Hausdorff dimension and subgroups of $SU(2)$

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Abstract

We prove that any Borel measurable proper dense subgroup of $SU(2)$ has Hausdorff dimension zero.

1 Introduction

In 1966, Erdős and Volkmann [8], after constructing additive subgroups of $\mathbb{R}$ of arbitrary dimension, made the following conjecture: "There is no proper Borel measurable subring of $\mathbb{R}$ of positive Hausdorff dimension." Note that without the measurability assumption, this conjecture is false, as was first observed by Davies [6], who constructed non measurable subrings of $\mathbb{R}$ of arbitrary dimension, using the continuum hypothesis (see also [10]).

Some partial progress on the Erdős-Volkmann Ring Conjecture was made in 1984 by Falconer [9], but it was only in 2002 that a complete solution was given by Edgar and Miller [7]. Shortly after their paper appeared, Bourgain [1, 2], building on previous work of Katz and Tao [11], found an alternative approach to the Ring Conjecture. A byproduct of Bourgain's proof, which is more difficult than that of Edgar and Miller, is an important result sometimes refereed to as Bourgain's Discretized Sum-Product Theorem, which has found many applications, in particular in the work of Bourgain-Gamburd [3, 4] on spectral gap for averaging operators on compact semisimple Lie groups and in the work of Bourgain, Furman, Lindenstrauss and Mozes [5] on quantitative equidistribution of orbits of semigroups on the torus.

Here we prove an analog of the Erdős-Volkmann Ring Conjecture for the group $SU(2)$:

Theorem 1.1. *Any Borel measurable proper dense subgroup of $SU(2)$ has Hausdorff dimension zero.*

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In fact, the proof yields a more precise statement, a dimensional inequality on product sets:

Proposition 1.2. *Let $\alpha \in (0, 3)$. There exists $\epsilon > 0$ such that if A is a symmetric Borel subset of G of Hausdorff dimension α which is not included in a torus, then the product set A^{32} has Hausdorff dimension at least $\alpha + \epsilon$. Moreover, ϵ remains bounded away from zero when α varies in a compact subset of* $(0, 3)$.

Those statements can be put into contrast with the results of [13], where subgroups of arbitrary dimension are constructed in nilpotent Lie groups. In the case we consider here, however, the semisimple structure provides much less flexibility.

In our proof we use a basic ingredient of the paper [3]: Bourgain-Gamburd's Product Theorem in $SU(2)$. This theorem is used by Bourgain and Gamburd to prove an L^2-Flattening Lemma regarding the L^2-norm of a convolution $\mu * \mu$ of probability measures on the group. We also want to apply an L^2-flattening lemma, but unfortunately, as it stands, the measures we need to consider (Frostman measures on an arbitrary positive dimensional Borel set A) need not satisfy the conditions imposed in Bourgain and Gamburd's flattening lemma unless $\dim_H A > 1$. In that case, a straightforward application of the results of [3] give that if A is a Borel subset of $SU(2)$ with $\dim_H A > 1$, then $\dim_H AA > \dim_H A$ and consequently, $SU(2)$ has no proper Borel measurable subgroup of dimension larger than 1.

To prove Theorem 1.1, starting from the Bourgain-Gamburd Product Theorem, we need to push one step further the combinatorial analysis. By a careful choice of an appropriate "expanding word" and making use (among other ingredients) of the Balog-Szemerédi-Gowers Lemma already at the combinatorial level, we get a statement strong enough to imply flattening for any measure satisfying a Frostman condition. In order to avoid the obvious complication of the convolved measure remaining trapped on a subgroup, our expanding word involves two parameters — elements of G, considered as fixed, in general position. Once the appropriate flattening statement is proven, Theorem 1.1 follows easily, by standard Hausdorff dimension techniques.

The structure of the paper is as follows. In section 2, we give the general setting of our work, together with some elementary observations on the geometry of $SU(2)$. Then, we prove a discretized combinatorial statement, using the Bourgain-Gamburd Product Theorem in $SU(2)$ as well as some additional basic results of additive combinatorics; this can be considered the core of the proof, and is contained in section 3. Finally, after establishing an appropriate L^2-flattening statement (Proposition 4.2) in section 4, we explain in section 5 how to translate it into statements about Hausdorff dimension of product sets.

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2 General setting and preliminary lemmas

We will denote by G the group $SU(2)$, endowed with the invariant metric d defined by

$$d(x, y) = \|x - y\|,$$

where, for any complex numbers a, b, c and d,

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = (|a|^2 + |b|^2 + |c|^2 + |d|^2)^{\frac{1}{2}}.$$

This metric induces a metric on the space of closed subsets of G, called the Hausdorff metric, also denoted by d, and defined by

$$d(X, Y) = \max\{\min_{x\in X} d(x, Y), \min_{y\in Y} d(y, X)\}.$$

The group G is compact and hence admits an invariant Haar measure $|\cdot|$, normalized by $|G| = 1$. As G is also a Lie group of dimension 3, we have, for $r \in (0, 1)$, for any x in G

$$|B(x, r)| \simeq r^3.$$

If X is a closed subset of G, we will denote by $X^{(\rho)}$ the ρ-neighborhood of X, i.e.

$$X^{(\rho)} = \{x \in G \,|\, d(x, X) \leq \rho\}.$$

Lemma 2.1 (Commutators and distance to a torus)**.** *For $a \neq I$ and b in G, denote $[a, b] = aba^{-1}b^{-1}$. We have*

$$d(a, \{\pm I\}) \cdot d(b, T_a) \leq d([a, b], I),$$

where T_a is the unique maximal torus containing a.

