Borelian subgroups of simple Lie groups

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Abstract

We prove that in a simple real Lie group, there is no Borel measurable dense subgroup of intermediate Hausdorff dimension.

1 Introduction

The main purpose of the present paper is to prove the following theorem.

Theorem 1.1. *Let G be a connected simple real Lie group endowed with a Riemannian metric. There is no Borel measurable dense subgroup of G with Hausdorff dimension strictly between 0 and $\dim G$.*

For the group $SU(2)$, Theorem 1.1 was proved by Lindenstrauss and Saxcé [11]. In contrast, it is shown in [13] that in a connected nilpotent Lie group G there exist dense Borel measurable subgroups of arbitrary dimension between 0 and $\dim G$.

The study of subgroups of Lie groups with intermediate Hausdorff dimension started with the work of Erdős and Volkmann [7], who constructed additive subgroups of the real line with arbitrary Hausdorff dimension between 0 and 1, and conjectured that any Borel subring of the reals has Hausdorff dimension 0. This conjecture was settled by Edgar and Miller [6] in 2002, and shortly afterwards, Bourgain [1, 2] provided an independent and more quantitative solution.

The proof of Theorem 1.1 given in this paper follows the strategy of "discretization" used by Bourgain in its solution to the Erdős-Volkmann Conjecture, and also yields the following more precise theorem.

Theorem 1.2. *Let G be a connected simple real Lie group endowed with a Riemannian metric. There exists a neighborhood U of the identity in G and a positive integer k such that for all $\sigma > 0$, there exists $\epsilon = \epsilon(\sigma) > 0$ such that the following holds.*
Suppose A is a Borel subset of U generating a dense subgroup of G and with Hausdorff dimension $\dim_H A \in [\sigma, \dim G - \sigma]$, then

$$\dim_H A^k \geq \dim_H A + \epsilon,$$

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where A^k denotes the set of all elements of G that can be written as products of k elements of A.

It should be noted that the assumption that the set A is Borel measurable cannot be omitted. Indeed, Davies [5] showed that there exist non-Borel subfields of the real line of arbitrary Hausdorff dimension (see also [8]); it is then easy to check that if F is a subfield of $\mathbb{R}$ of Hausdorff dimension α, then the subgroup $SL(2, F)$ in $SL(2, \mathbb{R})$ has Hausdorff dimension 3α.

The idea of "discretization" is to translate problems about Hausdorff dimension into combinatorial problems about covering numbers of sets by balls of some small fixed radius δ. For that, Katz and Tao [9] introduced the notion of (σ, ϵ)-set at scale δ, which is the natural discretized analog of sets of Hausdorff dimension σ. The study of Hausdorff dimension of product sets then consists into three steps: first, one proves a combinatorial statement about covering numbers of (σ, ϵ)-sets at scale δ, then one deduces from it a flattening statement for measures, and finally, using Frostman's Lemma, on derives an inequality on Hausdorff dimensions.

In the proof of Theorems 1.1 and 1.2, the combinatorial part is based on a discretized Product Theorem for simple Lie groups [14, Theorem 1.1]. A key point in this combinatorial analysis is to understand the set Ξ of "troublemakers" of a subset A in G. Roughly speaking, those are the elements ξ such that there exist large subsets A' and B' in A such that the product set $A'\xi B'$ is not much larger than A. By controlling the structure of approximate subgroups in G, we will show that if A is a (σ, ϵ)-set at scale δ, then the set Ξ is included in a union of few neighborhoods of cosets of closed subgroups of G. This observation will allow us to prove the expansion statement needed to derive flattening of measures.

The plan of the paper is as follows. In Section 2, we investigate the structure of approximate subgroups of G and derive some elementary lemmas about subgroup chunks. Section 3 is devoted to the proof of the combinatorial discretized version of Theorem 1.2. Finally, in Section 4, we prove a Flattening Lemma for Frostman measures, and carry out the applications to Hausdorff dimension of product sets.

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2 Approximate subgroups and subgroup chunks

2.1 Controlling approximate subgroups

We start by recalling some elementary facts from additive combinatorics. If A and B are subsets of a group G, we denote by AB the product set of A and B, i.e.

$$AB = \{ab\,;\, a \in A,\ b \in B\}.$$

Similarly, for $k \geq 1$, A^k denotes the set of elements that can be written as the product of k elements of A. An important definition for us will be that of an approximate subgroup, due to Tao [15].

Definition 2.1. Let G be a metric group, and $K \geq 1$ a parameter. A K-approximate subgroup of G is a subset of G satisfying

- A is symmetric and contains the identity.
- There exists a finite set X of cardinality at most K such that $AA \subset XA$.

In this paper, G will always denote a connected simple Lie group, endowed with a left-invariant Riemannian metric. If A is a bounded subset of G, and $\delta > 0$ is some small scale, we denote by $N(A, \delta)$ the minimal number of balls of radius δ needed to cover A.

For the application to the study of Hausdorff dimension of product sets, the following definition, due to Katz and Tao [9] is appropriate.

Definition 2.2. Let G be a real Lie group of dimension d. Given $\sigma \in (0, d)$ and $\epsilon > 0$, we say that a subset A in G is a (σ, ϵ)*-set at scale* δ if it satisfies

1. $N(A, \delta) \leq \delta^{-\sigma-\epsilon}$
2. For all $\rho \geq \delta$, for all x in G, $N(A \cap B(x, \rho), \delta) \leq \rho^{\sigma}\delta^{-\epsilon}N(A, \delta)$.

Remark 1. One should think of (σ, ϵ)-sets at scale δ as sets of Hausdorff dimension σ discretized at scale δ. The parameter ϵ quantifies what we lose in the discretization process.

Example 1. For $\sigma = \frac{\log 2}{\log 3}$ and any $\epsilon > 0$, the usual triadic Cantor set is a (σ, ϵ)-set at scale δ for all δ sufficiently small.

Given a connected simple Lie group G endowed with a Riemannian metric, we want to describe the structure of (σ, ϵ)-sets in G that are also $\delta^{-\epsilon}$-approximate subgroups. For that purpose, we make the following definition.

Definition 2.3. Let G be a Lie group, and fix O a neighborhood of 0 in the Lie algebra $\mathfrak{g}$ on which the exponential map is injective. Given a symmetric neighborhood U of the identity such that $U \subset \exp O$, we define a *subgroup chunk in* U to be a set of the form $U \cap \exp(O \cap \mathfrak{h})$, for some Lie subalgebra $\mathfrak{h} < \mathfrak{g}$.
Similarly, a *coset chunk in* U is a set of the form $U \cap g\exp(O \cap \mathfrak{h})$, for some Lie subalgebra $\mathfrak{h} < \mathfrak{g}$ and some element g in U.

Throughout the paper, if X is any subset of G and ρ some positive number, $X^{(\rho)}$ denotes the ρ-neighborhood of X in G, i.e.

$$X^{(\rho)} = \{x \in G \mid d(x, X) \leq \rho\}.$$

What will allow us to control approximate subgroups with subgroup chunks is the following discretized Product Theorem [14, Theorem 1.1].

Theorem 2.4 (Product Theorem). *Let G be a simple real Lie group of dimension d. There exists a neighborhood U of the identity in G such that the following holds.*
Given $\sigma \in (0, d)$, there exists $\tau = \tau(\sigma) > 0$ and $\epsilon_0 = \epsilon_0(\sigma) > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$, for all $\delta > 0$ sufficiently small, if $A \subset U$ is a set satisfying

1. $N(A, \delta) \leq \delta^{-\sigma-\epsilon}$
2. $\forall \rho \geq \delta$, $N(A, \rho) \geq \delta^{\epsilon}\rho^{-\sigma}$
3. $N(AAA, \delta) \leq \delta^{-\epsilon}N(A, \delta)$

then there exists a closed connected subgroup $H \subset G$, such that

$$A \subset H^{(\delta^{\tau})}.$$

Moreover, τ and ϵ_0 remain bounded away from zero when σ varies in a compact subset of $(0, d)$.

Remark 2. Note that if A is a (σ, ϵ)-set at scale δ, then it necessarily satisfies the first two conditions of the Product Theorem.

Remark 3. In the conclusion of the Product Theorem, we may of course assume that the closed connected subgroup H is maximal. If this is the case, then we know [14, Proposition 2.1] that, provided U has been chosen small enough, $H \cap U$ is just the subgroup chunk in U with Lie algebra $\mathfrak{h} = \operatorname{Lie} H$.

Given a (σ, ϵ)-set $\widetilde{H}$ that is also a $\delta^{-\epsilon}$-approximate subgroup, we know from the Product Theorem that $\widetilde{H}$ is included in a small neighborhood of a proper subgroup chunk. The purpose of the following lemma is to allow us to choose the subgroup chunk H' of minimal dimension that can control $\widetilde{H}$.

Lemma 2.5. *Let G be a simple Lie group of dimension d. There exists a neighborhood U of the identity in G such that the following holds.*
Given $\sigma \in (0, d)$ and $b \in (0, 1)$, there exist constants K_ℓ and $\tau_\ell = \tau_\ell(\sigma, b) > 0$, for $\ell \in \{1, \ldots, d-1\}$, and $\epsilon_0 = \epsilon_0(\sigma) > 0$ such that the following holds for any $\epsilon \in (0, \epsilon_0)$ and any $\delta > 0$ small enough.
Suppose $\widetilde{H} \subset U$ is a (σ, ϵ)-set at scale δ and a $\delta^{-\epsilon}$-approximate subgroup.
There exists ℓ in $\{1, \ldots, d-1\}$, a subgroup chunk H' in U^4 of dimension ℓ and a subset $\widetilde{H}' \subset H'^{(\tau_\ell)}$, such that:

1. *There is a finite set X of cardinality at most $\delta^{-K_\ell\epsilon}$ such that* $\widetilde{H} \subset X\widetilde{H}' \cap \widetilde{H}'X$.
2. *If D is any coset chunk in U such that* $\dim D < \ell$*, then* $N(\widetilde{H}' \cap D^{(\delta^{b\tau_\ell})}, \delta) \leq \delta^{8K_\ell\epsilon}N(\widetilde{H}, \delta)$.

