NON-UNIFORM STABILITY FOR BOUNDED SEMI-GROUPS ON BANACH SPACES

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ABSTRACT. Let $S(t)$ be a bounded strongly continuous semi-group on a Banach space B and $-A$ be its generator. We say that $S(t)$ is semi-uniformly stable when $S(t)(A+1)^{-1}$ tends to 0 in operator norm. This notion of asymptotic stability is stronger than pointwise stability, but strictly weaker than uniform stability, and generalizes the known logarithmic, polynomial and exponential stabilities.

In this note we show that if S is semi-uniformly stable then the spectrum of A does not intersect the imaginary axis. The converse is already known, but we give an estimate on the rate of decay of $S(t)(A+1)^{-1}$, linking the decay to the behaviour of the resolvent of A on the imaginary axis. This generalizes results of Lebeau and Burq (in the case of logarithmic stability) and Liu-Rao and Bátkai-Engel-Prüss-Schnaubelt (in the case of polynomial stability).

1. BACKGROUND

Consider a strongly continuous semi-group $S(t)$ on a Banach space B, with generator $-A$. Assume that $S(t) = e^{-tA}$ is bounded, i.e.

$$\sup_{t\geq 0} \left\| e^{-tA} \right\| = \widetilde{C} < \infty. \tag{1}$$

(Throughout the article, a semi-group will be strongly continuous on $[0,\infty)$, i.e., a C_0-semigroup. Moreover, $\|\cdot\|$ will denote both the norm on B and the operator norm from B to B.) The operator A is closed and densely defined, and we denote by $D(A)$ its domain, $\sigma(A)$ its spectrum and $\rho(A)$ its resolvent set. It is a well-known property that if (1) holds, then the left open half-plane $\{\operatorname{Re} z < 0\}$ is included in $\rho(A)$ (see [25, 11]).

In 1988, Lyubich and Vũ [20] and Arendt and Batty [1] have shown that if $\sigma(A) \cap i\mathbb{R}$ is countable and $\sigma(A^*) \cap i\mathbb{R}$ contains no eigenvalue (here A^* is the adjoint of A), then the semi-group is (pointwise) strongly stable, that is

$$\forall u_0 \in B, \quad \lim_{t\to+\infty} \left\| e^{-tA} u_0 \right\| = 0.$$

For surveys of this and other results concerning strong stability, see [4], [8][1].

In this note we investigate the rate of decay, as t tends to $+\infty$, of the norm $\left\| e^{-tA} u_0 \right\|$. The semi-group is said to be **uniformly stable** if

$$\lim_{t\to+\infty} \left\| e^{-tA} \right\| = 0.$$

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[1] We will only address strong stability concepts, and refer to [10] for a recent survey on different types of weak stability.

In this case, the semi-group property implies that this decay is at least exponential. Uniform stability of semi-groups has been intensively studied by many authors (see [25], [30, Chapter 3], [11, V.1.b], [2, Chapter 5] and references therein), and we are interested in the following weaker notion of stability:

$$\lim_{t\to+\infty}\left\|e^{-tA}(1+A)^{-k}\right\|=0, \tag{2}$$

where $k\in\mathbb{N}^*$. Observe that $\|(1+A)^k\cdot\|$ defines a norm on $D(A^k)$ which is equivalent to the natural norm. Thus (2) is equivalent to the fact that the norm of e^{-tA}, as a bounded operator from $D(A^k)$ to B, tends to 0 as t tends to infinity. Furthermore, by density of $D(A^k)$ in B, (2) implies strong stability.

The aim of this work is to give conditions on the spectrum of A to satisfy (2) and to estimate the rate of convergence in (2) in terms of the resolvent of A. Our main motivation is that (2) often appears in applications to linear partial differential equations. Observe that the semi-group property does not imply anymore that the rate of convergence in (2) is exponential, and there are many practical examples where the decay is only logarithmic or polynomial (see e.g. [15, 16, 17, 28, 6, 19, 26, 32, 9, 12]).

The following result shows that (2) may be characterized in terms of the intersection of $\sigma(A)$ with the imaginary axis $i\mathbb{R}$.

Theorem 1. *Let e^{-tA} be a bounded semi-group on a Banach space B. The following properties are equivalent:*

(a) *There exists $k\geq 1$ such that* (2) *holds.*
(b) *For all $k\geq 1$,* (2) *holds.*
(c) *$i\mathbb{R}\cap\sigma(A)=\emptyset$.*

Definition 2. We will say that the bounded semi-group e^{-tA} is **semi-uniformly stable** if one of the equivalent properties of Theorem 1 hold.

A few remarks are in order.

The equivalence between (a) and (b) is elementary (see also the quantitative statement (4) and (5) below). This equivalence and the fact that these properties imply (c) was observed in [3] in the particular case where $\left\|e^{-tA}(1+A)^{-k}\right\|\leq\frac{C}{(1+t)^\alpha}$, where α and C are positive constants.