Proof. Choose a basis in which

$$a = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

with $\theta \in (0, \pi)$. In that basis, write

$$b = \begin{pmatrix} x & y \\ z & t \end{pmatrix},$$

so that

$$d([a,b],I) = d(aba^{-1},b) = \left\| \begin{pmatrix} 0 & (e^{2i\theta}-1)y \\ (e^{-2i\theta}-1)z & 0 \end{pmatrix} \right\|$$
$$= 2|\sin\theta|\sqrt{y^2+z^2}$$
$$\geq d(a,\{\pm I\}).\, d(b,T_a).$$

□

Lemma 2.2. *If two tori T_1 and T_2 satisfy $d(T_1,T_2) \geq r$, then the intersection $T_1^{(\rho)} \cap T_2^{(\rho)}$ is contained in the union of the two balls of radius $\rho(1+\frac{3}{r})$ centered at I and $-I$.*

Proof. First, fix $b \in T_2$ such that $d(b,T_1) \geq r$. Now assume $x \in T_1^{(\rho)} \cap T_2^{(\rho)}$. Let $x_1 \in T_1$ such that $d(x,x_1) \leq \rho$. Now write

$$\begin{aligned} r.d(x_1,\{\pm I\}) &\leq d(b,T_1).d(x_1,\{\pm I\}) \\ &\leq d([b,x_1],I) \\ &\leq d([b,x],I) + 2\rho \\ &\leq 3\rho, \end{aligned}$$

so that

$$d(x,\{\pm I\}) \leq d(x_1,\{\pm I\}) + \rho \leq \rho.(1+\frac{3}{r}).$$

□

Lemma 2.3 (Intersection of neighborhoods of cosets of tori)**.** *Let $r > 8\rho > 0$. If C_1 and C_2 are two cosets of tori with $d(C_1,C_2) > r$, then the intersection $C_1^{(\rho)} \cap C_2^{(\rho)}$ is contained in the union of two balls of radius $2\rho(1+\frac{12}{r})$.*

Proof. Write $C_1 = x_1T_1$ and $C_2 = x_2T_2$. We distinguish two cases.
First case: $d(T_1,T_2) < \frac{r}{4}$
In this case we must have $d(x_1,x_2) > \frac{3r}{4}$, so that $x_1T_1^{(\frac{3r}{8})}$ and $x_2T_1^{(\frac{3r}{8})}$ are disjoint. But, as $d(T_1,T_2) \leq \frac{r}{4}$, we have

$$x_2T_2^{(\rho)} \subset x_2T_1^{(\frac{r}{4}+\rho)} \subset x_2T_1^{(\frac{3r}{8})}$$

so that

$$C_1^{(\rho)} \cap C_2^{(\rho)} = \emptyset.$$

Second case: $d(T_1,T_2) \geq \frac{r}{4}$
Assuming $C_1^{(\rho)} \cap C_2^{(\rho)}$ is nonempty, fix a point x in it. Now suppose y also is in $C_1^{(\rho)} \cap C_2^{(\rho)}$. Write,

$$x = x_1t_1 = x_2t_2 \text{ and } y = x_1s_1 = x_2s_2$$

with $t_1, s_1 \in T_1^{(\rho)}$ and $t_2, s_2 \in T_2^{(\rho)}$.
Then we get that $x^{-1}y = t_1^{-1}s_1 = t_2^{-1}s_2$ is in the intersection $T_1^{(2\rho)} \cap T_2^{(2\rho)}$. As the distance from T_1 to T_2 is at least $\frac{r}{4}$, this intersection is included in $B(\pm I, 2\rho(1+\frac{12}{r}))$. Hence $C_1^{(\rho)} \cap C_2^{(\rho)}$ is contained in $B(\pm x, 2\rho(1+\frac{12}{r}))$. $\square$

Notation. If A is a subset of a metric space, for $\delta > 0$, $N(A,\delta)$ denotes the minimal number of balls of radius δ needed to cover A. For basic properties of that quantity in this context, we refer the reader to [15]. For a (Borel) set $A \subset G$ we will write $|A|$ to denote the Haar measure of A, normalized to be a probability measure.

For convenience, we make the following definition.

Definition 2.4. A small scale $\delta > 0$ being fixed, a subset A of G will be called a (α,κ,ϵ)-*set* if it satisfies the following

1. $\delta^{-\alpha+\epsilon} < N(A,\delta) < \delta^{-\alpha-\epsilon}$;
2. $\forall x \in G, \forall r \in (\delta, 1)$, $N(A \cap B(x,r), \delta) \leq r^{\kappa}\delta^{-\epsilon}N(A,\delta)$.

Finally, we will also make use of two classical notations.

- The Landau notation: $O(\epsilon)$ stands for a quantity bounded in absolute value by $C.\epsilon$, for some absolute constant C. Slightly abusing this notation, we will denote by $O(\epsilon)$ a quantity that is bounded by an absolute constant only when δ (which throughout the paper will denote the scale in which we shall be working) is small enough.
- The Vinogradov notation: we write $x \ll y$ if, $x \leq C.y$ for some absolute constant C, and $x \ll_p y$ if the constant C depends on some parameter p. We will also write $x \simeq y$ if $x \ll y$ and $x \gg y$, and similarly for $x \simeq_p y$. For two real valued functions φ and ψ on G, we write $\varphi \ll \psi$ if there exists an absolute constant C such that for all x in G, $\varphi(x) \leq C \cdot \psi(x)$.

3 Combinatorial statement

The combinatorial lemma we are aiming at is the following:

Proposition 3.1. *Let $\alpha \in (0, \dim G)$ and $\kappa > 0$, and fix $\{g_1, g_2\}$ a pair of non-commuting elements of G. There exists $\epsilon > 0$, such that, for $\delta > 0$ small enough, if A is a (α,κ,ϵ)-set and $\Omega \subset A^8$ satisfies $N(\Omega,\delta) \geq \delta^{-8\alpha+\epsilon}$, then*

$$N(w(\Omega),\delta) \geq \delta^{-\alpha-\epsilon},$$

where $w : (a_1,\ldots,a_8) \mapsto w(a_1,\ldots,a_8,g_1,g_2)$ is a word in the letters $\{a_i\} \cup \{g_1,g_2\}$.

The main ingredient in the proof of that lemma will be the Bourgain-Gamburd Product Theorem:

Theorem 3.2 (Bourgain-Gamburd, [3]). *Given $\alpha \in (0,3)$ and $\kappa > 0$, there exists $\tau > 0$ and $\epsilon > 0$ such that, for $\delta > 0$ sufficiently small, if $A \subset G$ is a $(\alpha, \kappa, \epsilon)$-set satisfying $N(AAA, \delta) \leq \delta^{-\epsilon} N(A, \delta)$, then A is included in a δ^{τ}-neighborhood of a torus.*

For a $(\alpha, \kappa, \epsilon)$-set A, this theorem enables us to control the set of "trouble-makers", elements ξ in G for which there exists a large part of $A \times A$ whose image under the map $m_\xi : (a, b) \mapsto a\xi b$ is not much larger than A:

Corollary 3.3. *Let $\alpha \in (0,3)$ and $\kappa > 0$. There exists $\eta = \eta(\alpha, \kappa) > 0$ and $\epsilon_0 = \epsilon_0(\alpha, \kappa) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the following holds.*
Suppose A is a $(\alpha, \kappa, \epsilon)$-set and denote Ξ the set of elements ξ in G such that there exists $\Omega \subset A \times A$ with $N(\Omega, \delta) \geq \delta^{\epsilon} N(A, \delta)^2$ and $N(m_\xi(\Omega), \delta) \leq \delta^{-\epsilon} N(A, \delta)$. Then, Ξ is included in a union of at most $\delta^{-O(\epsilon)}$ neighborhoods of cosets of tori of size δ^{η}.

