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présentée par

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Formules d'intégration par parties pour les lois des ponts de Bessel, et EDP stochastiques associées

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Résumé

Dans cette thèse, nous obtenons des formules d'intégration par parties pour les lois de ponts de Bessel de dimension $\delta > 0$, étendant ainsi les formules précédemment obtenues par Zambotti dans le cas $\delta \geq 3$. Ceci nous permet d'identifier la structure de certaines EDP stochastiques (EDPS) ayant la loi d'un pont de Bessel de dimension $\delta \in (0, 3)$ pour mesure invariante, et qui étendent de manière naturelle les EDPS considérées précédemment par Zambotti dans le cas $\delta \geq 3$. Nous nommons ces équations EDPS de Bessel, et les écrivons à l'aide de temps locaux renormalisés. Dans les cas particuliers $\delta = 1, 2$, en utilisant la théorie des formes de Dirichlet, nous construisons une solution d'une version faible de ces EDPS. Nous prouvons également plusieurs résultats partiels qui suggèrent que les EDPS de Bessel de paramètre $\delta < 3$ possèdent certaines propriétés importantes: propriété de Feller forte, existence de temps locaux. Enfin, nous considérons différents modèles de pinning critiques dynamiques, discret et continu, et prouvons un résultat de tension. Nous conjecturons que ces modèles ont une même limite en loi décrite par l'EDPS de Bessel associée à $\delta = 1$.

Mots-clés: Formules d'intégration par parties, ponts de Bessel, EDP stochastiques avec réflexion, temps locaux, renormalisation, modèles de pinning.

Abstract

In this thesis, we derive integration by parts formulae (IbPF) for the laws of Bessel bridges of dimension $\delta > 0$, thus extending previous formulae obtained by Zambotti in the case $\delta \geq 3$. This allows us to identify the structure of some stochastic PDEs (SPDEs) having the law of a Bessel bridge of dimension $\delta < 3$ as invariant measure, and which extend in a natural way the family of SPDEs previously considered by Zambotti for $\delta \geq 3$. We call these equations Bessel SPDEs, and write them using renormalized local times. In the particular cases $\delta = 1, 2$, using Dirichlet forms, we construct a solution to a weak version of these SPDEs. We also provide several partial results suggesting that the SPDEs associated with $\delta < 3$ should have several important properties: strong Feller property, existence of local times. Finally, we consider dynamical critical wetting models, in the discrete and in the continuum, and prove a tightness result. We conjecture that these models have a common limit in law which should be described by the Bessel SPDE associated with $\delta = 1$.

Keywords: Integration by parts formulae, Bessel bridges, stochastic PDEs with reflection, local times, renormalization, pinning models.

Contents

Chapter 1

Introduction

In 1827, the Scottish botanist Robert Brown observed that some organic particles on a droplet he was looking at through his microscope were moving in a chaotic way along unusual and irregular paths. This movement, which was later baptized Brownian motion, was understood only much later, due to the work, at the turn of the 20th century, of Einstein and Smoluchowski, who argued that the chaotic trajectories one could see on large scales were due to the presence of highly-energetic atoms hitting the particles in all directions. They also predicted that the mean squared displacement of the particles should be proportional to $\sqrt{t}$, where t is the time parameter: one speaks of a *diffusive* behavior. Later, the French physicist Jean Perrin confirmed experimentally this prediction, thus establishing for the first time the existence of atoms.

Ever since these breakthroughs, Brownian motion has become one of the most famous probabilistic objects, which today finds applications in areas as diverse as physics, biology, finance and computer sciences. The study of its fine properties has given birth to a vast litterature: see e.g. [KS88], [RY13], [MY08], [RW00]. It also plays an important role in the theory of stochastic calculus due to Kiyoshi Itô, which was developped as a tool to define and solve stochastic differential equations (SDEs), in which one replaces standard integrals by stochastic integrals, and which allow to construct continuous-time Markov processes.

From a probabilistic point of view, one of the major interests of Brownian motion is its universal character: by Donsker's theorem, it arises as the limit in law of a large family of discrete random walks, when rescaled diffusively. This fact implies the invariance in law of Brownian motion under diffusive rescaling, i.e. if $(B_t)_{t\geq 0}$ is a standard Brownian motion, then

$$\forall \lambda > 0, \qquad B_{\lambda^2 t} \overset{(d)}{=} \lambda\, B_t, \tag{1.1}$$

where the equality is an identity in law of the entire processes. It also explains why Brownian motion appears when one looks at the large-scale behaviour of a small particle floating on water.

One can now wonder what trajectory such a particle will follow if one imposes an *obstacle*: for instance, in the one-dimensional situation, one may consider the origin 0 as an obstacle, and restrict our attention to real-valued stochastic processes which take only nonnegative values. Mathematically, the question can then be formulated as follows: what Markov process on $\mathbb{R}_+$ does there exist, that are furthermore invariant under diffusive scaling ? As is well-known, there exists a one-parameter family of nonnegative stochastic processes, called *Bessel processes*, which satisfy this property.

Bessel processes share many similarities with Brownian motion: in particular, they scale diffusively as in (1.1). However, in contrast with Brownian motion, they take nonnegative values. Moreover, they appear as the limit in law of various *non-negative* discrete random walks rescaled diffusively. In particular, they describe the large-scale fluctuations of several wetting models: see [DGZ05], [Fun05, Chapter 7], and [DO18], as well as Chapter 9 below.

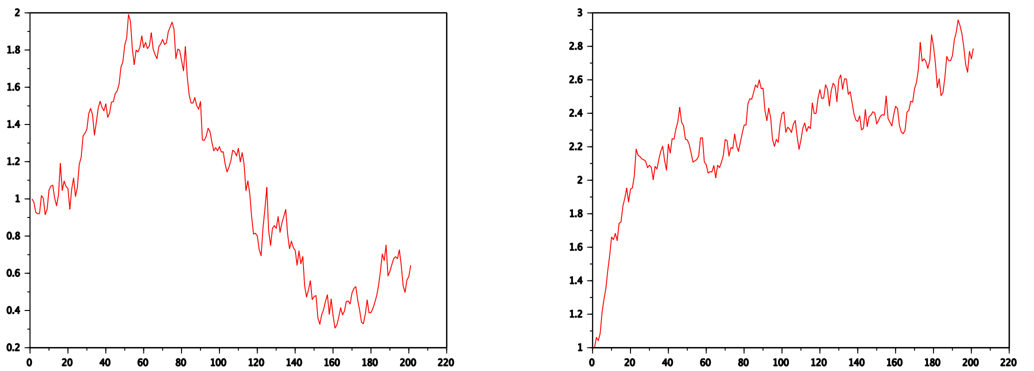

Figure 1.1: A 1-Bessel process (left), and a 3-Bessel process (right), arise as scaling limits of critical (resp. subcritical) wetting models

Beside this statistical mechanical interest, Bessel processes are omnipresent within the study of Brownian motion, for instance in the Ray-Knight theorems, see Chapter XI in [RY13]. They also appear when one considers the Euclidean norm of a Brownian motion in $\mathbb{R}^d$, for any integer $d \geq 1$, a fact that will be of interest for us: see Chapter 6 below. Let us also mention that, because of their numerous remarkable properties, Bessel processes find applications in various different fields. Thus, squared Bessel processes, which are obtained from Bessel processes by taking the square, arise as scaling limits of Galton-Watson processes with migration. Furthermore, Bessel processes are related to a family of random planar curves called Schramm-Loewner Evolution (SLE): in particular, the study of Bessel processes allows to derive non-trivial properties or formulae for SLE and, in turn, to better uderstand the scaling limit of several discrete models such as critical percolation in 2 dimensions, see [Kat16] for an introduction. Finally, Bessel processes also arise in mathematical finance because of their relation with geometric Brownian motion and CIR processes, see [GJY03].

The interest of Bessel processes is also theoretical. Indeed, their dynamics, which can be modeled by SDEs with a singular drift, are complicated and, in the soft-repulsion regime, involve a subtle renormalization phenomenon which, still today, seems to raise highly non-trivial questions: see Section 1.1 below. However, one can bypass these theoretical difficulties by exploiting the extraordinary power of stochastic calculus which allows to derive countless properties, see for instance [RY13]. Thus, studying the dynamics of Bessel processes is a first step to understanding solutions to similar, but more general equations, which are not amenable to stochastic calculus tools : this approach is followed, for instance, in Chapter 7 below.

In this thesis, rather than modeling the trajectory of particles, we aim at modeling the motion of random continuous *interfaces* evolving in time. Mathematically speaking, we aim at constructing Markov processes on an -infinite-dimensional - space of continuous functions (on the interval $[0, 1]$, say). A standard way of constructing such processes is by solving appropriate *stochastic partial differential equations* (SPDEs).

SPDEs were invented around fifty years ago as a natural function-valued analog of SDEs. We refer to the classical monographs [DPZ14] and [Wal86] for the foundations of this theory, which is by now a well-established field and is increasingly active and lively. SPDEs arise naturally in several contexts: for instance, they describe the large-scale fluctuations of various dynamical discrete interface models, see [Fun05].

In the past few years, SPDEs driven by a space-time white noise have received much attention, because they are naturally associated with *ultraviolet divergences* and *renormalization*, phenomena which are now mathematically well-understood thanks to the recent theories of regularity structures [Hai14, BHZ18] and paracontrolled distributions [GIP15]. These phenomena arise, for instance, when one attempts to generalize classical stochastic calculus results for semimartingales and SDEs in the framework of SPDEs driven by space-time white noise: see for instance [Zam06] and the more recent article (see [Bel18]), where several analogs for SPDEs of Itô-Tanaka type results are proposed. Despite these attempts, the theory of stochastic calculus does not yet nicely generalize to such SPDEs, because of the divergences created by the white noise: only partial results exist which are less powerful and effective than in the case of classical SDEs. A substitute to this theory is provided by the Fukushima stochastic calculus associated with Dirichlet forms [FOT10, MR92], but the formulae that it leads to are often less explicit than one would hope. Thus, the marvellous power of Itô calculus for the study of fine properties of semimartingales remains without proper analog in genuinely infinite-dimensional processes.

A fundamental example of SPDE is the additive stochastic heat equation (SHE) on $\mathbb{R}_+ \times [0,1]$ with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \xi \\ \\ u(t,0) = u(t,1) = 0. \end{cases} \tag{1.2}$$

Here, ξ is a space-time white noise, that is - loosely stated - a centered Gaussian process on $\mathbb{R}_+ \times [0,1]$ with covariance

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y), \quad t,s \geq 0, x,y \in [0,1]. \tag{1.3}$$

This SPDE is linear and can be solved explicitly. Its solution $(u_t)_{t\geq 0}$ is a Gaussian as well as a Markov process on $C([0,1])$, the space of real-valued, continuous functions on $[0,1]$. Moreover, (1.2) admits a unique invariant probability measure given by the law of a standard Brownian bridge on $[0,1]$, that is the law of a Brownian motion $(B_t)_{t\geq 0}$ on $[0,1]$ conditioned on the event $\{B_1 = 0\}$. An important property of (1.2) is its invariance under a $1-2-4$ scaling, which generalizes the diffusive scale invariance (1.1) of Brownian motion. Indeed, by the structure (1.3) for the covariance of ξ, one has

$$\forall \lambda > 0, \qquad \xi(\lambda t, \lambda^2 x) \overset{(d)}{=} \lambda^{-3/2}\xi(t,x).$$

Hence, if u is a solution to (1.2) then, for all $\lambda > 0$

$$u(\lambda^4 t, \lambda^2 x) \overset{(d)}{=} \lambda\, \tilde{u}(t,x), \tag{1.4}$$

where $\tilde{u}$ is a solution to the stochastic heat equation on $[0, \lambda^{-2}]$. Actually, the equation (1.2) has a universal feature: it describes the large-scale fluctuations of various models of discrete, symmetric (or weakly asymmetric) interfaces, see for instance [EL15] for an example. Finally, note that (1.2) is invariant in law upon changing u with $-u$.

In this thesis, we are interested in random dynamical interfaces evolving above an obstacle. Namely, we consider SPDEs which have the same scale invariance (1.4) as the stochastic heat equation (1.2), but which furthermore have *nonnegative* solutions. As it will turn out, this will lead us to consider a one-parameter family of equations which seem to be a natural analog of Bessel processes in the context of SPDEs driven by a space-time white noise. As suggested above, the standard approach to Bessel processes - relying on stochastic calculus - breaks down for these Bessel SPDEs, and we will have to apply a different method, with necessarily weaker results, at least in comparison with the finite-dimensional situation. We will mainly rely on integration by parts formulae on path spaces and on Dirichlet forms methods. The former will include distributional terms - rather than σ-finite measures - as in the theory of white noise calculus [HKPS93], a feature which will raise considerable difficulties, and which is at the same time the source of subtle and intriguing renormalization phenomena.

The processes that we consider have interesting path properties, as it is the case for Bessel processes, but with the enhanced richness of infinite-dimensional objects, see e.g. [Zam17] for a recent account. In particular, the hitting properties of these processes is more sophisticated and fascinating. We hope that this work will further motivate the study of infinite-dimensional stochastic calculus, which is still in its infancy.

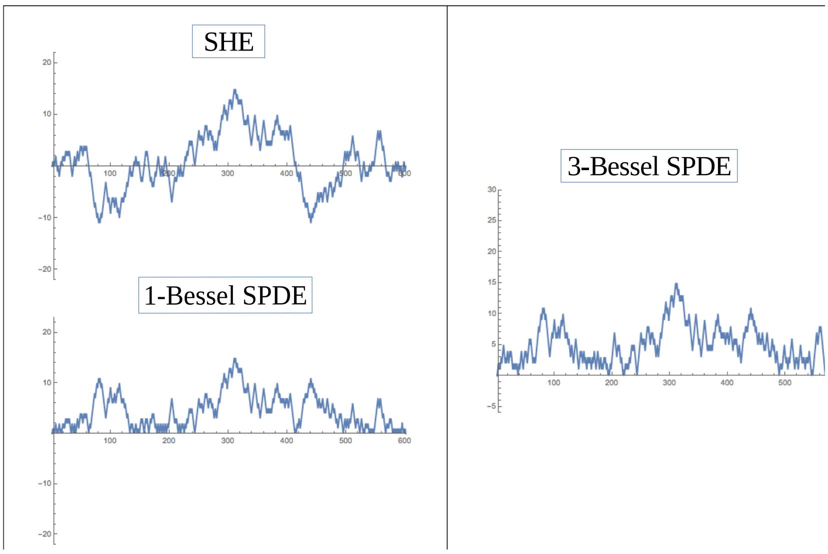

Figure 1.2: Solutions of the stochastic heat equation, and of Bessel SPDEs with respective parameter $\delta = 1$ and $\delta = 3$, at a given fixed time

1.1 From Bessel SDEs to Bessel SPDEs

A squared Bessel process of dimension $\delta \geq 0$ is defined as a continuous process $(X_t)_{t\geq 0}$ solving the SDE

$$X_t = X_0 + \int_0^t 2\sqrt{|X_s|}\,\mathrm{d}B_s + \delta\, t, \quad t \geq 0, \qquad (\delta \geq 0) \tag{1.5}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. The existence of a unique solution $(X_t)_{t\geq 0}$ to the SDE (1.5) follows from the classical Yamada-Watanabe theorem [RY13, Theorem IX.3.5]. By a comparison theorem [RY13, Theorem IX.3.7], it further holds that, almost-surely, $X_t \geq 0$ for all $t \geq 0$.

As first noted by Shiga and Watanabe, squared Bessel processes enjoy a remarkable additivity property (see [SW73] and (2.8) below): this property is essential, since it allows numerous explicit computations on the laws of these processes, a feature on which we will heavily rely.

On the other hand, a *Bessel process* is given by $\rho_t := \sqrt{X_t}$, $t \geq 0$, where X is a solution to (1.5) above. Using Itô's lemma, one can in turn write ρ as a solution to another SDE: that SDE turns out however to be more involved. Thus, for $\delta > 1$, by the Itô formula, ρ is solution to the SDE with a singular drift

$$\rho_t = \rho_0 + \frac{\delta - 1}{2}\int_0^t \frac{1}{\rho_s}\,\mathrm{d}s + B_t, \quad t \geq 0. \qquad (\delta > 1) \tag{1.6}$$

This equation satisfies pathwise uniqueness and existence of strong solutions, since the drift is given by the function $x \mapsto \frac{\delta-1}{2x}$ which is monotone decreasing on $(0,\infty)$. For $\delta = 1$, ρ is the solution to

$$\rho_t = \rho_0 + L_t + B_t, \qquad t \geq 0, \qquad (\delta = 1)$$

where $(L_t)_{t\geq 0}$ is continuous and monotone non-decreasing with $L_0 = 0$, and

$$\rho \geq 0, \qquad \int_0^\infty \rho_s \,\mathrm{d}L_s = 0. \tag{1.7}$$

In other words ρ is a reflecting Brownian motion, and the above equation has a unique solution by the Skorokhod Lemma [RY13, Lemma VI.2.1].

For $\delta \in (0,1)$, the situation is substantially more difficult and it turns out that the relation (1.6) is not valid anymore in that regime. Indeed, in that case, with positive probability, $\int_0^t \frac{1}{\rho_s}\,\mathrm{d}s = \infty$ so the equation for ρ can be formally written

$$\rho_t = \rho_0 + \frac{\delta - 1}{2}\left(\int_0^t \frac{1}{\rho_s}\,\mathrm{d}s - \infty\right) + B_t.$$

This is an instance of renormalization which is reminiscent of those happening in singular SPDEs: those can be treated by the recent theories of regularity structures or paracontrolled distributions. However, the type of renormalization entering here into play is quite different from the schemes generally used in these theories, since it applies to local times of the solution rather than the solution itself. One can show, see e.g. [Zam17, Proposition 3.12], that ρ admits *diffusion local times*, namely a continuous process $(\ell_t^a)_{t\geq 0, a\geq 0}$ such that

$$\int_0^t \varphi(\rho_s)\,\mathrm{d}s = \int_0^\infty \varphi(a)\,\ell_t^a\,a^{\delta-1}\,\mathrm{d}a, \tag{1.8}$$

for all Borel $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, and that ρ satisfies

$$\rho_t = \rho_0 + \frac{\delta-1}{2}\int_0^\infty \frac{\ell_t^a - \ell_t^0}{a}\,a^{\delta-1}\,\mathrm{d}a + B_t, \quad t\geq 0, \qquad (0<\delta<1). \tag{1.9}$$

Note that by the occupation time formula (1.8) we have

$$\int_0^\infty \frac{\ell_t^a - \ell_t^0}{a}\,a^{\delta-1}\,\mathrm{d}a = \lim_{\varepsilon\downarrow 0}\int_\varepsilon^\infty \frac{\ell_t^a - \ell_t^0}{a}\,a^{\delta-1}\,\mathrm{d}a =$$
$$= \lim_{\varepsilon\downarrow 0}\left(\int_0^t \mathbb{1}_{(X_s\geq\varepsilon)}\frac{1}{X_s}\,\mathrm{d}s - \ell_t^0\int_\varepsilon^\infty a^{\delta-2}\,\mathrm{d}a\right)$$

and in the latter expression both terms diverge as $\varepsilon\downarrow 0$ with positive probability. However, the difference converges. Indeed, one can prove that $|\ell_t^a - \ell_t^0| \lesssim a^{1-\frac{\delta}{2}-\kappa}$ with $\kappa > 0$, which ensures the integral in the right-hand side of (1.9) to be convergent. This is why we speak of *renormalized local times*.

The formula (1.9) is not really an SDE, because of the form of the drift, which involves local times of the solution ρ rather than ρ itself. However, it is tempting to try to characterize ρ as the unique process satisfying (1.9) as well as the occupation times formula (1.8). Unfortunately, there does not yet seem to exist a theory ensuring pathwise well-posedness of such systems of equations: several obstacles arise when trying to build such a theory, in particular the notion of local times, which for the moment seems to be best provided by stochastic calculus and to lie outside the scope of pathwise theories such as rough paths or regularity structures.

Thus, even in this simple SDE context, Bessel processes satisfy highly non-trivial dynamics involving, in the soft-repulsion regime $\delta < 1$, a renormalization phenomenon. The renormalzation entering here into play is of order 0, since one has to substract, to the local time term ℓ_t^a, the value of the local time at 0, ℓ_t^0. We will encounter similar, but higher-order renormalizations in the case of Bessel SPDEs, for which one will have to substract to the local time its Taylor polynomial of order 0 or 2, according to the strength of the repulsion.

The SPDEs in question have first been proposed by Lorenzo Zambotti in a series of papers [Zam01, Zam02, Zam03, Zam04b]. In these articles, Zambotti considered parabolic SPDEs which admit nonnegative solutions, and have properties analogous to those of Bessel processes. For a parameter $\delta > 3$ the equation, that we call *δ-Bessel SPDE*, is given by

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\kappa(\delta)}{2\,u^3} + \xi \tag{1.10}$$

where $u \geq 0$ is continuous, ξ is a space-time white noise on $\mathbb{R}_+ \times [0,1]$, and

$$\kappa(\delta) := \frac{(\delta-3)(\delta-1)}{4} > 0. \tag{1.11}$$

Hence, (1.10) is a stochastic heat equation with repulsion, the singular drift $\frac{\kappa(\delta)}{2\,u^3}$ acting as a repulsion away from 0. As $\delta \downarrow 3$, the solution to (1.10) converges to the solution of the Nualart-Pardoux equation [NP92], namely the random obstacle problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \eta + \xi \\ \\ u \geq 0, \; d\eta \geq 0, \; \int_{\mathbb{R}_+\times[0,1]} u \, \mathrm{d}\eta = 0. \end{cases} \qquad (\delta = 3) \tag{1.12}$$

In that equation, the unknown is a couple (u, η), where u is a continuous, nonnegative function on $\mathbb{R}_+ \times [0,1]$, and η is a nonnegative Radon measure on $[0,\infty) \times (0,1)$ which is supported in the set of all space-time points $(t,x) \in \mathbb{R}_+ \times [0,1]$ such that $u(t,x) = 0$. As shown in [NP92], for any initial condition $u_0 \in C([0,1], \mathbb{R}_+)$, and upon imposing suitable boundary conditions, there exists a unique such couple (u,η) satisfying, in the weak sense, the equality of the first line in (1.12).

If one imposes Dirichlet boundary conditions $u(t,0) = u(t,1) = a$ for some $a \geq 0$, then for all $\delta > 3$, the unique invariant measure of (1.10) is given by the law of a Bessel bridge of dimension δ from a to a on $[0,1]$. In other words, the invariant measure has the law of a solution ρ on $[0,1]$ to (1.6) with $\delta > 3$ started from a and conditioned on the event $\{\rho_1 = a\}$. On the other hand, the unique invariant measure of (1.12) is given by the law of a Bessel bridge of dimension 3, which in the case $a = 0$ coincides with the law of a normalized Brownian excursion, see Chapter 3 in [Zam17]. Actually, these SPDEs are of gradient type, noting that for all $\delta \geq 3$ and all $a > 0$, the law, on $L^2([0,1])$, of a δ-dimensional Bessel bridge from a to a on $[0,1]$ is absolutely continuous with respect to the law of a Brownian bridge from a to a on $[0,1]$, with a Radon-Nikodym derivative given by

$$\frac{1}{Z_{\delta,a}} \exp\left(-V_\delta(\zeta)\right), \qquad \zeta \in L^2([0,1]), \tag{1.13}$$

where $Z_{\delta,a}$ is a normalisation constant, and the function $V_\delta : L^2([0,1]) \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$V_\delta(\zeta) := \frac{\kappa(\delta)}{2} \int_0^1 \frac{\mathrm{d}s}{\zeta_s^2} + \mathbf{1}_{\zeta \notin K}\ \infty, \quad \zeta \in L^2([0,1]),$$

where $K := \{\zeta \in L^2([0,1]),\ \zeta \geq 0 \text{ a.e.}\}$. Then, formally, the SPDEs (1.10) and (1.12) above all take the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\nabla V_\delta(u) + \xi.$$

The essential feature allowing to solve these SPDEs is the fact that V_δ is convex on $L^2([0,1])$. In particular, the laws of Bessel bridges of dimension $\delta \geq 3$ are *log-concave*. As a consequence, classical SPDE tools apply to solve and study equations (1.12) and (1.10), see [Zam17]. See also [ASZ09] which proves general results for gradient SPDEs associated with log-concave probability measures.

We stress the analogy between the above SPDEs and the SDEs satisfies by Bessel processes of dimension $\delta \geq 1$. Thus, the equation (1.10) for $\delta > 3$ is an SPDE with additive space-time white noise and a singular, repulsive drift: it is thus an SPDE analog of (1.6) for $\delta > 1$. On the other hand, (1.12) can be seen as an infinite-dimensional Skorohod problem: it is thus an SPDE analog of (1.7). These analogies are further justified in terms of scale invariance. Indeed, recalling the scaling property of space-time white noise

$$\forall \lambda > 0, \qquad \xi(\lambda t, \lambda^2 x) \overset{(d)}{=} \lambda^{-3/2} \xi(t,x),$$

it follows that (1.10) and (1.12), like the stochastic heat equation (1.2), are invariant under a $1-2-4$ scaling, that is

$$\forall \lambda > 0, \qquad u(\lambda^4 t, \lambda^2 x) \overset{(d)}{=} \lambda\, \tilde{u}(t,x),$$

where $\tilde{u}$ is solution to the same equation, with the space interval changed from $[0,1]$ to $[0,\lambda^{-2}]$. This generalizes the scale invariance of Bessel processes.

Finally, the above SPDEs arise as scaling limits of various discrete random interface models. Thus, (1.12) describes the fluctuations of an effective $(1+1)$ interface model near a wall [FO01, Fun05] and it also appears as the scaling limit of several weakly asymmetric interface models, see [EL15]. On the other hand, (1.10) describes the fluctuations of an effective $(1+1)$ interface model with repulsion, see [Zam04a].

1.2 The main problem: what SPDEs for $\delta < 3$?

It has been an open problem for over 15 years to complete the above picture. Namely, what is an SPDE whose invariant measure is the law of a Bessel bridge of dimension $\delta < 3$? Is it an SPDE analog of (1.9)?

The difficulty of this problem lies in the fact that for $\delta < 3$, the laws of δ-dimensional Bessel bridges are no longer log-concave. Indeed, for $\delta \in [2, 3)$, these laws still admit the density (1.13) with respect to the law of a Brownian bridge, but in that regime $\kappa(\delta) < 0$, so the potential V_δ is extremely non-covex. On the other hand, for $\delta \in (0, 2)$, such an absolute continuity relation no longer holds, and log-concavity is also lost. As a consequence, while equations (1.10) and (1.12) satisfy nice properties (pathwise uniqueness, continuity with respect to initial data, the strong Feller property) due to the *dissipativity* of the drift, such features may a priori break down in the regime $\delta < 3$. Actually, in that regime, it is not even clear what SPDE to consider, and what is the correct notion of solution. This is similar to the situation encountered with Bessel processes in the regime $\delta < 1$.

This problem is particularly interesting for $\delta = 1$, which corresponds to the law of the modulus of a Brownian bridge as invariant measure. Indeed, the latter arises as the scaling limit of critical wetting models, see [DGZ05]. Several works have constructed, in different ways, reversible dynamics associated with these wetting models, see [Fun05, Chapter 15.2], as well as [FGV16, GV18] for a construction using Dirichlet form techniques, and [DO18] for a straightforward SDE construction made possible by considering a pinning model with a shrinking strip. These various dynamical pinning models are believed to have a scaling limit, which would be an infinite-dimensional diffusion having the law of a reflecting Brownian motion, or the modulus of a Brownian bridge, as reversible measure. What kind of SPDE that limit should satisfy has however remained a very open question so far.

As noted above, in the case of Bessel processes, one can bypass most of the difficulties by considering their square. It would be natural to try to carry over this trick to the case of Bessel SPDEs: considering for instance a solution u of (1.10), in view of the form of the drift u^-3, it would be natural to set $v := u^4$, and hope that v satisfies a nicer SPDE than u allowing a natural extension to $\delta < 3$. However, this does not seem to simplify the problem because one obtains rather frightening equations of the form

$$\frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + 2\, \kappa(\delta) - \frac{3}{8} \frac{1}{\sqrt{v}} : \left(\frac{\partial v}{\partial x} \right)^2 : + 4 v^{\frac{3}{4}}\, \xi$$

where the : : notation denotes a KPZ-type renormalization. Even the theory of regularity structures does not cover this kind of equations, due to the non-Lipschitz character of the coefficients. One could hope that a Yamada-Watanabe result could be proved for this class of equations, noting that the exponent $\frac{3}{4}$ in the noise term is known to be critical for pathwise uniqueness of parabolic SPDEs (without the KPZ-type term), see [MP11, MMP14]. This approach is, at present, completely out of reach.

1.3 Integration by parts formulae for the laws of Bessel bridges

As explained above, extending the family of SPDEs given by (1.10) and (1.12) to the case $\delta < 3$ is a very hard problem. In this thesis, we provide a partial solution by extending, instead, the corresponding *integration by parts fomulae.*

Integration by parts plays a fundamental role in analysis, and most notably in stochastic analysis. For instance, it lies at the core of Malliavin Calculus and the theory of Dirichlet forms, see e.g. [Nua09, FOT10, MR92].

While it is relatively easy in finite dimension, where the standard rules of calculus apply, obtaining integration by parts formulae (IbPF for short) for probability measures on infinite-dimensional spaces can be a difficult task, due to the lack of absolute continuity relations in that context. One of the most famous examples is the IbPF associated with the law of a Brownian motion (or a Brownian bridge) B on the interval $[0,1]$, which reads

$$E\left[\partial_h \Phi(B)\right] = -E\left[\langle h'', B\rangle\, \Phi(B)\right],$$

valid for all bounded, Fréchet differentiable $\Phi : L^2(0,1) \to \mathbb{R}$ with bounded differential and all $h \in C_c^2(0,1)$, and where $\langle \cdot, \cdot \rangle$ denotes the canonical scalar product in $L^2(0,1)$. This formula follows for instance from the quasi-invariance property of the Wiener measure on $[0,1]$ along the Cameron-Martin space.

In [Zam02] and [Zam03], Zambotti derived integration by parts formulae for the laws of Bessel bridges of dimension $\delta \geq 3$. The techniques used there rely essentially on the representation of the law of a δ-dimensional Bessel bridge for $\delta \geq 3$ as as a Gibbs measure with respect to the law of a Brownian bridge with density given by (1.13) (cf. Prop 3.23 in [Zam17]). Then, for all $a \geq 0$, denoting by $P^\delta_{a,a}$ the law, on the space of continuous real-valued functions on $[0,1]$, of a δ-dimensional Bessel bridge from a to a over the interval $[0,1]$, and writing $E^\delta_{a,a}$ for the corresponding expectation operator, these IbPF read as follows. For all $\delta > 3$,

$$E^\delta_{a,a}\left[\partial_h \Phi(X)\right] + E^\delta_{a,a}\left[\langle h'', X\rangle\, \Phi(X)\right] = -\kappa(\delta)\, E^\delta_{a,a}\left[\langle h, X^{-3}\rangle\, \Phi(X)\right] \tag{1.14}$$

where $\kappa(\delta)$ is defined in (1.11), and

$$\begin{aligned} &E^3_{a,a}\left[\partial_h \Phi(X)\right] + E^3_{a,a}\left[\langle h'', X\rangle\, \Phi(X)\right] = \\ &= -\int_0^1 \mathrm{d}r\, h_r \gamma(r,a)\, E^3_{a,a}\left[\Phi(X) \,|\, X_r = 0\right], \end{aligned} \tag{1.15}$$

where Φ and h are as above, and where

$$\gamma(r,a) := \frac{1}{\sqrt{2\pi r^3(1-r)^3}} \left(\mathbf{1}_{a=0} + \mathbf{1}_{a>0} \frac{2a^2 \exp\left(-\frac{a^2}{2r(1-r)}\right)}{1-\exp(-2a^2)} \right), \qquad r \in (0,1),$$

see (6.7) and (6.24) in [Zam17]. Note that $\kappa(\delta) > 0$ for $\delta > 3$, and κ vanishes at $\delta = 3$. At the same time, the quantity $\langle |h|, X^{-3} \rangle$ is integrable with respect to $P^\delta_{a,a}$ for $\delta > 3$, but is non-integrable with respect to $P^3_{a,a}$ for h that is not identically 0. Thus, the right-hand side of (1.14) is, in the limit $\delta \searrow 3$, an indeterminate form $0 \times \infty$ which happens to converge to the non-trivial quantity in the right-hand side of (1.15). Formula (1.15) also possesses a geometric-measure theory interpretation as a Gauss-Green formula in an infinite-dimensional space, the term in the right-hand side corresponding to a boundary term (see Chapter 6.1.2 in [Zam17]).

What can we say for Bessel bridges of dimension $\delta < 3$? Recall that, in such a regime, the absolute continuity relations mentioned above, or the requested convexity property, fall apart, and consequently the techniques used in [Zam03] break down. Hence, the problem of finding IbPF for the laws of Bessel bridges of dimension $\delta < 3$ had remained open until recently, except for the value $\delta = 1$ corresponding to the law of the modulus of a Brownian bridge, for which some IbPF have been obtained, see [Zam05] for the case of the reflected Brownian motion and [GV16] for the case of a genuine bridge. In these articles, the proofs relied on the underlying Gaussian structure and on explicit computations using the Cameron-Martin formula. In particular, the formulae are written in terms of Hida's renormalization of the squared derivative of the underlying Gaussian process. Therefore, they do not generalize in a natural way the IbPF for $\delta \geq 3$ obtained previously, and do not seem to help with a construction of the corresponding SPDE: see Remark 3.1.6 below for a discussion. Moreover, such constructions do not generalize to generic values of δ, for which a similar Gaussian representation does not hold. However, they are related to Itô-Tanaka formulae for solutions to stochastic heat equations: see Chapter 6 below.

1.4 The IbPF for the laws of Bessel bridges of dimension $\delta < 3$

Here and below, let $C([0,1]) := C([0,1],\mathbb{R})$ be the space of continuous real-valued functions on $[0,1]$. In this thesis, we obtain IbPF for the laws $P^\delta_{a,a'}$ of Bessel bridges of dimension δ from a to a' over $[0,1]$, for any $\delta > 0$ and any $a, a' \geq 0$. Our formulae hold for a large class of functionals $\Phi : C([0,1]) \to \mathbb{R}$. More precisely, we consider linear combinations of functionals of the form

$$\Phi(\zeta) = \exp(-\langle m, \zeta^2 \rangle), \quad \zeta \in C([0,1]), \tag{1.16}$$

with m a finite Borel measure on $[0,1]$, and where $\langle m, \zeta^2 \rangle := \int_0^1 \zeta_t^2 \, m(\mathrm{d}t)$. Our method is based on deriving semi-explicit expressions for quantities of the form

$$E^\delta_{a,a'} \left[\Phi(X) \right] \qquad \text{and} \qquad E^\delta_{a,a'} \left[\Phi(X) \,|\, X_r = b \right], \quad b \geq 0, \, r \in (0,1),$$

where $E^\delta_{a,a'}$ denotes the expectation operator associated with $P^\delta_{a,a'}$, using solutions to some second-order differential equations, and exploiting the nice computations done in Chapter XI of [RY13]. The fundamental property enabling these computations is the additivity property of squared Bessel processes, which in particular implies that both quantities above can be computed explicitly, see (2.24) and (2.25) below. In particular, for $P^\delta := P^\delta_{0,0}$, these quantities factorize in a very specific way and, for functionals as above, all the IbPF for P^δ, $\delta \geq 3$ are just multiples of a single differential relation which does not depend on δ (see Lemma 3.2.1 below), the dependence in δ entering only through the multiplying constant which involves some Γ values. When $\delta \geq 3$, expressing these Γ values as integrals, and performing a change of variable, we retrieve the formulae already obtained in [Zam02] and [Zam03]. On the other hand, when $\delta < 3$, one of the Γ values appearing is negative, so we cannot express it using the usual integral formula, but must rather use *renormalized* integrals.

As a result, in the case of homogeneous boundary values $a = a' = 0$, when $\delta \in (1,3)$, the IbPF can be written

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X \rangle \, \Phi(X)) = \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \Big[\Sigma^\delta_r \left(\Phi(X) \,|\, b \right) - \Sigma^\delta_r \left(\Phi(X) \,|\, 0 \right) \Big] \, \mathrm{d}b \, \mathrm{d}r, \end{aligned} \tag{1.17}$$

where, for all $b \geq 0$, $\Sigma^\delta_r \left(\mathrm{d}X \,|\, b \right)$ is a measure on $C([0,1])$ proportional to the law of the Bessel bridge conditioned to hit b at r, see (2.21). Thus, the left-hand side is the same as for (1.14) and (1.15), but the right-hand side now contains Taylor remainders at order 0 of the functions $b \mapsto \Sigma^\delta_r \left(\Phi(X) \,|\, b \right)$. When $\delta \in (0,1)$, this renormalization phenomenon becomes even more acute. Indeed, in that case, the IbPF are similar to (1.17), but the right-hand side is replaced by

$$- \kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \left[\varphi(b) - \varphi(0) - \frac{b^2}{2} \varphi''(0) \right] \mathrm{d}b \, \mathrm{d}r, \tag{1.18}$$

where $\varphi(b) := \Sigma^\delta_r \left(\Phi(X) \,|\, b \right)$, and where we see Taylor remainders at order 2 appearing. An important remark is that the terms of order 1 vanish

$$\varphi'(0) = \frac{\mathrm{d}}{\mathrm{d}b} \Sigma^\delta_r \left(\Phi(X) \,|\, b \right) \Big|_{b=0} = 0, \quad r \in (0,1), \tag{1.19}$$

so we do not see them in the above Taylor remainders. Finally, in the critical case $\delta = 1$, we obtain the fomula

$$E^1(\partial_h \Phi(X)) + E^1(\langle h'', X\rangle\, \Phi(X)) = \frac{1}{4}\int_0^1 h_r \,\frac{\mathrm{d}^2}{\mathrm{d}b^2}\, \Sigma_r^1(\Phi(X)\,|\,b)\bigg|_{b=0} \mathrm{d}r. \qquad (1.20)$$

Above we have stated the IbPF for the case $a = a' = 0$ for convenience, but they actually generalize very naturally to any boundary values $a, a' \geq 0$: see Theorem 3.1.1 below.

One important, expected feature is the transition that occurs at the critical values $\delta = 3$ and $\delta = 1$. Another important but less expected feature is the absence of transition at $\delta = 2$, as well as the related remarkable fact that the functions $b \mapsto \Sigma_r^\delta(\Phi(X)\,|\,b)$ are, for all $r \in (0,1)$, smooth functions in b^2, so that all their odd-order derivatives vanish at 0. This is the reason why there only ever appear derivatives of even order in our formulae. An objection to this observation might be that the class of functionals (1.16) is too restrictive. However, in Section 3.4 below, we prove that the same IbPF hold for another class of test functionals, namely the space generated by functionals of the form $X \mapsto \int_0^1 \zeta_r X_r \,\mathrm{d}r$, where $\zeta \in C([0,1])$. For these functionals, the proof of the IbPF is, however, different, and relies on properties of hypergeometric functions. Moreover, in Chapter 5 below, we show that, in the case $a = a' = 0$, the IbPF obtained here still hold for a class of very general functionals. In particular, for $1 < \delta < 3$, the IbPF (1.17) holds for any $\Phi : L^2(0,1) \to \mathbb{R}$ which is C^1, bounded, with bounded Fréchet differential, see Theorem 5.1.3. Furthermore, we establish the vanishing of the first-order derivatives at $b = 0$ of the function $b \mapsto \Sigma_r^\delta(\Phi(X)\,|\,b)$, for any such Φ, which confirms the absence of transition at $\delta = 2$ observed in Theorem 3.1.1 : see Prop 5.4.1. We also extend the IbPF (1.20) for $\delta = 1$ and the IbPF (1.18) for $\delta \in (0,1)$, although, for these small values of δ, we have to assume some more regularity on the test functionals : see Theorems 5.1.6 and 5.1.8.

Note that all the IbPF above can be written in a unified way. Indeed, let us introduce, for all $\alpha \in \mathbb{R}$, the following distribution on $[0,\infty)$:

- if $\alpha = -k$ with $k \in \mathbb{N}$, then

$$\langle \mu_\alpha, \varphi\rangle := (-1)^k \varphi^{(k)}(0), \qquad \forall\, \varphi \in S([0,\infty))$$

- otherwise,

$$\langle \mu_\alpha, \varphi\rangle := \int_0^{+\infty} \left(\varphi(b) - \sum_{0\leq j\leq -\alpha} \frac{b^j}{j!}\,\varphi^{(j)}(0)\right) \frac{b^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}b, \quad \forall\, \varphi \in S([0,\infty)),$$

where $S([0,\infty))$ is the space of rapidly decreasing functions on $[0,\infty)$, see Definition 2.1.2 below. Note that, for all $\alpha \in \mathbb{R}$, μ_α coincides with the distribution $\frac{x_+^{\alpha-1}}{\Gamma(\alpha)}$ considered in Section 3.5 of [GS77]. In particular, for all $\varphi \in S([0,\infty))$, the map $\alpha \mapsto \langle \mu_\alpha, \varphi \rangle$ is analytic. Moreover, the last term in all the IbPF above can be written

$$- \frac{\Gamma(\delta)}{4(\delta-2)} \int_0^1 \langle \mu_{\delta-3}, \Sigma_r^\delta(\Phi(X) \,|\, \cdot) \rangle. \tag{1.21}$$

Note that, by (1.19) above, we have $\langle \mu_{-1}, \Sigma_r^1(\Phi(X) \,|\, \cdot) \rangle = 0$, so that the singularity at $\delta = 2$ of the denominator in (1.21) is compensated by the vanishing of $\langle \mu_{\delta-3}, \Sigma_r^\delta(\Phi(X) \,|\, \cdot) \rangle$ at $\delta = 2$. The formula (1.21) bears out the idea that the new IbPF for Bessel bridges of dimension $\delta < 3$ are given by the unique analytic continuation of those for $\delta \geq 3$, at least for suitable test functionals Φ as in (1.16). Indeed, every term in the IbPF depends in an analytic way on δ, see (3.11) below.

1.5 The structure of the SPDEs for $\delta < 3$

There is a precise relation - given by the Revuz correspondence (see Section 4.1 below) - between the IbPF for the laws of Bessel bridges and the corresponding SPDEs: informally speaking, the IbPF is an infinitesimal version of the SPDE. Thus for $\delta > 3$, the IbPF (1.14) corresponds to the SPDE (1.10): the term $E^3 \left[\langle h'', X \rangle \, \Phi(X) \right]$ in (1.14) encodes the Laplacian term $\frac{1}{2} \frac{\partial^2 u}{\partial x^2}$ in (1.10), while the term $\kappa(\delta) \, E^\delta \left[\langle h, X^{-3} \rangle \, \Phi(X) \right]$ encodes the non-linear drift $\frac{\kappa(\delta)}{2u^3}$ in (1.10). A similar correspondence holds for $\delta = 3$, where the boundary measure appearing in (1.15) encodes the reflection measure η in (1.12).

The IbPF (1.17), (1.18) and (1.20) above suggest that the gradient dynamics associated with the laws of Bessel bridges of dimension $\delta < 3$ should have the same structure as the SPDEs (1.10) and (1.12), but with a drift that now contains *renormalized local times* of the solution u. More precisely, for all $x \in (0,1)$, we conjecture the existence of a process $(\ell^b_{t,x})_{b \geq 0}$ satisfying

$$\int_0^t \varphi(u(s,x)) \, \mathrm{d}s = \int_0^\infty \varphi(b) \, \ell^b_{t,x} \, b^{\delta-1} \, \mathrm{d}b, \tag{1.22}$$

for all Borel $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. Then, for all $r \in (0,1)$ and $b \geq 0$ the measure $\Sigma_r^\delta(\cdot \,|\, b)$ defined in (2.21) below should be the Revuz measure associated with the additive functional $(\ell^b_{t,r})_{t \geq 0}$: see the Conjecture 4.1.1 below. Moreover, the equality (1.19) above suggests that the local times of u should have a vanishing first-order derivative at 0, that is

$$\left. \frac{\partial}{\partial b} \ell^b_{t,x} \right|_{b=0} = 0, \qquad t \geq 0, \quad x \in (0,1). \tag{1.23}$$

Finally, the integration by parts formulae that we find enable us to identify the corresponding Bessel SPDEs. For $1 < \delta < 3$, the SPDE should have the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\kappa(\delta)}{2}\frac{\partial}{\partial t}\int_0^\infty \frac{1}{b^3}\left(\ell^b_{t,x} - \ell^0_{t,x}\right) b^{\delta-1}\,\mathrm{d}b + \xi, \qquad (1 < \delta < 3), \tag{1.24}$$

where, as before, ξ is a space-time white noise on $\mathbb{R}_+ \times [0,1]$. Note that (1.24) is the SPDE analog of (1.9). On the other hand, for $\delta = 1$, the SPDE should be of the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{8}\frac{\partial}{\partial t}\frac{\partial^2}{\partial b^2}\,\ell^b_{t,x}\Big|_{b=0} + \xi, \qquad (\delta = 1), \tag{1.25}$$

while for $0 < \delta < 1$

$$\begin{aligned}\frac{\partial u}{\partial t} = & \ \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \xi \qquad\qquad (0 < \delta < 1)\\ & + \frac{\kappa(\delta)}{2}\frac{\partial}{\partial t}\int_0^\infty \frac{1}{b^3}\left(\ell^b_{t,x} - \ell^0_{t,x} - \frac{b^2}{2}\frac{\partial^2}{\partial b^2}\,\ell^b_{t,x}\Big|_{b=0}\right) b^{\delta-1}\,\mathrm{d}b.\end{aligned} \tag{1.26}$$

In (1.24), as in (1.9), we have a Taylor expansion at order 0 of the functions $b \mapsto \ell^b_{t,x}$. By contrast, equations (1.25) and (1.26) have no analog in the context of one-dimensional Bessel processes. In (1.26) the Taylor expansion is at order 2, while (1.25) is a limit case, like (1.12).

All the above SPDEs can be written in a unified way. Indeed, the non-linear terms in (1.10)-(1.24)-(1.25)-(1.26) can all be written as

$$\frac{\Gamma(\delta)}{8(\delta-2)}\langle \mu_{\delta-3}, \ell^{\cdot}_{t,x}\rangle. \tag{1.27}$$

where μ_α, $\alpha \in \mathbb{R}$ are as in Section 1.4 above, noting here again that the singularity at $\delta = 2$ is just apparent, due to the fact that $\langle \mu_{-1}, \ell^{\cdot}_{t,x}\rangle = 0$ because of the cancellation (1.23). Note the correspondence between (1.27) and (1.21) above. Thus, the expression (1.27) encapsulates, in a unified way, the non-linearities of (1.10)-(1.24)-(1.25)-(1.26). It is also coherent with the SPDE (1.12): for $\delta = 3$, it equals $\frac{1}{4}\ell^0_{t,x}$, which is coherent with the results on the structure of the reflection measure η in (1.12) proved in [Zam04b] and showing that a.s.

$$\eta([0,t] \times \mathrm{d}x) = \frac{1}{4}\,\ell^0_{t,x}\,\mathrm{d}x.$$

Thus, at least formally, the δ-Bessel SPDEs for $\delta < 3$ correspond to the unique analytic continuation of the δ-Bessel SPDEs for $\delta \geq 3$. The above conjectures are all explained in detail in Section 4.1, and are summarized in the chart below:

Regime	Drift term in SPDE (in integral form)
$\delta > 3$	$\frac{\kappa(\delta)}{2}\int_0^t u(s,x)^{-3}\,\mathrm{d}s$
$\delta = 3$	$\frac{1}{4}\ell^0_{t,x}$
$\delta \in (1,3)$	$\frac{\kappa(\delta)}{2}\int_0^\infty b^{-3}(\ell^b_{t,x}-\ell^0_{t,x})b^{\delta-1}\,\mathrm{d}b$
$\delta = 1$	$-\frac{1}{8}\frac{\partial^2}{\partial b^2}\,\ell^b_{t,x}\,\big\vert_{b=0}$
$\delta \in (0,1)$	$\frac{\kappa(\delta)}{2}\int_0^\infty b^{-3}\left(\ell^b_{t,x}-\ell^0_{t,x}-\frac{b^2}{2}\frac{\partial^2}{\partial b^2}\,\ell^b_{t,x}\,\big\vert_{b=0}\right)b^{\delta-1}\,\mathrm{d}b$

For the moment, the formulae (1.24), (1.25) and (1.26) remain highly conjectural: it is not yet known whether a Markov process on $C([0,1])$ can be constructed on $[0,1]$ with a family of local times satisfying (1.8) as well as one of these SPDEs. However, in Sections 4.2 and 4.3 below, using Dirichlet form methods, and thanks to the IbPF (1.20) for $\delta = 1$ as well as (1.17) for $\delta = 2$, we do construct a Markov process $(u_t)_{t\geq 0}$ with the law of a δ-dimensional Bessel bridge as reversible measure for $\delta = 1, 2$. We also show that it satisfies a weak version of equation (1.25) in the case $\delta = 1$, and of (1.24) in the case $\delta = 2$. More precisely, for $\delta = 1$, we show that the process thus constructed satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{4}\lim_{\epsilon\to 0}\rho''_\epsilon(u) + \xi, \qquad (\delta = 1) \tag{1.28}$$

where $\rho_\epsilon(x) = \frac{1}{\epsilon}\rho(\frac{x}{\epsilon})$ is a smooth approximation of the Dirac measure at 0, see Theorem 4.2.8 for the precise statements. We also show (see Theorem 4.2.9) that the Markov process $(u_t)_{t\geq 0}$ is not identical to the process obtained from the solution to the stochastic heat equation (1.2) by taking its modulus, as could have been naturally guessed in view of the analogous results for the invariant measures. For $\delta = 2$, we show that the associated Markov process satisfies

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{8}\lim_{\epsilon\to 0}\lim_{\eta\to 0}\left(\frac{\mathbf{1}_{u\geq\epsilon}}{u^3} - \frac{2}{\epsilon}\frac{\rho_\eta(u)}{u}\right) + \xi, \qquad (\delta = 2), \tag{1.29}$$

see Theorem 4.3.7 for the precise statement. We stress that the approach using Dirichlet forms was already used in the thesis [Voß16], which provided a construction of the Markov process for $\delta = 1$, but not the SPDE.

