EQUIDISTRIBUTION IN THE SPACE OF 3-LATTICES AND DIRICHLET-IMPROVABLE VECTORS ON PLANAR LINES

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Abstract. Let $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$, and $g_t = \mathrm{diag}(e^{2t}, e^{-t}, e^{-t})$. Let ν denote the push-forward of the normalized Lebesgue measure on a segment of a straight line in the expanding horosphere of $\{g_t\}_{t>0}$, under the map $h \mapsto h\,\mathrm{SL}_3(\mathbb{Z})$ from $\mathrm{SL}_3(\mathbb{R})$ to X. We give explicit necessary and sufficient Diophantine conditions on the line for equidistribution of each of the following families of measures on X:

(1) g_t-translates of ν as $t \to \infty$.

(2) averages of g_t-translates of ν over $t \in [0, T]$ as $T \to \infty$.

(3) g_{t_i}-translates of ν for some $t_i \to \infty$.

We apply this dynamical result to show that Lebesgue-almost every point on the planar line $y = ax + b$ is not Dirichlet-improvable if and only if $(a, b) \notin \mathbb{Q}^2$.

1. Introduction

1.1. **Equidistribution of expanding translates of curves.** Let $G = \mathrm{SL}_{n+1}(\mathbb{R})$, $\Gamma = \mathrm{SL}_{n+1}(\mathbb{Z})$ and $X = G/\Gamma$. Let μ_X denote the unique G-invariant probability measure on X. Let $g_t = \mathrm{diag}(e^{nt}, e^{-t}, \dots, e^{-t})$, so that the expanding horospherical subgroup of G associated to g_1 is

$$U^+ = \{g \in G : g_{-t} g g_t \to e, t \to +\infty\} = \left\{ \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \right\} \cong \mathbb{R}^n.$$

Let $x_0 = e\Gamma \in X$. Using the Margulis thickening method (see e.g. [KM96]), one can show that the g_t-translates of the horosphere $U^+ x_0$ get equidistributed in X. One may ask what happens if we replace the whole horosphere with a bounded piece of a real-analytic submanifold therein. We note that X can be identified with the space of unimodular lattices in $\mathbb{R}^{n+1}$, hence numerous applications of dynamics on this space to Diophantine approximation, see e.g. [Dan85, KM98, KW08].

If the analytic submanifold is *non-degenerate*, i.e. is not contained in a proper affine subspace, then a result of the third named author in [Sha09a] tells us that equidistribution still holds. See also [SY18] for a generalization to differentiable submanifolds. It is thus a natural question to ask for conditions for equidistribution

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of degenerate submanifolds, such as proper affine subspaces of U^+. One expects these conditions to be expressed in terms of Diophantine properties of the affine subspaces.

The main goal of this article is to give a complete solution to this problem in the case $n = 2$, that is, study the case of straight lines in $\mathbb{R}^2$. In this case we will show that the dynamics is completely controlled by Diophantine conditions on the parameters of the straight line, and will give criteria for different types of equidistribution phenomena.

In what follows we will specialize to $n = 2$; that is, let $G = \mathrm{SL}_3(\mathbb{R})$, $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, and $g_t = \mathrm{diag}(e^{2t}, e^{-t}, e^{-t})$, so that the expanding horospherical subgroup of G associated to g_1 is

$$U^+ = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{pmatrix} \right\} \cong \mathbb{R}^2.$$

As before, we let $x_0 = e\Gamma \in X = G/\Gamma$. Note that, under the identification of X with the space of unimodular lattices in $\mathbb{R}^3$, x_0 corresponds to the standard lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$.

Let $\mathcal{W}_2$ denote the set of vectors $(a, b) \in \mathbb{R}^2$ for which there exists $C > 0$ such that the system of inequalities[1]

$$\begin{cases} |qb + p_1| \le C|q|^{-2} \\ |qa + p_2| \le C|q|^{-2} \end{cases} \tag{1.1}$$

has infinitely many solutions $(p_1, p_2; q) \in \mathbb{Z}^2 \times \mathbb{N}$.

Similarly, let $\mathcal{W}_2'$ denote the set of vectors for which (1.1) has a non-zero solution $(p_1, p_2; q) \in \mathbb{Z}^2 \times \mathbb{N}$ for every $C > 0$.

1.1.1. *Remark.* One has an obvious inclusion $\mathcal{W}_2' \subset \mathcal{W}_2$. It can be deduced from **[Roy15**, Theorem 1.3**]** that this inclusion is strict, even though both sets have Hausdorff dimension equal to 1, see **[Dod92]**.

Throughout the paper, $I = [s_0, s_1]$ denotes an arbitrary compact interval with non-empty interior, i.e. $s_0 < s_1$. For $(a, b) \in \mathbb{R}^2$, let $\phi_{a,b} : I \to U^+$ be the line segment defined by

$$\phi_{a,b}(s) = \begin{pmatrix} 1 & s & as+b \\ & 1 & \\ & & 1 \end{pmatrix}, \ \forall s \in I.$$

Let $\lambda_{a,b}$ denote the push-forward of the normalized Lebesgue measure on I under the map $s \mapsto \phi_{a,b}(s)x_0$ from I to X, and for any $t \in \mathbb{R}$, let $g_t\lambda_{a,b}$ denote the translate of $\lambda_{a,b}$ by g_t; that is, for any $f \in C_c(X)$,

$$\int_X f \, \mathrm{d}\lambda_{a,b} = \fint_I f(\phi_{a,b}(s)x_0) \, \mathrm{d}s := \frac{1}{|I|} \int_I f(\phi_{a,b}(s)x_0) \, \mathrm{d}s, \tag{1.2}$$

$$\int_X f \, \mathrm{d}(g_t\lambda_{a,b}) = \fint_I f(g_t\phi_{a,b}(s)x_0) \, \mathrm{d}s, \tag{1.3}$$

where $|A|$ is the Lebesgue measure of A for any measurable subset A of $\mathbb{R}$.

[1] Here and throughout the paper, the notation with a brace and several inequalities is used to indicate that *all* of the inequalities hold simultaneously.

We say that a family $\{\lambda_i\}_{i\in\mathcal{I}}$ of probability measures on X has ***no escape of mass*** if for every $\varepsilon > 0$ there exists a compact subset K of X such that $\lambda_i(K) > 1-\varepsilon$ for all $i \in \mathcal{I}$.

We will prove the following criterion for non-escape of mass.

Theorem 1.1. *The translates $\{g_t\lambda_{a,b}\}_{t>0}$ have no escape of mass if and only if*

$$(a, b) \notin \mathcal{W}_2'.$$

Furthermore, for $\mathcal{I} = \mathbb{N}$ or $\mathbb{R}_{>0}$, we say that a family $\{\lambda_i\}_{i\in\mathcal{I}}$ of probability measures on X gets ***equidistributed*** in X if

$$\int f \,\mathrm{d}\lambda_i \stackrel{i\to\infty}{\longrightarrow} \int f \,\mathrm{d}\mu_X, \ \forall f \in C_c(X);$$

that is, λ_i converges to μ_X with respect to the weak-* topology as $i \to \infty$.

Theorem 1.2. *The translates $\{g_t\lambda_{a,b}\}_{t>0}$ get equidistributed in X if and only if*

$$(a, b) \notin \mathcal{W}_2.$$

Since the set $\mathcal{W}_2$ has Lebesgue measure 0 in $\mathbb{R}^2$, a typical line gets equidistributed under the flow g_t.

Chow and Yang **[CY19]** proved effective equidistribution for translates of a Diophantine line by diagonal elements near $\mathrm{diag}(e^t, e^{-t}, 1)$. Unfortunately their method does not seem to apply to the flow $g_t = \mathrm{diag}(e^{2t}, e^{-t}, e^{-t})$ here. We will instead use Ratner's measure rigidity theorem for unipotent flows **[Rtn91]**, and tools from geometric invariant theory.

1.2. **Averaging over the time parameter.** Define

$$\mathcal{W}_2^+ = \left\{ (a, b) \in \mathbb{R}^2 : \limsup_{(p_1,p_2;q)\in\mathbb{Z}^2\times\mathbb{N}} \frac{-\log\max\{|qb + p_1|, |qa + p_2|\}}{\log q} > 2 \right\}.$$

In other words, $\mathcal{W}_2^+$ consists of vectors $(a, b) \in \mathbb{R}^2$ for which there exists $\varepsilon > 0$ such that the system of inequalities

$$\begin{cases} |qb + p_1| \le q^{-(2+\varepsilon)} \\ |qa + p_2| \le q^{-(2+\varepsilon)} \end{cases}$$

has infinitely many solutions $(p_1, p_2; q) \in \mathbb{Z}^2 \times \mathbb{N}$.

1.2.1. *Remark.* In view of Remark 1.1.1, we have strict inclusions

$$\mathcal{W}_2^+ \subsetneq \mathcal{W}_2' \subsetneq \mathcal{W}_2,$$

even though all of these sets have Hausdorff dimension equal to 1 (see **[Dod92]**). The strictness of the inclusion $\mathcal{W}_2^+ \subset \mathcal{W}_2'$ can be derived from a zero–infinity law for Hausdorff measures of those sets with appropriate dimension functions, see **[DV97]**.

We are also interested in the limit distributions of the ***averages*** of g_t-translates of $\lambda_{a,b}$, namely the family $\left\{\frac{1}{T}\int_0^T g_t\lambda_{a,b}\,\mathrm{d}t\right\}_{T>0}$ of probability measures on X. Similar questions have been considered in **[SW17]** and from the measure rigidity point of view in **[ES19]**.

Theorem 1.3. *The following are equivalent:*

(1) *The averages* $\{\frac{1}{T}\int_0^T g_t\lambda_{a,b}\,\mathrm{d}t\}_{T>0}$ *get equidistributed in* X.
(2) *The averages* $\{\frac{1}{T}\int_0^T g_t\lambda_{a,b}\,\mathrm{d}t\}_{T>0}$ *have no escape of mass.*
(3) $(a,b)\notin\mathcal{W}_2^+$.

1.2.2. *Remark.* It is shown in **[Kle03]** that the planar line $\{y = ax + b\}$ is *extremal* if and only if $(a,b)\notin\mathcal{W}_2^+$.

1.3. **Equidistribution along a sequence.** We are also interested in understanding when $\{g_t\lambda_{a,b}\}_{t>0}$ equidistributes along some subsequence $t_i\to\infty$.

Theorem 1.4. *Let* $(a,b)\in\mathbb{R}^2$. *Then the following are equivalent:*

(1) $(a,b)\notin\mathbb{Q}^2$.
(2) *The closure of* $\{g_t\lambda_{a,b}\}_{t\geq 0}$ *contains* μ_X *with respect to the weak-* topology.*
(3) *For almost every* $s\in\mathbb{R}$, *the trajectory* $\{g_t\phi_{a,b}(s)x_0\}_{t\geq 0}$ *is dense in* X.

1.3.1. *Remark.* Suppose $(a,b)\in\mathbb{Q}^2$, then $g_t\lambda_{a,b}$ will diverge, i.e. eventually leave any fixed compact set, see Remark 3.0.2. Hence Theorem 1.4 gives us a dichotomy which was somewhat unexpected: the $g_t\lambda_{a,b}$ *either* diverge as $t\to\infty$, *or* get equidistributed along some sequence $t_i\to\infty$.

1.3.2. *Remark.* We also have dual versions of Theorem 1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4 above. Let $\tilde{g}_t = \operatorname{diag}(e^t, e^t, e^{-2t})$, consider the map $\tilde{\phi}_{a,b}\colon s\mapsto\begin{pmatrix}1 & & as+b\\ & 1 & s\\ & & 1\end{pmatrix}$, and let $\tilde{\lambda}_{a,b}$ denote the push-forward of the normalized Lebesgue measure on I under the map $s\mapsto\tilde{\phi}_{a,b}(s)x_0$ from I to X. Then all the four theorems are still valid for $\tilde{g}_t$ in place of g_t and $\tilde{\lambda}_{a,b}$ in place of $\lambda_{a,b}$. Indeed, it suffices to consider the outer automorphism $g\mapsto w\cdot{}^tg^{-1}\cdot w$ of G, where $w=\begin{pmatrix} & & 1\\ & -1 & \\ 1 & & \end{pmatrix}$; under this automorphism g_t is sent to $\tilde{g}_t$, $\lambda_{a,b}$ is sent to $\tilde{\lambda}_{a,b}$, and μ_X is preserved. See **[Sha09a**, Page 511**]**.

1.3.3. *Remark.* It is also worthwhile to point out that even though the measures $\lambda_{a,b}$ depend on the choice of $I=[s_0,s_1]$, the criteria in all the theorems stated above do not; that is, the limiting behavior of these measures is the same for all nontrivial intervals simultaneously. Using the arguments of this article, one can see that for a given sequence $t_i\to\infty$, if $g_{t_i}\lambda_{a,b}\to\mu_X$, then for every finite interval J with nonempty interior, we have $g_{t_i}\lambda_{a,b}^J\to\mu_X$.

1.4. **Dirichlet-improvable vectors on planar lines.** The motivation for our study came from Diophantine approximation. Denote by $\|\cdot\|$ the supremum norm on $\mathbb{R}^n$ (unless specified otherwise, all the norms on $\mathbb{R}^n$ will be taken to be the supremum norm). Following Davenport and Schmidt **[DS70b]**, for $0<\delta<1$ we say that a vector $\mathbf{x} \in \mathbb{R}^n$ is ***δ-improvable*** if for every sufficiently large T, the system of inequalities

$$\begin{cases} \|q\mathbf{x}+\mathbf{p}\| \leq \delta T^{-1} \\ |q| \leq T^n \end{cases}$$

has a solution $(\mathbf{p}, q)$, where $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{Z} \setminus \{0\}$. One says that $\mathbf{x}$ is ***Dirichlet-improvable*** if it is δ-improvable for some $0 < \delta < 1$, and that it is ***singular*** if it is δ-improvable for all $0 < \delta < 1$.

Similarly, a real linear form on $\mathbf{q} \in \mathbb{Z}^n$ is given by $\mathbf{q} \mapsto \mathbf{x} \cdot \mathbf{q}$, parametrized by $\mathbf{x} \in \mathbb{R}^n$. We say that this linear form is ***δ-improvable*** if there exists $0 < \delta < 1$ such that for every sufficiently large T, the system of inequalities

$$\begin{cases} |\mathbf{x} \cdot \mathbf{q} + p| \leq \delta T^{-n} \\ \|\mathbf{q}\| \leq T \end{cases} \tag{1.4}$$

has a solution $(p, \mathbf{q})$, where $p \in \mathbb{Z}$ and $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$.

The notation $\mathrm{DI}(n,1)$ and $\mathrm{DI}(1,n)$ (resp., $\mathrm{Sing}(n,1)$ and $\mathrm{Sing}(1,n)$) is used in the literature to denote the set of Dirichlet-improvable (resp., singular) vectors and linear forms. It is known that $\mathrm{DI}(n,1) = \mathrm{DI}(1,n)$ and $\mathrm{Sing}(n,1) = \mathrm{Sing}(1,n)$, see **[DS70a]** and **[Cas57**, Chapter V, Theorem XII**]** respectively.

The readers who would like to know more background information are referred to **[KW08, Sha09a]** and references therein for Dirichlet-improvable vectors, and **[CC16, Dan85]** and references therein for singular vectors.

Now let us again specialize to $n = 2$, and take $\mathbf{x}$ of the form $(s, as + b)$. In the simplest possible case $(a, b) \in \mathbb{Q}^2$ it is very easy to see that every point on the planar line

$$L_{a,b} = \{(x, y) \in \mathbb{R}^2 : y = ax + b\}$$

is singular: indeed, take $a = k/m$ and $b = \ell/m$ and notice that one has

$$(x, y) \cdot (-k, m) = \left(s, \tfrac{k}{m}s + \tfrac{\ell}{m}\right) \cdot (-k, m) = \ell;$$

thus one can always find $(p, \mathbf{q})$ such that the left hand side of the first inequality in (1.4) iz zero.

