Rational approximation on quadrics: a simplex lemma and its consequences

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Abstract

We give elementary proofs of stronger versions of several recent results on intrinsic Diophantine approximation on rational quadric hypersurfaces $X \subset \mathbb{P}^n(\mathbb{R})$. The main tool is a refinement of the simplex lemma, which essentially says that rational points on X which are sufficiently close to each other must lie on a totally isotropic rational subspace of X.

1 Introduction

The classical theory of Diophantine approximation studies the way points $\mathbf{x} \in \mathbb{R}^n$ are approximated by rational points $\frac{\mathbf{p}}{q} \in \mathbb{Q}^n$, taking into account the trade-off between the size of q and the distance between $\frac{\mathbf{p}}{q}$ and $\mathbf{x}$; see [5, 23] for a general introduction. Sometimes $\mathbf{x}$ is assumed to lie on a certain subset of $\mathbb{R}^n$, for example a smooth manifold X; this leads to the theory of *Diophantine approximation on manifolds*, in which there is no distinction between rational points which do or do not lie in X (this is referred to as *ambient* approximation).

Let now X be a rational quadric hypersurface of $\mathbb{R}^n$, let $\mathbf{x} \in X$ and let $\frac{\mathbf{p}}{q} \in \mathbb{Q}^n$ be such that the distance between $\mathbf{x}$ and $\frac{\mathbf{p}}{q}$ is less than $\psi(q)$, where ψ is decaying fast enough, namely $\lim_{t\to\infty} t^2\psi(t) = 0$. Then $\frac{\mathbf{p}}{q}$ must lie on X whenever q is large enough! This elementary observation, due to Dickinson and Dodson [10] for $n = 2$ and more generally to Druţu, see [11, Lemma 4.1.1], has in part motivated a new field of *intrinsic approximation*, which examines the quality to which points on a manifold are approximated by rational points lying on that same manifold. The paper [18] studies the case $X = \mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$. Later in [12] the results of [18] were significantly strengthened and extended to the case of X being an arbitrary rational quadric hypersurface. An even more general framework was developed in [13]. Roughly speaking, in order to exhibit points on submanifolds $X \subset \mathbb{R}^n$ which are close enough to rational points of X, one has to make use of the structure of X (indeed, in general it is not even guaranteed that $X \cap \mathbb{Q}^n$ is not empty). On the other hand, it is shown in [13] that to prove some *negative* results, that is, to show that many points of X are not too close to rational points, one often does not need to know much about X. The main tool on which the argument of [13] is based is the *Simplex Lemma* originating in Davenport's work [9]. The version presented in [13, Lemma 4.1] is very general – it applies to any manifold embedded in $\mathbb{R}^n$ –

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and at the same time precise enough to yield some satisfying theorems in the case of quadric hypersurfaces.

The purpose of this note is to show that in the special case where X is a rational quadric hypersurface, one can give more elementary and more geometric proofs of the results of [13]. This new approach will also yield more precise theorems. The main point is that one can prove a version of the simplex lemma with arbitrary hyperplanes replaced by $\mathbb{Q}$-isotropic subspaces of X; this, in turn, yields refined information on the diophantine properties of X.

A detailed account of the results that are derived here is given in the next section. After that, in §3 we prove the simplex lemma for quadrics, Lemma 3.1, which is central in all the subsequent developments. Applications of the simplex lemma to Diophantine approximation on quadrics are presented in §4. Those results are proved along the same lines as the analogous statements for Diophantine approximation in the Euclidean space $\mathbb{R}^n$, but the proofs are included to make the paper self-contained. Finally, in §5 we discuss some open problems and possible further directions for the study of intrinsic Diophantine approximation on projective varieties.

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2 General setting and main results of the paper

Since it will make the proofs more transparent, we shall from now on always work in the projective setting. We denote by $\mathbb{P}^n(\mathbb{R})$ the n-dimensional real projective space. The natural map from $\mathbb{R}^{n+1}$ to $\mathbb{P}^n(\mathbb{R})$ will be denoted by $\mathbf{x} \mapsto [\mathbf{x}]$. We now endow $\mathbb{R}^{n+1}$ with the standard Euclidean norm $\|\cdot\|$, and explain how this defines a distance on $\mathbb{P}^n(\mathbb{R})$. The distance between two elements x and y in $\mathbb{P}^n(\mathbb{R})$ is equal to the sine of the angle between the two lines in $\mathbb{R}^{n+1}$:

$$\operatorname{dist}(x,y) := |\sin(x,y)|.$$

Equivalently,

$$\operatorname{dist}(x,y) = \frac{\|\mathbf{v}_x \wedge \mathbf{v}_y\|}{\|\mathbf{v}_x\|\|\mathbf{v}_y\|},$$

where $\mathbf{v}_x$ and $\mathbf{v}_y$ are any nonzero vectors on x and y respectively, $\mathbf{v}_x \wedge \mathbf{v}_y$ is the exterior product of $\mathbf{v}_x$ and $\mathbf{v}_y$, and the Euclidean norm is naturally extended to $\wedge^2(\mathbb{R}^{n+1})$ so that $\|\mathbf{v}_x \wedge \mathbf{v}_y\|$ is the area of the parallelogram spanned by $\mathbf{v}_x$ and $\mathbf{v}_y$.

If $v = [\mathbf{v}] \in \mathbb{P}^n(\mathbb{Q})$, where $\mathbf{v} = (v_1, \ldots, v_{n+1})$ is an integer vector with coprime coordinates, the *height* of v is simply

$$H(v) := \max_{1 \le i \le n+1} |v_i|.$$

Given a point x in $\mathbb{P}^n(\mathbb{R})$ we want to study how well x is approximated by points v in $\mathbb{P}^n(\mathbb{Q})$.