1.6 Properties of the SPDEs for $\delta < 3$

The reason why the SPDEs (1.24), (1.25) and (1.26) still remain highly conjectural is because they lie at the crossroads of several major open problems, such as the strong Feller property for equations with a non-dissipative drift, and the existence of local times for solutions to SPDEs.

The strong Feller property is an important notion in the study of Markov semigroups. While there are systematic tools, such as the Bismut-Elworthy-Li formula, to derive this property for solutions to SDEs or SPDEs with a dissipative drift, such tools generally break down in the non-dissipative case. An example of non-dissipative SDE is given by the equation (1.9) satisfied by Bessel processes of dimension $\delta < 1$. On the other hand, an example of non-dissipative SPDEs is given by the δ-Bessel SPDEs for $\delta < 3$ (1.24), (1.25) and (1.26) above. Thus, while it is easy to prove that the semigroup of the Bessel SPDE has the strong Feller property for $\delta \geq 3$ (see Section 5.4 in [Zam17]), it is an open problem for $\delta < 3$, because the drift of the SPDE becomes highly non-dissipative. However, in Chapter 7 below, we show that Bessel processes have the strong Feller property regardless of the dimension, in spite of the loss of dissipativity for dimensions $\delta < 1$. Indeed, denoting by $(\mathbf{P}_t^\delta)_{t\geq 0}$ the Markov semigroup of a δ-dimensional Bessel process on $\mathbb{R}^+$, we prove, by explicit computations, that for all $T > 0$ the following equality holds

$$\frac{d}{dx}\mathbf{P}_T^\delta F(x) = \frac{x}{T}\left(\mathbf{P}_T^{\delta+2}F(x) - \mathbf{P}_T^\delta F(x)\right),$$

for any $F : \mathbb{R}_+ \to \mathbb{R}$ bounded and Borel. As a consequence, we obtain at once the following estimate: for all, $R > 0$, $x, y \in [0, R]$ and $F : \mathbb{R}_+ \to \mathbb{R}$ bounded and Borel, we have

$$|\mathbf{P}_T^\delta F(x) - \mathbf{P}_T^\delta F(y)| \leq \frac{2R||F||_\infty}{T}|y - x|,$$

whence it in particular follows that the function $\mathbf{P}_T^\delta F$ is continuous, with a modulus of continuity which is independent of δ. Thus, the strong Feller property holds for the semigroup $(\mathbf{P}_t^\delta)_{t\geq 0}$, independently of $\delta \geq 0$. More surprisingly, one also obtains a Bismut-Elworthy-Li formula which holds for all $\delta > 0$. Namely, for all $\delta > 0$, $T > 0$ and $x \geq 0$

$$\frac{d}{dx}\mathbf{P}_T^\delta F(x) = \frac{1}{T}\,\mathbb{E}\left[F(\rho_t(x))\left(\int_0^T \eta_s(x)dB_s\right)\right],$$

where $(\rho_t(x))_{t,x\geq 0}$ is the flow of Bessel processes generated by (1.5) above and, for all $t, x \geq 0$, $\eta_t(x) := \frac{\mathrm{d}\rho_t}{\mathrm{d}x}$, see Theorem 7.6.2 below. This formula in turn allows us to slightly improve the Lipschitz estimate above, at least for a certain range of δ, see (7.25). Although these results are obtained by resorting to remarkable properties which are specific to Bessel processes, they provide a nice example of highly non-dissipative system which however satisfies the strong Feller property. Note that similar results have been recently obtained in the framework of singular SPDEs: thus, Tsatsoulis and Weber [TW18] have proved that the 2-dimensional stochastic quantization equation satisfies the strong Feller property, although it is an equation which needs renormalization; moreover, Hairer and Mattingly [HM18] have proved that property for a large class of equations with renormalized drifts. All this suggests that there may be hope that this technically very useful property holds also for δ-Bessel SPDEs with $\delta < 3$. Finally, we stress that the strong Feller bounds obtained in Chapter 7 are interesting in their own right, with potential applications to the study of Schramm-Loewner evolution.

As suggested above, the construction of δ-Bessel SPDEs for $\delta < 3$ should rely heavily on the existence of local times satisfying (1.22). While the existence of local times is systematic in the framework of SDEs, for solutions of SPDEs driven by a space-time white noise, which are typically not semimartingales, these objects are much more complicated to obtain. In certain specific cases, one can rely on ad hoc criteria such as provided in [GH80]: these results apply, fo instance, in the case of Gaussian processes such as the solution to (SHE), but they generally break down as soon as one adds a singular enough drift. In Chapter 8 below, we propose a few techniques to establish the existence of local times beyond the Gaussian case. These results do not yet cover the case of a singular drift as in the Bessel SPDEs we are interested in, but we hope they could be a first in that direction.

1.7 Application to scaling limits of dynamical critical wetting models

As mentioned above, we conjecture that several dynamical critical wetting models have a scaling limit described by the Bessel SPDE of parameter $\delta = 1$ (1.25) above. In Chapter 9 below, we focus on the dynamics associated with a wetting model with a shrinking strip as constructed in [DO18]. We also propose a natural version in the continuum of the wetting model, given by the law of Brownian meander tilted by its local time at a small level $\eta > 0$, and show that it converges as $\eta \to 0$ to the law of a reflecting Brownian motion, as for the discrete case. For both models - discrete and continuous - we address the tightness of the corresponding equilibrium dynamics. We conjecture that the limit should be described by a weak version of the 1-Bessel SPDE, such as equation (1.28) above.

This thesis is organized as follows. In Chapter 2, we recall and prove several basic facts about Bessel processes, Bessel bridges, and their squares. In Chapter 3 we state and prove the integration by parts fomulae for the laws of Bessel bridges (and also Bessel processes) of all positive dimension. Conjectures for the structure of the corresponding SPDEs are formulated in Chapter 4, where a rigorous construction using Dirichlet form techniques is also provided for the cases $\delta = 1, 2$. In Chapter 5, we extend the IbPF obtained in Chapter 3 to a larger space of test functionals, using Taylor estimates for the laws of pinned Bessel bridges. Chapter 6 is devoted to the IbPF in the case of integer dimensions. In Chapter 7, we prove the strong Feller property for Bessel processes, thus providing an example of non-dissipative system which nevertheless satisfies this important property. In Chapter 8, we prove some existence results for local times of solutions to parabolic SPDEs driven by a space-time white noise. Finally, Chapter 9 is dedicated to dynamical wetting models and associated tightness results.

Chapter 2

Bessel processes and Bessel bridges

In the present chapter we recall the definitions of squared Bessel processes, Bessel processes, as well as the corresponding bridges, and state some useful facts. The content of this chapter, as well as Chapters 3 and 4 below, is based on the articles [EAZ18] and [EAa].

2.1 An important family of distributions

In this section we consider a family of distributions on $\mathbb{R}_+$ which can be seen as a simplified version of the laws of Bessel processes or bridges. More than a toy model, these objects will be an essential tool in the proof of the IbPF below. For $\alpha \geq 0$, we set

$$\mu_\alpha(\mathrm{d}x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}x, \quad \alpha > 0, \qquad \mu_0 = \delta_0,$$

where δ_0 denotes the Dirac measure at 0. A simple change of variable yields the Laplace transform of the measures μ_α, $\alpha \geq 0$

$$\int \exp(-\lambda x)\,\mu_\alpha(\mathrm{d}x) = \lambda^{-\alpha}, \qquad \lambda > 0,\ \alpha \geq 0. \tag{2.1}$$

Remark 2.1.1. As suggested by formula (2.1), the family $(\mu_\alpha)_{\alpha\geq 0}$ forms a semi-group of measures, that is, the following holds:

$$\forall \alpha, \alpha' \geq 0, \quad \mu_{\alpha+\alpha'} = \mu_\alpha * \mu_{\alpha'}.$$

This is a baby version of the additivity property of squared Bessel processes, see (2.8) below.

It turns out that the family of measures $(\mu_\alpha)_{\alpha\geq 0}$ can be extended in a natural way to a family of *distributions* $(\mu_\alpha)_{\alpha\in\mathbb{R}}$. We first define the appropriate space of test functions on $[0,\infty)$.

Definition 2.1.2. Let $S([0,\infty))$ be the space of C^∞ functions $\varphi : [0,\infty) \to \mathbb{R}$ such that, for all $k, l \geq 0$, there exists $C_{k,\ell} \geq 0$ such that

$$\forall x \geq 0, \quad |\varphi^{(k)}(x)|\, x^\ell \leq C_{k,\ell}. \tag{2.2}$$

For $\alpha < 0$, we will define μ_α as a distribution, using a *renormalization* procedure based on Taylor polynomials. To do so, for any smooth function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, for all $n \in \mathbb{Z}$, and all $x \geq 0$, we set

$$\mathcal{T}_x^n \varphi := \varphi(x) - \sum_{0\leq j\leq n} \frac{x^j}{j!}\, \varphi^{(j)}(0). \tag{2.3}$$

In words, if $n \geq 0$ then $\mathcal{T}_x^n\varphi$ is the Taylor remainder based at 0, of order $n+1$, of the function φ, evaluated at x; if $n < 0$ then $\mathcal{T}_x^n\varphi$ is simply the value of φ at x.

Definition 2.1.3. For $\alpha < 0$, we define the distribution μ_α as follows

- if $\alpha = -k$ with $k \in \mathbb{N}$, then

$$\langle \mu_\alpha, \varphi\rangle := (-1)^k \varphi^{(k)}(0), \qquad \forall\, \varphi \in \mathcal{S}([0,\infty)) \tag{2.4}$$

- if $-k-1 < \alpha < -k$ with $k \in \mathbb{N}$, then

$$\langle \mu_\alpha, \varphi\rangle := \int_0^{+\infty} \mathcal{T}_x^k \varphi\, \frac{x^{\alpha-1}}{\Gamma(\alpha)}\, \mathrm{d}x, \qquad \forall\, \varphi \in \mathcal{S}([0,\infty)). \tag{2.5}$$

Note that formula (2.5) defines a bona fide distribution on $\mathcal{S}([0,\infty))$. Indeed, by Taylor's theorem, the integrand is of order $x^{k+\alpha}$ near 0, therefore integrable there, while it is dominated by $x^{k+\alpha-1}$ near $+\infty$, so is integrable at infinity as well. We note that μ_α is equal to the generalized function $\frac{x_+^{\alpha-1}}{\Gamma(\alpha)}$ of Section 3.5 of [GS77].

Remark 2.1.4. Note that for all $\alpha > 0$ and all Borel function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, the integral $\int_0^\infty \varphi(x)\, \mu_\alpha(\mathrm{d}x)$ coincides with $\Gamma(\alpha)^{-1}\mathcal{M}\varphi(\alpha)$, where $\mathcal{M}\varphi(\alpha)$ is the value of the Mellin transform of the function φ computed at α. Definition 2.1.3 thus provides an extension of the Mellin transform of a function $\varphi \in \mathcal{S}([0,\infty))$ to the whole real line. In particular, equality (2.4) is natural in view of Ramanujan's Master Theorem, which allows to see the successive derivatives at 0 of an analytic function as the values, for non-positive integers, of the analytic extension of its

Mellin transform. We refer to [AEG$^+$12] for more details on this theorem. We also stress that the renormalization procedure used in equation (2.5) to define μ_α for $\alpha < 0$ is very natural, and can also be used to extend the domain of validity of Ramanujan's Master Theorem, see Theorem 8.1 in [AEG$^+$12].

Remark 2.1.5. For $k \in \mathbb{N}$ and α such that $-k-1 < \alpha < -k$, and for all $\varphi \in \mathcal{S}([0,\infty))$, we obtain after $k+1$ successive integration by parts the equality

$$\langle \mu_\alpha, \varphi \rangle := (-1)^{k+1} \int_0^{+\infty} \varphi^{(k+1)}(x)\, \mu_{\alpha+k+1}(\mathrm{d}x), \tag{2.6}$$

which can be interpreted as a variant of the Caputo differential, at order $-\alpha$, of φ, see e.g. (1.17) in [GM08].

We recall the following basic fact, which is easily proven (see e.g. (5) in Section 3.5 of [GS77]). It can be seen as a toy-version of the integration by parts formulae of Theorem 3.1.1 below.

Proposition 2.1.6. *For all $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{S}([0,\infty))$*

$$\langle \mu_\alpha, \varphi' \rangle = -\langle \mu_{\alpha-1}, \varphi \rangle.$$

Thus, for all $\alpha \in \mathbb{R}$, μ_α admits $\mu_{\alpha-1}$ as its distributional derivative. In particular, for $\alpha \in (0,1)$ the distributional derivative of the measure μ_α is given by the genuine distribution $\mu_{\alpha-1}$.

Remark 2.1.7. As a consequence of Proposition 2.1.6, we deduce that the expression (2.1) for the Laplace tranform of μ_α remains true also for negative α. Indeed, for such α, picking $k \in \mathbb{N}$ such that $\alpha + k > 0$, we have, for all $\lambda > 0$

$$\langle \mu_\alpha, e^{-\lambda \cdot} \rangle = (-1)^k \langle \mu_{\alpha+k}, \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-\lambda \cdot} \rangle = \lambda^k \langle \mu_{\alpha+k}, e^{-\lambda \cdot} \rangle = \lambda^k \lambda^{-\alpha-k} = \lambda^{-\alpha}.$$

As shown by Prop 2.1.6, the family of distributions $(\mu_\alpha)_{\alpha\in\mathbb{R}}$ behaves nicely under differentiation. Actually it also behaves nicely under multiplication by x, as shown by the following result:

Lemma 2.1.8. *For all $\alpha \in \mathbb{R}$ and $f \in \mathcal{S}([0,\infty))$, the following relation holds:*

$$\langle \mu_\alpha(x), x f(x) \rangle = \alpha \langle \mu_{\alpha+1}, f \rangle.$$

Here we wrote a dummy variable x to indicate which variable is being integrated, a convention we will also use below.

Proof. Assume first that $\alpha > 0$. Then, we have

$$\begin{aligned}\langle \mu_\alpha(x), xf(x)\rangle &= \int_0^\infty f(x)\frac{x^\alpha}{\Gamma(\alpha)}\,\mathrm{d}x \\ &= \alpha\int_0^\infty f(x)\frac{x^\alpha}{\Gamma(\alpha+1)}\,\mathrm{d}x \\ &= \alpha\langle \mu_{\alpha+1}, f\rangle,\end{aligned}$$

Now, if $\alpha \leq 0$, picking an integer n such that $\alpha + n > 0$, we have, by n successive applications of Thm 2.1.6

$$\langle \mu_\alpha(x), xf(x)\rangle = (-1)^n \left\langle \mu_{\alpha+n}(x), \frac{\mathrm{d}^n}{\mathrm{d}x^n}(xf(x))\right\rangle$$

But, by the Leibniz rule, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(xf(x)) = xf^{(n)}(x) + nf^{(n-1)}(x), \quad x \geq 0.$$

Hence

$$\begin{aligned}\langle \mu_\alpha(x), xf(x)\rangle &= (-1)^n \left(\left\langle \mu_{\alpha+n}(x), xf^{(n)}(x)\right\rangle + n\left\langle \mu_{\alpha+n}, f^{(n-1)}\right\rangle\right) \\ &= (-1)^n \left((\alpha+n)\left\langle \mu_{\alpha+n+1}, f^{(n)}\right\rangle + n\left\langle \mu_{\alpha+n}, f^{(n-1)}\right\rangle\right)\end{aligned}$$

where we applied the first point to obtain the second equality. Invoking again Thm 2.1.6, we can rewrite the latter quantity as

$$(\alpha+n)\langle \mu_{\alpha+1}, f\rangle - n\langle \mu_{\alpha+1}, f\rangle = \alpha\langle \mu_{\alpha+1}, f\rangle.$$

This yields the claim. □

Finally the family $(\mu_\alpha)_\alpha$ possesses the nice following property, which further justifies the idea that it extends the convolution semigroup of measures $(\mu_\alpha)_{\alpha\geq 0}$ considered above:

Proposition 2.1.9. *The family $(\mu_\alpha)_{\alpha\in\mathbb{R}}$ is a convolution group of distributions, i.e., for all $\varphi \in \mathcal{S}([0,\infty))$ and all $\alpha, \alpha' \in \mathbb{R}$, the following formula holds:*

$$\langle \mu_\alpha(x), \langle \mu_{\alpha'}(y), \varphi(x+y)\rangle\rangle = \langle \mu_{\alpha+\alpha'}(z), \varphi(z)\rangle \tag{2.7}$$

Remark 2.1.10. The integration by parts formula stated in Thm2.1.6 actually corresponds to the equality (2.7) with $\alpha' = -1$.

The equality 2.7 is essentially a consequence of the semigroup property of $(\mu_\alpha)_{\alpha\geq 0}$, as well as Prop 2.1.6 above. One however has to argue that the test functions appearing in (2.7) belong to the correct spaces. Since we shall not need this result in the sequel, we prefer to postpone the proof of Prop 2.1.9 to the Section 2.3.

2.2 Bessel processes and associated bridges

Here and below, for all $I \subset \mathbb{R}_+$, we shall denote by $C(I)$ the space of continuous, real-valued functions on I. Moreover, for all $\alpha \geq 0$ and $\theta > 0$, we denote by $\Gamma(\alpha, \theta)$ the Gamma probability law on $\mathbb{R}_+$

$$\Gamma(\alpha,\theta)(\mathrm{d}x) = \frac{\theta^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}\, e^{-\theta x}\, \mathbf{1}_{x>0}\, \mathrm{d}x, \quad \alpha > 0, \qquad \Gamma(0,\theta) := \delta_0.$$

2.2.1 Squared Bessel processes and Bessel processes

For all $x, \delta \geq 0$, denote by Q^δ_x the law, on $C(\mathbb{R}_+, \mathbb{R}_+)$, of the δ-dimensional squared Bessel process started at x, namely the unique solution to the SDE (1.5) with $Y_0 = x$, see Chapter XI of [RY13]. We denote by $(X_t)_{t\geq 0}$ the canonical process

$$X_t : C([0,1]) \to \mathbb{R}, \qquad X_t(\omega) := \omega_t, \quad \omega \in C([0,1]).$$

Definition 2.2.1. For any interval $I \subset \mathbb{R}_+$, and any two probability laws μ, ν on $C(I, \mathbb{R}_+)$, let $\mu * \nu$ denote the convolution of μ and ν, i.e. the image of $\mu \otimes \nu$ under the addition map

$$C(I,\mathbb{R}_+) \times C(I,\mathbb{R}_+) \to C(I,\mathbb{R}_+), \quad (x,y) \mapsto x + y.$$

The family of probability measures $\left(Q^\delta_x\right)_{\delta, x\geq 0}$ satisfies the following well-known additivity property, first observed by Shiga and Watanabe in [SW73].

Proposition 2.2.2. *For all $x, x', \delta, \delta' \geq 0$, we have the following equality of laws on $C(\mathbb{R}_+, \mathbb{R}_+)$*

$$Q^\delta_x * Q^{\delta'}_{x'} = Q^{\delta+\delta'}_{x+x'} \tag{2.8}$$

We recall that squared Bessel processes are homogeneous Markov processes on $\mathbb{R}_+$. Exploiting the additivity property (2.8), one can compute (see e.g. section XI of [RY13]) the explicit expressions of their transition densities $\left(q^\delta_t(x,y)\right)_{t>0,x,y\geq 0}$. When $\delta > 0$, these are given by

$$q^\delta_t(x,y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} \exp\left(-\frac{x+y}{2t}\right) I_\nu\left(\frac{\sqrt{xy}}{t}\right), \quad t > 0,\ x > 0 \tag{2.9}$$

and

$$q^\delta_t(0,y) = (2t)^{-\frac{\delta}{2}}\, \Gamma\left(\delta/2\right)^{-1} y^{\delta/2-1} \exp\left(-\frac{y}{2t}\right), \quad t > 0. \tag{2.10}$$

Above, $\nu := \delta/2 - 1$ and I_ν is the modified Bessel function of index ν. In particular, for $x = 0$, we have

$$q^\delta_t(0,y)\,\mathrm{d}y = \Gamma\left(\frac{\delta}{2}, \frac{1}{2t}\right)(\mathrm{d}y).$$

We also denote by P_x^δ the law of the δ-Bessel process, image of $Q_{x^2}^\delta$ under the map

$$C(\mathbb{R}_+, \mathbb{R}_+) \ni \omega \mapsto \sqrt{\omega} \in C(\mathbb{R}_+, \mathbb{R}_+). \tag{2.11}$$

We shall denote by $\left(p_t^\delta(a,b)\right)_{t>0,\, a,b\geq 0}$ the transition densities of a δ-Bessel process. They are given in terms of the densities of the squared Bessel process by the relation

$$\forall t > 0, \quad \forall a, b \geq 0, \quad p_t^\delta(a,b) = 2\, b\, q_t^\delta(a^2, b^2). \tag{2.12}$$

Note that the measure μ_δ defined above is reversible for the δ-dimensional Bessel process. Indeed, the following detailed balance condition holds:

$$\forall t > 0, \forall a, b \geq 0, \quad a^{\delta-1} p_t^\delta(a,b) = b^{\delta-1} p_t^\delta(b,a)$$

In section XI of [RY13], Revuz and Yor provided semi-explicit expressions for the Laplace transforms of squared Bessel processes (and also the corresponding bridges). Their proof is based on the fact that, for all $\delta, x \geq 0$, and all finite Borel measure m on $[0,1]$, the measure $\exp\left(-\langle m, X\rangle\right) Q_x^\delta$ possesses a nice probabilistic interpretation, where we use the notation

$$\langle m, f\rangle := \int_0^1 f(r)\, m(\mathrm{d}r)$$

for any Borel function $f : [0,1] \to \mathbb{R}_+$. This remarkable fact is used implicitly in [RY13] (see e.g. the proof of Theorem (3.2) of Chap XI.3), where the authors compute the one-dimensional marginal distributions of this measure. By contrast, in the proof of Lemma 2.2.6 below, we will need to compute higher-dimensional marginals. We thus need a way to interpret the measure $\exp\left(-\langle m, X\rangle\right) Q_x^\delta$ probabilistically. To do so, we first introduce some notations.

Let m be a finite, Borel measure on $[0,1]$. As in Chap. XI of [RY13], we consider the unique solution $\phi = (\phi_r, r \geq 0)$ on $\mathbb{R}_+$ of the following problem

$$\begin{cases} \phi''(\mathrm{d}r) = 2\mathbf{1}_{[0,1]}(r)\, \phi_r\, m(\mathrm{d}r) \\ \phi_0 = 1,\ \phi > 0,\ \phi' \leq 0 \text{ on } \mathbb{R}_+, \end{cases} \tag{2.13}$$

where the first is an equality of measures (see Appendix 8 of [RY13] for existence and uniqueness of solutions to this problem). Note that the above function ϕ coincides with the function ϕ_μ of Chap XI.1 of [RY13], with $\mu := 2\mathbf{1}_{[0,1]}\, m$.

Lemma 2.2.3. *Let m be a finite, Borel measure on $[0,1]$, and let ϕ be the unique solution of (2.13). Then, for all $x, \delta \geq 0$, the measure R_x^δ on $C([0,1])$ defined by*

$$R_x^\delta := \exp\left(-\frac{x}{2}\phi_0'\right) \phi_1^{-\frac{\delta}{2}}\ e^{-\langle m, X\rangle}\ Q_x^\delta \tag{2.14}$$

is a probability measure, equal to the law of the process

$$\left(\phi_t^2 \, Y_{\varrho_t}\right)_{t\in[0,1]},$$

where $Y \overset{(d)}{=} Q_x^\delta$ *and* ϱ *is the deterministic time change*

$$\varrho_t = \int_0^t \phi_u^{-2} \, \mathrm{d}u, \quad t \geq 0. \tag{2.15}$$

Proof. We proceed as in the proofs of Theorem (1.7) and (3.2) in Chapter XI of [RY13]. Let $x, \delta \geq 0$. Under Q_x^δ, $M_t := X_t - \delta t$ is a local martingale, so we can define an exponential local martingale by setting

$$Z_t = \mathscr{E}\left(\frac{1}{2}\int_0^\cdot \frac{\phi_s'}{\phi_s} \, \mathrm{d}M_s\right)_t.$$

As established in the proof of Theorem (1.7) of [RY13], we have

$$\begin{aligned} Z_t &= \exp\left(\frac{1}{2}\left(\frac{\phi_t'}{\phi_t}X_t - \phi_0' x - \delta \ln \phi_t\right) - \int_0^t X_s \, m(\mathrm{d}s)\right) = \\ &= \exp\left(-\frac{x}{2}\phi_0'\right) \phi_t^{-\frac{\delta}{2}} \exp\left(\frac{1}{2}\frac{\phi_t'}{\phi_t}X_t - \int_0^t X_s \, m(\mathrm{d}s)\right), \end{aligned}$$

recalling that the measure μ considered in [RY13] is given in our case by $2\,\mathbf{1}_{[0,1]}\, m$. In particular, we deduce that the measure R_x^δ defined by (2.14) coincides with $Z_1 Q_x^\delta$ (note that $\phi_1' = 0$ as a consequence of (2.13)). Moreover, by the above expression, $(Z_t)_{t\in[0,1]}$ is uniformly bounded by $\exp\left(-\frac{1}{2}\phi_0'\right) \phi_1^{-\frac{\delta}{2}}$, so it is a martingale on $[0,1]$. Hence, R_x^δ defines a probability measure.

There remains to give a description of R_x^δ. By Girsanov's theorem, under R_x^1, $(X_t)_{t\in[0,1]}$ solves the following SDE on $[0,1]$

$$X_t = x + 2\int_0^t \sqrt{X_s} \, \mathrm{d}B_s + 2\int_0^t \frac{\phi_s'}{\phi_s} X_s \, \mathrm{d}s + t. \tag{2.16}$$

But a weak solution to this SDE is provided by $(H_t^2)_{t\in[0,1]}$, where

$$H_t := \phi_t \left(x + \int_0^t \phi_s^{-1} \, \mathrm{d}W_s\right),$$

where W is a standard Brownian motion. By strong and therefore weak uniqueness of solutions to equation (2.16), see [RY13, Theorem IX.3.5], we deduce that X is equal in law to the process $(H_t^2)_{t\in[0,1]}$. On the other hand, by Lévy's theorem, we have

$$(H_t)_{t\in[0,1]} \overset{(d)}{=} (\phi_t\, \gamma_{\varrho t})_{t\in[0,1]}\,,$$

where γ is a standard Brownian motion started at x. Hence we deduce that

$$(H_t^2)_{t\in[0,1]} \overset{(d)}{=} \left(\phi_t^2\, Y_{\varrho t}\right)_{t\in[0,1]}\,,$$

where $Y \overset{(d)}{=} Q_x^1$. Therefore, under R_x^1, we have

$$X \overset{(d)}{=} \left(\phi_t^2\, Y_{\varrho t}\right)_{t\in[0,1]}\,.$$

The claim is thus proven for $\delta = 1$ and for any $x \geq 0$. Now, by the additivity property (2.8) satisfied by $\left(Q_x^\delta\right)_{\delta,x\geq 0}$, there exist $A, B > 0$ such that, for all $x, \delta \geq 0$, and all finite Borel measure ν on $[0,1]$, we have

$$Q_x^\delta\left[\exp\left(-\int_0^1 \phi_t^2\, X_{\varrho t}\, \nu(\mathrm{d}t)\right)\right] = A^x B^\delta,$$

which can be proved exactly as Corollary 1.3 in Chapter XI of [RY13]. Note now that the family of probability laws $\left(R_x^\delta\right)_{\delta,x\geq 0}$ satisfies the same additivity property

$$\forall\ \delta, \delta', x, x' \geq 0, \quad R_x^\delta * R_{x'}^{\delta'} = R_{x+x'}^{\delta+\delta'}.$$

Hence, there also exist $\tilde{A}, \tilde{B} > 0$ such that, for all $x, \delta \geq 0$, and μ as above

$$R_x^\delta\left[\exp\left(-\int_0^1 X_t\, \nu(\mathrm{d}t)\right)\right] = \tilde{A}^x \tilde{B}^\delta.$$

By the previous point, evaluating at $\delta = 1$, we obtain

$$\forall x \geq 0, \quad A^x B = \tilde{A}^x \tilde{B}.$$

Hence $A = \tilde{A}$ and $B = \tilde{B}$, whence we deduce that, for all $\delta, x \geq 0$

$$Q_x^\delta\left[\exp\left(-\int_0^1 \phi_t^2\, X_{\varrho t}\, \nu(\mathrm{d}t)\right)\right] = R_x^\delta\left[\exp\left(-\int_0^1 X_t\, \nu(\mathrm{d}t)\right)\right].$$

Since this holds for any finite measure ν on $[0,1]$, by injectivity of the Laplace transform, the claimed equality in law holds for all $\delta, x \geq 0$. □

2.2.2 Squared Bessel bridges and Bessel bridges

For all $\delta, x, y \geq 0$, we denote by $Q^\delta_{x,y}$ the law, on $C([0,1])$, of the δ-dimensional squared Bessel bridge from x to y over the interval $[0,1]$. In other words, $Q^\delta_{x,y}$ is the law of of a δ-dimensional squared Bessel bridge started at x, and conditioned to hitting y at time 1. A rigourous construction of these probability laws is provided in Chap. XI.3 of [RY13] (see also [PY82] for a discussion on the particular case $\delta = y = 0$). An important feature is the following continuity property: for all $\delta > 0$, the map $(x,y) \mapsto Q^\delta_{x,y}$ is continuous on $\mathbb{R}^2_+$ for the weak topology on probability measures (see Chap. XI.3 in [RY13]).

In the sequel we shall often consider the case $x = y = 0$, and will write Q^δ instead of $Q^\delta_{0,0}$.

We recall that, if $X \overset{(d)}{=} Q^\delta_{x,y}$, and $r \in (0,1)$, then the random variable X_r admits the density $q^{\delta,r}_{x,y}$ on $\mathbb{R}_+$, where

$$q^{\delta,r}_{x,y}(z) := \frac{q^\delta_r(x,z)q^\delta_{1-r}(z,y)}{q^\delta_1(x,y)}, \quad z \geq 0 \tag{2.17}$$

see Chap. XI.3 of [RY13]. In particular, if $X \overset{(d)}{=} Q^\delta$, then, for all $r \in (0,1)$, the distribution of the random variable X_r is given by $\Gamma(\frac{\delta}{2}, \frac{1}{2r(1-r)})$, so it admits the density q^δ_r given by

$$q^\delta_r(z) := \frac{z^{\delta/2-1}}{(2r(1-r))^{\frac{\delta}{2}}\Gamma(\delta/2)} \exp\left(-\frac{z}{2r(1-r)}\right), \quad z \geq 0. \tag{2.18}$$

In the same way as one constructs the laws of squared Bessel bridges $Q^\delta_{x,y}$, $\delta, x, y \geq 0$, one can also construct the laws of Bessel bridges. In the following, for any $\delta, a, a' \geq 0$, we shall denote by $P^\delta_{a,a'}$ the law, on $C([0,1])$, of a δ-dimensional Bessel bridge from a to a' over the time interval $[0,1]$, that is the law of a δ-dimensional Bessel process started at a and conditioned to hit a' at time 1. We shall denote by $E^\delta_{a,a'}$ the expectation operator for $P^\delta_{a,a'}$. Note that, for all $r \in (0,1)$, under the law $P^\delta_{a,a'}$, the random variable X_r admits the density $p^{\delta,r}_{a,a'}$ on $\mathbb{R}_+$, where

$$p^{\delta,r}_{a,a'}(b) := \frac{p^\delta_r(a,b)p^\delta_{1-r}(b,a')}{p^\delta_1(a,a')}, \qquad b \geq 0 \tag{2.19}$$

When $a = a' = 0$, we drop the subindices and use the compact notations P^δ, E^δ and p^δ_r.

Note that, for all $a, a' \geq 0$, $P^\delta_{a,a'}$ is the image of $Q^\delta_{a^2,a'^2}$ under the map $\omega \mapsto \sqrt{\omega}$. In particular, under the measure P^δ, for all $r \in (0,1)$, X_r admits the density p^δ_r

on $\mathbb{R}_+$, where by (2.18)

$$p_r^\delta(a) = 2a\, q_r^\delta(a^2) = \frac{a^{\delta-1}}{2^{\frac{\delta}{2}-1}\,\Gamma(\frac{\delta}{2})(r(1-r))^{\delta/2}} \exp\left(-\frac{a^2}{2r(1-r)}\right), \quad a \geq 0. \quad (2.20)$$

2.2.3 Pinned bridges

Let $\delta \geq 0$. For all $x, y, z \geq 0$ and $r \in (0,1)$, we denote by $Q^\delta_{x,y}[\,\cdot\,|\,X_r = z]$ the law, on $C([0,1])$, of a δ-dimensional squared Bessel bridge between x and y, pinned at z at time r, that is conditioned to hit z at time r. Such a probability law can be constructed using the same conditioning procedure as for the construction of squared Bessel bridges. One similarly defines, for all $a, b, c \geq 0$ and $r \in (0,1)$, the law $P^\delta_{a,b}[\,\cdot\,|\,X_r = c]$ of a δ-dimensional Bessel bridge between a and b pinned at c at time r. Note that the latter probability measure is the image of $Q^\delta_{a^2,b^2}[\,\cdot\,|\,X_r = c^2]$ under the map (2.11). We shall write $Q^\delta[\,\cdot\,|\,X_r = z]$ instead of $Q^\delta_{0,0}[\,\cdot\,|\,X_r = z]$, and $P^\delta[\,\cdot\,|\,X_r = c]$ instead of $P^\delta_{0,0}[\,\cdot\,|\,X_r = c]$

With these notations at hand, we now define a family of measures which will play an important role in the IbPF for Bessel bridges.

Definition 2.2.4. For all $a, a', b \geq 0$ and $r \in (0,1)$, we set

$$\Sigma^{\delta,r}_{a,a'}(\mathrm{d}X\,|\,b) := \frac{p^{\delta,r}_{a,a'}(b)}{b^{\delta-1}}\, P^\delta_{a,a'}[\,\mathrm{d}X\,|\,X_r = b], \quad (2.21)$$

where $p^{\delta,r}_{a,a'}$ is the probability density function of X_r under $P^\delta_{a,a'}$, see (2.19). In the particular case $a = a' = 0$, we shall write $\Sigma^\delta_r(\mathrm{d}X\,|\,b)$ intstead of $\Sigma^{\delta,r}_{0,0}(\mathrm{d}X\,|\,b)$.

The definition of the measure $\Sigma^{\delta,r}_{a,a'}(\,\cdot\,|\,b)$ is motivated by the idea that the solution u to the SPDE associated with $P^\delta_{a,a'}$ should have the following feature : for all $r \in (0,1)$, the process $(u(t,r))_{t\geq 0}$ should admit a family of *diffusion local times* $(\ell^b_{t,x})_{b,t\geq 0}$. Then, at least formally, for all $b \geq 0$, $\Sigma^{\delta,r}_{a,a'}(\,\cdot\,|\,b)$ would be the Revuz measure associated with the process $(\ell^b_{t,x})_{t\geq 0}$: see Section 4.1 below for an explanation and a development of this idea.

Remark 2.2.5. The equalities (2.21) and (2.19) above *do* also include the cases $b = 0$ and $a' = 0$. Indeed, note that, as a consequence of the expressions (2.9), (2.10) and (2.12), $\frac{p^\delta_r(a,b)}{b^{\delta-1}}$ can be extended to an analytic function of b at $b = 0$, and, for all $a, b \geq 0$, the function

$$a' \to \frac{p^\delta_{1-r}(b,a')}{p^\delta_1(a,a')}$$

can be extended in an analytic way at $a' = 0$. In the sequel, we will systematically consider these analytic extensions. For instance, in the particular case $a = a' = 0$, by (2.20), we have

$$\frac{p_r^\delta(b)}{b^{\delta-1}} = \frac{1}{2^{\frac{\delta}{2}-1}\,\Gamma(\frac{\delta}{2})(r(1-r))^{\delta/2}} \exp\left(-\frac{b^2}{2r(1-r)}\right), \quad b \geq 0.$$

which is indeed well-defined also for $b = 0$.

To keep the formulae synthetic, for all $r \in (0,1)$ and $a, a', b \geq 0$, and all Borel function $\Phi : C([0,1]) \to \mathbb{R}_+$, we shall write with a slight abuse of language

$$\Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,b) := \int \Phi(X)\ \Sigma^{\delta,r}_{a,a'}(\mathrm{d}X\,|\,b).$$

In the sequel we will have to compute quantities of the form

$$\Sigma^{\delta,r}_{a,a'}\left[\exp(-\langle m,\, X^2\rangle)\,|\,b\right]$$

for m a finite Borel measure on $[0,1]$, and where we use the shorthand notation $\langle m,\, X^2\rangle := \int_0^1 X_t^2\, m(\,\mathrm{d}t)$. In that perspective, we introduce some further notations. Given m as before, following the notation used in [PY82] (see also Exercise (1.34), Chap. XI, of [RY13]), we denote by ψ the function on $[0,1]$ given by

$$\psi_r := \phi_r \int_0^r \phi_u^{-2}\,\mathrm{d}u = \phi_r \varrho_r, \qquad r \in [0,1], \tag{2.22}$$

where ϱ is as in (2.15). Note that ψ is the unique solution on $[0,1]$ of the Cauchy problem

$$\begin{cases} \psi''(\mathrm{d}r) = 2\,\psi_r\, m(\mathrm{d}r) \\ \psi_0 = 0, \quad \psi'_0 = 1. \end{cases}$$

Moreover, we denote by $\hat{\psi}$ the function on $[0,1]$ given by

$$\hat{\psi}_r := \phi_1\phi_r(\varrho_1 - \varrho_r) = \psi_1\phi_r - \psi_r\phi_1, \quad r \in [0,1]. \tag{2.23}$$

Note that $\hat{\psi}$ satisfies the following problem on $[0,1]$

$$\begin{cases} \hat{\psi}''(\mathrm{d}r) = 2\,\hat{\psi}_r\, m(\mathrm{d}r) \\ \hat{\psi}_1 = 0, \quad \hat{\psi}'_1 = -1. \end{cases}$$

Note that the functions ϕ, ψ and $\hat{\psi}$ take positive values on $(0,1)$.

Lemma 2.2.6. *For all $r \in (0,1)$, $\delta > 0$ and $a, a,', b \geq 0$, the following holds*

$$
\begin{aligned}
&\int \exp(-\langle m, X^2\rangle)\ \Sigma^{\delta,r}_{a,a'}(\mathrm{d}X \,|\, b) \\
&= 2\exp\left(\frac{a^2}{2}\phi_0'\right)\phi_1^{\delta/2-2}\phi_r^{-2}\frac{q^\delta_{\varrho_r}\left(a^2, \frac{b^2}{\phi_r^2}\right) q^\delta_{\varrho_1-\varrho_r}\left(\frac{b^2}{\phi_r^2}, \frac{a'^2}{\phi_1^2}\right)}{b^{\delta-2}\, q_1^\delta(a^2, a'^2)} \\
&= 2\exp\left(\frac{a^2}{2}\phi_0'\right)\phi_1^{\delta/2-2}\phi_r^{-\delta}\varrho_1^{-\delta-1}\frac{q^{\delta,t}_{x,y}(z)}{z^{\delta/2-1}},
\end{aligned}
\tag{2.24}
$$

where $x = \frac{a^2}{\varrho_1}$, $y = \frac{a'^2}{\varrho_1\phi_1}$, $z = \frac{b^2}{\varrho_1\phi_r^2}$, and $t = \frac{\varrho_r}{\varrho_1} \in [0,1]$. Here, $q^{\delta,t}_{x,y}$ denotes the density of the random variable X_t, when $X \overset{(d)}{=} Q^\delta_{x,y}$, see (2.17). In particular, for $a = a' = 0$, we have

$$
\int \exp(-\langle m, X^2\rangle)\ \Sigma^{\delta,r}_{0,0}(\mathrm{d}X \,|\, b) = \frac{1}{2^{\frac{\delta}{2}-1}\,\Gamma(\frac{\delta}{2})}\ \exp\left(-\frac{b^2}{2}C_r\right) D_r^{\delta/2}, \tag{2.25}
$$

where

$$
C_r = \frac{\psi_1}{\psi_r\hat{\psi}_r}, \qquad D_r = \frac{1}{\psi_r\hat{\psi}_r}.
$$

Remark 2.2.7. The above lemma shows that, for all measure m as above, all $a, a' \geq 0$ and $r \in (0,1)$, the function

$$
b \to \int \exp(-\langle m, X^2\rangle)\ \Sigma^{\delta,r}_{a,a'}(\mathrm{d}X \,|\, b)
$$

is a smooth (actually analytic) function of b^2. In particular

$$
\frac{\mathrm{d}}{\mathrm{d}b}\left(\int \exp(-\langle m, X^2\rangle)\ \Sigma^{\delta,r}_{a,a'}(\mathrm{d}X \,|\, b)\right)\bigg|_{b=0} = 0. \tag{2.26}
$$

Proof. First note that by the relation (2.12), and by the expression (2.21), we have

$$
\begin{aligned}
&\int \exp(-\langle m, X^2\rangle)\ \Sigma^{\delta,r}_{a,a'}(\mathrm{d}X \,|\, b) = \\
&= 2\frac{q_r^\delta(a^2, b^2) q^\delta_{1-r}(b^2, a'^2)}{b^{\delta-2}\, q_1^\delta(a^2, a'^2)} Q^\delta_{a^2,a'^2}[\exp(-\langle m, X\rangle) \,|\, X_r = b^2].
\end{aligned}
\tag{2.27}
$$

To obtain the claim, it therefore suffices to compute

$$
Q^\delta_{a^2,a'^2}[\exp(-\langle m, X\rangle) \,|\, X_r = b^2].
$$

To do so, consider two Borel functions $f, g : \mathbb{R}_+ \to \mathbb{R}_+$. We have

$$\int_0^\infty \int_0^\infty Q^\delta_{a^2}[\exp(-\langle m, X\rangle) \,|\, X_r = x, X_1 = y]\, q^\delta_r(a^2, x) q^\delta_{1-r}(x, y) f(x) g(y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= Q^\delta_{a^2}\left[\exp(-\langle m, X\rangle) f(X_r) g(X_1)\right]$$
$$= \exp\left(\frac{a^2}{2}\phi_0'\right) \phi_1^{\delta/2} Q^\delta_{a^2}\left[f\left(\phi_r^2 X_{\varrho_r}\right) g\left(\phi_1^2 X_{\varrho_1}\right)\right]$$
$$= \exp\left(\frac{a^2}{2}\phi_0'\right) \phi_1^{\delta/2-2}\phi_r^{-2} \int_0^\infty \int_0^\infty q^\delta_{\varrho_r}\left(a^2, \frac{x}{\phi_r^2}\right) q^\delta_{\varrho_1-\varrho_r}\left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right) f(x) g(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Here, we used Lemma 2.2.3 to obtain the third line. Since the functions f and g are arbitrary we deduce that

$$Q^\delta_{a^2}[\exp(-\langle m, X\rangle) \,|\, X_r = x, X_1 = y] =$$
$$= \exp\left(\frac{a^2}{2}\phi_0'\right) \phi_1^{\delta/2-2}\phi_r^{-2} \frac{q^\delta_{\varrho_r}\left(a^2, \frac{x}{\phi_r^2}\right) q^\delta_{\varrho_1-\varrho_r}\left(\frac{x}{\phi_r^2}, \frac{y}{\phi_1^2}\right)}{q^\delta_r(a^2, x)\, q^\delta_{1-r}(x, y)}$$

$\mathrm{d}x\,\mathrm{d}y$ a.e. on ${\mathbb{R}_+^*}^2$, hence everywhere by continuity. Applying this equality to $x = b^2$ and $y = a'^2$, and replacing in (2.27), we obtain (2.24)

Now assume that $a = a' = 0$. Recalling the expressions (2.9) and (2.10), we obtain

$$\int \exp(-\langle m, X^2\rangle)\; \Sigma^{\delta,r}_{0,0}(\mathrm{d}X \,|\, b) =$$
$$= \frac{1}{2^{\frac{\delta}{2}-1}\,\Gamma(\frac{\delta}{2})} \exp\left(-\frac{b^2\varrho_1}{2\phi_r^2\varrho_r(\varrho_1-\varrho_r)}\right) \left(2\phi_r^2\phi_1\varrho_r(\varrho_1-\varrho_r)\right)^{-\delta/2}.$$

The second claim then follows by the relations (2.22) and (2.23) defining ψ and $\hat\psi$. □

Remark 2.2.8. Along the proof of the above Proposition, for $\delta > 0$, $a \geq 0$, $r \in (0,1)$ and m as above, we also obtained the following, useful expression

$$Q^\delta\left[\exp(-\langle m, X\rangle) \,|\, X_r = a^2\right] = E^\delta[\exp(-\langle m, X^2\rangle) \,|\, X_r = a]$$
$$= \exp\left(-\frac{a^2}{2}\left(\frac{\psi_1}{\psi_r\hat\psi_r} - \frac{1}{r(1-r)}\right)\right)\left(\frac{r(1-r)}{\psi_r\hat\psi_r}\right)^{\delta/2}. \tag{2.28}$$

2.3 Proof of Prop 2.1.9

In this section we give the proof of Prop 2.1.9 above.

Proof of Proposition 2.1.9. We first prove that the right-hand side of equality (2.7) actually makes sense. To do so, first note that, given $x \geq 0$ fixed, the function $\varphi(x+\cdot)$ is indeed an element of $\mathcal{S}([0,\infty))$. Indeed, for all $k, \ell \geq 0$ and $y \geq 0$

$$|\varphi^{(k)}(x+y)|y^{\ell} \leq |\varphi^{(k)}(x+y)|(x+y)^{\ell} \leq C_{k,\ell}.$$

Hence $\langle \mu_\alpha(y), \varphi(x+y)\rangle$ is well-defined. Moreover, the function

$$x \mapsto \langle \mu_\alpha(y), \varphi(x+y)\rangle$$

is in $\mathcal{S}([0,\infty))$. Indeed, there exists $k \geq 0$ such that $\alpha + k > 0$. Then, by Theorem 2.1.6, for all $x \geq 0$ we have

$$\begin{aligned}\langle \mu_\alpha(y), \varphi(x+y)\rangle &= (-1)^k \langle \mu_{\alpha+k}(y), \varphi^{(k)}(x+y)\rangle \\ &= (-1)^k \int_0^{+\infty} \varphi^{(k)}(x+y) \frac{y^{\alpha+k-1}}{\Gamma(\alpha+k)} \,\mathrm{d}x\end{aligned}$$

By differentiating under the sum, and since $\varphi \in \mathcal{S}([0,\infty))$, we easily deduce therefrom that the function

$$y \mapsto \langle \mu_\alpha(y), \varphi(x+y)\rangle$$

is smooth, with derivatives given by

$$\forall n \geq 0, \quad \frac{\mathrm{d}^n}{\mathrm{d}x^n}\langle \mu_\alpha(y), \varphi(x+y)\rangle = \langle \mu_\alpha(y), \varphi^{(n)}(x+y)\rangle.$$

We also deduce that this function satisfies the inequalities (2.2), and hence lies in $\mathcal{S}([0,\infty))$. As a consequence, the quantity

$$\langle \mu_{\alpha'}(x), \langle \mu_\alpha(y), \varphi(x+y)\rangle\rangle$$

is indeed well-defined.

There remains to prove that equality (2.7) holds. To do so, we take $k, k' \geq 0$ such that $\alpha + k \geq 0$ and $\alpha' + k' \geq 0$. Then, by applying Thm 2.1.6 k times on μ_α, and then k' times on $\mu_{\alpha'}$, we have

$$\begin{aligned}\langle \mu_{\alpha'}(x), \langle \mu_\alpha(y), \varphi(x+y)\rangle\rangle &= (-1)^k \langle \mu_{\alpha'}(x), \langle \mu_{\alpha+k}(y), \varphi^{(k)}(x+y)\rangle\rangle \\ &= (-1)^{k+k'} \langle \mu_{\alpha'+k'}(x), \frac{\mathrm{d}^{k'}}{\mathrm{d}x^{k'}}\langle \mu_{\alpha+k}(y), \varphi^{(k)}(x+y)\rangle\rangle \\ &= (-1)^{k+k'} \langle \mu_{\alpha'+k'}(x), \langle \mu_{\alpha+k}(y), \varphi^{(k+k')}(x+y)\rangle\rangle\end{aligned}$$

Now, since $\alpha + k \geq 0$ and $\alpha' + k' \geq 0$, by Remark 2.1.1, we see that this equals

$$(-1)^{k+k'} \langle \mu_{\alpha+k+\alpha'+k'}, \varphi^{(k+k')}\rangle$$

which, by a new application of Thm 2.1.9, rewrites

$$\langle \mu_{\alpha+\alpha'}, \varphi\rangle.$$

This yields the claim. □

Chapter 3

Integration by parts formulae for the laws of Bessel bridges

In this chapter, we prove integration by parts formulae (IbPF) for the laws of Bessel bridges of any dimension $\delta > 0$.