Our next main theorem is the following stronger converse to the above computation:

Theorem 1.5. *Let $(a, b) \in \mathbb{R}^2$. If $(a, b) \notin \mathbb{Q}^2$, then almost every point on the planar line $L_{a,b}$ is not Dirichlet-improvable.*

The deduction of Theorem 1.5 from Theorem 1.4 uses Dani's correspondence, and has become a standard argument. We give the proof below for completeness.

Proof of Theorem 1.5 assuming Theorem 1.4. For $0 < \delta < 1$, let $K_\delta \subset X$ denote the set of unimodular lattices in $\mathbb{R}^3$ whose shortest non-zero vector has norm at least δ. Then K_δ contains an open neighborhood of x_0, and is compact by Mahler's compactness criterion. For $0 < \delta < 1$, let D_δ denote the set of $s \in \mathbb{R}$ such that

$(s, as+b)$ is a δ-improvable linear form. By Dani's correspondence **[Dan85]** and **[KW08**, Proposition 2.1**]**,

$$D_\delta = \{s \in \mathbb{R} : g_t\phi_{a,b}(s)x_0 \notin K_{\delta^{1/3}} \text{ for all large } t\}.$$

In particular, $\{g_t\phi_{a,b}(s)x_0\}_{t\geq 0}$ is not dense in G/Γ for all $s \in D_\delta$. By Theorem 1.4 we have $|D_\delta| = 0$ for all $0 < \delta < 1$. Finally, we conclude the proof by noting that the set of s such that $(s, as+b)$ is a Dirichlet-improvable point on $L_{a,b}$ equals $\bigcup_{m\geq 1} D_{\frac{m-1}{m}}$, and hence has Lebesgue measure 0. □

1.5. **Strategy of the proof.** Given any sequence $t_i \to \infty$, after passing to a subsequence we obtain that $g_{t_i}\lambda_{a,b}$ converges to a measure, say μ, on X with respect to the weak-* topology. It is straightforward to see that μ must be invariant under a non-trivial unipotent subgroup of G (Proposition 6.1). There are two possibilities: if μ is not a probability measure then we apply the Dani-Margulis non-divergence criterion (Proposition 5.1), and if μ is positive and not G-invariant then we apply Ratner's description of ergodic invariant measures for unipotent flows, combined with the linearization technique (Theorem 4.1). In both cases we obtain the following *linear dynamical obstruction to equidistribution:* There exist a finite-dimensional representation V of G over $\mathbb{Q}$, a non-zero vector $v_0 \in V(\mathbb{Q})$, a constant $R > 0$, and a sequence $\{\gamma_i\} \subset \Gamma = \mathrm{SL}_3(\mathbb{Z})$ such that for each i,

$$\sup_{s\in I} \|g_{t_i}\phi_{a,b}(s)\gamma_i v_0\| \leq R. \tag{1.5}$$

The major effort involved in this proof is to analyze this linear dynamical obstruction and show that (a, b) must satisfy certain Diophantine approximation condition.

Using Kempf's numerical criterion in geometric invariant theory, when the Gv_0 is not Zariski closed, we reduce the obstruction to the case of v_0 being a highest weight vector (Theorem 2.1). Then we further reduce to the case of v_0 being a highest weight vector of a fundamental representation of G, namely the standard representation $\mathbb{R}^3$, or its exterior power $\bigwedge^2 \mathbb{R}^3$ (Lemma 2.2). It is straightforward to show that the obstruction (1.5) does not arise for the exterior representation (Lemma 2.3), so we are left only with the case of V being the standard representation. In the case of the standard representation the dynamical obstruction leads to the Diophantine condition that $(a, b) \in \mathcal{W}_2$ (Lemma 3.1).

We are left with the case of Gv_0 being Zariski closed. Using explicit descriptions of finite-dimensional irreducible representations of SL_2 and SL_3, we show that in this case after passing to a further subsequence $\{\gamma_i v_0\}$ is constant and $(a, b) \in \mathbb{Q}^2$ (Theorem 4.1).

We remark that for $G = \mathrm{SL}_n$ for $n > 3$, analyzing the Zariski closed orbit case involves much greater complexities, and the above strong conclusion about (a, b) is not possible.

1.6. **Comparison with previous work.** We remark that Theorem 1.3 and Theorem 1.5 sharpen the main results of Shi and Weiss **[SW17]**. More precisely, it was shown in **[SW17]** that the *averages* of g_t-translates of $\lambda_{a,b}$ get equidistributed in X if the line $\{y = ax + b\}$ contains a badly approximable vector. By Remark 3.0.1

below, this condition implies in particular that $(a,b) \notin \mathcal{W}_2'$, and so $(a,b) \notin \mathcal{W}_2^+$. Our results give sharp conditions of non-escape of mass and equidistribution for not only averages, but also pure translates which was not considered in **[SW17]**. This is why we are able to prove the much stronger Theorem 1.5.

1.7. **Future directions.** Instead of g_t, one may consider more general flows. It seems that our method is also applicable to the study of translates by elements in a Weyl chamber, at least for certain cones. Then the non-effective version of Theorem 1.1 of **[CY19]** will be recovered. One would also be able to say something about improvability of weighted Dirichlet Theorem, and the readers are referred to **[Sha10]** for a detailed introduction to this subject.

One may also ask what happens to other Lie groups, e.g. $G = \mathrm{SL}_n(\mathbb{R})$ for $n > 3$. When $n = 4$, things already become more complicated. Roughly speaking, $\mathrm{SL}_3(\mathbb{R})$ is small and one does not have many choices of possible intermediate subgroups. However, in $\mathrm{SL}_n(\mathbb{R})$, where $n > 3$, there are more possibilities of intermediate subgroups; see the follow-up paper **[SY21]** for more details.

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2. Instability and invariant theory

To analyze limiting distributions of sequences of translates of measures on homogeneous spaces a technique has been developed, where one applies the Dani-Margulis and Kleinbock-Margulis non-divergence criteria, Ratner's theorem and the linearization method, and reduces the problem to dynamics of subgroup actions on finite-dimensional representations of semisimple groups, see **[Sha09a, SY20]**. In **[Yan20]**, this kind of linear dynamics was analysed in a very general situation using invariant theory results due to Kempf **[Kem78]**. We follow the same approach, and this section is devoted to describing the basic tools from geometric invariant theory that we shall need in our argument.

Let G be a reductive real algebraic group defined over $\mathbb{Q}$, and $\rho : G \to \mathrm{GL}(V)$ a linear representation of G defined over $\mathbb{Q}$. We say that a nonzero vector $v \in V$ is ***unstable*** if the Zariski closure of the orbit Gv contains the origin. Hilbert-Mumford's unstability criterion states that a nonzero vector v is unstable if and only if there exists a cocharacter $\lambda\colon \mathbb{G}_m \to G$ such that $\lambda(t)v \xrightarrow{t\to 0} 0$. Kempf **[Kem78]** refined this criterion by studying the set of cocharacters, up to scaling, λ such that $\lambda(t)$ bring v to 0 at maximal speed as $t \to 0$. Let us briefly recall his results.

Write $X_*(G)$ for the set of $\mathbb{Q}$-cocharacters of G. For any nonzero $v \in V(\mathbb{Q})$ and any nontrivial cocharacter λ in $X_*(G)$, one can write $v = \sum_{i\in\mathbb{Z}} v_i$, where $\lambda(t)v_i = t^i v_i$

for all i. Let $m(v,\lambda) = \min\{i \in \mathbb{Z} : v_i \neq 0\}$. Then

$$v = v_{m(v,\lambda)} + \sum_{i > m(v,\lambda)} v_i.$$

Thus for any $g \in G(\mathbb{Q})$, $m(gv, g\lambda g^{-1}) = m(v,\lambda)$.

For any $\lambda \in X_*(G)$, the group

$$P(\lambda) = \Big\{p \in G : \lim_{t\to 0} \lambda(t)p\lambda(t)^{-1} \text{ exists in } G\Big\}$$

is a parabolic subgroup of G defined over $\mathbb{Q}$, and

$$R_u(P(\lambda)) = \Big\{u \in G : \lim_{t\to 0} \lambda(t)u\lambda(t)^{-1} = e\Big\}$$

is the unipotent radical of $P(\lambda)$ defined over $\mathbb{Q}$. Also $P(\lambda) = Z_G(\lambda)R_u(P(\lambda))$, and this product holds over $\mathbb{Q}$-points, where $Z_G(\lambda)$ is the centralizer of the image of λ in G.

We note that if $u \in R_u(P(\lambda))$, then

$$uv = v_{m(v,\lambda)} + \sum_{i > m(v,\lambda)} (uv)_i. \tag{2.1}$$

Let S be a maximal $\mathbb{Q}$-split torus in G, we fix a positive definite integral bilinear form $(\cdot,\cdot)$ on the free abelian group $X_*(S)$ of $\mathbb{Q}$-cocharacters on S which is invariant under the Weyl group $N_G(S)/Z_G(S)$; it induces a norm on $X_*(S)$ defined by $\|\lambda\| = \sqrt{(\lambda,\lambda)}$. This norm extends uniquely to a norm on the set $X_*(G)$ of $\mathbb{Q}$-cocharacters of G which is invariant under the conjugation by $G(\mathbb{Q})$.

Kempf's Theorem [Kem78, Theorem 4.2]. *Let $v \in V(\mathbb{Q})$ be a nonzero unstable vector. Then the following hold:*

a) *Let $B_v = \sup\big\{m(v,\lambda)/\|\lambda\| : \lambda \in X_*(G) \text{ nontrivial}\big\}$. Then $B_v > 0$.*

b) *Let Λ_v = the set of indivisible $\lambda \in X_*(G)$ such that $m(v,\lambda) = B_v \cdot \|\lambda\|$. Then,*

(1) *Λ_v is non-empty.*
(2) *There exists a $\mathbb{Q}$-parabolic subgroup P_v of G such that $P_v = P(\lambda)$ for all $\lambda \in \Lambda_v$.*
(3) *The set Λ_v is a principle homogeneous space under conjugation by $\mathbb{Q}$-points of the unipotent radical of P_v. In particular, $P_v(\mathbb{Q})$ acts transitively on Λ_v under conjugation.*
(4) *For any maximal torus of P_v, which is defined over $\mathbb{Q}$, contains the image of a unique member of Λ_v.*

In the above result we observe that for any $g \in G(\mathbb{Q})$, gv is also unstable, $B_v = B_{gv}$, $\Lambda_{gv} = g\Lambda_v g^{-1}$, and $P_{gv} = gP_v g^{-1}$. Therefore, if $g \in P_v(\mathbb{Q})$, then by (3) of b) above, $\Lambda_{gv} = \Lambda_v$, and in particular, $m(gv,\lambda) = m(v,\lambda)$ for all $\lambda \in \Lambda_v$.

We will apply Kempf's theorem to the group $G = \mathrm{SL}_3(\mathbb{R})$. In that case, the maximal torus S is chosen to be the subgroup of $G = \mathrm{SL}_3(\mathbb{R})$ consisting of diagonal matrices. Then $\delta \in X_*(S)$ means that there exists a unique $(a,b,c) \in \mathbb{Z}^3$ such that $a+b+c = 0$ and $\delta(t) = \mathrm{diag}(t^a, t^b, t^c)$ for all $t \neq 0$. The Euclidean inner-product on

$\mathbb{Z}^3$ restricts to a Weyl group invariant inner-product on $X_*(S)$ with respect to this identification. So for δ as above, $\|\delta\| = \sqrt{a^2+b^2+c^2}$. Let $X^*(S)$, denote the abelian group of $\mathbb{Q}$-characters on S. Define a bilinear pairing $\langle\cdot,\cdot\rangle : X^*(S)\times X_*(S) \to \mathbb{Z}$ of the $\mathbb{Z}$-modules such that for any $\chi \in X^*(S)$ and $\delta \in X_*(S)$, we have $\chi(\delta(t)) = t^{\langle \chi,\delta\rangle}$ for all $t \neq 0$. Let $\delta^\vee \in X^*(S)$ denote the dual to δ in the following sense: $\langle \delta^\vee, \lambda\rangle = (\delta, \lambda)$ for all $\lambda \in X_*(S)$. So for the δ described as above, $\delta^\vee(\mathrm{diag}(t_1,t_2,t_3)) = t_1^a t_2^b t_3^c$ for all $\mathrm{diag}(t_1,t_2,t_3) \in S$.

Choose simple roots

$$\alpha_1\colon \mathrm{diag}(t_1,t_2,t_3) \mapsto t_1 t_2^{-1} \text{ and } \alpha_2\colon \mathrm{diag}(t_1,t_2,t_3) \mapsto t_2 t_3^{-1}.$$

The corresponding fundamental weights are

$$\omega_1\colon \mathrm{diag}(t_1,t_2,t_3) \mapsto t_1 \text{ and } \omega_2\colon \mathrm{diag}(t_1,t_2,t_3) \mapsto t_1 t_2.$$

For any non-negative integers n_1 and n_2, there exists a unique irreducible representation of G with highest weight $n_1\omega_1 + n_2\omega_2$, where we use the additive notation for $X^*(S)$ (see **[FH91]**[Theorem 13.1]).

The standard parabolic subgroups of G are

$$P_0 = \left\{\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}\right\}, \quad P_1 = \left\{\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}\right\}, \quad P_2 = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}\right\}.$$

We shall also use the following algebraic subgroups of G:

$$Q_0 = \left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\}, \quad Q_1 = \left\{\begin{pmatrix} 1 & * & * \\ & * & * \\ & * & * \end{pmatrix}\right\}, \quad Q_2 = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ & & 1 \end{pmatrix}\right\}.$$

$$S_0 = \left\{\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}\right\}, \quad S_1 = \left\{\begin{pmatrix} t^2 & & \\ & t^{-1} & \\ & & t^{-1} \end{pmatrix}\right\}, \quad S_2 = \left\{\begin{pmatrix} t & & \\ & t & \\ & & t^{-2} \end{pmatrix}\right\}.$$

$$H_0 = \{1\}, \quad H_1 = \left\{\begin{pmatrix} 1 & & \\ & * & * \\ & * & * \end{pmatrix}\right\}, \quad H_2 = \left\{\begin{pmatrix} * & * & \\ * & * & \\ & & 1 \end{pmatrix}\right\}.$$

$$U_0 = \left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\}, \quad U_1 = \left\{\begin{pmatrix} 1 & * & * \\ & 1 & \\ & & 1 \end{pmatrix}\right\}, \quad U_2 = \left\{\begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\}.$$

One has $P_i = S_i Q_i = S_i H_i U_i$, for $i = 0,1,2$.

2.1. **Reduction to a highest weight vector.** The next result (Theorem 2.1) provides a powerful new technique that allows one to reduce the study of linear dynamics of an arbitrary vector in an arbitrary representation to that of a highest weight vector. From this we will further reduce the study to fundamental representations (Lemma 2.2), opening the doors to directly relate the linear dynamics to Diophantine properties of vectors (Lemma 3.1).

The proof of the following result was motivated by **[Yan20**, Proposition 2.4**]**.

Throughout this article, we will assume that all the finite dimensional vector spaces are equipped with some norm, denoted by $\|\cdot\|$.