The proof of that corollary is based on the following lemma.

Lemma 3.4. *Let $\alpha \in (0,3)$ and $\kappa > 0$. There exists $\epsilon_0 = \epsilon_0(\alpha, \kappa) > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $\tau > 0$, the following holds at any scale $\delta > 0$ sufficiently small.*
Let A be a $(\alpha, \kappa, \epsilon)$-set and denote C the set of "rich cosets":

$$\mathcal{C} = \{C \,|\, C \text{ is a left coset of a torus and } N(C^{(\delta^{\tau})} \cap A, \delta) \geq \delta^{\epsilon} N(A, \delta)\}.$$

Then $N(\mathcal{C}, \delta^{\frac{\tau}{2}}) \leq \delta^{-O(\epsilon)}$.

Proof. Choose in $\mathcal{C}$ a $\delta^{\frac{\tau}{2}}$-separated subset $\{C_i\}_{1 \leq i \leq n}$ of maximal cardinality. By Lemma 2.3 applied to $\rho = \delta^{\tau}$ and $r = \delta^{\frac{\tau}{2}}$, for $i \neq j$, $C_i^{(\delta^{\tau})} \cap C_j^{(\delta^{\tau})}$ is included in two balls of radius $25\delta^{\frac{\tau}{2}}$ so that, using the fact that A is a $(\alpha, \kappa, \epsilon)$-set, we have

$$N(A \cap C_i^{(\delta^{\tau})} \cap C_j^{(\delta^{\tau})}, \delta) \leq \delta^{\frac{\kappa\tau}{2} - O(\epsilon)} N(A, \delta).$$

Using this bound, we now count the points of A that belong to some $C_i^{(\delta^{\tau})}$.

$$\begin{aligned} N(A, \delta) &\geq N(\bigcup_{i=1}^{n} A \cap C_i^{(\delta^{\tau})}, \delta) \\ &\geq N(A \cap C_1^{(\delta^{\tau})}, \delta) + (N(A \cap C_2^{(\delta^{\tau})}, \delta) - \delta^{\frac{\kappa\tau}{2} - O(\epsilon)} N(A, \delta)) \\ &\quad + (N(A \cap C_3^{(\delta^{\tau})}, \delta) - 2\delta^{\frac{\kappa\tau}{2} - O(\epsilon)} N(A, \delta)) + \ldots \\ &\geq \delta^{\epsilon} N(A, \delta) + \delta^{\epsilon} N(A, \delta)(1 - \delta^{\frac{\kappa\tau}{2} - O(\epsilon)}) + \delta^{\epsilon} N(A, \delta)(1 - 2\delta^{\frac{\kappa\tau}{2} - O(\epsilon)}) + \ldots \end{aligned}$$

The right-hand side of the last inequality is a sum of $\min\{n, \delta^{-\frac{\kappa\tau}{2} + O(\epsilon)}\}$ non-negative terms forming an arithmetic progression, so we get the lower bound

$$N(A, \delta) \geq \frac{\delta^{\epsilon} N(A, \delta)}{2} \min\{n, \delta^{-\frac{\kappa\tau}{2} + O(\epsilon)}\}.$$

If $\epsilon > 0$ is small enough, this forces $n \leq \delta^{-\frac{\kappa\tau}{2} + O(\epsilon)}$ and in turn

$$n = N(\mathcal{C}, \delta^{\frac{\tau}{2}}) \leq 2\delta^{-\epsilon}.$$

$\square$

Proof of corollary 3.3. Suppose ξ is given, satisfying the above condition, for some large set Ω in $A \times A$. Then, by the non-commutative version of the Balog-Szemerédi-Gowers Lemma (Tao [15], Theorem 6.10), there exist two subsets A_1 and B_1 in A such that:

- $N(A_1, \delta) \geq \delta^{O(\epsilon)} N(A, \delta)$ and $N(B_1, \delta) \geq \delta^{O(\epsilon)} N(A, \delta)$
- $N(A_1 \xi B_1, \delta) \leq \delta^{O(\epsilon)} N(A, \delta)$.

Then, by the structure of small doubling sets (Tao [15], Theorem 6.9) we know that there exists a $\delta^{O(\epsilon)}$-approximate subgroup H, together with a finite set X such that

- $N(A, \delta) = \delta^{O(\epsilon)} N(H, \delta)$
- $\operatorname{card} X \leq \delta^{-O(\epsilon)}$
- $A_1 \subset XH$ and $\xi B_1 \subset HX$.

In particular, there exist x and y in X such that $N(xH \cap A, \delta) \geq \delta^{O(\epsilon)} N(A, \delta)$ and $N(\xi^{-1} H y \cap A, \delta) \geq \delta^{O(\epsilon)} N(A, \delta)$.
Finally, by the Product Theorem 3.2, the approximate subgroup H must be included in T, a δ^{τ}-neighborhood of a torus, for which we therefore have

$$N(A \cap xT, \delta) \geq \delta^{O(\epsilon)} N(A, \delta) \quad \text{ and } \quad N(A \cap \xi^{-1} T y, \delta) \geq \delta^{O(\epsilon)} N(A, \delta).$$

From Lemma 3.4, the set of ξ satisfying such a condition is included in a union of at most $\delta^{-O(\epsilon)}$ neighborhoods of cosets of tori of size $\delta^{\frac{\tau}{2}}$. This concludes the proof, taking $\eta = \frac{\tau}{2}$.

□

We are now ready to show that, using appropriate products, one can escape the set Ξ of "trouble-makers" for A. Fix $\{g_1, g_2\}$ a pair of non-commuting elements of G. For $i \in \{0, 1, 2\}$ (with $g_0 = 1$), we define the map

$$\pi_i : \begin{array}{ccc} G \times G & \to & G \\ (x, y) & \mapsto & x g_i y. \end{array}$$

We also denote $\pi_3(x, y) = x$ and $\pi_4(x, y) = y$.