Remark 2.1. In order to go back to the setting of Diophantine approximation in $\mathbb{R}^n$, one can consider an affine chart from an open subset of $\mathbb{P}^n(\mathbb{R})$ to $\mathbb{R}^{n+1}$. For example, if $U = \{[(x_1,\dots,x_{n+1})] \ : \ x_{n+1} \neq 0\}$, one can use the chart

$$\begin{array}{ccc} U & \to & \mathbb{R}^n, \\ [(x_1,\dots,x_{n+1})] & \mapsto & (\frac{x_1}{x_{n+1}},\dots,\frac{x_n}{x_{n+1}}). \end{array}$$

We consider a projective rational quadric X, given as the set of zeros of a rational quadratic form Q in $n+1$ variables. Namely, for such Q let us consider

$$X = [Q^{-1}(0)] = \left\{x \in \mathbb{P}^n(\mathbb{R}) : x = [\mathbf{x}] \text{ with } Q(\mathbf{x}) = 0\right\}. \tag{2.1}$$

Let us say that a subspace $E \subset \mathbb{R}^{n+1}$ is *totally isotropic* if $Q|_E \equiv 0$. If E is as above, the projection $[E] \subset X$ of E onto $\mathbb{P}^n(\mathbb{R})$ will be referred to as a totally isotropic projective subspace. Recall that the $\mathbb{Q}$*-rank* $\operatorname{rk}_{\mathbb{Q}} X$ of the quadric X is the maximal dimension of a totally isotropic rational subspace of $\mathbb{R}^{n+1}$. If $\operatorname{rk}_{\mathbb{Q}} X > 0$, this is the same as the maximal dimension of a totally isotropic rational projective subspace of X plus one. In particular, $\operatorname{rk}_{\mathbb{Q}} X > 0$ if and only if $X(\mathbb{Q}) \neq \varnothing$.

Given a point x in X, we shall be interested in the quality of rational approximations $v \in X(\mathbb{Q})$ to x. The basic theory of such approximations has been developed in [12]. In particular it was proved there [12, Theorem 5.1] that if

$$\operatorname{rk}_{\mathbb{Q}} X > 0 \quad \text{ and } \quad X \text{ is nonsingular} \tag{2.2}$$

(recall that a quadric hypersurface X is said to be *nonsingular* if the quadratic form that defines it is nondegenerate, i.e. has nonzero discriminant[1]), then for every $x \in X$ there exists $C_x > 0$ and a sequence $(v_k)_1^\infty$ in $X(\mathbb{Q})$ such that

$$v_k \to x \quad \text{ and } \quad \operatorname{dist}(v_k, x) \leq \frac{C_x}{H(v_k)}. \tag{2.3}$$

Thus if one defines the *Diophantine exponent* of x by

$$\beta(x) := \inf\left\{\beta > 0 \mid \exists\, c > 0 : \forall\, v \in X(\mathbb{Q}),\, \operatorname{dist}(x,v) \geq cH(v)^{-\beta}\right\}, \tag{2.4}$$

then it follows that under the assumption (2.2), $\beta(x) \geq 1$ for all $x \in X$.

On the other hand, it is shown in [13, Theorem 1.5] that the opposite inequality $\beta(x) \leq 1$ is true for Lebesgue-almost every $x \in X$ in the generality when X is not just a rational quadric but an arbitrary non-degenerate hypersurface. Moreover, the same is true if the Lebesgue measure is replaced by an *absolutely decaying* measure (see §4.1 for definitions and more detail).

This naturally leads to a question of exhibiting other measures μ on X such that $\beta(x) \leq 1$ for μ-almost all $x \in X$. This is reminiscent to the subject of Diophantine approximation on manifolds and fractals, which has been extensively developed during recent decades for ambient approximation in $\mathbb{R}^n$, see [3], [17] and [16], for example. Measures satisfying the above property are usually called *extremal*. We shall also say that a submanifold $Y \subset X$ is extremal if so is the Lebesgue measure on Y (by which we mean the restriction to Y of the k-dimensional Hausdorff measure where $k = \dim Y$).

Our first theorem, which is actually a special case of a more general result, Theorem 4.2, refines [13, Theorem 1.5] for rational quadrics X as follows:

[1] This is also equivalent to X being nonsingular as a projective algebraic variety.

Theorem 2.2 (Extremality of submanifolds of large dimension)**.** *Let X be a rational quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$, and let Y be a smooth submanifold of X with $\dim Y \geq \mathrm{rk}_{\mathbb{Q}} X$. Then $\beta(x) \leq 1$ for Lebesgue-almost every $x \in Y$.*

In the case where X has $\mathbb{Q}$-rank one, the above theorem provides a very simple and satisfactory answer to the problem of Diophantine approximation on submanifolds of X: any positive-dimensional submanifold $Y \subset X$ is extremal. Note that there is no non-degeneracy condition on the submanifold Y. This comes in contrast to the case of approximation in $\mathbb{R}^n$, where one has to require that the submanifold is not included in an affine subspace.

In view of Theorem 2.2, it is natural to ask, given a submanifold Y of X of dimension at least $\mathrm{rk}_{\mathbb{Q}} X$ and a fixed $\beta > 1$, how large the intersection $Y \cap W_\beta$ can be, where W_β denotes the set of points in X whose Diophantine exponent is at least β. Note that it was proved in [12, Theorem 6.4] that whenever X satisfies (2.2), the Hausdorff dimension of W_β is equal to $\frac{n-1}{\beta}$. Also in [14] some upper estimates for the Hausdorff dimension of $Y \cap W_\beta$ were obtained in the case when Y supports an absolutely decaying and Ahlfors-regular measure (see §4.2 for details). Our second application of the simplex lemma strengthens the main result of [14]. Here is a special case of a more general result, Theorem 4.6:

Theorem 2.3 (β-approximable points on submanifolds of large dimension)**.** *Let X be a rational quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$, and let Y be a k-dimensional smooth submanifold of X with $k \geq \mathrm{rk}_{\mathbb{Q}} X$. Then one has*

$$\dim_H(Y \cap W_\beta) \leq k - (k + 1 - \mathrm{rk}_{\mathbb{Q}} X)(1 - \tfrac{1}{\beta}).$$

As the third application of our simplex lemma, we study the winning property of the set BA_X of badly approximable points on X. Schmidt introduced games in his landmark paper [22] in order to study the set of badly approximable numbers in $\mathbb{R}^n$. He defined a *winning* property for subsets of $\mathbb{R}^n$, and showed the following:

- Any countable intersection of winning sets is winning ;
- If S is winning and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-diffeomorphism, then $f(S)$ is winning;
- If $S \subset \mathbb{R}^n$ is winning then it has Hausdorff dimension n.