The equivalence between the decay property (b) and the purely spectral property (c) is specific to semi-uniform stability. The sufficient condition of strong stability given by the Theorem of Arendt, Batty, Lyubich and Vũ is not necessary, as shown by the example of the strongly stable semigroup $S(t)$ defined by $S(t)f(x)=f(t+x)$ on $L^2(0,+\infty)$. In this example $A=-\frac{d}{dx}$ and $\sigma(A)=\{\operatorname{Re} z\geq 0\}$. Thus strong stability of e^{-tA} cannot be characterized solely in terms of the spectrum of A. See also Example 8 for the case of uniform stability.

That (c) implies (a) (with $k=1$) was implicit in [1] (see the argument leading to the inductive statement on p.843 in the very special case when $i\mathbb{R}\cap\sigma(A)=\emptyset$). It was explicitly presented in [4, pp.40,41]. In fact, it is shown that

$$\lim_{t\to+\infty}\left\|e^{-tA}A^{-1}\right\|=0,$$

and (2) follows (for $k=1$) since $A(1+A)^{-1}$ is a bounded operator. This proof has its roots in a Tauberian theorem due to Ingham [13] and elementary expositions by Newman [24] and

Korevaar [14]. That (c) implies (a) (and (b)) is the most important part of Theorem 1 in applications, as it reduces the proof of a stability property to a simple spectral criterion. Such spectral criteria have also proved efficient in control theory [29, 7, 21, 27]. We shall give a quantitative statement and proof of that implication. It seems to be a new result that (a) implies (c), and we shall give a quantitative statement and proof of that as well. We first introduce some notation.

Let

$$m_k(t) = \sup_{s\geq t} \left\| e^{-sA}(1+A)^{-k} \right\|, \quad k \in \mathbb{N},\ t \geq 0. \tag{3}$$

It is easy to see that m_k is bounded by the constant $\widetilde{C}$ appearing in (1), non-increasing, and continuous on $[0,+\infty)$. The property (2) is obviously equivalent to the fact that $m_k(t)$ tends to 0 at infinity.

By a straightforward interpolation argument, there is a constant $C>0$ (depending only on $\widetilde{C}$ defined in (1)) such that

$$m_1(t) \leq C\left(m_k(t)\right)^{1/k}, \quad k \geq 1,\ t \geq 0. \tag{4}$$

This inequality has the following converse, given by a simple iteration argument,

$$m_k(t) \leq \left(m_1\left(\frac{t}{k}\right)\right)^k, \quad k \geq 1,\ t \geq 0. \tag{5}$$

In practical cases (for example if m_1 is equivalent to a negative power of t or a negative power of the logarithm of t), $\left(m_1\left(\frac{t}{k}\right)\right)^k$ decreases with k: smoother trajectories tend to have a faster decay to 0.

Observe that if (c) holds, the function $\tau \mapsto \|(A-i\tau)^{-1}\|$ is continuous on $\mathbb{R}$ (here $\|\cdot\|$ is the operator norm on B). Define the continuous positive increasing function

$$M(\xi) = \sup_{-\xi\leq\tau\leq\xi} \|(i\tau + A)^{-1}\|, \quad \xi \geq 0. \tag{6}$$

One may give bounds on the function M in terms of the functions m_k. We give a result in the case $k=1$ without essential loss of generality (see (4) and (5)) and with a gain in simplicity.

Proposition 3 (Necessary condition for semi-uniform stability). *Let e^{-tA} be a bounded semi-group on a Banach space B. Assume that*

$$\lim_{t\to+\infty} m_1(t) = 0.$$

Then $i\mathbb{R}\cap\sigma(A)=\emptyset$, and there exist constants $\xi_0, C>0$ (depending only on m_1 and the constant $\widetilde{C}$ in (1)*), such that*

$$\forall \xi \geq \xi_0, \quad M(\xi) \leq 1 + C m_{1r}^{-1}\left(\frac{1}{2(\xi+1)}\right),$$

where m_{1r}^{-1} is a right inverse of the function m_1, which maps $(0, m_1(0)]$ onto $[0,+\infty)$.

Example 4. If $m_1(t) \leq C\exp(-ct)$ for some positive constants c, C (exponential stability) the proposition shows that there exists a positive constant C' such that for large ξ, $M(\xi) \leq C'|\log\xi|$. If $m_1(t) \leq \frac{C}{t^\alpha}$, where α is positive (polynomial stability) we get the bound $M(\xi) \leq C'\xi^{1/\alpha}$. Finally if $m_1(t) \leq \frac{C}{\log t}$ (logarithmic stability), then $M(\xi) \leq C'\exp(C'\xi)$. We will come back to these special cases in Examples 6, 7 and 8.