Proof. Choose a symmetric neighborhood U, and parameters τ and ϵ_0 for which the Product Theorem 2.4 holds. We also assume that U^4 is still an exponential neighborhood of the identity in G.
Then, let $K_\ell = 2 \cdot 10^{d-\ell}$ and $\tau_\ell = (\frac{b}{3})^{d-\ell}\tau$, and choose ℓ maximal such that for any coset chunk D of dimension less than ℓ,

$$N(\widetilde{H} \cap D^{(\delta^{\frac{b\tau_\ell}{2}})}, \delta) \leq \delta^{9K_\ell \epsilon} N(\widetilde{H}, \delta).$$

By the Product Theorem 2.4 and Remark 3, there exists a proper subgroup chunk H_0 in U such that $\widetilde{H} \subset H_0^{(\delta^\tau)}$. This shows that $\ell \leq d-1$.
On the other hand, coset chunks of dimension 0 are just points, and using that $\widetilde{H}$ is a (σ, ϵ)-set at scale δ, we see that, provided ϵ is sufficiently small, for any x in G, one has,

$$N(\widetilde{H} \cap B(x, \delta^{\frac{b\tau_1}{2}}), \delta) \leq \delta^{\sigma\frac{b\tau_1}{2} - \epsilon} N(\widetilde{H}, \delta) \leq \delta^{9K_1\epsilon} N(\widetilde{H}, \delta).$$

So we also have $\ell \geq 1$.
By maximality of ℓ, there exists an ℓ-dimensional coset chunk C in U such that

$$N(\widetilde{H} \cap C^{(\delta^{\frac{b\tau_{\ell+1}}{2}})}, \delta) = N(\widetilde{H} \cap C^{(\delta^{\frac{3\tau_\ell}{2}})}, \delta) \geq \delta^{9K_{\ell+1}\epsilon} N(\widetilde{H}, \delta).$$

Writing $C = gH'$ for some subgroup chunk H' and some element g in U, one readily sees that, for some constant L depending only on U,

$$N(\widetilde{H}^2 \cap H'^{(L\delta^{\frac{3\tau_\ell}{2}})}, \delta) \geq \delta^{9K_{\ell+1}\epsilon} N(\widetilde{H}, \delta) \geq \delta^{\frac{9}{10}K_\ell \epsilon} N(\widetilde{H}, \delta).$$

Note that we allow ourselves here a slight abuse of notation, denoting by H' both the subgroup chunk in U and the subgroup chunk in U^2.
Let $A = \widetilde{H}^2 \cap H'^{(L\delta^{\frac{3\tau_\ell}{2}})}$ and $B = \widetilde{H}$. We have

$$N(AB, \delta) \leq N(\widetilde{H}^3, \delta) \leq \delta^{\epsilon} N(\widetilde{H}, \delta) \leq \delta^{(K_\ell - 1)\epsilon} N(A, \delta),$$

so that by Rusza's Covering Lemma (see below Lemma 2.6), we find that there exists a finite set X of cardinality at most $\delta^{-K_\ell \epsilon}$ such that

$$\widetilde{H} \subset X\widetilde{H}' \cap \widetilde{H}'X,$$

where $\widetilde{H}'$ is a neighborhood of size $O(\delta)$ of the set $(\widetilde{H}^2 \cap H'^{(L\delta^{\frac{3\tau_\ell}{2}})})^2$. Provided δ is sufficiently small, we have $\widetilde{H}' \subset H'^{(\delta^{\tau_\ell})}$, where H' now stands for the subgroup chunk in U^4.

It remains to check Condition 2. Using that $\widetilde{H}$ is a $\delta^{-\epsilon}$-approximate subgroup, one sees that $\widetilde{H}'$ can be covered by at most $\delta^{-4\epsilon}$ translates of neighborhoods of $\widetilde{H}$ of size $O(\delta)$:

$$\widetilde{H}' \subset \bigcup_{i=1}^{\delta^{-4\epsilon}} x_i \widetilde{H}^{(O(\delta))}.$$

Let D be any coset chunk in U of dimension less than ℓ.
For each i, we have,

$$N(x_i\widetilde{H}^{(O(\delta))}\cap D^{(\delta^{b\tau_\ell})},\delta) = N(\widetilde{H}^{(O(\delta))}\cap x_i^{-1}D^{(\delta^{b\tau_\ell})},\delta) \leq N(\widetilde{H}\cap x_i^{-1}D^{(\delta^{\frac{b\tau_\ell}{2}})},\delta),$$

and therefore, by assumption on ℓ,

$$N(x_i\widetilde{H}^{(O(\delta))}\cap D^{(\delta^{b\tau_\ell})},\delta) \leq \delta^{9K_\ell\epsilon}N(\widetilde{H},\delta).$$

This shows that

$$N(\widetilde{H}'\cap D^{(\delta^{b\tau_\ell})},\delta) \leq \delta^{-4\epsilon}\delta^{9K_\ell\epsilon}N(\widetilde{H},\delta) \leq \delta^{8K_\ell\epsilon}N(\widetilde{H},\delta).$$

□

For convenience of the reader, we now give the version of Ruzsa's Covering Lemma we used in the above proof.

Lemma 2.6 (Ruzsa Covering Lemma)**.** *Let G be a Lie group and U a compact neighborhood of the identity. There exists a positive constant L such that the following holds for any parameter $K \geq 1$.*
Suppose A and B are subsets of U such that $N(AB,\delta) \leq KN(A,\delta)$. Then there exists a finite set X of cardinality at most LK such that B is included in the neighborhood of size $L\delta$ of $A^{-1}AX$. Similarly, if $N(BA,\delta) \leq KN(A,\delta)$, there exists a finite set Y of cardinality at most LK such that B is included in the neighborhood of size $L\delta$ of YAA^{-1}.

Proof. Let $X = \{x_1,\dots,x_s\}$ be maximal among subsets of B such that for each $i \neq j$, the translates Ax_i and Ax_j are away from each other by at least 2δ, in the sense that

$$\forall x \in Ax_i,\ \forall y \in Ax_j,\ d(x,y) > 2\delta.$$

Let $L_1 > 0$ such that left and right translations by elements of U are L_1-bi-Lipschitz on U. For each i, we have

$$N(Ax_i,\delta) \geq N(A,L_1\delta) \gg N(A,\delta).$$

The set AB contains all the translates Ax_i, and those are 2δ separated, so we find

$$N(AB,\delta) \geq \sum N(Ax_i,\delta) \gg (\operatorname{card} X)N(A,\delta),$$

and therefore,

$$\operatorname{card} X \ll K.$$

On the other hand, by maximality of X, if b is any element of B, there exists an element x_i is X such that Ab meets the neighborhood of size 2δ of Ax_i. This shows that $d(b,A^{-1}Ax_i) \leq 2L_1\delta$ and thus,

$$B \subset A^{-1}AX^{(2L_1\delta)}.$$

□

2.2 Intersections of neighborhoods

For us, an important property of neighborhoods of coset chunks is that they are stable under intersection. Recall that if X is a subset of G, then $X^{(\rho)}$ denotes the ρ-neighborhood of X. The lemma we will need is as follows.

Lemma 2.7. *Let G be a real Lie group. There exists a neighborhood U of the identity in G and constants $a, b > 0$ such that for all $\rho > 0$ sufficiently small, for any two coset chunks C_1 and C_2 in U, satisfying $C_1 \not\subset C_2^{(\rho^a)}$, we have*

$$C_1^{(\rho)} \cap C_2^{(\rho)} \subset C_0^{(\rho^b)},$$

for some coset chunk C_0 in U with $\dim C_0 < \dim C_1$.

The proof goes into three steps. First, we study intersections of linear subspaces in a Euclidean space, then we consider intersections of subalgebras of a Lie algebra, and finally, we prove Lemma 2.7.

Definition 2.8. Given two subspaces V_1 and V_2 of a Euclidean space E, we define the distance from V_1 to V_2 by

$$d(V_1, V_2) = \sup\{d(v, V_2)\,;\, v \text{ unit vector in } V_1\}.$$

Note that d does not define a distance on the set of subspaces of E, as $d(V_1, V_2) = 0$ just means that V_1 is included in V_2.

Lemma 2.9. *Let d be a positive integer. There exists a constant $c_0 = c_0(d) > 0$ such that if E is a Euclidean space of dimension d, the following holds for any $\rho > 0$ small enough.*
Suppose V_1 and V_2 are two proper subspaces of E such that $d(V_1, V_2) \geq \rho^{c_0}$. Then there exists a nonnegative integer $\ell < \dim V_1$, a constant $c \geq c_0$, and an orthogonal family $(u_i)_{1\leq i\leq \ell}$ of unit vectors such that,

$$\forall i,\ u_i \in V_1 \cap V_2^{(\rho^c)} \tag{1}$$

and

$$B_E(0,1) \cap V_1 \cap V_2^{(\rho^{\frac{3c}{4}})} \subset V^{(\rho^{\frac{c}{6}})},$$

where $V = \operatorname{Span}(u_i)_{1\leq i\leq \ell}$.

Proof. We will prove the lemma with constant $c_0 = 2^{-d-1}$.
Let $\ell \geq 0$ be maximal such that there exists an orthonormal family $(u_i)_{1\leq i\leq \ell}$ of vectors in V_1 such that

$$\forall i \in \{1, \dots, \ell\}, \quad d(u_i, V_2) \leq \rho^{2^{-\ell}}.$$

The assumption $d(V_1, V_2) \geq \rho^{c_0}$ ensures that $\ell < \dim V_1$. Choosing $c = 2^{-\ell}$, the u_i's certainly satisfy condition (1).

