{t\to\infty} \frac{1}{t} \log \lambda_i(a_t s_x \mathbb{Z}^d) = \Lambda_i$, where

$$\Lambda_i = \frac{1}{d-\ell} - (\frac{1}{\ell} + \frac{1}{d-\ell}) \frac{\phi_M(T_s) - \phi_M(T_{s-1})}{\dim T_s - \dim T_{s-1}}.$$

Theorem 12 (Subvarieties defined over $\overline{\mathbb{Q}}$)**.** *Let $M \subset X_\ell(\mathbb{R})$ be a connected analytic submanifold whose Zariski closure is defined over $\overline{\mathbb{Q}}$. The rational flag $\{0\} = T_0 < T_1 < \cdots < T_{r+1} = \mathbb{Z}^d$ associated to M determines all exponents $\beta_k(M)$: setting, for $s = 1, \dots, r$,*

$$i_s = \dim T_s \quad \textit{and} \quad j_s = \phi_M(T_s) = \min_{x \in M} \dim x \cap T_s,$$

one has, for $k = 1, \dots, \ell$, denoting $k_s = \min(j_s, k)$,

$$\gamma_k(x) = -\sum_{s=0}^{r} (k_{s+1} - k_s) \Lambda_{i_s+1} = \frac{-k}{d-\ell} + (\frac{1}{\ell} + \frac{1}{d-\ell}) \sum_{s=0}^{r} \frac{(k_{s+1} - k_s)(j_{s+1} - j_s)}{i_{s+1} - i_s}.$$

Proof. The proof is analogous to that of Theorem 8. The only difference is that one uses the above description of the asymptotic behavior of $(a_t s_x \mathbb{Z}^d)_{t>0}$ instead of Theorem 7. □

Conclusion

We conclude with several problems that could be studied to generalize the results obtained above. The first two already appear in Schmidt's original paper.

Number fields One may replace the field $\mathbb{Q}$ of rational numbers by an arbitrary real number field K, and study approximations of a real subspace by subspaces defined over K. Once a height has been defined on subspaces defined over K, the diophantien exponents $\beta_k(x)$, $k = 1, \dots, d-1$ of a point $x \in X_\ell(\mathbb{R})$ are defined exactly as in the case $K = \mathbb{Q}$. In order to adapt our method to this setting, one replaces the space of lattices $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{R})$ by the homogeneous space $\mathrm{SL}_d(K_S)/\mathrm{SL}_d(\mathcal{O}_K)$, where $\sigma\colon K \to K_S \simeq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ denotes the canonical embedding of K, and the ring of integers $\mathcal{O}_K$ of K is embedded in K_s via σ. This allows one to derive a version of Dirichlet's principle analogous to Theorem 1, and one then observes that the critical exponent $\frac{d}{k(d-\ell)}$ for β_k is independent of the number field K, a statement conjectured by Schmidt. Moreover, it is a symmetric function of k and ℓ. (This does not appear in the formula because we assumed from the ouset that $k \leq \ell$.)

Approximation on an intermediate dimension As before, we assume $1 \leq k \leq \ell < d$. Fix an integer $j \leq k$. For x in $X_\ell(\mathbb{R})$ and y in $X_k(\mathbb{R})$, Schmidt defines

$$\psi_j(y, x) = \min\{d(y^j, x) \mid y^j \subset y \text{ of dimension } j\}.$$

The quantities $\psi_j(y, x)$, $j = 1, \dots, k$, give a full description of the position of y with respect to x. Schmidt suggested to study diophantine approximation in terms of the functions $\psi_j(y,x)$, $j=1,\dots,k$. In the case $j=k$, one of course has $\psi_k(y,x)=d(y,x)$, so this corresponds to the problem we studied here. But in general, the question of the optimal exponent in Dirichlet's principle is largely open. We note that Elio Joseph [14, théorème 3.3] recently solved this problem in the case $d=4$, $\ell=k=2$ et $j=1$, and that Moshchevitin [23] has obtained some first results for metric diophantine approximation in this setting. We refer the reader to Joseph [14, chapitre 1] for a detailed historical introduction to these problems.

Spectrum of exponents We saw that the set of values taken by the exponent $\beta_k(x)$ as x varies in $X_\ell(\mathbb{R})$ is equal to the interval $[\frac{d}{k(d-\ell)},+\infty[$. More generally, it would be interesting to determine the set of values that can be achieved by the d-tuple of exponents $(\beta_k(x))_{1\leq k\leq d}$ as x varies in $X_\ell(\mathbb{R})$. Proposition 2 gives a partial result in that direction.

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