Then, Schmidt also showed that the set of badly approximable numbers in $\mathbb{R}^n$ is winning. Variants of the Schmidt game were subsequently studied in numerous papers, among which [4] is the most relevant for the present purposes.

In our setting, the set of badly approximable points on the quadric X is

$$\mathrm{BA}_X := \{x \in X \mid \exists\, c > 0 : \forall\, v \in X(\mathbb{Q}),\ \mathrm{dist}(x, v) \geq cH(v)^{-1}\}. \tag{2.5}$$

We define in § 4.3 a version of Schmidt's game, and show the associated winning property for the set BA_X. As a corollary of this *isotropically winning* property, we get the following.

Theorem 2.4 (Thickness of BA_X on submanifolds of large dimension)**.** *Let X be a rational quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$. Then for any C^1 submanifold $Y \subset X$ of dimension at least $\mathrm{rk}_{\mathbb{Q}} X$,*

$$\dim_H(\mathrm{BA}_X \cap Y) = \dim Y.$$

The properties of the set BA_X have been studied in [13]. In particular, it was shown [13, Theorem 4.3] that BA_X is hyperplane absolute winning (see §4.3 for the definition and more detail); this gave the conclusion of the above theorem for $Y = X$. The refined version given above has the advantage that it is optimal: indeed, if Y is any totally isotropic rational projective subspace of X of dimension $\mathrm{rk}_{\mathbb{Q}} X - 1$, then $\mathrm{BA}_X \cap Y = \varnothing$.

3 Diagonal flows and the simplex lemma

The purpose of this section is to derive a simplex lemma, Lemma 3.1, for rational points on a rational quadric hypersurface $X \subset \mathbb{R}^{n+1}$. For the proof, we shall relate good rational approximations to $x \in X$ to the behavior of some diagonal orbit in the space of lattices in $\mathbb{R}^{n+1}$.

Recall that the classical simplex lemma states that for each $n \in \mathbb{N}$ there exists $c = c(n) > 0$ such that if $\mathbf{x}$ is a point in $\mathbb{R}^n$ and $\rho \in (0,1)$, then there exists an affine hyperplane containing all rational points with denominator at most $c\rho^{-\frac{n}{n+1}}$ inside the ball $B(\mathbf{x}, \rho)$. The proof is based on the observation that any affinely independent $n+1$ rational points with denominators at most D define inside $B(\mathbf{x}, \rho)$ a simplex whose volume can be bounded below by $\frac{1}{n!D^{n+1}}$. Therefore, one must have $\frac{1}{n!D^{n+1}} \leq \mathrm{Vol}\big(B(\mathbf{x}, \rho)\big) = v_n \rho^n$, where v_n is the volume of the unit ball in $\mathbb{R}^n$, and hence $D \geq (n! v_n)^{-\frac{1}{n+1}} \rho^{-\frac{n}{n+1}}$. For a detailed proof, we refer the reader to [20, Lemma 4]. The simplex appearing in the proof gave its name to the lemma.

Here we consider a rational quadratic form Q on $\mathbb{R}^d$ and study rational points on X as in (2.1).

Lemma 3.1 (Simplex lemma for quadric hypersurfaces)**.** *Let X be a rational quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$. Then there exists $c > 0$ such that for every ball $B_\rho \subset X$ of radius $\rho \in (0,1)$ the set*

$$B_\rho \cap \{v \in X(\mathbb{Q}) \mid H(v) \leq c\rho^{-1}\}$$

is contained in a totally isotropic rational projective subspace of X.

Let F_Q be the symmetric bilinear form associated to the quadratic form Q defining X. The *kernel* of Q is defined by

$$\ker Q = \{x = [\mathbf{x}] \in \mathbb{P}^n(\mathbb{R}) \mid \forall\, \mathbf{y} \in \mathbb{R}^{n+1},\ F_Q(\mathbf{x}, \mathbf{y}) = 0\}.$$

Assuming that $X(\mathbb{Q}) \smallsetminus \ker Q$ is non-empty, we may write, in some rational basis of $\mathbb{R}^{n+1}$,

$$Q(x_1, \ldots, x_{n+1}) = 2x_1 x_{n+1} + \tilde{Q}(x_2, \ldots, x_n), \tag{3.1}$$

where $\tilde{Q}$ is a quadratic form in $n-1$ variables. Let $G = \mathrm{SO}_Q(\mathbb{R})$ be the group of unimodular linear transformations of $\mathbb{R}^{n+1}$ preserving the quadratic form Q. The group G acts transitively on $X \smallsetminus \ker Q$, which may be identified with the quotient space $X \simeq P\backslash G$, where P is the stabilizer of the isotropic line $[\mathbf{e}_1]$ in the standard representation. In fact, for $x \in X \smallsetminus \ker Q$, we may choose $u_x \in G \cap \mathrm{O}_{n+1}(\mathbb{R})$ such that $u_x x = [\mathbf{e}_1]$.