Now let v be a vector in $B_E(0,1) \cap V_1 \cap V_2^{(\rho^{\frac{3c}{4}})}$. Write $v = \lambda_1 u_1 + \cdots + \lambda_\ell u_\ell + v'$, with v' in $V_1 \cap (u_1, \ldots, u_\ell)^\perp$. We have, provided ρ is small enough,

$$d(v', V_2) = d(v - \sum \lambda_i u_i, V_2) \leq \rho^{\frac{3c}{4}} + \ell \rho^c \leq \rho^{\frac{2c}{3}},$$

and therefore, by maximality of ℓ,

$$\rho^{\frac{2c}{3}} \geq \|v'\| d(\frac{v'}{\|v'\|}, V_2) \geq \|v'\| \rho^{\frac{c}{2}}$$

which implies

$$d(v, V_2) = \|v'\| \leq \rho^{\frac{c}{6}}.$$

This shows that $B_E(0,1) \cap V_1 \cap V_2^{(\rho^{\frac{3c}{4}})} \subset V^{(\rho^{\frac{c}{6}})}$. $\square$

The next step, passing from linear subspaces to Lie subalgebras, is an application of Łojasiewicz's inequality.

Lemma 2.10. *Let $\mathfrak{g}$ be a real Lie algebra endowed with a Euclidean metric. There exist positive constants a and b such that for all $\rho > 0$ small enough, we have the following. Let $\mathfrak{h}_1$ and $\mathfrak{h}_2$ be two Lie subalgebras of $\mathfrak{g}$, and assume that $d(\mathfrak{h}_1, \mathfrak{h}_2) \geq \rho^a$. Then, there exists a Lie subalgebra $\mathfrak{h}$ such that* $\dim \mathfrak{h} < \dim \mathfrak{h}_1$ *and*

$$B_{\mathfrak{g}}(0,1) \cap \mathfrak{h}_1^{(\rho)} \cap \mathfrak{h}_2^{(\rho)} \subset \mathfrak{h}^{(\rho^b)}.$$

Proof. For each ℓ in $\{1, \ldots, d\}$, the variety M_ℓ of orthogonal ℓ-tuples of unit vectors in $\mathfrak{g}$ is compact and real analytic. We define a real-valued function f on M_ℓ by

$$f(u_1, \ldots, u_\ell) = \sum_{1 \leq i < j \leq \ell} d([u_i, u_j], \mathrm{Span}(u_i)_{1 \leq i \leq \ell})^2,$$

where $[,]$ denotes the Lie bracket in $\mathfrak{g}$.
Note that $f(u_1, \ldots, u_\ell) = 0$ if and only if $\mathrm{Span}(u_i)$ is stable under Lie brackets, i.e. if and only if the ℓ-tuple (u_i) is the basis of a Lie subalgebra of $\mathfrak{g}$. The function f is real-analytic so that by the Łojasiewicz inequality [10, Théorème 2, page 62], there exists a constant C such that for r small enough,

$$f(u_1, \ldots, u_\ell) \leq r \quad \Longrightarrow \quad d((u_i), Z_f) \leq r^{\frac{1}{C}},$$

where Z_f is the zero set of f. In other terms, if $f(u_1, \ldots, u_\ell) \leq r$, then there exists a Lie subalgebra $\mathfrak{h}$ of dimension ℓ such that $d(\mathrm{Span}(u_i), \mathfrak{h}) \leq r^{\frac{1}{C}}$.
Let c_0 be the constant from Lemma 2.9, and let $a = c_0$.
Now suppose $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two subalgebras as in the lemma. Choose $\ell < \dim \mathfrak{h}_1$, $c \geq c_0$ and a orthonormal family $(u_i)_{1 \leq i \leq \ell}$ as given by Lemma 2.9. For each $i < j$, we have u_i and u_j are in $\mathfrak{h}_1$, so that $[u_i, u_j] \in \mathfrak{h}_1$. Moreover, u_i and u_j are in $\mathfrak{h}_2^{(\rho^c)}$ so, for some constant L depending only on $\mathfrak{g}$, we have $[u_i, u_j] \in \mathfrak{h}_2^{(L\rho^c)}$. Thus, for $\rho > 0$ small enough, $[u_i, u_j] \in \mathfrak{h}_1 \cap \mathfrak{h}_2^{(\rho^{\frac{3c}{4}})} \subset (\mathrm{Span}(u_i))^{(\rho^{\frac{c}{6}})}$, and

$$f(u_1, \ldots, u_\ell) \leq d^2 \rho^{\frac{c}{6}} \leq \rho^{\frac{c}{7}}.$$

Therefore, there exists a Lie subalgebra $\mathfrak{h}$ of dimension ℓ such that $d(\mathrm{Span}(u_i), \mathfrak{h}) \leq \rho^{\frac{c}{7C}}$. However, by definition of the u_i's, the intersection $B_{\mathfrak{g}}(0,1) \cap \mathfrak{h}_1^{(\rho)} \cap \mathfrak{h}_2^{(\rho)}$ is included in $(\mathrm{Span}(u_i))^{(\rho+\rho^{\frac{c}{6}})}$, so that, setting $b = \frac{c_0}{8C}$, we indeed get, provided ρ is small enough,

$$\mathfrak{h}_1^{(\rho)} \cap \mathfrak{h}_2^{(\rho)} \subset \mathfrak{h}^{(\rho^b)}.$$

□

The above Lemma 2.10 can of course be reformulated in terms of subgroup chunks, and thus allows to prove Lemma 2.7.

Proof of Lemma 2.7. Suppose without loss of generality that the intersection $g_1 H_1^{(\rho)} \cap g_2 H_2^{(\rho)}$ is nonempty, and fix $g_0 \in g_1 H_1^{(\rho)} \cap g_2 H_2^{(\rho)}$. Then we have, for some constant L depending only on the compact neighborhood U,

$$g_0 H_1^{(L\rho)} \supset g_1 H_1^{(\rho)} \quad \text{and} \quad g_0 H_2^{(L\rho)} \supset g_2 H_2^{(\rho)}.$$

All we have to show is that $g_0 H_1^{(L\rho)} \cap g_0 H_2^{(L\rho)}$ is included in the ρ^b-neighborhood of some coset chunk. From $g_1 H_1 \not\subset g_2 H_2^{(\rho^a)}$, we have $g_0 H_1 \not\subset g_0 H_2^{(\frac{\rho^a}{2})}$ whence $H_1 \not\subset H_2^{(\rho^{\frac{\rho^a}{2}})}$. Therefore, by Lemma 2.10 – adjusting slightly the values of a and b –, there exists a subgroup chunk H_0 of dimension less than $\dim H_1$ such that

$$H_1^{(L\rho)} \cap H_2^{(L\rho)} \subset H_0^{(\rho^b)},$$

and this allows us to concude that

$$g_1 H_1^{(\rho)} \cap g_2 H_2^{(\rho)} \subset g_0 H_0^{(\rho^b)}.$$

□

3 The set Ξ of troublemakers

Let G be a Lie group and U a compact neighborhood of the identity. If $A \subset U$ is a (σ, ϵ)-set at scale δ in a Lie group G, we associate to it the set Ξ of *troublemakers* for A, defined as

$$\Xi = \left\{ \xi \in U \;\middle|\; \exists \Omega \subset A \times A \text{ with } \begin{array}{l} N(\Omega, \delta) \geq \delta^{\epsilon} N(A,\delta)^2 \\ \text{and } N(\pi_\xi(\Omega), \delta) \leq \delta^{-\epsilon} N(A, \delta) \end{array} \right\}, \qquad (2)$$

where $\pi_\xi : (x, y) \mapsto x\xi y$. Roughly speaking, an element ξ is a troublemaker for the set A if there exist large portions A' and B' of A such that the product set $A'\xi B'$ is not much larger than A.

Example 2. Suppose $A = H \cap U$ for some closed subgroup of the Lie group G. If $\xi \in U$ is any element of the normalizer $N_G(H)$ of H, we have $A\xi A \subset H\xi \cap U^3$, which has roughly the same size as A. So Ξ contains $N_G(H) \cap U$.

3.1 Controlling the troublemakers

The purpose of this subsection is to show that if A is a (σ, ϵ)-set in a simple Lie group G, then the set of troublemakers for A is included in a small number of neighborhoods of cosets of proper closed subgroups of G. Our aim is the following.

Proposition 3.1. *Let G be a simple Lie group. There exists a neighborhood U of the identity in G such that, given $\sigma \in (0, \dim G)$, there exist constants $\eta = \eta(\sigma) > 0$ and $\epsilon_1 = \epsilon_1(\sigma) > 0$ such that the following holds for any $\epsilon \in (0, \epsilon_1)$ and any $\delta > 0$ small enough.*
If $A \subset U$ is a (σ, ϵ)-set at scale δ, then the set Ξ of troublemakers for A, defined as in (2), is included in a union of at most $\delta^{-O(\epsilon)}$ neighborhoods of size δ^{η} of coset chunks in U.
Moreover, η and ϵ_1 remain bounded away from 0 when σ varies in a compact subset of $(0, \dim G)$.

First, we recall the following proposition on "almost stabilizers" of subspaces in the adjoint representation of a simple Lie group [14, Proposition 2.7].

Proposition 3.2. *Let G be a simple Lie group with trivial center. There exists a neighborhood U of the identity in G, and a constant $c > 0$ such that for all $\rho > 0$ small enough, the following holds.*
For each proper subspace $V < \mathfrak{g}$, there exists a proper closed connected subgroup S_V such that for all ξ in U,

$$d((Ad\xi)V, V) \leq \rho \quad \Longrightarrow \quad d(\xi, S_V) \leq \rho^c.$$

That proposition has the following corollary.