Theorem 2.1. *Let V be a representation of G defined over $\mathbb{Q}$. Let v be an unstable vector in $V(\mathbb{Q})$. Then there exists an irreducible representation W of G defined over $\mathbb{Q}$, a highest weight vector $w' \in W(\mathbb{Q})$, an element $g_0 \in G(\mathbb{Q})$, a real number $\beta > 0$, and a real number $C > 0$ such that for any $g \in G$ one has*

$$\|gg_0 w'\| \leq C\|gv\|^\beta.$$

Proof. Without loss of generality, we may assume that the norms are $K := \mathrm{SO}(3)$-invariant. Given the unstable $v \in V(\mathbb{Q})$, let $B_v > 0$, $\Lambda_v \subset X*(G)$, and a $\mathbb{Q}$-parabolic subgroup P_v of G be as given by Kempf's theorem. By **[Bor91**, Proposition 21.12**]**, there exist $g_0 \in G(\mathbb{Q})$ and $j \in \{0,1,2\}$ such that $P_v = g_0 P_j g_0^{-1}$. Let $v' = g_0^{-1} v$. Then $v' \in V(\mathbb{Q})$ is also unstable, and by Kempf's theorem, $P_{v'} = g_0^{-1} P_v g_0 = P_j$, and since $S \subset P_j = P_{v'}$, we have that $X_*(S)$ contains a unique member, say δ, of $\Lambda_{v'}$. Therefore

$$P(\delta) = P_{v'} = P_j.$$

Hence, $\operatorname{Im}\delta$ is contained in S_j, and $\delta(t) = \operatorname{diag}(t^a, t^b, t^c)$ for all $t \neq 0$ such that $(a,b,c) \in \mathbb{Z}^3$, $a+b+c=0$ and $a \geq b \geq c$.

Now let $\delta^\vee \in X^*(S)$ be dual to δ. Let S_δ denote the $\mathbb{Q}$-subtorus of S_j which is the identity component of the kernel of $\delta^\vee$ in S_j. We have that $\operatorname{Im}\delta \cap S_\delta$ is finite and $(\operatorname{Im}\delta)S_\delta = S_j$.

We have that $\delta^\vee = (a-b)\omega_1 + (b-c)\omega_2$, with $a-b \geq 0$ and $b-c \geq 0$. Therefore there exists an irreducible representation W of G defined over $\mathbb{Q}$ with the highest weight $\delta^\vee$. Let $w' \in W(\mathbb{Q})$ be a highest weight vector. Let

$$\beta = \frac{(\delta,\delta)}{m(v',\delta)} = \frac{1}{B_{v'}} > 0.$$

Now it suffices to show that there exists $C > 0$ such that for any $g \in G$ we have

$$\|gw'\| \leq C \|gv'\|^\beta.$$

To argue by contradiction, suppose that there exists a sequence $\{g_i\} \subset G$ such that

$$\lim_{i\to\infty} \frac{\|g_i v'\|^\beta}{\|g_i w'\|} = 0.$$

We note that $P(\delta) = P_j = S_j Q_j$, $Q_j = H_j U_j$ fixes w', and S_j acts on w' via the character $\delta^\vee$. Since $G = KP_j = KS_jH_jU_j$, we can write $g_i = k_i s_i h_i u_i$ where $k_i \in K$, $s_i \in S_j$, $h_i \in H_j$ and $u_i \in U_j$. Since the norms are K-invariant, we may assume that $k_i = e$ for all i. Now $s_i h_i u_i w' = \delta^\vee(s_i) w'$. Hence $\|g_i w'\| = |\delta^\vee(s_i)| \|w'\|$. Since $S_j = (\operatorname{Im}\delta)S_\delta$, we can write $s_i = \delta(\tau_i)\sigma_i$, where $\tau_i \in \mathbb{R}^\times$ and $\sigma_i \in S_\delta$. Then

$$\|g_i w'\| = |\delta^\vee(\delta(\tau_i))| \|w'\| = |\tau_i|^{(\delta,\delta)} \|w'\|. \tag{2.2}$$

We consider the weight space decomposition $V = \oplus V_\chi$, where S acts on V_χ by multiplication via the character χ of S, where each V_χ is defined over $\mathbb{Q}$ as S is a $\mathbb{Q}$-split torus. Let

$$\tilde{V} = \{x \in V : \delta(t)x = t^{m(v',\delta)} x\} = \oplus\{V_\chi : \chi \in X^*(S) \text{ and } \langle \chi, \delta\rangle = m(v',\delta)\}.$$

Let $\pi : V \to \tilde{V}$ denote the natural projection defined over $\mathbb{Q}$. Then $\pi(v') \in \tilde{V}(\mathbb{Q})$. Since $\delta \in \Lambda_{v'}$, we have that $\pi(v') = v'_{m(v',\delta)} \neq 0$. Since $S_j H_j$ is contained in the centralizer of δ, we have that π is $S_j H_j$-equivariant.

There exists $C_1 > 0$ such that $\|\pi(x)\| \leq C_1 \|x\|$ for all $x \in V$. It follows that

$$\frac{\|\pi(g_i v')\|^\beta}{\|g_i w'\|} \leq C_1^\beta \frac{\|g_i v'\|^\beta}{\|g_i w'\|} \xrightarrow[i\to\infty]{} 0. \tag{2.3}$$

For any $u \in U_j$, $\delta(t)u\delta(t)^{-1} \to e$ as $t \to 0$, so by (2.1), $\pi(uv') = \pi(v')$. Since $g_i = \delta(\tau_i)\sigma_i h_i u_i$,

$$\|\pi(g_i v')\| = \|\delta(\tau_i)\pi(\sigma_i h_i u_i v')\| = |\tau_i|^{m(v',\delta)}\|\sigma_i h_i \pi(v')\|. \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), since $m(v',\delta)\beta = (\delta,\delta)$, we get $\|\sigma_i h_i \pi(v')\| \to 0$. Since $\sigma_i \in S_\delta$ and $h_i \in H_j$, we conclude that $\pi(v')$ is $S_\delta H_j$-unstable in $\tilde{V}$.

Thus $S_\delta H_j$ is a reductive $\mathbb{Q}$-group acting on $\tilde{V}$ over $\mathbb{Q}$ and $\pi(v') \in \tilde{V}(\mathbb{Q}) \setminus \{0\}$ is an unstable vector for this action. Therefore by (a) of Kempf's theorem, there exists $\lambda \in X_*(S_\delta H_j)$ such that $\lambda(t)\pi(v') \to 0$ as $t \to 0$.

Since $S_\delta H_j \cap S$ is a maximal $\mathbb{Q}$-split torus of $S_\delta H_j$, by the conjugacy of maximal $\mathbb{Q}$-split tori, there exists $l \in (S_\delta H_j)(\mathbb{Q})$ such that $\delta_l := l\lambda l^{-1} \in X_*(S_\delta H_j \cap S)$. So $\delta_l(t)(l\pi(v')) \to 0$ as $t \to 0$. Now $l\pi(v') = \pi(lv')$. So for any $\chi \in X^*(S)$,

$$\text{if } (\pi(lv'))_\chi \neq 0, \text{ then } \langle \chi, \delta_l \rangle > 0.$$

Since $S_\delta H_j \cap S = \ker \delta^\vee$, we have $1 = \delta^\vee(\delta_l(t)) = t^{\langle \delta^\vee, \delta_l \rangle}$ for all $t \neq 0$, and hence

$$(\delta, \delta_l) = \langle \delta^\vee, \delta_l \rangle = 0. \tag{2.5}$$

Since $l \in P_j(\mathbb{Q}) = P_{v'}(\mathbb{Q})$, as we noted after the statement of Kempf's theorem,

$$\delta \in \Lambda_{lv'} \text{ and } m(lv', \delta) = m(v', \delta).$$

For a positive integer N, let $\delta_N = N\delta + \delta_l \in X_*(S)$, in the additive notation. For N large enough, we claim that

$$\frac{m(lv', \delta_N)}{\|\delta_N\|} > \frac{m(\delta, lv')}{\|\delta\|} = B_{lv'}, \tag{2.6}$$

which will contradict the maximality of $B_{lv'}$.

For any $w \in V$, we write $w = \sum_{\chi \in X^*(S)} w_\chi$, where $w_\chi \in V_\chi$. Note that for any nonzero $w \in V$ and $\lambda \in X_*(S)$, we have

$$m(w, \lambda) = \min\{\langle \chi, \lambda \rangle : \chi \in X_*(S),\, w_\chi \neq 0\}.$$

So to prove (2.6), we pick any $\chi \in X^*(S)$ such that $(lv')_\chi \neq 0$, and we will show that for all sufficiently large N,

$$\frac{\langle \chi, \delta_N \rangle}{\|\delta_N\|} > \frac{m(\delta, lv')}{\|\delta\|}. \tag{2.7}$$

By definition $m(lv', \delta) \leq \langle \chi, \delta \rangle$. First suppose that $\langle \chi, \delta \rangle > m(lv', \delta)$. Then

$$\lim_{N \to \infty} \frac{\langle \chi, \delta_N \rangle}{\|\delta_N\|} = \frac{\langle \chi, \delta \rangle}{\|\delta\|} > \frac{m(lv', \delta)}{\|\delta\|},$$

because $\langle \chi, \delta_l \rangle < \infty$. Therefore (2.7) follows for all sufficiently large N.

Now suppose that $\langle \chi, \delta \rangle = m(lv', \delta)$. Since $m(lv', \delta) = m(v', \delta)$, we have $(lv')_\chi \in \tilde{V}$. Since π is S-equivariant, we have $(\pi(lv'))_\chi = (lv')_\chi \neq 0$. Therefore $\langle \chi, \delta_l \rangle > 0$.

To prove (2.7), we define an auxiliary function:

$$f(s) = \frac{\langle \chi, \delta + s \cdot \delta_l \rangle^2}{\|\delta + s \cdot \delta_l\|^2} := \frac{\langle \chi, \delta \rangle^2 + 2s\langle \chi, \delta \rangle \langle \chi, \delta_l \rangle + s^2 \langle \chi, \delta_l \rangle^2}{(\delta, \delta) + 2s(\delta, \delta_l) + s^2(\delta_l, \delta_l)}, \forall s \in \mathbb{R}.$$

Compute its derivative at 0:

$$f'(0) = \frac{2\langle\chi,\delta\rangle\langle\chi,\delta_l\rangle(\delta,\delta) - 2\langle\chi,\delta\rangle^2(\delta,\delta_l)}{(\delta,\delta)^2}.$$

We have $\langle\chi,\delta\rangle = m(lv',\delta) > 0$ and $\langle\chi,\delta_l\rangle > 0$. Also $(\delta,\delta_l) = 0$ by (2.5). Therefore $f'(0) > 0$. Hence for N large we have

$$f(1/N) > f(0). \tag{2.8}$$

Now (2.7) follows because each side of (2.8) is the square of each corresponding side of (2.7). Therefore (2.6) holds, contradicting the maximality of $B_{lv'}$. □

2.2. **Reduction to fundamental representations.** Let $W_1 = \mathbb{R}^3$ and $W_2 = \bigwedge^2 \mathbb{R}^3$. Let $w_1 = e_1 \in W_1$ and $w_2 = e_1 \wedge e_2 \in W_2$. Let ω_1 and ω_2 be the highest weights of W_1 and W_2. Then ω_1 and ω_2 are the fundamental weights of G, and any dominant integral weight is a non-negative integral linear combination of ω_1 and ω_2.

Lemma 2.2. *Let W be an irreducible representation of G with highest weight $\omega = n_1\omega_1 + n_2\omega_2$, where n_1, n_2 are non-negative integers, and let $w \in W$ be a highest weight vector. Then for any real-analytic map $\psi\colon I \to G$, where $I \subset \mathbb{R}$ is a nontrivial compact interval, there exists a constant $c > 0$ such that for any $h_1, h_2 \in G$,*

$$\sup_{s\in I}\|h_1\psi(s)h_2 w\| \geq c \cdot \left(\min_{1\leq i\leq 2}\sup_{s\in I}\|h_1\psi(s)h_2 w_i\|\right)^{n_1+n_2}.$$

Proof. Let the notation be as in the beginning of this section. We have $G = KS_0U_0$. Hence for $g \in G$, we can write $g = ktu$ for $k \in K, t \in S_0 = S$ and $u \in U_0$. We note that w_1 and w_2 are both fixed by U_0. Taking any K-invariant norms on W_1 and W_2, we have $\|gw_1\| = |\omega_1(t)|\|w_1\|$ and $\|gw_2\| = |\omega_2(t)|\|w_2\|$. We take a K-invariant norm on W such that $\|w\| = \|w_1\|^{n_1}\|w_2\|^{n_2}$. Then for any $g \in G$, $\|gw\| = \|gw_1\|^{n_1}\|gw_2\|^{n_2}$. Now let

$$F(g) = \|h_1 g h_2 w\|^2 \quad \text{and} \quad F_i(g) = \|h_1 g h_2 w_i\|^2, \quad i = 1, 2.$$

Then F, F_1, F_2 are regular functions on G, and

$$F(g) = F_1(g)^{n_1} F_2(g)^{n_2}.$$

Let Z be the Zariski closure of $\psi(I)$ in G. Since ψ is analytic, Z is an irreducible algebraic set. We use the norm

$$\|F\| = \sup_{s\in I}|F(\psi(s))|$$

on the space of regular functions on Z. We claim that for any positive integers d_1 and d_2, there exists a constant $c = c(d_1, d_2) > 0$ such that for any polynomials E_1 and E_2 of degrees d_1 and d_2 respectively on Z, we have $\|E_1E_2\| \geq c\|E_1\|\|E_2\|$. Indeed, by homogeneity we only need to check this for $\|E_1\| = \|E_2\| = 1$, and then the possible values of $\|E_1E_2\|$ form a compact subset of $\mathbb{R}_{>0}$. Therefore,

$$\|F\| \geq c\|F_1\|^{n_1}\|F_2\|^{n_2} \geq c \cdot \left(\min_{1\leq i\leq 2}\|F_i\|\right)^{n_1+n_2}.$$

□

2.3. **From fundamental representations to the standard representation.** Let the notation be as before: We fix $(a,b) \in \mathbb{R}^2$, $I = [s_0, s_1] \subset \mathbb{R}$ for some $s_0 < s_1$, and for any $s \in I$ and $t \in \mathbb{R}$, we have

$$\phi_{a,b}(s) = \begin{pmatrix} 1 & s & as+b \\ & 1 & \\ & & 1 \end{pmatrix} \text{ and } g_t = \begin{pmatrix} e^{2t} & & \\ & e^{-t} & \\ & & e^{-t} \end{pmatrix}.$$

The following observation allows us to reduce our possibilities from all fundamental representations to only the standard representation.