Lemma 3.5. *Let $\alpha \in (0, 3)$ and $\kappa > 0$. Then there exists $\epsilon > 0$ such that, for $\delta > 0$ sufficiently small, if A is a $(\alpha, \kappa, \epsilon)$-set, and $\Omega \subset A \times A$ satisfies $N(\Omega, \delta) \geq \delta^{\epsilon} N(A, \delta)^2$, then there exists $i \in \{0, \dots, 4\}$ such that $\pi_i(\Omega)$ is not included in Ξ.*

Proof. Let $\rho = \delta^{\eta}$, where $\eta > 0$ is the parameter given by corollary 3.3. We know that Ξ is included in a union of few ρ-neighborhoods of cosets of tori:

$$\Xi \subset \bigcup_{k=1}^{\delta^{-O(\epsilon)}} C_k^{(\rho)}.$$

If either $\pi_3(\Omega)$ or $\pi_4(\Omega)$ is not included in Ξ, then we are done. Otherwise, there must exist k and l such that

$$N(\Omega \cap (C_k^{(\rho)} \times C_l^{(\rho)}), \delta) \geq \delta^{O(\epsilon)} N(\Omega, \delta) \geq \delta^{O(\epsilon)} N(A, \delta)^2.$$

Write $C_k = x_k T_k$ and $C_l = x_l T_l$. As g_1 and g_2 do not commute with each other, for at least one g_i, $i \in \{0,1,2\}$, the distance between the two tori $g_i T_k g_i^{-1}$ and T_l is larger than $c = c(g_1, g_2) > 0$. We will now check that for such i, $\pi_i(\Omega)$ cannot be included in Ξ, or rather, that it cannot be covered by as few as $\delta^{-O(\epsilon)}$ neighborhoods of size ρ of cosets of tori.
Let μ be the uniform probability measure on a maximal δ-separated set of A, and let $\Omega' = \Omega \cap (C_k^{(\rho)} \times C_l^{(\rho)})$. We define a probability measure ν supported on $\pi_i(\Omega)$ by

$$\nu(X) = \frac{1}{\mu \otimes \mu(\Omega')} \mu \otimes \mu(\Omega' \cap \pi_i^{-1}(X)),$$

and want to show that for any left coset $C = aT$,

$$\nu(C^{(\rho)}) \leq \delta^{\kappa\eta - O(\epsilon)}.$$

For that, we first write

$$\begin{aligned}
\nu(C^{(\rho)}) &= \frac{1}{\mu \otimes \mu(\Omega')} \mu \otimes \mu(\Omega' \cap \pi_i^{-1}(C^{(\rho)})) \\
&\leq \delta^{-O(\epsilon)} \mu \otimes \mu(\Omega' \cap \pi_i^{-1}(C^{(\rho)})) \\
&\leq \delta^{-O(\epsilon)} \mu \otimes \mu((C_k^{(\rho)} \times C_l^{(\rho)}) \cap \pi_i^{-1}(C^{(\rho)}).
\end{aligned}$$

As the distance between T_k and $g_i T_l g_i^{-1}$ is bounded below by $c(g_1, g_2)$, the torus T is away from one of these two by at least $\frac{c(g_1,g_2)}{2}$. Without loss of generality, we assume

$$d(T, T_l) \geq \frac{c(g_1, g_2)}{2}. \tag{1}$$

Then, we write

$$\mu \otimes \mu((C_k^{(\rho)} \times C_l^{(\rho)}) \cap \pi_i^{-1}(C^{(\rho)}) \leq \int \mu(C_l^{(\rho)} \cap x^{-1} g_i^{-1} C^{(\rho)}) d\mu(x).$$

Now from (1) and Lemma 2.3, the intersection $C_l^{(\rho)} \cap x^{-1} g_i^{-1} C^{(\rho)}$ is included in two balls of radius ρ and thus, using also that A is a $(\alpha, \kappa, \epsilon)$-set,

$$\begin{aligned}
\mu(C_l^{(\rho)} \cap x^{-1} C^{(\rho)}) &\leq \delta^{-O(\epsilon)} \rho^{\kappa} \\
&\leq \delta^{\kappa\eta - O(\epsilon)}
\end{aligned}$$

Integrating over x, we find

$$\nu(C^{(\rho)}) \leq \delta^{\kappa\eta - O(\epsilon)}.$$

As ν is a probability measure supported on $\pi_i(\Omega)$ this implies that $\pi_i(\Omega)$ cannot be covered by fewer than $\delta^{-\kappa\eta + O(\epsilon)}$ and in particular is not included in Ξ, provided ϵ is sufficiently small. □

Now that we explained how to escape from Ξ, we can obtain the combinatorial statement announced at the beginning of the section.

Proposition 3.6. *Let $\alpha \in (0,3)$ and $\kappa > 0$. There exists $\epsilon > 0$, such that, for $\delta > 0$ sufficiently small (depending on g_1, g_2), if A is a $(\alpha, \kappa, \epsilon)$-set and $\Omega \subset A^8$ satisfies $N(\Omega, \delta) \geq \delta^{-8\alpha+\epsilon}$, then*

$$N(w(\Omega), \delta) \geq \delta^{-\alpha-\epsilon},$$

where

$$w(\omega) = \omega_3 \pi_0(\omega_1, \omega_2)\omega_4 \pi_1(\omega_1, \omega_2)\omega_5 \pi_2(\omega_1, \omega_2)\omega_6 \pi_3(\omega_1, \omega_2)\omega_7 \pi_4(\omega_1, \omega_2)\omega_8.$$

Notation. If $I \subset \{1, \dots, 8\}$, we denote by p_I the projection map

$$p_I : \begin{array}{ccc} A^8 & \to & A^I \\ (a_i)_{1 \leq i \leq 8} & \mapsto & (a_i)_{i \in I} \end{array}$$

Also, for $(\omega_i)_{i \in I} \in p_I(\Omega)$ we denote by $\Omega_{(\omega_i)_{i\in I}}$ the set

$$\Omega_{(\omega_i)_{i \in I}} = \{(\omega_i)_{i \notin I} \,|\, (\omega_i)_{1 \leq i \leq 8} \in \Omega\}.$$