For a quantitative version of the implication (c)⇒(a) of Theorem 1, and with a view to practical applications, we would like to establish a converse of Proposition 3 such as

$$\exists t_0,\ C>0,\ \forall t\geq t_0,\quad m_1(t)\leq \frac{C}{M^{-1}(t/C)}, \tag{7}$$

where M^{-1} is the inverse function of M (assuming for simplicity that M is strictly increasing). Partial results are already known in the case where M is polynomially or exponentially bounded. We recall these results in the next two Theorems.

Theorem A (Lebeau, Burq). *Assume that B is a Hilbert space. If $M(\xi)\leq C\exp(C\xi)$ for some constant $C>0$, then*

$$\forall k\geq 1,\ \exists C_k,\quad m_k(t)\leq \frac{C_k}{\log^k(2+t)}. \tag{8}$$

In [15], Gilles Lebeau showed Theorem A with an additional $\log\log$ factor in the right hand side of (8). The exact bound (8) is obtained in [5, Th. 3], in a more general setting also adapted to local energy-decay estimates. Theorem A has important applications in linear partial differential equations, where exponential bounds on the resolvent often appear as a consequence of Carleman estimates (see e.g. [16]).

The previously known polynomial decay results are as follows:

Theorem B (Bátkai-Engel-Prüss-Schnaubelt, Liu-Rao). *If $M(\xi)\leq C(1+\xi)^\alpha$ for some constants $C,\alpha>0$, then (see* [3]*)*

$$\forall k\geq 1,\ \forall\varepsilon>0,\ \exists C_{k,\varepsilon},\quad m_k(t)\leq \frac{C_{k,\varepsilon}}{t^{\frac{k}{\alpha}-\varepsilon}}.$$

If furthermore B is a Hilbert space, then (see [18]*):*

$$\forall k\geq 1,\ \exists C_k,\quad m_k(t)\leq \frac{C_k\log^{\frac{k}{\alpha}+1}(2+t)}{t^{\frac{k}{\alpha}}}.$$

Thus if B is a Hilbert space, (7) holds if M grows exponentially. If M grows only polynomially, it holds up to a logarithmic correction. We will state a general result, unifying the points of view of Theorems A and B and showing that the Hilbert space assumption is unnecessary. We will denote

$$M_{\log}(\eta)=M(\eta)\big[\log(1+M(\eta))+\log(1+\eta)\big], \tag{9}$$

which tends to infinity when η tends to infinity. Up to a logarithmic term, $M_{\log}$ is of the same order as M. Let $M_{\log}^{-1}$ be the inverse of $M_{\log}$, which maps $(T,+\infty)$ onto $(0,+\infty)$, where $T=M_{\log}(0)$. Then we have the following Theorem.

Theorem 5 (Sufficient condition for semi-uniform stability). *Let e^{-tA} be a bounded semi-group on a Banach space B such that $i\mathbb{R}\cap\sigma(A)=\emptyset$. Let M and $M_{\log}$ be defined by* (6) *and* (9)*. Let $k\in\mathbb{N}^*$. Then there are constants C_k, T_k, depending only on k, $\widetilde{C}$ and M, such that :*

$$\forall t\geq T_k,\quad \left\|e^{-tA}(A+1)^{-k}\right\|\leq \frac{C_k}{\left(M_{\log}^{-1}\left(\frac{t}{C_k}\right)\right)^k}. \tag{10}$$

In other words, under the assumptions of Theorem 5,

$$\forall t \geq T_k, \quad m_k(t) \leq \frac{C_k}{\left(M_{\log}^{-1}\left(\frac{t}{C_k}\right)\right)^k}. \tag{11}$$

This does not quite achieve (7) because of the logarithmic term, but the following examples show that it includes Theorem A and Theorem B.

Example 6. If $M(\xi) = \beta e^{\alpha\xi}$, $\alpha, \beta > 0$, then $M_{\log}^{-1}(t) \sim \frac{1}{\alpha}\log(t)$ as t tends to $+\infty$. We recover the theorem of G. Lebeau and N. Burq mentioned above. In this particular case, the logarithmic factors in the definition of $M_{\log}$ are not seen at first order. In the theorem of N. Burq, designed to show a local energy decay result, multiplication by bounded operators was allowed. We give a generalization of Theorem 5 in this spirit in Section 4 (see Corollary 11 and Proposition 12).

Example 7. If $M(\xi) = \beta(1+\xi)^\alpha$, $\alpha, \beta > 0$, then

$$M_{\log}^{-1}(t) \sim C_{\alpha,\beta}\left(\frac{t}{\log(t)}\right)^{\frac{1}{\alpha}}, \ t \to +\infty$$

This generalizes to Banach spaces the result of B. Rao and Z. Liu on Hilbert spaces, with a slight improvement on the logarithmic loss.