We shall consider the diagonal subgroup $a_t = \mathrm{diag}(e^{-t}, 1, \ldots, 1, e^t)$ in G, and if $x \in X$, let

$$g_t^x = u_x^{-1} a_t u_x.$$

The lemma below is due to Kleinbock-Merrill [18] in the case of projective spheres, and to Fishman-Kleinbock-Merrill-Simmons [12, Lemma 7.1] in the general case. To make the paper self-contained, we provide a proof here.

Lemma 3.2 (Dani correspondence for quadric hypersurfaces)**.** *Let Q be as in (3.1), and write X for the associated rational quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$. With the above notation, there exists $C > 0$ such that for $x \in X$ and $v \in X$, we have, for all $t \in \mathbb{R}$,*

$$\|g_t^x \mathbf{v}\| \leq C \max(e^{-t} H(v), H(v) \operatorname{dist}(x,v), e^t H(v) \operatorname{dist}(x,v)^2),$$

where $\mathbf{v} \in \mathbb{Z}^{n+1}$ is a representative of v with coprime integer coordinates.

Proof. Fix $C_0 \geq 2$ larger than $\max_{\|\mathbf{w}\|=1} |\tilde{Q}(\mathbf{w})|$, so that for all $\mathbf{w}$ in $\mathbb{R}^{n-1}$, $|\tilde{Q}(\mathbf{w})| \leq C_0 \|\mathbf{w}\|^2$. With u_x as above, write

$$u_x \mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_{n+1} \mathbf{e}_{n+1}.$$

Letting $\mathbf{w} = v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n$, we have

$$u_x g_t^x \mathbf{v} = e^{-t} v_1 \mathbf{e}_1 + \mathbf{w} + e^t v_{n+1} \mathbf{e}_{n+1},$$

and therefore, since u_x is in $\mathrm{O}_{n+1}(\mathbb{R})$,

$$\|g_t^x \mathbf{v}\| \leq 3 \max(e^{-t} |v_1|, \|\mathbf{w}\|, e^t |v_{n+1}|). \tag{3.2}$$

Now note that $|v_1| \leq H(v)$ and $H(v) \geq \frac{1}{\sqrt{n+1}} \|\mathbf{v}\|$, so

$$\begin{aligned} \sqrt{n+1} H(v) \operatorname{dist}(x,v) &\geq \|u_x^{-1} \mathbf{e}_1 \wedge \mathbf{v}\| = \|\mathbf{e}_1 \wedge u_x \mathbf{v}\| \\ &= \|\mathbf{e}_1 \wedge (\mathbf{w} + v_{n+1} \mathbf{e}_{n+1})\| = \|\mathbf{w} + v_{n+1} \mathbf{e}_{n+1}\| \geq \|\mathbf{w}\|. \end{aligned}$$

Moreover, $Q(u_x \mathbf{v}) = 0$ yields

$$|v_{n+1}| = \frac{|\tilde{Q}(\mathbf{w})|}{2|v_1|} \leq \frac{C_0 \|\mathbf{w}\|^2}{2|v_1|},$$

so that, provided $\operatorname{dist}(x,v) \leq \frac{\sqrt{2}}{2}$,

$$|v_{n+1}| \leq \frac{C_0}{2} \frac{H(v) \operatorname{dist}(x,v)^2}{\sqrt{1 - \frac{\operatorname{dist}(x,v)^2}{H(v)^2}}} \leq C_0 H(v) \operatorname{dist}(x,v)^2.$$

Of course, if $\operatorname{dist}(x,v) \geq \frac{\sqrt{2}}{2}$, we also have $|v_{n+1}| \leq H(v) \leq C_0 H(v) \operatorname{dist}(x,v)^2$, because $C_0 \geq 2$. Going back to (3.2), we find the desired inequality, with $C = \max(3C_0, \sqrt{n+1})$. ∎

We can now prove the simplex lemma.

Proof of Lemma 3.1. Let Q be a quadratic form defining the hypersurface X. The result is obvious if $X(\mathbb{Q}) \subset \ker Q$, so we may assume that $X(\mathbb{Q}) \smallsetminus \ker Q$ is non-empty. Then, replacing Q if necessary by an integer multiple, we may find an integer basis of $\mathbb{R}^{n+1}$ in which Q has the form (3.1).

Let $C_1 = \max_{\|\mathbf{v}\|=1} |Q(\mathbf{v})|$, so that for all $\mathbf{v} \in \mathbb{R}^{n+1}$, $|Q(\mathbf{v})| \le C_1 \|\mathbf{v}\|^2$, and let $c = \frac{1}{C\sqrt{5C_1}}$, where C is the constant given by Lemma 3.2. We need to show that any family $v_1, \dots, v_s$ of points in $X(\mathbb{Q}) \cap B(x,\rho)$ satisfying $H(v_i) \le c\rho^{-1}$, $i = 1, \dots, s$, generates a totally isotropic subspace. For each v_i, we take a representant $\mathbf{v}_i$ in $\mathbb{Z}^{n+1}$ with coprime integer coordinates. It is enough to show that for all i and j, $Q(\mathbf{v}_i \pm \mathbf{v}_j) = 0$, and since the quadratic form Q takes integer values at integer points, it suffices to check that for all i and j, $|Q(\mathbf{v}_i \pm \mathbf{v}_j)|$ is less than 1.