Proof of Proposition 3.6. First note that we may assume without loss of generality that Ω is a union of balls of radius δ. Then, losing a harmless factor $\frac{1}{2}$ in the cardinality of Ω, we may restrict Ω to the set

$$\Omega' = \{(\omega_i)_{1 \leq i \leq 8} \,|\, N(\Omega_{(\omega_1, \omega_2)}, \delta) \geq \frac{\delta^{-6\alpha+\epsilon}}{2}\}. \tag{2}$$

Now $p_{12}(\Omega)$ is a subset of $A \times A$ with $N(p_{12}(\Omega), \delta) \geq \delta^{-2\alpha+\epsilon}$, so, by Lemma 3.5, there exists $(\omega_1, \omega_2) \in p_{12}(\Omega)$ and $i \in \{0, \dots, 4\}$ such that $\pi_i(\omega_1, \omega_2) \notin \Xi$. Assume for simplicity that $i = 4$.
By (2),

$$N(\Omega_{(\omega_1, \omega_2)}, \delta) \geq \frac{\delta^{-6\alpha+\epsilon}}{2},$$

so we may choose $(\omega_3, \omega_4, \omega_5, \omega_6)$ such that

$$N(\Omega_{(\omega_1, \dots, \omega_6)}, \delta) \geq \frac{\delta^{-2\alpha+\epsilon}}{2}.$$

Then, as $\xi := \pi_i(\omega_1, \omega_2) \notin \Xi$, we have

$$N(m_\xi(\Omega_{(\omega_1, \dots, \omega_6)}), \delta) \geq \delta^{-\epsilon} N(A, \delta).$$

As

$$\omega_3 \pi_0(\omega_1, \omega_2)\omega_4 \pi_1(\omega_1, \omega_2)\omega_5 \pi_2(\omega_1, \omega_2)\omega_6 \pi_3(\omega_1, \omega_2) m_\xi(\Omega_{(\omega_1, \dots, \omega_6)}) \subset w(\Omega),$$

this implies

$$N(w(\Omega), \delta) \geq \delta^{-\epsilon} N(A, \delta).$$

$\square$

4 Flattening of measures

Mimicking the proof of Bourgain-Gamburd's l^2-Flattening Lemma, but starting from the combinatorial statement above, we prove in this section the flattening of measures announced in the introduction.

Notation. For $\delta > 0$, we denote $P_\delta = \frac{\mathbb{1}_{B(I,\delta)}}{|B(I,\delta)|}$ the normalized indicator function of the ball of radius δ centered at I. If μ is a Borel measure on G, we also denote

$$\mu_\delta = \mu * P_\delta.$$

The measure μ_δ is absolutely continuous with respect to the Haar measure. With a slight abuse of notation, we will also write μ_δ for its density function.

Definition 4.1. Let $\alpha \in (0,3)$. A Borel measure on G will be called *α-Frostman* if it satisfies, for some $C \geq 0$, for all $r > 0$, for all $x \in G$,

$$\mu(B(x,r)) \leq C.r^\alpha.$$

The flattening occurs for α-Frostman measures to which one applies the map w of Proposition 3.6:

Lemma 4.2. *Let $\alpha \in (0,3)$ and $\{g_1, g_2\}$ a pair of non-commuting elements in G. There exists $\epsilon = \epsilon(\alpha) > 0$ and a neighborhood V of I in G such that the following holds:*
Let μ be a α-Frostman Borel probability measure supported on V. Then the pushforward ν of the measure $\mu^{\otimes 8}$ under the map w defined in Proposition 3.6 above, satisfies, for all $\delta > 0$ sufficiently small (depending on g_1, g_2, ϵ)

$$\|\nu_\delta\|_2^2 \leq \delta^{3-\alpha+\epsilon}.$$

The proof will be by contradiction: starting from a measure μ such that for some $\delta > 0$ (arbitrarily small), and for some small $\epsilon > 0$,

$$\|\nu_\delta\|_2^2 \geq \delta^{3-\alpha+\epsilon},$$

we will construct a set Ω violating Proposition 3.6. We start by some elementary observations on how to approximate μ_δ by dyadic level sets.

Definition 4.3. We will say that a family of subsets $(S_i)_{i\in I}$ in G has *bounded multiplicity* if there exists an absolute constant K such that for any $\{i_1, \dots, i_K\} \subset I$, $\cap_{k=1}^K S_{i_k} = \emptyset$. Alternatively, we will also say that the union $\bigcup_{i\in I} S_i$ is *essentially disjoint.*

Lemma 4.4. *Let μ be a α-Frostman Borel probability measure on G and $\delta > 0$. There exist subsets A_i, $1 \leq 2^i \ll \delta^{-3+\alpha}$ such that*

1. $\mu_\delta \ll \sum_i 2^i \mathbb{1}_{A_i} \ll \mu_{4\delta}$
2. *Each A_i is an essentially disjoint union of balls of radius δ.*

Moreover, if for some i and some $\epsilon > 0$, $2^i|A_i| \geq \delta^\epsilon$ and $2^i \geq \delta^{-3+\alpha+\epsilon}$, then A_i is a $(\alpha, \alpha, O(\epsilon))$-set.

Proof. Take a $\mathcal{C}$ a maximal δ-separated subset of G. Then, the collection of balls $\{B(x,\delta)\}_{x\in\mathcal{C}}$ covers G. This gives

$$\mu_\delta \leq \sum_{x\in\mathcal{C}} \mu_\delta \cdot \mathbb{1}_{B(x,\delta)}.$$

Now, for $\xi \in B(x,\delta)$, we have

$$\mu_\delta(\xi) = \frac{\mu(B(\xi,\delta))}{|B(I,\delta)|} \leq \frac{\mu(B(x,2\delta))}{|B(I,\delta)|} \ll \frac{\mu(B(x,2\delta))}{|B(I,2\delta)|} = \mu_{2\delta}(x),$$

so that

$$\mu_\delta \ll \sum_{x\in\mathcal{C}} \mu_{2\delta}(x)\mathbb{1}_{B(x,\delta)} \tag{3}$$