Corollary 3.3. *Let G be a simple Lie group. There exists a neighborhood U of the identity and a constant $c > 0$ such that the following holds for any $\rho > 0$ sufficiently small.*
For any two proper subgroup chunks H and R of same dimension in U, there exists a coset chunk C in U such that, for all ξ in U,

$$\xi H \xi^{-1} \subset R^{(\rho)} \quad \Longrightarrow \quad \xi \in C^{(\rho^c)}.$$

Proof. Since the statement only involves a neighborhood of the identity, it is enough to prove it in the case the group G has trivial center. Choose a neighborhood U of the identity and a constant $c > 0$ such that Proposition 3.2 holds for UU^{-1} and $2c$.
Given two subgroup chunks H and R having the same dimension, assume that for some ξ_0 in U, $\xi_0 H \xi_0^{-1}$ is included in $R^{(\rho)}$. If ξ is another element satisfying $\xi H \xi^{-1} \subset R^{(\rho)}$, we have, for some constant L depending only on U,

$$(\xi_0^{-1}\xi)H(\xi\xi_0^{-1})^{-1} \subset H^{(L\rho)}.$$

and this implies, if $\mathfrak{h}$ denotes the Lie algebra of H,

$$d((\mathrm{Ad}\,\xi_0^{-1}\xi)\mathfrak{h}, \mathfrak{h}) \leq L\rho.$$

From Proposition 3.2, it follows that there exists a closed subgroup S such that for ρ small enough,

$$\xi H\xi^{-1} \subset R^{(\rho)} \quad \Longrightarrow \quad \xi \in \xi_0 S^{(L\rho^{2c})} \subset \ \xi_0 S^{(\rho^c)}.$$

This proves the lemma. $\square$

The proof of Proposition 3.1 is based on the following lemma, which is an application of the inclusion-exclusion principle.

Lemma 3.4. *Let G be a Lie group, and fix a neighborhood U of the identity and constants a and b as given by Lemma 2.7.*
Let $\sigma \in (0,d)$ and fix an integer $\ell \in \{1,\dots,d-1\}$. Let A be any subset of U, and $\delta > 0$ some small scale. Given two parameters K and $\tau > 0$, we are interested in subsets $\widetilde{C}$ of U such that:

1. $\delta^{K\epsilon} N(A,\delta) \leq N(\widetilde{C}\cap A,\delta)$
2. *We have $\widetilde{C} \subset C^{(\delta^\tau)}$, for some coset chunk C of dimension ℓ.*
3. *If D is any coset chunk such that $\dim D < \ell$, then* $N(\widetilde{C}\cap D^{(\delta^{b\tau})},\delta) \leq \delta^{4K\epsilon} N(A,\delta)$.

Let $\mathcal{C}$ be the union of all such sets $\widetilde{C}$. Then, there exists a family $(C_i)_{1\leq i\leq 2\delta^{-K\epsilon}}$ of coset chunks of dimension ℓ such that

$$\mathcal{C} \subset \bigcup_{i=1}^{2\delta^{-K\epsilon}} C_i^{(\delta^{a\tau})}.$$

Proof. Choose successively sets $\widetilde{C}_i$, $i \geq 1$ satisfying all requirements of the lemma – in particular $\widetilde{C}_i \subset C_i^{(\delta^\tau)}$ for some ℓ-dimensional coset chunk C_i – and such that for each i,

$$C_{i+1}^{(\delta^\tau)} \not\subset \bigcup_{k=1}^{i} C_k^{(\delta^{a\tau})}.$$

Clearly, this procedure must stop, and when it does, we obtain a finite family $(C_i)_{1\leq i\leq N}$ of coset chunks such that

$$\mathcal{C} \subset \bigcup_{i=1}^{N} C_i^{(\delta^{a\tau})}.$$

It remains to check that $N \leq 2\delta^{-K\epsilon}$. For that, first note that for all $1 \leq i < j \leq N$, by Lemma 2.7, there exists a coset chunk D with $\dim D < \ell$ such that $C_i^{(\delta^\tau)} \cap C_j^{(\delta^\tau)} \subset D^{(\delta^{b\tau})}$. In particular,

$$N(\widetilde{C}_i \cap \widetilde{C}_j,\delta) \leq N(\widetilde{C}_j \cap C_i^{(\delta^\tau)} \cap C_j^{(\delta^\tau)},\delta) \leq N(\widetilde{C}_j \cap D^{(\delta^{b\tau})},\delta),$$

whence, using the third assumption on $\widetilde{C_j}$,

$$N(\widetilde{C}_i \cap \widetilde{C}_j,\delta) \leq \delta^{4K\epsilon} N(A,\delta).$$

Now, as A certainly contains $\bigcup_{i=1}^N A\cap\widetilde{C}_i$, we find

$$\begin{aligned}N(A,\delta) &\geq N(A\cap\widetilde{C}_1,\delta)+(N(A\cap\widetilde{C}_2,\delta)-N(A\cap\widetilde{C}_1\cap\widetilde{C}_2,\delta))\\ &\quad+(N(A\cap\widetilde{C}_3,\delta)-N(A\cap\widetilde{C}_1\cap\widetilde{C}_3,\delta)-N(A\cap\widetilde{C}_2\cap\widetilde{C}_3,\delta))+\dots\\ &\geq \delta^{K\epsilon}N(A,\delta)\left[1+(1-\delta^{3K\epsilon})+(1-2\delta^{3K\epsilon})+\dots\right],\end{aligned}$$

keeping only the first $\min(N,\delta^{-3K\epsilon})$ terms in the sum. The terms on the right-hand side of the above inequality are non-negative and form an arithmetic progression, so that we get the lower bound

$$N(A,\delta)\geq \delta^{K\epsilon}\frac{1}{2}\min(N,\delta^{-3K\epsilon})N(A,\delta).$$

This forces $\min(N,\delta^{-3K\epsilon})=N$ and in turn,

$$N\leq 2\delta^{-K\epsilon}.$$

□

Proof of Proposition 3.1. A. Choose a symmetric neighborhood U such that both Lemma 2.5 and Lemma 3.7 hold in the neighborhood U^4. Let ξ be an element of Ξ. From the non-commutative version of the Balog-Szemerédi-Gowers Lemma, due to Tao [15, Theorem 6.10], there exists a constant $K\geq 2$ such that there exists a $\delta^{-K\epsilon}$-approximate subgroup $\widetilde{H}$ and elements x,y in G such that

$$\delta^{K\epsilon}N(A,\delta)\leq N(x\widetilde{H}\cap A,\delta)\leq N(\widetilde{H},\delta)\leq \delta^{-K\epsilon}N(A,\delta)$$

and

$$\delta^{K\epsilon}N(A,\delta)\leq N(\xi^{-1}\widetilde{H}y\cap A,\delta)\leq N(\widetilde{H},\delta)\leq \delta^{-K\epsilon}N(A,\delta).$$

B. First, we claim that $\widetilde{H}$ is a $(\sigma,3K\epsilon)$-set at scale δ.
Indeed, suppose for a contradiction that for some ball B_ρ of radius $\rho\geq\delta$, we have

$$N(\widetilde{H}\cap B_\rho,\delta)>\delta^{-3K\epsilon}\rho^\sigma N(\widetilde{H},\delta).$$

Then,

$$N((\widetilde{H}\cap B_\rho)\widetilde{H},\delta)\leq N(\widetilde{H}^2,\delta)\leq\delta^{-K\epsilon}N(\widetilde{H},\delta),$$

so that

$$N((\widetilde{H}\cap B_\rho)\widetilde{H},\delta)\leq\delta^{2K\epsilon}\rho^{-\sigma}N(\widetilde{H}\cap B_\rho,\delta).$$

Applying the Covering Lemma 2.6 to the sets $\widetilde{H}\cap B_\rho$ and $\widetilde{H}$, we find that for some constant L depending on U only, there is a set W of cardinality at most $L\delta^{2K\epsilon}\rho^{-\sigma}$ such that,

$$\widetilde{H}\subset W\cdot(\widetilde{H}^2\cap B(1,L\rho))\subset\bigcup_{w\in W}B(w,L\rho).$$

Recalling that $N(x\widetilde{H}\cap A,\delta)\geq\delta^{K\epsilon}N(A,\delta)$, we see that for some w in W, we have

$$N(A\cap B(xw,L\rho),\delta)\geq\frac{1}{\operatorname{card}W}\delta^{K\epsilon}N(A,\delta)\geq\frac{1}{L}\delta^{-K\epsilon}\rho^\sigma N(A,\delta),$$

contradicting the fact that A is (σ,ϵ)-set at scale δ, since for δ small enough,

$$\frac{1}{L}\delta^{-K\epsilon}\rho^{\sigma} > (L\rho)^{\sigma}\delta^{-\epsilon}.$$

C. Now, let ϵ_0, K_ℓ and τ_ℓ, $1 \le \ell \le d-1$, be as in Lemma 2.5. Provided $\epsilon < \epsilon_1 := \frac{\epsilon_0}{2K}$, Lemma 2.5 shows that there is an integer $\ell \in \{1,\dots,d-1\}$ and a subgroup chunk H' of dimension ℓ such that, for some set $\widetilde{H}' \subset H'^{(\tau_\ell)}$,

1. There is a finite set X of cardinality at most $\delta^{-3K_\ell K\epsilon}$ such that $\widetilde{H} \subset X\widetilde{H}' \cap \widetilde{H}'X$.
2. If D is any coset chunk in U such that $\dim D < \ell$, then $N(\widetilde{H}' \cap D^{(\delta^{b\tau_\ell})},\delta) \le \delta^{24K_\ell K\epsilon}N(\widetilde{H},\delta) \le \delta^{23K_\ell K\epsilon}N(A,\delta)$.