Now, choosing $t > 0$ such that $e^t = \rho^{-1}$, Lemma 3.2 shows that $\|g_t^x \mathbf{v}_i\| \le Cc$. Then, we write

$$Q(\mathbf{v}_i \pm \mathbf{v}_j) = Q(g_t^x \mathbf{v}_i \pm g_t^x \mathbf{v}_j)) \le C_1 \|g_t^x \mathbf{v}_i \pm g_t^x \mathbf{v}_j\|^2 \le 4C_1 (Cc)^2 = \frac{4}{5}.$$

This implies what we want. □

spheres **Remark 3.3.** In the case when $X = \mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere, identified with the subset of $\mathbb{R}^n$ defined by the equation $x_1^2 + \cdots + x_n^2 = 1$, one can give a more direct proof of the simplex lemma. Indeed, if $\frac{\mathbf{p}_1}{q_1}$ and $\frac{\mathbf{p}_2}{q_2}$ are two distinct rational points on $\mathbb{S}^{n-1}$ of height at most $\frac{\rho^{-1}}{2}$, we have

$$\left\| \frac{\mathbf{p}_1}{q_1} - \frac{\mathbf{p}_2}{q_2} \right\|^2 = 2 - \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{q_1 q_2} \ge \frac{1}{q_1 q_2} \ge 4\rho^2,$$

so that any open ball of radius ρ contains at most one rational point of height at most $\frac{\rho^{-1}}{2}$. In fact, such a direct computation can also be made for a general quadric hypersurface, but we chose to give a more geometric proof of Lemma 3.1.

Remark 3.4. When the quadratic form Q has $\mathbb{Q}$-rank one, the only isotropic rational projective subspaces are points in $X(\mathbb{Q})$. This makes the consequences of the simplex lemma more spectacular in the particular case of $\mathbb{Q}$-rank one.

4 Applications to Diophantine approximation

sec:diophantine

In this section, as before, X is a rational quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$ defined by a rational quadratic form Q. We are concerned with *intrinsic* Diophantine approximation on X, which is the study of the quality of approximations of a point x in X by rational points v *lying on* X. On that matter, the simplex lemma has several simple consequences, which we now explain.

4.1 Extremality

sec:diophantine1

Recall that the Diophantine exponent of a point $x \in X$ was defined by (2.4). Our next theorem generalizes Theorem 2.2 using the following definition.

def-ad **Definition 4.1.** Given a positive parameter α, a finite Borel measure μ on the quadric hypersurface X will be called *α-isotropically absolutely decaying*, abbreviated as *α-IAD*, if there exists a constant $C > 0$ such that for every $x \in X$ and every totally isotropic rational projective subspace $L \subset X$,

$$\forall \varepsilon > 0\ \forall \rho \in (0,1), \quad \mu\big(B(x,\rho) \cap L^{(\varepsilon\rho)}\big) \le C\varepsilon^\alpha \mu\big(B(x,\rho)\big), \tag{4.1}$$ eq:ad

where $L^{(\tau)}$ denotes the neighborhood of size τ of the set L. We shall say that μ is *isotropically absolutely decaying* (IAD) if it is α-IAD for some $\alpha > 0$.

Theorem 4.2 (IAD measures are extremal)**.** *Let X be a rational quadric in $\mathbb{P}^n(\mathbb{R})$, and let μ be an IAD measure on X. Then $\beta(x) \leq 1$ for μ-almost every $x \in X$.*

Remark 4.3. Recall that a measure μ is called α-*absolutely decaying* if (4.1) holds for some $C > 0$, every $x \in X$ and *every* subspace $L \subset \mathbb{P}^n(\mathbb{R})$, and *absolutely decaying* if it is α-absolutely decaying for some $\alpha > 0$. It follows from [13, Theorem 1.5] that for any absolutely decaying measure μ on X one has $\beta(x) \leq 1$ for μ-almost every $x \in X$. In fact it holds more generally when X is not just a rational quadric but an arbitrary non-degenerate smooth hypersurface.

Absolutely decaying measures are IAD but not vice versa. In particular, the Lebesgue measure on a smooth proper submanifold Y of X with $\dim Y \geq \mathrm{rk}_{\mathbb{Q}} X$ is not absolutely decaying but α-IAD with $\alpha = \dim Y - \mathrm{rk}_{\mathbb{Q}} X + 1$; so Theorem 2.2 is a corollary from Theorem 4.2.

Proof of Theorem 4.2. The argument follows the lines of the proof of [21, Theorem 1], see also [24] for a one-dimensional version. By the Borel–Cantelli lemma, it is enough to check that for all $\varepsilon > 0$,

$$\sum_{k \geq 1} \mu \left(\left\{ x \in X \;\middle|\; \exists v \in X(\mathbb{Q}) : \begin{array}{c} 2^k \leq H(v) < 2^{k+1} \\ \mathrm{dist}(x, v) \leq 2^{-k(1+\varepsilon)} \end{array} \right\} \right) < \infty.$$

Fix $k \geq 1$. There exists an integer K such that we may cover X by a family of balls $B_i = B(x_i, 2^{-k(1+\frac{\varepsilon}{2})})$, $i = 1, \ldots, N$, so that any intersection of more than K distinct balls is empty. By Lemma 3.1, for k large enough, for each i, the set of points $v \in X(\mathbb{Q}) \cap B_i$ satisfying $2^k \leq H(v) < 2^{k+1}$ is contained in a totally isotropic rational subspace L_i, and therefore, by the IAD property of μ for some $C, \alpha > 0$ one has

$$\begin{aligned} \mu \left(\left\{ x \in B_i \;\middle|\; \exists v \in X(\mathbb{Q}) : \begin{array}{c} 2^k \leq H(v) < 2^{k+1} \\ \mathrm{dist}(x, v) \leq 2^{-k(1+\varepsilon)} \end{array} \right\} \right) &\leq \mu \left(B_i \cap L_i^{(2^{-k(1+\varepsilon)})} \right) \\ &\leq C 2^{-k\alpha\frac{\varepsilon}{2}} \mu(B_i). \end{aligned}$$

Summing over all balls B_i, and using the fact that the cover $(B_i)_{i \in \mathbb{N}}$ has multiplicity at most K, we get

$$\mu \left(\left\{ x \in X \;\middle|\; \exists v \in X(\mathbb{Q}) : \begin{array}{c} 2^k \leq H(v) < 2^{k+1} \\ \mathrm{dist}(x, v) \leq 2^{-k(1+\varepsilon)} \end{array} \right\} \right) \leq K C 2^{-k\alpha\frac{\varepsilon}{2}}.$$

Since this last bound is summable in k, this concludes the proof of the theorem. □

Remark 4.4. When $\mathrm{rk}_{\mathbb{Q}}(X) = 1$, all the subspaces L appearing in Definition 4.1 are zero-dimensional, and isotropic absolute decay coincides with weak absolute decay as defined in [2]. Moreover, in the case where X is a sphere, Theorem 4.2 can be viewed as a corollary of [2, Theorem 2].