Example 8. If $M(\xi)$ is bounded we obtain

$$M_{\log}^{-1}(t) \sim Ce^t, \quad t \to +\infty,$$

and the norm $\|e^{-tA}u_0\|$ decays exponentially whenever $u_0 \in D(A)$, but in contrast with the uniform decay case, Theorem 5 does not imply a uniform bound with respect to $\|u_0\|$. This result is already known, as it follows directly from a more general stability result in semi-group theory (see e.g. [2, Thm 5.1.9]). In the Hilbert space case, it is not an optimal statement. Indeed, by a theorem of Gearhart, Prüss and Greiner, uniform boundedness of the resolvent on the imaginary axis is equivalent to uniform exponential decay of the semi-group (see [11, V.1.11], [2, Thm 5.1.12]). However that is not the case in general Banach spaces, as shown by the example of the semi-group $S(t)$ defined by

$$S(t)f(x) = e^{t/q}f(xe^t), \quad B = L^p(0,+\infty) \cap L^q(0,+\infty), \quad 1 < p < q < \infty.$$

(see [30, Example 1.4.4], [2, Example 5.2.2]).

Remark 9. As shown by Batkai, Engel, Prüss and Schnaubelt [3], the spectral theorem for normal operators implies that (7) holds when B is a Hilbert space, the operator A is normal and M grows polynomially. Thus, in this particular case, one may replace $M_{\log}^{-1}$ by M^{-1} in Theorem 5. The necessity of the logarithmic correction in the general case, even if B is assumed to be a Hilbert space, is to our knowledge completely open. As seen in Example 8, when M is constant at infinity, the logarithmic factor is natural for a general Banach space, but the bound (11) is not optimal for Hilbert space. We conjecture that the logarithmic correction may be dropped, or at least replaced by a smaller rectification, in the case of Hilbert space, but cannot be forgotten in general Banach spaces.

Section 2 is devoted to the proof of Proposition 3. In section 3, we prove Theorem 5. Both proofs are quite elementary. In particular, use of the method devised by Newman [24] and Korevaar [14], as in [4], simplifies the proofs given in [3], [5], [15], [18]. In Section 4, we extend Theorem 5 to general Laplace transforms, thereby generalizing results in [5] and [15] involving cut-off operators.

2. Necessary condition for stability

In this section we show Proposition 3 (and thus the implication (b)⇒(c) of Theorem 1). Let

$$\begin{aligned} G(\xi) &= m_{1r}^{-1}\left(\frac{1}{2(\xi+1)}\right) && \text{if } \xi>0 \text{ and } \frac{1}{2(\xi+1)} \leq m_1(0),\\ G(\xi) &= 0 && \text{if } \xi>0 \text{ and } \frac{1}{2(\xi+1)} > m_1(0). \end{aligned}$$

Under the assumptions of Proposition 3, we will show that $i\mathbb{R}\subset\rho(A)$ and

$$\forall \tau\in\mathbb{R},\quad \|(i\tau-A)^{-1}\|\leq 1+2\widetilde{C}G(|\tau|),$$

where $\widetilde{C}$ is given by (1).

We denote by $\sigma_r(A)$ the residual spectrum of A, which is the set of $z\in\mathbb{C}$ such that the range of $A-z$ is not dense in B. It is known that $z\in\sigma_r(A)$ if and only if z is an eigenvalue of the adjoint A^*. Let us first show:

$$\sigma_r(A)\cap i\mathbb{R}=\emptyset.$$

Indeed assume that $i\tau\in\sigma_r(A)$ for some $\tau\in\mathbb{R}$. Let ϕ be an eigenfunction of A^* for the eigenvalue $i\tau$. Let $u_0\in B$ such that $(\phi,u_0)\neq 0$. If $u(t)=e^{-tA}u_0$

$$\forall t>0,\quad \frac{d}{dt}\left[e^{it\tau}\left(\phi,u(t)\right)\right]=e^{it\tau}\left(-A^*\phi+i\tau\phi,u(t)\right)=0,$$

which contradicts the fact that e^{-tA} is strongly stable.

Let $u_0\in D(A)$, $\tau\in\mathbb{R}$ and $f_0=(A-i\tau)u_0$. Let

$$v(t)=e^{-it\tau}u_0.$$

Then

$$\partial_t v+Av=e^{-it\tau}(A-i\tau)u_0=e^{-it\tau}f_0,\quad v(0)=u_0.$$

By the Duhamel formula

$$v(t)=e^{-tA}u_0+\int_0^t e^{(s-t)A}e^{-is\tau}f_0\,ds.$$

By the boundedness of the semi-group and the definition of m_1,

$$\|u_0\|=\|v(t)\|\leq m_1(t)\|(1+A)u_0\|+\widetilde{C}t\|f_0\|\leq m_1(t)\left[(|\tau|+1)\|u_0\|+\|f_0\|\right]+\widetilde{C}t\|f_0\|.$$