3.1 The statement

Here and in the sequel, we denote by $\mathcal{S}$ the linear span of all functionals on $C([0,1])$ of the form

$$C([0,1]) \ni X \mapsto \exp\left(-\langle m, X^2\rangle\right) \in \mathbb{R} \tag{3.1}$$

where m is a finite Borel measure on $[0,1]$. The space $\mathcal{S}$ is the space of functionals on which we will derive our IbPF for the laws of Bessel bridges.

Recalling the definition

$$\kappa(\delta) := \frac{(\delta-3)(\delta-1)}{4}, \qquad \delta \in \mathbb{R},$$

here is the one of the main theorems of this thesis:

Theorem 3.1.1. *Let $a, a' \geq 0$, $\delta \in (0,\infty) \setminus \{1,3\}$, and $k := \lfloor \frac{3-\delta}{2} \rfloor$. Then, for all $h \in C_c^2(0,1)$ and $\Phi \in \mathcal{S}$*

$$\begin{aligned} &E^\delta_{a,a'}(\partial_h \Phi(X)) + E^\delta_{a,a'}(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \Big[\mathcal{T}_b^{2k}\, \Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,\cdot\,)\Big]\, \mathrm{d}b\, \mathrm{d}r. \end{aligned} \tag{3.2}$$

On the other hand, when $\delta \in \{1,3\}$, the following formulae hold: for all $\Phi \in \mathcal{S}$,

$$E^3_{a,a'}(\partial_h \Phi(X)) + E^3_{a,a'}(\langle h'', X\rangle\, \Phi(X)) = -\frac{1}{2}\int_0^1 h_r\, \Sigma^{3,r}_{a,a'}(\Phi(X)\,|\,0\,)\, \mathrm{d}r, \tag{3.3}$$

and

$$E^1_{a,a'}(\partial_h \Phi(X)) + E^1_{a,a'}(\langle h'', X\rangle \, \Phi(X)) = \frac{1}{4}\int_0^1 h_r \, \frac{\mathrm{d}^2}{\mathrm{d}b^2}\, \Sigma^{1,r}_{a,a'}(\Phi(X)\,|\,b)\Big|_{b=0} \mathrm{d}r. \quad (3.4)$$

Concerning this statement, a few remarks are in order:

Remark 3.1.2. Note that the last integral in (3.2) is indeed convergent. Indeed, recall that $\mathcal{T}^{2k}_b\, \Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,\cdot\,)$ is the Taylor remainder of a smooth, even function. Hence, near 0, the integrand is of order $O(b^{\delta+2k-2})$. Since, $\delta + 2k - 2 > -1$, the integral is convergent at 0. On the other hand, the integrand is of order $O(b^{\delta+2k-4})$ as $b \to \infty$. Hence, since $\delta + 2k - 4 < -1$, integrability also holds at $+\infty$.

Remark 3.1.3. For all $\delta \in (1, 3)$ the right-hand side in the IbPF (3.2) takes the form

$$-\kappa(\delta)\int_0^1 h_r \int_0^\infty b^{\delta-4}\Big[\Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,b) - \Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,0)\Big]\,\mathrm{d}b\,\mathrm{d}r.$$

Note the absence of transition at the threshold $\delta = 2$. This might seem surprising given the transition that the Bessel bridges undergo at $\delta = 2$, which is the smallest value of δ satisfying

$$P^\delta_{a,a'}\,[\exists r \in]0,1[\, :\, X_r = 0] = 0.$$

This lack of transition is related to the fact that, as a consequence of Lemma 2.2.6, we have for all $\Phi \in \mathcal{S}$

$$\frac{\mathrm{d}}{\mathrm{d}b}\, \Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,b)\Big|_{b=0} = 0.$$

However, we do conjecture that a transition should occur for $\delta = 2$ at the level of the SPDEs: see Section 4.5 below.

Remark 3.1.4. Recalling the definition 2.1.3 of μ_α for $\alpha < 0$, we can write all the above IbPF above in a unified way as follows:

$$\begin{aligned} &E^\delta_{a,a'}(\partial_h \Phi(X)) + E^\delta_{a,a'}(\langle h'', X\rangle \, \Phi(X)) \\ &= -\frac{\Gamma(\delta)}{2(\delta-2)}\int_0^1 h_r \,\langle \mu_{\delta-3}, \Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,\cdot\,)\rangle\,\mathrm{d}r. \end{aligned} \quad (3.5)$$

Note that the pole at $\delta = 2$ is compensated by the zero of $\langle \mu_{\delta-3}, \Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,\cdot\,)\rangle$ at $\delta = 2$ in virtue of (2.26). Actually the proof of the formulae Theorem 3.1.1 will be based on rewriting both sides of the equalities using μ_α, for some $\alpha < 0$: see Lemma 3.3.5 and its proof. Note that in that lemma there appears $\mu_{\frac{\delta-3}{2}}$ rather than $\mu_{\delta-3}$ because, for convenience, we work there with squared Bessel processes rather than Bessel processes.

As a consequence of the above theorem, we retrieve the following known results (see Chapter 6 of [Zam17]):

Proposition 3.1.5. *Let $\Phi \in \mathcal{S}$ and $h \in C_c^2(0,1)$. Then, for all $a \geq 0$ and $\delta > 3$, the following IbPF holds*

$$E^{\delta}_{a,a}(\partial_h \Phi(X)) + E^{\delta}_{a,a}(\langle h'', X\rangle\, \Phi(X)) = -\kappa(\delta)\, E^{\delta}_{a,a}(\langle h, X^{-3}\rangle\, \Phi(X)). \tag{3.6}$$

Moreover, for $\delta = 3$, the following IbPF holds

$$\begin{aligned} &E^{3}_{a,a}(\partial_h \Phi(X)) + E^{3}_{a,a}(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\int_0^1 \mathrm{d}r\, h_r\, \gamma(r,a)\, E^{3}_{a,a}[\Phi(X) \,|\, X_r = 0] \end{aligned} \tag{3.7}$$

where, for all $(r,a) \in (0,1) \times \mathbb{R}_+$

$$\gamma(r,a) := \frac{1}{\sqrt{2\pi r^3(1-r)^3}} \left(\mathbf{1}_{a=0} + \mathbf{1}_{a>0} \frac{2a^2 \exp\left(-\frac{a^2}{2r(1-r)}\right)}{1 - e^{-2a^2}} \right).$$

Proof. For $\delta > 3$ we have $k := \lfloor \frac{3-\delta}{2} \rfloor < 0$, and by (3.2)

$$\begin{aligned} &E^{\delta}_{a,a}(\partial_h \Phi(X)) + E^{\delta}_{a,a}(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4}\, \Sigma^{\delta,r}_{a,a}(\Phi(X) \,|\, b)\, \mathrm{d}b\, \mathrm{d}r \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{-3}\, p^{\delta,r}_{a,a}(b)\, E^{\delta}_{a,a}[\Phi(X) \,|\, X_r = b]\, \mathrm{d}b\, \mathrm{d}r \\ &= -\kappa(\delta)\, E^{\delta}_{a,a}(\langle h, X^{-3}\rangle\, \Phi(X)). \end{aligned}$$

For $\delta = 3$, it suffices to note that, for all $r \in (0,1)$

$$\frac{1}{2} \lim_{\epsilon \to 0} \frac{p^{3,r}_{a,a}(\epsilon)}{\epsilon^2} = \gamma(r,a),$$

so that

$$\frac{1}{2}\, \Sigma^{3,r}_{a,a}(\Phi(X) \,|\, 0\,) = \gamma(r,a) E^{3}_{a,a}[\Phi(X) \,|\, X_r = 0].$$

and the proof is complete. □

Remark 3.1.6. In [Zam05] for the reflecting Brownian motion, and then in [GV16] for the Reflecting Brownian bridge, a different formula was proved in the case

$\delta = 1$. In our present notations, for $(\beta_r)_{r\in[0,1]}$ a Brownian bridge and $X := |\beta|$, the formula reads

$$\mathbb{E}(\partial_h \Phi(X)) + \mathbb{E}(\langle h'', X\rangle \, \Phi(X)) = \lim_{\epsilon\to 0} 2\,\mathbb{E}\left(\Phi(X)\int_0^1 h_r \left[\left(\dot{\beta}^\epsilon_r\right)^2 - c^\epsilon_r\right] \mathrm{d}L^0_r\right), \quad (3.8)$$

where $\Phi : H \to \mathbb{R}$ is any Lipschitz function, $h \in C^2_0(0,1)$, L^0 is the standard local time of β at 0 and for some even smooth mollifier ρ_ϵ we set

$$\beta^\epsilon := \rho_\epsilon * \beta, \qquad c^\epsilon_r := \frac{\|\rho\|^2_{L^2(0,1)}}{\epsilon}.$$

The reason why (3.8) is strictly weaker than (3.15), is that the former depends explicitly on β, while the latter is written only in terms of X. This will become crucial when we compute the SPDE satisfied by u for $\delta = 1$ in Theorem 4.2.8 below. However, (3.8) possesses a dynamical interpretation in terms of an infinite-dimensional Itô-Tanaka formula, see [Zam06]. It also possesses a natural generalization to all integer values of δ, see Chapter 6 below: these higher-dimensional counterparts also differ - at first sight - from the corresponding IbPF stated in Theorem 3.1.1 above.

3.2 Proof of the IbPF : the case of homogeneous Dirichlet boundary values

Before giving the general proof of Theorem 3.1.1 above, we first consider the case of Bessel bridges from 0 to 0, P^δ, for $\delta > 0$. Indeed, in this case, the proof is much easier and possesses a nice interpretation in terms of solutions to Sturm-Liouville equations. The general case will be treated separately.

We first state a differential relation satisfied by the product of the functions ψ and $\hat{\psi}$ associated as above with a finite Borel measure m on $[0,1]$. This relation is the skeleton of all the IbPF for P^δ, $\delta > 0$: the latter will all be deduced from the former with a simple multiplication by a constant (depending on the parameter δ).

Lemma 3.2.1. *Let m be a finite Borel measure on $[0,1]$, and consider the functions ψ and $\hat{\psi}$ as in (2.22) and (2.23). Then, for all $h \in C^2_c(0,1)$ and $\delta > 0$, the following equality holds*

$$\int_0^1 \sqrt{\psi_r \hat{\psi}_r}\,(h''_r \,\mathrm{d}r - 2h_r\, m(\mathrm{d}r)) = -\frac{1}{4}\psi_1^2 \int_0^1 h_r (\psi_r \hat{\psi}_r)^{-\frac{3}{2}} \,\mathrm{d}r. \qquad (3.9)$$

Proof. Performing an integration by parts, we can rewrite the left-hand side as

$$\int_0^1 h_r \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\, m(\mathrm{d}r) \right) \left(\psi_r \hat{\psi}_r \right)^{\frac{1}{2}}.$$

Note that here we are integrating wrt the signed measure

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\, m(\mathrm{d}r) \right) \left(\psi_r \hat{\psi}_r \right)^{\frac{1}{2}} = \frac{\mathrm{d}^2}{\mathrm{d}r^2} \left(\psi_r \hat{\psi}_r \right)^{\frac{1}{2}} - 2 \left(\psi_r \hat{\psi}_r \right)^{\frac{1}{2}} m(\mathrm{d}r).$$

Now, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2} \left(\psi \hat{\psi} \right)^{\frac{1}{2}} = \frac{1}{2} \frac{\psi'' \hat{\psi} + 2\psi' \hat{\psi}' + \psi \hat{\psi}''}{(\psi \hat{\psi})^{\frac{1}{2}}} - \frac{1}{4} \frac{(\psi' \hat{\psi} + \psi \hat{\psi}')^2}{(\psi \hat{\psi})^{3/2}}.$$

Recalling that $\psi'' = 2\psi\, m$ and $\hat{\psi}'' = 2\hat{\psi}\, m$, we obtain

$$\begin{aligned} \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\, m(\mathrm{d}r) \right) \left(\psi \hat{\psi} \right)^{\frac{1}{2}} &= \frac{\psi' \hat{\psi}' \psi \hat{\psi} - \frac{1}{4} (\psi' \hat{\psi} + \psi \hat{\psi}')^2}{(\psi \hat{\psi})^{3/2}} \\ &= - \frac{1}{4} \frac{(\psi' \hat{\psi} - \psi \hat{\psi}')^2}{(\psi \hat{\psi})^{3/2}}. \end{aligned}$$

Using the expressions (2.22) and (2.23) for ψ and $\hat{\psi}$, we easily see that

$$\psi_r' \hat{\psi}_r - \psi \hat{\psi}_r' = \psi_1, \qquad r \in (0,1). \tag{3.10}$$

Hence, we obtain the following equality of signed measures:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\, m \right) \left(\psi \hat{\psi} \right)^{\frac{1}{2}} = -\frac{1}{4} \frac{\psi_1^2}{(\psi_r \hat{\psi}_r)^{3/2}}\, \mathrm{d}r.$$

Consequently, the left-hand side in (3.9) is equal to

$$-\frac{1}{4} \psi_1^2 \int_0^1 \mathrm{d}r\ h_r \left(\psi_r \hat{\psi}_r \right)^{-3/2}.$$

The claim follows. ☐

As a consequence, we obtain the following preliminary result.

Lemma 3.2.2. *Let m be a finite measure on $[0,1]$, and let $\Phi : C([0,1]) \to \mathbb{R}$ be the functional thereto associated as in (3.1). Then, for all $\delta > 0$ and $h \in C_c^2(0,1)$,*

$$\begin{aligned} & E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X \rangle\, \Phi(X)) = \\ & = -\frac{\Gamma(\frac{\delta+1}{2})}{2^{\frac{3}{2}}\, \Gamma(\frac{\delta}{2})}\, \psi_1^{-\frac{\delta-3}{2}} \int_0^1 h_r \left(\psi_r \hat{\psi}_r \right)^{-\frac{3}{2}} \mathrm{d}r, \end{aligned} \tag{3.11}$$

where ψ and $\hat{\psi}$ are associated with m as in (2.22) and (2.23).

Proof. By the expression (3.1) for Φ, we have

$$\partial_h \Phi(X) = -2\langle Xh, m\rangle\, \Phi(X).$$

Therefore

$$E^\delta(\partial_h\Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = Q^\delta\left[\left(\langle h'', \sqrt{X}\rangle - 2\langle h\sqrt{X}, m\rangle\right)\, e^{-\langle m,X\rangle}\right] =$$
$$= \int_0^1 (h''_r\,\mathrm{d}r - 2h_r\, m(\mathrm{d}r)) \int_0^{+\infty} \Gamma\left(\frac{\delta}{2}, \frac{1}{2r(1-r)}\right)(\mathrm{d}b)\sqrt{b}\, Q^\delta\left[e^{-\langle m,X\rangle} \mid X_r = b\right].$$

By (2.28) we obtain

$$E^\delta(\partial_h\Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) =$$
$$= \int_0^1 (h''_r\,\mathrm{d}r - 2h_r\, m(\mathrm{d}r))\, \frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta}{2})} \left(\frac{C_r}{2}\,\psi_1^\delta\right)^{-\frac{1}{2}} \int_0^{+\infty} \Gamma\left(\frac{\delta+1}{2}, \frac{C_r}{2}\right)(\mathrm{d}b)$$
$$= \sqrt{2}\, \frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta}{2})} \psi_1^{-\frac{\delta+1}{2}} \int_0^1 \mathrm{d}r\ (h''_r\,\mathrm{d}r - 2h_r\, m(\mathrm{d}r))\, \sqrt{\psi_r\hat{\psi}_r}.$$

Finally, by (3.9), the latter expression is equal to

$$-\frac{\Gamma(\frac{\delta+1}{2})}{2^{\frac{3}{2}}\,\Gamma(\frac{\delta}{2})}\, \psi_1^{-\frac{\delta-3}{2}} \int_0^1 h_r \left(\psi_r\hat{\psi}_r\right)^{-\frac{3}{2}}\, \mathrm{d}r$$

and the proof is complete. □

Apart from the above lemma, the proof of the IbPF for P^δ, $\delta > 0$, will require integral expressions for negative Gamma values. For all $x \in \mathbb{R}$ we set $\lfloor x \rfloor := \sup\{k \in \mathbb{Z} : k \le x\}$. We also use the notation $\mathbb{Z}^- := \{n \in \mathbb{Z} : n \le 0\}$.

Lemma 3.2.3. *For all $x \in \mathbb{R} \setminus \mathbb{Z}^-$*

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathcal{T}_t^{\lfloor -x \rfloor}(e^{-\,\cdot\,})\,\mathrm{d}t.$$

Proof. By Remark 2.1.7 we have

$$\int_0^\infty t^{x-1}\mathcal{T}_t^{\lfloor -x \rfloor}(e^{-\,\cdot\,})\,\mathrm{d}t = \Gamma(x)\,\langle \mu_\alpha, e^{-\cdot}\rangle = \Gamma(x)\,1^x = \Gamma(x),$$

and the claim follows. □

From Lemma 3.2.3 we obtain for all $C > 0$, $x \in \mathbb{R} \setminus \mathbb{Z}^-$

$$\Gamma(x)\, C^{-x} = 2^{1-x} \int_0^{+\infty} b^{2x-1} \left(e^{-C\frac{b^2}{2}} - \sum_{0 \le j \le \lfloor -x \rfloor} \frac{(-C)^j b^{2j}}{2^j j!} \right) \mathrm{d}b \tag{3.12}$$

by a simple change of variable $t = Cb^2/2$. Then (3.12) can be rewritten as follows:

$$\Gamma(x)\, C^{-x} = 2^{1-x} \int_0^{+\infty} b^{2x-1}\, \mathcal{T}_b^{2\lfloor -x \rfloor} \left(e^{-C\frac{(\cdot)^2}{2}} \right) \mathrm{d}b, \quad x \in \mathbb{R} \setminus \mathbb{Z}^-. \tag{3.13}$$

Theorem 3.2.4. *Let $\delta > 0$, $\delta \notin \{1,3\}$, and $k := \lfloor \frac{3-\delta}{2} \rfloor \le 1$. Then*

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \Big[\mathcal{T}_b^{2k}\, \Sigma_r^\delta(\Phi(X) \,|\, \cdot) \Big] \mathrm{d}b\, \mathrm{d}r. \end{aligned} \tag{3.14}$$

Proof of Theorem 3.2.4. Let $\delta > 0$ and $\delta \notin \{1,3\}$. Then by (3.11)

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2}\, \Gamma(\frac{\delta}{2})} \int_0^1 h_r \left(\frac{\psi_1}{\psi_r \hat{\psi}_r} \right)^{\frac{3-\delta}{2}} \left(\psi_r \hat{\psi}_r \right)^{-\frac{\delta}{2}} \mathrm{d}r \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2}\, \Gamma(\frac{\delta}{2})} \int_0^1 h_r\, C_r^{\frac{3-\delta}{2}}\, D_r^{\delta/2}\, \mathrm{d}r \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{2^{3/2} \Gamma(\frac{\delta}{2})} \frac{2^{\frac{5-\delta}{2}}}{\Gamma\left(\frac{\delta-3}{2}\right)} \int_0^1 h_r\, D_r^{\delta/2} \int_0^\infty b^{\delta-4}\, \mathcal{T}_b^{2k} e^{-\frac{C_r}{2}(\cdot)^2}\, \mathrm{d}b\, \mathrm{d}r, \end{aligned}$$

where we used (3.13) with $C = C_r$ and $x = \frac{\delta-3}{2}$ to obtain the last line. Recalling the expression (2.25) for $\Sigma_r^\delta(\Phi(X) \,|\, b)$, we thus obtain

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &= -\frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta-3}{2})} \int_0^1 h_r \int_0^\infty b^{\delta-4}\, \mathcal{T}_{2b}^k\, \Sigma_r^\delta(\Phi(X) \,|\, b)\, \mathrm{d}b\, \mathrm{d}r. \end{aligned}$$

Now, since $\delta \notin \{1,3\}$,

$$\Gamma(\tfrac{\delta+1}{2}) = \tfrac{\delta-1}{2}\, \Gamma(\tfrac{\delta-1}{2}) = \tfrac{\delta-1}{2}\, \tfrac{\delta-3}{2}\, \Gamma(\tfrac{\delta-3}{2}) = \kappa(\delta)\, \Gamma(\tfrac{\delta-3}{2}).$$

Therefore $\frac{\Gamma(\frac{\delta+1}{2})}{\Gamma(\frac{\delta-3}{2})} = \kappa(\delta)$ and we obtain the claim. □

There remains to treat the critical cases $\delta \in \{1,3\}$.

Theorem 3.2.5. *Let $\Phi \in \mathcal{S}$ and $h \in C_c^2(0,1)$. The following IbPF holds*

$$E^3(\partial_h \Phi(X)) + E^3(\langle h'', X\rangle \, \Phi(X)) = -\frac{1}{2}\int_0^1 \mathrm{d}r\, h_r\, \Sigma_r^3(\Phi(X)\,|\,0),$$

$$E^1(\partial_h \Phi(X)) + E^1(\langle h'', X\rangle \, \Phi(X)) = \frac{1}{4}\int_0^1 \mathrm{d}r\, h_r\, \frac{\mathrm{d}^2}{\mathrm{d}b^2}\, \Sigma_r^1(\Phi(X)\,|\,b)\bigg|_{b=0}. \tag{3.15}$$

Proof. By linearity, we may assume that Φ is of the form (3.1). For $\delta = 3$ we have by (3.11)

$$E^3(\partial_h \Phi(X)) + E^3(\langle h'', X\rangle \, \Phi(X)) = -\frac{1}{2^{\frac{3}{2}}\,\Gamma(\frac{3}{2})}\int_0^1 h_r \left(\psi_r \hat{\psi}_r\right)^{-\frac{3}{2}} \mathrm{d}r.$$

By (2.25) this equals

$$-\frac{1}{2}\int_0^1 \mathrm{d}r\, h_r\, \Sigma_r^3(\Phi(X)\,|\,0)$$

and the proof is complete. For $\delta = 1$, by (3.11), we have

$$E^1(\partial_h \Phi(X) + \langle h'', X\rangle \, \Phi(X)) = -\frac{1}{2\sqrt{2\pi}}\,\psi_1 \int_0^1 h_r \left(\psi_r \hat{\psi}_r\right)^{-\frac{3}{2}} \mathrm{d}r.$$

But by (2.25) we have, for all $r \in (0,1)$

$$\frac{\mathrm{d}^2}{\mathrm{d}b^2}\, \Sigma_r^1(\Phi(X)\,|\,b)\bigg|_{b=0} = -\frac{C_r\, D_r^{\frac{1}{2}}}{2^{-\frac{1}{2}}\,\Gamma(\frac{1}{2})} = -\sqrt{\frac{2}{\pi}}\,\psi_1 \left(\psi_r \hat{\psi}_r\right)^{-\frac{3}{2}}. \tag{3.16}$$

The claimed IbPF follows. □

3.3 The case of general boundary values

This section are devoted to the proof of the above IbPF in the general case. We shall actually first state and prove similar IbPF for the laws of Bessel processes, which are nicer to handle, and then obtain the results for Bessel bridges by conditioning.

3.3.1 Case of unconstrained Bessel processes

Definition 3.3.1. For all $a, b \geq 0$ and $r \in (0,1)$, we consider the measure $\Sigma_a^{\delta,r}(\mathrm{d}X\,|\,b)$ on $C([0,1])$ defined by

$$\Sigma_a^{\delta,r}(\mathrm{d}X\,|\,b) := \frac{p_r^{\delta}(a,b)}{b^{\delta-1}}\, E_a^{\delta}[\,\cdot\,|\,X_r = b]. \tag{3.17}$$

Lemma 3.3.2. *For all $r \in (0,1)$, $\delta > 0$ and $a, b \geq 0$, the following holds*

$$\int \exp(-\langle m, X^2 \rangle)\ \Sigma_a^{\delta,r}(\mathrm{d}X \,|\, b) = 2 \exp\left(\frac{a^2}{2}\phi_0'\right) \phi_1^{\delta/2} \phi_r^{-2} \frac{q_{\varrho_r}^{\delta}\left(a^2, \frac{b^2}{\phi_r^2}\right)}{b^{\delta-2}} \tag{3.18}$$

In particular, for $a = 0$, we have

$$\int \exp(-\langle m, X^2 \rangle)\ \Sigma_0^{\delta,r}(\mathrm{d}X \,|\, b) = \frac{1}{2^{\frac{\delta}{2}-1}\,\Gamma(\frac{\delta}{2})} \exp\left(-\frac{b^2}{2\phi_r^2 \varrho_r}\right) \left(\frac{\phi_1}{\phi_r^2 \varrho_r}\right)^{\delta/2}.$$

Proof. These equalities follow from Lemma 2.2.6 upon noticing that, for all $a \geq 0$

$$\Sigma_a^{\delta,r}(\mathrm{d}X \,|\, b) = \int_0^\infty \Sigma_{a,a'}^{\delta,r}(\mathrm{d}X \,|\, b)\, p_1^{\delta}(a, a')\ \mathrm{d}a'.$$

□

Theorem 3.3.3. *Let $a \geq 0$, $\delta \in (0,\infty) \setminus \{1,3\}$, and $k := \lfloor \frac{3-\delta}{2} \rfloor \leq 1$. Then, for all $\Phi \in \mathcal{S}$*

$$\begin{aligned} &E_a^\delta(\partial_h \Phi(X)) + E_a^\delta(\langle h'', X \rangle\, \Phi(X)) = \\ &-\kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \left[\mathcal{T}_b^{2k}\, \Sigma_a^{\delta,r}(\Phi(X) \,|\, \cdot\,)\right] \mathrm{d}b\, \mathrm{d}r. \end{aligned} \tag{3.19}$$

On the other hand, when $\delta \in \{1,3\}$, the following formulae hold: for all $\Phi \in \mathcal{S}$,

$$E_a^3(\partial_h \Phi(X)) + E_a^3(\langle h'', X \rangle\, \Phi(X)) = -\frac{1}{2} \int_0^1 h_r\, \Sigma_a^{3,r}(\Phi(X) \,|\, 0\,)\, \mathrm{d}r, \tag{3.20}$$

and

$$E_a^1(\partial_h \Phi(X)) + E_a^1(\langle h'', X \rangle\, \Phi(X)) = \frac{1}{4} \int_0^1 h_r \frac{\mathrm{d}^2}{\mathrm{d}b^2}\, \Sigma_a^{1,r}(\Phi(X) \,|\, b)\bigg|_{b=0} \mathrm{d}r. \tag{3.21}$$

Proof. By linearity, it suffices to prove formula (3.19) for Φ of the form (3.1). So let m be a finite Borel measure on $[0,1]$, and let Φ be the functional thereto associated. We have

$$\begin{aligned} &E_a^\delta(\partial_h \Phi(X)) + E_a^\delta(\langle h'', X \rangle\, \Phi(X)) = E_a^\delta\left(\langle h'' - 2m, hX \rangle \Phi(X)\right) \\ &= \int_0^1 \left(\mathrm{d}r\, h_r \frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2m(\,\mathrm{d}r)\, h_r\right) E_a^\delta[X_r \exp\left(-\langle m, X^2 \rangle\right)]. \end{aligned}$$

Lemma 3.3.4.

$$E_a^\delta[X_r \exp(-\langle m, X^2 \rangle)] = \exp\left(\frac{a^2}{2}\phi'(0)\right) \phi_1^{\delta/2}\, \phi_r\, E_a^\delta\left(X_{\varrho_r}\right). \tag{3.22}$$

Proof. We have

$$E^\delta_a[X_r \exp(-\langle m, X^2\rangle)] = Q^\delta_{a^2}\left[\sqrt{X_r}\exp(-\langle m, X\rangle)\right]$$
$$= \exp\left(\frac{a^2}{2}\phi'(0)\right)\phi_1^{\delta/2}\,\phi_r\,Q^\delta_{a^2}\left(\sqrt{X_{\varrho_r}}\right),$$

where we used Lemma 2.2.3 to obtain the second equality. The claim follows. □

We rewrite (3.22) as follows

$$E^\delta_a[X_r \exp(-\langle m, X^2\rangle)] = K(a,m)\,\phi_r\,\zeta_{\varrho_r},$$

where

$$K(a,m) := \exp\left(\frac{a^2}{2}\phi'(0)\right)\phi_1^{\delta/2}$$

is a constant which does not depend on r and

$$\zeta_t := Q^\delta_{a^2}\left(\sqrt{X_t}\right) = E^\delta_a\left(X_t\right), \quad t \geq 0.$$

To compute the left-hand side of (3.19), it therefore suffices to compute the following distribution on $(0,1)$:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\,m(\,\mathrm{d}r)\right)(\phi_r\zeta_{\varrho_r}).$$

We recall that by (2.15)

$$\varrho_r' = \phi_r^{-2}.$$

By the Leibniz formula, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}(\phi_r\zeta_{\varrho_r}) = \phi_r'\zeta_{\varrho_r} + \phi_r\frac{\zeta_{\varrho_r}'}{\phi_r^2} = \phi_r'\zeta_{\varrho_r} + \frac{\zeta_{\varrho_r}'}{\phi_r},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(\phi_r\zeta_{\varrho_r}) = \phi_r''\zeta_{\varrho_r} + \phi_r'\frac{\zeta_{\varrho_r}'}{\phi_r^2} - \phi_r'\frac{\zeta_{\varrho_r}'}{\phi_r^2} + \frac{\zeta_{\varrho_r}''}{\phi_r^3} = \phi_r''\zeta_{\varrho_r} + \frac{\zeta_{\varrho_r}''}{\phi_r^3}.$$

Consequently, recalling that $\phi'' = 2\phi\,m$, we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - 2\,m(\,\mathrm{d}r)\right)(\phi_r\zeta_{\varrho_r}) = \frac{\zeta_{\varrho_r}''}{\phi_r^3}.$$

Finally, we thus obtain the following expression for the left-hand sides of (3.19), (3.20) and (3.21):

$$E^\delta_a(\partial_h\Phi(X)) + E^\delta_a(\langle h'', X\rangle\,\Phi(X))$$
$$= K(a,m)\int_0^1 \mathrm{d}r\,h_r\,\phi_r^{-3}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}E^\delta_a\left(X_t\right)\Big|_{t=\varrho_r}. \tag{3.23}$$

We now compute their respective right-hand sides. Recall that, by (3.18), we have, for all $b \geq 0$

$$\Sigma_a^{\delta,r}(\Phi(X)\,|\,b) = 2\exp\left(\frac{a^2}{2}\phi_0'\right)\phi_1^{\delta/2}\phi_r^{-2}\frac{q_{\varrho_r}^{\delta}\left(a^2,\frac{b^2}{\phi_r^2}\right)}{b^{\delta-2}}$$
$$= 2K(a,m)\,\phi_r^{-2}\,\frac{q_{\varrho_r}^{\delta}\left(a^2,\frac{b^2}{\phi_r^2}\right)}{b^{\delta-2}}.$$

Therefore, denoting by f the function defined by

$$f(y) := \frac{q_t^{\delta}(a^2,y)}{y^{\delta/2-1}}, \quad y > 0,$$

and extended by continuity at 0, we have

$$\Sigma_a^{\delta,r}(\Phi(X)\,|\,b) = 2K(a,m)\,\phi_r^{-\delta}\,f\left(\frac{b^2}{\phi_r^2}\right), \tag{3.24}$$

for all $b \geq 0$. Now, we first assume that $\delta \notin \{1,3\}$, and compute the right-hand side of (3.19). Note that, by (3.24), and performing the change of variable $y := \frac{b^2}{\phi_r^2}$, we obtain

$$\int_0^{\infty} \mathrm{d}b\; b^{\delta-4}\left[\mathcal{T}_b^{2k}\,\Sigma_a^{\delta,r}(\Phi(X)\,|\,\cdot\,)\right] = K(a,m)\,\phi_r^{-3}\int_0^{\infty}\mathrm{d}y\; y^{\frac{\delta-3}{2}-1}\,\mathcal{T}_y^{k} f, \tag{3.25}$$

Recalling the definition of $\mu_{\frac{\delta-3}{2}}$, we can rewrite the last integral of (3.25) as

$$\Gamma\left(\frac{\delta-3}{2}\right)\left\langle \mu_{\frac{\delta-3}{2}}(y), y^{1-\delta/2}q_t^{\delta}(a^2,y)\right\rangle.$$

Since $\Gamma\left(\frac{\delta+1}{2}\right) = \kappa(\delta)\,\Gamma\left(\frac{\delta-3}{2}\right)$, we thus deduce that the right-hand side of (3.19) equals

$$-\,K(a,m)\,\phi_r^{-3}\,\Gamma\left(\frac{\delta+1}{2}\right)\left\langle \mu_{\frac{\delta-3}{2}}(y), y^{1-\delta/2}q_t^{\delta}(a^2,y)\right\rangle. \tag{3.26}$$

Supposing now that $\delta = 3$, by the expression (3.24), we see that the right-hand side of (3.20) equals

$$-K(a,m)\,\phi_r^{-3}\,f(0),$$

which is precisely the quantity (3.26) with $\delta = 3$.

Finally, supposing that $\delta = 1$, by (3.24), we see that the right-hand side of (3.21) equals

$$K(a,m)\,\phi_r^{-3}\,f'(0),$$

which also coincides with the quantity (3.26) with $\delta = 1$. □

In conclusion, comparing the expressions (3.23) and (3.26), we see that the claimed IbPF will be proven, provided that we show the following result:

Lemma 3.3.5. *For all $t > 0$ and $a \geq 0$, we have*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} E_a^\delta (X_t) = -\Gamma\left(\frac{\delta+1}{2}\right) \left\langle \mu_{\frac{\delta-3}{2}}(y)\, ,\, \frac{q_t^\delta(x,y)}{y^{\delta/2-1}} \right\rangle,$$

where $x := a^2$.

Proof. We have

$$\begin{aligned} E_a^\delta (X_t) = Q_x^\delta\left(\sqrt{X_t}\right) &= \int_0^\infty y^{\frac{\delta+1}{2}-1} \left(y^{1-\delta/2} q_t^\delta(x,y)\right) \mathrm{d}z \\ &= \Gamma\left(\frac{\delta+1}{2}\right) \left\langle \mu_{\frac{\delta+1}{2}}, y^{1-\delta/2} q_t^\delta(x,y) \right\rangle. \end{aligned}$$

Therefore, differentiating in t, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_x^\delta\left(\sqrt{X_t}\right) = \Gamma\left(\frac{\delta+1}{2}\right) \left\langle \mu_{\frac{\delta+1}{2}}, \partial_t \left(y^{1-\delta/2} q_t^\delta(x,y)\right) \right\rangle.$$

Note that the interversion of $\frac{\mathrm{d}}{\mathrm{d}t}$ and the brackets is allowed. Indeed, the function

$$y \mapsto \partial_t \left(y^{1-\delta/2} q_t^\delta(x,y)\right)$$

is in $\mathcal{S}([0,\infty))$, so that

$$\int_0^\infty y^{\frac{\delta+1}{2}-1} \left|\partial_t \left(y^{1-\delta/2} q_t^\delta(x,y)\right)\right| \mathrm{d}y < \infty.$$

Now, we intend to re-express the time-derivative of $q_t^\delta(x,y)$. To do so, we recall that the following equation holds:

$$\partial_t q_t^\delta(x,y) = (4-\delta)\, \partial_y q_t^\delta(x,y) + 2y\, \partial_y^2 q_t^\delta(x,y). \tag{3.27}$$

Therefore, by the Leibniz formula, we have

$$\partial_t \left(\frac{q_t^\delta(x,y)}{y^{\delta/2-1}}\right) = 2\, \partial_y \left(y^{2-\delta/2}\, \partial_y q_t^\delta(x,y)\right). \tag{3.28}$$

Hence, applying the distribution $\mu_{\frac{\delta+1}{2}}$, and recalling Theorem 2.1.6, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_x^\delta\left(\sqrt{X_t}\right) = -2\,\Gamma\left(\frac{\delta+1}{2}\right) \left\langle \mu_{\frac{\delta-1}{2}}, y^{2-\delta/2}\, \partial_y q_t^\delta(x,y) \right\rangle. \tag{3.29}$$

We now intend to re-express the right-hand side of (3.29) in order to get rid of the derivative ∂_y. To do so, we note that

$$y^{2-\delta/2}\,\partial_y q_t^\delta(x,y) = \partial_y\left(y^{2-\delta/2} q_t^\delta(x,y)\right) - \left(2-\frac{\delta}{2}\right) y^{1-\delta/2} q_t^\delta(x,y). \tag{3.30}$$

Hence, by Theorem 2.1.6, we have

$$\left\langle \mu_{\frac{\delta-1}{2}}, y^{2-\delta/2}\,\partial_y q_t^\delta(x,y)\right\rangle = -\left\langle \mu_{\frac{\delta-3}{2}}, y^{2-\delta/2} q_t^\delta(x,y)\right\rangle - \left(2-\frac{\delta}{2}\right)\left\langle \mu_{\frac{\delta-1}{2}}, y^{1-\delta/2} q_t^\delta(x,y)\right\rangle. \tag{3.31}$$

Now, applying Lemma 2.1.8 with $\alpha = \frac{\delta-3}{2}$ and f the smooth function defined by

$$f(y) := y^{2-\delta/2} q_t^\delta(x,y), \quad y \in \mathbb{R}_+,$$

we can rewrite equation (3.31) as

$$\begin{aligned}\left\langle \mu_{\frac{\delta-1}{2}}, y^{2-\delta/2}\,\partial_y q_t^\delta(x,y)\right\rangle &= -\left(\frac{\delta-3}{2}+2-\frac{\delta}{2}\right)\left\langle \mu_{\frac{\delta-1}{2}}, y^{1-\delta/2} q_t^\delta(x,y)\right\rangle \\ &= -\frac{1}{2}\left\langle \mu_{\frac{\delta-1}{2}}, y^{1-\delta/2} q_t^\delta(x,y)\right\rangle.\end{aligned} \tag{3.32}$$

We thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\, Q_x^\delta\left(\sqrt{X_t}\right) = \Gamma\left(\frac{\delta+1}{2}\right)\left\langle \mu_{\frac{\delta-1}{2}},\, y^{1-\delta/2} q_t^\delta(x,y)\right\rangle$$

Hence, differentiating in t a second time, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\, Q_{x,y}^\delta\left(\sqrt{X_t}\right) = \Gamma\left(\frac{\delta+1}{2}\right)\left\langle \mu_{\frac{\delta-1}{2}}, \partial_t\left(y^{1-\delta/2} q_t^\delta(x,y)\right)\right\rangle \tag{3.33}$$

The fact that we can differentiate in t inside the brackets is justified as before. Now, recalling the differential relation (3.28), and applying Theorem 2.1.6, we obtain

$$\left\langle \mu_{\frac{\delta-1}{2}}, \partial_t\left(y^{1-\delta/2} q_t^\delta(x,y)\right)\right\rangle = -2\left\langle \mu_{\frac{\delta-3}{2}}, y^{2-\delta/2}\,\partial_y q_t^\delta(x,y)\right\rangle$$

As previously, we re-express the right-hand side to get rid of the derivative. By (3.30), and applying Thm 2.1.6, we see that

$$\left\langle \mu_{\frac{\delta-3}{2}}, y^{2-\delta/2}\,\partial_y q_t^\delta(x,y)\right\rangle = -\left\langle \mu_{\frac{\delta-5}{2}}, y^{2-\delta/2} q_t^\delta(x,y)\right\rangle - \left(2-\frac{\delta}{2}\right)\left\langle \mu_{\frac{\delta-3}{2}}, y^{1-\delta/2} q_t^\delta(x,y)\right\rangle.$$

Upon applying Lemma 2.1.8 to the first term in the right-hand side, we thus obtain

$$\begin{aligned}\left\langle \mu_{\frac{\delta-3}{2}}, y^{2-\delta/2}\,\partial_y q_t^\delta(x,y)\right\rangle &= -\left(\frac{\delta-5}{2}+2-\frac{\delta}{2}\right)\left\langle \mu_{\frac{\delta-3}{2}}, y^{1-\delta/2} q_t^\delta(x,y)\right\rangle \\ &= \frac{1}{2}\left\langle \mu_{\frac{\delta-3}{2}}, y^{1-\delta/2} q_t^\delta(x,y)\right\rangle\end{aligned}$$

Finally, we thus obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} Q_x^\delta \left(\sqrt{X_t}\right) = -\Gamma\left(\frac{\delta+1}{2}\right) \left\langle \mu_{\frac{\delta-3}{2}}, y^{1-\delta/2} q_t^\delta(x,y) \right\rangle,$$

which yields the claim. □

3.3.2 The case of bridges

Now we finally prove the IbPF associated with Bessel bridges stated in Theorem 3.1.1. This will just follow from Theorem 3.3.3 by conditioning on the value of X_1.

Proof of Theorem 3.1.1. Let $\Phi \in \mathcal{S}$ and $h \in C_c^2(0,1)$. Then, for any $\lambda \geq 0$, we consider the functional $\Psi : C([0,1]) \to \mathbb{R}$ defined as

$$\Psi(X) := \Phi(X) e^{-\lambda X_1^2}, \quad X \in C([0,1]).$$

Note that Ψ is an element of $\mathcal{S}$, since one can write $X_1^2 := \int_0^1 X_t^2 \, \mathrm{d}m(X)$, where $m := \delta_1$ is the Dirac measure at 1. Therefore, Ψ satisfies the IbPF stated in Theorem 3.3.3. Moreover, since $h_1 = 0$, we have

$$\forall X \in C([0,1]), \quad \partial_h \Psi(X) = \partial_h \Phi(X) e^{-\lambda X_1^2}.$$

Therefore, assuming for example that $\delta \notin \{1,3\}$, it holds that

$$\begin{aligned} &E_a^\delta(\partial_h \Phi(X) e^{-\lambda X_1^2}) + E_a^\delta(\langle h'', X\rangle \, \Phi(X) e^{-\lambda X_1^2}) = \\ &- \kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \Big[\mathcal{T}_b^{2k} \, \Sigma_a^{\delta,r}(\Phi(X) e^{-\lambda X_1^2} \,|\, \cdot\,)\Big] \, \mathrm{d}b \, \mathrm{d}r. \end{aligned} \tag{3.34}$$

By conditioning on the value of X_1, the left-hand side of this equality can be rewritten as

$$\int_0^\infty p_1^\delta(a,a') e^{-\lambda a'^2} \left(E_{a,a'}^\delta(\partial_h \Phi(X)) + E_{a,a'}^\delta(\langle h'', X\rangle \, \Phi(X))\right) \mathrm{d}a'.$$

On the other hand, for all $r \in (0,1)$ and $b \geq 0$, we have, by the same type of conditioning

$$E_a^\delta(\Phi(X) e^{-\lambda X_1^2} \,|\, X_r = b) = \int_0^\infty p_{1-r}^\delta(b,a') \, e^{-\lambda a'^2} \, E_{a,a'}^\delta(\Phi(X) | X_r = b) \, \mathrm{d}a',$$

whence we deduce that

$$\Sigma_a^{\delta,r}(\Phi(X) e^{-\lambda X_1^2} \,|\, b\,) = \int_0^\infty p_1^\delta(a,a') e^{-\lambda a'^2} \, \Sigma_{a,a'}^{\delta,r}(\Phi(X) \,|\, b\,) \, \mathrm{d}a'.$$

Consequently, Equation (3.34) above can be rewritten

$$\int_0^\infty p_1^\delta(a,a')e^{-\lambda a'^2}\left(E^\delta_{a,a'}(\partial_h\Phi(X)) + E^\delta_{a,a'}(\langle h'', X\rangle\,\Phi(X))\right)\mathrm{d}a' =$$
$$-\kappa(\delta)\int_0^\infty p_1^\delta(a,a')e^{-\lambda a'^2}\left(\int_0^1 h_r\int_0^\infty b^{\delta-4}\Big[\mathcal{T}_b^{2k}\,\Sigma^{\delta,r}_{a,a'}(\Phi(X)\,|\,\cdot)\Big]\,\mathrm{d}b\,\mathrm{d}r\right)\mathrm{d}a'.$$

Note that this equality holds true for any $\lambda \geq 0$. Hence the functions

$$x \mapsto \frac{p_1^\delta(a,\sqrt{x})}{\sqrt{x}}\left(E^\delta_{a,\sqrt{x}}(\partial_h\Phi(X)) + E^\delta_{a,\sqrt{x}}(\langle h'', X\rangle\,\Phi(X))\right)$$

and

$$x \mapsto -\kappa(\delta)\,\frac{p_1^\delta(a,\sqrt{x})}{\sqrt{x}}\left(\int_0^1 h_r\int_0^\infty b^{\delta-4}\Big[\mathcal{T}_b^{2k}\,\Sigma^{\delta,r}_{a,\sqrt{x}}(\Phi(X)\,|\,\cdot)\Big]\,\mathrm{d}b\,\mathrm{d}r\right)$$

have the same Laplace transform. Since they are continuous on $\mathbb{R}_+$, they must coincide. This yields the claimed IbPF for the Bessel bridges of dimension $\delta \notin \{1,3\}$. The cases $\delta \in \{1,3\}$ are treated in the same way.

3.4 The IbPF via hypergeometric functions

In this section we provide a proof of the above IbPF for a different class of test functionals, in the case of homogeneous boundary values $a = a' = 0$. Namely, given a function $\zeta \in C([0,1])$, we consider the functional Φ defined on $L^2([0,1])$ by

$$\Phi(X) := \langle \zeta, X\rangle$$

Note that, when ζ is not identically 0, Φ may not be written as a function of the form $\exp(-\langle m, X^2\rangle)$, with m a finite measure on $[0,1]$, so the above results do not apply directly. However, it turns out that the IbPF still hold for such a functional Φ. More interesting than the result is the proof, which provides an interpretation of the IbPF using properties of hypergeometric functions. The proof relies on the fact that correlation functions of Bessel processes involve hypergeomteric functions: this fact is reminiscent of Cardy's formula for Bessel processes which, for the special value $\delta = 5/3$, admits an interpretation in terms of the crossing probability for a critical percolation model: see 1.3 in [Kat16]

Proposition 3.4.1. *For all $\delta > 0$ and $h \in C_c^2([0,1])$, we have*

$$E^\delta(\partial_h\Phi(X)) = -E^\delta[\langle h'', X\rangle\Phi(X)]$$
$$- \frac{\Gamma(\delta)}{4(\delta-2)}\int_0^1 dr\,h(r)\,\langle \mu_{\delta-3}(\,\mathrm{d}b), \Sigma^\delta_r(\Phi|b)\rangle$$

In the remainder of this section, we prove this result. Note that given the particular form of our test function $\Phi = \langle \zeta, \cdot \rangle$, the above formula can be rewritten in the following way:

$$\begin{aligned}\langle \zeta, h \rangle = &- \int_0^1 \zeta(s) \int_0^1 h''(r) E^\delta \left[X_s X_r \right] \, \mathrm{d}s \, \mathrm{d}r \\ &- \frac{\Gamma(\delta)}{4(\delta-2)} \int_0^1 \mathrm{d}s \, \zeta(s) \int_0^1 \mathrm{d}r \, h(r) \, \langle \mu_{\delta-3}(\, \mathrm{d}b), \Sigma_r^\delta (X_s | b) \rangle.\end{aligned} \tag{3.35}$$

In the last line, we interverted the integral in the variable s and the action of the distribution $\mu_{\delta-3}$. To justify this operation, we use the following result:

Lemma 3.4.2. *For all $r, s \in (0,1)$, $r \neq s$, and $b \geq 0$, we have*

$$\Sigma_r^\delta (X_s | b) = \frac{1}{2^{\delta/2-1} (r(1-r))^{\delta/2}} \exp \left(- \frac{D(s,r)}{2} b^2 \right) \sum_{k=0}^\infty C_k f_k(s,r) \, b^{2k}, \tag{3.36}$$

where

$$D(s,r) := \mathbf{1}_{s<r} \frac{1-s}{(r-s)(1-r)} + \mathbf{1}_{s>r} \frac{s}{r(s-r)},$$

and, for all $k \geq 0$

$$C_k := \frac{\Gamma(k + \frac{\delta+1}{2})}{\Gamma(\delta/2) \Gamma(k+\delta/2) k!},$$

and

$$f_k(s,r) = \frac{\mathbf{1}_{s<r}}{(2\,(r-s))^{k-\frac{1}{2}}} \left(\frac{s}{r} \right)^{k+1/2} + \frac{\mathbf{1}_{s>r}}{(2\,(s-r))^{k-\frac{1}{2}}} \left(\frac{1-s}{1-r} \right)^{k+1/2}.$$

Proof. The proof follows by a direct computation from the expression of the transition densities of Bessel processes. □

As a consequence, we claim that the following equality holds for all $r \in (0,1)$:

$$\langle \mu_{\delta-3}(\, \mathrm{d}b), \Sigma_r^\delta (\Phi(X) | b) \rangle = \int_0^1 \mathrm{d}s \, \zeta(s) \, \langle \mu_{\delta-3}(\, \mathrm{d}b), \Sigma_r^\delta (X_s | b) \rangle.$$

Indeed, we have

$$\begin{aligned}\langle \mu_{\delta-3}(\, \mathrm{d}b), \Sigma_r^\delta (\Phi(X) | b) \rangle &= - \langle \mu_\delta (\, \mathrm{d}b), \frac{\mathrm{d}^3}{\mathrm{d}b^3} \Sigma_r^\delta (\Phi(X) | b) \rangle \\ &= - \frac{1}{\Gamma(\delta)} \int_0^\infty \mathrm{d}b \, b^{\delta-1} \frac{\mathrm{d}^3}{\mathrm{d}b^3} \Sigma_r^\delta (\Phi(X) | b).\end{aligned}$$

and, by (3.36) above, one can check that

$$\int_0^\infty \mathrm{d}b\, b^{\delta-1} \left| \frac{\mathrm{d}^3}{\mathrm{d}b^3} \Sigma_r^\delta(\Phi(X)|b) \right| < \infty.$$

Hence, by Fubini, we deduce that

$$\begin{aligned}\langle \mu_{\delta-3}(\mathrm{d}b), \Sigma_r^\delta(\Phi(X)|b)\rangle &= -\langle \mu_\delta(\mathrm{d}b), \frac{\mathrm{d}^3}{\mathrm{d}b^3}\Sigma_r^\delta(\Phi(X)|b)\rangle \\ &= -\int_0^1 \mathrm{d}s\, \zeta(s) \frac{1}{\Gamma(\delta)} \int_0^\infty \mathrm{d}b\, b^{\delta-1} \frac{\mathrm{d}^3}{\mathrm{d}b^3} \Sigma_r^\delta(X_s|b) \\ &= -\int_0^1 \mathrm{d}s\, \zeta(s) \langle \mu_\delta, \frac{\mathrm{d}^3}{\mathrm{d}b^3}\Sigma_r^\delta(X_s|b)\rangle \\ &= \int_0^1 \mathrm{d}s\, \zeta(s) \langle \mu_{\delta-3}, \Sigma_r^\delta(X_s|b)\rangle,\end{aligned}$$

and the claim follows. Therefore, it suffices to prove equality (3.35). To do so, it suffices to prove that the following equality holds ds-almost everywhere:

$$\begin{aligned} h(s) = &-\int_0^1 h''(r) E^\delta\left[X_s X_r\right] \mathrm{d}r \\ &- \frac{\Gamma(\delta)}{4(\delta-2)} \int_0^1 \mathrm{d}r\, h(r)\, \langle \mu_{\delta-3}(\mathrm{d}b), \Sigma_r^\delta(X_s|b)\rangle.\end{aligned}$$

Hence, to prove Prop 3.4.1, it is sufficient to show that, for all $s \in (0,1)$, the function $r \mapsto E^\delta\left[X_r X_s\right]$ satisfies the following equality of distributions on $(0,1)$:

$$\begin{aligned}\frac{d^2}{dr^2} E^\delta\left[X_r X_s\right] = &-\delta_s(r) \\ &- \frac{\Gamma(\delta)}{4(\delta-2)} \int_0^1 \mathrm{d}r\, h(r)\, \langle \mu_{\delta-3}(\mathrm{d}b), \Sigma_r^\delta(X_s|b)\rangle.\end{aligned} \tag{3.37}$$

The proof of (3.37) will rely on the explicit computation of moments of Bessel bridges using hypergeometric functions.