Remark 4.5. We could have stated a slightly stronger version of the theorem, in the form of a Khintchine-type theorem: if μ is α-IAD, and if $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing function satisfying

$$\sum_{k\in\mathbb{N}} k^{\alpha-1}\psi(k)^\alpha < \infty,$$

then for μ-almost every x in X, there exists $c > 0$ such that

$$\forall v \in X(\mathbb{Q}),\ \operatorname{dist}(x, v) \geq c\psi\big(H(v)\big).$$

The proof, based on the easy half of the Borel–Cantelli lemma, is essentially the same as the one presented above.

4.2 Hausdorff dimension and Diophantine exponents

As a complement to the above study of the extremality problem, we explain here how the simplex lemma can be used to give a simple proof of a recent result of Fishman-Merrill-Simmons [14]. Once again, X denotes a rational quadric projective hypersurface of dimension n. Given $\beta > 0$, we shall be concerned with the set

$$W_\beta = \{x \in X \mid \beta(x) \geq \beta\}.$$

Given a subset K in X, our goal will be to bound the Hausdorff dimension of the intersection $K \cap W_\beta$; we shall be able to do so if K is the support of a sufficiently regular measure.

For $\delta > 0$, a Borel measure μ on a metric space X is said to be *Ahlfors-regular* of dimension δ if we have, for some constant $A > 0$,

$$\forall\, x \in X\, \forall\, r\, \in (0, 1], \quad \frac{1}{A} r^\delta \leq \mu\big(B(x, r)\big) \leq Ar^\delta.$$

We now present a short proof of a strengthening of [14, Theorem 1.2], using Lemma 3.1.

Theorem 4.6. *Let X be a rational quadric projective hypersurface. Let μ be an Ahlfors-regular measure of dimension δ on X, and let $K = \operatorname{Supp}\mu$. If μ is α-IAD, then we have, for all $\beta \geq 1$,*

$$\dim_H(K \cap W_\beta) \leq \delta - \alpha\left(1 - \frac{1}{\beta}\right). \tag{4.2}$$

Remark 4.7. Under a stronger assumption that μ is α-absolutely decaying (4.2) is established in [14, Theorem 1.2]. However in our decay condition we only have to consider totally isotropic subspaces. In particular, Theorem 4.6 covers the case where K is a smooth submanifold of X of dimension at least $\operatorname{rk}_Q(X)$, and therefore generalizes Theorem 2.3.

The proof of Theorem 4.6 is a straightforward adaptation of that of [21, Theorem 2]. We shall use the easy Hausdorff–Cantelli lemma stated below.

Lemma 4.8 (Hausdorff–Cantelli)**.** *Let $(B_i)_{i\geq 0}$ be a family of balls in a metric space, and assume that $\sum_{i\geq 0}(\operatorname{diam} B_i)^s < \infty$. Then,*

$$\dim_H(\limsup B_i) \leq s.$$

Proof. Left as an exercise, see Bernik-Dodson [3, Lemma 3.10]. $\square$

Proof of Theorem 4.6. If $\beta = 1$, there is nothing to prove, so we assume $\beta > 1$ and fix $\gamma \in (1, \beta)$. For $p \geq 0$, let

$$A_p = \left\{ x \in X \;\middle|\; \exists\, v \in X(\mathbb{Q}) : \begin{array}{c} 2^p \leq H(v) < 2^{p+1} \\ \operatorname{dist}(x, v) \leq 2^{-\gamma p} \end{array} \right\}.$$

Taking a maximal 2^{-p}-separated subset $\{x_i\}_{1 \leq i \leq \ell_p}$ of $K \cap A_p$, the collection of balls $\mathcal{C}_p = \big(B(x_i, 2^{-p})\big)_{1 \leq i \leq \ell_p}$ covers $K \cap A_p$ and has multiplicity bounded above by some constant C depending only on X. Using the Ahlfors regularity of μ, this implies $\ell_p 2^{-p\delta} \leq AC\mu(X) = AC$, i.e. $\ell_p \leq AC2^{p\delta}$.

Since $\gamma > 1$, Lemma 3.1 shows that for p large enough, for each ball $B \in \mathcal{C}_p$, there exists a totally isotropic subspace L_B of X such that $A_p \cap B \subset L_B^{(2^{-\gamma p})}$. So the decay condition on μ yields, up to multiplicative constants depending only on X and μ, that

$$\mu(A_p \cap B) \ll 2^{-(\gamma-1)\alpha p} \mu(B) \asymp 2^{-p[\delta + (\gamma-1)\alpha]}.$$

Next, take a minimal cover $\mathcal{D}_B = (B_i)_{i \in I_B}$ of the set $K \cap A_p \cap B$ by balls of radius $2^{-\gamma p}$ centered on $K \cap A_p \cap B$. Just as above, the Ahlfors regularity of μ shows that

$$\# I_B \ll 2^{\delta\gamma p} \mu(A_p \cap B) \ll 2^{p\gamma\delta} 2^{-p[\delta + (\gamma-1)\alpha]}.$$