Apply the preceding inequality with $t=G(|\tau|)$. In particular

$$m_1(t)(|\tau|+1)\leq\frac{1}{2},$$

which yields

$$\frac{1}{2}\|u_0\|\leq\left(\frac{1}{2(|\tau|+1)}\right)\|f_0\|+\widetilde{C}G(|\tau|)\|f_0\|\leq\left(\frac{1}{2}+\widetilde{C}G(|\tau|)\right)\|f_0\|. \tag{12}$$

Recall that $\sigma(A)$ is the union of $\sigma_r(A)$ and $\sigma_a(A)$, where the set $\sigma_a(A)$ of approximate eigenvalues is the set of λ such that there exists a sequence $\{x_n\}_n \in B^{\mathbb{N}}$ with $\|x_n\| = 1$ and $(A-\lambda)x_n$ tends to 0 (see e.g. [11, Lemma IV 1.9]). By (12), $i\tau \notin \sigma_a(A)$, and, as we have already shown that $i\tau \notin \sigma_r(A)$ we get that $i\tau \in \rho(A)$. Furthermore, (12) also gives

$$\left\|(i\tau - A)^{-1}\right\| \leq 1 + 2\widetilde{C}G(|\tau|), \tag{13}$$

which concludes the proof of Proposition 3. $\square$

3. Sufficient condition for stability

In this section we prove Theorem 5. We use the elementary method devised by Newman [24] and Korevaar [14], exactly as in [4]. This involves a partial inversion of a Laplace transform by means of a finite contour integral, but now we estimate the terms more precisely. We obtain the desired result by optimizing the choice of a parameter R which is the radius of a semicircular part of the contour.

By (5), it suffices to prove (11) for $k = 1$. Let $t \geq 0$ and $R > 0$, and let γ be the contour consisting of the right-hand half of the circle $|z| = R$ and any path γ' in $\{z \in \rho(-A) : \operatorname{Re} z < 0\}$ from iR to $-iR$. By Cauchy's Theorem,

$$e^{-tA}A^{-1} = \frac{1}{2\pi i}\int_\gamma \left(1 + \frac{z^2}{R^2}\right)(z+A)^{-1}e^{-tA}\,\frac{dz}{z}.$$

We first show:

$$\left\|e^{-tA}A^{-1}\right\| \leq \frac{2\widetilde{C}}{R} + \frac{1}{2\pi}\left\|\int_{\gamma'}\left(1 + \frac{z^2}{R^2}\right)(z+A)^{-1}\frac{e^{tz}}{z}\,dz\right\|. \tag{14}$$

The proof is exactly as in [4, p.40]. We sketch it for the sake of completeness. If $z = Re^{i\theta}$, $-\pi/2 < \theta < \pi/2$, we have, by (1),

$$\left\|(z+A)^{-1}e^{-tA}\right\| = \left\|e^{tz}\int_t^{+\infty} e^{-(z+A)s}\,ds\right\| \leq \frac{\widetilde{C}}{R\cos\theta}.$$

Noticing that $\left|1 + z^2/R^2\right| = 2|\cos\theta|$, we get the bound

$$\left\|\int_{\substack{|z|=R\\ \operatorname{Re} z>0}}\left(1 + \frac{z^2}{R^2}\right)(z+A)^{-1}e^{-tA}\,\frac{dz}{z}\right\| \leq \frac{2\pi\widetilde{C}}{R}. \tag{15}$$

Next, define the analytic function of z

$$h_t(z) = \int_0^t e^{(t-s)z}e^{-sA}ds = (z+A)^{-1}e^{tz} - (z+A)^{-1}e^{-tA}. \tag{16}$$

Cauchy's Theorem and similar estimate as for (15) yield

$$\left\|\int_{\gamma'}\left(1 + \frac{z^2}{R^2}\right)h_t(z)\frac{dz}{z}\right\| = \left\|\int_{\substack{|z|=R\\ \operatorname{Re} z<0}}\left(1 + \frac{z^2}{R^2}\right)h_t(z)\frac{dz}{z}\right\| \leq \frac{2\pi\widetilde{C}}{R}. \tag{17}$$

Together with (16) and (17), we get the estimate (14).

