Algebraic Topology

This series of four second-year master courses aims at offering an introduction to modern algebraic topology. The first semester will start with elementary homological algebra and singular (co-)homology, Poincaré duality, and will continue with homotopy theory, simplicial sets and (rational) homotopy theory. During the second semester, more advanced topics will be presented, including model categories and derived functors, infinity-categories, E_n-algebras, and factorization homology, aiming at Lurie's result on non-abelian Poincaré duality.

The prerequisites of the first lecture are general topology and basic commutative algebra, while each later lecture builds on the contents of the previous one.

1. Homology Theory (Christian Ausoni, ausoni@math.univ-paris13.fr) : 9ECTS, 1st semester, 1st period, 24 hours of lectures and 12 hours of exercise sessions.

In this lecture course, we will begin by introducing basic category theory, chain complexes and elementary homological algebra, including resolutions and the derived functors Tor and Ext. We will then define and study the singular homology and cohomology of spaces, review the Eilenberg-Steenrod axioms, as well as the Künneth and universal coefficient theorems. The end of the lecture will be dedicated to the cup product and Poincaré duality.

References : [Hat02], [M92], [ML95], [ML98], [tD08], [Wei94].

2. Homotopy Theory (Bruno Vallette, vallette@math.univ-paris13.fr) : 9 ECTS, 1st semester, 2nd period, 24 hours of lectures and 12 hours of exercise sessions.

The goal of this lecture will be to present various "concrete" homotopy theories. We will start with the classical homotopy theory of topological spaces (higher homotopy groups, cellular complexes, Whitehead and Hurewicz theorems, Eilenberg—MacLane spaces, fibrations, and Postnikov towers). Then we will move to the homotopy theory of simplicial sets (definitions, simplex category, adjunction and cosimplicial objects, examples, fibrations, Kan complexes, and simplicial homotopy). Finally, we will study the rational homotopy theory via the homotopy theory of differential graded Lie or commutative (co)algebras (Sullivan approach: minimal model, Quillen approach: Whitehead Lie bracket, bar and cobar constructions, complete Lie algebra-Hopf algebras-groups).

This presentation opens the door to the axiomatic treatment of homotopy theory done by Quillen and treated in details in the next course.

Prerequisites : Basic notions of category theory, chain complexes, exact sequence, singular (co)homology (all being covered by the previous course).

References: [FHT01], [GM13], [GJ99], [Hat02], [Whi78].

3. Homotopical Algebra (Grégory Ginot, ginot@math.univ-paris13.fr) : 9 ECTS, 2nd semester, 1st period, 24 hours of lectures and 12 hours of exercise sessions.

The aim of this lecture is to introduce Quillen's abstract homotopy theory, which permits to define homotopy categories in many different contexts, for example in homological algebra or in topology, and to relate them. It will include a treatment of model categories and their homotopy categories, Quillen functors and derived functors, and will make precise the comparison of simplicial sets and topological spaces. The model category of chain complexes will also be treated as well as possibly some other algebraic categories. The lecture will end with an introduction to infinity-categories.

References : [DS95], [GJ99], [Hov99], [LUR09], [Wei94], [Whi78].

4. Higher Algebra (Yonatan Harpaz, harpaz@math.univ-paris13.fr) : 9 ECTS, 2nd semester, 2nd period, 24 hours of lectures (possibly with 12h of exercise sessions).

This is an advanced course in homotopy theory intended for students with background in the field. In particular, we will assume basic familiarity with infinity-cateogories and key examples such as spaces and chain-complexes. The course will focus on the notion of E_n-algebras, a homotopy coherent algebraic structure interpolating between associative and commutative algebras. We will begin by introducing topological and infinity-operads, focusing on the example of the little n-cube operads, whose algebras are the E_n-algebras. We will discuss the tensor product of infinity-operad and the additivity theorem for little cube operads. We will also describe some naturally occuring examples of E_n-algebras, focusing on the cases of spaces and chain-complexes. In the later parts of the course we will discuss factorization homology, a fundamental invariant of E_n-algebras which is a vast generalization of Hochschild homology for associative algebras. Our goal at the end of the course is to build up towards a proof of non-abelian Poincaré duality, following the approach of Lurie.

References : The majority of the material will be taken from chapter 5 of [Lur14].

References.

[DS95] W. Dwyer, J. Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology, 73–126, North-Holland, Amsterdam, 1995.

[FHT01] Y. Félix, S. Halperin, and J.-C. Thomas. *Rational homotopy theory*, Graduate Texts in Mathematics, 205,. Springer, 2001.

[GJ99] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, reprint of the 1999 edition, Birkhauser, 2009.

[GM13] P. Griffiths, J. Morgan, *Rational Homotopy Theory and Differential forms*, second edition, Birkhauser, 2013.

[Hat02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[Hov99] M. Hovey, *Model categories*, Mathematical Surveys and Monographs, 63, American Mathematical Society, Providence, RI, 1999.

[Lur09] J. Lurie, *Higher Topos Theory,* Annals of mathematical Studies no. 170, 2009.

[Lur 14] J. Lurie, *Higher Algebra*, http://www.math.harvard.edu/~lurie/papers/HA.pdf

[M92] J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Maths, University of Chicago Press, 1992.

[ML95] S. Mac Lane, *Homology*, Classics in Mathematics, reprint of the 1975 edition, Springer, 1995.

[ML98] S. Mac Lane, *Categories for the working mathematician*, Second edition. Graduate Texts in Mathematics, 5, Springer, 1998.

[tD08] T. tom Dieck, *Algebraic Topology*, EMS Textbook in Mathematics, 2008.

[Wei94] Ch. Weibel, *An introduction to homological algebra,* Cambridge Studies in Advanced Mathematics, 38, Cambridge University Press, Cambridge, 1994.

[Whi78] G. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics,61. Springer-Verlag, New York-Berlin, 1978.