Thus, we find for every $s > 0$,

$$\sum_{B \in \mathcal{C}_p} \sum_{i \in I_B} (\operatorname{diam} B_i)^s \ll 2^{p\delta} 2^{p(\gamma-1)(\delta-\alpha)} 2^{-p\gamma s} = 2^{-p[s\gamma - \gamma\delta + \alpha(\gamma-1)]}$$

If $s > \delta - \alpha\big(1 - \frac{1}{\gamma}\big)$, then the family of balls $(B_i)_{i \in I_B,\, B \in \mathcal{C}_p,\, p \in \mathbb{N}}$ satisfies the assumption of the Hausdorff–Cantelli lemma, and therefore, letting

$$s \to \delta - \alpha \left(1 - \frac{1}{\gamma}\right)$$

we find that $\dim_H(\limsup B_i) \leq \delta - \alpha\big(1 - \frac{1}{\gamma}\big)$. Now, since $\gamma < \beta$, we have $K \cap W_\beta \subset (\limsup B_i)$, hence letting $\gamma \to \beta$, we can conclude that the Hausdorff dimension of $K \cap W_\beta$ is not greater than $\delta - \alpha\big(1 - \frac{1}{\beta}\big)$. $\square$

In the case of $\mathbb{Q}$-rank one, any Ahlfors-regular measure of dimension δ is δ-IAD, so we get the following corollary, which applies in particular when $X = \mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$:

Corollary 4.9. *Let X be a rational quadric hypersurface of $\mathbb{Q}$-rank one, and let μ be an Ahlfors-regular measure of dimension δ on X. Writing $K = \operatorname{Supp} \mu$, we have, for every $\beta \geq 1$, $\dim_H(K \cap W_\beta) \leq \frac{\delta}{\beta}$.*

4.3 Badly approximable points

Recall the definition (2.5) of the set BA_X of intrinsically badly approximable points in X. As was mentioned in Section 2, it is known [13] to satisfy some winning properties in the sense of Schmidt's games. Our goal will now be to give a more elementary proof of a refinement of the winning property, again using the simplex lemma.

We now explain the principles of our version of Schmidt's game. As before, X is a rational quadric hypersurface of $\mathbb{P}^n(\mathbb{R})$. There are two players, Alice and Bob, and some parameter $\beta \in (0, \frac{1}{3})$. To start, Bob chooses a ball $B_0 = B(x_0, \rho_0)$ in X. Then, at each stage of the game, after Bob has chosen a ball $B_i = B(x_i, \rho_i)$, Alice chooses a totally isotropic rational subspace L of X and deletes its neighborhood of size ε, with $0 < \varepsilon \leq \beta\rho_i$.

A set S is *isotropically β-winning* if Alice can make sure that

$$\bigcap B_i \cap S \neq \varnothing.$$

Finally, S is *isotropically winning* if it is isotropically β-winning for arbitrarily small $\beta > 0$. Our game is inspired by Broderick, Fishman, Kleinbock, Reich and Weiss [4], where the authors define the notion of *k-dimensionally absolute winning* using exactly the same game, except that Alice is allowed to delete neighborhoods of arbitrary k-dimensional subspaces. In particular, we have the following properties of isotropically winning sets.

Proposition 4.10 (Properties of winning sets)**.** *Let X be a projective quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$.*

1. *If S is isotropically winning on X, then S is dense and* $\dim_H S = \dim X$.
2. *If $(S_i)_{i\in\mathbb{N}}$ is a countable family of isotropically winning sets on X, then $\bigcap_{i\in\mathbb{N}} S_i$ is isotropically winning.*

Proof. Let $k = \operatorname{rk} X - 1$. Any isotropically winning set is k-dimensionally absolute winning in the sense of [4, page 323], so that the first item follows from the analogous property for k-dimensional absolute winning [4, Proposition 2.3]. Alternatively, one may adapt the proof of Schmidt [22, Theorem 2]. The proof of the second item is identical to the analogous statement for k-dimensional absolute winning, see [4]. □

Remark 4.11. We warn the reader that the image of an isotropically winning set under a C^1 diffeomorphism of X may not be isotropically winning. However, by [4, Proposition 2.3.(c)], it will certainly be k-dimensionally absolute winning, and therefore dense and with maximal Hausdorff dimension.

The following theorem is a refinement of [13, Theorem 4.3]:

Theorem 4.12 (Badly approximable points on X are winning)**.** *Let X be a rational quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$. Then the set BA_X is isotropically winning.*

Proof. Fix $\beta \in (0, \frac{1}{3})$. Bob first picks a ball $B_0 = B(x_0, \rho_0)$. By Lemma 3.1, there exists a constant $c > 0$ depending only on X such that all rational points v in $2B_0$ satisfying $H(v) \leq c\rho_0^{-1}$ are included in some totally isotropic rational subspace L_0. Alice deletes $L_0^{(\beta\rho_0)}$. Similarly, once Bob has chosen a ball $B_i = B(x_i, \rho_i)$, the rational points $v \in 2B_i$ such that $H(v) \leq c\rho_i^{-1}$ all lie on a hyperplane L_i, and Alice deletes $L_i^{(\beta\rho_i)}$. If there is no rational point of small height in B_i, then Alice can delete a ball of radius $\beta\rho_i$ around the center. This ensures that $\rho_i \to 0$.