We have $x\widetilde{H} \subset xX\widetilde{H}'$, and recalling that $N(x\widetilde{H}\cap A,\delta) \ge \delta^{K\epsilon}N(A,\delta)$, we find that for some x' in xX,

$$N(x'\widetilde{H}'\cap A,\delta) \ge \frac{1}{\operatorname{card} X}\delta^{K\epsilon}N(A,\delta) \ge \delta^{4K_\ell K\epsilon}N(A,\delta).$$

Similarly, there exists y' such that

$$N(\xi^{-1}\widetilde{H}'y'\cap A,\delta) \ge \delta^{4K_\ell K\epsilon}N(A,\delta).$$

Denote by $\mathcal{C}$ the union of all subsets $\widetilde{C}$ of U^4 satisfying the conditions of Lemma 3.4, with constants $K' = 4K_\ell K$ and $\tau = \tau_\ell$. By Lemma 3.4, there is a family of coset chunks $(C_i)_{1\le i\le 2\delta^{-4K_\ell K\epsilon}}$ such that

$$\mathcal{C} \subset \bigcup_{i=1}^{2\delta^{-4K_\ell K\epsilon}} C_i^{(\delta^{a\tau})}.$$

D. As $x'\widetilde{H}'$ and $\xi^{-1}\widetilde{H}'y'$ both satisfy the conditions of Lemma 3.4, there must exist indices i and j such that

$$x'\widetilde{H}' \subset C_i^{(\delta^{a\tau})} \quad\text{and}\quad \xi^{-1}\widetilde{H}'y' \subset C_j^{(\delta^{a\tau})}. \tag{3}$$

Denote by H_i the left-direction of C_i, i.e. the subgroup chunk such that there exists x_i such that $C_i = x_iH_i$, and by R_j the right-direction of C_j. From (3), we get

$$\xi^{-1}H_i\xi \subset R_j^{(\delta^{a\tau})},$$

and therefore, by Corollary 3.3, for some small $c>0$ depending only on U,

$$\xi \in C_{i,j}^{(\delta^{ca\tau})},$$

where $C_{i,j}$ is a left-coset of some proper maximal closed subgroup in G.
Letting $\eta = ca\tau_{d-1} = \min_{1\le\ell\le d-1} ca\tau_\ell$ and considering all (at most $\delta^{-O(\epsilon)}$) cosets $C_{i,j}$ arising for some dimension $\ell\in\{1,\dots,d-1\}$, this proves the proposition. $\square$

3.2 Escaping from the troublemakers

We start by a lemma that will allow us to escape from hyperplanes in the adjoint representation.

Lemma 3.5. *Let G be a connected simple real Lie group of dimension d. There exist integers k and s such that for any set A containing 1 and generating a dense subgroup of G, there exists a finite family $(a_i)_{1\leq i\leq k}$ of elements of the product set A^s such that for any nonzero vector $v \in \mathfrak{g}$ and any hyperplane $V < \mathfrak{g}$, there exists an index i for which*

$$(\mathrm{Ad}\, a_i)v \notin V.$$

Proof. Let A be a topologically generating set of G, and fix (a, b) a topologically generating pair of elements of A (for the existence of such a pair, see for example [4]). By induction on $\ell < d$ we will show that for any nonzero vector v in $\mathfrak{g}$ and any subspace V of dimension ℓ, there exist elements $a_1, \dots, a_\ell$ in $\{1, a, b\}$ such that

$$a_\ell \dots a_1 \cdot v \notin V.$$

$\underline{\ell = 1}$
If $v \notin V$, just take $a_1 = 1$. Otherwise, $V = \mathbb{R}v$. The stabilizer $\mathrm{Stab}\, V$ of the line V is a proper closed subgroup of G and therefore we must have $a \notin \mathrm{Stab}\, V$ or $b \notin \mathrm{Stab}\, V$. This shows that $av \notin V$ or $bv \notin V$.
$\underline{\ell \to \ell + 1}$
Suppose we know the result for subspaces of dimension at most ℓ, and let V be a proper subspace of $\mathfrak{g}$ of dimension $\ell + 1 < d$. Let $V_1 = V \cap a^{-1}V \cap b^{-1}V$. Either a or b is out of the proper closed subgroup $\mathrm{Stab}\, V$, so we must have $\dim V_1 \leq \ell$. By the induction hypothesis, there exist elements $a_1, \dots, a_\ell$ in $\{1, a, b\}$ such that $a_\ell \dots a_1 \cdot v \notin V_1$. By definition of V_1 this shows that for some $a_{\ell+1}$ in $\{1, a, b\}$, we must have

$$a_{\ell+1} a_\ell \dots a_1 \cdot v \notin V.$$

This proves the lemma, with constants $s = d - 1$ and $k = 2^d - 1$ (the number of words of length at most $d - 1$ in a and b). □

The goal of this subsection is to show that one can always escape from the set Ξ of troublemakers of a (σ, ϵ)-set A, in the following precise sense.

Proposition 3.6. *Let G be a simple Lie group. There exists a neighborhood U of the identity such that, given $\sigma \in (0, d)$, there exists $\epsilon = \epsilon(\sigma) > 0$ such that the following holds.*
Suppose $\{a_i\}_{1\leq i\leq k}$ is a finite set of elements satisfying the conclusion of Lemma 3.5, let $n = d^d$ and consider the set Π of all projections $\pi : G^{\times n} \to G$ of the form

$$\pi(x_1, \dots, x_n) = x_{i_1} a_{j_1} x_{i_2} a_{j_2} \dots a_{j_{m-1}} x_{i_m},$$

where $\{i_1 < i_2 < \dots < i_m\} \subset \{1, \dots, n\}$ and $(j_1, \dots, j_{m-1}) \in \{1, \dots, k\}^{m-1}$. Then, for all $\delta > 0$ small enough, if $A \subset U$ is a (σ, ϵ)-set at scale δ, and if Ω

is any subset of the Cartesian product set $A^{\times n}$ such that $N(\Omega,\delta) \geq \delta^{\epsilon} N(A,\delta)^n$, then there exists some π in Π such that

$$\pi(\Omega) \not\subset \Xi,$$

where Ξ is the set of troublemakers for A, as defined in (2).

The proof of Proposition 3.6 will be based on a repeated application of the following lemma.

Lemma 3.7. *Let G be a simple Lie group. There exist a neighborhood U of the identity and a constant $b > 0$ such that given $c_0 > 0$, for all $\rho > 0$ small enough (in terms of c_0), the following holds.*
Let A be a subset of U, and Ω a subset of the Cartesian product set $A^{\times n}$ $(n \geq d)$. Assume that there exist coset chunks C_i, $1 \leq i \leq d$ in U such that

$$\Omega \subset C_1^{(\rho)} \times \cdots \times C_d^{(\rho)} \times U \times \cdots \times U.$$

For each i, write $C_i = g_i H_i$, for some subgroup chunk H_i and some element g_i in U, and suppose that there exist elements a_i, $1 \leq i \leq d-1$ in U and unit vectors $v_i \in \mathfrak{h}_i$, such that, denoting $t_i = a_i g_{i+1} a_{i+1} g_{i+2} \dots a_{d-1} g_d$, we have, for each i in $\{1, \dots, d-1\}$,

$$d((\operatorname{Ad} t_i^{-1}) v_i, \bigoplus_{j=i+1}^{d} \mathbb{R}(\operatorname{Ad} t_j^{-1}) v_j) \geq c_0. \tag{4}$$

Finally, let π be the map $G^{\times n} \to G$ defined by

$$\pi(x_1, \dots, x_n) = x_1 a_1 \dots x_{d-1} a_{d-1} x_d$$

and assume that for some proper coset chunk C in U,

$$\pi(\Omega) \subset C^{(\rho)}.$$

Then there exists an index i_0 in $\{1, \dots, d\}$ and elements $u_i \in U$, $1 \leq i \leq d$, $i \neq i_0$, such that the set

$$\Omega' = \{(x_{i_0}, x_{d+1}, \dots, x_n) \in A^{\times n-d+1} \,|\, (u_1, \dots, u_{i_0-1}, x_{i_0}, u_{i_0+1}, \dots, u_d, x_{d+1}, \dots, x_n) \in \Omega\}$$

satisfies

1. $N(\Omega', \delta) \geq \frac{N(\Omega,\delta)}{N(A,\delta)^{d-1}}$
2. $\Omega' \subset C'^{(\rho^b)} \times U \times \cdots \times U$, *for some coset chunk C' in U satisfying* $\dim C' < \max_{1 \leq i \leq d} \dim C_i$.

Proof. Choose U such that Lemma 2.7 holds.
Let $c_1 > 0$ be as in Lemma 3.8 below. Write $C = gH$ for some g in U and some subgroup chunk H with Lie algebra $\mathfrak{h}$. By Lemma 3.8 applied to the family $(u_i) = ((\operatorname{Ad} t_i)^{-1} v_i)$, there exists an index i in $\{1, \dots, d\}$ for which

$$d((\operatorname{Ad} t_i^{-1}) v_i, \mathfrak{h}) \geq c_1.$$

Fix a large constant $L \geq 2$, and let $i_0 \in \{1, \dots, d\}$ be maximal such that

$$d((\operatorname{Ad} t_{i_0}^{-1})\mathfrak{h}_{i_0}, \mathfrak{h}) \geq \frac{c_1}{L^{i_0}}.$$

Then choose any elements u_i, $i \neq i_0$, $1 \leq i \leq d$ so that the set Ω' defined in the lemma satisfies

$$N(\Omega', \delta) \geq \frac{N(\Omega, \delta)}{N(A, \delta)^{d-1}}.$$

Now let $(x_{i_0}, x_{d+1}, \dots, x_n)$ be an element of Ω'. We want to show that x_{i_0} stays in a neighborhood of size ρ^b of a coset chunk C' satisfying $\dim C' < \dim C_{i_0}$. This will follow from

$$x_{i_0} \in g_{i_0} H_{i_0}^{(\rho)} \quad \text{and} \quad u_1 a_1 \dots a_{i_0 - 1} x_{i_0} a_{i_0} \dots u_d \in g H^{(\rho)}.$$

For simplicity, denote $t = t_{i_0}$ and $s = t_{i_0}^{-1} a_{i_0} \dots u_d$, so that the above can be rewritten, for some u in U,

$$x_{i_0} \in g_{i_0} H_{i_0}^{(\rho)} \quad \text{and} \quad x_{i_0} \in u t s H^{(\rho)} s^{-1} t^{-1}.$$