Lemma 2.3. *There exists a constant C_I depending only on I such that for any non-zero $v \in W_2(\mathbb{Z}) = \bigwedge^2 \mathbb{Z}^3$ and $t \geq 0$,*

$$\sup_{s\in I} \|g_t \phi_{a,b}(s) v\| \geq C_I e^t. \tag{2.9}$$

Proof. Let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$ (and $\mathbb{Z}^3$). We write $e_{ij} = e_i \wedge e_j$ for the standard basis of $\bigwedge^2 \mathbb{R}^3$. For any $s \in I$ and $t \geq 0$, one can readily compute the matrix of $g_t \phi_{a,b}(s)$ in the standard basis (e_{23}, e_{13}, e_{12}):

$$\bigwedge^2 g_t \phi_{a,b}(s) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ se^t & e^t & 0 \\ -e^t(as+b) & 0 & e^t \end{pmatrix}.$$

So, for any $v = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ in $\bigwedge^2 \mathbb{R}^3$, $s \in I$ and $t \geq 0$,

$$g_t \phi_{a,b}(s) v = \begin{pmatrix} e^{-2t} p \\ e^t(sp+q) \\ e^t[-(as+b)p+r] \end{pmatrix}.$$

Now observe that

$$\max\{|s_0 p + q|, |s_1 p + q|\} \geq \min\left\{ \frac{|p|(s_1 - s_0)}{2}, \frac{|q|(s_1 - s_0)}{|s_0| + |s_1|} \right\},$$

so that if $(p,q,r) \in \mathbb{Z}^3 \setminus \{0\}$, then

$$\max_{s\in\{s_0,s_1\}} \{|sp+q|, |-(as+b)p+r|\} \geq C_I,$$

where

$$C_I = \min\left\{ \frac{s_1 - s_0}{2}, \frac{s_1 - s_0}{|s_0| + |s_1|}, 1 \right\} > 0,$$

because if $(p,q) = (0,0)$ then $|-(as+b)p+r| = |r| \geq 1$. So (2.9) follows. □

By combining the above results we obtain the following:

Proposition 2.4. *Let V be a finite-dimensional representation of G defined over $\mathbb{Q}$ and let $v_0 \in V(\mathbb{Q}) \setminus \{0\}$. Suppose that Gv_0 is not Zariski closed. Then given $C > 0$ there exists $R > 0$ such that the following holds: There exists $t_0 > 0$ such that for any $t > t_0$ and any $\gamma \in \Gamma$, if*

$$\sup_{s\in I} \|g_t \phi_{a,b}(s) \gamma v_0\| \leq C, \tag{2.10}$$

then there exists $v \in \mathbb{Z}^3 \setminus \{0\}$ such that

$$\sup_{s\in I}\|g_t\phi_{a,b}(s)v\| \le R. \tag{2.11}$$

Proof. Let S denote the boundary of Gv_0. By **[Kem78**, Lemma 1.1**]**, there exists a representation V' of G and a G-equivariant polynomial map $f\colon V \to V'$, both defined over $\mathbb{Q}$, such that $S = f^{-1}(0)$. It follows that $f(v_0) \in V'(\mathbb{Q})$ is unstable in V', and there exists a constant $C' > 0$ and a norm on V' such that (2.10) holds for $(f(v_0), V', C')$ in place of (v_0, V, C). Hence by replacing v_0 with $f(v_0)$ we may assume that v_0 is unstable in V.

Now we can apply Theorem 2.1, and conclude that there exists an irreducible representation W of G defined over $\mathbb{Q}$, a highest weight vector $w \in W(\mathbb{Q})$, an element $g_0 \in G(\mathbb{Q})$, and a constant $D > 0$ such that for any $t \ge 0$ and $\gamma \in \Gamma$ if (2.10) holds, then

$$\sup_{s\in I}\|g_t\phi_{a,b}(s)\gamma g_0 w\| \le D.$$

Combined with Lemma 2.2, this implies that there exists $D' > 0$, such that for any $t \ge 0$ and $\gamma \in \Gamma$, if (2.10) holds, then

$$\min_{1\le j\le 2}\sup_{s\in I}\|g_t\phi_{a,b}(s)\gamma g_0 w_j\| \le D'. \tag{2.12}$$

Since $g_0 \in G(\mathbb{Q})$, there exists $N \in \mathbb{N}$ such that $N \cdot \Gamma g_0 w_1 \subset W_1(\mathbb{Z}) = \mathbb{Z}^3$ and $N \cdot \Gamma g_0 w_2 \subset W_2(\mathbb{Z}) = \bigwedge^2 \mathbb{Z}^3$.

By Lemma 2.3, for any $t \ge 0$ and $\gamma \in \Gamma$, since $v_2 := N\gamma g_0 w_2 \in \bigwedge^2 \mathbb{Z}^3 \setminus \{0\}$, we have $\sup_{s\in I}\|g_t\phi_{a,b}(s)v_2\| \ge C_I e^t$. Set $R = ND'$ and $t_0 := \log RC_I^{-1}$. Then for any $t > t_0$ and $\gamma \in \Gamma$, we have

$$\sup_{s\in I}\|g_t\phi_{a,b}(s)(\gamma g_0 w_2)\| > R/N = D';$$

and hence if (2.12) holds, then $\sup_{s\in I}\|g_t\phi_{a,b}(s)(\gamma g_0 w_1)\| \le D'$.

Therefore, for any $t \ge t_0$ and $\gamma \in \Gamma$, if (2.10) holds, then (2.12) holds, so the non-zero vector $v = N\gamma g_0 w_1 \in \mathbb{Z}^3$ satisfies (2.11), as desired. ☐

3. Dynamics in the standard representation and Diophantine conditions

In this section we relate asymptotic dynamics of the g_t-action on the curves $\{\phi_{a,b}(s)v : s \in I\}$ for nonzero $v \in \mathbb{Z}^3$ in the standard representation, with some Diophantine approximation properties of the vector (a, b). Our first lemma characterizes the condition $(a, b) \in \mathcal{W}_2$ in terms of vectors of bounded size in the lattices $g_t\phi_{a,b}(s)\mathbb{Z}^3$, $s \in I$.

Lemma 3.1. *The following are equivalent:*

(1) $(a, b) \in \mathcal{W}_2$.

(2) *There exist $t_i \to \infty$, $\{v_i\} \subset \mathbb{Z}^3 \setminus \{0\}$ and $R > 0$ such that for all i,*

$$\sup_{s\in I}\|g_{t_i}\phi_{a,b}(s)v_i\| \le R. \tag{3.1}$$

Proof. We write $v_i = \left(\begin{smallmatrix} p_{1,i} \\ p_{2,i} \\ q_i \end{smallmatrix}\right) \in \mathbb{Z}^3 \setminus \{0\}$. It follows that

$$g_{t_i}\phi_{a,b}(s)v_i = \begin{pmatrix} e^{2t_i}\Big((bq_i+p_{1,i})+(aq_i+p_{2,i})s\Big) \\ e^{-t_i}p_{2,i} \\ e^{-t_i}q_i \end{pmatrix}. \tag{3.2}$$

(2)⇒(1): Let $s_0 < s_1$ such that $I = [s_0, s_1]$ and (3.1) holds for some $R > 0$. Let $R_1 = \left\| \left(\begin{smallmatrix} 1 & s_0 \\ 1 & s_1 \end{smallmatrix}\right)^{-1} \right\| \cdot R$, where $\|\cdot\|$ denotes the operator norm with respect to the sup-norm. Then for all i, the following system of inequalities hold:

$$\begin{cases} |q_i b + p_{1,i}| \le R_1 e^{-2t_i}, \\ |q_i a + p_{2,i}| \le R_1 e^{-2t_i}, \\ |q_i| \le R e^{t_i}. \end{cases} \tag{3.3}$$

Case 1. Suppose a subsequence of $\{q_i\}$ is bounded.

After passing to a subsequence, we may assume that $q_i = q$ is a constant. Since qa and qb are fixed, $\mathbb{Z}$ is discrete and $R_1 e^{-2t_i} \to 0$, the first two equations from (3.3) force that $qb + p_{1,i} = 0$ and $qa + p_{2,i} = 0$ for all large i. Since $(p_{1,i}, p_{2,i}, q_i) \neq 0$, we conclude that $q \neq 0$ and $(a, b) \in \mathbb{Q}^2$.

Case 2. Suppose $|q_i| \to \infty$ as $i \to \infty$.

Put $R_2 = R_1 R^2$. Then (3.3) shows that for all i,

$$\begin{cases} |q_i b + p_{1,i}| \le R_2 |q_i|^{-2} \\ |q_i a + p_{2,i}| \le R_2 |q_i|^{-2}. \end{cases} \tag{3.4}$$

Therefore $(a, b) \in \mathcal{W}_2$. Combining both cases we proved that (2)⇒(1).

(1)⇒(2): Suppose $(a, b) \in \mathcal{W}_2$. By (1.1) we pick a sequence $(p_{1,i}, p_{2,i}, q_i) \in \mathbb{Z}^3$ such that $0 \neq q_i \to \infty$ and (3.4) holds for some $R_2 \ge 0$. Then for $t_i = \log|q_i|$, we get (3.3) for $R_1 = R_2$ and $R = 1$ and $|p_{2,i}| \le R_1 e^{-2t_i} + |a| e^{t_i}$. So in view of (3.2), we get that (3.1) holds for $R = (|s_0| + |s_1| + 1)R_1 + |a| + 1$, where $I = [s_0, s_1]$. This completes the proof of (1)⇒(2). □

The second lemma shows that $(a, b) \in \mathcal{W}_2'$ if and only if the curve $\phi_{a,b}$ is entirely sent to the cusp under the action of g_t in the space of lattices, along a subsequence of times t_i going to infinity.

Lemma 3.2. *We have $(a, b) \in \mathcal{W}_2'$ if and only if there exist $t_i \to \infty$ and $\{v_i\} \subset \mathbb{Z}^3 \setminus \{0\}$ such that*

$$\sup_{s \in I} \|g_{t_i}\phi_{a,b}(s)v_i\| \to 0.$$

Proof. The proof is identical to that of Lemma 3.1, and we leave it to the reader. □

3.0.1. *Remark.* In view of the identification between X and the space of unimodular lattices in $\mathbb{R}^3$, given a compact set $K \subset X$, there exists $\delta > 0$ such that for any $g \in G$, if $g\mathbb{Z}^3 = gx_0 \in K$, then $\|gv\| \ge \delta$ for every $v \in \mathbb{Z}^3 \setminus \{0\}$.

Suppose $(a, b) \in \mathcal{W}_2'$. By Lemma 3.2, there exists sequences $t_i \to \infty$ and $v_i \in \mathbb{Z}^3 \setminus \{0\}$ and $i_0 \in \mathbb{N}$ such that $\sup_{s \in I} \|g_{t_i}\phi_{a,b}(s)v_i\| < \delta$ for all $i \ge i_0$. Therefore

$g_{t_i}\phi_{a,b}(s)x_0 \notin K$ for all $s \in I$, and hence $g_{t_i}\lambda_{a,b}(K) = 0$ for all $i \geq i_0$. So we say that the measures $g_{t_i}\lambda_{a,b}$ escape to infinity as $i \to \infty$.

In particular, $\{g_t\phi_{a,b}(s)x_0 : t \geq 0\}$ is unbounded in X for every $s \in \mathbb{R}$. Hence by Dani correspondence **[Dan85]**, $(s, as+b)$ is not badly approximable for any $s \in \mathbb{R}$.

The stronger condition that the measures $g_t\lambda_{a,b}$ go to the cusp for *all large* t can only be satisfied if $(a,b) \in \mathbb{Q}^2$; this is the content of the following lemma.

Lemma 3.3. *The following statements are equivalent:*

(1) $(a,b) \in \mathbb{Q}^2$.
(2) There exists $v \in \mathbb{Z}^3 \setminus \{0\}$ *such that* $\sup_{s\in I}\|g_t\phi_{a,b}(s)v\| \to 0$ *as* $t \to \infty$.
(3) There exists $R > 0$ *such that for each large* $t > 0$*, there exists* $v \in \mathbb{Z}^3 \setminus \{0\}$ *such that*

$$\sup_{s\in I}\|g_t\phi_{a,b}(s)v\| \leq R. \tag{3.5}$$

Proof. (1)⇒(2): Suppose that $a = p_1/q$ and $b = p_2/q$ for some $p_1, p_2 \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $v = \left(\begin{smallmatrix} -p_2 \\ -p_1 \\ q \end{smallmatrix}\right) \in \mathbb{Z}^3 \setminus \{0\}$. Then for any $s \in \mathbb{R}$ and $t \geq 0$,

$$g_t\phi_{a,b}(s)v = g_t \begin{pmatrix} 1 & s & as+b \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} -p_2 \\ -p_1 \\ q \end{pmatrix} = g_t \begin{pmatrix} -p_2 - sp_1 + (as+b)q \\ -p_2 \\ q \end{pmatrix} = e^{-t} \begin{pmatrix} 0 \\ -p_2 \\ q \end{pmatrix}.$$

Therefore (2) holds.

(2)⇒(3): This is is obvious.

(3)⇒(1): We observe using (3.3) that (3) implies the following: for any $c > 0$, and all sufficiently large enough $T > 0$, setting $t = \log T - \log R_1$, there exists $(p_1, p_2, q) \in \mathbb{Z}^3 \setminus \{0\}$ such that, all the following inequalities hold:

$$\begin{cases} |q| \leq R_1 e^t = T, \\ |qb + p_1| \leq R_1 e^{-2t} \leq cT^{-1}, \\ |qa + p_2| \leq R_1 e^{-2t} \leq cT^{-1}. \end{cases}$$

This implies that a and b are both singular real numbers. But singular real numbers are rational, see **[Khi26]** or **[Cas57**, Remark before Theorem XIV**]**. ☐

3.0.2. *Remark.* Suppose $(a,b) \in \mathbb{Q}^2$. Given a compact set $K \subset X$, let $\delta > 0$ be as in Remark 3.0.1. By the proof of (1)⇒(2) in Lemma 3.3, there exists $v \in \mathbb{Z}^3 \setminus \{0\}$ and $t_0 > 0$ such that $\|g_t\phi_{a,b}(s)v\| < \delta$ for all $t \geq t_0$ and all $s \in \mathbb{R}$. Therefore $g_t\phi_{a,b}(s)x_0 \notin K$ for all $t \geq t_0$ and for all $s \in \mathbb{R}$. Hence $g_t\lambda_{a,b}(K) = 0$ for all $t \geq t_0$. Therefore $g_t\lambda_{a,b}$ escapes to infinity as $t \to \infty$.

Finally, we have a version of Lemma 3.1 for the behavior on average of the measures $g_t\lambda_{a,b}$; this will relate to the set $\mathcal{W}_2^+$. For $R > 0$, define

$$\mathcal{I}_R = \left\{ t \in [0, +\infty) : \exists\, v \in \mathbb{Z}^3 \setminus \{0\} \text{ such that } \sup_{s\in I}\|g_t\phi_{a,b}(s)v\| < R \right\}. \tag{3.6}$$

Lemma 3.4. *The following are equivalent:*

(1) *For every* $R > 0$,

$$\limsup_{T\to\infty} \frac{|\mathcal{I}_R \cap [0,T]|}{T} > 0. \tag{3.7}$$

(2) *There exists* $R > 0$ *such that* (3.7) *holds.*
(3) $(a, b) \in \mathcal{W}_2^+$.

We defer the proof of this result to Section 8.

4. Reduction of linear dynamics to the standard representation

The following is one of the main technical results in this article. It shows that Proposition 2.4 still holds even if the orbit $G \cdot v_0$ is closed, as long as it is not reduced to $\{v_0\}$.

Theorem 4.1. *Let* V *be a finite-dimensional representation of* G *over* $\mathbb{Q}$ *and* $v_0 \in V(\mathbb{Q}) \setminus \{0\}$ *such that* v_0 *is not* G*-fixed. Then given* $C > 0$ *there exists* $R > 0$ *and* $t_0 > 0$ *such that the following holds: For every* $t > t_0$*, if there exists* $\gamma \in \Gamma$ *such that*

$$\sup_{s\in I} \|g_t \phi_{a,b}(s)\gamma v_0\| \le C,$$

then there exists $v \in \mathbb{Z}^3 \setminus \{0\}$ *such that*

$$\sup_{s\in I} \|g_t \phi_{a,b}(s) v\| \le R.$$

For the proof of the above theorem we introduce some notation and make some observations.

4.1. **Linear dynamics of g_t and $\phi_{a,b}(s)$ actions.** Let V be an irreducible real representation of $G = \mathrm{SL}_3(\mathbb{R})$ over $\mathbb{Q}$. Since G is $\mathbb{Q}$-split, $V \otimes \mathbb{C}$ is G-irreducible over $\mathbb{C}$.