We start by showing that, for all $s \in (0,1)$ and $r \in (0,1) \setminus \{s\}$, the function $r \mapsto E^\delta\left[X_r X_s\right]$ is twice differentiable at s, and that

$$\frac{d^2}{dr^2} E^\delta\left[X_r X_s\right] = -\frac{\Gamma(\delta)}{4(\delta-2)} \int_0^1 \mathrm{d}r\, h(r)\, \langle \mu_{\delta-3}(\mathrm{d}b), \Sigma_r^\delta(X_s|b)\rangle. \tag{3.38}$$

Assume for instance that $0 < s < r < 1$. Then a direct computation using the explicit transition densities of a δ-dimensional Bessel bridge shows that the following equality holds:

$$E^\delta[X_sX_r] = 2\frac{\Gamma\left(\frac{\delta+1}{2}\right)^2}{\Gamma\left(\frac{\delta}{2}\right)^2}\frac{(r-s)^{\delta/2+1}\left(s(1-r)\right)^{1/2}}{(r(1-s))^{\frac{\delta+1}{2}}}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\delta}{2},\frac{s(1-r)}{r(1-s)}\right). \tag{3.39}$$

On the other hand, by equality (3.36), the right-hand side of (3.38) equals

$$-\frac{1}{2}\frac{\Gamma\left(\frac{\delta+1}{2}\right)^2}{\Gamma\left(\frac{\delta}{2}\right)^2}\frac{(r-s)^{\delta/2-1}s^{1/2}}{(1-r)^{3/2}r^{\frac{\delta+1}{2}}(1-s)^{\frac{\delta-3}{2}}}\,{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-3}{2},\frac{\delta}{2},\frac{s(1-r)}{r(1-s)}\right) \tag{3.40}$$

where ${}_2F_1$ denotes the hypergeometric function. Recall that the hypergeometric function ${}_2F_1$ is defined, for all $a, b, c \in \mathbb{C}\setminus\mathbb{Z}_-$, and all $z \in \mathbb{C}$ such that $|z| < 1$, by

$${}_2F_1(a,b,c,z) := \sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{k!(c)_k}z^k$$

where, for any $\alpha > 0$ and $k \in \mathbb{N}$, $(\alpha)_k := \begin{cases} 1, & \text{if } k = 0 \\ \alpha(\alpha+1)\dots(\alpha+k-1), & \text{if } k \geq 1 \end{cases}$.

Note that the "b-parameter" of the hypergeometric function appearing in (3.39), $\frac{\delta+1}{2}$, differs by 2 from the one appearing in (3.40), $\frac{\delta-3}{2} = \frac{\delta+1}{2} - 2$. Hence, in order to prove the quality (3.38), we would need to exploit a differential relation linking ${}_2F_1(a,b,c,z)$ to ${}_2F_1(a,b-2,c,z)$. This relation is provided by the following, classical fact: for all $a, b, c \in \mathbb{C}\setminus\mathbb{Z}_-$,

$$\frac{d}{dz}\left(z^{c-b}(1-z)^{a+b-c}{}_2F_1(a,b,c,z)\right) = (c-b)z^{c-b-1}(1-z)^{a+b-c-1}{}_2F_1(a,b-1,c,z). \tag{3.41}$$

Now let $s \in (0,1)$, and $r \in (s,1)$. Setting $z := \frac{s(1-r)}{r(1-s)}$, we have

$$1 - z = \frac{r-s}{r(1-s)}$$

Therefore, equality (3.39) can be rewritten as follows:

$$E^\delta[X_sX_r] = K(\delta)s(1-r)z^{-1/2}(1-z)^{\delta/2+1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\delta}{2},z\right)$$

where

$$K(\delta) := 2\frac{\Gamma\left(\frac{\delta+1}{2}\right)^2}{\Gamma\left(\frac{\delta}{2}\right)^2}$$

Therefore, for all $r \in (s,1)$, we obtain, by the Leibniz formula and the chain rule

$$\frac{d}{dr}E^{\delta}\left[X_rX_s\right] = -K(\delta)sz^{-1/2}(1-z)^{\delta/2+1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\delta}{2},z\right)$$
$$+ K(\delta)s(1-r)\frac{dz}{dr}\frac{d}{dz}\left(z^{-1/2}(1-z)^{\delta/2+1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\delta}{2},z\right)\right)$$

But $\frac{dz}{dr} = -\frac{s}{r^2(1-s)}$, and, by (3.41), it holds that

$$\frac{d}{dz}\left(z^{-1/2}(1-z)^{\delta/2+1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\delta}{2},z\right)\right) = -\frac{1}{2}z^{-3/2}(1-z)^{\delta/2}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\delta}{2},z\right)$$

Hence we obtain

$$\begin{aligned}\frac{d}{dr}E^{\delta}\left[X_rX_s\right] &= -K(\delta)sz^{-1/2}(1-z)^{\delta/2+1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\delta}{2},z\right)\\
&- K(\delta)s(1-r)\frac{s}{r^2(1-s)}\left(-\frac{1}{2}z^{-3/2}(1-z)^{\delta/2}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\delta}{2},z\right)\right)\\
&= -K(\delta)sz^{-1/2}(1-z)^{\delta/2+1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\delta}{2},z\right)\\
&+ K(\delta)\frac{1}{2}\frac{1-s}{1-r}z^{1/2}(1-z)^{\delta/2}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\delta}{2},z\right)\end{aligned}$$

Differentiating with respect to r a second time, we obtain

$$\begin{aligned}\frac{d^2}{dr^2}E^{\delta}\left[X_rX_s\right] &= -K(\delta)s\frac{dz}{dr}\frac{d}{dz}\left\{z^{-1/2}(1-z)^{\delta/2+1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta+1}{2},\frac{\delta}{2},z\right)\right\}\\
&+ \frac{1}{2}K(\delta)\frac{1-s}{(1-r)^2}z^{1/2}(1-z)^{\delta/2}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\delta}{2},z\right)\\
&+ \frac{1}{2}K(\delta)\frac{1-s}{1-r}\frac{dz}{dr}\frac{d}{dz}\left\{z^{1/2}(1-z)^{\delta/2}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\delta}{2},z\right)\right\}\end{aligned}$$

Using again the expression for $\frac{dz}{dr}$, as well as (3.41), we deduce that

$$\begin{aligned}\frac{d^2}{dr^2}E^{\delta}\left[X_rX_s\right] &= K(\delta)s\frac{(1-r)}{r^2(1-s)}\left\{-\frac{1}{2}z^{-3/2}(1-z)^{\delta/2}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\delta}{2},z\right)\right\}\\
&+ K(\delta)\frac{1}{2}\frac{1-s}{(1-r)^2}z^{1/2}(1-z)^{\delta/2}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-1}{2},\frac{\delta}{2},z\right)\\
&- K(\delta)\frac{1}{2}\frac{1-s}{1-r}\frac{s}{r^2(1-s)}\left\{\frac{1}{2}z^{-1/2}(1-z)^{\delta/2-1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-3}{2},\frac{\delta}{2},z\right)\right\}\end{aligned}$$

The first two terms are easily shown to cancel out, so that we obtain

$$\frac{d^2}{dr^2}E^\delta\left[X_rX_s\right] = -\frac{K(\delta)}{4}\frac{s}{r^2(1-r)}z^{-1/2}(1-z)^{\delta/2-1}{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-3}{2},\frac{\delta}{2},z\right)$$
$$= -\frac{K(\delta)}{4}\frac{(r-s)^{\delta/2-1}s^{1/2}}{(1-r)^{3/2}r^{\frac{\delta+1}{2}}(1-s)^{\frac{\delta-3}{2}}}\;{}_2F_1\left(\frac{\delta+1}{2},\frac{\delta-3}{2},\frac{\delta}{2},\frac{s(1-r)}{r(1-s)}\right)$$

and, by (3.40), the last expression is equal to

$$-\frac{\Gamma(\delta)}{4(\delta-2)}\int_0^1 \mathrm{d}r\, h(r)\,\langle \mu_{\delta-3}(\,\mathrm{d}b), \Sigma_r^\delta(X_s|b)\rangle.$$

This yields the claimed equality (3.38).

We now prove that equality (3.37) holds. More precisely, for any test function $h \in C_c^2(0,r)$, we compute

$$\int_0^1 h''(r)E^\delta[X_rX_s]dr$$

Performing two successive integration by parts on the intervals $(0,s)$ and $(s,1)$, and reminding that h has compact support in $(0,1)$ and is continuous at s, we obtain that

$$\int_0^1 h''(r)E^\delta[X_rX_s]dr = h(s)\left\{\frac{d^+}{dr}E^\delta[X_rX_s] - \frac{d^-}{dr}E^\delta[X_rX_s]\right\} + \int_0^1 h''(r)\frac{d^2}{dr^2}E^\delta[X_rX_s]dr \tag{3.42}$$

where

$$\frac{d^+}{dr}E^\delta[X_rX_s] := \lim_{r\searrow s}\frac{d}{dr}E^\delta[X_rX_s] \tag{3.43}$$

and

$$\frac{d^-}{dr}E^\delta[X_rX_s] := \lim_{r\nearrow s}\frac{d}{dr}E^\delta[X_rX_s] \tag{3.44}$$

(the existence of these limits will be justified herebelow). By the first step, we readily know that the second term in the RHS above equals

$$-\frac{\Gamma(\delta)}{4(\delta-2)}\int_0^1 \mathrm{d}r\, h(r)\,\langle \mu_{\delta-3}(\,\mathrm{d}b), \Sigma_r^\delta(X_s|b)\rangle.$$

So there remains to establish the existence of and compute the limits (3.43) and (3.44). For this, we use the following lemma:

Lemma 3.4.3. *Let $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\gamma \notin \mathbb{Z}_-$, and $\gamma - \alpha - \beta \in \mathbb{R}_-^* \setminus \mathbb{Z}$. Then, for $z \in (0,1)$ tending to 1*

$$ {}_2F_1(\alpha, \beta, \gamma, z) \underset{z \to 1}{\sim} \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta} $$

Proof. By Thm 8.5 in [Vio16], the following equality holds for all $z \in (0,1)$:

$$ \begin{aligned} {}_2F_1(\alpha, \beta, \gamma, z) =& \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} {}_2F_1(\alpha, \beta, \alpha+\beta-\gamma-1, 1-z) \\ &+ \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, 1-z) \end{aligned} $$

Now, the functions ${}_2F_1(\alpha, \beta, \alpha+\beta-\gamma-1, \cdot)$ and ${}_2F_1(\gamma-\alpha, \gamma-\beta, \gamma-\alpha-\beta+1, \cdot)$ are continuous at 0 and equal to 1 there, while $(1-z)^{\gamma-\alpha-\beta} \to +\infty$ as $z \to 1$, since $\gamma - \alpha - \beta < 0$. The claim follows. □

Now, recalling the computations done in the first step, we have, for all $r > s$

$$ \begin{aligned} \frac{d}{dr} E^\delta [X_r X_s] =& - K(\delta) s z^{-1/2}(1-z)^{\delta/2+1} {}_2F_1\left(\frac{\delta+1}{2}, \frac{\delta+1}{2}, \frac{\delta}{2}, z\right) \\ &+ K(\delta) \frac{1}{2} \frac{1-s}{1-r} z^{1/2}(1-z)^{\delta/2} {}_2F_1\left(\frac{\delta+1}{2}, \frac{\delta-1}{2}, \frac{\delta}{2}, z\right) \end{aligned} $$

where $z := \frac{s(1-r)}{r(1-s)} \in (0,1)$. Therefore, letting $r \searrow s$ and using the lemma we see that

$$ \begin{aligned} \lim_{r \searrow s} \frac{d}{dr} E^\delta [X_r X_s] =& - K(\delta) \frac{\Gamma\left(\frac{\delta}{2}\right)\Gamma(\frac{\delta}{2}+1)}{\Gamma\left(\frac{\delta+1}{2}\right)^2} s + \frac{1}{2} K(\delta) \frac{\Gamma\left(\frac{\delta}{2}\right)^2}{\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{\delta-1}{2}\right)} \\ =& - \delta s + \frac{\delta-1}{2} \end{aligned} $$

Similarly, for all $r < s$, we can show that

$$ \begin{aligned} \frac{d}{dr} E^\delta [X_r X_s] =& K(\delta)(1-s) z^{-1/2}(1-z)^{\delta/2+1} {}_2F_1\left(\frac{\delta+1}{2}, \frac{\delta+1}{2}, \frac{\delta}{2}, z\right) \\ &- \frac{1}{2} K(\delta) \frac{1}{2} \frac{s}{r} z^{1/2}(1-z)^{\delta/2} {}_2F_1\left(\frac{\delta+1}{2}, \frac{\delta-1}{2}, \frac{\delta}{2}, z\right) \end{aligned} $$

where $z := \frac{r(1-s)}{s(1-r)} \in (0,1)$. Therefore, letting $r \nearrow s$ and using the lemma we see that

$$ \begin{aligned} \lim_{r \nearrow s} \frac{d}{dr} E^\delta [X_r X_s] =& K(\delta) \frac{\Gamma\left(\frac{\delta}{2}\right)\Gamma(\frac{\delta}{2}+1)}{\Gamma\left(\frac{\delta+1}{2}\right)^2}(1-s) - \frac{1}{2} K(\delta) \frac{\Gamma\left(\frac{\delta}{2}\right)^2}{\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{\delta-1}{2}\right)} \\ =& \delta(1-s) - \frac{\delta-1}{2} \end{aligned} $$

Therefore, $\frac{d^+}{dr}E^\delta[X_rX_s]$ and $\frac{d^-}{dr}E^\delta[X_rX_s]$ do indeed exist, and they satisfy

$$\begin{aligned}\frac{d^+}{dr}E^\delta\left[X_rX_s\right] - \frac{d^-}{dr}E^\delta\left[X_rX_s\right] &= \left(-\delta s + \frac{\delta-1}{2}\right) - \left(\delta(1-s) - \frac{\delta-1}{2}\right)\\ &= -1\end{aligned}$$

Hence, (3.42), finally becomes

$$\begin{aligned}\int_0^1 h''(r)E^\delta[X_rX_s]dr = &-h(s)\\ &- \frac{\Gamma(\delta)}{4(\delta-2)}\int_0^1 \mathrm{d}r\, h(r)\left\langle \mu_{\delta-3}(\,\mathrm{d}b), \Sigma_r^\delta(X_s|b)\right\rangle.\end{aligned}$$

Proposition 3.4.1 is proved.

3.5 A slightly more general class of functionals

Generalizing the approach taken in the above section, given two functions $\zeta : [0,1] \to \mathbb{R}$ and $\theta : [0,1] \to \mathbb{R}_+$ bounded and Borel, we can consider the functional Φ defined on $L^2([0,1])$ by

$$\Phi(X) := \langle \zeta, X\rangle \exp\left(-\langle \theta, X^2\rangle\right), \quad X \in L^2(0,1). \tag{3.45}$$

Note that

$$\nabla\Phi(X) = (\zeta - 2\langle \zeta, X\rangle \theta X)\exp\left(-\langle \theta, X^2\rangle\right), \quad X \in L^2(0,1).$$

In particular, as soon as $\zeta \neq 0$, $\nabla\Phi(0) \neq 0$, so also in this case $\Phi \notin \mathcal{S}$.

Then, upon invoking lemma 2.2.3, one can use the same computations as in the case $\theta = 0$ to show that the IbPF above also hold for Φ of the form (3.45). Since the techniques are the same as the ones presented above, but the computations much lenghtier, we do not provide the proof of this fact.

□

Chapter 4

Bessel SPDEs: conjectures, and existence of solutions for $\delta = 1, 2$

4.1 From the IbPF to the SPDEs

The IbPF we obtained above allow us to conjecture the structure of some reversible dynamics associated with the laws of Bessel bridges with arbitrary boundary values $a, a' \geq 0$.

To obtain the conjecture, we argue as follows. Recall that, if $f : \mathbb{R} \to \mathbb{R}$ is a bounded, globally Lipschitz continuous function, then the SPDE

$$\begin{cases} \partial_t u = \dfrac{1}{2}\partial_x^2 u + \dfrac{1}{2} f(u) + \xi, \\ \\ u(t,0) = a, \quad u(t,1) = a'. \end{cases} \tag{4.1}$$

where ξ is a space-time white noise on $\mathbb{R}_+ \times [0,1]$, is well-posed in the space $C([0,1] \times \mathbb{R}_+, \mathbb{R})$ (see e.g. Chapter 5 in [Zam17]). It also admits a unique invariant probability measure $\mathbb{P}$ on $L^2([0,1])$ given by

$$\mathbb{P}(\mathrm{d}\zeta) = \frac{1}{\mathrm{Z}} \exp\left(- \int_0^1 \mathrm{F}(\zeta_\mathrm{r}) \, \mathrm{dr} \right) \mathbb{W}(\mathrm{d}\zeta),$$

where $F : \mathbb{R} \to \mathbb{R}$ is such that $F' = f$, $\mathbb{W}$ is the law of a Brownian bridge from a to a' on $[0,1]$, and Z is a normalization constant. The measure $\mathbb{P}$ satisfies the IbPF

$$\mathbb{E}[\partial_h \Phi(X)] = -\mathbb{E}[\langle h'', X \rangle \Phi(X)] - \mathbb{E}[\langle h, f(X) \rangle \Phi(X)], \tag{4.2}$$

for all $h \in C_c^2(0,1)$. Note the correspondence between the SPDE (4.1) and the IbPF (4.2): the terms $\frac{1}{2}\partial_x^2 u$ and $\frac{1}{2} f(u)$ in the SPDE (4.1) are respectively encoded in the first and the second term in the right-hand side of the IbPF

(4.2). Using the terminology of Markov process, for all $h \in C_c^2(0,1)$, the measure $\langle h'', X\rangle \, \mathbb{P}(\mathrm{dX})$ is the Revuz measure associated with the additive functional $\int_0^t \int_0^1 h_x \, \partial_x^2 u(s,x) \, \mathrm{d}x \, \mathrm{d}s$, while the measure $\langle h, f(X)\rangle \, \mathbb{P}(\mathrm{dX})$ is the Revuz measure associated with $\int_0^t \int_0^1 h_x \, f(u(s,x)) \, \mathrm{d}x \, \mathrm{d}s$.

Coming back to the case of Bessel bridges, that is $\mathbb{P} = P^\delta_{a,a'}$ with $\delta \in (0,3)$, $a, a' \geq 0$, we have now an IbPF of the form (3.5). This IbPF is similar to (4.2), except that the last term is no longer given by a smooth measure, but rather by a generalized functional in the sense of Schwartz:

$$-\frac{\Gamma(\delta)}{4(\delta-2)} \int_0^1 h_r \langle \mu_{\delta-3}, \Sigma^{\delta,r}_{a,a'}(\Phi|\cdot)\rangle \, \mathrm{d}r. \tag{4.3}$$

Note however that, for all $b \geq 0$, $\Sigma^{\delta,r}_{a,a'}(\mathrm{dX}|\mathrm{b})$ is indeed a measure. We actually formulate the following conjecture concerning the relation between $\Sigma^{\delta,r}_{a,a'}(\mathrm{dX}|\mathrm{b})$ and a hypothetical Markov process $(u_t)_{t\geq 0}$ on $C([0,1])$ with invariant measure $P^\delta_{a,a'}$:

Conjecture 4.1.1. *There exists a family of additive functionals $(\ell^b_{t,r})_{t\geq 0}$, $b \geq 0, r \in (0,1)$ satisfying the occupation times formula*

$$\int_0^t \varphi(u(s,r)) \, \mathrm{d}s = \int_0^\infty \varphi(b) \, \ell^b_{t,r} \, b^{\delta-1} \, \mathrm{d}b, \tag{4.4}$$

for all Borel $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$. For all $r \in (0,1)$, and $b \geq 0$, $\Sigma^{\delta,r}_{a,a'}(\mathrm{dX}|\mathrm{b})$ is the Revuz measure associated with $(\ell^b_{t,r})_{t\geq 0}$.

We refer to Section 5.1 in [FOT10] for the definition of additive functionals and their Revuz measures. The relation between $(\ell^b_{t,r})_{t\geq 0}$ and $\Sigma^{\delta,r}_{a,a'}(\mathrm{dX}|\mathrm{b})$ claimed above is justified as follows. For all $F : C([0,1]) \to \mathbb{R}_+$ and $g : \mathbb{R}_+ \to \mathbb{R}$ Borel, one would have

$$\begin{aligned}
&\int_0^\infty \int_{C([0,1])} \mathbb{E}_x \left[\int_0^t F(u_s) \, \mathrm{d}\ell^b_{s,r} \right] P^\delta_{a,a'}(\mathrm{dx}) \, \mathrm{g(b)} \, \mathrm{b}^{\delta-1} \, \mathrm{db} \\
&= \int_{C([0,1])} \mathbb{E}_x \left[\int_0^\infty \mathrm{d}b \, g(b) \, b^{\delta-1} \int_0^t F(u_s) \, \mathrm{d}\ell^b_{s,r} \right] P^\delta_{a,a'}(\mathrm{dx}) \\
&= \int_{C([0,1])} \mathbb{E}_x \left[\int_0^t F(u_s) \, g(u(s,r)) \, \mathrm{d}s \right] P^\delta_{a,a'}(\mathrm{dx}),
\end{aligned}$$

where we used the occupation times formula (4.4) to obtain the third line. But, since the law $P^\delta_{a,a'}$ is supposed to be invariant for $(u_t)_{t\geq 0}$, we have

$$\begin{aligned}
\int_{C([0,1])} \mathbb{E}_x\left[\int_0^t F(u_s)\, g(u(s,r))\,\mathrm{ds}\right] P^\delta_{a,a'}(\mathrm{dx}) =& t\, E^\delta_{a,a'}\left[F(X)\, g(X_r)\right] \\
=& t \int_0^\infty \mathrm{d}b\, p^{\delta,r}_{a,a'}(b)\, g(b)\, E^\delta_{a,a'}\left[F(X) | X_r = b\right] \\
=& t \int_0^\infty \mathrm{d}b\, g(b)\, b^{\delta-1}\, \Sigma^{\delta,r}_{a,a'}\left(F|b\right),
\end{aligned}$$

where the second line follows by conditioning on the value of X_r. Hence, we obtain

$$\int_0^\infty \int_{C([0,1])} \mathbb{E}_x\left[\int_0^t F(u_s)\,\mathrm{d}\ell^b_{s,r}\right] P^\delta_{a,a'}(\mathrm{dx})\, \mathrm{g(b)\, b^{\delta-1}\, db} = \mathrm{t} \int_0^\infty \mathrm{db\, g(b)\, b^{\delta-1}}\, \Sigma^{\delta,r}_{\mathrm{a,a'}}\left(\mathrm{F|b}\right).$$

The map g introduced above being arbitrary, we would thus have

$$\int_{C([0,1])} \mathbb{E}_x\left[\int_0^t F(u_s)\,\mathrm{d}\ell^b_{s,r}\right] P^\delta_{a,a'}(\mathrm{dx}) = \mathrm{t}\, \Sigma^{\delta,\mathrm{r}}_{\mathrm{a,a'}}\left(\mathrm{F|b}\right) \tag{4.5}$$

for a.e. $b \geq 0$, and actually for all $b \geq 0$ if one admits the reasonable assumption that both terms are continuous in b. Since equality (4.5) holds for arbitrary F, we deduce that $\Sigma^{\delta,r}_{a,a'}\left(\mathrm{d}X|b\right)$ is indeed the Revuz measure associated with $(\ell^b_{t,r})_{t\geq 0}$.

As a consequence of this fact, the property (2.26) on $\Sigma^{\delta,r}_{a,a'}(\mathrm{dX|b})$ suggests that we should have

$$\left.\frac{\partial}{\partial b}\ell^b_{t,r}\right|_{b=0} = 0, \qquad t \geq 0.$$

Moreover, by the expression (4.3) for the last term in the IbPF, and in analogy with the classical case, we conjecture that the SPDE corresponding to $P^\delta_{a,a'}$ should be formally given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\Gamma(\delta)}{8(\delta-2)}\langle \mu_{\delta-3}(\mathrm{db}), \ell^{\mathrm{b}}_{\mathrm{t,x}}\rangle + \xi \\ u(t,0) = a, \quad u(t,1) = a'. \end{cases}$$

For $\delta \in (1,3)$, the SPDE would thus be given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\kappa(\delta)}{2}\dfrac{\partial}{\partial t}\displaystyle\int_0^\infty \frac{1}{b^3}\left(\ell^b_{t,x} - \ell^0_{t,x}\right) b^{\delta-1}\,\mathrm{d}b + \xi \\ u(t,0) = a, \quad u(t,1) = a'. \end{cases} \qquad (1 < \delta < 3), \tag{4.6}$$

On the other hand, for $\delta = 1$, the SPDE would take the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{8}\dfrac{\partial}{\partial t}\dfrac{\partial^2}{\partial b^2}\,\ell^b_{t,x}\Big|_{b=0} + \xi \\ \\ u(t,0) = a, \quad u(t,1) = a' \end{cases}, \quad (\delta = 1) \tag{4.7}$$

while for $\delta \in (0,1)$, it would be given by

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \xi + \dfrac{\kappa(\delta)}{2}\dfrac{\partial}{\partial t}\displaystyle\int_0^\infty \dfrac{1}{b^3}\left(\ell^b_{t,x} - \ell^0_{t,x} - \dfrac{b^2}{2}\dfrac{\partial^2}{\partial b^2}\,\ell^b_{t,x}\Big|_{b=0}\right) b^{\delta-1}\,\mathrm{d}b \\ \\ u(t,0) = a, \quad u(t,1) = a'. \qquad\qquad (0 < \delta < 1) \end{cases} \tag{4.8}$$

In all the SPDEs above, the unknown would be the couple (u, ℓ), where u is a continuous nonnegative function on $\mathbb{R}_+ \times (0,1)$, and, for all $x \in (0,1)$, $(\ell^b_{t,x})_{b,t\geq 0}$ is a family of occupation times satisfying (4.4).

These SPDEs are still very conjectural, see Section 4.5 below for a discussion. However, it turns out that in the particular cases $\delta = 1, 2$, a construction of weak solutions via Dirichlet form techniques is possible. This is the content of the two following sections.

4.2 The case $\delta = 1$

In this section we exploit the IbPF obtained above to construct a weak version of the gradient dynamics associated with P^1, using the theory of Dirichlet forms. The reason for considering this particular Bessel bridge is that for integer values of δ, and for zero boundary conditions, we can exploit a representation of the Bessel bridge in terms of a Brownian bridge, for which the corresponding gradient dynamics is well-known and corresponds to a linear stochastic heat equation. Such a representation actually still holds for any integer-dimensional Bessel bridge from a to a' when either a or a' vanishes, but it fails when $a, a' > 0$, see [YZ04]. Here, we shall consider the case $a = a' = 0$ and $\delta = 1$, and the case $\delta = 2$ will be considered in Section 4.3 below.

The representation of a 1-Bessel bridge from 0 to 0 in terms of a Brownian bridge allows us to construct a quasi-regular Dirichlet form associated with P^1, a construction which does not follow from the IbPF (3.2.5) due to the distributional character of its last term. The IbPF (3.2.5) is then exploited to prove that the associated Markov process, at equilibrium, satisfies (1.28). The treatment of the particular value $\delta = 1$ is also motivated by potential applications to scaling limits of dynamical critical pinning models, see e.g. [Voß16] and [DO18].

For the sake of our analysis, instead of working on the Banach space $C([0,1])$, it shall actually be more convenient to work on the Hilbert space $H := L^2(0,1)$ endowed with the L^2 inner product

$$\langle f, g \rangle = \int_0^1 f_r \, g_r \, \mathrm{d}r, \quad f, g \in H.$$

We shall denote by $\| \cdot \|$ the corresponding norm on H. Moreover we denote by μ the law of β on H, where β is a Brownian bridge from 0 to 0 over the interval $[0,1]$. We shall use the shorthand notation $L^2(\mu)$ for the space $L^2(H,\mu)$. We also consider the closed subset $K \subset H$ of nonnegative functions

$$K := \{z \in H, \ z \geq 0 \text{ a.e.}\}.$$

Note that K is a Polish space. We further denote by ν the law, on K, of the 1-Bessel bridge from 0 to 0 on $[0,1]$ (so that P^1 is then the restriction of ν to $C([0,1])$). We shall use the shorthand $L^2(\nu)$ to denote the space $L^2(K,\nu)$. Denoting by $j : H \to K$ the absolute value map

$$j(z) := |z|, \quad z \in H, \tag{4.9}$$

we remark that the map $L^2(\nu) \ni \varphi \mapsto \varphi \circ j \in L^2(\mu)$ is an isometry.

4.2.1 The one-dimensional random string

Consider the Ornstein-Uhlenbeck semigroup $(\mathbf{Q}_t)_{t \geq 0}$ on H defined, for all $F \in L^2(\mu)$ and $z \in H$, by

$$\mathbf{Q}_t F(z) := \mathbb{E}\left[F(v_t(z))\right], \quad t \geq 0,$$

where $(v_t(z))_{t\geq 0}$ is the solution to the stochastic heat equation on $[0,1]$ with initial condition z, and with homogeneous Dirichlet boundary conditions

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \xi \\ v(0,x) = z(x), & x \in [0,1] \\ v(t,0) = v(t,1) = 0, & t > 0 \end{cases} \tag{4.10}$$

with ξ a space-time white noise on $\mathbb{R}_+ \times [0,1]$. Recall that v can be written explicitly in terms of the fundamental solution $(g_t(x,x'))_{t\geq 0,\, x,x' \in (0,1)}$ of the stochastic heat equation with homogeneous Dirichlet boundary conditions on $[0,1]$, which by definition is the unique solution to

$$\begin{cases} \frac{\partial g}{\partial t} = \frac{1}{2} \frac{\partial^2 g}{\partial x^2} \\ g_0(x,x') = \delta_x(x') \\ g_t(x,0) = g_t(x,1) = 0. \end{cases} \tag{4.11}$$

Recall further that g can be represented as follows:

$$\forall t > 0, \quad \forall x, x' \geq 0, \quad g_t(x, x') = \sum_{k=1}^{\infty} e^{-\frac{\lambda_k}{2} t} e_k(x) e_k(x'), \tag{4.12}$$

where $(e_k)_{k\geq 1}$ is the complete orthonormal system of H given by

$$e_k(x) := \sqrt{2} \sin(k\pi x), \quad x \in [0, 1], \quad k \geq 1$$

and $\lambda_k := k^2\pi^2$, $k \geq 1$. We can then represent u as follows:

$$v(t, x) = z(t, x) + \int_0^t \int_0^1 g_{t-s}(x, x') \, \xi(\mathrm{d}s, \, \mathrm{d}x'), \tag{4.13}$$

where the double integral is a stochastic convolution, and

$$z(t, x) := \int_0^1 g_t(x, x') z(x') \, \mathrm{d}x'. \tag{4.14}$$

In particular, it follows from this formula that v is a Gaussian process. An important role will be played by its covariance function. Namely, for all $t \geq 0$ and $x, x' \in (0, 1)$, we set

$$q_t(x, x') := \mathrm{Cov}(v(t, x), v(t, x')) = \int_0^t g_{2\tau}(x, x') \, \mathrm{d}\tau.$$

We also set

$$q_\infty(x, x') := \int_0^\infty g_{2\tau}(x, x') \, \mathrm{d}\tau = \mathbb{E}[\beta_x \beta_{x'}] = x \wedge x' - xx'.$$

For all $t \geq 0$, we set moreover

$$q^t(x, x') := q_\infty(x, x') - q_t(x, x') = \int_t^\infty g_{2\tau}(x, x') \, \mathrm{d}\tau.$$

When $x = x'$, we will use the shorthand notations $q_t(x), q_\infty(x)$ and $q^t(x)$ instead of $q_t(x, x), q_\infty(x, x)$ and $q^t(x, x)$ respectively. Finally, we denote by $(\Lambda, D(\Lambda))$ the Dirichlet form associated with $(\mathbf{Q}_t)_{t\geq 0}$ in $L^2(H, \mu)$, and which is given by

$$\Lambda(F, G) = \frac{1}{2} \int_H \langle \nabla F, \nabla G \rangle \, \mathrm{d}\mu, \quad F, G \in D(\Lambda) = W^{1,2}(\mu).$$

Here, for all $F \in W^{1,2}(\mu)$, $\nabla F : H \to H$ is the gradient of F, see [DPZ02]. The corresponding family of resolvents $(\mathbf{R}_\lambda)_{\lambda > 0}$ is then given by

$$\mathbf{R}_\lambda F(z) = \int_0^\infty e^{-\lambda t} \mathbf{Q}_t F(z) \, \mathrm{d}t, \quad z \in H, \, \lambda > 0, \qquad F \in L^2(\mu).$$

4.2.2 Gradient Dirichlet form associated with the 1-dimensional Bessel bridge

In this section we consider the Dirichlet form associated with our equation (1.25), and the associated Markov process $(u_t)_{t\geq 0}$. We stress that a construction of the Dirichlet form and the Markov process was already provided in [Voß16].

Let $\mathcal{F}\mathcal{C}_b^\infty(H)$ denote the space of all functionals $F : H \to \mathbb{R}$ of the form

$$F(z) = \psi(\langle l_1, z\rangle, \ldots, \langle l_m, z\rangle), \quad z \in H, \tag{4.15}$$

with $m \in \mathbb{N}$, $\psi \in C_b^\infty(\mathbb{R}^m)$, and $l_1, \ldots, l_m \in \mathrm{Span}\{e_k, k \geq 1\}$. Recalling that $K := \{z \in H, z \geq 0\}$, we also define

$$\mathcal{F}\mathcal{C}_b^\infty(K) := \left\{ F\big|_K, \ F \in \mathcal{F}\mathcal{C}_b^\infty(H) \right\}.$$

Moreover, for $f \in \mathcal{F}\mathcal{C}_b^\infty(K)$ of the form $f = F\big|_K$, with $F \in \mathcal{F}\mathcal{C}_b^\infty(H)$, we define $\nabla f : K \to H$ by

$$\nabla f(z) = \nabla F(z), \quad z \in K,$$

where this definition does not depend on the choice of F. We denote by $\mathcal{E}$ the bilinear form defined on $\mathcal{F}\mathcal{C}_b^\infty(K)$ by

$$\mathcal{E}(f, g) := \frac{1}{2} \int \langle \nabla f, \nabla g\rangle \, \mathrm{d}\nu, \qquad f, g \in \mathcal{F}\mathcal{C}_b^\infty(K),$$

Proposition 4.2.1. *The form $(\mathcal{E}, \mathcal{F}\mathcal{C}_b^\infty(K))$ is closable. Its closure $(\mathcal{E}, D(\mathcal{E}))$ is a local, quasi-regular Dirichlet form on $L^2(\nu)$. Moreover, for all $f \in D(\mathcal{E})$, $f \circ j \in D(\Lambda)$, and we have*

$$\forall f, g \in D(\mathcal{E}), \quad \mathcal{E}(f, g) = \Lambda(f \circ j, g \circ j) \tag{4.16}$$

The proof of Proposition 4.2.1 is postponed to Section 4.2.6 below.

Let $(Q_t)_{t\geq 0}$ be the contraction semigroup on $L^2(K, \nu)$ associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$, and let $(R_\lambda)_{\lambda > 0}$ be the associated family of resolvents. Let also $\mathcal{B}_b(K)$ denote the set of Borel and bounded functions on K. As a consequence of Prop. 4.2.1, in virtue of Thm IV.3.5 and Thm V.1.5 in [MR92], we obtain the following result.

Corollary 4.2.2. *There exists a diffusion process $M = \{\Omega, \mathcal{F}, (u_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in K}\}$ properly associated to $(\mathcal{E}, D(\mathcal{E}))$, i.e. for all $\varphi \in L^2(\nu) \cap \mathcal{B}_b(K)$, and for all $t > 0$, $E_x(\varphi(u_t))$, $x \in K$, defines an $\mathcal{E}$ quasi-continuous version of $Q_t\varphi$. Moreover, the process M admits the following continuity property*

$$\mathbb{P}_x[t \mapsto u_t \textit{ is continuous on } \mathbb{R}_+] = 1, \quad \textit{for } \mathcal{E} - \text{q.e.}\, x \in K.$$

The rest of this section will be devoted to show that for $\mathcal{E}$-q.e. $x \in K$, under $\mathbb{P}_x$, $(u_t)_{t\geq 0}$ solves (1.25), or rather its weaker form (1.28).

In the sequel, we set $\Lambda_1 := \Lambda + (\cdot,\cdot)_{L^2(\mu)}$ and $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_{L^2(\nu)}$, which are inner products for the Hilbert spaces $D(\Lambda)$ and $D(\mathcal{E})$ respectively.

Since the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular, by the transfer method stated in VI.2 of [MR92], we can apply several results of [FOT10] in our setting. An important technical point is the density of the space $\mathcal{S}$ introduced in Section 3.1 above in the Dirichlet space $D(\mathcal{E})$. To state this precisely, we consider $\mathscr{S}$ to be the vector space generated by functionals $F : H \to \mathbb{R}$ of the form

$$F(\zeta) = \exp(-\langle \theta, \zeta^2 \rangle), \quad \zeta \in H,$$

for some $\theta : [0,1] \to \mathbb{R}_+$ Borel and bounded. Note that $\mathscr{S}$ may be seen as a subspace of the space $\mathcal{S}$ of Section 3.1 in the following sense: for any $F \in \mathscr{S}$, $F|_{C([0,1])} \in \mathcal{S}$. We also set

$$\mathscr{S}_K := \{F\big|_K, \ F \in \mathscr{S}\}.$$

Lemma 4.2.3. *$\mathscr{S}_K$ is dense in $D(\mathcal{E})$.*

The proof of Lemma 4.2.3 is postponed to Section 4.2.6 below.

4.2.3 Convergence of one-potentials

The key tool in showing that the Markov process constructed above defines a solution of (1.28) is the IbPF (3.15). The rule of thumb is that the last term in the IbPF yields the expression of the drift in the SPDE. Recall however that, for any fixed $h \in C^2_c(0,1)$, the last term in (3.15) is given by

$$\frac{1}{4}\int_0^1 \mathrm{d}r\, h_r \, \frac{\mathrm{d}^2}{\mathrm{d}a^2}\, \Sigma^1_r(\Phi(X)\,|\,a)\bigg|_{a=0}, \quad \Phi \in \mathcal{S},$$

which defines a generalized functional in the sense of Schwartz, rather than a genuine measure, on $C([0,1])$. It is therefore not immediate to translate the IbPF in terms of the corresponding dynamics. The strategy we follow to handle this difficulty consists in approximating the above generalized functional by a sequence of measures admitting a smooth density w.r.t. the law of the reflecting Brownian bridge, and showing that the corresponding one-potentials converge in the Dirichlet space $D(\mathcal{E})$. This will imply that the associated additive functionals converge to the functional describing the drift in the SPDE.

More precisely, let ρ be a smooth function supported on $[-1,1]$ such that

$$\rho \geq 0, \quad \int_{-1}^1 \rho = 1, \quad \rho(y) = \rho(-y), \quad y \in \mathbb{R}.$$

For all $\epsilon > 0$, let

$$\rho_\epsilon(y) := \frac{1}{\epsilon}\,\rho\left(\frac{y}{\epsilon}\right), \quad y \in \mathbb{R}. \tag{4.17}$$

Then, for all $\Phi \in \mathcal{S}$ and $h \in C_c^2(0,1)$, the right-hand side of the IbPF (3.15) can be rewritten as follows

$$\frac{1}{4}\int_0^1 h_r \,\frac{\mathrm{d}^2}{\mathrm{d}a^2}\,\Sigma_r^1(\Phi(X)\,|\,a)\bigg|_{a=0} \mathrm{d}r = \frac{1}{2}\lim_{\epsilon\to 0}\mathbb{E}\left[\Phi(|\beta|)\int_0^1 h_r\,\rho_\epsilon''(\beta_r)\,\mathrm{d}r\right]. \tag{4.18}$$

Indeed, starting from the right-hand side, by conditioning on the value of $|\beta_r|$, and recalling that $|\beta| \overset{(d)}{=} \nu$, the equality follows at once.

We will now show that the convergence of measures (4.18) can be enhanced to a convergence in the Dirichlet space $D(\Lambda)$ of the associated one-potentials. We henceforth fix a function $h \in C_c^2(0,1)$. For all $\epsilon > 0$, let $G_\epsilon : H \to \mathbb{R}$ be defined by

$$G_\epsilon(z) := \frac{1}{2}\int_0^1 h_r\,\rho_\epsilon''(z_r)\,\mathrm{d}r, \quad z \in H. \tag{4.19}$$

For all $t > 0$ and $z \in H$, we have

$$\mathbf{Q}_t G_\epsilon(z) = \int_0^1 \frac{h_r}{2\sqrt{2\pi q_t(r)}}\int_{\mathbb{R}} \rho_\epsilon''(a)\exp\left(-\frac{(a - z(t,r))^2}{2q_t(r)}\right)\mathrm{d}a\,\mathrm{d}r,$$

which, after two successive integration by parts, can be also written

$$\int_0^1 \frac{h_r}{2\sqrt{2\pi q_t(r)}}\int_{\mathbb{R}} \rho_\epsilon(b)\left[\left(\frac{b - z(t,r)}{q_t(r)}\right)^2 - \frac{1}{q_t(r)}\right]\exp\left(-\frac{(b - z(t,r))^2}{2q_t(r)}\right)\mathrm{d}b\,\mathrm{d}r,$$

where $z(t,\cdot)$ depends on z via (4.14). Let also $G^{(t)} : H \to \mathbb{R}$ be the functional defined by

$$G^{(t)}(z) := \int_0^1 \frac{h_r}{2\sqrt{2\pi q_t(r)}}\left[\left(\frac{z(t,r)}{q_t(r)}\right)^2 - \frac{1}{q_t(r)}\right]\exp\left(-\frac{(z(t,r)^2}{2q_t(r)}\right)\mathrm{d}r.$$

For all $\epsilon > 0$, we define the functional $U_\epsilon : H \to \mathbb{R}$ by

$$U_\epsilon(z) = \int_0^\infty e^{-t}\,\mathbf{Q}_t G_\epsilon(z)\,\mathrm{d}t, \quad z \in H.$$

Note that U_ϵ is the one-potential of the additive functional

$$\int_0^t G_\epsilon(v(s,\cdot))\,\mathrm{d}s, \qquad t \geq 0,$$

associated with the Markov process $(v(t,\cdot))_{t\geq 0}$ in H defined in (4.10) (see Section 5 of [FOT10] for this terminology). In particular, $U_\epsilon \in D(\Lambda)$.

Proposition 4.2.4. *The functional $U : H \to \mathbb{R}$ defined by*

$$U(z) := \int_0^\infty e^{-t}\, G^{(t)}(z)\,\mathrm{d}t, \qquad z \in H, \tag{4.20}$$

belongs to $D(\Lambda)$. Moreover, $U_\epsilon \underset{\epsilon\to 0}{\longrightarrow} U$ in $D(\Lambda)$.