On the other hand, the balls $\{B(x,\frac{\delta}{2})\}_{x\in\mathcal{C}}$ are disjoint, so that a ball of radius 2δ can contain at most $C = \frac{|B(I,2\delta)|}{|B(I,\frac{\delta}{2})|}$ of them. Therefore, a point y in G cannot belong to more than C balls of the cover $\{B(x,\delta)\}_{x\in\mathcal{C}}$. It follows that

$$\sum_{x\in\mathcal{C}} \mu_{2\delta}(x)\mathbb{1}_{B(x,\delta)} \ll \sum_{x\in\mathcal{C}} \mu_{4\delta}\mathbb{1}_{B(x,\delta)} \ll \mu_{4\delta}. \tag{4}$$

Finally, set

$$\mathcal{C}_i = \left\{x \in \mathcal{C} \,|\, 2^i \leq \mu_{2\delta}(x) < 2^{i+1}\right\}$$

and

$$A_i = \bigcup_{x\in\mathcal{C}_i} B(x,\delta).$$

Equations (3) and (4) imply that indeed

$$\mu_\delta \ll \sum_i 2^i \mathbb{1}_{A_i} \ll \mu_{4\delta}.$$

Now suppose that for some i, $2^i|A_i| \geq \delta^\epsilon$ and $2^i \geq \delta^{-3+\alpha+\epsilon}$. By construction, we must also have $2^i \ll \delta^{-3+\alpha}$, so that

$$\delta^{-\alpha-\epsilon} \gg N(A_i,\delta) \simeq \delta^{-3}|A_i| \gg \delta^{-\alpha+\epsilon}.$$

Moreover, for $r \in (\delta, 1)$ and $x \in G$,

$$\begin{aligned} N(A_i \cap B(x,r),\delta) &\leq \delta^{-3}|A_i \cap B(x,r+\delta)| \\ &\leq \delta^{-3}|A_i \cap B(x,2r)| \\ &\ll \delta^{-3}2^{-i}\mu(A_i \cap B(x,3r)), \end{aligned}$$

which implies, as μ is α-Frostman,

$$\begin{aligned} N(A_i \cap B(x,r),\delta) &= \delta^{-\alpha+O(\epsilon)}(3r)^\alpha \\ &= r^\alpha \delta^{O(\epsilon)} N(A_i,\delta), \end{aligned}$$

and so, A_i is a $(\alpha,\alpha,O(\epsilon))$-set. $\square$

Before starting the proof of Proposition 4.2, we give some technical estimates on the pushforward of μ under the map w.
Denote

$$w'(a_1,\dots,a_7) = a_3\pi_0(a_1,a_2)a_4\pi_1(a_1,a_2)a_5\pi_2(a_1,a_2)a_6\pi_3(a_1,a_2)a_7\pi_4(a_1,a_2),$$

so that

$$w(a_1,\dots,a_8) = w'(a_1,\dots,a_7)a_8.$$

Lemma 4.5. *There exists a neighborhood V of I in G, and an absolute constant $C > 0$ such that if g_1 and g_2 are in V, then for all $1 \le j \le 7$, for all $\{a_i\}_{i\neq j} \in V^6$ and x in G, the map*

$$\varphi_j : a_j \mapsto w'(a_1,\dots,a_7)^{-1}x,$$

is a diffeomorphism from V on its image with Jacobian J_{φ_j} satisfying

$$\forall a_j \in V, \qquad \frac{1}{C} \le |J_{\varphi_j}(a_j)| \le C.$$

Proof. This is an immediate consequence of the Local Inverse Theorem, and of the fact that the map $(a_1,\dots,a_7,g_1,g_2,x) \mapsto J_\varphi(a_j)$ is continuous, and takes a nonzero integer value (concretely, 1 or 4 depending on j) on $(I,\dots,I,x)$ for any $x \in G$. □

If μ is a Borel probability measure on G, we can estimate:

$$\begin{aligned}
I(x) &= \int_V \mu^{\otimes 7}(\{(a_2,\dots,a_8)\,|\,w(a_1,\dots,a_8) \in B(x,\delta)\})\,da_1 \\
&= \int_{G^6}\left(\int_V \mu(\{a_8\,|\,w(a_1,\dots,a_8)\in B(x,\delta)\})\,da_1\right) d\mu^{\otimes 6}(a_2,\dots,a_7) \\
&= \int_{G^6}\left(\int_V \mu(B(w'(a_1,\dots,a_7)^{-1}x,\delta))\,da_1\right) d\mu^{\otimes 6}(a_2,\dots,a_7) \\
&= \int_{G^6}\left(\int_{\varphi(V)} \mu(B(u,\delta))|J_{\varphi^{-1}}(u)|\,du\right) d\mu^{\otimes 6}(a_2,\dots,a_7),
\end{aligned}$$

so that if μ is supported on V, the previous lemma implies, uniformly in x,

$$I(x) \le \int_{G^6} C\delta^3\,d\mu^{\otimes 6}(a_2,\dots,a_7) = C.\delta^3. \tag{5}$$

Similarly, one bounds, uniformly in a_1,

$$J(a_1) = \int_G \mu^{\otimes 7}(\{(a_2,\dots,a_8)\,|\,w(a_1,\dots,a_8)\in B(x,\delta)\})\,dx \le C.\delta^3. \tag{6}$$

We can now make our first step in the direction of Proposition 4.2:

Lemma 4.6. *Let $\alpha \in (0,3)$ and $\epsilon > 0$. For $\delta > 0$ sufficiently small (in terms of ϵ), we have the following.*
Let μ be a α-Frostman Borel probability measure on G such that $\mu_{4\delta}$ is supported on V, and write $\mu_\delta \ll \sum_i 2^i \mathbb{1}_{A_i} \ll \mu_{4\delta}$ as in Lemma 4.4. If $\nu = w_*(\mu^{\otimes 8})$ satisfies $\|\nu_{8\delta}\|_2^2 \geq \delta^{-3+\alpha+\epsilon}$, then there exists $(i_1, \ldots, i_8)$ such that,

$$\|w_*(2^{i_1}\mathbb{1}_{A_{i_1}} \otimes \cdots \otimes 2^{i_8}\mathbb{1}_{A_{i_8}})\|^2 \geq \delta^{-3+\alpha+O(\epsilon)},$$

and each A_{i_k} is a $(\alpha, \alpha, O(\epsilon))$-set.