To conclude, we want to apply Lemma 2.7, but for that, we need to check that the two coset chunks above are away from one another. For each i, write $u_i = g_i h_i$, for some h_i is the subgroup chunk H_i, so that

$$s = t_{i_0+1}^{-1} h_{i_0+1} t_{i_0+1} \dots t_{d-1}^{-1} h_{d-1} t_{d-1} h_d.$$

By maximality of i_0, we have, for each $i > i_0$, $d((\operatorname{Ad} t_i)^{-1}\mathfrak{h}_i, \mathfrak{h}) \leq \frac{c_1}{L^i}$, so that in particular, for some constant L_0 depending only on U,

$$\forall i > i_0, \quad d(t_i^{-1} h_i t_i, H) \leq \frac{L_0 c_1}{L^i}.$$

This shows that s is quite close to H:

$$d(s, H) \leq \sum_{i > i_0} \frac{L_0 c_1}{L^i} = \frac{2 L_0 c_1}{L^{i_0+1}},$$

which implies, for some constant L_1 depending only on U,

$$d(sHs^{-1}, H) \leq \frac{L_1 c_1}{L^{i_0+1}}.$$

On the other hand, by our choice of i_0, we have, for some constant L_2 depending on U only,

$$d(t^{-1} H_{i_0} t, H) \geq \frac{c_1}{L_2 L^{i_0}}.$$

Now, for some constant L_3 depending only on U,

$$\begin{aligned} d(H_{i_0}, tsHs^{-1}t^{-1}) &\geq \frac{1}{L_3} d(t^{-1} H_{i_0} t, sHs^{-1}) \\ &\geq \frac{1}{L_3} [d(t^{-1} H_{i_0} t, H) - d(H, sHs^{-1})], \end{aligned}$$

so that

$$
\begin{aligned}
d(H_{i_0}, tsHs^{-1}t^{-1}) &\geq \frac{1}{L_3}\left[\frac{c_1}{L_2 L^{i_0}} - \frac{L_1 c_1}{L^{i_0+1}}\right] \\
&\geq \frac{c_1}{L^{i_0+1}}
\end{aligned}
$$

provided L has been chosen larger than $L_2(L_3 + L_1)$.

To conclude, let a and b be the constants from Lemma 2.7. The above inequality ensures that for $\rho > 0$ sufficiently small,

$$d(H_{i_0}, tsHs^{-1}t^{-1}) \geq \rho^a,$$

so that there exists a coset chunk C' with $\dim C' < \dim C_{i_0}$ and

$$g_{i_0} H_{i_0}^{(\rho)} \cap utsH^{(\rho)}s^{-1}t^{-1} \subset C'^{(\rho^b)}.$$

Thus,

$$\Omega' \subset C'^{(\rho^b)} \times A \times \cdots \times A,$$

and the lemma is proven. □

At the beginning of the above proof, we made use of the following easy lemma.

Lemma 3.8. *Let E be a Euclidean vector space of dimension d. Given $c_0 > 0$, there exists a constant $c_1 > 0$ such that the following holds.*
Suppose $(u_i)_{1\leq i\leq d}$ is a family of unit vectors of E such that

$$\forall i \in \{1, \ldots, d-1\}, \quad d(u_{i+1}, \mathrm{Span}(u_j)_{1\leq j\leq i}) \geq c_0. \tag{5}$$

Then, for all proper linear subspace $W < E$, there exists an index i for which

$$d(u_i, W) \geq c_1.$$

Proof. Given a d-tuple (u_i) of elements of $\mathfrak{g}$ and a hyperplane $W < \mathfrak{g}$, we define

$$\varphi(u_1, \ldots, u_d, W) = \max_{1\leq i\leq d} d(u_i, W).$$

The map φ is continuous, and strictly positive whenever (u_i) is a basis for $\mathfrak{g}$. The set E_{c_0} of d-tuples (u_i) of unit vectors satisfying (5) is compact, and so is the Grassmannian variety $\mathcal{G}$ of hyperplanes of $\mathfrak{g}$. This proves the lemma, with constant c_1 equal to the minimal value of φ on the compact set $E_{c_0} \times \mathcal{G}$. □

For simplicity, if Ω is a subset of the Cartesian product set $G^{\times n}$, we will say that a set Ω' *comes from* Ω if there exist indices $i_1 < i_2 < \cdots < i_r$ in $\{1, \ldots, n\}$ and elements $x_{i_1}, \ldots, x_{i_r}$ such that

$$\Omega' \subset \left\{ (x_i)_{\substack{1\leq i\leq n \\ i\notin\{i_1,\ldots,i_r\}}} \;\middle|\; (x_i)_{1\leq i\leq n} \in \Omega \right\}.$$

We now turn to the proof of Proposition 3.6.

Proof of Proposition 3.6. The neighborhood U is chosen such that Lemma 3.7 holds.
Recall from Proposition 3.1 that there exists a finite family $(C_i)_{1\leq i\leq \delta^{-O(\epsilon)}}$ of coset chunks in U such that

$$\Xi \subset \bigcup_{1\leq i\leq \delta^{-O(\epsilon)}} C_i^{(\delta^\eta)}. \tag{6}$$

Let Ω be a subset of the Cartesian product $A^{\times n}$ such that $N(\Omega,\delta)\geq \delta^\epsilon N(A,\delta)^n$, and assume for a contradiction that for all π in Π,

$$\pi(\Omega)\subset \Xi. \tag{7}$$

For $s\in\{0,\dots,d-1\}$, we let $n_s=d^{d-s}$ and $\eta_s=b^s\eta$. To reach a contradiction, we apply Lemma 3.7 inductively. We decompose the reasoning into (at most) d steps.
Step 0
Just using the inclusion (7) for all projections on the coordinates, and recalling that Ξ is controlled by (6), we see that Ω is included in a union of at most $\delta^{-O(\epsilon)}$ sets of the form $C_1^{(\delta^\eta)}\times\cdots\times C_n^{(\delta^\eta)}$, where the C_i's are proper coset chunks in U. By the pigeonhole principle, there must exist coset chunks $C_{01},\dots,C_{0n}$ of dimension at most $d-1$ and a set $\Omega_0\subset\Omega$ such that

1. $N(\Omega_0,\delta)\geq \delta^{O(\epsilon)}N(\Omega,\delta)$
2. $\Omega_0\subset C_{01}^{(\delta^\eta)}\times\cdots\times C_{0n}^{(\delta^\eta)}$

Step $s+1$, $s\geq 0$
Suppose we have constructed a set $\Omega_s\subset A^{\times n_s}$ coming from Ω, and coset chunks $C_{s1},\dots,C_{sn_s}$ of dimension at most $d-1-s$ and at least 1 such that

1. $N(\Omega_s,\delta)\geq \delta^{O(\epsilon)}N(A,\delta)^{n_s}$
2. $\Omega_s\subset C_{s1}^{(\delta^{\eta_s})}\times\cdots\times C_{sn_s}^{(\delta^{\eta_s})}$

For each $i\in\{1,\dots,n_s\}$, write $C_{si}=g_iH_i$ for some subgroup chunk H_i and some element g_i in U. By assumption on the family $\{a_i\}$, there exists a constant $c_0>0$ such that for all unit vector $v\in\mathfrak{g}$ and all hyperplane $W<\mathfrak{g}$, there exists an element a_i such that $d((\mathrm{Ad}\,a_i)v,W)\geq c_0$. This allows us to choose $a_1,\dots,a_d$ among the a_i's so that condition (4) of Lemma 3.7 is satisfied (for some constant c_0 depending only on the set of parameters $\{a_i\}$). Denote by π the associated projection. From the inclusions $\pi(\Omega_s)\subset\Xi$ and (6), we see by the pigeonhole principle that there exists a coset chunk $C=gH$ in U and a subset $\Omega_s'\subset\Omega_s$ such that $N(\Omega_s',\delta)\geq\delta^{O(\epsilon)}N(\Omega_s,\delta)$ and $\pi(\Omega_s')\subset C^{(\delta^\eta)}$.
We now apply Lemma 3.7 to Ω_s', at scale $\rho=\delta^{\eta_s}$, and get a set $\Omega_{s1}\subset A^{\times n_s-d+1}$ coming from Ω_s and a coset chunk $C_{(s+1)1}$ in U of dimension at most $d-2-s$ such that

1. $N(\Omega_{s1},\delta)\geq\delta^{O(\epsilon)}N(A,\delta)^{n_s-d+1}$
2. $\Omega_{s1}\subset C_{(s+1)1}^{(\delta^{\eta_{s+1}})}\times C_{s(d+1)}^{(\delta^{\eta_s})}\cdots\times C_{sn_s}^{(\delta^{\eta_s})}$

Repeating this argument with the next d coordinates, and then again with the d following, etc., we finally get a set Ω_{s+1} coming from Ω_s and included in the Cartesian product $A^{\times n_{s+1}}$, and coset chunks $C_{(s+1)1}, \dots, C_{(s+1)n_{s+1}}$ of dimension at most $d-2-s$ in U such that

1. $N(\Omega_{s+1}, \delta) \geq \delta^{O(\epsilon)} N(A, \delta)^{n_{s+1}}$
2. $\Omega_{s+1} \subset C_{(s+1)1}^{(\delta^{\eta_{s+1}})} \times \cdots \times C_{(s+1)n}^{(\delta^{\eta_{s+1}})}$

As the dimensions of the coset chunks C_{si} are bounded above by $d-s-1$, we must obtain, for some $s \leq d-1$ and some $i \in \{1, \dots, n_s\}$ that $\dim C_{si} = 0$. In other terms, the set C_{si} is reduced to a point, so that the projection S of Ω_s on its i-th coordinate is included in a ball of radius δ^{η_s}. By construction, S is included in A, so that recalling that A is a (σ, ϵ)-set at scale δ, we find

$$N(S, \delta) \leq \delta^{\sigma \eta_s - \epsilon} N(A, \delta).$$

However, from the lower bound $\delta^{O(\epsilon)} N(A, \delta)^{n_s}$ on the cardinality of Ω_s, it is readily seen that

$$N(S, \delta) \geq \delta^{O(\epsilon)} N(A, \delta),$$

which yields the desired contradiction, provided ϵ has been chosen small enough. $\square$

Let $n = d^d$, and Π be the set of projections $G^{\times n} \to G$ as defined in Proposition 3.6. Let N denote the cardinality of Π, and consider the map

$$\begin{array}{llcl} w: & G^{\times n+N+1} & \to & G \\ & (x_1, \dots, x_n, y_0, \dots, y_N) & \mapsto & y_0 \pi_1(x_1, \dots, x_n) y_1 \dots y_{N-1} \pi_N(x_1, \dots, x_n) y_N \end{array} \tag{8}$$

Proposition 3.6 has the following corollary on expansion of (σ, ϵ)-sets in the simple Lie group G.