Proof. First note that $U_\epsilon \underset{\epsilon\to 0}{\longrightarrow} U$ in $L^2(\mu)$. Indeed, for all fixed $t > 0$ and $z \in H$, we have

$$|\mathbf{Q}_t G_\epsilon(z) - G^{(t)}(z)| \le \int_0^1 \frac{|h_r|}{2\sqrt{2\pi q_t(r)^3}} \int_{\mathbb{R}} \rho(x) \left| F\left(\frac{\epsilon x - z(t,r)}{\sqrt{q_t(r)}}\right) - F\left(\frac{z(t,r)}{\sqrt{q_t(r)}}\right)\right| \mathrm{d}x\,\mathrm{d}r,$$

where the function $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(y) = (y^2 - 1)\exp(-y^2/2), \qquad y \in \mathbb{R}.$$

Since F is continuous and bounded, by dominated convergence, we deduce that, for all $r \in (0,1)$ and $x \in \mathbb{R}$

$$\left\| F\left(\frac{\epsilon x - z(t,r)}{q_t(r)}\right) - F\left(\frac{z(t,r)}{q_t(r)}\right)\right\|_{L^2(\mu)} \underset{\epsilon\to 0}{\longrightarrow} 0.$$

Therefore, again by dominated convergence, we have

$$\begin{aligned}
&\|\mathbf{Q}_t G_\epsilon - G^{(t)}\|_{L^2(\mu)} \\
&\le \int_0^1 \frac{|h_r|}{2\sqrt{2\pi q_t(r)^{3/2}}} \int_{\mathbb{R}} \rho(x) \left\| F\left(\frac{\epsilon x - z(t,r)}{q_t(r)}\right) - F\left(\frac{z(t,r)}{q_t(r)}\right)\right\|_{L^2(\mu)} \mathrm{d}x\,\mathrm{d}r \\
&\underset{\epsilon\to 0}{\longrightarrow} 0.
\end{aligned}$$

Henceforth we fix $\delta \in (0,1)$ such that h is supported in $[\delta, 1-\delta]$. As showed in the proof of Proposition 1 in [Zam04b], there exists $C_\delta > 0$ such that, for all $r \in (\delta, 1-\delta)$ and $t > 0$

$$q_t(r) \ge C_\delta(\sqrt{t} \wedge 1). \tag{4.21}$$

In the following, we will denote by C_δ any constant depending only on δ, and whose value may change from line to line. Thanks to (4.21), we obtain the bound

$$\|\mathbf{Q}_t G_\epsilon - G^{(t)}\|_{L^2(\mu)} \le C_\delta \|F\|_\infty \frac{\|h\|_\infty}{t^{3/4} \wedge 1},$$

where the right-hand side is integrable w.r.t. the measure $e^{-t}\,\mathrm{d}t$ on $\mathbb{R}_+$. Hence, by dominated convergence,

$$\begin{aligned}
\|U_\epsilon - U\|_{L^2(\mu)} &\le \int_0^\infty e^{-t}\, \|P_t G_\epsilon - G^{(t)}\|_{L^2(\mu)}\,\mathrm{d}t \\
&\underset{\epsilon\to 0}{\longrightarrow} 0,
\end{aligned}$$

whence the claim. Now, we show that $U \in D(\Lambda)$. Note that, for all $t > 0$ and $\epsilon > 0$, we have

$$\nabla \mathbf{Q}_t G_\epsilon(z) = \frac{1}{2} \int_0^1 h_r \, g_t(r, \cdot) \, \mathbb{E}[\rho_\epsilon^{(3)}(v(t,r))] \, \mathrm{d}r,$$

where v is given by (4.13) and where we are taking expectation with respect to the white noise ξ. Therefore, denoting by $\| \cdot \|_{L^2}$ the norm in $L^2(H, \mu; H)$, we have

$$\|\nabla \mathbf{Q}_t G_\epsilon\|_{L^2}^2 =$$
$$\frac{1}{4} \int_{[0,1]^2} h_r h_s \langle g_t(r, \cdot), g_t(s, \cdot) \rangle \int_H \mathbb{E}[\rho_\epsilon^{(3)}(v(t,r))] \, \mathbb{E}[\rho_\epsilon^{(3)}(v(t,s))] \; \mathrm{d}\mu(z) \; \mathrm{d}r \, \mathrm{d}s$$

where the integral in $\mathrm{d}\mu(z)$ is taken with respect to $v(0, \cdot) = z$. Hence

$$\|\nabla P_t G_\epsilon\|_{L^2}^2 =$$
$$\int_{[0,1]} \frac{h_r h_s \langle g_t(r, \cdot), g_t(s, \cdot) \rangle}{4} \int_{\mathbb{R}^2} \rho_\epsilon^{(3)}(x) \, \rho_\epsilon^{(3)}(y) \, \Gamma_{r,s}(x,y) \; \mathrm{d}x \, \mathrm{d}y \; \mathrm{d}r \, \mathrm{d}s =$$
$$\int_{[0,1]} \frac{h_r h_s \langle g_t(r, \cdot), g_t(s, \cdot) \rangle}{4} \int_{\mathbb{R}^2} \rho_\epsilon(x) \, \rho_\epsilon(y) \, \frac{\partial^6 \Gamma_{r,s}}{\partial x^3 \partial y^3}(x,y) \; \mathrm{d}x \, \mathrm{d}y \; \mathrm{d}r \, \mathrm{d}s,$$

where, for all $(r,s) \in [0,1]$ and $(x,y) \in \mathbb{R}^2$

$$\Gamma_{r,s}(x,y) := \mathbb{E}\left[\frac{1}{2\pi \sqrt{q_t(r) q_t(s)}} \exp\left(-\frac{(x - z(t,r))^2}{2 q_t(r)} - \frac{(y - z(t,s))^2}{2 q_t(s)} \right) \right]. \quad (4.22)$$

Reasoning as in Section 6 of [Zam05], we see that $\Gamma_{r,s}$ is the density of the centered Gaussian law on $\mathbb{R}^2$ with covariance matrix

$$M = \begin{pmatrix} q_\infty(r) & q^t(r,s) \\ q^t(r,s) & q_\infty(s) \end{pmatrix}.$$

Similarly, we have

$$\|\nabla G^{(t)}\|_{L^2}^2 = \int_0^1 \int_0^1 \frac{h_r h_s \langle g_t(r, \cdot), g_t(s, \cdot) \rangle}{4} \frac{\partial^6 \Gamma_{r,s}}{\partial x^3 \partial y^3}(0,0) \; \mathrm{d}r \, \mathrm{d}s.$$

So there remains to obtain a bound on

$$\sup_{\mathbb{R}^2} \left| \frac{\partial^6 \Gamma_{r,s}}{\partial x^3 \partial y^3} \right|,$$

for all $(r,s) \in [0,1]^2$. To do so, we use the following lemma:

Lemma 4.2.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the density of a centered Gaussian law on $\mathbb{R}^2$ with non-degenerate covariance matrix M satisfying $|M_{i,j}| \leq 1$ for all $i, j \in \{1, 2\}$. Then, for all $k, \ell \in \mathbb{N}$ and $(x, y) \in \mathbb{R}^2$*

$$\left| \frac{\partial^{k+\ell} f}{\partial^k x \, \partial^\ell y} \right| \leq A_{k,\ell} \ \det(M)^{-\frac{1+k+\ell}{2}}$$

where $A_{k,\ell} > 0$ is a constant depending only on k and ℓ.

Proof. Setting

$$M = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

we can express the eigenvalues λ and μ of M as

$$\lambda = \frac{a+c}{2} + \sqrt{\left(\frac{a-c}{2}\right)^2 + b^2}$$

and

$$\mu = \frac{a+c}{2} - \sqrt{\left(\frac{a-c}{2}\right)^2 + b^2}.$$

Hence, since a, b and c are bounded by 1, we deduce that λ and μ are bounded by some universal constant $C > 0$. Let now P be an orthogonal matrix such that $M = P^T D P$, where P^T denotes the transposed of the matrix P, and where

$$D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

Then, for all $u \in \mathbb{R}^2$

$$f(u) = \frac{1}{2\pi\sqrt{\det(M)}} \, g(Pu) \tag{4.23}$$

where

$$g(v) := \exp\left(-\frac{1}{2} v^T D^{-1} v\right) = \exp\left(-\frac{x^2}{2\lambda} - \frac{y^2}{2\mu}\right)$$

for all $v = (x, y) \in \mathbb{R}^2$. Since the function $u \mapsto e^{-\frac{u^2}{2}}$ is bounded on $\mathbb{R}$ with all its derivatives, we deduce that for all $k, \ell \in \mathbb{N}$, there exists $C_{k,\ell} > 0$ depending only on k and ℓ such that

$$\left| \frac{\partial^{k+\ell} g}{\partial x^k \partial y^\ell} \right| \leq C_{k,\ell} \, \lambda^{-k/2} \mu^{-\ell/2}.$$

Therefore, since λ and μ are bounded by C, and noting that $\det(M) = \lambda\,\mu$, setting $C'_{k,\ell} := C_{k,\ell}\, C^{\frac{k+\ell}{2}}$ we have

$$\left|\frac{\partial^{k+\ell} g}{\partial x^k \partial y^\ell}\right| \le C'_{k,\ell}\ \det(M)^{-\frac{k+\ell}{2}}.$$

Hence, by the relation (4.23) and the chain rule, and since the coefficients of the orthogonal matrix P are all bounded aby 1, we obtain the claim. □

We now apply the Lemma to the Gaussian density function $\Gamma_{r,s}$ for all $(r,s) \in (0,1)$. Note that $q_\infty(r) \le 1$ and $q_\infty(s) \le 1$, so all coefficients of its covariance matrix M are indeed bounded by 1 as requested. Therefore

$$\sup_{\mathbb{R}^2} \left|\frac{\partial^6 \Gamma_{r,s}}{\partial x^3 \partial y^3}\right| \le A\ \det(M)^{-7/2},$$

where $A \in (0,\infty)$ is a universal constant. Now

$$\det(M) = q_\infty(r) q_\infty(s) - q^t(r,s)^2, \tag{4.24}$$

But

$$\begin{aligned} q_\infty(r) q_\infty(s) - q^t(r,s)^2 &\ge q_\infty(r) q_\infty(s) - q_\infty(r,s)^2 \\ &= r(1-r)s(1-s) - (r \wedge s - rs)^2 \\ &= s \wedge r(1 - s \vee r)|s-r|, \end{aligned}$$

so we obtain the lower bound

$$\det(M) \ge \delta^2 |r-s| \tag{4.25}$$

for all $r,s \in [\delta, 1-\delta]$. On the other hand, reasoning as in Section 6 of [Zam05], we can show that there exists $c_\delta > 0$ depending only on δ such that, for all $r,s \in [\delta, 1-\delta]$

$$q_\infty(r) q_\infty(s) - q^t(r,s)^2 \ge c_\delta\, (t \wedge 1)^{1/2},$$

which yields the lower bound

$$\det(M) \ge c_\delta\, (t \wedge 1)^{1/2}. \tag{4.26}$$

As a consequence, for all $r,s \in [\delta, 1-\delta]$, interpolating (4.25) and (4.26), we thus obtain

$$\left|\frac{\partial^6 \Gamma_{r,s}}{\partial^3 x\, \partial^3 y}\right| \le C_\delta\, (t \wedge 1)^{-\gamma/2} |r-s|^{-(7/2-\gamma)}, \tag{4.27}$$

for any $\gamma \in (5/2, 3)$, where $C_\delta > 0$ is a constant depending only on δ. Note also that, for some universal constant $C > 0$, we have

$$\forall r, s \in [\delta, 1-\delta], \qquad \langle g_t(r,\cdot), g_t(s,\cdot)\rangle = g_{2t}(r,s) \le C\, t^{-1/2}, \tag{4.28}$$

see e.g. Exercise 4.16 in [Zam17]. Therefore

$$\|\nabla G^{(t)}\|_{L^2}^2 \le C_\delta\, \|h\|_\infty^2 (t \wedge 1)^{-(1+\gamma)/2} \int_0^1 \int_0^1 |r-s|^{-(7/2-\gamma)}\, \mathrm{d}r\, \mathrm{d}s,$$

and the last integral is finite due to the choice of γ. Therefore, we deduce that $\|\nabla G^{(t)}\|_{L^2} \le C(\delta, h, \gamma)\, t^{-(1+\gamma)/4}$, where the constant $C(\delta, h, \gamma)$ does not depend on t. Since $(1+\gamma)/4 < 1$, it follows that

$$\int_0^\infty e^{-t}\, \|\nabla G^{(t)}\|_{L^2}\, \mathrm{d}t < \infty,$$

so that $\nabla U \in L^2(H, \mu; H)$. Therefore $U \in D(\Lambda)$ as claimed. There remains to prove that $U_\epsilon \underset{\epsilon \to 0}{\longrightarrow} U$ in $D(\Lambda)$. Note that, for all $t > 0$ and $\epsilon > 0$,

$$\begin{aligned} &\|\mathbf{Q}_t G_\epsilon - G^{(t)}\|_{L^2}^2 \\ &= \int_{[0,1]^2} \mathrm{d}r\, \mathrm{d}s\, \frac{h_r h_s \langle g_t(r,\cdot), g_t(s,\cdot)\rangle}{4} \int_{\mathbb{R}^2} \mathrm{d}x\, \mathrm{d}y\, \rho(x)\, \rho(y)\, \Gamma_{r,s}^{(3;3)}(\epsilon x, \epsilon y), \end{aligned}$$

where for all $(u, v) \in \mathbb{R}^2$,

$$\Gamma_{r,s}^{(3;3)}(u,v) := \frac{\partial^6 \Gamma_{r,s}}{\partial^3 x\, \partial^3 y}(u,v) - \frac{\partial^6 \Gamma_{r,s}}{\partial^3 x\, \partial^3 y}(u,0) - \frac{\partial^6 \Gamma_{r,s}}{\partial^3 x\, \partial^3 y}(0,v) + \frac{\partial^6 \Gamma_{r,s}}{\partial^3 x\, \partial^3 y}(0,0).$$

By (4.27) and (4.28) we deduce that

$$\|\mathbf{Q}_t G_\epsilon - G^{(t)}\|_{L^2(H,\mu;H)}^2 \le C_\delta\, \|h\|_\infty^2\, (t \wedge 1)^{-(1+\gamma)/2} \int_0^1 \int_0^1 |r-s|^{-(7/2-\gamma)}\, \mathrm{d}r\, \mathrm{d}s,$$

so that:

$$\|\mathbf{Q}_t G_\epsilon - G^{(t)}\|_{L^2} \le C(\delta, h, \gamma)\, (t \wedge 1)^{-(1+\gamma)/4},$$

where $C(\delta, h, \gamma) > 0$ is independent of ϵ and t. Recall that the right-hand side above is integrable with respect to $e^{-t}\, \mathrm{d}t$. Moreover, since $\Gamma_{r,s}$ is continuous, it follows that for all $t > 0$,

$$\|\mathbf{Q}_t G_\epsilon - G^{(t)}\|_{L^2} \underset{\epsilon \to 0}{\longrightarrow} 0.$$

Hence, by dominated convergence, we deduce that

$$\|\nabla U_\epsilon - \nabla U\|_{L^2} \le \int_0^t e^{-t}\, \|\nabla \mathbf{Q}_t G_\epsilon - \nabla G^{(t)}\|_{L^2}\, \mathrm{d}t \underset{\epsilon \to 0}{\longrightarrow} 0.$$

Hence $U_\epsilon \underset{\epsilon \to 0}{\longrightarrow} U$ in $D(\Lambda)$, and the Proposition is proved. □

4.2.4 The dynamics for $\delta = 1$

Note that in the above section we worked in the Gaussian Dirichlet space $D(\Lambda)$. For our dynamical problem, we shall however need to transfer the above results to the Dirichlet space $D(\mathcal{E})$. To do so, we invoke the following projection principle, which was first used in [Zam04b] for the case of a 3-Bessel bridge (see Lemma 2.2 therein).

Lemma 4.2.6. *There exists a unique bounded linear operator* $\Pi : D(\Lambda) \to D(\mathcal{E})$ *such that, for all* $F, G \in D(\Lambda)$ *and* $f \in D(\mathcal{E})$

$$\Lambda_1(F, f \circ j) = \mathcal{E}_1(\Pi F, f),$$

where j *is as in* (4.9). *Moreover, we have*

$$\mathcal{E}_1(\Pi F, \Pi F) \leq \Lambda_1(F, F).$$

Proof. We use the same arguments as in the proof of Lemma 2 in [Zam04b]. Let $\mathcal{D} := \{\varphi \circ j, \quad \varphi \in D(\mathcal{E})\}$. By Proposition 4.2.1, $\mathcal{D}$ is a linear subspace of $D(\Lambda)$ which is isometric to $D(\mathcal{E})$. In particular, it is a closed subspace of the Hilbert space $D(\Lambda)$. Hence, we may consider the orthogonal projection operator $\hat{\Pi}$ onto $\mathcal{D}$. Then, for all $F \in D(\Lambda)$, let ΠF be the unique element of $D(\mathcal{E})$ such that $\hat{\Pi} F = (\Pi F) \circ j$. It then follows that Π possesses the required properties. □

We obtain the following refinement of the IbPF (3.4) for P^1.

Corollary 4.2.7. *Let* U *be as in* (4.20). *For all* $f \in D(\mathcal{E})$ *and* $h \in C_c^2(0,1)$, *we have*

$$\mathcal{E}\left(\langle h, \cdot \rangle - \frac{1}{2}\Pi U \,,\, f\right) = -\frac{1}{2}\int_K \left(\langle h'', \zeta\rangle - \Pi U(\zeta)\right) f(\zeta)\, \mathrm{d}\nu(\zeta). \tag{4.29}$$

Proof. By the density of $\mathscr{S}_K$ in $D(\mathcal{E})$ proved in Lemma 4.2.3, it is enough to consider $f \in \mathscr{S}_K$. By (4.18)

$$\frac{1}{4}\int_0^1 \mathrm{d}r\, h_r \frac{\mathrm{d}^2}{\mathrm{d}a^2} \Sigma_r^1\left(f(X)\,|\,a\right)\Big|_{a=0} = \frac{1}{2}\lim_{\epsilon\to 0} \mathbb{E}\left[f(|\beta|)\int_0^1 h_r\, \rho_\epsilon''(\beta_r)\, \mathrm{d}r\right]$$
$$= \lim_{\epsilon\to 0}\int (f\circ j)\, G_\epsilon\, \mathrm{d}\mu = \lim_{\epsilon\to 0} \Lambda_1(f\circ j,\, U_\epsilon) = \Lambda_1(f\circ j,\, U) = \mathcal{E}_1(f,\, \Pi U).$$

Therefore, for all $f \in \mathscr{S}_K$, the IbPF (3.15) can be rewritten

$$2\mathcal{E}(\langle h, \cdot\rangle, f) = -\int_K \langle h'', \zeta\rangle\, f(\zeta)\, \mathrm{d}\nu(\zeta) + \mathcal{E}_1(f,\, \Pi U),$$

that is

$$\mathcal{E}\left(\langle h, \cdot \rangle - \frac{1}{2}\Pi U, f\right) = -\frac{1}{2}\int_K \left(\langle h'', \zeta\rangle - \Pi U(\zeta)\right) f(\zeta)\, \mathrm{d}\nu(\zeta).$$

The proof is complete. □

Recall that $M = (\Omega, \mathcal{F}, (u_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in K})$ denotes the Markov process properly associated with the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ constructed above. Note that, by Theorem 5.2.2 in [FOT10], for all $F \in D(\mathcal{E})$, we can write in a unique way

$$F(u_t) - F(u_0) = M_t^{[F]} + N_t^{[F]}, \quad t \geq 0, \tag{4.30}$$

$\mathbb{P}_\nu$ a.s., where $M^{[F]}$ is a martingale additive functional, and $N^{[F]}$ is an additive functional of zero energy. Using this fact we can thus write u as the weak solution to some SPDE, but with coefficients that are not explicit. However the formula (4.29) above will allow us to identify these coefficients.

We can now finally state the result justifying that that the Markov process constructed above satisfies the SPDE (1.28) above.

Theorem 4.2.8. *For all $h \in C_c^2(0,1)$, we have*

$$\langle u_t, h\rangle - \langle u_0, h\rangle = M_t + N_t, \qquad \mathbb{P}_{u_0} - a.s., \quad q.e.\ u_0 \in K.$$

Here $(N_t)_{t\geq 0}$ is a continuous additive functional of zero energy satisfying

$$N_t - \frac{1}{2}\int_0^t \langle h'', u_s\rangle \, \mathrm{d}s = \lim_{\epsilon\to 0} N_t^\epsilon, \qquad N_t^\epsilon := -\frac{1}{4}\int_0^t \langle \rho_\epsilon''(u_s), h\rangle \, \mathrm{d}s,$$

in $\mathbb{P}_\nu$-probability, uniformly in t on finite intervals. Moreover, $(M_t)_{t\geq 0}$ is a martingale additive functional whose sharp bracket has the Revuz measure $\|h\|_H^2\,\nu$. Finally we also have

$$N_t - \frac{1}{2}\int_0^t \langle h'', u_s\rangle \, \mathrm{d}s = \lim_{k\to\infty} N_t^{\epsilon_k}$$

along a subsequence $\epsilon_k \to 0$ in $\mathbb{P}_{u_0}$-probability, for q.e. $u_0 \in K$.

Proof. On the one hand, by (4.29), we can write

$$\langle u_t, h\rangle - \frac{1}{2}\Pi U(u_t) - \left(\langle u_0, h\rangle - \frac{1}{2}\Pi U(u_0)\right) = N_t^{(1)} + M_t^{(1)}, \tag{4.31}$$

where $N^{(1)}$ is the continuous additive functional of zero energy given by

$$N_t^{(1)} = \frac{1}{2}\int_0^t \left(\langle h'', u_s\rangle - \Pi U(u_s)\right) \mathrm{d}s, \quad t \geq 0$$

and where $M^{(1)}$ defined by (4.31) is a martingale additive functional. On the other hand, for all $\epsilon > 0$, by definition of U_ϵ, we have for G_ϵ as in (4.19)

$$\Lambda_1(U_\epsilon, \Phi) = \int_H G_\epsilon \, \Phi \, \mathrm{d}\mu, \quad \Phi \in D(\Lambda).$$

Hence, remarking that $G_\epsilon = g_\epsilon \circ j$, where $g_\epsilon : K \to \mathbb{R}$ is the functional defined by

$$g_\epsilon(z) := \frac{1}{2}\int_0^1 h_r\, \rho_\epsilon''(z_r)\,\mathrm{d}r = \frac{1}{2}\langle \rho_\epsilon''(z), h\rangle,$$

by Lemma 4.2.6, we obtain for all $f \in D(\mathcal{E})$

$$\mathcal{E}_1(\Pi U_\epsilon, f) = \int_K f(z)\, g_\epsilon(z)\,\mathrm{d}\nu(z),$$

that is:

$$\mathcal{E}(\Pi U_\epsilon, f) = -\int_K f(z)(\Pi U_\epsilon(z) - g_\epsilon(z))\,\mathrm{d}\nu(z). \tag{4.32}$$

As a consequence, we have the decomposition

$$\frac{1}{2}\Pi U_\epsilon(u_t) - \frac{1}{2}\Pi U_\epsilon(u_0) = N_t^{(2,\epsilon)} + M_t^{(2,\epsilon)}, \tag{4.33}$$

where $N^{(2,\epsilon)}$ is the continuous additive functional of zero energy given by

$$N_t^{(2,\epsilon)} = \frac{1}{2}\int_0^t (\Pi U_\epsilon(u_s) - g_\epsilon(u_s))\,\mathrm{d}s, \quad t \geq 0$$

and where $M^{(2,\epsilon)}$ defined by (4.33) is a martingale additive functional. Since $U_\epsilon \underset{\epsilon\to 0}{\longrightarrow} U$ in $D(\Lambda)$ by Proposition 4.2.4, by the continuity of $\Pi : D(\Lambda) \to D(\mathcal{E})$, we have the convergence $\Pi U_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \Pi U$ in $D(\mathcal{E})$. Therefore, setting

$$M_t^{(2)} = M_t^{[\Pi U]}, \qquad N_t^{(2)} := N_t^{[\Pi U]},$$

then, by (5.1.1), (5.2.22) and (5.2.25) in [FOT10], we have

$$\Pi U_\epsilon(u_t) - \Pi U_\epsilon(u_0) \underset{\epsilon\to 0}{\longrightarrow} \Pi U(u_t) - \Pi U(u_0), \quad M_t^{(2,\epsilon)} \underset{\epsilon\to 0}{\longrightarrow} M_t^{(2)}, \quad N_t^{(2,\epsilon)} \underset{\epsilon\to \infty}{\longrightarrow} N_t^{(2)}$$

in $\mathbb{P}_\nu$-probability, for the topology of uniform convergence on finite intervals of $t \in \mathbb{R}_+$. Adding equality (4.33) to (4.31) yields

$$\langle u_t, h\rangle - \langle u_0, h\rangle = M_t + N_t,$$

with $M_t = M_t^1 + M_t^2$ and

$$\begin{aligned} N_t = N_t^1 + N_t^2 &= \frac{1}{2}\int_0^t (\langle h'', u_s\rangle - \Pi U(u_s))\,\mathrm{d}s + \lim_{\epsilon\to 0}\frac{1}{2}\int_0^t (\Pi U_\epsilon(u_s) - g_\epsilon(u_s))\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \langle h'', u_s\rangle\,\mathrm{d}s - \lim_{\epsilon\to 0}\frac{1}{2}\int_0^t g_\epsilon(u_s)\,\mathrm{d}s, \end{aligned}$$

Moreover, note that $M = M^{[F_h]}$, where $F_h \in D(\mathcal{E})$ is given by

$$F_h(z) := \langle z, h\rangle, \quad z \in K.$$

Hence, by Theorem 5.2.3 in [FOT10], $\mu_{<M>}$ is given by $\|h\|^2_{L^2(0,1)} \cdot \nu$. For the last statement, we apply [FOT10, Corollary 5.2.1]. □

4.2.5 A distinction result

As a consequence of our IbPF and the above constructions, we can prove that the Markov process $(u_t)_{t\geq 0}$ obtained above is not identically equal in law to the process corresponding to the modulus of the solution $(v_t)_{t\geq 0}$ to the stochastic heat equation, as might suggest the analogous relation between the invariant measures μ and ν.

Let $K^{\mathbb{R}_+}$ denote the space of functions from $\mathbb{R}_+$ to K, endowed with the product σ-algebra. For all $x \in K$, let P_x be the law, on $K^{\mathbb{R}_+}$, of the Markov process $(u_t)_{t\geq 0}$ associated with $\mathcal{E}$, started from x. Similarly, for all $z \in H$, let $\mathbf{P}_z$ be the law, on $K^{\mathbb{R}_+}$, of $(|v_t|)_{t\geq 0}$, where $(v_t)_{t\geq 0}$ is the solution of the stochastic heat equation (4.10) with $v_0 = z$.

Theorem 4.2.9.

$$\mu\left(\{z \in H : P_{|z|} \neq \mathbf{P}_z\}\right) > 0.$$

Proof. Assume by contradiction that $P_{|z|} = \mathbf{P}_z$ for μ-a.e. $z \in H$. Then, recalling that $(\mathbf{Q}_t)_{t\geq 0}$ denotes the semigroup associated with Λ, and $(Q_t)_{t\geq 0}$ the semigroup associated with $\mathcal{E}$, we would have

$$\mathbf{Q}_t(f \circ j) = (Q_t f) \circ j, \quad \mu - \text{a.e.},$$

for all $t \geq 0$ and $f \in L^2(\nu)$. Therefore, the corresponding families of resolvents $(\mathbf{R}_\lambda)_{\lambda>0}$ and $(R_\lambda)_{\lambda>0}$ would satisfy, for all $f \in L^2(\nu)$

$$\mathbf{R}_1(f \circ j) = (R_1 f) \circ j,$$

where the equality holds in $L^2(\mu)$. In particular, this shows that $(R_1 f) \circ j \in D(\Lambda)$ for any f as above. We then claim that, for all $F \in D(\Lambda)$

$$\Pi F(y) = \mathbb{E}[F(\beta) \,|\, |\beta| = y\,], \tag{4.34}$$

for ν-a.e $y \in K$, where β is a Brownian bridge from 0 to 0 on $[0, 1]$. Indeed, by the previous observations, for all $f \in L^2(\nu)$, it holds

$$\begin{aligned}\int_H (f \circ j)(z) F(z)\, \mathrm{d}\mu(z) &= \Lambda_1(\mathbf{R}_1(f \circ j), F) = \Lambda_1((R_1 f) \circ j, F) \\ &= \mathcal{E}_1(R_1 f, \Pi F) = \int_K f(x)(\Pi F)(x)\, \mathrm{d}\nu(x),\end{aligned} \tag{4.35}$$

i.e. $\Pi F(y) = \mathbb{E}[F(\beta) \,|\, |\beta| = y\,]$ for ν-a.e. $y \in K$, as claimed. By (4.35) and the first equality in Lemma 4.2.6, we deduce that, for all $f \in D(\mathcal{E})$ and $F \in D(\Lambda)$

$$\Lambda(F, f \circ j) = \mathcal{E}(\Pi F, f).$$

Consider now the process $(v_t)_{t\geq 0}$ associated with Λ and started from $v_0 = \beta$, where β is a Brownian bridge on $[0,1]$. Consider also the process $(u_t)_{t\geq 0}$ associated with $\mathcal{E}$ under the law $\mathbb{P}_\nu$ (so that, in particular, $u_0 \overset{(d)}{=} |\beta|$). Thus the processes v and u are stationary, and $|v| \overset{(d)}{=} u$ by our assumption. Let us set

$$A_t := \langle |v_t|, h\rangle - \langle |v_0|, h\rangle - \frac{1}{2}\int_0^t \langle |v_s|, h''\rangle \, \mathrm{d}s,$$

$$C_t := \langle u_t, h\rangle - \langle u_0, h\rangle - \frac{1}{2}\int_0^t \langle u_s, h''\rangle \, \mathrm{d}s.$$

Let further $k \in C^2([0,1])$ with $k(0) = k(1) = 0$, and consider the functionals $\Psi_k : H \to \mathbb{R}$ and $\tilde{\Psi}_k : K \to \mathbb{R}$ given by

$$\Psi_k(z) := \exp(\langle k, z\rangle), \qquad z \in H$$

$$\tilde{\Psi}_k(y) := \mathbb{E}\left[\Psi_k(\beta) \,|\, |\beta| = y\right], \qquad y \in K.$$

Note that $\Psi_k \in D(\Lambda)$, and recall that, by the above remarks, $\tilde{\Psi}_k = \Pi\Psi_k$ ν-a.e., so in particular $\tilde{\Psi}_k \in D(\mathcal{E})$. We then have

$$\begin{aligned}
&J(t) := -\frac{d}{dt}\mathbb{E}\left[A_t\, \tilde{\Psi}_k(|v_0|)\right] = \\
&= -\frac{d}{dt}\mathbb{E}\left[(\langle u_t, h\rangle - \langle u_0, h\rangle)\, \tilde{\Psi}_k(u_0)\right] + \frac{1}{2}\frac{d}{dt}\mathbb{E}\left[\int_0^t \langle h'', |v_s|\rangle \, \mathrm{d}s\, \Psi_k(\beta)\right] \\
&= \mathcal{E}(\langle \cdot, h\rangle\,,\, \tilde{\Psi}_k) + \frac{1}{2}\mathbb{E}[\langle h'', |\beta|\rangle \Psi_k(\beta)] = \Lambda(\langle |\cdot|, h\rangle\,,\, \Psi_k) + \frac{1}{2}\mathbb{E}[\langle h'', |\beta|\rangle \Psi_k(\beta)] \\
&= \frac{1}{2}\mathbb{E}[\langle \nabla\Psi_k(\beta), \mathrm{sign}(\beta)\, h\rangle + \langle h'', |\beta|\rangle \Psi_k(\beta)] = \mathbb{E}\left[\Psi_k(\beta)\int_0^1 h \,:\dot{\beta}^2:\, \mathrm{d}L^0\right]
\end{aligned}$$

by (3.10) in [Zam05], or rather its analogue for the Brownian bridge as stated in Remark 1.3 of [GV16]. But, by [Zam05, Corollary 3.4] and [GV16, Theorem 3.2], the last quantity equals

$$\frac{1}{\sqrt{2\pi}}\, e^{\frac{1}{2}\langle Qk,k\rangle} \int_0^1 \frac{h_r}{\sqrt{r(1-r)}} \exp\left(-\frac{K_r^2}{2r(1-r)}\right) \lambda(K_r', -K_r, r)\, \mathrm{d}r,$$

where $K = Qk$, with Q the covariance operator of β,

$$(Qk)_r = \int_0^1 (r \wedge \sigma - r\sigma)\, k_\sigma \, \mathrm{d}\sigma, \qquad r \in [0,1],$$

and $\lambda : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ is defined by

$$\lambda(x,y,r) := x^2 + xy\frac{1-2r}{r(1-r)} + y^2\frac{(1-2r)^2}{4r^2(1-r)^2} - \frac{1}{4r(1-r)}, \quad x, y \in \mathbb{R},\ r \in [0,1].$$

Hence,

$$J(t) = \sqrt{\frac{1}{2\pi}} e^{\frac{1}{2}\langle Qk,k\rangle} \int_0^1 \frac{h_r}{\sqrt{r(1-r)}} \exp\left(-\frac{K_r^2}{2r(1-r)}\right) \lambda(K_r', -K_r, r)\, \mathrm{d}r. \quad (4.36)$$

On the other hand

$$\begin{aligned} L(t) &:= -\frac{d}{dt}\mathbb{E}\left[C_t\, \tilde{\Psi}_k(|v_0|)\right]\bigg|_{t=0} = \mathcal{E}(\Pi\Psi_k, \langle\cdot, h\rangle) + \frac{1}{2}\mathbb{E}[\langle h'', |\beta|\rangle\, \Pi\Psi_k(|\beta|)] \\ &= \frac{1}{2}\mathcal{E}(\Pi U, \Pi\Psi_k) = \frac{1}{4}\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^1 h_r\, \rho_\epsilon''(|\beta_r|)\, \mathrm{d}r\, \Pi\Psi_k(|\beta|)\right], \end{aligned}$$

where we used (4.29) to obtain the second equality, and the fact that $U = \lim_{\epsilon\to 0} U_\epsilon$ in $D(\mathcal{E})$, combined with (4.32), to obtain the third one. Therefore, recalling (4.34), we have

$$L(t) = \frac{1}{4}\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^1 h_r\, \rho_\epsilon''(|\beta_r|)\, \mathrm{d}r\, \Psi_k(\beta)\right] = \frac{1}{4}\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^1 h_r\, \rho_\epsilon''(\beta_r)\, \mathrm{d}r\, e^{\langle k,\beta\rangle}\right].$$

By the Cameron-Martin formula, for all $\epsilon > 0$

$$\begin{aligned} &\frac{1}{4}\mathbb{E}\left[\int_0^1 h_r\, \rho_\epsilon''(\beta_r)\, \mathrm{d}r\, e^{\langle k,\beta\rangle}\right] = \\ &= \frac{1}{4}e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}}\int_{\mathbb{R}} \rho_\epsilon''(a)\exp\left(-\frac{(a-K_r)^2}{2r(1-r)}\right)\mathrm{d}a\,\mathrm{d}r \\ &\underset{\epsilon\to 0}{\to} \frac{1}{4}e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}}\left[\frac{K_r^2 - r(1-r)}{r^2(1-r)^2}\right]\exp\left(-\frac{K_r^2}{2r(1-r)}\right)\mathrm{d}r. \end{aligned}$$

Hence we obtain

$$L(t) = \frac{1}{4}e^{\frac{1}{2}\langle Qk,k\rangle}\int_0^1 \frac{h_r}{\sqrt{2\pi r(1-r)}}\left[\frac{K_r^2 - r(1-r)}{r^2(1-r)^2}\right]\exp\left(-\frac{K_r^2}{2r(1-r)}\right)\mathrm{d}r. \quad (4.37)$$

Since $|v|$ and u have the same law, $J(t) = L(t)$ and therefore the right-hand sides of (4.36) and (4.37) above are equal. This being true for any $h \in C_c^2(0,1)$, we deduce that

$$\frac{K_r^2 - r(1-r)}{4r^2(1-r)^2} = \lambda(K_r', -K_r, r),$$

for a.e. $r \in (0,1)$, hence for all r by continuity. We thus deduce that

$$(K_r')^2 - \frac{1-2r}{r(1-r)}K_r K_r' - \frac{1}{r(1-r)}K_r^2 = 0, \qquad \forall\, r \in (0,1).$$

Since we can choose $k \in C_c^2(0,1)$ such that $K = Qk$ does not satisfy the above equation, we obtain a contradiction. □

4.2.6 Proofs of two technical results

Proof of Proposition 4.2.1. Since $D(\Lambda)$ contains all globally Lipschitz functions on H, for all $f \in \mathcal{FC}_b^\infty(K)$ we have $f \circ j \in D(\Lambda)$. A simple calculation shows that for any $f \in \mathcal{FC}_b^\infty(K)$ of the form (4.15) we have

$$\nabla(f \circ j)(z) = \nabla f(j(z)) \operatorname{sgn}(z). \tag{4.38}$$

Hence, for all $f, g \in \mathcal{FC}_b^\infty(K)$, we have

$$\begin{aligned}\mathcal{E}(f,g) &= \frac{1}{2}\int \langle \nabla f(x), \nabla g(x)\rangle \, \mathrm{d}\nu(x) = \frac{1}{2}\int \langle \nabla f(j(z)), \nabla g(j(z))\rangle \, \mathrm{d}\mu(z) \\ &= \frac{1}{2}\int \langle \nabla (f\circ j)(z), \nabla (g \circ j)(z)\rangle \, \mathrm{d}\mu(z) = \Lambda(f \circ j, g \circ j),\end{aligned}$$

where the third equality follows from (4.38). This shows that the bilinear symmetric form $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ admits as an extension the image of the Dirichlet form $(\Lambda, D(\Lambda))$ under the map j. Since $\mathcal{FC}_b^\infty(K)$ is dense in $L^2(\nu)$, this extension is a Dirichlet form. In particular, $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ is closable, its closure $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form, and we have the isometry property (4.16).

There remains to prove that the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Since it is the closure of $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$, it suffices to show that the associated capacity is tight. Since K is separable, we can find a countable dense subset $\{y_k,\ k \in \mathbb{N}\} \subset K$ such that $y_k \neq 0$ for all $k \in \mathbb{N}$. Let now $\varphi \in C_b^\infty(\mathbb{R})$ be an increasing function such that $\varphi(t) = t$ for all $t \in [-1,1]$ and $\|\varphi'\|_\infty \leq 1$. For all $m \in \mathbb{N}$, we define the function $v_m : K \to \mathbb{R}$ by

$$v_m(z) := \varphi(\|z - y_m\|), \quad z \in K.$$

Moreover, we set, for all $n \in \mathbb{N}$

$$w_n(z) := \inf_{m \leq n} v_m(z), \quad z \in K.$$

We claim that $w_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, and that $w_n \underset{n\to\infty}{\longrightarrow} 0$, $\mathcal{E}$ quasi-uniformly in K. Assuming this claim for the moment, for all $k \geq 1$ we can find a closed subset F_k of K such that $\mathrm{Cap}(K \setminus F_k) < 1/k$, and $w_n \underset{n\to\infty}{\longrightarrow} 0$ uniformly on F_k. Hence, for all $\epsilon > 0$, we can find $n \in \mathbb{N}$ such that $w_n < \epsilon$ on F_k. Therefore

$$F_k \subset \bigcup_{m \leq n} B(y_m, \epsilon)$$

where $B(y, r)$ is the open ball in K centered at $y \in K$ with radius $r > 0$. This shows that F_k is totally bounded. Since it is, moreover, complete as a closed subspace of a complete metric space, it is compact, and the tightness of Cap follows.

We now justify our claim. For all $i \in \mathbb{N}$, we set $l_i := \|y_i\|^{-1} y_i$. Then for all $i \geq 1$, $l_i \in K$, $\|l_i\| = 1$ and, for all $z \in K$

$$\|z\| = \sup_{i \geq 0} \langle l_i, z \rangle.$$

Let $m \in \mathbb{N}$ be fixed. For all $i \geq 0$, let $u_i(z) := \sup_{j \leq i} \varphi(\langle l_j, z - y_m \rangle)$, $z \in K$. We have $u_i \in D(\mathcal{E})$, and, for ν - a.e. $z \in K$

$$\sum_{k=1}^{\infty} \frac{\partial u_i}{\partial e_k}(z)^2 \leq \sup_{j \leq i} \left(\sum_{k=1}^{\infty} \varphi'(\langle l_j, z - y_m \rangle)^2 \langle l_j, e_k \rangle^2 \right) \leq 1,$$

whence $\mathcal{E}(u_i, u_i) \leq 1$. By the definition of v_m, as $i \to \infty$, $u_i \uparrow v_m$ on K, hence in $L^2(K, \nu)$. By [MR92, I.2.12], we deduce that $v_m \in D(\mathcal{E})$, and that $\mathcal{E}(v_m, v_m) \leq 1$. Therefore, for all $n \in \mathbb{N}$, $w_n \in D(\mathcal{E})$, and $\mathcal{E}(w_n, w_n) \leq 1$. But, since $\{y_k,\, k \in \mathbb{N}\}$ is dense in K, as $n \to \infty$, $w_n \downarrow 0$ on K. Hence $w_n \underset{n \to \infty}{\longrightarrow} 0$ in $L^2(K, \nu)$. This and the previous bound imply, by [MR92, I.2.12], that the Cesàro means of some subsequence of $(w_n)_{n \geq 0}$ converge to 0 in $D(\mathcal{E})$. By [MR92, III.3.5], some subsequence thereof converges $\mathcal{E}$ quasi-uniformly to 0. But, since $(w_n)_{n \geq 0}$ is non-increasing, we deduce that it converges $\mathcal{E}$-quasi-uniformly to 0. The claimed quasi-regularity follows.

There finally remains to check that $(\mathcal{E}, D(\mathcal{E}))$ is local in the sense of Definition [MR92, V.1.1]. Let $u, v \in D(\mathcal{E})$ satisfying $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$. Then, $u \circ j$ and $v \circ j$ are two elements of $D(\Lambda) = W^{1,2}(\mu)$ with disjoint supports, and, recalling (4.16), we have

$$\mathcal{E}(u, v) = \Lambda(u \circ j, v \circ j) = \frac{1}{2} \int_H \nabla(u \circ j) \cdot \nabla(v \circ j) \, \mathrm{d}\mu = 0.$$

The claim follows. □

Proof of Lemma 4.2.3. Recall that $D(\mathcal{E})$ is the closure under the bilinear form $\mathcal{E}_1$ of the space $\mathcal{FC}_b^\infty(K)$ of functionals of the form $F = \Phi\big|_K$, where $\Phi \in \mathcal{FC}_b^\infty(H)$. Therefore, to prove the claim, it suffices to show that for any functional $\Phi \in \mathcal{FC}_b^\infty(H)$ and all $\epsilon > 0$, there exists $\Psi \in \mathscr{S}$ such that $\mathcal{E}_1(\Phi - \Psi, \Phi - \Psi) < \epsilon$.

Let $\Phi \in \mathcal{FC}_b^\infty(H)$. We set for all $\epsilon > 0$

$$\Phi_\epsilon(\zeta) := \Phi(\sqrt{\zeta^2 + \epsilon}), \quad \zeta \in H.$$

A simple calculation shows that $\Phi_\epsilon \underset{\epsilon \to 0}{\longrightarrow} \Phi$ and $\nabla \Phi_\epsilon \underset{\epsilon \to 0}{\longrightarrow} \nabla \Phi$ pointwise, with uniform bounds $\|\Phi_\epsilon\|_\infty \leq \|\Phi\|_\infty$ and $\|\nabla \Phi_\epsilon\|_\infty \leq \|\nabla \Phi\|_\infty$. Hence, by dominated convergence, $\mathcal{E}_1(\Phi_\epsilon - \Phi, \Phi_\epsilon - \Phi) \underset{\epsilon \to 0}{\longrightarrow} 0$. Then, introducing for all $d \geq 1$ $(\zeta_i^d)_{1 \leq i \leq d}$ the orthonormal family in $L^2(0,1)$ given by

$$\zeta_i^d := \sqrt{d}\, \mathbf{1}_{[\frac{i-1}{d}, \frac{i}{d}[}, \quad i = 1, \ldots, d,$$

and setting

$$\Phi_\epsilon^d(\zeta) := \Phi_\epsilon \left(\left(\sum_{i=1}^d \langle \zeta_i^d, \zeta^2 \rangle \right)^{\frac{1}{2}} \right) = \Phi \left(\left(\sum_{i=1}^d \langle \zeta_i^d, \zeta^2 \rangle + \epsilon \right)^{\frac{1}{2}} \right), \quad \zeta \in H,$$

again we obtain the convergence $\mathcal{E}_1(\Phi_\epsilon^d - \Phi_\epsilon, \Phi_\epsilon^d - \Phi_\epsilon) \underset{d \to \infty}{\longrightarrow} 0$.

There remains to show that any fixed functional of the form

$$\Phi(\zeta) = f\left(\langle \zeta_1, \zeta^2 \rangle, \ldots, \langle \zeta_d, \zeta^2 \rangle \right), \quad \zeta \in H$$

with $d \geq 1$, $f \in C_b^1(\mathbb{R}_+^d)$, and $(\zeta_i)_{i=1,\ldots,d}$ a family of elements of K, can be approximated by elements of $\mathscr{S}$. Again by dominated convergence, we can suppose that f has compact support in $\mathbb{R}_+^d$. We define $g \in C_b^1([0,1]^d)$,

$$g(y) := f(-\ln(y_1), \cdots, -\ln(y_d)), \quad y \in]0,1]^d,$$

and $g(y) := 0$ if $y_i = 0$ for any $i = 1, \ldots, d$. By a differentiable version of the Weierstrass Approximation Theorem (see Theorem 1.1.2 in [Lla86]), there exists a sequence $(p_k)_{k \geq 1}$ of polynomial functions converging to g for the C^1 topology on $[0,1]^d$. Defining for all $k \geq 1$ the function $f_k : \mathbb{R}_+^d \to \mathbb{R}$ by

$$f_k(x) = p_k(e^{-x_1}, \cdots, e^{-x_d}), \quad x \in \mathbb{R}_+^d,$$

we define $\Phi_k \in \mathscr{S}$ by

$$\Phi_k(\zeta) = f_k\left(\langle \zeta_1, \zeta^2 \rangle, \ldots, \langle \zeta_d, \zeta^2 \rangle \right), \quad \zeta \in H.$$

Since $p_k \underset{k \to \infty}{\longrightarrow} g$ for the C^1 topology on $[0,1]^d$, $f_k \underset{k \to \infty}{\longrightarrow} f$ uniformly on $\mathbb{R}_+^d$ together with its first order derivatives. Hence, it follows that $\Phi_k \underset{k \to \infty}{\longrightarrow} \Phi$ pointwise on K together with its gradient. It also follows that there exists $C > 0$ such that for all $k \geq 1$

$$\forall \zeta \in K, \quad |\Phi_k(\zeta)|^2 + \|\nabla \Phi_k(\zeta)\|^2 \leq C(1 + \|\zeta\|^2).$$

Since the quantity in the right-hand side is ν integrable in ζ, it follows by dominated convergence that $\mathcal{E}_1(\Phi_k - \Phi, \Phi_k - \Phi) \underset{k \to \infty}{\longrightarrow} 0$. This yields the claim. □

4.3 The case $\delta = 2$

We now use the same techniques as above to construct a weak version of the gradient dynamics associated with the law of a 2-dimensional Bessel bridge from 0 to 0 over $[0,1]$.

We denote by ν_2 the law, on K, of the 2-Bessel bridge from 0 to 0 on $[0,1]$ (so that P^2 is then the restriction of ν_2 to $C([0,1])$). We shall use the shorthand $L^2(\nu)$ to denote the space $L^2(K,\nu)$.

4.3.1 The 2-dimensional random string

Consider the space $\mathbb{H}_2 := L^2([0,1],\mathbb{R}^2)$ endowed with the component-wise L^2 product. Let μ_2 denote the law, on $\mathbb{H}_2$, of a two-dimensional Brownian bridge from 0 to 0. We shall use the shorthand notation $L^2(\mu_2)$ for the space $L^2(\mathbb{H}_2,\mu_2)$. Consider moreover the semigroup $(\mathbf{Q}^2_t)_{t\geq 0}$ on $\mathbb{H}_2$ defined, for all $F \in L^2(\mathbb{H}_2,\mu_2)$, and $z=(z_1,z_2)\in\mathbb{H}_2$, by

$$\mathbf{Q}^2_t F(z) := \mathbb{E}\left[F(v_t(z))\right], \quad t\geq 0,$$

where $(v_t(z))_{t\geq 0}$ is the solution to the 2-dimensional stochastic heat equation with initial condition z, and with homogeneous Dirichlet boundary conditions

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \xi \\ v(0,x) = z(x), \qquad x\in[0,1], \\ v(t,0) = v(t,1) = 0, \end{cases}$$

where $\xi := (\xi_1,\xi_2)$, with ξ_1,ξ_2 two independent space-time white noises on $\mathbb{R}_+\times[0,1]$. More precisely

$$v(t,x) = (v_1(t,x), v_2(t,x)), \quad t\geq 0,\ x\in[0,1],$$

where, for $i=1,2$

$$v_i(t,x) = z_i(t,x) + \int_0^t\int_0^1 g_{t-s}(x,x')\,\xi_i(\mathrm{d}s,\,\mathrm{d}x'),$$

with $z_i(t,x) := \int_0^1 g_t(x,x')z_i(x')\,\mathrm{d}x'$. In words, v is the vector composed of two independent copies of a solution to the one-dimensional stochastic heat equation, with respective intial data z_1 and z_2. In particular, it follows from this fomula that v is a Gaussian process and, for all $t\geq 0$ and $x,x'\in(0,1)$

$$\mathrm{Cov}(v_1(t,x),v_1(t,x')) = \mathrm{Cov}(v_2(t,x),v_2(t,x')) = q_t(x,x') := \int_0^t g_{2\tau}(x,x')\,\mathrm{d}\tau,$$

We denote by $(\Lambda^2, D(\Lambda^2))$ the Dirichlet form generated by $(\mathbf{Q}^2_t)_{t\geq 0}$ in $L^2(\mathbb{H}_2, \mu_2)$, and which is given by

$$\Lambda^2(F,G) = \frac{1}{2}\int_{\mathbb{H}_2} \langle \overline{\nabla} F, \overline{\nabla} G\rangle_{\mathbb{H}_2} \, \mathrm{d}\mu_2, \quad F, G \in D(\Lambda^2) = W^{1,2}(\mu_2),$$

where, for all $F \in W^{1,2}(\mu_2)$, $\overline{\nabla} F : \mathbb{H}_2 \to \mathbb{H}_2$ denotes the gradient of F in $\mathbb{H}_2$, see [DPZ02].

4.3.2 Gradient Dirichlet form associated with the 2-dimensional Bessel bridge

Recalling the definition of $\mathcal{FC}^\infty_b(K)$ (see Section 4.2.2 above), we denote by $\mathcal{E}^2$ the bilinear form defined on $\mathcal{FC}^\infty_b(K)$ by

$$\forall f, g \in \mathcal{FC}^\infty_b(K), \quad \mathcal{E}^2(f,g) := \frac{1}{2}\int \langle \nabla f, \nabla g\rangle \, \mathrm{d}\nu_2.$$

We also denote by $j_2 : \mathbb{H}_2 \to K$ the map

$$j_2(z) := \|z\| = \sqrt{(z_1)^2 + (z_2)^2}, \quad z = (z_1, z_2) \in \mathbb{H}_2 \tag{4.39}$$

Note that

$$\nu_2 = \mu_2 \circ j_2^{-1}, \tag{4.40}$$

so that the map

$$\begin{cases} L^2(\nu_2) \to L^2(\mu_2) \\ \varphi \mapsto \varphi \circ j_2 \end{cases}$$

is an isometry. We have the following result, the proof of which is postponed to Section 4.4 below.