We claim that this strategy forces $\bigcap_{i\geq 0} B_i \subset \mathrm{BA}_X$. To see this, let $x \in \bigcap B_i$ and $v \in X(\mathbb{Q})$. Choose i such that

$$c\rho_{i-1}^{-1} \leq H(v) \leq c\rho_i^{-1}. \tag{4.3}$$

If $v \notin 2B_i$, then, using $x \in B_i$, we find

$$\mathrm{dist}(x,v) \geq \rho_i \geq \beta\rho_{i-1} \geq \beta c H(v)^{-1}.$$

And if $v \in 2B_i$, then (4.3) implies that $v \in L_i$, and since $x \in B_{i+1}$,

$$\mathrm{dist}(x,v) \geq \beta\rho_i \geq \beta^2\rho_{i-1} \geq \beta^2 c H(v)^{-1}.$$

Taking $c_0 = c\beta^2$, we find

$$\forall\, v \in X(\mathbb{Q}),\ \mathrm{dist}(x,v) \geq c_0 H(v)^{-1},$$

so $x \in \mathrm{BA}_X$. □

As is the case with the k-dimensional absolute game, the advantage of the isotropic game is the inheritance of winning properties to sufficiently regular subsets. More precisely, given a compact subset $K \subset X$, we may consider the isotropic game played on K. The rules are the same as before, but the ambient metric space is now K: at each stage, Bob chooses a ball $B(x_i, \rho_i)$ centered on K, and Alice deletes the intersection of K with the neighborhood of size $\beta\rho_i$ of a rational isotropic subspace. Naturally, we shall say that a set S is *isotropically winning on* K if $S \cap K$ is winning for the isotropic game on K.

Following Broderick, Fishman, Kleinbock, Reich and Weiss [4], let us say that a subset $K \subset X$ is *isotropically diffuse* if there exists $\beta, \rho_K > 0$ such that for every $\rho \in (0, \rho_K)$, $x \in K$, and every totally isotropic rational subspace L, the set

$$K \cap B(x,\rho) \smallsetminus L^{(\beta\rho)}$$

is non-empty. This is a quantitative way to say that K is nowhere included in a small neighborhood of a totally isotropic subspace. The next lemma is a straightforward analogue of [4, Proposition 4.9].

Lemma 4.13. *Let X be a rational quadric hypersurface in $\mathbb{P}^n(X)$. If $L \subset K$ are two isotropically diffuse subsets of X, and $S \subset X$ is isotropically winning on K, then S is isotropically winning on L.*

The proof is very similar to the one presented in [4], once one has replaced the notions of k-dimensionally diffuse and k-dimensionally winning by those of isotropically diffuse and isotropically winning. We refer the reader to [4, Section 4] for details.

It follows from the above lemma and Theorem 4.12 that BA_X is isotropically winning on any isotropically diffuse subset of X. This in particular applies to smooth submanifolds Y of X of dimension not less than $\mathrm{rk}_{\mathbb{Q}}(X)$, which are isotropically diffuse. Furthermore, the Lebesgue measure on Y as above is Ahlfors-regular of dimension equal to $\dim Y$. Therefore, in view of [4, Lemma 5.3], for every open subset U of X such that $U \cap Y \neq \varnothing$, one has

$$\dim_H(Y \cap \mathrm{BA}_X \cap U) = \dim Y,$$

which implies Theorem 2.4.

Remark 4.14. In the case of $X = \mathbb{S}^{n-1}$, or more generally of a rational quadric of $\mathbb{Q}$-rank one, the above shows that BA_X is winning on any positive-dimensional submanifold of X. This can be compared with a similar question for Diophantine approximation in Euclidean spaces, for which it is still open, despite recent progress of Beresnevich [1] and Yang [25].

5 Further directions and open problems

Khintchine's theorem. It would be interesting to use the geometric observations of this note to give an elementary proof of Khintchine's theorem on quadric hypersurfaces, due to Fishman, Kleinbock, Merrill and Simmons [12, Theorem 6.3].

Singular points. Given a rational quadric X in $\mathbb{P}^n(X)$, one may define, for $c > 0$,

$$D(c) = \left\{ x \in X \;\middle|\; \begin{array}{c} \exists N_0 : \forall N \geq N_0 \; \exists v \in X(\mathbb{Q}) \text{ such that} \\ H(v) \leq N \text{ and } \operatorname{dist}(x,v) \leq \dfrac{c}{\sqrt{NH(v)}} \end{array} \right\},$$

and call a point $x \in X$ *singular* if $x \in \bigcap_{c>0} D(c)$. If X has $\mathbb{Q}$-rank 1, it follows from Dani's work [8] that x is singular if and only if $x \in X(\mathbb{Q})$. In fact, one can show that if X has $\mathbb{Q}$-rank 1, $D(c) = X(\mathbb{Q})$ for $c > 0$ small enough. This follows for example from the following strengthening of Lemma 3.1, whose proof is identical up to some minor changes. See also [19, Theorem 3] for an alternative proof.

Lemma 5.1 (A stronger simplex lemma for quadric hypersurfaces)**.** *Let X be a rational quadric hypersurface in $\mathbb{P}^n(\mathbb{R})$. Then there exists $c > 0$ such that, for every $x \in X$ and any $\rho \in (0,1)$, the set*

$$\left\{ v \in X(\mathbb{Q}) \;\middle|\; H(v) \leq c\rho^{-1},\; \operatorname{dist}(x,v) \leq \sqrt{\frac{\rho}{H(v)}} \right\}$$

is contained in a totally isotropic rational subspace $L \subset X$.

When the quadric X has $\mathbb{Q}$-rank at least 2, it is natural to expect that there exist some nontrivial singular points. It might then be interesting to compute the Hausdorff dimension of the set of singular points on X, similarly to what has been done in [6, 7] for Diophantine approximation in the Euclidean space.

Extremality. In view of the definitive results in the area of Diophantine approximation on manifolds and fractals obtained in [17], it is natural to attempt to weaken the condition of isotropic absolute decay of μ as in Theorem 4.2, and conjecture that on a general quadric hypersurface, any analytic submanifold that is not included in an isotropic subspace is extremal. In fact, by analogy with [15], one can guess that an analytic submanifold on a quadric hypersurface inherits its Diophantine exponent from the smallest totally isotropic subspace in which it is contained.

Other projective varieties. One may wonder how general is the approach presented here, and whether it can be used to study intrinsic Diophantine approximation on varieties that are not quadric hypersurfaces.

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