Proof. By definition of the A_i's we have

$$\begin{aligned}\nu_{8\delta} &\ll w_*(\mu_\delta \otimes \cdots \otimes \mu_\delta) \\ &\ll \sum_{i_1,\ldots,i_8} w_*(2^{i_1}\mathbb{1}_{A_{i_1}} \otimes \cdots \otimes 2^{i_8}\mathbb{1}_{A_{i_8}}).\end{aligned}$$

Thus, the lower bound on $\|\nu_{8\delta}\|_2$ together with the triangle inequality imply that there exists $(i_1, \ldots, i_8)$ such that

$$\|2^{i_1+\cdots+i_8} w_*(\mathbb{1}_{A_{i_1}} \otimes \cdots \otimes \mathbb{1}_{A_{i_8}})\|_2^2 \geq \frac{1}{(\log \frac{1}{\delta})^8}\delta^{-3+\alpha+\epsilon} \geq \delta^{-3+\alpha+O(\epsilon)}. \tag{7}$$

Remains to show that for each $k = 1, \ldots, 8$, the set A_{i_k} is a $(\alpha, \alpha, O(\epsilon))$-set. By Lemma 4.4, it suffices to check that for each k,

$$2^{i_k}|A_{i_k}| \geq \delta^{O(\epsilon)} \quad \text{and} \quad 2^{i_k} \geq \delta^{-3+\alpha+O(\epsilon)}. \tag{8}$$

For notational convenience, we consider the case $k = 1$, the other cases can be handled in the same way, mutatis mutandis. Write

$$\begin{aligned}\delta^{-3+\alpha+O(\epsilon)} &\leq \|w_*(2^{i_1}\mathbb{1}_{A_{i_1}} \otimes \cdots \otimes 2^{i_8}\mathbb{1}_{A_{i_8}})\|_2^2 \\ &\leq \|w_*(2^{i_1}\mathbb{1}_{A_{i_1}} \otimes \mu_{4\delta} \otimes \cdots \otimes \mu_{4\delta})\|_2^2 \\ &\leq 2^{2i_1}\delta^{-6}\int_G w_*(\mathbb{1}_{A_{i_1}} \otimes \mu \otimes \cdots \otimes \mu)(B(x, 28\delta))^2\, dx \\ &= 2^{2i_1}\delta^{-6}\int_G \left(\int_V \mathbb{1}_{A_{i_1}}(a_1)\mu^{\otimes 7}(\{(a_2, \ldots, a_8) \,|\, w(a_1, \ldots, a_8) \in B(x, 28\delta)\})\, da_1\right)^2 dx.\end{aligned}$$

By Jensen's inequality applied to the inner squared integral, this gives

$$\delta^{-3+\alpha+O(\epsilon)} \leq 2^{2i_1}\delta^{-6}\int_G\int_V \mathbb{1}_{A_{i_1}}(a_1)\mu^{\otimes 7}(\{(a_2, \ldots, a_8) \,|\, w(a_1, \ldots, a_8) \in B(x, 28\delta)\})I(x)\, da_1\, dx,$$

which yields, using successively estimates (5) and (6) above,

$$\begin{aligned}\delta^{-3+\alpha+O(\epsilon)} &\ll 2^{2i_1}\delta^{-3}\int_G\int_V \mathbb{1}_{A_{i_1}}(a_1)\mu^{\otimes 7}(\{(a_2, \ldots, a_8) \,|\, w(a_1, \ldots, a_8) \in B(x, 28\delta)\})\, da_1\, dx \\ &\ll 2^{2i_1}\int_G \mathbb{1}_{A_{i_1}}(a_1)\, da_1,\end{aligned}$$

whence

$$\delta^{-3+\alpha+O(\epsilon)} \leq 2^{2i_1}|A_{i_1}|. \tag{9}$$

As $2^{i_1}\mathbb{1}_{A_{i_1}} \ll \mu_{4\delta}$, taking the integral, we find $2^{i_1}|A_{i_1}| \ll 1$, and so,

$$2^{i_1} \geq \delta^{-3+\alpha+O(\epsilon)}.$$

By construction of the A_i's we have $2^{i_1} \ll \delta^{-3+\alpha}$, and therefore (9) also implies

$$2^{i_1}|A_{i_1}| \geq \delta^{O(\epsilon)}.$$

□

We are finally ready to prove our flattening statement.

Proof of Proposition 4.2. Under the hypothesis of the proposition, assume for contradiction that for $\epsilon > 0$ arbitrarily small,

$$\|\nu_{8\delta}\|_2^2 \geq \delta^{-3+\alpha+\epsilon}.$$

Let $A_{i_1}, \dots, A_{i_8}$ be the $(\alpha, \alpha, O(\epsilon))$-sets given by the previous lemma. Denote $\tilde{\nu} = w_*(2^{i_1}\mathbb{1}_{A_{i_1}} \otimes \cdots \otimes 2^{i_8}\mathbb{1}_{A_{i_8}})$ and φ its density function. On one hand we have

$$\|\varphi\|_2^2 \geq \delta^{-3+\alpha+O(\epsilon)},$$

and on the other hand,

$$\tilde{\nu} \ll w_*(\underbrace{\mu_{4\delta} \otimes \cdots \otimes \mu_{4\delta}}_{8 \text{ times}}) = w'_*(\underbrace{\mu_{4\delta} \otimes \cdots \otimes \mu_{4\delta}}_{7 \text{ times}}) * \mu_{4\delta},$$

whence, using the fact that $\mu_{4\delta}$ is α-Frostman,

$$\|\varphi\|_\infty \ll \delta^{-3+\alpha}.$$

Also, as $\mu_{4\delta}$ is a probability measure $\|\varphi\|_1 \ll 1$. Now let

$$D = \left\{ x \in G \,|\, \varphi(x) \geq \frac{\|\varphi\|_2^2}{2\|\varphi\|_1} \right\}.$$

We have,

$$\delta^{-3+\alpha+O(\epsilon)} \leq \|\varphi\|_2^2$$

and

$$\begin{aligned}\|\varphi\|_2^2 &\leq \int_D \varphi(x)^2\,dx + \int_{D^c} \frac{\|\varphi\|_2^2\,\varphi(x)}{2\|\varphi\|_1}\,dx \\ &\leq \int_D \varphi(x)^2\,dx + \frac{\|\varphi\|_2^2}{2}\end{aligned}$$

so that

$$\tilde{\nu}(D) = \int_D \varphi(x)\,dx \geq \frac{1}{\|\varphi\|_\infty}\int_D \varphi(x)^2\,dx \geq \frac{\|\varphi\|_2^2}{2\|\varphi\|_\infty} \geq \delta^{O(\epsilon)}.$$