By means of standard Neumann series, we may take γ' to be the union of γ_0, γ_+ and γ_-, where

$$\begin{aligned} \gamma_0(\tau) &= -\frac{1}{2M(|\tau|)} + i\tau \qquad (-R \le \tau \le R), \\ \gamma_\pm(s) &= s \pm iR \qquad (-(2M(R))^{-1} \le s < 0). \end{aligned}$$

Although γ_0 may not be piecewise smooth, it can be approximated by smooth paths and (14) remains valid. Moreover,

$$\begin{gathered} \left\|(\gamma_0(\tau) + A)^{-1}\right\| \le 2M(|\tau|), \\ \left\|(\gamma_\pm(s) + A)^{-1}\right\| \le 2M(R). \end{gathered}$$

Now assume that $R > 1$. On $\gamma_\pm$, $|1 + z^2/R^2| \le C/R$, so we can estimate the norms of the integral in (14) over $\gamma_\pm$ by

$$C \int_0^{(2M(R))^{-1}} \frac{1}{R}\, M(R)\, \frac{e^{-ts}}{R}\, ds \le \frac{CM(R)}{R^2 t},$$

where C is a constant depending only on M. On γ_0, $1 + z^2/R^2$ and $1/z$ are bounded independently of R, and $|e^{tz}| \le e^{-t/2M(R)}$. The length of γ_0 is at most $2(M(0) + R)$. Hence, we can estimate the norm of the integral over γ_0 by

$$CM(R)(1 + R)e^{-t/2M(R)},$$

where again C depends only on M.

Thus we have the estimate

$$\begin{aligned} \|e^{-tA}A^{-1}\| &\le C\left(\frac{1}{R} + \frac{M(R)}{R^2 t} + M(R)(1 + R)e^{-t/2M(R)}\right) \\ &\le C\left(\frac{1}{R} + \frac{M(R)}{R^2 t} + \frac{(1 + M(R))^2(1 + R)^2}{R}\, e^{-t/2M(R)}\right). \end{aligned}$$

Here, C depends on $\widetilde{C}$ as well as M. Given $t > 4M_{\log}(1)$, choose $R = M_{\log}^{-1}(t/4) > 1$. Then

$$\begin{gathered} \frac{M(R)}{R^2 t} = \frac{1}{4R^2 \log\big((1 + M(R))(1 + R)\big)} \le \frac{C}{R}\,, \\ (1 + M(R))^2(1 + R)^2 e^{-t/2M(R)} = 1. \end{gathered}$$

So

$$\left\|e^{-tA}A^{-1}\right\| \le \frac{C}{R} = \frac{C}{M_{\log}^{-1}(t/4)}\,.$$

Hence,

$$m_1(t) \le \frac{C\|A(1 + A)^{-1}\|}{M_{\log}^{-1}(t/4)}\,.$$

□

4. LAPLACE TRANSFORMS

The proof given in Section 3 uses the semi-group property of e^{-tA}, or equivalently the properties of the resolvent of A, at just two points. One occurrence is where the resolvent is extended into the left half-plane by means of the Neumann series, and the other is the deduction of the case when $k > 1$ from the case when $k = 1$.

The result of Burq [5, Théorème 3] includes cut-off operators which destroy the semi-group property. In this section, we shall give a version of Theorem 5 for Laplace transforms of bounded functions, and decay results in the spirit of [5] (see Corollary 11 and Proposition 12).

We will have to *assume* that the Laplace transform extends to a suitable region with suitable bounds. Then the argument of Section 3 for $k = 1$ needs no significant changes. However further estimates are needed for the higher-order cases.

Let $f : [0,\infty) \to B$ be a bounded measurable function with Laplace transform $\hat{f}$ defined on the right half-plane in $\mathbb{C}$. Let $M : [0,\infty) \to (0,\infty)$ be a continuous increasing function. We assume throughout this section that $\hat{f}$ has a holomorphic extension to the region

$$\left\{ z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{M(|\operatorname{Im} z|)} \right\}$$

such that

$$\left\| \hat{f}(z) \right\| \le M(|\operatorname{Im} z|)$$

throughout that region. Let $F_0 = f$ and

$$F_k(t) = \frac{1}{(k-1)!} \int_0^t (t-s)^{k-1} f(s)\, ds, \quad t \ge 0, k \ge 1.$$

Note that

$$F_k(t) = \int_0^t F_{k-1}(s)\, ds, \qquad \hat{F}_k(z) = \frac{\hat{f}(z)}{z^k}.$$

Theorem 10. *Let $k \in \mathbb{N}^*$. Then there are constants C_k, T_k, depending only on k, $\|f\|_\infty$ and M, such that*

$$\forall t \ge T_k, \quad \left\| F_k(t) - \sum_{j=0}^{k-1} \frac{t^j}{j!} \hat{f}^{(k-1-j)}(0) \right\| \le \frac{C_k}{\left(M_{\log}^{-1}\left(\frac{t}{C_k}\right)\right)^k}. \tag{18}$$