We express

$$g_t = c_t b_t, \text{ where } b_t = \mathrm{diag}(e^{t/2}, e^{t/2}, e^{-t}) \text{ and } c_t = \mathrm{diag}(e^{3t/2}, e^{-3t/2}, 1).$$

Let

$$u_{23}(s) = \begin{pmatrix} 1 & & \\ & 1 & s \\ & & 1 \end{pmatrix},\ u_{12}(s) = \begin{pmatrix} 1 & s & \\ & 1 & \\ & & 1 \end{pmatrix},\ h_{a,b} = \begin{pmatrix} 1 & & b \\ & 1 & a \\ & & 1 \end{pmatrix}. \tag{4.1}$$

Then we have $\phi_{a,b}(s) = u_{23}(-a)u_{12}(s)h_{a,b}$. Since g_t commutes with $u_{23}(-a)$ and b_t commutes with $u_{12}(s)$,

$$g_t\phi_{a,b}(s) = u_{23}(-a)g_t u_{12}(s)h_{a,b} \text{ and } g_t u_{12}(s)h_{a,b} = c_t u_{12}(s) b_t h_{a,b}. \tag{4.2}$$

Let $H = H_2 = \begin{pmatrix} \mathrm{SL}_2(\mathbb{R}) & \\ & 1 \end{pmatrix} < G$, and consider V as the restricted representation of H. We have a decomposition $V = V_1 \oplus V_2$, where $V_1 = \{v \in V \mid \forall h \in H,\ hv = v\}$ is the subspace fixed by H, and V_2 is the complement of V_1 stable under the action of H. Let π_1 and π_2 be the H-equivariant projections from V to V_1 and V_2 respectively. Since b_t centralizes H, π_1 and π_2 are also b_t-equivariant.

The following observation is based on [**Sha09b**, Lemma 2.3].

Lemma 4.2 (Linear dynamics of an SL_2 action)**.** *For any $m \geq 0$, let W denote the $(m+1)$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. Let*

$$a_t = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \text{ and } u(s) = \left(\begin{smallmatrix} 1 & s \\ & 1 \end{smallmatrix}\right).$$

Then there exists $C_I > 0$ such that for any $w \in W$ and $t \in \mathbb{R}$,

$$\sup_{s \in I} \|a_t u(s) w\| \geq e^{mt} (C_I m)^{-m} \|w\|.$$

Proof. Let $e_0, \ldots, e_m$ denote a basis of W such that $a_t e_k = e^{(m-2k)t} e_k$ for all k. For any $w \in W$, we express $w = \sum_{k=0}^m w_k e_k$, where $w_k \in \mathbb{R}$. Then

$$(u(s)w)_0 = \sum_{k=0}^m w_k s^k.$$

Let $\|w\| = \max_{0 \leq k \leq m} |w_k|$. Recall that $I = [s_0, s_1]$ and let $\tau_j = s_0 + (j/m)(s_1 - s_0)$ for $0 \leq j \leq m$. Then

$$\sup_{s \in I} |(u(s)w)_0| \geq \max_{0 \leq j \leq m} \left| \sum_{k=0}^m w_k \tau_j^k \right| \geq (C_I m)^{-m} \|w\|, \tag{4.3}$$

where $C_I = (1 + \max\{|s_0|, |s_1|\})/(s_1 - s_0)$, from an estimate for the norm of the inverse of the $(m+1) \times (m+1)$-Vandermonde matrix (τ_j^k) **[Gau62**, Theorem 1**]**. □

Corollary 4.3. *There exist constants $C_2 > 0$ and $\beta \geq 3/2$ such that for all $v \in V_2$ and all $t \geq 0$,*

$$\sup_{s \in I} \|c_t u_{12}(s) v\| \geq C_2 e^{\beta t} \|v\|. \tag{4.4}$$

Proof. Consider the action of $H \cong \mathrm{SL}_2(\mathbb{R})$ on any irreducible component W of V_2. By definition of V_2, the representation W is non-trivial, i.e. $\dim W = m + 1$, with $m \geq 1$. Under the identification of H with $\mathrm{SL}_2(\mathbb{R})$, we have $c_t = a_{3t/2}$ and $u_{12}(s) = u(s)$. Therefore Lemma 4.2 shows that

$$\sup_{s \in I} \|c_t u_{12}(s) w\| \geq C_2 e^{\frac{3m}{2} t} \|w\|,$$

where $C_2 = (C_I m)^{-m} > 0$. Since this holds for every H-irreducible component $W \subset V_2$, we indeed obtain the desired inequality for all v in V_2, with

$$\beta = \min \left\{ \frac{3m}{2} \ ; \ m = \dim W - 1, \ W \subset V_2 \text{ irreducible} \right\}.$$

□

Let $U_{12} = \{u_{12}(s)\}_{s \in \mathbb{R}}$, and $V^{U_{12}}$ be the subspace of U_{12}-fixed vectors in V. Let $\pi_{U_{12}} : V \to V^{U_{12}}$ denote the $\{c_t\}_{t \in \mathbb{R}}$-equivariant projection. We apply (4.3) of Lemma 4.2 to each H-irreducible component of V to obtain the following:

Corollary 4.4. *There exists $C_4 = C_4(I, V) > 0$ such that for any $v \in V$,*

$$\sup_{s \in I} \|\pi_{U_{12}}(u_{12}(s) v)\| \geq C_4 \|v\|.$$

4.2. **Consequences of description of irreducible representations of** $\mathrm{SL}(3,\mathbb{R})$**.** Let $\omega = n_1\omega_1 + n_2\omega_2$ be the highest weight of V, where n_1, n_2 are non-negative integers.

Lemma 4.5. $\dim(V_1) = 1$ *and its weight is* $-(n_1 - n_2)\omega_2$*. In particular,* b_t *acts on* V_1 *as the scalar multiplication by* $e^{-(n_1-n_2)t}$*.*

Proof. We consider the weight diagram and multiplicities of the weights of an irreducible SL_3-representation as in Figure 4.1; see **[FH91**, §13.2**]**.

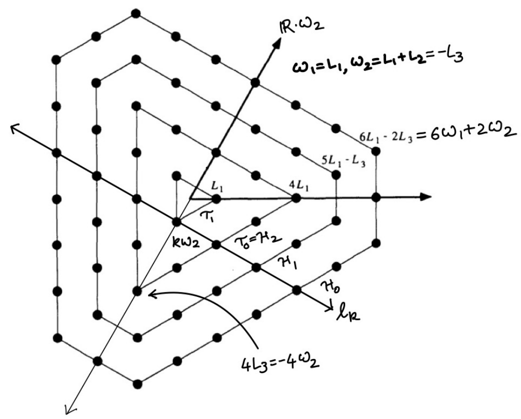

Figure 4.1. SL_3-representation with the highest weight $6\omega_1 + 2\omega_2$. (Based on **[FH91**, Figure (13.6)**]**.)

The weights of V lie on hexagons $\mathcal{H}_0, \dots, \mathcal{H}_{m-1}$, where $m = \min(n_1, n_2)$, and triangles $\mathcal{T}_0, \dots, \mathcal{T}_{\lfloor |n_1 - n_2|/3 \rfloor}$. We set $\mathcal{H}_m = \mathcal{T}_0$ and also call it a hexagon, which is degenerate. The multiplicity of a weight on any $\mathcal{H}_i$ is $i + 1$ and on any triangle is $m + 1$.

Consider any weight of V_1. Then it is fixed by the Weyl reflection corresponding to H, so it must be $k\omega_2$ for some $k \in \mathbb{Z}$. Let ℓ_k be the line perpendicular to ω_2 and passing through $k\omega_2$. Then $\ell_k \cap \mathcal{H}_i \neq \emptyset$ if and only if $0 \le i \le j_k - 1$, where j_k is the multiplicity of $k\omega_2$ in V, and for each such i, $\ell_k \cap \mathcal{H}_i$ contains the highest and the lowest weights of an irreducible representation of H containing $k\omega_2$.

In the above description, there is exactly one case when we have a trivial H-representation; that is, when $k\omega_2$ is a vertex of the triangle $\mathcal{T}_0 = \mathcal{H}_m$ and $\ell_k \cap \mathcal{H}_m = \{k\omega_2\}$. In particular, $\dim V_1 = 1$. The dominant vertex of $\mathcal{T}_0$ is $(n_1 - n_2)\omega_1$ or $(n_2 - n_1)\omega_2$. A Weyl reflection sends ω_1 to $-\omega_2$. So in both cases, $-(n_1 - n_2)\omega_2$ is a vertex of $\mathcal{T}_0$ and $k = -(n_1 - n_2)$. This proves the claim. ☐

Lemma 4.6. *Suppose* $n_1 \le n_2$*. Then all the weights occurring in* $V^{U^{12}}$ *are non-negative for the Lie algebra element* $\mathrm{diag}(2, -1, -1)$ *corresponding to* g_t*.*

In particular, by Corollary 4.4, for any $v \in V$ *and* $t \ge 0$*,*

$$\sup_{s \in I} \|g_t u_{12}(s) v\| \ge C_4 \|v\|. \tag{4.5}$$

Proof. We consider the weight diagram of an irreducible SL_3-representation as in Figure 4.2; see **[FH91**, Proposition 12.18**]**.

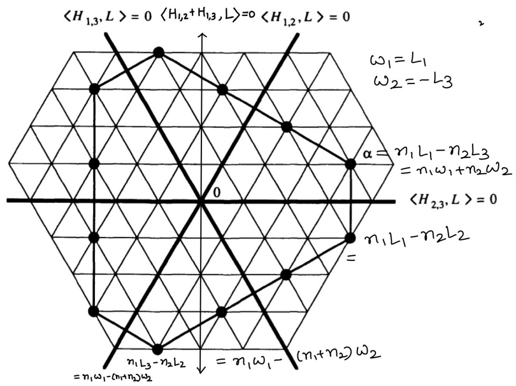

FIGURE 4.2. Arrangement of weights of an SL_3-representation. (Based on **[FH91**, Figure (12.14)**]**.)

We note that the weights on the vertical line passing through the origin in the weight diagram (see Figure 4.2) vanish on $\mathrm{diag}(2,-1,-1)$ and those on the right half take positive values.

The weights occurring in $V^{U_{12}}$ are the highest weights of irreducible representations of H in V, and they lie on two of the sides of the hexagons $\mathcal{H}_i$, where $0 \leq i \leq \min(n_1, n_2) = n_1$; if one draws the set of weights as in Figure 4.2, then for each i, one of the sides is a vertical segment from the dominant weight $(n_1 - i)L_1 - (n_2 - i)L_3$ to the weight $(n_1 - i)L_1 - (n_2 - i)L_2$, and the other side is the segment joining the last weight to $(n_1 - i)L_3 - (n_2 - i)L_2$. For $\mathrm{diag}(2,-1,-1)$, all the weights on the vertical segment have constant non-negative value

$$2(n_1 - i) + (n_2 - i) \geq n_2 - n_1,$$

and the weight $(n_1 - i)L_3 - (n_2 - i)L_2$ has the value

$$(n_1 - i)(-1) - (n_2 - i)(-1) = n_2 - n_1 \geq 0.$$

So all the weights on both segments have a non-negative value for $\mathrm{diag}(2,-1,-1)$. (Note that this is not the case in Figure 4.2, where $n_1 = 3 > 1 = n_2$.) $\square$

4.3. **Proof of Theorem 4.1.** If Gv_0 is not Zariski closed, then the result follows from Proposition 2.4. So we assume that Gv_0 is Zariski closed.

If the theorem fails to hold, then there exist sequences $t_i \to \infty$ and $\gamma_i \in \Gamma$ such that

$$\sup_{s\in I} \|g_{t_i}\phi_{a,b}(s)\gamma_i v_0\| \leq C, \tag{4.6}$$

and for every sequence $v_i \in \mathbb{Z}^3 \setminus \{0\}$,

$$\sup_{s\in I} \|g_{t_i}\phi_{a,b}(s) v_i\| \to \infty. \tag{4.7}$$

Write $\|\cdot\|_V$ to denote the operator norm on the linear space V. By (4.6) and (4.2), for $C_1 = \frac{C}{\|u_{23}(-a)\|_V} > 0$, for all i,

$$C_1 \geq \sup_{s\in I}\|g_{t_i}u_{12}(s)(h_{a,b}\gamma_i v_0)\| = \sup_{s\in I}\|c_{t_i}u_{12}(s)(b_{t_i}h_{a,b}\gamma_i v_0)\|. \tag{4.8}$$

We recall that $V = V_1 \oplus V_2$, where H acts trivially on V_1, and V_2 is a sum of non-trivial irreducible representations of H, and for $j = 1, 2$, $\pi_j : V \to V_j$ is the corresponding projection. By Corollary 4.3, this implies that there exist $C_3 > 0$ and $\beta \geq 3/2$ such that

$$\forall i, \quad \|\pi_1(b_{t_i}h_{a,b}\gamma_i v_0)\| \leq C_3 \quad \text{and} \quad \|\pi_2(b_{t_i}h_{a,b}\gamma_i v_0)\| \leq C_3 e^{-\beta t_i}. \tag{4.9}$$

There are two cases. We will show that each will lead to a contradiction.

Case 1: $\{\gamma_i v_0\}_{i\in\mathbb{N}}$ is unbounded in V.

Let n_1 and n_2 be non-negative integers such that the highest weight of the irreducible G-representation V is $n_1\omega_1 + n_2\omega_2$.

First suppose that $n_1 > n_2$. The highest eigenvalue of b_t on V is $e^{(n_1/2+n_2)t}$. We pick $\varepsilon > 0$ such that $\varepsilon(n_1/2 + n_2) < \beta$. By Lemma 4.5, b_t acts on V_1 by the scalar $e^{-(n_1-n_2)t}$. Therefore, by (4.9), for all i,

$$\|b_{\varepsilon t_i}\pi_1(b_{t_i}h_{a,b}\gamma_i v_0)\| \leq C_3 e^{-\varepsilon(n_1-n_2)t_i} \text{ and}$$
$$\|b_{\varepsilon t_i}\pi_2(b_{t_i}h_{a,b}\gamma_i v_0)\| \leq C_3 e^{(-\beta+\varepsilon(n_1/2+n_2))t_i}.$$

So $b_{(1+\varepsilon)t_i}h_{a,b}\gamma_i v_0 \to 0$ as $i \to \infty$. This contradicts the fact that Gv_0 is Zariski closed.

Hence we must have $n_1 \leq n_2$. Then by (4.5) and (4.8) we get $\{h_{a,b}\gamma_i v_0\}_{i\in\mathbb{N}}$ is bounded, and it follows that $\{\gamma_i v_0\}_{i\in\mathbb{N}}$ is bounded. This contradicts the assumption of Case 1.

Case 2: $\{\gamma_i v_0\}_{i\in\mathbb{N}}$ is bounded in V.

In this case using (4.6) we will deduce that $(a, b) \in \mathbb{Q}^2$, which contradicts (4.7) by Lemma 3.3 .

Since Γv_0 a discrete subset of V and $\{\gamma_i v_0\}_{i\in\mathbb{N}}$ is bounded, there exists $j \in \mathbb{N}$ such that $\gamma_i v_0 = \gamma_j v_0$ for infinitely many $i \in \mathbb{N}$. Therefore, replacing v_0 with $\gamma_j v_0$ and γ_i with $\gamma_i\gamma_j^{-1} \in \Gamma$ for each i, after passing to a subsequence we may assume that $\gamma_i v_0 = v_0$ for all i and (4.6) holds. Let

$$F = \operatorname{Stab}_G(v_0).$$

Since $v_0 \in V(\mathbb{Q})$ and v_0 is not G-fixed, F is a proper algebraic subgroup of G defined over $\mathbb{Q}$. Since Gv_0 is Zariski closed, $G/F \cong Gv_0$ is an affine variety. So by Matsushima's criterion **[Bor69**, §7.10**]**, F is a reductive subgroup of G. Thus F is a proper reductive algebraic subgroup of G defined over $\mathbb{Q}$.