Corollary 3.9. *Let G be a simple Lie group. There exists a neighborhood U of the identity such that, given $\sigma \in (0, d)$, there exists $\epsilon = \epsilon(\sigma) > 0$ such that the following holds.*
Suppose (a_i) is a family of elements of U satisfying the conclusion of Lemma 3.5, and let $w : G^{\times n+N+1} \to G$ be the associated map, as defined above.
For all $\delta > 0$ sufficiently small, if $A \subset U$ is a (σ, ϵ)-set at scale δ and Ω is a subset of the Cartesian product $A^{\times n+N+1}$ satisfying $N(\Omega, \delta) \geq \delta^{\epsilon} N(A, \delta)^{n+N+1}$, then

$$N(w(\Omega), \delta) \geq \delta^{-\epsilon} N(A, \delta).$$

Proof. For a n-tuple $x = (x_1, \dots, x_n)$ of elements of U, we denote

$$\Omega_x = \{y = (y_0, \dots, y_N) \,|\, (x_1, \dots, x_n, y_0, \dots, y_N) \in \Omega\}.$$

Let

$$\Omega' = \{(x, y) \in \Omega \,|\, N(\Omega_x, \delta) \geq \frac{\delta^{\epsilon}}{2} N(A, \delta)^{N+1}\}.$$

One has

$$\delta^{\epsilon} N(A, \delta)^{n+N+1} \leq N(\Omega, \delta) \leq N(\Omega', \delta) + N(A, \delta)^n \frac{\delta^{\epsilon}}{2} N(A, \delta)^{N+1},$$

so that

$$N(\Omega', \delta) \geq \frac{\delta^\epsilon}{2} N(A, \delta)^{N+n+1}.$$

This shows that we may assume without loss of generality that for all x in the projection of Ω onto the first n coordinates,

$$N(\Omega_x, \delta) \geq \delta^\epsilon N(A, \delta)^{N+1}.$$

Now, provided ϵ is small enough, we may apply Proposition 3.6 to the projection of Ω to the first n coordinates, and we obtain an index $i \in \{1, \ldots, N\}$ and $x^0 = (x_1^0, \ldots, x_n^0)$ such that

$$\xi = \pi_i(x_1^0, \ldots, x_n^0) \notin \Xi.$$

As $N(\Omega_{x^0}, \delta) \geq \delta^\epsilon N(A, \delta)^{N+1}$, we may find elements y_j^0, $j \notin \{i-1, i\}$ such that denoting

$$\Omega_1 = \{(y_{i-1}, y_i) \,|\, (x_1^0, \ldots, x_n^0, y_1^0, \ldots, y_{i-2}^0, y_{i-1}, y_i, y_{i+1}^0, \ldots, y_N^0) \in \Omega\},$$

we have

$$N(\Omega_1, \delta) \geq \delta^\epsilon N(A, \delta)^2.$$

By definition of the set Ξ of troublemakers, $\xi \notin \Xi$ implies that

$$N(\pi_\xi(\Omega_1), \delta) \geq \delta^{-\epsilon} N(A, \delta),$$

where $\pi_\xi : (x, y) \mapsto x\xi y$. However, it is readily seen that for some elements u and v in U^L (where L is the total length of the word w), we have $u\pi_\xi(\Omega_1)v \subset w(\Omega)$, and therefore,

$$N(w(\Omega), \delta) \gg \delta^{-\epsilon} N(A, \delta).$$

□

4 Flattening and dimension increment

It is now time to translate the combinatorial results of the previous section into statements about measures, and in turn, about Hausdorff dimension of product sets.

Definition 4.1. A Borel probability measure on the Lie group G is called σ-*Frostman* if it satisfies, for all $\delta > 0$ sufficiently small, and all x in G,

$$\mu(B(x, \delta)) \leq \delta^\sigma.$$

The importance of this definition lies in the following lemma (see Mattila [12, Chapter 8]).

Lemma 4.2 (Frostman's Lemma)**.** *Let G be a Lie group of dimension d, and $\sigma \in (0, d)$.*

- *Suppose μ is a σ-Frostman measure on G, and A is a Borel subset of G such that $\mu(A) > 0$. Then $\dim_H A \geq \sigma$.*
- *Conversely, if A is a Borel subset of G satisfying $\dim_H A > \sigma$, then there exists a σ-Frostman measure μ whose support in included in A.*

The goal of this section is to prove the following Flattening Lemma, in the spirit of Bourgain-Gamburd [3, Proposition 1].

Lemma 4.3. *Let G be a connected simple Lie group of dimension d. There exists a neighborhood U of the identity in G such that, given $\sigma \in (0, d)$, there exists $\epsilon_1 = \epsilon_1(\sigma) > 0$ such that the following holds.*
Suppose $\{a_i\}$ is a family of elements of U satisfying the conclusion of Lemma 3.5, and let $w : G^{\times p} \to G$ be the associated map, as defined in (8).
If μ is a σ-Frostman finite measure supported on U and ν is the pushforward of $\mu^{\otimes p}$ under the map w, then $\nu * \nu$ is $(\sigma + \epsilon_1)$-Frostman.
Moreover, ϵ_1 is bounded away from 0 if σ varies in a compact subset of $(0, d)$.

From the flattening lemma, it is easy to prove the results announced in the introduction:

Theorem 4.4. *Let G be a connected simple Lie group of dimension d. There exists a neighborhood U of the identity in G and a positive integer k such that given $\sigma > 0$, there exists $\epsilon = \epsilon(\sigma) > 0$ such that if $A \subset U$ is any Borel measurable topologically generating set of Hausdorff dimension $\alpha \in [\sigma, d - \sigma]$ then*

$$\dim_H A^k \geq \epsilon + \dim_H A.$$

Proof. Choose a neighborhood U of the identity and $\epsilon_1 > 0$ such that Lemma 4.3 holds, and let $\epsilon = \frac{\epsilon_1}{2}$.
The set A is topologically generating, so we may choose in a product set A^s a finite collection of elements $\{a_i\}$ satisfying the conclusion of Lemma 3.5.
By Frostman's Lemma, there exists a Borel probability measure μ which is $(\alpha - \epsilon)$-Frostman and whose support is included in A.
Let ν be the image measure $\nu = w_*(\mu^{\otimes p})$. All the a_i's are in a product set A^s and the measure μ is supported on A, so there exists an integer k (depending only on G) such that $\nu * \nu$ is supported on the product set A^k.
By Lemma 4.3, we know that, provided we have chosen ϵ small enough, the measure $\nu * \nu$ is $(\alpha + \epsilon)$-Frostman, and this shows that $\dim_H A^k \geq \alpha + \epsilon$. □

As a corollary, we obtain:

Corollary 4.5. *Let G be a connected simple real Lie group. Any dense Borel measurable sub-semigroup of G has Hausdorff dimension 0 or $\dim G$.*

Before we turn to the proof of Lemma 4.3, we record the following elementary lemma.

Lemma 4.6. *Let ν be a finite measure on a measurable space T, let U be an open subset of $\mathbb{R}^d$, and μ be a Borel measure on U with square integrable density.*

Suppose $w : U \times T \to \mathbb{R}^d$ is a measurable map such that for each t in T, the partial application $w_t : u \mapsto w(u,t)$ is injective and differentiable, with Jacobian J_{w_t}. If C is a positive constant such that,

$$\forall t, u, \quad |J_{w_t}(u)| \geq \frac{1}{C},$$

then the measure $w_*(\mu \otimes \nu)$ has square integrable density, and

$$\|w_*(\mu \otimes \nu)\|_2 \leq C^{\frac{1}{2}} \nu(T) \|\mu\|_2.$$

Proof. Denoting by f the density of μ, it is readily checked that the measure $w_*(\mu \otimes \nu)$ has density θ given by

$$\theta(z) = \int_T \mathbb{1}_{\{z \in w_t(U)\}} f(w_t^{-1}(z)) |J_{w_t^{-1}}(z)| \, d\nu(t).$$

By Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
\|w_*(\mu \otimes \nu)\|_2^2 &= \int_{\mathbb{R}^d} \left(\int_T \mathbb{1}_{\{z \in w_t(U)\}} f(w_t^{-1}(z)) |J_{w_t^{-1}}(z)| \, d\nu \right)^2 dz \\
&\leq \nu(T) \int_{\mathbb{R}^d} \int_T \mathbb{1}_{\{z \in w_t(U)\}} f(w_t^{-1}(z))^2 |J_{w_t^{-1}}(z)|^2 \, d\nu \, dz
\end{aligned}$$

By assumption, we have for all t and z, $|J_{w_t^{-1}}(z)| \leq C$, and therefore, using also Fubini's Theorem and the obvious change of variables,

$$\begin{aligned}
\|w_*(\mu \otimes \nu)\|_2^2 &\leq C\nu(T) \int_{\mathbb{R}^d} \int_T \mathbb{1}_{\{z \in w_t(U)\}} f(w_t^{-1}(z))^2 |J_{w_t^{-1}}(z)| \, d\nu \, dz \\
&= C\nu(T)^2 \|\mu\|_2^2.
\end{aligned}$$

□

We will apply the above lemma to the map w defined in (8). By the following lemma, this will be possible, provided we restrict to a suitable neighborhood of the identity.