Proof. The proof will be by induction on k. For $k = 1$, the proof is very similar to Theorem 5. Let

$$g_t(z) = \int_0^t e^{-zs} f(s)\, ds,$$

so that $F_1(t) = g_t(0) - \hat{f}(0)$. Let $R > 1$ and γ' be as in the proof of Theorem 5. Then

$$\begin{aligned} F_1(t) &= \frac{1}{2\pi i} \int_{\substack{|z|=R \\ \operatorname{Re} z > 0}} \left(1 + \frac{z^2}{R^2}\right) \left(g_t(z) - \hat{f}(z)\right) e^{zt} \frac{dz}{z} \\ &\quad + \frac{1}{2\pi i} \int_{\substack{|z|=R \\ \operatorname{Re} z < 0}} \left(1 + \frac{z^2}{R^2}\right) g_t(z) e^{zt} \frac{dz}{z} - \frac{1}{2\pi i} \int_{\gamma'} \left(1 + \frac{z^2}{R^2}\right) \hat{f}(z) e^{zt} \frac{dz}{z}. \end{aligned} \tag{19}$$

Now, similarly to Section 2 or as in [2, pp.276,277] (the estimation of $I_1(t)$ and $I_4(t)$) but ignoring terms involving η or ε, the norms of each of the first two terms can be estimated by $\|f\|_\infty / R$;

in fact, the first term can be estimated by $\sup_{s\geq t}\|f(s)\|/R$. The norm of the third term can be estimated exactly as in the proof of Theorem 5 and the rest of that proof is unchanged.

Now suppose that (18) holds for $k=1,\dots,n$. For simplicity of presentation, assume first that

$$\hat{f}^{(j)}(0)=0, \quad j=0,1,\dots,n. \tag{20}$$

Then $\hat{F}_n(z)=\hat{f}(z)/z^n$ extends holomorphically to the same region as $\hat{f}$, with the same bounds (up to a constant multiple). In particular, (19) holds with f replaced by F_n, F_1 replaced by F_{n+1}, and $g_t(z)=\int_0^t e^{-zs}F_n(s)\,ds$. For $t>T_n$, the first term in the resulting expression for $F_{n+1}(t)$ is estimated by

$$\frac{\sup_{s\geq t}\|F_n(s)\|}{R}\leq \frac{C_n}{RM_{\log}^{-1}\left(\frac{t}{C_n}\right)^n},$$

by the inductive hypothesis. By repeated integration by parts,

$$\begin{aligned} e^{zt}g_t(z) &= \int_0^t e^{z(t-s)}F_n(s)\,ds \\ &= -\frac{F_n(t)}{z}+\frac{1}{z}\int_0^t e^{z(t-s)}F_{n-1}(s)\,ds \\ &= -\sum_{j=1}^{n}\frac{F_j(t)}{z^{n+1-j}}+\frac{1}{z^n}\int_0^t e^{z(t-s)}f(s)\,ds. \end{aligned}$$

By the inductive hypothesis and a simple estimate for the final integral, when $z=Re^{i\theta}$ and $t>\max(T_1,\dots,T_n)$,

$$\left\|e^{zt}g_t(z)\right\|\leq \sum_{j=1}^{n}\frac{C_j}{R^{n+1-j}\left(M_{\log}^{-1}\left(\frac{t}{C_j}\right)\right)^j}+\frac{\|f\|_\infty}{R^{n+1}|\cos\theta|}.$$

As in [2, p.277] (the estimation of $I_4(t)$), it follows that the second term in the expression for $F_{n+1}(t)$ can be estimated by

$$\sum_{j=1}^{n}\frac{C_j}{R^{n+1-j}\left(M_{\log}^{-1}\left(\frac{t}{C_j}\right)\right)^j}+\frac{\|f\|_\infty}{R^{n+1}}.$$

Finally the third term can be estimated as for f but using the estimate $\|\hat{F}_n(z)\|\leq M(R)/R^n$ on $\gamma_\pm$. We conclude that

$$\begin{aligned} \|F_{n+1}(t)\|\leq{}& \frac{C_n}{RM_{\log}^{-1}\left(\frac{t}{C_n}\right)^n}+\sum_{j=1}^{n}\frac{C_j}{R^{n+1-j}\left(M_{\log}^{-1}\left(\frac{t}{C_j}\right)\right)^j}+\frac{\|f\|_\infty}{R^{n+1}} \\ &+C\frac{M(R)}{R^{n+2}t}+C\frac{(1+M(R))^2(1+R)^2}{R}\,e^{-t/2M(R)}, \end{aligned}$$

for some C. Let $C_{n+1}=\max(C_1,\dots,C_n,2(n+2))$. For sufficiently large t, we may choose $R=M_{\log}^{-1}(t/C_{n+1})$. After increasing C_{n+1} if necessary, we obtain (18) for $k=n+1$, under the assumptions (20) on f.

For a general f, let $h : [0,\infty) \to B$ be a measurable function of compact support such that

$$\hat{h}^{(j)}(0) = \hat{f}^{(j)}(0), \quad j = 0,1,\dots,n.$$

Applying the previous special case to $f - h$ gives (18). This completes the proof by induction. □

We deduce from Theorem 10 the following generalization of Theorem 5.