Now (4.9) implies that $\{b_{t_i}h_{a,b}v_0\}_{i\in\mathbb{N}}$ is bounded. Therefore after passing to a subsequence we may assume that

$$b_{t_i}h_{a,b}v_0 \to v_\infty \quad \text{for some } v_\infty \in V \setminus \{0\}.$$

Since Gv_0 is Zariski closed, we may write $v_\infty = gv_0$ for some $g \in G$. Let

$$L = \mathrm{Stab}_G(v_\infty).$$

Then $L = gFg^{-1}$ and $L \neq G$. It is clear from (4.9) that $v_\infty \in V_1$ is fixed by H, and we also know that v_∞ is fixed by $\{b_t\}_{t\in\mathbb{R}}$ by definition of v_∞. Hence L is a proper reductive subgroup of G which contains the group generated by H and $\{b_t\}_{t\in\mathbb{R}}$; the normalizer $N_G(H)$ of H. But $N_G(H)$ is a maximal reductive subgroup of G. Hence

$$L = N_G(H).$$

Claim. We can pick $h_0 \in G(\mathbb{Q})$ such that $v_\infty = h_0 v_0$.

To prove the claim, consider the center of F, denoted $Z(F)$. Then

$$gZ(F)g^{-1} = Z(L) = Z(N_G(H)) = S_2 = \left\{ \left(\begin{smallmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{smallmatrix} \right) \right\}.$$

Therefore, since F is an algebraic subgroup of G defined over $\mathbb{Q}$, $Z(F)$ is a one-dimensional $\mathbb{R}$-split torus in G defined over $\mathbb{Q}$. In particular, $Z(F)(\mathbb{Q})$ is Zariski dense in $Z(F)$, and hence a single element, say $\gamma \in Z(F)(\mathbb{Q})$, generates a Zariski dense subgroup of $Z(F)$. Since $g\gamma g^{-1} \in S_2$, the roots of the characteristic polynomial of γ are t, t, and t^{-2} for some $t \in \mathbb{R} \setminus \{0\}$. Since $\gamma \in \mathrm{SL}(3,\mathbb{Q})$, these roots permute under the Galois action. We conclude that t is fixed by this action, so $t \in \mathbb{Q}$. Hence there exists $h_0 \in G(\mathbb{Q})$ such that

$$h_0 \gamma h_0^{-1} = \mathrm{diag}(t, t, t^{-2}) \in S_2.$$

Therefore $h_0 Z(F) h_0^{-1} \subset S_2$. Hence $h_0 Z(F) h_0 = S_2$. The centralizers of S_2 in G is $N_G(H) = L$. Therefore the centralizer of $Z(F)$ in G is conjugate to F and contains F, so it equals F. Therefore $h_0 F h_0^{-1} = L$. Since $gFg^{-1} = L$, we have $h_0 g^{-1} \in N_G(L) = L$, as $L = N_G(H)$ is a maximal subgroup. Hence $h_0 v_0 = h_0 g^{-1} v_\infty = v_\infty$. This proves the claim.

Thus $b_{t_i} h_{a,b} v_0 = b_{t_i} h_{a,b}(h_0^{-1} v_\infty) \to v_\infty$. Since $\mathrm{Stab}_G(v_\infty) = N_G(H)$ and the orbit Gv_∞ is locally compact, the map $g[N_G(H)] \mapsto gv_\infty$ from $G/N_G(H) \to V$ is a homeomorphism onto its image. Therefore

$$b_{t_i} h_{a,b} h_0^{-1} [N_G(H)] \to [N_G(H)]$$

in $G/N_G(H)$ as $i \to \infty$. Consider the standard projective action of G on $\mathbb{P}(\mathbb{R}^3)$. Then $N_G(H)$ fixes $\langle e_3 \rangle$. So

$$b_{t_i} h_{a,b} h_0^{-1} \langle e_3 \rangle \to \langle e_3 \rangle$$

as $i \to \infty$. Since $b_t = \mathrm{diag}(e^{t/2}, e^{t/2}, e^{-t})$, we conclude that $h_{a,b} h_0^{-1} \langle e_3 \rangle = \langle e_3 \rangle$. So $h_0^{-1} e_3 = \lambda h_{a,b}^{-1} e_3$ for some $\lambda \neq 0$. Since $h_0^{-1} \in G(\mathbb{Q})$, by (4.1) we get $\lambda(-b, -a, 1) \in \mathbb{Q}^3$. So $\lambda \in \mathbb{Q}$, and hence $(a, b) \in \mathbb{Q}^2$.

As noted earlier, $(a, b) \in \mathbb{Q}^2$ contradicts (4.7) in view of by Lemma 3.3. $\square$

5. The Dani-Margulis criterion for non-escape of mass

Let $\lambda_{a,b}$ be the probability measure on X given by (1.2). The goal of this section is to give a necessary and sufficient condition for non-escape of mass for $\{g_{t_i}\lambda_{a,b}\}$ as $t_i \to \infty$.

Define the unipotent one-parameter subgroup

$$W = \{w(r) := \begin{pmatrix} 1 & r & ar \\ & 1 & \\ & & 1 \end{pmatrix} : r \in \mathbb{R}\} \subset G.$$

For any $t \in \mathbb{R}$ and $s \in I = [s_0, s_1]$, we have

$$g_t\phi_{a,b}(s) = w(r)h_t, \text{ where } r = e^{3t}(s - s_0),\ h_t = g_t\phi_{a,b}(s_0). \tag{5.1}$$

Hence the trajectory $\{g_t\phi_{a,b}(s)x_0 : s \in I\}$ equals $\{w(r)h_tx_0 : r \in [0, e^{3t}|I|]\}$, which is a segment of a unipotent orbit.

By a criterion due to Dani and Margulis for analyzing non-escape of mass for unipotent trajectories on the space of unimodular lattices, we obtain the following:

Proposition 5.1. *For any $\varepsilon > 0$ and $R > 0$, there exist a compact set $K \subset X$ and $t_I \geq 0$ such that for any $t \geq t_I$, one of the following two possibilities holds:*

(1) $|\{s \in I : g_t\phi_{a,b}(s)x_0 \in K\}| \geq (1 - \varepsilon)|I|$.
(2) *There exists $w \in \mathbb{Z}^3 \setminus \{0\}$ such that*

$$\sup_{s \in I} \|g_t\phi_{a,b}(s)w\| < R.$$

Proof. By the result of Dani and Margulis **[DM89**, 1.1. Theorem**]**, given any $\varepsilon > 0$ and $R > 0$, we can pick a compact set $K \subset X$ such that given any finite interval $I \subset \mathbb{R}$ and any $t \geq 0$ one of the following three statements holds: the above condition (1), or the above condition (2) or the following additional condition (3): there exists $w \in \bigwedge^2 \mathbb{Z}^3 \setminus \{0\}$ such that

$$\sup_{s \in I} \|g_t\phi_{a,b}(s)w\| \leq R.$$

Now if (3) holds for some t, then $C_Ie^t \leq R$ by Lemma 2.3. So the additional condition (3) will not occur for $t \geq t_I := \log(C_I^{-1}R)$. ☐

Proposition 5.2. *Let $\{t_i\}_{i \in \mathbb{N}}$ be a sequence of real numbers such that $t_i \to \infty$. The following are equivalent:*

(1) *For every compact set $K \subset X$, $g_{t_i}\phi_{a,b}(I) \cap K = \emptyset$ for all large i.*
(2) *There exists $\varepsilon > 0$ such that for every compact set $K \subset X$,*

$$g_{t_i}\lambda_{a,b}(K) \leq 1 - \varepsilon, \text{ for all large } i. \tag{5.2}$$

(3) *There exist vectors $\{v_i\} \subset \mathbb{Z}^3 \setminus \{0\}$ such that*

$$\sup_{s \in I} \|g_{t_i}\phi_{a,b}(s)v_i\| \to 0 \text{ as } i \to \infty. \tag{5.3}$$

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3): Fix $\varepsilon > 0$ so that (5.2) holds for every compact $K \subset X$. For each $j \in \mathbb{N}$ and $R = 1/j$, obtain a compact set $K_j \subset X$ as in Proposition 5.1. Then for all i,

$$g_{t_i}\lambda_{a,b}(K_j) = \frac{1}{|I|}|\{s \in I : g_{t_i}\phi_{a,b}(s)x_0 \in K_j\}|.$$

So, by (5.2) there exists $i_j \in \mathbb{N}$ such that the possibility (1) of Proposition 5.1 does not hold for all $i \geq i_j$; and hence its second assertion (2) must hold. So for all $i \geq i_j$, there exists $w_{j,i} \in \mathbb{Z}^3 \setminus \{0\}$ such that

$$\sup_{s\in I}\|g_{t_i}\phi_{a,b}(s)w_{j,i}\| \leq 1/j. \tag{5.4}$$

For each $i \geq i_1$, let j be maximal such that $i \geq i_j$, and put $v_i = w_{j,i}$. Then (5.3) follows from (5.4).

(3) $\Rightarrow$ (1) is a straightforward consequence of Mahler's compactness criterion. $\square$

Proof of Theorem 1.1. To say that the sequence of g_t-translates of $\lambda_{a,b}$ has no escape of mass means that there exists a sequence $t_i \to \infty$ such that condition (2), and hence equivalently condition (3), of Proposition 5.2 fails to hold. It remains to apply Lemma 3.2. $\square$

6. Ratner's theorem and a linear dynamical criterion for avoidance of singular sets

The collection of probability measures on the one-point compactification, say $\bar{X} = G/\Gamma \cup \{\infty\}$, of $X = G/\Gamma$ is compact with respect to the weak-* topology on $\bar{X}$. So given any sequence $t_i \to \infty$, after passing to a subsequence, we obtain that $g_{t_i}\lambda_{a,b}$ converges to a probability measure $\bar{\mu}$ on $\bar{X}$. Let μ denote the restriction of $\bar{\mu}$ to X. Then $g_{t_i}\lambda_{a,b}$ converges to μ with respect to the weak-* topology; that is, for all $f \in C_c(X)$, we have

$$\lim_{i\to\infty}\int_X f\,\mathrm{d}(g_{t_i}\lambda_{a,b}) = \int_X f\,\mathrm{d}\mu.$$

For the proposition below, recall that

$$W = \left\{w(r) = \begin{pmatrix} 1 & r & ar \\ & 1 & \\ & & 1\end{pmatrix} : \ r \in \mathbb{R}\right\}.$$

Proposition 6.1. *Suppose that μ is a weak-* limit of $g_{t_i}\lambda_{a,b}$ for a sequence $t_i \to \infty$. Then μ is invariant under the action of W.*

Proof. By (5.1), for any $t > 0$ and any $f \in C_c(X)$,

$$\int_X f\,\mathrm{d}(g_t\lambda_{a,b}) = \fint_I f(g_t\phi_{a,b}(s)x_0)\,\mathrm{d}s = \frac{1}{e^{3t}}\int_0^{e^{3t}} f(w(r)h_tx_0)\,\mathrm{d}r, \tag{6.1}$$

where $h_t = g_t\phi_{a,b}(s_0)$. So for any $r_0 \in \mathbb{R}$,

$$\begin{aligned}&\textstyle\int_X f(w(r_0)x)\,\mathrm{d}(g_t\lambda_{a,b})(x) = \frac{1}{e^{3t}}\int_0^{e^{3t}} f(w(r_0)w(r)h_tx_0)\,\mathrm{d}r\\ &\textstyle= \frac{1}{e^{3t}}\int_{r_0}^{r_0+e^{3t}} f(w(r)h_tx_0)\,\mathrm{d}r = \frac{1}{e^{3t}}\int_0^{e^{3t}} f(w(r)h_tx_0)\,\mathrm{d}r + \varepsilon_t\\ &\textstyle= \int_X f\,\mathrm{d}(g_t\lambda_{a,b}) + \varepsilon_t,\end{aligned} \tag{6.2}$$

where $|\varepsilon_t| \leq 2r_0 e^{-3t} \|f\|_\infty$. Then observe that the left-most (resp., right-most) term of (6.2) converges to $\int_X f(w(r_0)x)\,\mathrm{d}\mu(x)$ (resp., to $\int_X f\,\mathrm{d}\mu$) as $t = t_i \to \infty$. $\square$

Proposition 6.2. *Suppose that μ is a weak-* limit of $(1/T_i)\int_0^{T_i} g_t\lambda_{a,b}\,\mathrm{d}t$ for a sequence $T_i \to \infty$. Then μ is invariant under the action of W.*

Proof. We perform the average over $t \in [0, T_i]$ in (6.1) and (6.2) and take the limit as $i \to \infty$ to conclude that μ is $w(r_0)$-invariant. $\square$

With Proposition 6.1, we will be able to apply Ratner's description of ergodic invariant measures for actions of unipotent one-parameter subgroups on X to analyze the limiting distributions of $\{g_t\lambda_{a,b}\}$ as $t \to \infty$. For this purpose we will apply what is now called 'the linearization technique' **[DM93]**.

Let $\pi\colon G \to X$ denote the natural quotient map. Let $\mathcal{H}$ denote the collection of closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H, and such that a unipotent one-parameter subgroup contained in H acts ergodically with respect to the H-invariant probability measure on $H/H \cap \Gamma$. Then any $H \in \mathcal{H}$ is a real algebraic group defined over $\mathbb{Q}$ **[Sha91**, (3.2) Proposition**]**. In particular, $\mathcal{H}$ is a countable collection **[Rtn91]**.

Let W be a one-parameter unipotent subgroup of G. For a closed connected subgroup H of G, define

$$N(H, W) = \{g \in G\colon g^{-1}Wg \subset H\}.$$

Now, suppose that $H \in \mathcal{H}$. We define the associated singular set

$$S(H, W) = \bigcup_{F\in\mathcal{H},\, F\subsetneq H} N(F, W).$$

Note that $N_G(W)N(H, W) = N(H, W) = N(H, W)N_G(H)$. By **[MS95**, Proposition 2.1, Lemma 2.4**]**,

$$N(H, W) \cap N(H, W)\gamma \subset S(H, W),\ \forall\gamma \in \Gamma\backslash N_G(H).$$

By Ratner's theorem **[Rtn91**, Theorem 1**]**, as explained in **[MS95**, Theorem 2.2**]**, we have the following.

Theorem 6.3 (Ratner)**.** *Given a W-invariant probability measure λ on X, there exists $H \in \mathcal{H}$ such that*

$$\lambda(\pi(N(H, W))) > 0 \quad \textit{and} \quad \lambda(\pi(S(H, W))) = 0.$$

Moreover, almost every W-ergodic component of λ restricted to $\pi(N(H, W))$ is a measure of the form $g\mu_H$, where $g \in N(H, W)\backslash S(H, W)$ and μ_H is a finite H-invariant measure on $\pi(H) \cong H/H \cap \Gamma$.

Further, if H as above is a normal subgroup of G, then λ is H-invariant.

To justify the last sentence in Theorem 6.3, note that $\lambda(\pi(N(H, W))) > 0$, so $N(H, W) \neq \emptyset$. Since $N_G(H) = G$, we have $N(H, W) = N(H, W)N_G(H) = G$, and hence λ restricted to $\pi(N(H, W))$ equals λ. And for every $g \in G$, $g\mu_H$ is H-invariant. So almost every W-ergodic component of λ is H-invariant, so λ is H-invariant.