Proposition 4.3.1. *The form $(\mathcal{E}^2, \mathcal{FC}^\infty_b(K))$ is closable. Its closure $(\mathcal{E}^2, D(\mathcal{E}^2))$ is a local, quasi-regular Dirichlet form on $L^2(\nu_2)$. Moreover, for all $f \in D(\mathcal{E}^2)$, $f \circ j \in D(\Lambda^2)$, and we have*

$$\forall f, g \in D(\mathcal{E}^2), \quad \mathcal{E}^2(f,g) = \Lambda^2(f\circ j, g \circ j).$$

Let $(Q^2_t)_{t\geq 0}$ be the contraction semigroup on (K, ν_2) associated with the Dirichlet form $(\mathcal{E}^2, D(\mathcal{E}^2))$, and recall that $\mathcal{B}_b(K)$ denotes the set of Borel and bounded functions on K. As a consequence of Prop. 4.3.1, in virtue of Thm IV.3.5 and Thm V.1.5 in [MR92], we obtain the following:

Corollary 4.3.2. *There exists a Markov diffusion process*

$$M = \{\Omega, \mathcal{F}, (u_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in K}\}$$

properly associated to $(\mathcal{E}^2, D(\mathcal{E}^2))$*, i.e. for all* $\varphi \in L^2(\nu_2) \cap \mathcal{B}_b(K)$*, and for all* $t > 0$*,* $E_x(\varphi(u_t)), x \in K$ *defines an* $\mathcal{E}^2$ *quasi-continuous version of* $Q_t^2\varphi$*. Moreover, the process* M *admits the following continuity property*

$$P_x[t \mapsto u_t \text{ is continuous on } \mathbb{R}_+] = 1, \quad \text{for } \mathcal{E}^2 \text{ q.e. } x \in K.$$

Using the same techniques as for the case $\delta = 1$, we will show that for $\mathcal{E}^2$ q.e. $x \in K$, under $\mathbb{P}_x$, $(u_t)_{t\geq 0}$ solves (1.24) with $\delta = 2$, or rather its weaker form (1.29). As was done for $\delta = 1$, since the Dirichlet form $(\mathcal{E}^2, D(\mathcal{E}^2))$ is quasi-regular, by the transfer method stated in VI.2 of [MR92], we can apply several results of [FOT10] in our setting.

In the sequel, we set $\Lambda_1^2 := \Lambda^2 + (\cdot,\cdot)_{L^2(\mu_2)}$ and $\mathcal{E}_1^2 := \mathcal{E}^2 + (\cdot,\cdot)_{L^2(\nu_2)}$. As for $\delta = 1$, we have the following important technical point:

Lemma 4.3.3. $\mathscr{S}_K$ *is dense in* $D(\mathcal{E}^2)$*.*

Proof. The same arguments as for the proof of Lemma 4.2.3 apply here. Indeed, the only particular feature of the space $D(\mathcal{E})$ used in the proof of Lemma 4.2.3 is the fact that ν has finite second moments, that is $\int_K \|x\|^2 \,\mathrm{d}\nu(x) < \infty$. Since the same is true for ν_2 in place of ν, the same arguments apply for $D(\mathcal{E}^2)$ in place of $D(\mathcal{E})$, and the claim follows. □

4.3.3 Convergence of one-potentials

In order to write an SPDE associated with ν_2, we still need to give a more robust interpretation of the last term in the IbPF (3.2) for $\delta = 2$ (and $a = a' = 0$). Recall that we have the following equality in law on K

$$(\|\beta_t\|)_{0\leq t\leq 1} \overset{(d)}{=} \nu_2, \tag{4.41}$$

where $\beta = (\beta^1, \beta^2)$ is a two-dimensional Brownian bridge from 0 to 0, and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^2$. Let now $(\rho_\eta)_{\eta>0}$ be a family of mollifiers as in (4.17). Then, for any $\Phi \in \mathcal{S}$, the last term of the right-hand side in the IbPF (3.2) with $\delta = 2$ and $a = a' = 0$ can be re-expressed using the equality

$$\begin{aligned} &- \kappa(2) \int_0^1 h_r \int_0^\infty \mathrm{d}b\, b^{-2} \left(\Sigma_r^2(\Phi(X)\,|\,b) - \Sigma_r^2(\Phi(X)\,|\,0)\right) \mathrm{d}r \\ &= \frac{1}{4} \lim_{\epsilon\to 0}\lim_{\eta\to 0} \mathbb{E}\left[\Phi(\|\beta\|) \int_0^1 h_r \left(\frac{\mathbf{1}_{\|\beta_r\|\geq\epsilon}}{\|\beta_r\|^3} - \frac{2}{\epsilon}\frac{\rho_\eta(\|\beta_r\|)}{\|\beta_r\|}\right) \mathrm{d}r\right]. \end{aligned} \tag{4.42}$$

Indeed, the equality follows easily upon noting that $X := \|\beta\|$ is distributed according to ν_2, and by conditioning on the value of X_r, $r \in (0,1)$.

Let $\epsilon, \eta > 0$ with $\eta < \epsilon$. We consider the functional $G_{\epsilon,\eta} : \mathbb{H}_2 \to \mathbb{R}$ defined, for all $z = (z^1, z^2) \in \mathbb{H}_2$ by

$$G_{\epsilon,\eta}(z) := \int_0^1 \mathrm{d}r \ h_r \, f_{\epsilon,\eta}(\|z_r\|),$$

where

$$f_{\epsilon,\eta}(x) := \frac{1}{4}\left(\frac{\mathbf{1}_{x\geq\epsilon}}{x^3} - \frac{2}{\epsilon}\frac{\rho_\eta(x)}{x}\right), \quad x \geq 0.$$

For all $\epsilon > \eta > 0$, we then define the functional $V_{\epsilon,\eta} : \mathbb{H}_2 \to \mathbb{R}$ by

$$V_{\epsilon,\eta}(z) = \int_0^\infty e^{-t}\, \mathbf{Q}^2_t G_{\epsilon,\eta}(z)\, \mathrm{d}t, \quad z \in H.$$

Note that, in the language of [FOT10], Chap. 5, $V_{\epsilon,\eta}$ is the one-potential associated with the continuous additive functional

$$\int_0^t G_{\epsilon,\eta}(v(s,\cdot))\, \mathrm{d}s, \quad t \geq 0.$$

In particular, $V_{\epsilon,\eta} \in D(\Lambda^2)$.

We will show that $V_{\epsilon,\eta}$ converges in $D(\Lambda^2)$ as we send η and ϵ to 0. To do so, we remark that, for all $z \in \mathbb{H}_2$, we have

$$V_{\epsilon,\eta}(z) = \int_0^\infty \int_0^1 \frac{e^{-t}\, h_r}{8\pi q_t(r)}\, \mathrm{d}r \int_{\mathbb{R}^2} \left(\frac{\mathbf{1}_{\|a\|\geq\epsilon}}{\|a\|^3} - \frac{2}{\epsilon}\frac{\rho_\eta(\|a\|)}{\|a\|}\right) \exp\left(-\frac{\|a - z(t,r)\|^2}{2q_t(r)}\right) \mathrm{d}a,$$

where, for all $t > 0$ and $r \in (0,1)$, $z(t,r) := (z_1(t,r), z_2(t,r))$. We define also the functional $V_\epsilon : \mathbb{H}_2 \to \mathbb{R}$ by setting, for all $z \in \mathbb{H}_2$

$$V_\epsilon(z) := \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \frac{h_r}{8\pi q_t(r)}\, \mathrm{d}r \int_{\mathbb{R}^2} \mathrm{d}a \frac{\mathbf{1}_{\|a\|\geq\epsilon}}{\|a\|^3} \cdot \\ \cdot \left(\exp\left(-\frac{\|a - z(t,r)\|^2}{2q_t(r)}\right) - \exp\left(-\frac{\|z(t,r)\|^2}{2q_t(r)}\right)\right).$$

Note that, by splitting the domain $\mathbb{R}^2$ into four quadrants, we can rewrite

$$V_\epsilon(z) := \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \frac{h_r}{8\pi q_t(r)}\, \mathrm{d}r \int_{\mathbb{R}^2_+} \mathrm{d}a\, \frac{\mathbf{1}_{\|a\|\geq\epsilon}}{\|a\|^3} \cdot \\ \cdot \sum_{\alpha\in\{-1,1\}^2} \left(\exp\left(-\frac{\|\alpha a - z(t,r)\|^2}{2q_t(r)}\right) - \exp\left(-\frac{\|z(t,r)\|^2}{2q_t(r)}\right)\right), \tag{4.43}$$

where, for all $a \in \mathbb{R}^2$ and $\alpha \in \{-1,1\}^2$, we have set

$$\alpha\, a := (\alpha_1 a_1, \alpha_2 a_2).$$

Let us finally define the functional $V : \mathbb{H}_2 \to \mathbb{R}$ by setting, for all $z \in \mathbb{H}_2$

$$\begin{aligned} V(z) := \int_0^\infty \mathrm{d}t\, e^{-t} \int_0^1 \frac{h_r}{8\pi q_t(r)}\, \mathrm{d}r \int_{\mathbb{R}^2_+} \frac{\mathrm{d}a}{\|a\|^3} \cdot \\ \cdot \sum_{\alpha \in \{-1,1\}^2} \left(\exp\left(-\frac{\|\alpha a - z(t,r)\|^2}{2q_t(r)} \right) - \exp\left(-\frac{\|z(t,r)\|^2}{2q_t(r)} \right) \right). \end{aligned} \tag{4.44}$$

We can then state the following result, the proof of which is postponed to Section 4.4 below:

Proposition 4.3.4. *The functionals V_ϵ and V all belong to $D(\Lambda^2)$. Moreover*

$$\forall \epsilon > 0, \qquad \lim_{\eta \to 0} V_{\epsilon,\eta} = V_\epsilon,$$

and

$$\lim_{\epsilon \to 0} V_\epsilon = V,$$

where all convergences take place in $D(\Lambda^2)$.

4.3.4 The dynamics for $\delta = 2$

We now explain how the IbPF for P^2 can be used to analyze the Markov process M constructed above.

As for the case $\delta = 1$ considered above, we shall exploit a projection principle, the proof of which follows exactly as for Lemma 4.2.6 above .

Lemma 4.3.5. *There exists a unique bounded linear operator $\Pi : D(\Lambda^2) \to D(\mathcal{E}^2)$ such that, for all $F, G \in D(\Lambda^2)$ and $f \in D(\mathcal{E}^2)$*

$$\Lambda_1^2(F, f \circ j_2) = \mathcal{E}_1^2(\Pi F, f).$$

where j_2 is as in (4.39)*. Moreover, we have*

$$\mathcal{E}_1^2(\Pi F, \Pi F) \le \Lambda_1^2(F,F).$$

As a consequence, we can obtain a stronger version of the IbPF for P^2. Recall that, by Prop 4.3.4 and by the above definitions, $V = \lim_{\epsilon \to 0} \lim_{\eta \to 0} V_{\epsilon,\eta}$, where $V_{\epsilon,\eta} \in D(\Lambda^2)$ is the one-potential of the additive functional

$$\int_0^t \mathrm{d}s\, G_{\epsilon,\eta}(v(s,\cdot)) = \int_0^t \mathrm{d}s \int_0^1 \mathrm{d}r\, h_r f_{\epsilon,\eta}(\|v(s,r)\|)\, \mathrm{d}r\, \mathrm{d}s.$$

Therefore, combining the IbPF (3.2) with $\delta = 2$ and $a = a' = 0$, the equality (4.42), the density result 4.3.3 and the projection principle 4.3.5, and arguing as for the proof of Corollary 4.2.7, we obtain the following result:

Corollary 4.3.6. *For all $f \in D(\Lambda^2)$ and $h \in C_c^2(0,1)$, we have*

$$\mathcal{E}^2\left(\langle h, \cdot\rangle - \frac{1}{2}\Pi V\,,\, f\right) = -\frac{1}{2}\int_K (\langle h'', \zeta\rangle - \Pi V(\zeta)) f(\zeta)\, \mathrm{d}\nu_2(\zeta), \qquad (4.45)$$

where V is as in (4.44).

Recall that $M = \{\Omega, \mathcal{F}, (u_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in K}\}$ denotes the Markov process properly associated with the Dirichlet form $(\mathcal{E}^2, D(\mathcal{E}^2))$ constructed above. The following theorem states that the process $(u_t)_{t\geq 0}$ satisfies the SPDE (1.29) above, which is a weaker version of the Bessel SPDE (1.24) with $\delta = 2$.

Theorem 4.3.7. *For all $h \in C_c^2(0,1)$, we have, almost surely*

$$\langle u_t, h\rangle - \langle u_0, h\rangle = M_t + N_t, \quad t \geq 0, \qquad \mathbb{P}_{u_0} - a.s., \quad q.e.\ u_0 \in K..$$

Here $(N_t)_{t\geq 0}$ is a continuous additive functional of zero energy satisfying

$$N_t = \frac{1}{2}\int_0^t \langle h'', u_s\rangle\, \mathrm{d}s - \lim_{\epsilon\to 0}\lim_{\eta\to 0} N_t^{\epsilon,\eta},$$

where

$$N_t^{\epsilon,\eta} = \frac{1}{2}\int_0^t \langle f_{\epsilon,\eta}(u_s), h\rangle\, \mathrm{d}s,$$

and where the limit holds in $\mathbb{P}_{\nu_2}$ probability, for the topology of uniform convergence in t on each finite interval. Moreover, $(M_t)_{t\geq 0}$ is a martingale additive functional whose sharp bracket has Revuz measure $\|h\|_{L^2}^2\, \nu_2$.

Proof. The result follows from (4.45) using the same arguments as in the proof of Theorem 4.2.8. □

4.4 Proof of two technical result

Proof of Proposition 4.3.1. The following arguments were communicated to us by Rongchan Zhu and Xiangchan Zhu. Let $f \in \mathcal{F}C_b^\infty(K)$. We can write f as

$$f(z) = \psi(\langle l_1, jz\rangle, \ldots, \langle l_m, z\rangle), \quad z \in K,$$

with $m \in \mathbb{N}$, $\psi \in C_b^\infty(\mathbb{R}^m)$, and $l_1, \dots, l_m \in \mathrm{Span}\{e_k, k \geq 1\}$. Then we have, for all $z = (z_1, z_2) \in \mathbb{H}_2$

$$\overline{\nabla}(f \circ j_2)(z) = \sum_{i=1}^m \partial_i \psi(\langle l_1, j_2(z)\rangle, \dots, \langle l_m, j_2(z)\rangle)\, l_i \, \frac{z}{\|z\|},$$

that is

$$\overline{\nabla}(f \circ j_2)(z) = \nabla f(j_2(z))\, \frac{z}{\|z\|}, \tag{4.46}$$

where the right-hand side denotes the point-wise multiplication of the real-valued function $\nabla f(j(z))$ by the $\mathbb{R}^2$-valued function $\frac{z}{\|z\|}$. In particular, we deduce that

$$\int \|\overline{\nabla}(f \circ j_2)(z)\|_{\mathbb{H}_2}\, \mathrm{d}\mu_2(z) = \int \|\nabla f(j_2(z))\|_H \, \mathrm{d}\mu_2(z) < \infty,$$

so that $f \circ j_2 \in D(\Lambda^2)$. Moreover, for all $f, g \in \mathcal{FC}_b^\infty(K)$, we have

$$\begin{aligned}
\mathcal{E}^2(f,g) &= \frac{1}{2} \int \langle \nabla f(x), \nabla g(x)\rangle \, \mathrm{d}\nu_2(x) \\
&= \frac{1}{2} \int \langle \nabla f(j_2(z)), \nabla g(j_2(z))\rangle \, \mathrm{d}\mu_2(z) \\
&= \frac{1}{2} \int \langle \nabla(f \circ j_2)(z), \nabla(g \circ j_2)(z)\rangle \, \mathrm{d}\mu_2(z) \\
&= \Lambda^2(f \circ j_2, g \circ j_2),
\end{aligned}$$

where the second equality follows from (4.40), and the third equality folows from (4.46). This shows that the bilinear symmetric form $(\mathcal{E}^2, \mathcal{FC}_b^\infty(K))$ possesses the image of the Dirichlet form $(\Lambda^2, D(\Lambda^2))$ under the map j_2 as an extension. Since $\mathcal{FC}_b^\infty(K)$ is dense in $L^2(\nu_2)$, this extension is a Dirichlet form. In particular, $(\mathcal{E}^2, \mathcal{FC}_b^\infty(K))$ is closable, and its closure $(\mathcal{E}^2, D(\mathcal{E}^2))$ is a Dirichlet form.

The quasi-regularity and the local property of that form can be proven in exactly the same way as done for $(\mathcal{E}, D(\mathcal{E}))$ in the proof of Prop 4.2.1. □

Proof of Proposition 4.3.4. We first show that the sequence of functionals $(V_{\epsilon,\eta})_{0<\eta<\epsilon<1}$ is bounded in $L^2(\mu_2)$: the proof of the requested convergences will follow by similar arguments. For any $t > 0$, we have

$$\|\mathbf{Q}_t^2 G_{\epsilon,\eta}\|^2_{L^2(\mu)} = \int_{[0,1]^2} h_r\, h_s \int_{\mathbb{R}^2} f_{\epsilon,\eta}(\|a\|) f_{\epsilon,\eta}(\|b\|) \Gamma_{r,s}(a_1, b_1)\, \Gamma_{r,s}(a_2, b_2)\, \mathrm{d}a\, \mathrm{d}b, \tag{4.47}$$

where, for all $r, s \in (0,1)$, $\Gamma_{r,s}$ is the Gaussian density function given by (4.22). Therefore it suffices to bound, for all $(r,s) \in (0,1)^2$, the integral

$$I(\epsilon, \eta) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_{\epsilon,\eta}(||a||)\, f_{\epsilon,\eta}(||b||)\, \Gamma_{r,s}(a_1, b_1)\, \Gamma_{r,s}(a_2, b_2)\, \mathrm{d}a\, \mathrm{d}b.$$

As before, we may assume that $(r,s) \in [\delta, 1-\delta]^2$, where $\delta \in (0,1)$ is such that h is supported in $[\delta, 1-\delta]$. To bound $I(\epsilon,\eta)$, we first switch to polar coordinates by setting

$$a = (x\cos(\theta), x\sin(\theta)), \quad b = (y\cos(\varphi), y\sin(\varphi)),$$

with $x, y \geq 0$ and $\theta, \varphi \in [0, 2\pi]$. We then have

$$I(\epsilon,\eta) := \int_{\mathbb{R}_+\times\mathbb{R}_+} f_{\epsilon,\eta}(x)\, f_{\epsilon,\eta}(y)\, xy\, G(x,y)\, \mathrm{d}x\, \mathrm{d}y.$$

where

$$G(x,y) := \int_{[0,2\pi]^2} \Gamma_{r,s}(x\cos(\theta), y\cos(\varphi))\, \Gamma_{r,s}(x\sin(\theta), y\sin(\varphi))\ \mathrm{d}\theta\, \mathrm{d}\varphi. \tag{4.48}$$

Hence

$$\begin{aligned} I(\epsilon,\eta) &= 16 \int_{\mathbb{R}_+\times\mathbb{R}_+} \left(\frac{\mathbf{1}_{x\geq\epsilon}}{x^2} - \frac{2}{\epsilon}\rho_\eta(x)\right)\left(\frac{\mathbf{1}_{y\geq\epsilon}}{y^2} - \frac{2}{\epsilon}\rho_\eta(y)\right) G(x,y)\, \mathrm{d}x\, \mathrm{d}y \\ &= 16 \int_{[\epsilon,\infty)\times[\epsilon,\infty)} \frac{\mathrm{d}x}{x^2}\frac{\mathrm{d}y}{y^2} \Big(G(x,y) - \int_{[0,\eta]} 2\rho_\eta(w) G(x,w)\, \mathrm{d}w - \\ &- \int_{[0,\eta]} 2\rho_\eta(z) G(z,y)\, \mathrm{d}z + \int_{[0,\eta]^2} 4\rho_\eta(z)\rho_\eta(w) G(z,w)\, \mathrm{d}z\, \mathrm{d}w\Big) \\ &= (16)^2 \int_{[\epsilon,\infty)\times[\epsilon,\infty)} \frac{\mathrm{d}x}{x^2}\frac{\mathrm{d}y}{y^2} \int_{[0,\eta]^2} \mathrm{d}z\, \mathrm{d}w\, \rho_\eta(z)\, \rho_\eta(w)\, G_{z,w}(x,y), \end{aligned}$$

with

$$G_{z,w}(x,y) := G(x,y) - G(x,w) - G(z,y) + G(z,w). \tag{4.49}$$

Here we used the fact that $\int_0^\eta \rho_\eta(z)\, \mathrm{d}z = 1/2$ to obtain the last line. Hence, it suffices to bound, for all $x, y \geq \epsilon$ and $z, w \in [0,\eta]$, the quantity $G_{z,w}(x,y)$. By the triangular inequality, this is obviously bounded by:

$$4 \sup_{x,y\geq 0} |G(x,y)|.$$

In turn, by (4.48), we have

$$\begin{aligned} \sup_{x,y\geq 0} |G(x,y)| &\leq 4\pi^2 \sup_{z\in\mathbb{R}^2} |\Gamma_{r,s}(z)|^2 \\ &\leq 4\pi^2 \frac{1}{4\pi^2\ \det(M)} \\ &\leq C_\delta\, (t\wedge 1)^{-1/2} \end{aligned}$$

where M denotes the covariant matrix associated with $\Gamma_{r,s}$, and where the last bound follows from (4.26). Therefore, for all $x, y \geq \epsilon$ and $z, w \in [0, \eta]$

$$|G_{z,w}(x,y)| \leq C_\delta \, (t \wedge 1)^{-1/2}, \tag{4.50}$$

where $C_\delta > 0$ depends only on δ. Note that although this bound is sufficient when x and y are away from 0, say $x, y \geq 1$, it will not be satisfactory when either x or y tend to 0: in that regime, we need a stronger bound in order to cure the potential divergency created by the terms $\frac{1}{x^2}$ and $\frac{1}{y^2}$ in the integral $I(\epsilon, \eta)$. This will be done by harvesting the renormalizations appearing in (4.49). Note that this kind of reasoning is an instance, in a tremendously simpler context, of the sophisticated methods used to obtain bounds on Feynman integrals, for instance in the theory of regularity structure (see [HQ15, Appendix A], and [CH16]). First note that, for all $x, y \geq 0$, we have

$$\begin{aligned} \frac{\partial G}{\partial x}(x,y) = \int_{[0,2\pi]^2} \Big(&\cos(\theta) \frac{\partial \Gamma_{r,s}}{\partial x}(x\cos(\theta), y\cos(\varphi)) \, \Gamma_{r,s}(x\sin(\theta), y\sin(\varphi)) \\ &+ \sin(\theta)\Gamma_{r,s}(x\cos(\theta), y\cos(\varphi)) \, \frac{\partial \Gamma_{r,s}}{\partial x}(x\sin(\theta), y\sin(\varphi)) \Big) \, \mathrm{d}\theta \, \mathrm{d}\varphi, \end{aligned}$$

whence we in particular obtain

$$\frac{\partial G}{\partial x}(0,y) = 0.$$

Therefore, for all x, y, z, w as above, we have:

$$\begin{aligned} |G_{z,w}(x,y)| &\leq |G(x,y) - G(z,y)| + |G(x,w) - G(z,w)| \\ &\leq 2x^2 \sup_{\mathbb{R}^2} \left| \frac{\partial^2 G}{\partial x^2} \right| \end{aligned}$$

where the second inequality follows from the fact that $0 \leq z \leq x$ due to our assumptions on ϵ and η. But, by Lemma 4.2.5 above, for all $k \geq 0$, $\frac{\partial^k}{\partial x^k} \Gamma_{r,s}$ is bounded uniformly by:

$$A_k \det(M)^{-\frac{1+k}{2}},$$

where $A_k > 0$ depends only on k. Therefore, by the Leibniz formula, we deduce that there exists a universal constant $A > 0$ such that

$$\sup_{\mathbb{R}^2} \left| \frac{\partial^2 G}{\partial x^2} \right| \leq A \det(M)^{-2}.$$

Hence, by (4.26), we obtain

$$\sup_{\mathbb{R}^2} \left| \frac{\partial^2 G}{\partial x^2} \right| \leq C_\delta \, (t \wedge 1)^{-1}$$

where $C_\delta > 0$ is a constant depending only on δ. We thus obtain the bound

$$|G_{z,w}(x,y)| \le C_\delta \, x^2 \, (t \wedge 1)^{-1} \tag{4.51}$$

for $x \in [0,1]$. This bound is appropriate in the regime where y is large but x is small. In the same way, we obtain the bound

$$|G_{z,w}(x,y)| \le C_\delta \, y^2 \, (t \wedge 1)^{-1} \tag{4.52}$$

for $y \in [0,1]$, which takes care of the case when x is large but y is small. There remains to obtain a bound for x and y which are both small. To do so, note that

$$\begin{aligned} |G_{z,w}(x,y)| &= \left| \int_w^y \int_z^x \frac{\partial^2 G}{\partial x \partial y}(u,v) \, \mathrm{d}u \, \mathrm{d}v \right| \\ &\le xy \sup_{[0,x]\times[0,y]} \left| \frac{\partial^2 G}{\partial x \partial y} \right|. \end{aligned}$$

Now, differentiating the expression for G in (4.48), we easily find that, for all $x, y \ge 0$

$$\frac{\partial^2 G}{\partial x \partial y}(x,0) = \frac{\partial^2 G}{\partial x \partial y}(0,y) = 0.$$

Similarly, we have

$$\frac{\partial^3 G}{\partial x^2 \partial y}(x,0) = \frac{\partial^3 G}{\partial x \partial y^2}(0,y) = 0.$$

Therefore, we have, for all $x, y \in [0,1]$

$$\left| \frac{\partial^2 G}{\partial x \partial y}(x,y) \right| \le xy \sup_{[0,1]^2} \left| \frac{\partial^4 G}{\partial x^2 \partial y^2}(x,y) \right|,$$

whence we obtain

$$\begin{aligned} |G_{z,w}(x,y)| &\le x^2 y^2 \sup_{[0,1]^2} \left| \frac{\partial^4 G}{\partial x^2 \partial y^2} \right| \\ &\le K \, x^2 y^2 \, \det(M)^{-3}, \end{aligned}$$

where the last bound follows from Lemma 4.2.5 and the Leibniz formula, with $K > 0$ a universal constant. But, interpolating between the lower bounds provided by (4.25) and (4.26), we have, for any $\alpha \in (0,1)$:

$$\det(M)^{-3} \le (\delta^2 |r-s|)^{\alpha-1} (c_\delta \sqrt{t \wedge 1})^{-(\alpha+2)}$$

Choosing for example $\alpha = 1/2$, we obtain

$$\det(M)^{-3} \le C_\delta \, |r-s|^{-1/2} \, (t \wedge 1)^{-5/4}$$

where C_δ depends only on δ. Hence

$$|G_{z,w}(x,y)| \leq C_\delta\, x^2 y^2\, |r-s|^{-1/2}\, (t \wedge 1)^{-5/4} \tag{4.53}$$

for all $x, y \in [0,1]$. Finally, we can now bound $I(\epsilon, \eta)$. To do so, we decompose this integral as follows:

$$I(\epsilon,\eta) = I_{[1,\infty]^2} + I_{[\epsilon,1)\times[1,\infty]} + I_{[1,\infty]\times[\epsilon,1)} + I_{[\epsilon,1)^2},$$

where, for $A \subset \mathbb{R}_+^2$, I_A denotes the integral

$$\int_A \frac{\mathrm{d}x}{x^2}\frac{\mathrm{d}y}{y^2} \int_{[0,\eta]^2} \mathrm{d}z\, \mathrm{d}w\, \rho_\eta(z)\rho_\eta(w)\, G_{z,w}(x,y).$$

We start by obtaining a bound for $I_{[1,\infty]^2}$. By (4.50), and recalling that $\int_0^\eta \rho_\eta(x)\,\mathrm{d}x = 1/2$, we have

$$|I_{[1,\infty]^2}| \leq C_\delta\,(t\wedge 1)^{-1/2}.$$

On the other hand, by (4.51), we have

$$\begin{aligned}|I_{[\epsilon,1)\times[1,\infty]}| &\leq C_\delta\,(t\wedge 1)^{-1}\int_\epsilon^1 \mathrm{d}x \int_1^\infty \frac{\mathrm{d}y}{y^2}\\ &\leq C_\delta\,(t\wedge 1)^{-1}.\end{aligned}$$

Similarly, by (4.52), we have

$$|I_{[1,\infty]\times[\epsilon,1)}| \leq C_\delta\,(t\wedge 1)^{-1}.$$

Finally, by (4.53), we have

$$\begin{aligned}|I_{[\epsilon,1)\times[1,\infty]}| &\leq C_\delta\,|r-s|^{-1/2}\,(t\wedge 1)^{-5/4}\int_\epsilon^1 \mathrm{d}x \int_\epsilon^1 \mathrm{d}y\\ &\leq C_\delta\,|r-s|^{-1/2}(t\wedge 1)^{-5/4}.\end{aligned}$$

Putting these bounds together finally yields

$$|I(\epsilon,\eta)| \leq C_\delta\,|r-s|^{-1/2}\,(t\wedge 1)^{-5/4}$$

for all $\eta < \epsilon < 1$ and $t > 0$, where $C_\delta > 0$ depends only on δ. Therefore, recalling (4.47), we obtain

$$\|\mathbf{Q}_t^2 G_{\epsilon,\eta}\|_{L^2(\mu_2)}^2 \leq C_\delta\,(t\wedge 1)^{-5/4}\,\|h\|_\infty^2 \int_{[0,1]^2} |r-s|^{-1/2}\,\mathrm{d}r\ \mathrm{d}s,$$

where the last integral is finite. Hence

$$\|\mathbf{Q}_t^2 G_{\epsilon,\eta}\|_{L^2(\mu_2)} \leq C(\delta,h)\,(t\wedge 1)^{-5/8},$$

where $C(\delta, h)$ is a constant independent of t. Therefore

$$\|V_{\epsilon,\eta}\|_{L^2(\mu_2)} \leq \int_0^\infty e^{-t}\,(t\wedge 1)^{-5/8}\,\mathrm{d}t < \infty.$$

Thus, $(V_{\epsilon,\eta})_{0<\eta<\epsilon<1}$ is bounded in $L^2(\mu_2)$. Reasoning similarly to bound $V_{\epsilon,\eta} - V_\epsilon$ and $V_\epsilon - V$, we deduce by dominated convergence that

$$V_{\epsilon,\eta} \underset{\eta\to 0}{\longrightarrow} V_\epsilon$$

and

$$V_\epsilon \underset{\epsilon\to 0}{\longrightarrow} V$$

in $L^2(\mu_2)$.

There remains to show that these convergences hold in $D(\Lambda^2)$. To do so, for all $\eta < \epsilon < 1$, we bound $\|\overline{\nabla} V_{\epsilon,\eta}\|_{L^2}$, where $\|\cdot\|_{L^2}$ stands for the norm in $L^2(\mathbb{H}_2, \mu_2; \mathbb{H}_2)$. We have

$$\|\overline{\nabla} Q_t^2 G_{\epsilon,\eta}\|_{L^2}^2 = \int_{[0,1]^2} \mathrm{d}r\;\mathrm{d}s\, h_r\, h_s \int_{\mathbb{R}^2\times\mathbb{R}^2} \mathrm{d}a\;\mathrm{d}b\, f_{\epsilon,\eta}(\|a\|)\, f_{\epsilon,\eta}(\|b\|)$$
$$\left(\frac{\partial^2\Gamma_{r,s}}{\partial x\,\partial y}(a_1,b_1)\,\Gamma_{r,s}(a_2,b_2) + \Gamma_{r,s}(a_1,b_1)\,\frac{\partial^2\Gamma_{r,s}}{\partial x\,\partial y}(a_2,b_2)\right).$$

Therefore it suffices to bound, for all $(r,s) \in [0,1]^2$, the integral

$$I^{(1)}(\epsilon,\eta) := \int_{\mathbb{R}_+\times\mathbb{R}_+} f_{\epsilon,\eta}(x) f_{\epsilon,\eta}(y)\, xy\, G^{(1)}(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

where

$$G^{(1)}(x,y) := \int_{[0,2\pi]^2} \Big(\frac{\partial^2\Gamma_{r,s}}{\partial x\,\partial y}(x\cos(\theta), y\cos(\varphi))\,\Gamma_{r,s}(x\sin(\theta), y\sin(\varphi))$$
$$+ \Gamma_{r,s}(x\cos(\theta), y\cos(\varphi))\,\frac{\partial^2\Gamma_{r,s}}{\partial x\,\partial y}(x\sin(\theta), y\sin(\varphi))\Big)\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Reasoning as above, we can rewrite $I^{(1)}(\epsilon,\eta)$ as

$$I(\epsilon,\eta) = (16)^2 \int_{[\epsilon,\infty)\times[\epsilon,\infty)} \frac{\mathrm{d}x}{x^2}\frac{\mathrm{d}y}{y^2} \int_{[0,\eta]^2} \mathrm{d}z\,\mathrm{d}w\,\rho_\eta(z)\,\rho_\eta(w)\,G^{(1)}_{z,w}(x,y),$$

where

$$G^{(1)}_{z,w}(x,y) := G^{(1)}(x,y) - G^{(1)}(x,w) - G^{(1)}(z,y) + G^{(1)}(z,w).$$

By the triangular inequality we have, for all $x, y, z, w \in \mathbb{R}$

$$|G^{(1)}_{z,w}(x,y)| \leq 4 \sup_{\mathbb{R}^2_+} |G^{(1)}|.$$

Now, the supremum above is bounded by

$$2 \sup_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial x\, \partial y} \Gamma_{r,s} \right| \sup_{\mathbb{R}^2} |\Gamma_{r,s}| \,.$$

By Lemma 4.2.5, this in turn is bounded by

$$K \det(M)^{-2}$$

for some universal constant $K > 0$. In virtue of (4.26), we thus deduce that

$$|G^{(1)}_{z,w}(x,y)| \leq C_\delta \, (t \wedge 1)^{-1}$$

for some universal constant C_δ, as soon as $r, s \in [\delta, 1-\delta]$. On the other hand, noting that, for all $y \in \mathbb{R}$

$$\frac{\partial G^(1)}{\partial x}(0,y) = 0$$

we have, for all $x \in [0,1]$ and $y, z, w \in \mathbb{R}_+$, the bound

$$|G^{(1)}_{z,w}(x,y)| \leq 2x^2 \sup_{\mathbb{R}^2_+} \left| \frac{\partial^2 G^(1)}{\partial x^2} \right| .$$

But, by Lemma 4.2.5, the Leibniz formula, and the bound (4.26), we have

$$\sup_{\mathbb{R}^2_+} \left| \frac{\partial^2 G^(1)}{\partial x^2} \right| \leq C_\delta \, (t \wedge 1)^{-3/2}.$$

Hence, when $x \in [0,1]$, we have the bound

$$|G^{(1)}_{z,w}(x,y)| \leq C_\delta \, (t \wedge 1)^{-3/2} \, x^2.$$

Similarly, when $y \in [0,1]$, we have

$$|G^{(1)}_{z,w}(x,y)| \leq C_\delta \, (t \wedge 1)^{-3/2} \, y^2.$$

Finally, when $x, y \in [0,1]^2$, one has

$$|G^{(1)}_{z,w}(x,y)| \leq x^2y^2 \sup_{\mathbb{R}^2_+} \left|\frac{\partial^4 G^{(1)}}{\partial x^2 \partial y^2}\right| \\ \leq K\, x^2y^2\, \det(M)^{-4},$$

where the last bound follows from Lemma 4.2.5 and the Leibniz formula, and with $K > 0$ a universal constant. Now, by interpolation of (4.25) and (4.26), we have

$$\det(M)^{-4} \leq (\delta^2|r-s|)^{-1/2}(c_\delta\sqrt{t\wedge 1})^{-7/2}$$

provided that $r, s \in [\delta, 1-\delta]$. Therefore, for all such r and s, and for $x, y \in [0,1]$, we have

$$|G^{(1)}_{z,w}(x,y)| \leq C_\delta\,(t\wedge 1)^{-7/4}\,|r-s|^{-1/2}\,x^2y^2.$$

We now put together all these estimates by writing

$$I^{(1)}(\epsilon,\eta) = I^{(1)}_{[1,\infty]^2} + I^{(1)}_{[\epsilon,1)\times[1,\infty]} + I^{(1)}_{[1,\infty]\times[\epsilon,1)} + I^{(1)}_{[\epsilon,1)^2},$$

where, for $A \subset \mathbb{R}^2_+$, $I^{(1)}_A$ denotes the integral

$$\int_A \frac{\mathrm{d}x}{x^2}\frac{\mathrm{d}y}{y^2} \int_{[0,\eta]^2} \mathrm{d}z\,\mathrm{d}w\,\rho_\eta(z)\rho_\eta(w)\,G^{(1)}_{z,w}(x,y).$$

The previous estimates yield

$$I^{(1)}_{[1,\infty]^2} \leq C_\delta\,(t\wedge 1)^{-1}$$

as well as

$$I^{(1)}_{[\epsilon,1)\times[1,\infty]} \leq C_\delta\,(t\wedge 1)^{-3/2}, \quad I^{(1)}_{[1,\infty]\times[\epsilon,1)} \leq C_\delta\,(t\wedge 1)^{-3/2},$$

and

$$I^{(1)}_{[\epsilon,1)^2} \leq C_\delta\,(t\wedge 1)^{-7/4}\,|r-s|^{-1/2}.$$

Therefore, we obtain

$$I^{(1)}(\epsilon,\eta) \leq C_\delta\,(t\wedge 1)^{-7/4}\,|r-s|^{-1/2}$$

and, since $\int_{[0,1]^2} |r-s|^{-1/2}\,\mathrm{d}r\ \mathrm{d}s < \infty$, we deduce finally that

$$\|\overline{\nabla} Q_t^2 G_{\epsilon,\eta}\|^2_{L^2} \leq C(\delta,h)\,(t\wedge 1)^{-7/4},$$

where $C(\delta,h) \in (0,\infty)$ does not depend on t. Therefore,

$$\|\overline{\nabla} V_{\epsilon,\eta}\|_{L^2} \leq \sqrt{C(\delta,h)}\int_0^\infty e^{-t}(t\wedge 1)^{-7/8}\,\mathrm{d}t < \infty.$$

Hence, $(\overline{\nabla} V_{\epsilon,\eta})_{\epsilon>\eta>0}$ is uniformly bounded in $L^2(\mathbb{H}_2, \mu_2; \mathbb{H}_2)$. Similar bounds on $\nabla V_{\epsilon,\eta} - \nabla V_\epsilon$ and on $\nabla V_\epsilon - \nabla V$ yield, by dominated convergence,

$$\overline{\nabla} V_{\epsilon,\eta} \underset{\eta \to 0}{\longrightarrow} \nabla V_\epsilon$$

and

$$\overline{\nabla} V_\epsilon \underset{\epsilon \to 0}{\longrightarrow} \nabla V$$

in $L^2(\mathbb{H}_2, \mu_2; \mathbb{H}_2)$. The proposition is proved. □

4.5 Open problems

The Dirichlet form techniques used in Section 4.2 above to construct u in the cases $\delta = 1, 2$ a priori break down for other $\delta \in (0, 3)$. Indeed, for $\delta \in]0, 3[\setminus\{1, 2\}$, it is not even known whether the form which naturally generalizes $(\mathcal{E}, \mathcal{FC}_b^\infty(K))$ in Proposition 4.2.1 is closable and whether its closure is a quasi-regular Dirichlet form.

We recall the main result of [DMZ06]: for all $\delta \geq 3$, we set

$$\zeta(\delta) := \sup\{k \geq \mathbb{N} : \exists t > 0,\, 0 < x_1 < \ldots < x_k < 1,\, u(t, x_i) = 0 \quad i = 1, \ldots, k\},$$

where u is the solution to the δ-Bessel SPDE (1.10)-(1.12). Then we have

$$\mathbb{P}\left(\zeta(\delta) > \frac{4}{\delta - 2}\right) = 0. \tag{4.54}$$

In other words, a.s. u hits the obstacle 0 in at most $\lceil \frac{4}{\delta-2} \rceil$ space points simultaneously in time. It is very tempting to conjecture that (4.54) holds for all $\delta > 2$ in other words, the δ-Bessel SPDE would hit 0 at finitely many space points simultaneously in time for any $\delta > 2$, but the number of such hitting points would tend to $+\infty$ as $\delta \downarrow 2$. The fact that $\delta = 2$ is the critical value for this behaviour is clearly related to the fact that $\delta = 2$ is also the critical dimension for the probability that the δ-Bessel process or bridge hit 0. However, the precise behaviour at 0 of the Bessel SPDE for the threshold value $\delta = 2$ is not clear to predict, and seems a very subtle and intriguing question.

A related problem concerns the regularity properties of the Markov semigroup associated with the Bessel SPDEs (4.6) , (4.7) and (4.8) for $\delta < 3$, e.g. for the cases $\delta \in \{1, 2\}$ treated above. Thus, while it is easy to prove that the semigroup has the strong Feller property for $\delta \geq 3$ (see see Section 5.4 in [Zam17]), it is an open problem for $\delta < 3$, again because the drift of the SPDE becomes highly non-dissipative. Such a property would allow to strengthen much of the results obtained above for the cases $\delta = 1, 2$: in particular, it would imply that Theorems 4.2.8 and 4.3.7 remain true for every - rather than quasi-every - initial condition; it would also ensure that the convergence statements therein hold regardless of the intitial condition, rather than only at equilibrium. Although we miss the tools to establish this property, we may hope that it holds for all $\delta > 0$. Indeed, in Chapter 7 below, we show that Bessel processes have the strong Feller property regardless of the dimension, although, for dimensions $\delta < 1$, the drift is higly non-dissipative. Moreover Tsatsoulis and Weber [TW18] have proved that the 2-dimensional stochastic quantization equation satisfies a strong Feller property, although it is an equation which needs renormalization; also Hairer and Mattingly [HM18] have proved that property for a large class of equations with renormalised drifts. All this suggests that there may be hope that this technically very useful property holds also for δ-Bessel SPDEs with $\delta < 3$.

Chapter 5

Taylor estimates on the laws of pinned Bessel bridges, and application to integration by parts

In this chapter, we extend the integration by parts formulae for the laws of Bessel bridges (from 0 to 0) obtained in Chapter 3 above, by showing that these formulae hold for very general test functionals on $L^2(0,1)$. The main argument consists in proving Taylor estimates for the laws of Bessel bridges conditioned to take a prescribed value at a given point. The content of this chapter is based on the article [EAb] in preparation.

5.1 Statement of the results

In Chapter 3 above, we have derived IbPF for the laws P^δ of Bessel bridges of all dimension $\delta \in (0,3)$ from 0 to 0 on $[0,1]$, for a specific class of functionals. More precisely, we considered the vector space $\mathscr{S}$ generated by all functionals on $L^2(0,1)$ of the form

$$\begin{cases} L^2(0,1) \to \mathbb{R} \\ X \mapsto \exp\left(-\langle \theta, X^2 \rangle\right), \end{cases} \tag{5.1}$$

where, $\theta : [0,1] \to \mathbb{R}_+$ is Borel and bounded. Note that functionals of the form (3.1) play a special role. Indeed, as a consequence of the remarkable additivity property of squared Bessel bridges (see Prop 2.2.2 above), such functionals act on the laws of Bessel processes as a Girsanov transformation corresponding to a deternimistic time-change (see Lemma 2.2.3 above), thus allowing nice explicit computations. Thus, functionals of the type (5.1) play, in this context, the same role as functionals of the form $\exp(\langle k, X\rangle)$, $k \in C([0,1])$, in the papers [Zam05] and [GV16].

Recall that, for $\delta > 0$, $b \geq 0$ and $r \in (0,1)$, $\Sigma^\delta_r(\mathrm{d}X \,|\, b)$ denotes the finite measure on $C([0,1])$ given by

$$\Sigma^\delta_r(\mathrm{d}X \,|\, b) := \frac{p^\delta_r(b)}{b^{\delta-1}} \, E^\delta[\mathrm{d}X \,|\, X_r = b], \tag{5.2}$$

In the above, E^δ denotes the expectation operator corresponding to the probability measure P^δ on $C([0,1])$, while, for all $r \in (0,1)$, p^δ_r denotes the density of the law of X_r under P^δ. Recall that, for all $b > 0$

$$p^\delta_r(b) = \frac{b^{\delta-1}}{2^{\frac{\delta}{2}-1}(r(1-r))^{\delta/2}\Gamma(\frac{\delta}{2})} \exp\left(-\frac{b^2}{2r(1-r)}\right). \tag{5.3}$$

Recall also that, for any sufficiently differentiable function $f : \mathbb{R}_+ \to \mathbb{R}$, for all $n \in \mathbb{Z}$, and $b \geq 0$, we set

$$\mathcal{T}_b^{\,n} f := f(b) - \sum_{0 \leq j \leq n} \frac{b^j}{j!} f^{(j)}(0).$$

In words, for all $b \geq 0$, if $n \geq 0$ then $\mathcal{T}_b^{\,n} f$ is the Taylor remainder centered at 0, of order $n+1$, of the function f, evaluated at b; if $n < 0$ then $\mathcal{T}_b^{\,n} f$ is simply the value of f at b.

Finally, defining for all $\delta > 0$

$$\kappa(\delta) := \frac{(\delta-1)(\delta-3)}{4},$$

and setting

$$k := \left\lfloor \frac{3-\delta}{2} \right\rfloor \leq 1$$

the IbPF obtained in Section 3 above can be written as follows. For all $\delta \in (0,3)\backslash 1$, $\Phi \in \mathscr{S}$ and $h \in C^2_c(0,1)$, it holds

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle \, \Phi(X)) = \\ &- \kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \left[\mathcal{T}_b^{\,2k} \, \Sigma^\delta_r(\Phi(X) \,|\, \cdot\,)\right] \mathrm{d}b \, \mathrm{d}r, \end{aligned} \tag{5.4}$$

see Theorem 3.1.1 above. Recall also that the term

$$\mathcal{T}_b^{\,2k} \, \Sigma^\delta_r(\Phi(X) \,|\, \cdot\,)$$

appearing in the formulae is actually the Taylor remainder, centered at 0, of a smooth function of b^2. In particular, it is of order $b^{2(k+1)}$ as $b \to 0$, which ensures the integral to be convergent. In Theorem 3.1.1, we also obtained the following formula for the critical case $\delta = 1$:

$$E^1(\partial_h \Phi(X)) + E^1(\langle h'', X\rangle \, \Phi(X)) = \frac{1}{4} \int_0^1 \mathrm{d}r \, h_r \, \frac{\mathrm{d}^2}{\mathrm{d}b^2} \, \Sigma^1_r[\Phi(X) \,|\, b] \bigg|_{b=0}. \tag{5.5}$$

We stress that in both of the above propositions, the test functionals are assumed to lie in the space $\mathscr{S}$. Indeed, this assumption allows to perform the computations leading to these formulae, and ensures all the quantities (derivatives and integrals) involved to be well-defined.

In the present chapter, we aim at extending these formulae to a space of more general functionals on $L^2(0,1)$. We introduce the following definition:

Definition 5.1.1. For any Banach space $(B, \|\cdot\|)$, let $C^1_b(B)$ be the space of all $\Phi : B \to \mathbb{R}$ which are bounded, C^1, with bounded Fréchet differential. Moreover, let $C^{1,1}_b(B)$ be the set of all $\Psi \in C^1_b(B)$ such that there exists $L > 0$ satisfying

$$\forall Z, Z' \in L^1(0,1), \quad |||D\Psi(Z) - D\Psi(Z')||| \leq L\|Z - Z'\|. \tag{5.6}$$

where $|||\cdot|||$ denotes the operator norm on B'.

Remark 5.1.2. Note that for any $\Psi \in C^{1,1}_b(B)$ and L as in (5.6), we have

$$\forall x, y, z \in B, \quad |\Psi(x+y+z) - \Psi(x+y) - \Psi(x+z) + \Psi(x)| \leq L\|y\| \, \|z\|.$$

In [Zam02] and [Zam03], the IbPF for P^δ for $\delta \geq 3$ are established for any element of $C^1_b(L^2(0,1))$. Note that $\mathscr{S} \subsetneq C^1_b(L^2(0,1))$, and it is natural to ask whether the IbPF obtained in [EAZ18] can be generalized to $C^1_b(L^2(0,1))$, or at least to some large space of functionals containing $\mathscr{S}$. In this chapter, we prove that such an extension is possible. To do so, we use an approximation argument consisting of two ingredients:

1. the space $\mathscr{S}$ is dense in the space of functionals we are considering, for a certain topology to be specified,

2. the terms appearing in our formulae are all continuous w.r.t. the above mentioned topology.

Note that, in addressing the second point, we obtain rather strong estimates on the Taylor remainders at 0 of the laws of pinned Bessel bridges, which are interesting in their own right. In particular, we prove the remarkable fact that, for any $\Phi \in C_b^1(L^2(0,1))$ and $r \in (0,1)$, we have

$$\frac{d}{db} E^\delta[\Phi(X)|X_r = b]\bigg|_{b=0} = 0$$

(see Proposition 5.4.1).