Corollary 11. *Let e^{-tA} be a bounded semi-group on a Banach space B and let T_1 and T_2 be bounded operators on B. Let $M : [0,\infty) \to (0,\infty)$ be continuous and increasing. Suppose that $T_1(z+A)^{-1}T_2$ has a holomorphic extension G to the region $\{z \in \mathbb{C} : \operatorname{Re} z > -1/M(|\operatorname{Im} z|)\}$ and suppose that $\|G(z)\| \le M(|\operatorname{Im} z|)$ throughout the region. Then for each $k \ge 1$ there exists C_k such that*

$$\left\| T_1 e^{-tA}(1+A)^{-k} T_2 \right\| \le \frac{C_k}{\left(M_{\log}^{-1}\left(\frac{t}{C_k}\right)\right)^k} \tag{21}$$

for all sufficiently large t.

Proof. For simplicity, we will give the proof only for $k = 1$. The general case follows in a similar way from Theorem 10, but one needs more complicated forms of the resolvent identity. We remark that if A is invertible, a version of (21), with $(1+A)^{-k}$ replaced by A^{-k}, can be obtained more directly from Theorem 10 by taking $f(t) = T_1 e^{-tA} T_2 x$ where $x \in X$ with $\|x\| = 1$.

For such x, let

$$\begin{aligned} f(t) &= \frac{d}{dt}\left(T_1 e^{-tA}(1+A)^{-1}T_2 x\right) \\ &= T_1 e^{-tA}\left((1+A)^{-1} - 1\right) T_2 x. \end{aligned}$$

For $\operatorname{Re} z > 0$ and $z \ne 1$, the resolvent identity gives

$$\begin{aligned} \hat{f}(z) &= T_1\left((z+A)^{-1}(1+A)^{-1} - (z+A)^{-1}\right) T_2 x \\ &= \frac{z}{1-z} G(z)x - \frac{1}{1-z} T_1(1+A)^{-1}T_2 x. \end{aligned}$$

This formula provides a holomorphic extension of $\hat{f}$ to the same region as G satisfying $\|\hat{f}(z)\| \le cM(|\operatorname{Im} z|)$ throughout the region, where c is independent of x. Moreover

$$\int_0^t f(s)\,ds = T_1 e^{-tA}(1+A)^{-1}T_2 x - T_1(1+A)^{-1}T_2 x = T_1 e^{-tA}(1+A)^{-1}T_2 x + \hat{f}(0).$$

Thus (21) follows from Theorem 10. □

Let us give a typical example satisfying the assumptions of Corollary 11. Consider the semi-group associated to the linear wave equation outside a compact obstacle of the Euclidian space $\mathbb{R}^N$, $N \ge 2$, with Dirichlet boundary conditions, and assume that T_1 and T_2 are multiplications by cut-off functions. When the dimension N is odd, the assumptions of Corollary 11 are satisfied with $M(R) = Ce^{CR}$ for some $C > 0$ (see [5]) and, under a non-trapping assumption on the obstacle, with a constant function M (see the classical works [22, 23] and references therein, as well as [31] for more general geometries).

If the dimension N is even, the holomorphic extension of the resolvent associated to the preceding example admits a singularity at $z = 0$, and the assumptions of Corollary 11 are no longer satisfied. The proofs of Theorem 10 and Corollary 11 adapt readily to this case, but it is difficult to write a nice general result for all k. We conclude this work by giving a statement in the case $k=1$, which includes the preceding semi-group for even dimension and generalizes [5, Théorème 3], in the case $k=1$, to non-logarithmic decay and general Banach spaces. The proof uses a straightforward adjustment of the statement and proof of the case $k=1$ in Theorem 10 and we omit it.

Consider as before a function $M:[0,\infty)\to(0,\infty)$ which is continuous and increasing. Let $\mu:\mathbb{R}\to[0,\infty)$ be a continuous function which admits only a finite number of zeros, and such that

$$\exists \tau_0>0, \quad |\tau|\geq\tau_0 \Longrightarrow \mu(\tau)=\frac{1}{M(|\tau|)}.$$

Proposition 12. *Let e^{-tA} be a bounded semi-group on a Banach space B and let T_1 and T_2 be bounded operators on B. Suppose that $T_1(z+A)^{-1}T_2$ has a holomorphic extension G to the region $\{z\in\mathbb{C}: \operatorname{Re} z>-\mu(|\operatorname{Im} z|)\}$ and suppose that $\|G(z)\|\leq M(|\operatorname{Im} z|)$ throughout the region. Then there exists $C>0$ such that*

$$\left\|T_1e^{-tA}(1+A)^{-1}T_2\right\|\leq\frac{C}{M_{\log}^{-1}\left(\frac{t}{C}\right)}+\int_{-\tau_0}^{+\tau_0}M(|\tau|)e^{-t\mu(\tau)}d\tau.$$

for all sufficiently large t.

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