Denoting

$$\Omega = w^{-1}(D) \cap (A_{i_1} \times \cdots \times A_{i_8}),$$

this implies, using also the fact that for each k, $2^{i_k} \mathbb{1}_{A_{i_k}} \ll \mu_{4\delta}$,

$$\mu_{4\delta}^{\otimes 8}(\Omega) \geq \delta^{O(\epsilon)},$$

and as $\mu_{4\delta}$ is α-Frostman,

$$N(\Omega, \delta) \geq \delta^{-8\alpha + O(\epsilon)}.$$

On the other hand, by construction,

$$N(w(\Omega), \delta) \leq N(D, \delta)$$

and from the fact that on D, φ is larger than $\| \frac{\varphi\|_2^2}{2\|\varphi\|_1}$, we have

$$|D| \leq \| \frac{2\|\varphi\|_1^2}{\varphi\|_2^2} \ll \delta^{3-\alpha+O(\epsilon)}$$

and therefore

$$N(D, \delta) \ll \delta^{-3}|D| \leq \delta^{-\alpha+O(\epsilon)}.$$

Thus,

$$N(w(\Omega), \delta) \leq \delta^{-\alpha+O(\epsilon)}.$$

Choosing $\epsilon > 0$ (and hence $O(\epsilon)$) sufficiently small, this yields the desired contradiction with Proposition 3.6. □

5 Hausdorff dimension and product sets

Here we explain how to obtain results on the Hausdorff dimension of product sets in G from the flattening statement Proposition 4.2 of the preceding section. Recall (Definition 4.1) that a Borel measure μ on G is called α-Frostman if it satisfies, for some $C \geq 0$, for all $r > 0$, for all $x \in G$,

$$\mu(B(x, r)) \leq C.r^{\alpha}.$$

The basic tool will be the Frostman Theorem (see [12], Theorem 8.17, p. 120).

Theorem 5.1 (Frostman)**.** *Let $\alpha \in (0, 3)$ and let A be a Borel measurable subset of G. If $\dim_H A > \alpha$, then there exists an α-Frostman measure whose support is included in A.*

We will also use an L^2-version of the easy and well-known converse to this theorem (we recall the proof for the convenience of the reader):

Lemma 5.2. *Let $\alpha \in (0,3)$ and let A be a subset of G. If there exists a measure on G such that $\mu(A) = 1$ and for all $\delta > 0$ sufficiently small,*

$$\|\mu_\delta\|_2^2 \leq \delta^{-3+\alpha},$$

then

$$\dim_H A \geq \alpha.$$

Proof. Suppose $\dim_H A < \alpha$. One can write, for k_0 arbitrary large (and for some small $\epsilon > 0$, e.g. $\epsilon = \frac{\alpha - \dim_H A}{2}$)

$$A \subset \bigcup_{k \geq k_0} \bigcup_{x \in S_k} B(x, 2^{-k})$$

with S_k a 2^{-k}-separated set satisfying $\operatorname{card} S_k \leq 2^{(\alpha-\epsilon)k}$. Therefore,

$$\begin{aligned}
1 = \mu(A) &\leq \sum_{k \geq k_0} \sum_{x \in S_k} \mu(B(x, 2^{-k})) \\
&\leq \sum_{k \geq k_0} (\operatorname{card} S_k)^{\frac{1}{2}} \sqrt{\sum_{x \in S_k} \mu(B(x, 2^{-k}))^2} \\
&\leq \sum_{k \geq k_0} 2^{\frac{(\alpha-\epsilon)k}{2}} \sqrt{\sum_{x \in S_k} \mu(B(x, 2^{-k}))^2}
\end{aligned}$$

so there exists $k \geq k_0$ such that

$$\sqrt{\sum_{x \in S_k} \mu(B(x, 2^{-k}))^2} \geq k^{-2} 2^{-\frac{(\alpha-\epsilon)k}{2}} \geq 2^{-\frac{k\alpha}{2}},$$

assuming k_0 (and hence k) large enough.
Now let $\delta = 2^{-k}$, and write, using that S_k is δ-separated:

$$\begin{aligned}
\|\mu_\delta\|_2^2 &\gg \delta^{-3} \sum_{x \in S_k} \mu(B(x, \delta))^2 \\
&\geq \delta^{-3+\alpha}.
\end{aligned}$$

□

We are now ready to prove a dimensional growth for product sets.

Proposition 5.3. *Let $\alpha \in (0,3)$. There exists $\epsilon = \epsilon(\alpha) > 0$ such that the following holds for any symmetric Borel $A \subset G$ of Hausdorff dimension larger than α not included in a torus:*

$$\dim_H A^{32} \geq \alpha + \epsilon.$$

Proof. Suppose first that A is included in the neighborhood V of I defined in Lemma 4.2. By Frostman's Theorem, there exists a Borel probability measure supported on A which is α-Frostman. Choose also g_1 and g_2 two non-commuting elements in A. By Proposition 4.2, the pushforward ν of the measure $\mu^{\otimes 8}$ under the map w defined in Proposition 3.6 satisfies, for all $\delta > 0$ sufficiently small,

$$\|\nu_\delta\|_2^2 \leq \delta^{3-\alpha+\epsilon}.$$

Now, w is a word of length 16, so the support of ν is included in A^{16}, and we may apply Lemma 5.2 to get

$$\dim_H A^{16} \geq \alpha + \epsilon.$$

If A is not included in the neighborhood V, using that A is symmetric, one sees that $AA \cap V$ has Hausdorff dimension at least α. So we may apply the first part of the proof to this set, and find

$$\dim_H A^{32} \geq \alpha + \epsilon.$$

□

Remark. The symmetry assumption on A is not crucial, it only made it slightly easier to find a integer k such that $\dim_H A^k \cap V \geq \alpha$. In the general case, one could prove that there exists such a k, depending on the choice of the set V — which can easily be made explicit — and hence get the analogous growth statement, possibly with a worse exponent in the product set.

Corollary 5.4. *There is no dense Borel measurable proper sub-semigroup of $SU(2)$ of positive Hausdorff dimension.*

Proof. Theorem 5.3 certainly implies that there cannot exist a Borel measurable subgroup of Hausdorff dimension $\alpha \in (0, 3)$. Moreover, by [14], Théorème 4.4, any Borel subset of G of dimension larger than 2 generates G, so in particular G has no Borel subgroup of dimension 3. □

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