Combining the two points highlighted above, we are able to prove the following results:

Theorem 5.1.3. *Let $\delta \in (1,3)$. Then, for all $\Phi \in C_b^1(L^2(0,1))$ and all $h \in C_c^2(0,1)$, we have*

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle \, \Phi(X)) = \\ &- \kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \Big[\mathcal{T}_b^{\,0}\, \Sigma_r^\delta(\Phi(X)\,|\,\cdot\,)\Big] \, \mathrm{d}b \, \mathrm{d}r, \end{aligned} \tag{5.7}$$

Thus, for Bessel bridges of dimension strictly between 1 and 3, the formulae hold for any functional in $C_b^1(L^2(0,1))$. In lower dimensions, we can also generalize the formulae, but to some space distinct from $C_b^1(L^2(0,1))$. Indeed, in order to work, our arguments require some additional regularity on our functionals (see however Remark 5.4.10 below).

Definition 5.1.4. Let $\mathcal{S}C_b^1\,(L^1(0,1))$ be the set of functionals on $L^2(0,1)$ of the form

$$\Phi(X) = \Psi(X^2), \quad X \in L^2(0,1) \tag{5.8}$$

where $\Psi \in C_b^1\,(L^1(0,1))$

The following nice, remarkable property holds:

Proposition 5.1.5. *Let $\delta \geq 0$ and $\Phi \in \mathcal{S}C_b^1\,(L^1(0,1))$. Then, for all $r \in (0,1)$, the function*

$$\begin{cases} \mathbb{R}_+ \to \mathbb{R} \\ b \mapsto E^\delta[\Phi(X)|X_r = b] \end{cases}$$

is twice differentiable at 0.

For $\delta = 1$, the IbPF can be extended to all elements of $\mathcal{S}C_b^1\,(L^1(0,1))$:

Theorem 5.1.6. *For all $\Phi \in \mathcal{S}C_b^1\,(L^1(0,1))$ and $h \in C_c^2(0,1)$, we have*

$$E^1(\partial_h \Phi(X)) + E^1(\langle h'', X\rangle \, \Phi(X)) = \frac{1}{4} \int_0^1 \mathrm{d}r \, h_r \, \frac{\mathrm{d}^2}{\mathrm{d}b^2} \, \Sigma_r^1[\Phi(X)\,|\,b]\bigg|_{b=0}. \tag{5.9}$$

Finally, to state the result for $\delta \in (0,1)$, we need to introduce a more particular space:

Definition 5.1.7. Let $\mathcal{S}C_b^{1,1}(L^1(0,1))$ be the set of functionals on $L^2(0,1)$ of the form

$$\Phi(X) = \Psi(X^2), \quad X \in L^2(0,1) \tag{5.10}$$

where $\Psi \in C_b^{1,1}(L^1(0,1))$.

We prove the following result:

Theorem 5.1.8. *Let $\delta \in (0,1)$. Then, for all $\Phi \in \mathcal{S}C_b^{1,1}(L^1(0,1))$ and $h \in C_c^2(0,1)$, we have*

$$\begin{aligned} &E^\delta(\partial_h \Phi(X)) + E^\delta(\langle h'', X\rangle\, \Phi(X)) = \\ &- \kappa(\delta) \int_0^1 h_r \int_0^\infty b^{\delta-4} \Big[\mathcal{T}_b^{\,2}\, \Sigma_r^\delta(\Phi(X)\,|\,\cdot\,)\Big] \,\mathrm{d}b\,\mathrm{d}r, \end{aligned} \tag{5.11}$$

This chapter is organized as follows: after introducing the notations and stating some useful facts on the laws of squared Bessel bridges in Section 5.2, we prove density results for $\mathscr{S}$ in large spaces of functionals in Section 5.3, and establish Taylor estimates for the laws of pinned Bessel bridges in Section 5.4. Putting these results together, we proceed in Section 5.5 to the proofs of Theorems 5.1.3, 5.1.6 and 5.1.8.

5.2 Notations and basic facts

5.2.1 Notations

We need to introduce notations for the various norms and vector spaces that we will consider.

Notation 5.2.1. In the sequel, for all $p \in \{1,2\}$, we denote by $\|\cdot\|_p$ the L^p norm on $L^p(0,1)$. In the special case $p=2$, we will simply write $\|\cdot\|$ for $\|\cdot\|_2$, and denote by $\langle\cdot,\cdot\rangle$ the corresponding inner product. Moreover, for all $p \in \{1,2\}$ and all functional $\Phi : L^p(0,1) \to \mathbb{R}$, we set

$$\|\Phi\|_\infty := \sup_{X \in L^p(0,1)} |\Phi(X)|.$$

Furthermore, for all $\Phi \in C_b^1(L^p(0,1))$, we set

$$|||D\Phi|||_\infty := \sup_{X \in L^p(0,1)} |||D\Phi(X)|||,$$

where $|||\cdot|||$ denotes the norm on $(L^p(0,1))'$, and we set

$$\|\Phi\|_{C^1} := \|\Phi\|_\infty + |||D\Phi|||_\infty.$$

We also introduce the following shorthand notations:

Notation 5.2.2. For all $p \in [1,\infty]$, we denote by $L^p_+(0,1)$ the subset of nonnegative functions in $L^p(0,1)$. Moreover, we use the shorthand notation $C_+([0,1])$ for $C([0,1],\mathbb{R}_+)$.

5.2.2 Squared Bessel bridges and Bessel bridges

Recall that for all $\delta, x, y \geq 0$, $Q^\delta_{x,y}$ denotes the law, on $C_+([0,1])$, of the δ-dimensional squared Bessel bridge between x and y on the interval $[0,1]$. Recall also that we use the shorthand Q^δ for $Q^\delta_{0,0}$. Withe these notations, for all $a, b \geq 0$, the law $P^\delta_{a,b}$ of a δ-Bessel bridge from a to b on $[0,1]$ is the image of $Q^\delta_{a^2,b^2}$ under the map

$$\begin{cases} C_+([0,1]) \to C_+([0,1]) \\ X \mapsto \sqrt{X}. \end{cases} \tag{5.12}$$

The family of probability measures $\left(Q^\delta_{0,x}\right)_{\delta,x\geq 0}$ inherits the remarkable additivity property (2.8) of squared Bessel bridges. Recall the following:

Definition 5.2.3. For any two laws μ, ν on $C_+([0,1])$, let $\mu * \nu$ denote the convolution of μ and ν, i.e. the image of $\mu \otimes \nu$ under the addition map

$$C_+([0,1]) \times C_+([0,1]) \to C_+([0,1]), \quad (x,y) \mapsto x+y$$

The following statement is an equivalent of (2.8) for the bridges, and is a particular case of Theorem 5.8 in [PY82]:

Proposition 5.2.4. *For all x, x', δ, δ', we have the following equality of probability laws on $C_+([0,1])$:*

$$Q^\delta_{x,0} * Q^{\delta'}_{x',0} = Q^{\delta+\delta'}_{x+x',0}$$

Note that this relation in particular says that the families of probability measures $\left(Q^0_{0,x}\right)_{x\geq 0}$ and $\left(Q^\delta\right)_{\delta\geq 0}$ are convolution semi-groups on $C_+([0,1])$. In [PY82], the authors constructed the corresponding Lévy measures M_0 and N_0 on $C_+([0,1])$ (they actually provided an explicit construction in the case of squared Bessel processes, but stressed that the case of the bridges can be dealt similarly, see section (5.4) in that article). The measures M_0 and N_0 are characterized by the fact that $M_0(\{0\}) = N_0(\{0\}) = 0$ and, for all $\delta, x \geq 0$ and all $\theta : [0,1] \to \mathbb{R}_+$ bounded and Borel, we have

$$\begin{aligned} Q^\delta_{x,0}\left[\exp\left(-\langle \theta, X\rangle\right)\right] = \exp&\left(-x \int \left(1 - \exp\left(-\langle\theta, X\rangle\right)\right) dM_0(X)\right) \\ &\exp\left(-\delta \int \left(1 - \exp\left(-\langle\theta, X\rangle\right)\right) dN_0(X)\right). \end{aligned} \tag{5.13}$$

5.2.3 Laws of pinned squared Bessel bridges as a convolution semigroup on $C_+([0,1])$

For all $\delta \geq 0$, $x \geq 0$ and $r \in (0,1)$, we denote by $Q^\delta \left[\, \cdot \,|X_r = x\right]$ the law of a squared Bessel bridge between 0 and 0 conditioned on the event $\{X_r = 0\}$ (to which we shall also refer as the law of a *pinned squared Bessel bridge*). Note that $Q^\delta \left[\, \cdot \,|X_r = x\right]$ is the image of $Q^\delta_{x,0} \otimes Q^\delta_{x,0}$ under the reversal, scaling, and concatenation map $S_r : C_+([0,1]) \times C_+([0,1]) \to C_+([0,1])$ defined, for all $X, Y \in C_+([0,1])$, by

$$S_r(X,Y) : \tau \mapsto \begin{cases} rX\left(\frac{r-\tau}{r}\right), & \text{if } 0 \leq \tau \leq r \\ (1-r)Y\left(\frac{\tau-r}{1-r}\right), & \text{if } r < \tau \leq 1. \end{cases} \tag{5.14}$$

With this representation, we see that Proposition 5.2.4 implies the following:

Proposition 5.2.5. *For all $r \in (0,1)$ and all x, x', δ, δ', we have the following equality of probability laws on $C_+([0,1])$:*

$$Q^\delta \left[\, \cdot \,|X_r = x\right] * Q^{\delta'} \left[\, \cdot \,|X_r = x\right] = Q^{\delta+\delta'} \left[\, \cdot \,|X_r = x + x'\right].$$

A very important consequence for us will be the fact that $(Q^0 \left[\, \cdot \,|X_r = x\right])_{x\geq 0}$ forms a convolution semigroup of probability laws on $C_+([0,1])$. Exploiting the constructions of Pitman-Yor, we can furthermore exhibit the associated Lévy measure:

Proposition 5.2.6. *Let $r \in (0,1)$. Then there exists a measure M^r on $C_+([0,1])$ such that $M^r(\{0\}) = 0$ and, for all $x \geq 0$ and all $\theta : [0,1] \to \mathbb{R}_+$ bounded and Borel, we have*

$$Q^0 \left[\exp\left(-\langle \theta, X\rangle\right) |X_r = x\right] = \exp\left(-x \int \left(1 - \exp\left(-\langle \theta, X\rangle\right)\right) dM^r(X)\right) \tag{5.15}$$

Proof. Let $x \geq 0$ and $\theta : [0,1] \to \mathbb{R}_+$ bounded and Borel. Since $Q^\delta \left[\, \cdot \,|X_r = x\right]$ is the image of $Q^\delta_{x,0} \otimes Q^\delta_{x,0}$ under the map S_r defined by (5.14), we have

$$\begin{aligned} &Q^0[\exp(-\langle \theta, X\rangle |X_r = x] = \\ &Q^0_{\frac{x}{r},0}\left[\exp\left(-\int_0^1 \underline{\theta}(1-v)\ X_v\ dv\right)\right]\ Q^0_{\frac{x}{1-r},0}\left[\exp\left(-\int_0^1 \overline{\theta}(v)\ X_v\ dv\right)\right] \end{aligned}$$

where

$$\underline{\theta}(v) := r^2\ \theta(rv), \quad 0 \leq v \leq 1,$$

and

$$\overline{\theta}(v) := (1-r)^2\ \theta\left(r + v(1-r)\right), \quad 0 \leq v \leq 1.$$

Therefore, by (5.13), we obtain

$$Q^0[\exp(-\langle\theta, X\rangle|X_r = x] = \exp\Big[-\frac{x}{r}\int\Big(1-\exp\Big(-\int_0^1 \underline{\theta}(1-v)X_v dv\Big)\Big)\,dM_0(X) \\ -\frac{x}{1-r}\int\Big(1-\exp\Big(-\int_0^1 \overline{\theta}(v)X_v du\Big)\Big)\,dM_0(X)\Big].$$

Upon performing the changes of variable $u := r(1-v)$ in the first integral, and $u := r + v(1-r)$ in the second one, this yields

$$Q^0[\exp(-\langle\theta, X\rangle|X_r = x] = \exp\Big[-\frac{x}{r}\int\Big(1-\exp\Big(-\int_0^r \theta(u)\ rX_{\frac{r-u}{r}} du\Big)\Big)\,dM_0(X) \\ -\frac{x}{1-r}\int\Big(1-\exp\Big(-\int_r^1 \theta(u)\ (1-r)X_{\frac{u-r}{1-r}} du\Big)\Big)\,dM_0(X)\Big].$$

Therefore, denoting by M_1^r the image of M_0 under the map

$$\begin{cases} C_+([0,1]) & \to C_+([0,1]) \\ X & \mapsto \Big(rX_{\frac{r-u}{r}}\mathbf{1}_{[0,r]}(u)\Big)_{0\le u\le 1} \end{cases},$$

and by M_2^r the image of M_0 under the map

$$\begin{cases} C_+([0,1]) & \to C_+([0,1]) \\ X & \mapsto \Big((1-r)X_{\frac{u-r}{1-r}}\mathbf{1}_{[r,1]}(u)\Big) \end{cases},$$

and setting $M^r := \frac{1}{r}M_1^r + \frac{1}{1-r}M_2^r$, we deduce that $M^r(\{0\}) = 0$, and that (5.15) holds. $\square$

The above propositions will be very important for us in proving Taylor estimates for the laws of pinned Bessel bridges $P^\delta\,[\,\cdot\,|X_r = b]$, for $r \in (0,1)$ and $\delta, b \ge 0$, since $P^\delta\,[\,\cdot\,|X_r = b]$ is the image of $Q^\delta\,[\,\cdot\,|X_r = b^2]$ under the map (5.12).

5.3 Density of $\mathscr{S}$ in a large space of functionals on $L^2(0,1)$

In this section we prove that a large class of functionals $\Phi : L^2(0,1) \to \mathbb{R}$ can be approximated by elements of $\mathscr{S}$. We do not need convergence in a very strong sense: point-wise convergence with some uniform dominations on the functionals and their differentials will suffice for our purpose. More precisely, we introduce the following definition:

Definition 5.3.1. Let $p \in \{1, 2\}$, and let Φ_n $(n \geq 1)$ and Φ be functionals on $L^2(0, 1)$ which are differentiable at each element of $C_+([0, 1])$, along any direction in $C_c^2(0, 1)$. We say that the sequence $(\Phi_n)_{n\geq 1}$ converges to Φ *point-wise with its derivatives, with p-domination on increments* (PDI_p for short) if the following three conditions hold:

(P) for all $X \in C_+([0, 1])$, we have the convergence

$$\Phi_n(X) \underset{n\to\infty}{\longrightarrow} \Phi(X),$$

together with the domination

$$\forall n \geq 1, \quad |\Phi_n(X)| \leq \|\Phi\|_\infty,$$

(D) for all $h \in C_c^2(0, 1)$, and all $X \in C_+([0, 1])$, we have the convergence

$$\partial_h \Phi_n(X) \underset{n\to\infty}{\longrightarrow} \partial_h \Phi(X),$$

together with the domination

$$\forall n \geq 1, \quad |\partial_h \Phi_n(X)| \leq C\|h\|_\infty(1 + \|X\|),$$

where $C > 0$ is some contant,

(I_p) there exists $K > 0$ such that, for all $X, Y \in C_+([0, 1])$ and $n \geq 1$, we have

$$|\Phi_n(X) - \Phi_n(Y)| \leq K\|X^2 - Y^2\|_1^{1/p}.$$

We can now state a density result, which is close in spirit to the density Lemmas 4.2.3 and 4.3.3 above, but with a different topology, more appropriate to the aim of the present chapter.

Proposition 5.3.2. *Let $\Phi \in \mathcal{S}C_b^1\left(L^1(0, 1)\right)$. Then there exists a family $(\Phi_{n,k}^d)_{d,n,k\geq 1}$ of elements of $\mathscr{S}$ such that*

$$\lim_{d\to\infty} \lim_{n\to\infty} \lim_{k\to\infty} \Phi_{n,k}^d = \Phi \tag{5.16}$$

where all the convergences are PDI_1.

Remark 5.3.3. We stress that, in the above statement, the domination properties associated with the PDI_1 convergence are uniform only on one index, the other indices being fixed. For instance, for all $d, n \geq 1$, there exists $C(d, n) > 0$ such that

$$\forall k \geq 1, \quad |\partial_h \Phi_{n,k}^d(X)| \leq C(d, n)||h||_\infty||X||,$$

but we do not claim that the constants $C(d,n)$ are bounded uniformly in $d,n \geq 1$. However, such bounds will be sufficient for our purposes; indeed, the only reason we need them is in order to show that each term in the IbPF converges when we take the successive limits $k \to \infty$, $n \to \infty$ and $d \to \infty$. The domination properties stated above will precisely allow us to do that by applying the dominated convergence theorem three times, successively.

Proof. We will follow the same route as for the proof of Lemma 4.2.3 above. We construct sequences $(\Phi^d)_{d,\geq 1}$, $(\Phi^d_n)_{d,n\geq 1}$ and $(\Phi^d_{n,k})_{d,n,k\geq 1}$ of functionals on $L^2(0,1)$ such that $\Phi^d_{n,k} \in \mathscr{S}$ for all $d,n,k \geq 1$, with the following convergences holding PDI_1:

$$\Phi^d_{n,k} \underset{k\to\infty}{\longrightarrow} \Phi^d_n \underset{n\to\infty}{\longrightarrow} \Phi^d \underset{d\to\infty}{\longrightarrow} \Phi.$$

We start by constructing $(\Phi^d)_{d,\geq 1}$. Let $\Psi \in C^1_b\left(L^1(0,1)\right)$ such that $\Phi(X) = \Psi(X^2)$, for all $X \in L^2(0,1)$. Then, for any $d \geq 1$, we define $(\zeta^d_i)_{1\leq i\leq d}$ to be the orthonormal family in $L^2(0,1)$ given by

$$\zeta^d_i := \sqrt{d}\ \mathbf{1}_{[\frac{i-1}{d},\frac{i}{d}[}, \quad i = 1,\ldots,d, \tag{5.17}$$

and we define Φ^d by

$$\Phi^d(X) := \Psi\left(\sum_{i=1}^{d} \langle \zeta^d_i, X^2\rangle\ \zeta^d_i\right), \quad X \in L^2(0,1).$$

We check that Φ^d converges PDI_1 to Φ as $d \to \infty$. We first remark that, for all $X \in C([0,1])$, we have

$$\sum_{i=1}^{d} \langle \zeta^d_i, X^2\rangle\ \zeta^d_i \underset{d\to\infty}{\longrightarrow} X^2$$

uniformly on $(0,1)$, hence in particular in $L^1(0,1)$. Since $\Psi : L^1(0,1) \to \mathbb{R}$ is continuous, this implies that

$$\Phi^d(X) \underset{d\to\infty}{\longrightarrow} \Phi(X).$$

Moreover, we have the domination

$$\forall X \in L^2(0,1), \quad |\Phi^d(X)| \leq ||\Phi||_\infty,$$

as requested by condition (P) in Definition 5.3.1. Furthermore, for all $h \in C^2_c(0,1)$ and $X \in C([0,1])$, we have

$$\partial_h \Phi^d(X) = 2D\Psi\left(\sum_{i=1}^{d} \langle \zeta^d_i, X^2\rangle\ \zeta^d_i\right)\left(\sum_{i=1}^{d} \langle \zeta^d_i X, h\rangle \zeta^d_i\right).$$

Now, since Ψ is C^1 on $L^1(0,1)$, we have

$$D\Psi\left(\sum_{i=1}^{d}\langle\zeta_i^d, X^2\rangle\ \zeta_i^d\right) \underset{d\to\infty}{\longrightarrow} D\Psi(X^2) \quad \text{in } L^1(0,1)',$$

while, at the same time, we also have

$$\sum_{i=1}^{d}\langle\zeta_i^d X, h\rangle\ \zeta_i^d \underset{d\to\infty}{\longrightarrow} h\ X,$$

uniformly in $(0,1)$, hence in $L^1(0,1)$. Therefore

$$D\Psi\left(\sum_{i=1}^{d}\langle\zeta_i^d, X^2\rangle\ \zeta_i^d\right)\left(\sum_{i=1}^{d}\langle\zeta_i^d X, h\rangle\zeta_i^d\right) \underset{d\to\infty}{\longrightarrow} 2D\Psi(X^2)(hX),$$

i.e.

$$\partial_h\Phi^d(X) \underset{d\to\infty}{\longrightarrow} \partial_h\Phi(X).$$

Moreover, for all $d \geq 0$, we have

$$|\partial_h\Phi^d(X)| \leq 2|||D\Psi|||_\infty ||h||_\infty ||X||, \tag{5.18}$$

which provides the requested domination property for $\left(\partial_h\Phi^d\right)_{d\geq 1}$. Thus condition (D) is fulfilled as well. Finally, for all $X, Y \in C_+([0,1])$, we have

$$\begin{aligned}\left|\Phi^d(X) - \Phi^d(Y)\right| &= \left|\Psi(X^2) - \Psi(X^2)\right| \\ &\leq |||D\Psi|||_\infty ||X^2 - Y^2||_1,\end{aligned}$$

as requested by condition (I_1). Therefore, $\Phi^d \underset{d\to\infty}{\longrightarrow} \Phi$ in the PDI_1 sense.

We now fix $d \geq 1$ and, for all integer $n \geq 1$, we construct Φ_n^d. The latter will be a truncated version of Φ^d obtained as follows. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function with values in $[0,1]$, such that $\chi = 1$ on $(-\infty, -1]$ and $\chi = 0$ on $[0, +\infty)$. Set $\chi_n(\cdot) := \chi(\cdot - n)$ and let

$$\Phi_n^d(X) = \Phi^d(X)\prod_{i=1}^{d}\chi_n\left(\langle\zeta_i, X^2\rangle\right), \quad X \in L^2(0,1).$$

We check that Φ_n^d converges PDI_1 to Φ^d as $n \to \infty$. Since $\chi_n \underset{n\to\infty}{\longrightarrow} 1$ point-wise, we have

$$\Phi_n^d(X) \underset{n\to\infty}{\longrightarrow} \Phi^d(X)$$

for all $X \in L^2(0,1)$. Moreover, we have $|\Phi_n^d(X)| \le \|\Phi^d\|_\infty$ for all $n \ge 1$ and $X \in L^2(0,1)$. Hence, the convergence and domination assumptions in condition (1) do indeed hold. Turning to condition (D), we remark that, for all $n \in \mathbb{N}$, $h \in C_c^2(0,1)$ and $X \in L^2(0,1)$, we have

$$\begin{aligned}\partial_h \Phi_n^d(X) &= \partial_h \Phi^d(X) \prod_{i=1}^d \chi_n\left(\langle \zeta_i, X^2\rangle\right) \\ &\quad + \Phi^d(X) \sum_{i=1}^d \chi_n'\left(\langle \zeta_i, X^2\rangle\right) \prod_{j\neq i} \chi_n\left(\langle \zeta_j, X^2\rangle\right) \langle 2\zeta_i X, h\rangle\end{aligned} \tag{5.19}$$

Since $\chi_n \underset{n\to\infty}{\longrightarrow} 1$ and $\chi_n' \underset{n\to\infty}{\longrightarrow} 0$ point-wise, it holds that $\partial_h \Phi_n^d(X) \underset{n\to\infty}{\longrightarrow} \partial_h^d \Phi(X)$. Moreover, by equality (5.19), and recalling (5.18), we have

$$|\partial_h \Phi_n^d(X)| \le \|\Psi\|_{C^1} \left(1 + d\|\chi'\|_\infty \|h\|_\infty \|X\|\right),$$

which provides the requested domination property. Finally, for all $n \ge 1$ and $X, Y \in C_+([0,1])$, we have

$$\begin{aligned}|\Phi_n^d(X) - \Phi_n^d(Y)| &\le \|\Psi\|_{C^1} \left(1 + \sum_{i=1}^d \|\chi'\|_\infty \|\zeta_i^d\|_\infty\right) \|X^2 - Y^2\|_1 \\ &\le \|\Psi\|_{C^1}(1 + d^{3/2}\|\chi'\|_\infty)||X^2 - Y^2||_1,\end{aligned}$$

so condition (I_1) is fulfilled as well. Hence Φ_n^d converges PDI_1 to Φ^d as $n \to \infty$.

Finally, we fix $d, n \ge 1$, and construct the sequence $(\Phi_{n,k}^d)_{k\ge 1}$. Note that Φ_n^d is of the form

$$\Phi_n^d(X) = g_n\left(\langle \zeta_1, X^2\rangle, \ldots, \langle \zeta_d, X^2\rangle\right), \quad X \in L^2(0,1),$$

where $g_n : \mathbb{R}_+^d \to \mathbb{R}$ is the function given by

$$g_n(x) := \Psi\left(\sum_{i=1}^d x_i\ \zeta_i^d\right) \prod_{i=1}^d \chi(x_i - n), \quad x \in \mathbb{R}_+^d. \tag{5.20}$$

Remark that g_n is C^1 with bounded support in $[0,n]^d$. We now make use of an approximation result in $\mathbb{R}_+^d$. Denoting by $\cdot$ the standard inner product on $\mathbb{R}^d$, let $\mathscr{E}$ be the linear span of the functions

$$e^{-\lambda\cdot} : \begin{cases} \mathbb{R}_+^d \to \mathbb{R} \\ x \mapsto e^{-\lambda\cdot x} \end{cases},$$

for $\lambda \in \mathbb{R}_+^d$. We state the following approximation result, the proof of which is postponed to Section 5.6 below:

Lemma 5.3.4. *Given $M > 0$, let $h : \mathbb{R}_+^d \to \mathbb{R}$ be a C^1 function supported in $[0, M]^d$. Then there exists a sequence of functions $h_k \in \mathscr{E}$, $k \geq 1$, such that*

- *for all $x \in \mathbb{R}_+^d$, $\quad h_k(x) \underset{k\to\infty}{\longrightarrow} h(x)$ and $\nabla h_k(x) \underset{k\to\infty}{\longrightarrow} \nabla h(x)$,*
- *for all $k \geq 1$ and all $x \in \mathbb{R}_+^d$, we have*
$$|h_k(x)| \leq |h(x)|,$$
and
$$\forall i = 1 \dots d, \qquad |\partial_i h_k(x)| \leq C(M)\ |\partial_i h(x)|,$$
where $C(M) > 0$ is a constant depending only on M.

For all $n \geq 1$ fixed, let now $(g_{n,k})_{k\geq 1}$ be a sequence of elements of $\mathscr{E}$ approximating g_n as in Lemma 5.3.4, and set

$$\Phi_{n,k}^d(X) := g_{n,k}\left(\langle \zeta_1, X^2\rangle, \dots, \langle \zeta_d, X^2\rangle\right), \quad X \in L^2(0,1).$$

Then for all $k \geq 1$, the functional $\Phi_{n,k}^d$ lies in $\mathscr{S}$. There remains to show that the sequence $\left(\Phi_{n,k}^d\right)_{k\geq 1}$ converges PDI_1 to Φ_n^d. For all $X \in L^2(0,1)$, we have

$$\begin{aligned}\Phi_{n,k}^d(X) &= g_{n,k}\left(\langle \zeta_1, X^2\rangle, \dots, \langle \zeta_d, X^2\rangle\right)\\ &\underset{k\to\infty}{\longrightarrow} g_n\left(\langle \zeta_1, X^2\rangle, \dots, \langle \zeta_d, X^2\rangle\right)\\ &= \Phi_n^d(X),\end{aligned}$$

with the domination

$$|\Phi_{n,k}^d(X)| \leq |\Phi_n^d(X)| \leq ||\Phi_n^d||_{C^1},$$

valid for all $X \in L^2(0,1)$ and $k \geq 1$. Moreover, for all $h \in C([0,1])$, and $X \in C_+([0,1])$, we have

$$\partial_h \Phi_{n,k}^d(X) = 2\sum_{i=1}^d \partial_i g_{n,k}\left(\langle \zeta_1, X^2\rangle, \dots, \langle \zeta_d, X^2\rangle\right)\langle \zeta_i X, h\rangle,$$

so that

$$\partial_h \Phi_{n,k}^d(X) \underset{k\to\infty}{\longrightarrow} \partial_h \Phi_n^d(X),$$

and

$$\begin{aligned}|\partial_h \Phi_{n,k}^d(X)| &\leq 2\sum_{i=1}^d ||\partial_i g_{n,k}||_\infty ||h||_\infty ||X||\\ &\leq 2C(n)\sum_{i=1}^d ||\partial_i g_n||_\infty ||h||_\infty ||X||\end{aligned}$$

Finally, we have, for all $X, Y \in C_+([0,1])$ and $k \geq 1$

$$|\Phi_{n,k}^d(X) - \Phi_{n,k}^d(Y)| \leq 2C(n) \sum_{i=1}^{d} ||\partial_i g_n||_\infty ||\zeta_i^d||_\infty ||X^2 - Y^2||_1.$$

Thus the sequence $\left(\Phi_{n,k}^d\right)_{k\geq 1}$ converges PDI_1 to $\Phi_{n,k}^d$. The proposition is proved. □

In the proof of the IbPF for $\delta \in (0,1)$, we shall need a slight refinement of the above proposition, stating that, if $\Phi \in \mathcal{S}C_b^{1,1}(L^1(0,1))$, the approximating sequences converge in a stronger sense. More precisely, we introduce the following notion of convergence:

Definition 5.3.5. Let Φ_n $(n \geq 1)$ and Φ be functionals on $L^2(0,1)$ which are differentiable at each element of $C_+([0,1])$, along any direction in $C_c^2(0,1)$. We say that the sequence $(\Phi_n)_{n\geq 1}$ converges PDI_{lip} to Φ if both following conditions hold:

- $(\Phi_n)_{n\geq 1}$ converges PDI_1 to Φ,
- for all $n \geq 1$ and $X, Z, Z \in L_+^1(0,1)$, we have

$$\begin{aligned}&\left|\Phi_n(\sqrt{X+Z+Z'}) - \Phi_n(\sqrt{X+Z}) - \Phi_n(\sqrt{X+Z'}) + \Phi_n(\sqrt{X})\right| \\ &\leq L||Z||_1||Z'||_1,\end{aligned} \tag{5.21}$$

  where $L > 0$ is some constant.

Proposition 5.3.6. *Let $\Phi \in \mathcal{S}C_b^{1,1}(L^1(0,1))$. Then the approximating sequences of functionals given by Proposition 5.3.2 are such that the convergences* (5.16) *actually hold in the PDI_{lip} sense.*

Proof. Since we already now that the convergences (5.16) hold in the PDI_1 sense, there only remains to prove that these approximating sequences further satisfy condition (5.21).

Let Ψ as in (5.10). Since $\Psi \in C_{b,lip}^1$, there exists $L > 0$ satisfying (5.6). Moreover, since the function

$$\begin{cases} L^1(0,1) & \to L^1(0,1) \\ Z & \mapsto \sum_{i=1}^d \langle \zeta_i^d, Z\rangle \zeta_i^d \end{cases}$$

is Lipschitz continuous (with Lipschitz constant 1), we deduce that the functional $\Psi^d : Z \mapsto \Psi\left(\sum_{i=1}^d \langle \zeta_i^d, Z\rangle \zeta_i^d\rangle\right)$ also satisfies (5.6). As a consequence, by Remark 5.1.2, for all $X, Z, Z' \in L^1_+(0,1)$ and $d \geq 1$, we have

$$\begin{aligned}
&\left|\Phi^d(\sqrt{X+Z+Z'}) - \Phi^d(\sqrt{X+Z}) - \Phi^d(\sqrt{X+Z'}) + \Phi^d(\sqrt{X})\right| = \\
&\left|\Psi^d(X+Z+Z') - \Psi^d(X+Z) - \Psi^d(X+Z') + \Psi^d(X)\right| \\
&\leq L \,||Z||_1 \,||Z'||_1 \,.
\end{aligned}$$

Hence, the sequence $(\Phi^d)_{d\geq 1}$ satisfies the condition (5.21), so it converges PDI_{lip} to Φ.

Moreover, for all $d \geq 1$, $\Psi^d \in C^1_{b,lip}$ and χ' is globally Lipschitz continuous (it is smooth and compactly supported). Hence, for all $n \geq 1$, the functional Ψ^d_n given by

$$\Psi^d_n(Z) := \Psi^d(Z) \prod_{i=1}^d \chi_n\left(\langle \zeta_i, Z\rangle\right), \quad Z \in L^1(0,1),$$

satisfies (5.6), with some Lipschitz constant L' depending only on Ψ, χ and d. Therefore for all $n \geq 1$ and $X, Z, Z' \in C_+([0,1])$, we have

$$\begin{aligned}
&\left|\Phi^d_n(\sqrt{X+Z+Z'}) - \Phi^d_n(\sqrt{X+Z}) - \Phi^d_n(\sqrt{X+Z'}) + \Phi^d_n(\sqrt{X})\right| = \\
&\left|\Psi^d_n(X+Z+Z') - \Psi^d_n(X+Z) - \Psi^d_n(X+Z') + \Psi^d_n(X)\right| \\
&\leq L' \,||Z||_1 \,||Z'||_1 \,,
\end{aligned}$$

so $(\Phi^d_n)_{n\geq 1}$ satisfies the condition (5.21), and hence converges PDI_{lip} to Φ.

Finally, note that, for all $n \geq 1$, the function g_n defined by (5.20) satisfies

$$\forall i = 1, \ldots, d, \ \forall x, y \in \mathbb{R}^d_+, \quad |\partial_i g_n(x) - \partial_i g_n(y)| \leq L' \sum_{j=1}^d |x_j - y_j|,$$

where $L' > 0$ is as above. We now invoke the following refinement of Lemma 5.3.4, the proof of which is also postponed to Section 5.6 below:

Lemma 5.3.7. *Given $M > 0$, let $h : \mathbb{R}^d_+ \to \mathbb{R}$ be a C^1 function supported in $[0, M]^d$, and satisfying furthermore*

$$\forall x, y \in \mathbb{R}^d_+, \forall i = 1, \ldots, d, \quad |\partial_i h(x) - \partial_i h(y)| \leq L' \sum_{j=1}^d |x_j - y_j| \qquad (5.22)$$

for some constant $L' > 0$. Then the sequence of functions $(h_k)_{k\geq 0}$ given by Lemma 5.3.4 further satisfies the following: for all $k \geq 1$ and $i = 1, \ldots, d$, we have

$$\forall x, y \in \mathbb{R}_+^d, \quad |\partial_i h_k(x) - \partial_i h_k(y)| \leq C'(M)\,(L' + ||\partial_i h||_\infty) \sum_{j=1}^{d} |x_j - y_j|$$

where $C'(M) > 0$ is a constant depending only on M.

Let now $(g_{n,k})_{k\geq 1}$ the sequence of functions approximating g_n as in Lemmas 5.3.4 and 5.3.7. Then, for all $k \geq 1$, we have

$$\forall i = 1, \ldots, d,\ \forall x, y \in \mathbb{R}_+^d, \quad |\partial_i g_{n,k}(x) - \partial_i g_{n,k}(y)| \leq C'(n)\,(L' + ||\partial_i g_n||_\infty) \sum_{j=1}^{d} |x_j - y_j|.$$

As a consequence, for all $k \geq 1$ and $x, z, z' \in \mathbb{R}_+^d$, we have

$$\begin{aligned} &|g_{n,k}(x + z + z') - g_{n,k}(x + z) - g_{n,k}(x + z') + g_{n,k}(x)| \\ &\leq C'(n)\,(L' + ||\partial_i g_n||_\infty) \sum_{j=1}^{d} |z_j| \sum_{j=1}^{d} |z'_j|. \end{aligned}$$

From that inequality we deduce that, for all $X, Z, Z' \in L_+^1(0, 1)$

$$\begin{aligned} &\left|\Phi_{n,k}^d(\sqrt{X + Z + Z'}) - \Phi_{n,k}^d(\sqrt{X + Z}) - \Phi_{n,k}^d(\sqrt{X + Z'}) + \Phi_{n,k}^d(\sqrt{X})\right| \\ &\leq C'(n)\,(L' + ||\partial_i g_n||_\infty)\, ||Z||_1 ||Z'||_1, \end{aligned}$$

which proves that the sequence $\left(\Phi_{n,k}^d\right)_{k\geq 1}$ satisfies the condition (5.21), and hence the requested convergence holds.

□

Propositions 5.3.2 and 5.3.6 enable to approximate, by elements of $\mathscr{S}$, any functional Φ of the form

$$\Phi(X) = \Psi(X^2), \quad X \in L^2(0, 1),$$

where $\Psi \in C_b^1\left(L^1(0, 1)\right)$. Such an assumption may appear rather restrictive, since it in particular forces $D\Psi(0)$ to vanish. However, it turns out that for general functionals $\Phi \in C_b^1(L^2(0, 1))$, we can also obtain such an approximation result, but in a weaker sense.

Proposition 5.3.8. *Let $\Phi \in C_b^1\left(L^2(0, 1)\right)$. Then there exists a family $(\Phi_{n,k}^{m,d})_{m,d,n,k\geq 1}$ of elements of $\mathscr{S}$ such that*

$$\lim_{m\to\infty} \lim_{d\to\infty} \lim_{n\to\infty} \lim_{k\to\infty} \Phi_{n,k}^{m,d} = \Phi \tag{5.23}$$

where the first three limits are PDI_1, while the last limit (in m) is PDI_2.

Proof. As in the proof of Proposition 5.3.2, we will proceed in several steps, by constructing sequences $(\Phi^m)_{m\geq 1}$, $(\Phi^{m,d})_{d,\geq 1}$, $(\Phi^{m,d}_n)_{m,d,n\geq 1}$ and $(\Phi^{m,d}_{n,k})_{m,d,n,k\geq 1}$ of functionals on $L^2(0,1)$ such that $\Phi^{m,d}_{n,k} \in \mathscr{S}$ for all $m, d, n, k \geq 1$, with

$$\Phi^{m,d}_{n,k} \underset{k\to\infty}{\longrightarrow} \Phi^{m,d}_n \underset{n\to\infty}{\longrightarrow} \Phi^{m,d} \underset{d\to\infty}{\longrightarrow} \Phi^m \underset{m\to\infty}{\longrightarrow} \Phi,$$

where the first three convergences hold PDI_1, and the last one holds PDI_2.

We start by constructing $(\Phi^m)_{m\geq 1}$. For all $m \geq 1$, let Φ^m be the functional given by

$$\Phi^m(X) := \Phi\left(\sqrt{X^2 + \frac{1}{m}}\right), \quad X \in L^2(0,1).$$

We show that the sequence $(\Phi^m)_{m\geq 1}$ converges PDI_2 to Φ. Note that for all $X \in C_+([0,1])$,

$$\sqrt{X^2 + \frac{1}{m}} \underset{m\to\infty}{\longrightarrow} X$$

in $L^2(0,1)$. Hence, since $\Phi : L^2(0,1) \to \mathbb{R}$ is continuous, we have

$$\Phi^m(X) \underset{m\to\infty}{\longrightarrow} \Phi(X).$$

Moreover, we have $|\Phi^m(X)| \leq ||\Phi||_\infty$ for all $X \in L^2(0,1)$. Furthermore, for all $h \in C([0,1])$, and all $X \in C_+([0,1])$

$$\partial_h \Phi^m(X) = \left\langle \nabla\Phi\left(\sqrt{X^2 + \frac{1}{m}}\right), \frac{Xh}{\sqrt{X^2 + \frac{1}{m}}} \right\rangle.$$

Now, since $\sqrt{X^2 + \frac{1}{m}} \underset{m\to\infty}{\longrightarrow} X$ in $L^2(0,1)$, we have

$$\nabla\Phi\left(\sqrt{X^2 + \frac{1}{m}}\right) \underset{m\to\infty}{\longrightarrow} \nabla\Phi\left(X^2\right) \qquad \text{in} \quad L^2(0,1).$$

On the other hand, we have

$$\frac{Xh}{\sqrt{X^2 + \frac{1}{m}}} \underset{m\to\infty}{\longrightarrow} h \qquad \text{a.e. on} \quad [0,1];$$

since, moreover

$$\left|\frac{Xh}{\sqrt{X^2 + \frac{1}{m}}}\right| \leq ||h||_\infty \quad \text{a.e. on } (0,1),$$

by the dominated convergence theorem, we deduce that

$$\frac{Xh}{\sqrt{X^2 + \frac{1}{m}}} \underset{m\to\infty}{\longrightarrow} h \qquad \text{in} \quad L^2(0,1).$$

Therefore,

$$\partial_h \Phi^m(X) \underset{m\to\infty}{\longrightarrow} \langle \nabla\Phi\,(X)\,, h\rangle = \partial_h \Phi(X),$$

with the domination

$$\forall m \geq 1, \quad |\partial_h \Phi^m(X)| \leq ||\nabla\Phi||_\infty ||h||.$$

Finally, given $X, Y \in C_+(0,1)$, for all $m \geq 1$, we have

$$|\Phi^m(X) - \Phi^m(Y)| \leq ||\nabla\Phi||_\infty \left|\left|X^2 - Y^2\right|\right|_1^{1/2},$$

which provides the domination condition (I_2) in Def. 5.3.1. Hence, the sequence (Φ^m) converges PDI_2 to Φ.

For $m \geq 1$ fixed, the sequences $(\Phi^{m,d})_{m,d\geq 1}$, $(\Phi_n^{m,d})_{m,d,n\geq 1}$ and $(\Phi_{n,k}^{m,d})_{m,d,n,k\geq 1}$ can then be constructed from Φ^m in exactly the same way as $(\Phi^d)_{d,\geq 1}$, $(\Phi_n^d)_{d,n\geq 1}$ and $(\Phi_{n,k}^d)_{d,n,k\geq 1}$ were constructed from Φ in the proof of Proposition 5.3.2. The key remark is that, for all $X \in C_+(0,1)$, we have

$$\Phi^m(X) = \Psi^m(X^2),$$

where

$$\Psi^m : \begin{cases} L^1(0,1) & \to L^1(0,1) \\ Z & \mapsto \Phi\left(\sqrt{|Z + \frac{1}{m}|}\right) \end{cases}$$

Although Ψ^m is not C^1, it is Lipschitz continuous on $L^1(0,1)$, and, at each $X \in C_+([0,1])$, has directional derivatives in all directions $h \in C([0,1])$ satisfying the bound

$$|\partial_h \Psi^m(Z)| \leq \frac{m}{2} ||\Phi||_{C^1} ||h||.$$

Therefore, exactly as in the proof of Proposition 5.3.2, we can show that the sequences $(\Phi^{m,d})_{m,d\geq 1}$, $(\Phi_n^{m,d})_{m,d,n\geq 1}$ and $(\Phi_{n,k}^{m,d})_{m,d,n,k\geq 1}$ will satisfy all the requested convergence and domination properties. We thus get the claim.

□

5.4 Taylor estimates for the laws of pinned Bessel bridges

In the previous section, we have shown that rather general functionals can be approximated by sequences of functionals in $\mathscr{S}$, for which we readily know that the IbPF of Theorem 3.1.1 above hold. Hence, to generalize the IbPF for the former functionals, we need to show that the terms appearing in our formulae converge when we take such limits. Thus in the case $\delta \in (1,3)$, as suggested by (5.4), we need to control, for all $r \in (0,1)$ and $b > 0$, the quantity

$$\mathcal{T}^0_{0,b} E^\delta[\Phi(X) | X_r = \cdot],$$

while in the case $\delta \in (0,1)$, we need to control

$$\mathcal{T}^2_{0,b} E^\delta[\Phi(X) | X_r = \cdot],$$

for all sufficiently regular functional Φ on $L^2(0,1)$. Obtaining such estimates is the goal of the present section.

5.4.1 Taylor estimates at order 0

As recalled in Section 5.1 above, for all $\Phi \in \mathscr{S}$, $\delta \in (1,3)$ and $h \in C^2_c(0,1)$, the integral

$$\int_0^1 dr h(r) \int_0^\infty db \; p^\delta_r(b) \frac{1}{b^3} \mathcal{T}^0_{0,b} E^\delta[\Phi(X) | X_r = \cdot] \tag{5.24}$$

is convergent. This is due to the fact that, for all $r \in (0,1)$, the function $b \mapsto E^\delta[\Phi(X) | X_r = b]$ is smooth, with vanishing derivative at 0. Hence, as $b \to 0$

$$\mathcal{T}^0_{0,b} E^\delta[\Phi(X) | X_r = \cdot] = O(b^2),$$

(see Remarks 3.1.2 and 3.1.3 in Chapter 3 above). By contrast, for an arbirary $\Phi \in C^1_b(L^2(0,1))$, it is not clear a priori whether such an estimate holds. Actually it is not even clear whether the integral (5.24) converges. However, it turns out that we can obtain a domination on the quantity $\mathcal{T}^0_{0,b} E^\delta[\Phi(X) | X_r = \cdot]$, even for an arbitrary $\Phi \in C^1_b(L^2(0,1))$. This bound is a little worse than in b^2, but it is still sufficient to make the double integral (5.24) converge.

Proposition 5.4.1. *There exists a universal constant $M > 0$ such that the following holds: for all $\delta \geq 0$, $L > 0$ and all functional $\Phi : L^2(0,1) \to \mathbb{R}$ satisfying*

$$\forall X, Y \in L^2_+(0,1), \quad |\Phi(X) - \Phi(Y)| \leq L \left(||X^2 - Y^2||_1 \right)^{1/2} \tag{5.25}$$

we have

$$\forall r \in (0,1),\ \forall b > 0, \quad |\mathcal{T}^0_{0,b} E^\delta[\Phi(X)|X_r = \cdot]| \leq M L\, b^2(|\log(b)| + 1). \tag{5.26}$$

In particular, for all such Φ, and all $\delta > 1$, the function

$$(r,b) \mapsto \mathcal{T}^0_{0,b} E^\delta[\Phi(X)|X_r = \cdot]$$

is integrable with respect to the measure $\frac{p^\delta_r(b)}{b^3}\ dr\ db$ on $(0,1) \times \mathbb{R}^*_+$.

Remark 5.4.2. Let $\Phi \in C^1_b(L^2(0,1))$. Then (5.25) holds with $L = ||\Phi||_{C^1}$. Indeed, for all $X, Y \in L^2_+(0,1)$, we then have

$$\begin{aligned}|\Phi(X) - \Phi(Y)| &\leq ||\Phi||_{C^1} ||X - Y|| \\ &= ||\Phi||_{C^1} \left(\int_0^1 |X(u) - Y(u)|^2\ du \right)^{1/2}\end{aligned}$$

But, noting that $(x-y)^2 \leq |x^2 - y^2|$ for all $x, y \geq 0$, we deduce that

$$\begin{aligned}|\Phi(X) - \Phi(Y)| &\leq ||\Phi||_{C^1} \left(\int_0^1 |X(u)^2 - Y(u)^2|\ du \right)^{1/2} \\ &= ||\Phi||_{C^1} \left(||X^2 - Y^2||_1 \right)^{1/2}\end{aligned}$$

whence the claim.