Lemma 4.7. *Let G be a simple Lie group. There exists a neighborhood U of the identity in G and a constant C depending on G only such that the following holds.*
Suppose $\{a_i\}_{1 \leq i \leq k}$ is a finite set of elements of U satisfying the conclusion of Lemma 3.5, and let $w : U^p \to G$ be the corresponding map, defined as in (8). If $(t_i)_{\substack{1 \leq i \leq p \\ i \neq i_0}}$ is any family of elements of U, then the partial application

$$w_t : x \mapsto w(t_1, \ldots, t_{i_0 - 1}, x, t_{i_0 + 1}, \ldots, t_p)$$

is injective on U and its Jacobian satisfies

$$|J_{w_t}| \geq \frac{1}{C}.$$

Proof. Let $\widetilde{w}$ be the map

$$\begin{array}{ccc} G^{k+p} & \to & G \\ (\{a_i\},(x_i)) & \mapsto & w_{\{a_i\}}(x_1,\dots,x_p). \end{array}$$

Since $\widetilde{w}$ is a word in the a_i's and x_i's with only positive exponents, its derivative at the identity has the form

$$(n_1 I \,|\, n_2 I \,|\, \dots \,|\, n_{k+p} I),$$

where the n_i's are positive integers. The lemma easily follows from this observation, by continuity of the derivative of $\widetilde{w}$ and by a quantitative version of the Inverse Function Theorem (see e.g. [14, Theorem 2.11]). $\square$

For any small scale $\delta > 0$, we denote by P_δ the function $\frac{\mathbb{1}_{B(1,\delta)}}{|B(1,\delta|}$, and if μ is any Borel measure on the Lie group G, we write $\mu_\delta = \mu * P_\delta$.

The proof of Lemma 4.3 goes by approximating the measure μ_δ by dyadic level sets. We say that a collection of sets $\{X_i\}_{i\in I}$ is *essentially disjoint* if for some constant C depending only on the ambient group G, any intersection of more than C distinct sets X_i is empty. We will use the following lemma.

Lemma 4.8. *Let G be a real Lie group and U be a compact neighborhood of the identity in G. Suppose μ is a Borel probability measure on G and $\delta > 0$ is some small scale.*
Then, there exist subsets A_i, $0 \le i \ll \log \frac{1}{\delta}$ such that

1. $\mu_\delta \ll \sum_i 2^i \mathbb{1}_{A_i} \ll \mu_{4\delta}$
2. *Each A_i is an essentially disjoint union of balls of radius δ.*

Proof. A proof in the case $G = SU(2)$ is given in [11] and also applies in this more general setting, up to some minor changes. $\square$

Proof of Lemma 4.3. Let μ be a σ-Frostman probability measure supported on U, and assume for a contradiction that for some small $\epsilon > 0$ and some arbitrary small ball $B(x,\delta)$, we have

$$\nu * \nu(B(x,\delta)) \ge \delta^{\sigma+\epsilon}.$$

From

$$\nu * \nu(B(x,\delta)) \ll \delta^d \nu_\delta * \nu_\delta(x) \ll \delta^d \|\nu_\delta\|_2^2,$$

we find

$$\|\nu_\delta\|_2^2 \gg \delta^{-d+\sigma+\epsilon}. \tag{9}$$

Using Lemma 4.8, we approximate μ_δ by dyadic level sets:

$$\mu_\delta \ll \sum_i 2^i \mathbb{1}_{A_i} \ll \mu_{4\delta},$$

each A_i being an essentially disjoint union of balls of radius δ.
By inequality (9),

$$\delta^{\frac{-d+\sigma+\epsilon}{2}} \ll \|\nu_\delta\|_2 \leq \left\| \sum_{i_1,\ldots,i_p} w_*(2^{i_1}\mathbb{1}_{A_{i_1}} \otimes \cdots \otimes 2^{i_p}\mathbb{1}_{A_{i_p}}) \right\|_2$$
$$\leq \sum_{i_1,\ldots,i_p} \|w_*(2^{i_1}\mathbb{1}_{A_{i_1}} \otimes \cdots \otimes 2^{i_p}\mathbb{1}_{A_{i_p}})\|_2,$$

so there exist indices $i_1,\ldots,i_p$ such that

$$\|w_*(2^{i_1}\mathbb{1}_{A_{i_1}} \otimes \cdots \otimes 2^{i_p}\mathbb{1}_{A_{i_p}})\|_2 \geq \delta^{\frac{-d+\sigma}{2}+O(\epsilon)}. \tag{10}$$

Given ℓ in $\{1,\ldots,p\}$, Lemma 4.7 ensures that we may apply Lemma 4.6 to the map w and to the measures with density $2^{i_\ell}\mathbb{1}_{A_{i_\ell}}$ and $\bigotimes_{\ell'\neq\ell} 2^{i_{\ell'}}\mathbb{1}_{A_{i'_\ell}}$ and this yields

$$\delta^{\frac{-d+\sigma}{2}+O(\epsilon)} \ll \|2^{i_\ell}\mathbb{1}_{A_{i_\ell}}\|_2 \cdot \|\bigotimes_{\ell\neq\ell'} 2^{i_{\ell'}}\mathbb{1}_{A_{i_{\ell'}}}\|_1$$
$$= 2^{i_\ell}|A_{i_\ell}|^{\frac{1}{2}} \prod_{\ell'\neq\ell} 2^{i_{\ell'}}|A_{i_{\ell'}}|.$$

Using also that the definition of the A_i's implies that

$$2^i|A_i| \ll 1 \quad \text{and} \quad 2^i|A_i|^{\frac{1}{2}} \ll \|\mu_\delta\|_2,$$

the above forces

$$2^{i_\ell/2} \geq \delta^{\frac{-d+\sigma}{2}+O(\epsilon)} \quad \text{and} \quad \forall \ell'\neq\ell,\ 2^{i_{\ell'}}|A_{i_{\ell'}}| \geq \delta^{O(\epsilon)}.$$

This must hold for each ℓ, and therefore, for each ℓ,

$$2^{i_\ell} = \delta^{-d+\sigma+O(\epsilon)} \quad \text{and} \quad 2^{i_\ell}|A_{i_\ell}| = \delta^{O(\epsilon)}. \tag{11}$$

As the set A_{i_ℓ} is a union of ball of radius δ, this shows that

$$N(A_{i_\ell},\delta) \gg \delta^{-d}|A_{i_\ell}| \geq \delta^{-\sigma+O(\epsilon)}.$$

Moreover, as the measure μ is σ-Frostman, we have, for all $\rho \geq \delta$,

$$2^{i_\ell}|A_{i_\ell} \cap B(x,\rho)| \ll \mu(B(x,4\rho)) \ll \rho^\sigma,$$

whence

$$N(A_{i_\ell} \cap B(x,\rho),\delta) \ll \delta^{-d}|A_{i_\ell} \cap B(x,\rho)| \leq \rho^\sigma \delta^{-O(\epsilon)} N(A_{i_\ell},\delta).$$

Thus, each A_{i_ℓ} is a $(\sigma, O(\epsilon))$-set at scale δ, and therefore, so is

$$A := \bigcup_{\ell=1}^{p} A_{i_\ell}.$$

Now let φ be the density function of the measure $w_*(2^{i_1}\mathbb{1}_{A_{i_1}}\otimes\cdots\otimes 2^{i_p}\mathbb{1}_{A_{i_p}})$. On one hand, by (10), we have

$$\|\varphi\|_2^2=\delta^{-d+\sigma+O(\epsilon)}.$$

On the other hand, μ is σ-Frostman and ν can be written $\nu_1*\mu$ for some probability measure ν_1, so that ν is also σ-Frostman, which implies

$$\|\varphi\|_\infty\leq\delta^{-d+\sigma}.$$

Let

$$E=\{x\in G\,|\,\varphi(x)\geq\frac{\|\varphi\|_2^2}{2}\}.$$

We have

$$\|\varphi\|_2^2\leq\int_E\varphi^2+\int_{G\setminus E}\varphi^2\leq\|\varphi\|_\infty\int_E\varphi+\frac{\|\varphi\|_2^2}{2}\int_G\varphi\leq\|\varphi\|_\infty\int_E\varphi+\frac{\|\varphi\|_2^2}{2}$$

whence

$$\int_E\varphi\geq\frac{\|\varphi\|_2^2}{2\|\varphi\|_\infty}\geq\delta^{O(\epsilon)}.$$

Letting Ω be the inverse image $w^{-1}(E)$, the above inequality certainly implies that

$$\mu^{\otimes k}(\Omega)\geq\delta^{O(\epsilon)},$$

which, by the fact that μ is σ-Frostman, shows that

$$N(\Omega,\delta)\geq\delta^{-k\sigma+O(\epsilon)}\geq\delta^{O(\epsilon)}N(A,\delta)^k.$$

To obtain a contradiction, we will bound the size of $w(\Omega)=E$ using that φ takes large values on that set. First observe that isolating the last letter of w – in (8), the letter y_N – allows us to write φ as a convolution

$$\varphi=\varphi_1*(2^{i_p}\mathbb{1}_{A_p}).$$

Then, as A_p is a union of balls of radius δ, we have $\mathbb{1}_{A_p}\ll\mathbb{1}_{A_p}*P_{\frac{\delta}{2}}$ and therefore,

$$\varphi\ll\varphi*P_{\frac{\delta}{2}}.$$

In particular, for each x in E,

$$\frac{\|\varphi\|_2^2}{2}\leq\varphi(x)\ll\delta^{-d}\int_{B(x,\frac{\delta}{2})}\varphi,$$

and summing this inequality for x in a maximal δ-separated set in E, we find

$$\frac{\|\varphi\|_2^2}{2}N(E,\delta)\ll\delta^{-d}\int\varphi\leq\delta^{-d}.$$

Thus,

$$N(w(\Omega),\delta)\leq\delta^{-\sigma-O(\epsilon)}\leq\delta^{-O(\epsilon)}N(A,\delta),$$

which contradicts Corollary 3.9, provided we have chosen ϵ small enough. $\square$

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