Now let $H \in \mathcal{H}$ and put $d = \dim H$. Let $\mathfrak{g}$ denote the Lie algebra of G and take $V = \bigwedge^d \mathfrak{g}$. Then V admits a $\mathbb{Q}$-structure corresponding to the standard $\mathbb{Q}$-structure on $\mathfrak{g}$. Also G acts on V via the adjoint action of G on $\mathfrak{g}$. Since H is defined over $\mathbb{Q}$, its Lie algebra $\mathfrak{h}$ is a $\mathbb{Q}$-subspace of $\mathfrak{g}$. Fix $p_H \in \bigwedge^d \mathfrak{h}(\mathbb{Q})\setminus\{0\}$. Then the orbit Γp_H is a discrete subset of V. We note that for any $g \in N_G(H)$, $gp_H = \det(\operatorname{Ad} g|_{\mathfrak{h}})p_H$. Hence the stabilizer of p_H in G equals

$$N_G^1(H) := \{g \in N_G(H)\colon \det(\operatorname{Ad} g|_{\mathfrak{h}}) = 1\}.$$

Fix $w_0 \in \mathfrak{g}$ such that $\operatorname{Lie}(W) = \mathbb{R}w_0$, and for V as above define

$$\mathcal{A} = \{v \in V\colon v \wedge w_0 = 0\}.$$

Then $\mathcal{A}$ is a linear subspace of V and we observe that

$$N(H,W) = \{g \in G\colon g \cdot p_H \in \mathcal{A}\}.$$

By the linearization technique **[DM93**, Proposition 4.2**]** we obtain the following:

Proposition 6.4. *Let $\mathcal{C}$ be a compact subset of $N(H,W) \setminus S(H,W)$. Given $\varepsilon > 0$, there exists a compact set $\mathcal{D} \subset \mathcal{A}$ such that, given a neighborhood Φ of $\mathcal{D}$ in V, there exists a neighborhood $\mathcal{O}$ of $\pi(\mathcal{C})$ in X such that for any $t \in \mathbb{R}$ and any subinterval $J \subset I$, one of the following statements holds:*

(1) $|\{s \in J\colon g_t\phi_{a,b}(s)x_0 \in \mathcal{O}\}| \leq \varepsilon|J|$.
(2) *There exists $\gamma \in \Gamma$ such that $g_t\phi_{a,b}(s)\gamma p_H \in \Phi$ for all $s \in J$.*

Let $\lambda_{a,b}$ be as in (1.2).

Proposition 6.5. *Let μ be a weak-* limit of $g_{t_i}\lambda_{a,b}$ for a sequence $t_i \to \infty$. Suppose μ is not the G-invariant probability measure μ_X. Then there exists $R > 0$ and a sequence $\{v_i\} \subset \mathbb{Z}^3 \setminus \{0\}$ such that*

$$\sup_{s\in I} \|g_{t_i}\phi_{a,b}(s)v_i\| \leq R. \tag{6.3}$$

In particular $(a,b) \in \mathcal{W}_2$.

Proof. If μ is not a probability measure on X, then condition (2), and hence condition (3), of Proposition 5.2 hold. So (6.3) follows.

Therefore, we now assume that μ is a probability measure on X. By Proposition 6.1 μ is W-invariant. By Theorem 6.3, there exists $H \in \mathcal{H}$ such that

$$\mu(\pi(N(H,W))) > 0 \text{ and } \mu(\pi(S(H,W))) = 0,$$

and since μ is not G-invariant, $H \neq G$. So $\dim(H) < \dim(G)$.

We thus conclude that there exist $\varepsilon > 0$ and a compact set $\mathcal{C} \subset N(H,W)\setminus S(H,W)$ such that $\mu(\pi(\mathcal{C})) > \varepsilon$. By Proposition 6.4 applied to $\varepsilon/2$ in place of ε, we obtain a compact set $\mathcal{D} \subset \mathcal{A}$. Then we pick $R_1 > 0$ such that $\mathcal{D}$ is contained in the open norm-ball of radius R_1 in V, denoted Φ, and obtain a neighborhood $\mathcal{O}$ of $\pi(\mathcal{C})$ in G/Γ so that the conclusion of Proposition 6.4 holds.

Since $\mu(\pi(\mathcal{C})) > \varepsilon$, there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, $g_{t_i}\lambda_{a,b}(\mathcal{O}) > \varepsilon$. So for $i \geq i_0$, $t = t_i$ and $J = I$ condition (1) of the conclusion of Proposition 6.4 fails to hold, and hence condition (2) of the conclusion holds for some $\gamma_i \in \Gamma$; that is,

$$\sup_{s\in I}\|g_{t_i}\phi_{a,b}(s)(\gamma_i p_H)\| \leq R_1.$$

Therefore (6.3) follows from Theorem 4.1 for a choice of $R > 0$ depending on $p_H \in V(\mathbb{Q}) \setminus \{0\}$ and $R_1 > 0$. □

Proposition 6.6. *Let μ be a weak-* limit of $\mu_i := (1/T_i)\int_0^{T_i} g_{t_i}\lambda_{a,b}\,\mathrm{d}t$ for a sequence $T_i \to \infty$; here $0 \leq \mu(X) \leq 1$. Suppose $\mu \neq \mu_X$. Then there exists $R > 0$ such that*

$$\liminf_{I\to\infty} \frac{|\mathcal{I}_R \cap [0, T_i]|}{T_i} > 0, \tag{6.4}$$

where $\mathcal{I}_R$ is defined in (3.6).

Proof. There are two possibilities: $\mu(X) < 1$, or μ is a probability measure which is not G-invariant. By Proposition 6.2, μ is W-invariant. So there exists $\varepsilon > 0$ such that one of the following two possibilities occur:

(i) $\mu(X) < 1 - \varepsilon$;
(ii) or, by Theorem 6.3, there exists $H \in \mathcal{H}$ with $H \neq G$ and a compact set $\mathcal{C} \subset N(H,W) \setminus S(H,W)$ such that $\mu(\pi(\mathcal{C})) > \varepsilon$.

First suppose that possibility (i) occurs. Take any $R > 0$ and pick a compact $K \subset X$ given by Proposition 5.1 for $\varepsilon/2$ in place of ε. Then, for each non-negative $t \notin \mathcal{I}_R$, by definition (3.6), the possibility (2) of Proposition 5.1 does not hold, and hence its possibility (1) must hold; that is, $g_t\lambda_{a,b}(K) \geq 1 - \varepsilon/2$. Write $\kappa_i = |\mathcal{I}_R \cap [0, T_i]|/T_i$. So for all large i,

$$(1 - \kappa_i)(1 - \varepsilon/2) \leq \frac{1}{T_i}\int_0^{T_i} g_t\lambda_{a,b}(K)\,\mathrm{d}t \leq \mu_i(X) < 1 - \varepsilon;$$

and hence $\kappa_i > \varepsilon/2$. So (6.4) holds.

Now suppose possibility (ii) occurs. Then for any open neighborhood $\mathcal{O}$ of $\pi(\mathcal{C})$, $\mu(\mathcal{O}) > \varepsilon$, and so for all large i,

$$\frac{1}{T_i}\int_0^{T_i} g_t\lambda_{a,b}(\mathcal{O})\,\mathrm{d}t > \varepsilon.$$

Let

$$\mathcal{I}_{\mathcal{O}} = \{t \in [0,\infty) : g_t\lambda_{a,b}(\mathcal{O}) > \varepsilon/2\} \text{ and } \kappa_i = \frac{|\mathcal{I}_{\mathcal{O}} \cap [0, T_i]|}{T_i}.$$

Then for all large i,

$$(1 - \kappa_i)\varepsilon/2 + \kappa_i \geq \frac{1}{T_i}\int_0^{T_i} g_t\lambda_{a,b}(\mathcal{O})\,\mathrm{d}t > \varepsilon,$$

and hence $\kappa_i > \varepsilon/2$.

By Proposition 6.4 applied to the set $\mathcal{C} \subset N(H,W) \setminus S(H,W)$ and $\varepsilon/2$ in place of ε, we obtain a compact set $\mathcal{D} \subset \mathcal{A}$. Pick $R_1 > 0$ such that $\mathcal{D}$ is contained in the open norm-ball of radius R_1 in V, denoted Φ, and obtain a neighborhood $\mathcal{O}$ of $\pi(\mathcal{C})$ in G/Γ so that the conclusion of Proposition 6.4 holds.

Now suppose $t \in \mathcal{I}_{\mathcal{O}}$. Then for $J = I$, the condition (1) of Proposition 6.4 fails to hold, so condition (2) of Proposition 6.4 holds: there exists $\gamma \in \Gamma$ such that

$$\sup_{s \in I} \|g_t \phi_{a,b}(s) \gamma p_H\| < R_1. \tag{6.5}$$

Let $R > 0$ be the quantity given by Theorem 4.1 applied to $v_0 = p_H \in V(\mathbb{Q}) \setminus \{0\}$ and $C = R_1$. Under (6.5), Theorem 4.1 shows that there exists $v \in \mathbb{Z}^3 \setminus \{0\}$ such that $\sup_{s\in I} \|g_t \phi_{a,b}(s) v\| \leq R$; that is, $t \in I_R$. Thus $I_{\mathcal{O}} \subset I_R$. Therefore

$$|I_R \cap [0, T_i]|/T_i \geq \kappa_i > \varepsilon/2$$

for all large i, and (6.4) follows. □

7. Proof of Theorem 1.2 and Theorem 1.4

In this section we prove Theorem 1.2 and Theorem 1.4.

Proof of Theorem 1.2. First suppose that the g_t-translates of $\lambda_{a,b}$ do *not* get equidistributed in X as $t \to \infty$. Then there exist $t_i \to \infty$ such that $g_{t_i} \lambda_{a,b}$ weak-* converge to a measure which is not μ_X. So by Proposition 6.5 we have $(a, b) \in \mathcal{W}_2$.

Conversely, suppose $(a, b) \in \mathcal{W}_2$. We want to show that the g_t-translates of $\lambda_{a,b}$ do *not* get equidistributed in X.

By Lemma 3.1, there exist $R \geq 1$, $t_i \to \infty$ and $\{\gamma_i\} \subset \Gamma$ such that for all $i \in \mathbb{N}$,

$$\sup_{s \in I} \|g_{t_i} \phi_{a,b}(s) \gamma_i e_1\| \leq R. \tag{7.1}$$

Case 1: Suppose there exists $c > 0$ such that for all $i \geq 1$ and all $s \in I$,

$$\|g_{t_i} \phi_{a,b}(s) \gamma_i e_1\| \geq c.$$

Then by Proposition 5.2, after passing to a subsequence, we may assume that $g_{t_i} \lambda_{a,b}$ weak-* converge to a probability measure μ. It suffices to show that the support of μ is not full.

Let E denote the set of unimodular lattices in $\mathbb{R}^3$ containing a primitive vector whose (sup)norm is in the interval $[c, R]$. Then E is closed and contains the support of each $g_{t_i} \lambda_{a,b}$. Therefore the support of μ is also contained in E. But $X \setminus E$ is a nonempty open set, as E does not contain the unimodular lattice $\mathbb{Z}M^{-2}e_1 + \mathbb{Z}Me_2 + \mathbb{Z}Me_3$, for any $M > R$ such that $M^{-2} < c$. Thus the support of μ is not full.

Case 2: Suppose Case 1 does not occur. Then after passing to a subsequence, there exists a sequence $\{s_i\} \subset I$ such that

$$\|g_{t_i} \phi_{a,b}(s_i) \gamma_i e_1\| = c_i \to 0 \text{ as } i \to \infty. \tag{7.2}$$

We write $\gamma_i e_1 = \begin{pmatrix} p_{1,i} \\ p_{2,i} \\ q_i \end{pmatrix} \in \mathbb{Z}^3$. Then for all $s \in I$ and $t \in \mathbb{R}$

$$g_t \phi_{a,b}(s) \gamma_i e_1 = \begin{pmatrix} e^{2t} x_1(s) \\ e^{-t} p_{2,i} \\ e^{-t} q_i \end{pmatrix}, \tag{7.3}$$

where $x_1(s) = (aq_i + p_{2,i})s + (bq_i + p_{1,i})$. So by (7.2), $e^{-t_i}|p_{2,i}| \le c_i$ and $e^{-t_i}|q_i| \le c_i$, and by (7.1), $\left|e^{2t_i}x_1(s)\right| \le R$. We note that if $p_{2,i} = 0$ and $q_i = 0$, then $|x_1(s)| = |p_{1,i}| \ge 1$, and hence $e^{2t_i} \le R$ for all large i, which is absurd. Hence

$$c_i e^{t_i} \ge \max\{|p_{2,i}|, |q_i|\} \ge 1;$$

that is, $t_i + \log c_i \ge 0$ for all i. Let $t_i' := t_i + (1/3)\log c_i \ge -(2/3)\log c_i$, so $t_i' \to \infty$. By (7.3), for all $s \in I$,

$$\begin{aligned}\left\|g_{t_i'}\phi_{a,b}(s)\gamma_i e_1\right\| &\le \max\left\{\left|c_i^{2/3}e^{2t_i}x_1(s)\right|, \left|c_i^{-1/3}e^{-t_i}p_{2,i}\right|, \left|c_i^{-1/3}e^{-t_i}q_i\right|\right\}\\ &\le \max\left\{Rc_i^{2/3}, c_i^{-1/3}c_i\right\} \le Rc_i^{2/3}.\end{aligned}$$

Since $c_i \to 0$, by Proposition 5.2, for any compact $K \subset X$ we have $g_{t_i'}\lambda_{a,b}(K) = 0$ for all large i. □

Proof of Theorem 1.4. (1)⇒(2): Assume $(a,b) \notin \mathbb{Q}^2$. Then (3) of Lemma 3.3 fails to hold, so there exists a sequence $t_i \to \infty$ such that for every sequence $\{v_i\} \subset \mathbb{Z}^3 \setminus \{0\}$,

$$\sup_{s\in I}\|g_{t_i}\phi_{a,b}(s)v_i\| \to \infty \text{ as } i \to \infty.$$

So by Proposition 6.5, we conclude that $g_{t_i}\lambda_{a,b} \xrightarrow{\text{weak-*}} \mu_X$.

(2)⇒(1): If $(a,b) \in \mathbb{Q}^2$, then by Remark 3.0.2, the translated measure $g_t\lambda_{a,b}$ escapes to ∞ as $t \to \infty$. Therefore (2) fails to hold.

Thus (1) and (2) are equivalent. Next we will prove that (1)⇒(3) and (3)⇒(1).

(1)⇒(3): We assume (1) and argue by contradiction, supposing that the set

$$E = \{s \in \mathbb{R} : \{g_t\phi_{a,b}(s)x_0\}_{t\ge 0} \text{ is not dense in } X\}$$

has positive Lebesgue measure. We take a countable topological basis $\{B_i\}_{i\in\mathbb{N}}$ of X consisting of non-empty open subsets, and let

$$E_i = \{s \in \mathbb{R} : \{g_t\phi_{a,b}(s)x_0\}_{t\ge 0} \cap B_i = \emptyset\}.$$

One has $E = \bigcup_{i\in\mathbb{N}} E_i$. Since $|E| > 0$, there exists $i_0 \in \mathbb{N}$ such that $|E_{i_0}| > 0$. Without loss of generality we may assume that $|E_1| > 0$. By the Lebesgue density theorem, there exists a compact interval $I \subset \mathbb{R}$ with non-empty interior such that

$$\frac{|I \cap E_1|}{|I|} \ge 1 - \frac{\mu_X(B_1)}{2}, \tag{7.4}$$

as $\mu_X(B_1) > 0$. Because we assumed (1), and since we have proved that (1)⇒(2), there exists $t_i \to \infty$ such that $g_{t_i}\lambda_{a,b} \to \mu_X$ in the weak-* topology. Since B_1 is non-empty and open, for all large i,

$$\frac{1}{|I|}|\{s \in I : g_{t_i}\phi_{a,b}(s)x_0 \in B_1\}| > \frac{\mu_X(B_1)}{2},$$

which, by the definition of E_1, implies that

$$\frac{|I \setminus E_1|}{|I|} > \frac{\mu_X(B_1)}{2}.$$

This contradicts (7.4). Hence we must have $|E| = 0$.

(3)⇒(1): To prove this by contraposition, suppose that $(a, b) \in \mathbb{Q}^2$. Let $B \subset X$ be a non-empty relatively compact open set. By Remark 3.0.2, there exists $t_0 > 0$ such that $g_t\{\phi_{a,b}(s)x_0 : s \in \mathbb{R}\} \cap B = \emptyset$ for all $t > t_0$. If $q \in \mathbb{N}$ be such that $qa \in \mathbb{Z}$, then $\phi_{a,b}(s+q)\mathbb{Z}^3 = \phi_{a,b}(s)\mathbb{Z}^3$ for all $s \in \mathbb{R}$. Therefore $\{\phi_{a,b}(s)x_0 : s \in \mathbb{R}\}$ is compact. So $C := \cup_{0 \le t \le t_0} g_t\{\phi_{a,b}(s)x_0 : s \in \mathbb{R}\}$ is a compact subset of a 2-dimensional submanifold of X. So $B \setminus C$ is a non-empty open subset of X. Therefore for every $s \in \mathbb{R}$,

$$\{g_t\phi_{a,b}(s)x_0 : t \ge 0\} \cap (B \setminus C) = \emptyset;$$

in particular, $\{g_t\phi_{a,b}(s)x_0 : t \ge 0\}$ is not dense in X. So (3) fails to hold. □

8. Behavior on average — Proofs of Lemma 3.4 and Theorem 1.3

In this section, we discuss the averages of the g_t-translates, and prove Theorem 1.3. As Lemma 3.4 will be used in the proof of Theorem 1.3, we first provide its proof.

Proof of Lemma 3.4. (1)⇒(2) is obvious.

To prove that (2)⇒(3), we pick $R \ge 1$ such that (3.7) holds. For any $v = \begin{pmatrix} p_1 \\ p_2 \\ q \end{pmatrix} \in \mathbb{Z}^3 \setminus \{0\}$ and $t \ge 0$, by (3.3) we get

$$\sup_{s \in I} \|g_t\phi_{a,b}(s)v\| < R \Rightarrow \begin{cases} |qb + p_1| < R_1 e^{-2t} \\ |qa + p_2| < R_1 e^{-2t} \\ |q| < Re^t \end{cases} \Rightarrow \begin{cases} \langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle < R_1 e^{-2t} \\ |q| < Re^t, \end{cases} \tag{8.1}$$

where $R_1 = \left\| \left(\begin{smallmatrix} 1 & s_0 \\ 1 & s_1 \end{smallmatrix}\right)^{-1} \right\| R \ge R$, and $\langle \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \rangle$ denotes the sup-norm distance between $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$ and its nearest integral vector. Note that if $q = 0$ in (8.1), then $t < (1/2)\log R_1$.

For each $q \in \mathbb{N}$, define

$$\begin{aligned} E_q &= \{t > 0 : e^{-t}q < R \text{ and } e^{2t}\langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle < R_1\} \\ &= \left\{t > 0 : R^{-1}q < e^t < R_1^{1/2}\langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle^{-1/2}\right\} \\ &= \left(\log q - \log R, -\frac{1}{2}\log\langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle + \frac{1}{2}\log R_1\right) \cap (0, \infty). \end{aligned} \tag{8.2}$$

Now $t \in E_q$ if and only if the right-most term of (8.1) holds, so

$$E_q \ne \emptyset \iff \langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle < R_1(Rq^{-1})^2. \tag{8.3}$$

Let $\mathcal{P}(\mathbb{Z}^3)$ denote the set of primitive integral vectors in $\mathbb{Z}^3$. From (3.6), note that

$$\mathcal{I}_R = \left\{ t \in [0, +\infty) : \sup_{s \in I}\|g_t\phi_{a,b}(s)v\| < R, \text{for some } v \in \mathcal{P}(\mathbb{Z}^3) \setminus \{0\} \right\}. \tag{8.4}$$

Let $\mathcal{Q}$ be the collection of $q \in \mathbb{N}$ such that $E_q \ne \emptyset$, and

$$\langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle = \max\{|qb + p_1|, |qa + p_2|\} \text{ for some } p_1, p_2 \in \mathbb{Z} \text{ such that } \begin{pmatrix} p_1 \\ p_2 \\ q \end{pmatrix} \in \mathcal{P}(\mathbb{Z}^3).$$

Let $t_0 = (1/2)\log R_1$. Then by (8.4), (8.1) and (8.2) we get

$$\mathcal{I}_R \cap [t_0, \infty] \subset \bigcup_{q\in\mathcal{Q}} E_q. \tag{8.5}$$

We may assume that $(a,b) \notin \mathbb{Q}^2$, because otherwise $(a,b) \in \mathcal{W}_2^+$ and we are done. Then for any $q \in \mathcal{Q}$, E_q is a finite interval. Therefore by (3.7), $\mathcal{Q}$ is infinite. We write $\mathcal{Q} = \{q_1, q_2, \dots\}$, where $q_i < q_{i+1}$ for all i.

Claim 1. If E_q and $E_{q'}$ are both non-empty for some $q, q' \in \mathcal{Q}$ and $q' > q$, then $q^2 < Cq'$, where $C = 2R_1R^2 \geq 2$. In particular, for any $n \in \mathbb{N}$,

$$\log q_i < \log C + 2^{-(n-i)} \log q_n, \ \forall i < n. \tag{8.6}$$

Indeed, by definition of $\mathcal{Q}$ and (8.3), there exist $\begin{pmatrix} p_1 \\ p_2 \\ q \end{pmatrix}, \begin{pmatrix} p_1' \\ p_2' \\ q' \end{pmatrix} \in \mathcal{P}(\mathbb{Z}^3)$ such that

$$\left\| q\begin{pmatrix} b \\ a \end{pmatrix} + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\| < R_1R^2q^{-2} \text{ and } \left\| q'\begin{pmatrix} b \\ a \end{pmatrix} + \begin{pmatrix} p_1' \\ p_2' \end{pmatrix} \right\| < R_1R^2q'^{-2}.$$

By primitivity, $\frac{1}{q}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \neq \frac{1}{q'}\begin{pmatrix} p_1' \\ p_2' \end{pmatrix}$. Hence, by triangular inequality,

$$\frac{1}{qq'} \leq \left\| \frac{1}{q}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - \frac{1}{q'}\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} \right\| \leq R_1R^2(q^{-3} + q'^{-3}) < (2R_1R^2)q^{-3}.$$

Therefore $q^2 < (2R_1R^2)q'$. This proves the first part of the claim.

For the second assertion of the claim, we iteratively apply the inequality $q^2 < Cq'$ to $q = q_j$ and $q' = q_{j+1}$, for $j = i, \dots, n-1$ to get

$$\begin{aligned} \log q_i &< \frac{\log C}{2} + \frac{\log q_{i+1}}{2} \\ &< \frac{\log C}{2} + \frac{\log C}{4} + \frac{\log q_{i+2}}{4} \\ &< \dots \\ &< \log C + 2^{-(n-i)} \log q_n. \end{aligned}$$

Next, in view of (3.7) and (8.5), to achieve the quantity $\limsup_{T\to\infty} \frac{\left|[0,T]\cap\bigcup_{q\in\mathcal{Q}} E_q\right|}{T}$, it is enough to let T vary along the sequence $\{T_n\}$ of right endpoints of intervals E_{q_n}, which, by (8.2), can be rewritten $T_n = \log q_n - \log R + |E_{q_n}|$. Then,

$$\frac{\left|[0,T_n]\cap\bigcup_{q\in\mathcal{Q}} E_q\right|}{T_n} \leq \frac{\sum_{i=1}^n |E_{q_i}|}{\log q_n - \log R + |E_{q_n}|}.$$

Therefore we infer from (3.7) and (8.5) that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n |E_{q_i}|}{\log q_n - \log R + |E_{q_n}|} > 0.$$

It follows that there exists $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n |E_{q_i}|}{\log q_n} = 4\varepsilon > 0. \tag{8.7}$$

Claim 2. We claim that

$$\limsup_{n\to\infty} \frac{|E_{q_n}|}{\log q_n} > \varepsilon. \tag{8.8}$$

Indeed, suppose $\limsup_{n\to\infty} \frac{|E_{q_n}|}{\log q_n} \leq \varepsilon$. Then there exists $N > 0$ such that $q_N > C$ and for all $n \geq N$, $|E_{q_n}| < 2\varepsilon \log q_n$. Therefore

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n}|E_{q_i}|}{\log q_n} = \limsup_{n\to\infty} \frac{\sum_{i=N}^{n}|E_{q_i}|}{\log q_n} < \limsup_{n\to\infty} \frac{\sum_{i=N}^{n} 2\varepsilon \log q_i}{\log q_n}$$
$$\leq \limsup_{n\to\infty} \frac{2\varepsilon(n-N)\log C + 2\varepsilon \sum_{i=N}^{n} 2^{-(n-i)} \log q_n}{\log q_n} = 0 + 2\epsilon \sum_{i=N}^{n} 2^{-(n-i)} < 4\epsilon,$$

because by (8.6), for any $i < n$,

$$\log q_i < \log C + 2^{-(n-i)} \log q_n \text{ and } \log q_n > 2^{(n-N)}(\log q_N - \log C).$$

This contradicts (8.7), and proves Claim 2.

Now in view of (8.8), for any $Q > 0$ there exists $q > Q$ such that $|E_q| > \varepsilon \log q$. By (8.2), this means

$$(1/2)\log R_1 + \log R - \log q - \frac{1}{2}\log \langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle > \varepsilon \log q,$$

or equivalently,

$$\langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle < R_1 R^2 q^{-(2+2\varepsilon)}.$$

Hence $\langle q \left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right)\rangle \leq q^{-(2+\varepsilon)}$ has infinitely many solutions $q \in \mathbb{N}$, which means $(a,b) \in \mathcal{W}_2^+$. This proves (2)⇒(3).

Now to prove (3)⇒(1), suppose that $(a,b) \in \mathcal{W}_2^+$. Then there exists $\varepsilon > 0$ and an increasing sequence $\{q_n\}_{n\in\mathbb{N}}$ of positive integers such that

$$\begin{cases} |q_n b + p_{1,n}| \leq q_n^{-(2+\varepsilon)} \\ |q_n a + p_{2,n}| \leq q_n^{-(2+\varepsilon)} \end{cases}, \text{ for some } p_{1,n}, p_{2,n} \in \mathbb{Z}. \tag{8.9}$$

For each $n \in \mathbb{N}$, pick $v_n = \left(\begin{smallmatrix} p_{1,n} \\ p_{2,n} \\ q_n \end{smallmatrix}\right) \in \mathbb{Z}^3$ such that (8.9) holds. For any $t \in \mathbb{R}$,

$$g_t \phi_{a,b}(s) v_n = \begin{pmatrix} e^{2t}\big((bq_n + p_{1,n}) + (aq_n + p_{2,n})s\big) \\ e^{-t} p_{2,n} \\ e^{-t} q_n \end{pmatrix}. \tag{8.10}$$

Pick any constants $0 < c_1 < c_2 < 1/2$, independent of n. Let

$$t \in [(1 + c_1\varepsilon)\log q_n, (1 + c_2\varepsilon)\log q_n].$$

Then

$$q_n^{-(2+\varepsilon)} \leq e^{-\left(\frac{(1-2c_2)\varepsilon}{1+c_2\varepsilon}\right)t} e^{-2t} \text{ and } q_n \leq e^{-\left(\frac{c_1\varepsilon}{1+c_1\varepsilon}\right)t} e^{t}.$$

By (8.9),

$$|p_{2,n}| \leq q_n^{-(2+\varepsilon)} + |a| q_n \leq 1 + |a| q_n$$

and so, by (8.10),

$$\|g_t\phi_{a,b}(s)v_n\| \le C_1 e^{-\varepsilon_1 t}, \forall s \in I = [s_0, s_1], \tag{8.11}$$

where $\varepsilon_1 := \min\left\{\frac{c_1\varepsilon}{1+c_1\varepsilon}, \frac{(1-2c_2)\varepsilon}{1+c_2\varepsilon}\right\} > 0$ and $C_1 := (1+|s_0|+|s_1|+|a|)$.

Given any $R > 0$, let $N > 0$ such that for every $n > N$,

$$C_1 q_n^{-(1+c_1\varepsilon)\varepsilon_1} < R.$$

For $n > N$, by (8.11), one has $[(1+c_1\varepsilon)\log q_n, (1+c_2\varepsilon)\log q_n] \subset \mathcal{I}_R$. So, setting $T_n = (1+c_2\varepsilon)\log q_n$, we get

$$\frac{|\mathcal{I}_R \cap [0,T_n]|}{T_n} \ge \frac{|[(1+c_1\varepsilon)\log q_n, (1+c_2\varepsilon)\log q_n]|}{T_n} = \frac{(c_2-c_1)\varepsilon}{1+c_2\varepsilon}.$$

Therefore

$$\limsup_{T\to\infty} \frac{|\mathcal{I}_R \cap [0,T]|}{T} \ge \frac{(c_2-c_1)\varepsilon}{1+c_2\varepsilon} > 0.$$

This proves that (3)⇒(1). □

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. (1)⇒(2) is obvious.

To prove (2)⇒(3) by contrapositive, suppose that $(a,b) \in \mathcal{W}_2^+$. Let K be a compact subset of X, which, as we may recall, is identified with the space of unimodular lattices in $\mathbb{R}^3$. By Mahler's criterion, there exists $R > 0$ such that every nonzero vector in any lattice in K has norm at least R. So, by (3.6), for any $t \in \mathcal{I}_R$, we have $g_t\phi_{a,b}(s)\mathbb{Z}^n \notin K$ for all $s \in I$; in particular,

$$g_t\lambda_{a,b}(K) = 0. \tag{8.12}$$

Since $(a,b) \in \mathcal{W}_2^+$, Lemma 3.4 shows that there exists a sequence $T_n \to \infty$ and an $\varepsilon > 0$ such that for all n,

$$\frac{|\mathcal{I}_R \cap [0,T_n]|}{T_n} \ge \epsilon,$$

and hence, by (8.12),

$$\frac{1}{T_n}\int_0^{T_n} g_t\lambda_{a,b}(K)\,\mathrm{d}t \le 1-\epsilon,$$

where the ϵ is independent of K. Thus, the family of averages $\{\frac{1}{T}\int_0^T g_t\lambda_{a,b}\,\mathrm{d}t\}_{T>0}$ has escape of mass. This proves that (2)⇒(3).

To prove (3)⇒(1) by contraposition, suppose that (1) fails to hold. Then there exists a sequence $T_i \to \infty$ such that $\mu_i := (1/T_i)\int_0^{T_i} g_t\lambda_{a,b}\,\mathrm{d}t$ does not converge to μ_X. Since the μ_i are probability measures, by passing to a subsequence, without loss of generality we may assume that μ_i converges to a Borel measure μ on X which is not μ_X; here $0 \le \mu(X) \le 1$. Then by Proposition 6.6 there exists $R > 0$ such that

$$\liminf_{i\to\infty} \frac{|\mathcal{I}_R \cap [0,T_i]|}{T_i} > 0.$$

Then by Lemma 3.4, we get $(a,b) \in \mathcal{W}_2^+$, which contradicts (3). □

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