Proof. Let $\Phi : L^2(0,1) \to \mathbb{R}$ satisfying (5.25). We first assume the bound (5.26) to be true and check that the second statement holds. Let $\delta > 1$. Recalling (4.2.3), we have

$$\begin{aligned}&\int_0^1 \int_0^\infty \frac{p^\delta_r(b)}{b^3} |\mathcal{T}^0_{0,b} E^\delta[\Phi(X)|X_r = \cdot]|db\ dr \\ &\leq \int_0^1 \int_0^\infty \frac{b^{\delta-4}}{2^{\frac{\delta}{2}-1}(r(1-r))^{\delta/2}\Gamma(\frac{\delta}{2})} \exp\left(-\frac{b^2}{2r(1-r)}\right) M\ L\ b^2(|\log(b)|+1)\ \mathrm{d}b\ \mathrm{d}r \\ &= \frac{ML}{2^{\frac{\delta}{2}-1}\Gamma(\frac{\delta}{2})} \int_0^1 \frac{1}{(r(1-r))^{\delta/2}} \left(\int_0^\infty b^{\delta-2}(|\log(b)|+1) \exp\left(-\frac{b^2}{2r(1-r)}\right) \mathrm{d}b \right) \mathrm{d}r\end{aligned}$$

But, for all $r \in (0,1)$, performing the change of variable $a = \frac{b^2}{2r(1-r)}$, we obtain

$$\begin{aligned}&\int_0^\infty b^{\delta-2}(|\log(b)|+1) \exp\left(-\frac{b^2}{2r(1-r)}\right) \mathrm{d}b \\ &= (r(1-r))^{\frac{\delta-1}{2}}\, 2^{\frac{\delta-3}{2}} \int_0^\infty a^{\frac{\delta-3}{2}} e^{-a} \left(|\log(\sqrt{2r(1-r)a})| + 1 \right) da \\ &\leq (r(1-r))^{\frac{\delta-1}{2}}\, 2^{\frac{\delta-5}{2}} \left(\Gamma\left(\frac{\delta-1}{2}\right) |\log(2r(1-r))| + A \right)\end{aligned}$$

where $A := \int_0^\infty a^{\frac{\delta-3}{2}} e^{-a}(|\log(a)|+1)\ da \in (0,+\infty)$, since $\frac{\delta-3}{2} > -1$. Therefore

$$\begin{aligned}&\int_0^1 \int_0^\infty \frac{p_r^\delta(b)}{b^3} |\mathcal{T}_{0,b}^0 E^\delta[\Phi(X)|X_r = \cdot]| db\ dr \\ &\leq \frac{ML}{2^{3/2}\Gamma(\frac{\delta}{2})} \left\{ \Gamma\left(\frac{\delta-1}{2}\right) \int_0^1 \frac{|\log(2r(1-r))|}{(r(1-r))^{1/2}} dr + A \int_0^1 \frac{dr}{(r(1-r))^{1/2}} \right\}\end{aligned}$$

which is indeed finite, whence the claim.
We now prove that (5.26) indeed holds. By Proposition 5.2.5, for all $r \in (0,1)$ and $\delta, x \geq 0$, denoting by $Z_r(\delta, x)$ a random variable in $C_+([0,1])$ distributed according to $Q^\delta\ (\cdot|X_r = x)$, we have

$$Z_r(\delta, x) = Z_r(\delta, 0) + Z_r(0, x)$$

where $Z_r(\delta, 0)$ and $Z_r(0, x)$ are two independent random variables with laws given respectively by $Q^\delta\ (\cdot|X_r = 0)$ and $Q^0\ (\cdot|X_r = x)$. Therefore, for all functional $\Phi : L^2(0,1) \to \mathbb{R}$ satisfying (5.25), for all $r \in (0,1)$ and $b > 0$, we have

$$\begin{aligned} E^\delta[\Phi(X)|X_r = b] &= Q^\delta\ [\Phi(\sqrt{X})|X_r = b^2] \\ &= \mathbb{E}\left[\Phi\left(\sqrt{Z_r(\delta, b^2)}\right)\right] \\ &= \mathbb{E}\left[\Phi\left(\sqrt{Z_r(\delta, 0) + Z_r(0, b^2)}\right)\right] \end{aligned}$$

Hence

$$\begin{aligned} \left|\mathcal{T}_{0,b}^0 E^\delta[\Phi(X)|X_r = \cdot]\right| &= |E^\delta[\Phi(X)|X_r = b] - E^\delta[\Phi(X)|X_r = 0]| \\ &= \left|\mathbb{E}\left[\Phi\left(\sqrt{Z_r(\delta, 0) + Z_r(0, b^2)}\right) - \Phi\left(\sqrt{Z_r(\delta, 0)}\right)\right]\right| \\ &\leq \mathbb{E}\left[\left|\Phi\left(\sqrt{Z_r(\delta, 0) + Z_r(0, b^2)}\right) - \Phi\left(\sqrt{Z_r(\delta, 0)}\right)\right|\right]. \end{aligned}$$

But, by assumption (5.25), we have

$$\left|\Phi\left(\sqrt{Z_r(\delta, 0) + Z_r(0, b^2)}\right) - \Phi\left(\sqrt{Z_r(\delta, 0)}\right)\right| \leq L\left(||Z_r(0, b^2)||_1\right)^{1/2}.$$

Therefore

$$\begin{aligned} \left|\mathcal{T}_{0,b}^0 E^\delta[\Phi(X)|X_r = \cdot]\right| &\leq L\ \mathbb{E}\left[||Z_r(0, b^2)||_1^{1/2}\right] \\ &= L\ E^0\left[\ ||X^2||_1^{1/2}|X_r = b\right] \\ &= L\ E^0[\ ||X||\ |X_r = b], \end{aligned}$$

so there only remains to obtain a bound on $E^0[\,||X||\,|X_r = b]$. To do so, we exploit the knowledge of the quantity $E^0[\exp(-\lambda||X||^2)\,|X_r = b]$, for all $\lambda > 0$. Indeed, by equality (2.28) in Chapter 2, we have

$$E^0[\exp\left(-\lambda||X||^2\right)|X_r = b] = \exp\left[-C(r)\frac{b^2}{2}\right]$$

where

$$C(r) := \frac{\psi(1)}{\psi(r)\hat{\psi}(r)} - \frac{1}{r(1-r)},$$

with $\psi, \hat{\psi}$ associated via (2.22) and (2.23) with the measure $m(\mathrm{du}) = \lambda\,\mathrm{du}$ on $[0,1]$. One finds easily the following expressions for ψ and $\hat{\psi}$:

$$\psi(u) = \frac{1}{\sqrt{2\lambda}}\sinh\left(\sqrt{2\lambda}u\right), \quad \hat{\psi} = \frac{1}{\sqrt{2\lambda}}\sinh\left(\sqrt{2\lambda}\,(1-u)\right)$$

for all $u \in [0,1]$. In particular we obtain

$$\begin{aligned}\frac{\psi(1)}{\psi(r)\hat{\psi}(r)} &= \frac{\sqrt{2\lambda}\sinh\left(\sqrt{2\lambda}\right)}{\sinh\left(\sqrt{2\lambda}r\right)\sinh\left(\sqrt{2\lambda}(1-r)\right)} \\ &= \sqrt{2\lambda}\left(\coth(\sqrt{2\lambda}r) + \coth(\sqrt{2\lambda}(1-r))\right),\end{aligned}$$

where $\coth(x) := \frac{\cosh(x)}{sinh(x)}$ for all $x \neq 0$. Therefore, we have

$$\begin{aligned}C(r) &= \sqrt{2\lambda}\left(\coth(\sqrt{2\lambda}r) + \coth(\sqrt{2\lambda}(1-r))\right) - \frac{1}{r(1-r)} \\ &= \frac{1}{r}f(\sqrt{2\lambda}r) + \frac{1}{1-r}f\left(\sqrt{2\lambda}(1-r)\right)\end{aligned}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined for all $u \in \mathbb{R}$ by

$$f(u) = \begin{cases} u\,\coth(u) - 1, & \text{if } u \neq 0 \\ 0, & \text{if } u = 0. \end{cases}$$

We thus obtain the expression

$$E^0[\exp\left(-\lambda||X||^2\right)|X_r = b] = \exp\left[-\frac{b^2}{2}\left(\frac{1}{r}f(\sqrt{2\lambda}r) + \frac{1}{1-r}f\left(\sqrt{2\lambda}(1-r)\right)\right)\right]. \tag{5.27}$$

There now remains to deduce from (5.27) an expression for $E^0[\,||X||\,|X_r = b]$. To do so, we use the following lemma:

Lemma 5.4.3. *Let R be a nonnegative real variable such that $R > 0$ a.s. Then*

$$\mathbb{E}[R] = \frac{1}{2\sqrt{\pi}} \int_0^\infty \lambda^{-3/2} \left(1 - \mathbb{E}\left(\exp\left(-\lambda R^2\right)\right)\right) d\lambda.$$

Proof. By Fubini-Tonnelli, we have

$$\begin{aligned}
\int_0^\infty \lambda^{-3/2} \left(1 - \mathbb{E}\left(\exp\left(-\lambda R^2\right)\right)\right) d\lambda &= \mathbb{E}\left[\int_0^\infty \lambda^{-3/2} \left(1 - \exp\left(-\lambda R^2\right)\right) d\lambda\right] \\
&= \mathbb{E}[R] \int_0^\infty x^{-3/2}(1 - e^{-x}) dx,
\end{aligned}$$

where we performed the change of variable $x := R^2\lambda$ to obtain the last line (this is allowed, since $R > 0$ a.s.). But, by Lemma 3.2.3 in Chapter 2, the last integral equals $-\Gamma\left(-\frac{1}{2}\right) = 2\sqrt{\pi}$. The claim follows. □

Applying this lemma to the random variable $R := ||X||$ under the probability measure $E^0[\cdot|X_r = b]$ on $C_+(0,1)$, we obtain

$$\begin{aligned}
&E^0[\ ||X||\ |X_r = b] = \\
&\frac{1}{2\sqrt{\pi}} \int_0^\infty \lambda^{-3/2} \left(1 - \exp\left[-\frac{b^2}{2}\left(\frac{1}{r} f(\sqrt{2\lambda} r) + \frac{1}{1-r} f\left(\sqrt{2\lambda}(1-r)\right)\right)\right]\right) d\lambda.
\end{aligned}$$

Performing the change of variable $x = \sqrt{2\lambda}$, this yields

$$\begin{aligned}
&E^0[\ ||X||\ |X_r = b] = \\
&\sqrt{\frac{2}{\pi}} \int_0^\infty x^{-2} \left(1 - \exp\left[-\frac{b^2}{2}\left(\frac{1}{r} f(rx) + \frac{1}{1-r} f\left((1-r)x\right)\right)\right]\right) dx,
\end{aligned}$$

so it suffices to bound the latter integral. To do so, note that $f(u) = O(u^2)$ when $u \to 0$, whereas $f(u) = O(u)$ as $u \to +\infty$. Hence there exists a universal constant $C > 0$ such that

$$\forall u \geq 0, \quad f(u) \leq Cu \wedge u^2$$

Therefore

$$\begin{aligned}
\frac{1}{r} f(rx) + \frac{1}{1-r} f\left((1-r)x\right) &\leq \frac{C}{r}(rx) \wedge (rx)^2 + \frac{C}{1-r}\left((1-r)x\right) \wedge \left((1-r)x\right)^2 \\
&\leq \frac{C}{r}\ r\ x \wedge x^2 + \frac{C}{1-r}\ (1-r)\ x \wedge x^2 \\
&\leq 2C \quad x \wedge x^2.
\end{aligned}$$

Hence, we have

$$\int_0^{\infty} x^{-2}\left\{1-\exp\left[-\frac{b^2}{2}\left(\frac{1}{r}f(rx)+\frac{1}{1-r}f\left((1-r)x\right)\right)\right]\right\}dx$$
$$\leq \int_0^1 x^{-2}\left(1-\exp\left(-C\frac{b^2}{2}x^2\right)\right)dx+\int_1^{+\infty} x^{-2}\left(1-\exp\left(-C\frac{b^2}{2}x\right)\right)dx.$$

The first integral is bounded by

$$\int_0^1 x^{-2}C\frac{b^2}{2}x^2 \,\mathrm{d}x = C\frac{b^2}{2},$$

while the second one is seen, by a change of variable, to be equal to

$$\begin{aligned}\frac{Cb^2}{2}\int_{\frac{Cb^2}{2}}^{+\infty}\frac{1}{y^2}\left(1-e^{-y}\right)dy &\leq \frac{Cb^2}{2}\left\{\left|\int_{\frac{Cb^2}{2}}^{1}\frac{1}{y^2}\left(1-e^{-y}\right)dy\right|+\int_1^{+\infty}\frac{1}{y^2}\left(1-e^{-y}\right)dy\right\}\\ &\leq \frac{Cb^2}{2}\left\{\left|\int_{\frac{Cb^2}{2}}^{1}\frac{1}{y}dy\right|+\int_1^{+\infty}\frac{1}{y^2}dy\right\}\\ &= \frac{Cb^2}{2}\left(\left|\log\left(\frac{Cb^2}{2}\right)\right|+1\right).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\int_0^{\infty} x^{-2}\left\{1-\exp\left[-\frac{b^2}{2}\left(\frac{1}{r}f(rx)+\frac{1}{1-r}f\left((1-r)x\right)\right)\right]\right\}dx\\ &\leq Cb^2\left(\frac{1}{2}\left|\log\left(\frac{Cb^2}{2}\right)\right|+1\right)\\ &\leq C'b^2(|\log(b)|+1),\end{aligned}$$

where C' is some universal constant. Setting $M := \sqrt{\frac{2}{\pi}}C'$, the claim follows.

□

5.4.2 Differentiability properties of conditional expectations

In this section, we aim at proving that, for any $\delta \geq 0$, for a large class of functionals $\Phi : L^2(0,1) \to \mathbb{R}$, the function

$$\begin{cases}\mathbb{R}_+ \to \mathbb{R}\\ b \mapsto E^{\delta}[\Phi(X)|X_r = b]\end{cases}$$

is twice differentiable at 0. To do so we shall exploit Proposition 5.2.6 above, which provides the existence, for all $r \in (0,1)$, of a Lévy measure on $C_+([0,1])$ corresponding to the convolution semi-group $(Q^0[\,\cdot\,|X_r = x])_{x\geq 0}$. Note that the measures M^r, $r \in (0,1)$, are not finite. However they have the following important property:

Lemma 5.4.4. *For all $r \in (0,1)$,*

$$\int ||X||_1 \; dM^r(X) = \frac{1}{3} < \infty.$$

Proof. For all $x \geq 0$ and $\lambda > 0$, by (5.15), we have

$$Q^0\left[\exp\left(-\lambda ||X||_1\right) | X_r = x\right] = \exp\left(-x \int \left(1 - \exp\left(-\lambda ||X||_1\right)\right) dM^r(X)\right)$$

On the other hand, by (5.27), we have

$$\begin{aligned} Q^0\left[\exp\left(-\lambda ||X||_1\right) | X_r = x\right] &= E^0\left[\exp\left(-\lambda ||X||^2\right) | X_r = \sqrt{x}\right] \\ &= \exp\left[-\frac{x}{2}\left(\frac{1}{r} f(\sqrt{2\lambda} r) + \frac{1}{1-r} f\left(\sqrt{2\lambda}(1-r)\right)\right)\right] \end{aligned}$$

Therefore, we deduce that, for all $\lambda > 0$

$$\int \left(1 - \exp\left(-\lambda ||X||_1\right)\right) dM^r(X) = \frac{1}{2}\left(\frac{1}{r} f(\sqrt{2\lambda} r) + \frac{1}{1-r} f\left(\sqrt{2\lambda}(1-r)\right)\right) \tag{5.28}$$

But, by monotone convergence, we have

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \int \left(1 - \exp\left(-\lambda ||X||_1\right)\right) dM^r(X) = \int ||X||_1 \; dM^r(X).$$

On the other hand, since $f(x) = \frac{x^2}{3} + o(x^2)$ as $x \to 0$, we have

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\left(\frac{1}{r} f(\sqrt{2\lambda} r) + \frac{1}{1-r} f\left(\sqrt{2\lambda}(1-r)\right)\right) = \frac{2}{3}.$$

Therefore, dividing both sides of (5.28) by λ and taking the limit $\lambda \to 0$, we obtain

$$\int ||X||_1 \; dM^r(X) = \frac{1}{3},$$

which yields the claim. □

We are now in position to establish Proposition 5.1.5, which is an immediate consequence of the following result.

Proposition 5.4.5. *Let $\delta \geq 0$, and let Φ be a functional on $L^2(0,1)$ of the form*

$$\Phi(X) = \Psi(X^2), \quad X \in L^2(0,1) \tag{5.29}$$

where $\Psi : L^1(0,1) \to \mathbb{R}$ is bounded and globally Lipschitz continuous. Then, for all $r \in (0,1)$ and $b \geq 0$, we have

$$\begin{aligned} &E^\delta\left[\Phi(X)|X_r = b\right] = E^\delta\left[\Phi(X)|X_r = 0\right] \\ &+ 2\int_0^b a \int \left(E^\delta\left[\Phi\left(\sqrt{X^2+Z}\right)|X_r = a\right] - E^\delta\left[\Phi(X)|X_r = a\right]\right) \mathrm{d}M^r(Z)\,\mathrm{d}a \end{aligned} \tag{5.30}$$

In particular, the function

$$\begin{cases} \mathbb{R}_+ & \to \mathbb{R} \\ b & \mapsto E^\delta[\Phi(X)|X_r = b] \end{cases}$$

is twice differentiable at 0, and

$$\begin{aligned} &\frac{d^2}{db^2} E^\delta[\Phi(X)|X_r = b]\bigg|_{b=0} = \\ &2\int \left(E^\delta\left[\Phi\left(\sqrt{X^2+Z}\right)|X_r = 0\right] - E^\delta\left[\Phi(X)|X_r = 0\right]\right) \mathrm{d}M^r(Z) \end{aligned}$$

Remark 5.4.6. The idea behind this Proposition is the fact that for all $r \in (0,1)$

$$\left(Q^0\left[\,\cdot\,|X_r = x\right]\right)_{x \geq 0}$$

is a convolution semi-group, to which one could, using the same techniques as in [PY82], associate a subordinator with values in $C_+([0,1])$. That subordinator would be a compound Poisson point process with intensity $dt \otimes M^r$. For such a process one should have an Itô formula as in Theorem 5.1 of [IW14], from which formula (5.30) would then follow simply by taking expectations. Although such a strategy should be possible to implement using the constructions done in [PY82], since we do not need any pathwise statement, we prefer to resort to a more basic proof based on a density argument.

Proof. The second statement follows from equality (5.30). Indeed, for all fixed $Z \in C_+([0,1])$, the function

$$\begin{cases} \mathbb{R}_+ & \to \mathbb{R} \\ a & \mapsto E^\delta\left[\Phi\left(\sqrt{X^2+Z}\right)|X_r = a\right] - E^\delta\left[\Phi(X)|X_r = a\right] \end{cases}$$

is continuous. Moreover, it is dominated by $L||Z||_1$, where $L > 0$ is a Lipschitz constant for Ψ. Since $||Z||_1$ is integrable w.r.t. $M^r(dZ)$, we deduce that the function

$$F : \begin{cases} \mathbb{R}_+ & \to \mathbb{R} \\ a & \mapsto \int \left(E^\delta \left[\Phi\left(\sqrt{X^2+Z}\right)|X_r = a\right] - E^\delta \left[\Phi(X)|X_r = a\right]\right) M^r(dZ) \end{cases}$$

is continuous. But, by (5.30), we have, for all $b \geq 0$

$$E^\delta \left[\Phi(X)|X_r = b\right] = E^\delta \left[\Phi(X)|X_r = 0\right] + 2\int_0^b a\, F(a)\, \mathrm{d}a.$$

Hence, we deduce that $b \mapsto E^\delta \left[\Phi(X)|X_r = b\right]$ is twice differentiable at 0, with its derivative there given by $2F(0)$. This yields the second statement.

We now prove the first statement. We start by proving (5.30) for all $\Phi \in \mathscr{S}$. By linearity, we may assume that Ψ is of the form (3.1), which is tantamount to Φ satisfying (5.29), with Ψ given by

$$\Psi(Z) = \exp\left(-\langle\theta, Z\rangle\right), \quad Z \in L^1(0,1)$$

for some $\theta : [0,1] \to \mathbb{R}_+$ bounded and Borel. Note that, as a consequence of Proposition 5.2.5, for all $x \geq 0$, we have

$$Q^\delta[\Psi(X)|X_r = x] = Q^\delta[\Psi(X)|X_r = 0]\ Q^0[\Psi(X)|X_r = x],$$

so that, by (5.15)

$$Q^\delta[\Psi(X)|X_r = x] = Q^\delta[\Psi(X)|X_r = 0]\ \exp\left(x\int \left(\Psi(Z) - 1\right) dM^r(Z)\right).$$

Hence, differentiating in x, we have

$$\begin{aligned}
\frac{d}{dx}Q^\delta[\Psi(X)|X_r = x] &= \int \left(\Psi(Z) - 1\right) dM^r(Z)\ Q^\delta[\Psi(X)|X_r = x] \\
&= \int \left(Q^\delta[\Psi(Z)\Psi(X)|X_r = x] - Q^\delta[\Psi(X)|X_r = x]\right) dM^r(Z) \\
&= \int \left(Q^\delta[\Psi(X+Z)|X_r = x] - Q^\delta[\Psi(X)|X_r = x]\right) \mathrm{d}M^r(Z)
\end{aligned}$$

Hence, for all $x \geq 0$, we have

$$\begin{aligned}
Q^\delta[\Psi(X)|X_r = x] &= Q^\delta[\Psi(X)|X_r = 0] \\
&\quad + \int_0^x \int \left(Q^\delta[\Psi(X+Z)|X_r = y] - Q^\delta[\Psi(X)|X_r = y]\right) dM^r(Z)\, \mathrm{d}y.
\end{aligned}$$

Therefore, for all $b \geq 0$, we have

$$E^\delta[\Phi(X)|X_r = b^2] = E^\delta[\Phi(X)|X_r = 0]$$
$$+ \int_0^{b^2} \int \left(E^\delta[\Phi(\sqrt{X+Z})|X_r = \sqrt{y}] - E^\delta[\Phi(X)|X_r = \sqrt{y}] \right) \mathrm{d}M^r(Z)\, \mathrm{d}y,$$

so that, performing the change of variable $a := \sqrt{y}$, we obtain (5.30). Let now Φ be of the form (5.29), with $\Psi : L^1(0,1) \to \mathbb{R}$ bounded and globally Lipschitz continuous. We can construct a family of approximating functionals $(\Phi^d_{n,k})_{d,n,k\geq 1}$ as in the proof of Proposition 5.3.2. Although Ψ is not necessarily C^1, reasoning as in the proof of Proposition 5.3.2, we can check that these sequences will satisfy

$$\lim_{d\to\infty} \lim_{n\to\infty} \lim_{k\to\infty} \Phi^d_{n,k} = \Phi,$$

where each limit happens *almost* in the PDI_1 sense: conditions (P) and (I_1) of Definition 5.3.1 hold, and only condition (D) may not be satisfied. This will however suffice to conclude. Indeed, since for all $d, n, k \geq 1$, $\Phi^d_{n,k}$ lies in $\mathscr{S}$, by the previous point, we have

$$E^\delta \left[\Phi^d_{n,k}(X)|X_r = b\right] = E^\delta \left[\Phi^d_{n,k}(X)|X_r = 0\right]$$
$$+ 2\int_0^b a \int \left(E^\delta \left[\Phi^d_{n,k}\left(\sqrt{X^2+Z}\right)|X_r = a\right] - E^\delta \left[\Phi^d_{n,k}(X)|X_r = a\right]\right) dM^r(Z) da$$

Now, by virtue of the property (P) of Definition 5.3.1, we deduce that

$$\lim_{d,n,k\to\infty} E^\delta[\Phi^d_{n,k}(X)|X_r = b] = E^\delta[\Phi(X)|X_r = b],$$

and

$$\lim_{d,n,k\to\infty} E^\delta[\Phi^d_{n,k}(X)|X_r = 0] = E^\delta[\Phi(X)|X_r = 0].$$

We also deduce therefrom that, for all $Z \in C_+([0,1])$ and $a \in [0,b]$, we have

$$\lim_{d,n,k\to\infty} E^\delta \left[\Phi^d_{n,k}\left(\sqrt{X^2+Z}\right)\Big|X_r = a\right] - E^\delta \left[\Phi^d_{n,k}(X)|X_r = a\right] =$$
$$E^\delta \left[\Phi\left(\sqrt{X^2+Z}\right)\Big|X_r = a\right] - E^\delta \left[\Phi(X)|X_r = a\right],$$

and, by condition (I_1) in Definition 5.3.1, these three limits happen with uniform domination by $||Z||_1$. Since $||Z||_1$ is integrable with respect to $dM^r(Z)$ over $C_+([0,1])$, by three successive applications of the dominated convergence theorem, we deduce that

$$\lim_{d,n,k\to\infty} \int_0^b a \int \left(E^\delta \left[\Phi^d_{n,k}\left(\sqrt{X^2+Z}\right)\Big|X_r = a\right] - E^\delta \left[\Phi^d_{n,k}(X)|X_r = a\right]\right) dM^r(Z)\, \mathrm{d}a$$
$$= \int_0^b a \int \left(E^\delta \left[\Phi\left(\sqrt{X^2+Z}\right)\Big|X_r = a\right] - E^\delta \left[\Phi(X)|X_r = a\right]\right) dM^r(Z)\, \mathrm{d}a.$$

Hence, sending successively k, n and d to ∞ in (5.4.2), we deduce that Φ also satisfies (5.30). This yields the claim. $\square$

As a consequence of the above proposition, we deduce the following result, which improves the estimate (5.26) above for functionals of the form (5.29).

Proposition 5.4.7. *Let $\Phi : L^2(0,1) \to \mathbb{R}$ be a functional of the form* (5.29)*, with $\Psi : L^1(0,1) \to \mathbb{R}$ bounded and globally Lipschitz continous, with Lipschitz constant $L > 0$. Then, for all $\delta \geq 0$ the following holds:*

$$\forall r \in (0,1), \forall b > 0, \qquad \left|\mathcal{T}^0_{0,b} E^\delta[\Phi(X)|X_r = \cdot]\right| \leq \frac{L}{3}\, b^2. \tag{5.31}$$

In particular, for all $\delta > 1$, the function

$$(r,b) \mapsto \mathcal{T}^0_{0,b} E^\delta[\Phi(X)|X_r = \cdot]$$

is integrable with respect to the measure $\frac{p^\delta_r(b)}{b^3}\, dr\, db$ on $(0,1) \times \mathbb{R}^*_+$.

Proof. We first assume the bound (5.31) to be true and check that the second statement holds. Let $\delta > 1$. Recalling (5.3), we have

$$\begin{aligned}
&\int_0^1 \int_0^\infty \frac{p^\delta_r(b)}{b^3} \left|\mathcal{T}^0_{0,b} E^\delta[\Phi(X)|X_r = \cdot]\right| db\, dr \\
&\leq \int_0^1 \int_0^\infty \frac{b^{\delta-4}}{2^{\frac{\delta}{2}-1}(r(1-r))^{\delta/2}\Gamma(\frac{\delta}{2})} \exp\left(-\frac{b^2}{2r(1-r)}\right) \frac{L}{3}\, b^2\, db\, dr \\
&= \frac{L}{2^{\frac{\delta}{2}-1} 3\Gamma(\frac{\delta}{2})} \int_0^1 \frac{1}{(r(1-r))^{\delta/2}} \left(\int_0^\infty b^{\delta-2} \exp\left(-\frac{b^2}{2r(1-r)}\right) db\right) dr
\end{aligned}$$

But, for all $r \in (0,1)$, performing the change of variable $a = \frac{b^2}{2r(1-r)}$, we obtain

$$\begin{aligned}
\int_0^\infty b^{\delta-2} \exp\left(-\frac{b^2}{2r(1-r)}\right) db &= (r(1-r))^{\frac{\delta-1}{2}}\, 2^{\frac{\delta-3}{2}} \int_0^\infty a^{\frac{\delta-3}{2}} e^{-a} da \\
&= (r(1-r))^{\frac{\delta-1}{2}}\, 2^{\frac{\delta-3}{2}} \Gamma\left(\frac{\delta-1}{2}\right)
\end{aligned}$$

Therefore

$$\int_0^1 \int_0^\infty \frac{p^\delta_r(b)}{b^3} \left|\mathcal{T}^0_{0,b} E^\delta[\Phi(X)|X_r = \cdot]\right| db\, dr \leq \frac{L}{3\sqrt{2}\Gamma(\frac{\delta}{2})} \Gamma\left(\frac{\delta-1}{2}\right) \int_0^1 \frac{dr}{(r(1-r))^{1/2}}$$

which is indeed finite, whence the claim.

We now turn to the proof of the bound (5.31). By (5.30), for all $\delta \geq 0$, $r \in (0,1)$ and $b \geq 0$, we have

$$\begin{aligned}
&\left|\mathcal{T}^0_{0,b} E^\delta[\Phi(X)|X_r = \cdot]\right| = |E^\delta[\Phi(X)|X_r = b] - E^\delta[\Phi(X)|X_r = 0]| \\
&\leq 2\int_0^b a \int \left| E^\delta\left[\Phi\left(\sqrt{X^2+Z}\right)|X_r = a\right] - E^\delta\left[\Phi(X)|X_r = a\right]\right| dM^r(Z) da \\
&\leq 2\int_0^b a \int L||Z||_1 \, \mathrm{d}M^r(Z)\, \mathrm{d}a.
\end{aligned}$$

But, by Lemma 5.4.4, the last expression equals $2\int_0^b \ a \ \frac{L}{3} \ \mathrm{d}a = \frac{L}{3}b^2$, whence the claim. □

5.4.3 A second-order Taylor estimate

Proposition 5.4.8. *Let $\delta > 0$, and let Φ be a functional on $L^2(0,1)$ of the form*

$$\Phi(X) = \Psi(X^2), \quad X \in L^2(0,1)$$

where $\Psi : L^1(0,1) \to \mathbb{R}$ is bounded and globally Lipschitz-continuous. Assume furthermore that there exists $L > 0$ such that

$$\forall X, Z, Z' \in L^1(0,1), \quad |\Psi(X+Z+Z')-\Psi(X+Z)-\Psi(X+Z')+\Psi(X)| \leq L||Z||_1||Z'||_1.$$

Then, for all $r \in (0,1)$ and $b \geq 0$, the function

$$\begin{cases} \mathbb{R}_+ & \to \mathbb{R} \\ b & \mapsto E^\delta[\Phi(X)|X_r = b] \end{cases}$$

is twice differentiable at 0. Moreover, for all $b \geq 0$

$$\left|\mathcal{T}^2_{0,b} E^\delta[\Phi(X)|X_r = \cdot]\right| \leq L\, b^4, \tag{5.32}$$

In particular, the function

$$(r,b) \mapsto \mathcal{T}^2_{0,b} E^\delta[\Phi(X)|X_r = \cdot]$$

is integrable with respect to the measure $\frac{p^\delta_r(b)}{b^3}\, dr\, db$ on $(0,1) \times \mathbb{R}^*_+$.

Remark 5.4.9. This proposition applies in particular to any $\Phi \in \mathcal{SC}^1_{b,lip}(L^1(0,1))$.

Proof. The differentiability property follows from Proposition 5.4.5. Moreover, assuming the estimate (5.32) to be true, we have

$$\int_0^1 \int_0^\infty \frac{p_r^\delta(b)}{b^3} \left| \mathcal{T}_{0,b}^2 E^\delta[\Phi(X)|X_r = \cdot] \right| db\ dr$$
$$\leq \int_0^1 \int_0^\infty \frac{b^{\delta-4}}{2^{\frac{\delta}{2}-1}(r(1-r))^{\delta/2}\Gamma(\frac{\delta}{2})} \exp\left(-\frac{b^2}{2r(1-r)}\right)\ L\ b^4\ db\ dr$$
$$= \frac{L}{2^{\frac{\delta}{2}-1}\Gamma(\frac{\delta}{2})} \int_0^1 \frac{1}{(r(1-r))^{\delta/2}} \left(\int_0^\infty b^\delta \exp\left(-\frac{b^2}{2r(1-r)}\right) db \right) dr$$

But, for all $r \in (0,1)$, performing the change of variable $a = \frac{b^2}{2r(1-r)}$, we obtain

$$\begin{aligned}\int_0^\infty b^\delta \exp\left(-\frac{b^2}{2r(1-r)}\right) db &= (r(1-r))^{\frac{\delta+1}{2}}\, 2^{\frac{\delta-1}{2}} \int_0^\infty a^{\frac{\delta-1}{2}} e^{-a} da \\ &= (r(1-r))^{\frac{\delta+1}{2}}\, 2^{\frac{\delta-1}{2}} \Gamma\left(\frac{\delta+1}{2}\right)\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^1 \int_0^\infty \frac{p_r^\delta(b)}{b^3} \left| \mathcal{T}_{0,b}^2 E^\delta[\Phi(X)|X_r = \cdot] \right| \mathrm{d}b\ dr &\leq \sqrt{2} L \frac{\Gamma\left(\frac{\delta+1}{2}\right)}{\Gamma(\frac{\delta}{2})} \int_0^1 (r(1-r))^{1/2}\ \mathrm{d}r \\ &< \infty\end{aligned}$$

which proves the last statement. So there only remains to prove (5.32). To do so, remark that, for all $b \geq 0$, we have

$$E^\delta[\Phi(X)|X_r = b] = G(b^2),$$

where, for all $x \geq 0$, $G(x) := Q^\delta[\Psi(X)|X_r = x]$. As a consequence, we have

$$\mathcal{T}_{0,b}^2 E^\delta[\Phi(X)|X_r = \cdot] = G(b^2) - G(0) - b^2 G'(0).$$

Our claim will then follow from Taylor's theorem, once we have proved that G is C^2 on $\mathbb{R}_+$. To do so, note that, by equality (5.30), for all $x \geq 0$, we have

$$G(x) = G(0) + \int_0^x \int \left(Q^\delta[\Psi(X+Z)|X_r = y] - Q^\delta[\Psi(X)|X_r = y] \right) \mathrm{d}M^r(Z)\, \mathrm{d}y.$$

Now, by the Lipschitz property of Ψ, and since $\int ||Z||_1 M^r(\mathrm{dZ}) < \infty$, the function

$$\begin{cases} \mathbb{R}_+ & \to \mathbb{R} \\ y & \mapsto \int \left(Q^\delta \left[\Psi\left(X+Z\right)|X_r = y\right] - Q^\delta\left[\Psi(X)|X_r = y\right] \right) M^r(\mathrm{dZ}) \end{cases}$$

is continuous. Therefore, G is differentiable on $\mathbb{R}_+$, and, for all $x \geq 0$

$$G'(x) = \int \left(Q^\delta[\Psi(X+Z)|X_r = x] - Q^\delta[\Psi(X)|X_r = x]\right) \mathrm{d}M^r(Z). \tag{5.33}$$

By the same arguments, for all $Z \in C_+([0,1])$, the function

$$x \to Q^\delta[\Psi(X+Z)|X_r = x] - Q^\delta[\Psi(X)|X_r = x]$$

is differentiable on $\mathbb{R}_+$, with derivative given by

$$\begin{aligned}
&\int \left(Q^\delta[\Psi(X+Z+Z')|X_r = x] - Q^\delta[\Psi(X+Z)|X_r = x]\right) \mathrm{d}M^r(Z') \\
&- \int \left(Q^\delta[\Psi(X+Z')|X_r = x] - Q^\delta[\Psi(X)|X_r = x]\right) \mathrm{d}M^r(Z') \\
&= \int Q^\delta\left[\Psi(X+Z+Z') - \Psi(X+Z) - \Psi(X+Z') + \Psi(X)|X_r = x\right] \mathrm{d}M^r(Z')
\end{aligned}$$

Since for all $X, Z' \in C_+([0,1])$ we have

$$|\Psi(X+Z+Z') - \Psi(X+Z) - \Psi(X+Z') + \Psi(X)| \leq L||Z||_1||Z'||_1,$$

we deduce that

$$\begin{aligned}
\left|\frac{d}{dx}\left(Q^\delta[\Psi(X+Z)|X_r = x] - Q^\delta[\Psi(X)|X_r = x]\right)\right| &\leq L||Z||_1 \int ||Z'||_1 \,\mathrm{d}M^r(Z') \\
&= \frac{L}{3}||Z||_1,
\end{aligned}$$

Since $\int ||Z||_1 \,\mathrm{d}M^r(Z) < \infty$, we deduce that G' is differentiable on $\mathbb{R}_+$ and, for all $x \geq 0$, $G''(x)$ is given by

$$G''(x) = \int\int Q^\delta\left[\Psi(X+Z+Z') - \Psi(X+Z) - \Psi(X+Z') + \Psi(X)|X_r = x\right] \mathrm{d}M^r(Z')\,\mathrm{d}M^r(Z).$$

Note in particular that

$$||G''||_\infty \leq L \int ||Z||_1 \,\mathrm{d}M^r(Z) \int ||Z'||_1 \,\mathrm{d}M^r(Z') = \frac{L}{9},$$

Hence, by Taylor's theorem, we have, for all $x \geq 0$

$$|G(x) - G(0) - xG'(0)| \leq ||G''||_\infty \frac{x^2}{2} \leq L\,x^2$$

Therefore, for all $b \geq 0$, we have

$$|G(b^2) - G(0) - b^2 G'(0)| \leq L\,b^4.$$

This yields the claimed estimate. □

Remark 5.4.10. Propositions 5.4.5 and 5.4.8 above a priori apply for functionals Φ of the form

$$\Phi(X) = \Psi(X^2), \quad X \in L^2(0,1),$$

with $\Psi : L^1(0,1) \to \mathbb{R}$ sufficiently regular. It is not clear whether one could relax these conditions. For example, it is an open question whether these estimates would still hold for any $\Phi \in C_b^1(L^2(0,1))$, as is the case for the Taylor estimate at order 0 obtained in Proposition 5.4.1. If such were the case, then we could also relax the conditions on Φ in Theorems 5.1.6 and 5.1.8.

5.5 Extension of the integration by parts formulae to general functionals

We now turn to the proof of Theorems 5.1.3, 5.1.6, and 5.1.8, stating that the IbPF on P^δ for $\delta \in (0,3)$ extend to general, sufficiently regular functionals on $L^2(0,1)$. To do so, we will use the density results of Section 5.3 to approximate a general functional by elements of $\mathscr{S}$. Then we will use the estimates obtained in Section 5.5 to show that the last term appearing in the IbPF converges when we take such approximating sequences.

A little caveat here lies in the fact that our estimates concern Taylor remainders of the functions

$$b \mapsto E^\delta[\Phi(X)|X_r = b], \quad r \in (0,1)$$

while the last term in the IbPF contains Taylor remainders of the functions

$$b \mapsto \Sigma_r^\delta(\Phi(X)\,|\,\cdot\,), \quad r \in (0,1).$$

However, since the latter differs from the former only by a smooth function of b^2, we can re-express Taylor remainders of the latter as the sum of Taylor remainders of the former and some additional nicely-behaved terms. More precisely, the following holds:

Lemma 5.5.1. *Let $h \in C_c^2(0,1)$. Then, for all $\delta \in (1,3)$ and $\Phi \in C_b^1(L^2(0,1))$, we have*

$$\begin{aligned} &\int_0^1 h_r \int_0^\infty b^{\delta-4}\Big[\mathcal{T}_b^0\, \Sigma_r^\delta(\Phi(X)\,|\,\cdot\,)\Big]\,\mathrm{d}b\,\mathrm{d}r = \\ &\int_0^1 \mathrm{d}r\, h(r) \int_0^\infty \mathrm{d}b\, \frac{p_r^\delta(b)}{b^3} \mathcal{T}_b^0 E^\delta[\Phi(X)|X_r = \cdot\,] \\ &+ \frac{\Gamma(\frac{\delta-3}{2})}{\Gamma(\frac{\delta}{2})} \int_0^1 \mathrm{d}r \frac{h(r)}{(2r(1-r))^{3/2}} E^\delta[\Phi(X)|X_r = 0]. \end{aligned} \tag{5.34}$$

Moreover, for all $\Phi \in \mathcal{S}C^1_b\left(L^1(0,1)\right)$, *we have*

$$\frac{1}{4}\int_0^1 \mathrm{d}r\, h_r\, \frac{\mathrm{d}^2}{\mathrm{d}b^2}\,\Sigma^1_r[\Phi(X)\,|\,b]\Big|_{b=0} = -\frac{1}{2\sqrt{2\pi}}\int_0^1 dr\frac{h(r)}{(r(1-r))^{3/2}}E^1[\Phi(X)|X_r=0] + \frac{1}{2\sqrt{2\pi}}\int_0^1 \mathrm{d}r\frac{h(r)}{(r(1-r))^{1/2}}\frac{d^2}{db^2}E^1[\Phi(X)|X_r=b]\Big|_{b=0}. \tag{5.35}$$

Finally, for all $\delta \in (0,1)$ *and* $\Phi \in \mathcal{S}C^{1,1}_b\left(L^1(0,1)\right)$, *we have*

$$\int_0^1 h_r \int_0^\infty b^{\delta-4}\Big[\mathcal{T}^{\,2}_b\,\Sigma^\delta_r(\Phi(X)\,|\,\cdot\,)\Big]\,\mathrm{d}b\,\mathrm{d}r = \int_0^1 \mathrm{d}r\;h(r)\int_0^\infty \mathrm{d}b\;\frac{p^\delta_r(b)}{b^3}\mathcal{T}^2_b\left(E^\delta[\Phi(X)|X_r=\cdot]\right) + \sum_{0\le j\le 1}\frac{\Gamma\left(\frac{\delta-3}{2}+j\right)}{\Gamma\left(\frac{\delta}{2}\right)}\int_0^1 \mathrm{d}r\frac{h(r)}{2^{3/2}(r(1-r))^{3/2-j}}\frac{d^{2j}}{dx^{2j}}E^\delta[\Phi(X)|X_r=x]\Big|_{x=0}. \tag{5.36}$$

Proof. We prove only the equality for $\delta \in (1,3)$, since the other cases can be treated in the same way. For $h \in C^2_c(0,1)$ and $\Phi \in C^1_b(L^2(0,1))$, we have

$$\begin{aligned}&\int_0^1 h_r \int_0^\infty b^{\delta-4}\Big[\mathcal{T}^{\,0}_b\,\Sigma^\delta_r(\Phi(X)\,|\,\cdot\,)\Big]\,\mathrm{d}b\,\mathrm{d}r\\ &= \int_0^1 \mathrm{d}r h(r)\int_0^\infty \mathrm{d}b\;\frac{p^\delta_r(b)}{b^3}(E^\delta[\Phi(X)|X_r=b]-E^\delta[\Phi(X)|X_r=0])\\ &+\int_0^1 \mathrm{d}r h(r)\int_0^\infty \mathrm{d}b\; b^{\delta-4}\left(\gamma(r,b)-\gamma(r,0)\right)E^\delta[\Phi(X)|X_r=0],\end{aligned}$$

where, for $r\in(0,1)$ and $b\ge 0$, we have set $\gamma(r,b) := \frac{p^\delta_r(b)}{b^{\delta-1}}$. Note that the first integral in the right-hand side is aboslutely convergent by Prop. 5.4.1. Moreover, recall from (5.3) that

$$\gamma(r,b) = \frac{1}{2^{\frac{\delta}{2}-1}(r(1-r))^{\delta/2}\Gamma(\frac{\delta}{2})}\exp\left(-\frac{b^2}{2r(1-r)}\right),$$

so the second integral in the right-hand side is also absolutely convergent, and, hence, so is the integral in the left-hand side. Moreover, for all $r\in(0,1)$, applying equality (3.13) of Chapter 2 (with $x=\frac{\delta-3}{2}$ and $C=\frac{1}{r(1-r)}$), we have

$$\begin{aligned}\int_0^\infty \mathrm{d}b\; b^{\delta-4}\left(\gamma(r,b)-\gamma(r,0)\right) &= \int_0^\infty \frac{b^{\delta-4}}{2^{\frac{\delta}{2}-1}(r(1-r))^{\delta/2}\Gamma(\frac{\delta}{2})}\left(e^{-\frac{b^2}{2r(1-r)}}-1\right)\mathrm{d}b\\ &= \frac{\Gamma(\frac{\delta-3}{2})}{\Gamma(\frac{\delta}{2})}\frac{1}{(2r(1-r))^{3/2}}.\end{aligned}$$

We thus obtain the claim. □

5.5.1 Extension of the IbPF for $\delta \in (1,3)$

Proof of Theorem 5.1.3. Given $\Phi \in C_b^1(L^2(0,1))$, consider $(\Phi_{n,k}^{m,d})_{m,d,n,k\geq 1}$ approximating Φ as in Proposition 5.3.8. Then, for all $m,d,n,k \geq 1$, $\Phi_{n,k}^{m,d} \in \mathscr{S}$. Hence, by (5.4) and (5.34), we have

$$\begin{aligned} E^\delta(\partial_h \Phi_{n,k}^{m,d}(X)) = & -E^\delta(\langle h'', X\rangle \Phi_{n,k}^{m,d}(X)) \\ & - \kappa(\delta) \int_0^1 dr h(r) \int_0^\infty db\ p_r^\delta(b) \frac{1}{b^3} \mathcal{T}_{0,b}^0 E^\delta[\Phi_{n,k}^{m,d}(X) | X_r = \cdot] \\ & - \frac{\kappa(\delta)\Gamma(\frac{\delta-3}{2})}{\Gamma(\frac{\delta}{2})} \int_0^1 dr \frac{h(r)}{(2r(1-r))^{3/2}} E^\delta[\Phi_{n,k}^{m,d}(X)|X_r = 0]. \end{aligned} \tag{5.37}$$

Hence, to obtain the claim, it suffices to show that, as we send k, n, d and m to $+\infty$, each term appearing in (5.37) converges to the same term with $\Phi_{n,k}^{m,d}$ replaced with Φ.

Here, the convergence (5.16) comes into play. Indeed, as a consequence of condition (D) in Definition 5.3.1, and since $||h||_\infty(1+||X||)$ is integrable w.r.t. P^δ, by dominated convergence, we have

$$\lim_{m,d,n,k\to\infty} E^\delta(\partial_h \Phi_{n,k}^{m,d}(X)) = E^\delta(\partial_h \Phi(X)),$$

where we take the limits k, n, d and m successively. Moreover, by the condition (P), and since $|\langle h'', X\rangle| \leq ||h''||_\infty ||X||$ is integrable with respect to P^δ, by dominated convergence, we have

$$\lim_{m,d,n,k\to\infty} E^\delta(\langle h'', X\rangle \Phi_{n,k}^{m,d}(X)) = E^\delta(\langle h'', X\rangle \Phi(X)).$$

In a similar way, we obtain that

$$\begin{aligned} \lim_{m,d,n,k\to\infty} \int_0^1 dr \frac{h(r)}{(2r(1-r))^{3/2}} E^\delta[\Phi_{n,k}^{m,d}(X)|X_r = 0] = \\ \int_0^1 dr \frac{h(r)}{(2r(1-r))^{3/2}} E^\delta[\Phi(X)|X_r = 0]. \end{aligned}$$

Finally, for all $r \in (0,1)$, $b > 0$ and $X, Y \in C_+([0,1])$, by dominated convergence, we have

$$\lim_{m,d,n,k\to\infty} \mathcal{T}_{0,b}^0 E^\delta[\Phi_{n,k}^{m,d}(X)|X_r = \cdot] = \mathcal{T}_{0,b}^0 E^\delta[\Phi(X)|X_r = \cdot],$$

Moreover, as a consequence of the condition (I_p), $p = 1, 2$, in Definition 5.3.1, and by Lemmas 5.4.1 and 5.4.7, these convergences all happen with uniform domination by $b^2(|\log(b)|+1)$. Since the latter is integrable w.r.t. $p_r^\delta(b)\frac{1}{b^3}\, dr\, db$ on $(0,1)\times\mathbb{R}_+$, by dominated convergence, we obtain that

$$\lim_{m,d,n,k\to\infty}\int_0^1 dr h(r)\int_0^\infty db\ p_r^\delta(b)\frac{1}{b^3}\mathcal{T}_{0,b}^0 E^\delta[\Phi_{n,k}^{m,d}(X)|X_r=\cdot]$$
$$=\int_0^1 dr h(r)\int_0^\infty db\ p_r^\delta(b)\frac{1}{b^3}\mathcal{T}_{0,b}^0 E^\delta[\Phi(X)|X_r=\cdot].$$

We have thus proved that, when we send k,n,d and m to $+\infty$ in (5.37), all the terms converge to the same terms with $\Phi_{n,k}^{m,d}$ replaced with Φ. We thus obtain the claim. □

5.5.2 Extension of the IbPF for $\delta=1$

Proof of Theorem (5.1.6). Let $\Phi\in\mathcal{SC}_b^1\left(L^1(0,1)\right)$. Consider $(\Phi_{n,k}^d)_{d,n,k\geq 1}$ approximating Φ as in Proposition 5.3.2. Then, for all $d,n,k\geq 1$, $\Phi_{n,k}^d\in\mathscr{S}$ so, by (5.5) and (5.35), we have

$$\begin{aligned}E^1(\partial_h\Phi_{n,k}^d) =& -E^1(\langle h'',X\rangle\Phi_{n,k}^d)\\ &-\frac{1}{2\sqrt{2\pi}}\int_0^1 dr\frac{h(r)}{(r(1-r))^{3/2}}E^1[\Phi_{n,k}^d|X_r=0]\\ &+\frac{1}{2\sqrt{2\pi}}\int_0^1 dr\frac{h(r)}{(r(1-r))^{1/2}}\frac{d^2}{db^2}E^1[\Phi_{n,k}^d|X_r=b]\Big|_{b=0}.\end{aligned}\tag{5.38}$$

Here again, to conclude, it suffices to show that, as we send k,n and d to $+\infty$, each term appearing in (5.38) converges to the same term with $\Phi_{n,k}^d$ replaced with Φ. Reasoning as in the proof of Theorem 5.1.3, we obtain

$$\lim_{d,n,k\to\infty}E^1(\partial_h\Phi_{n,k}^d(X))=E^1(\partial_h\Phi(X)),$$

$$\lim_{d,n,k\to\infty}E^1(\langle h'',X\rangle\Phi_{n,k}^d(X))=E^1(\langle h'',X\rangle\Phi(X)),$$

and

$$\lim_{d,n,k\to\infty}\int_0^1 dr\frac{h(r)}{(r(1-r))^{3/2}}E^1[\Phi_{n,k}^d(X)|X_r=0]=\int_0^1 dr\frac{h(r)}{(r(1-r))^{3/2}}E^1[\Phi(X)|X_r=0],$$

where we take the limits k,n and d successively. Hence, there only remains to treat the last term in the right-hand side of (5.38).

For that term, note that, for all $d, n, k \geq 1$, $r \in (0,1)$ and $b \geq 0$, by Proposition 5.4.5, we have

$$\frac{d^2}{db^2} E^1[\Phi^d_{n,k} | X_r = b]\bigg|_{b=0} =$$
$$2 \int \left(E^1 \left[\Phi^d_{n,k} \left(\sqrt{X^2 + Z} \right) | X_r = 0 \right] - E^1 \left[\Phi^d_{n,k} | X_r = 0 \right] \right) dM^r(Z),$$

and, similarly

$$\frac{d^2}{db^2} E^1[\Phi(X) | X_r = b]\bigg|_{b=0} =$$
$$2 \int \left(E^1 \left[\Phi \left(\sqrt{X^2 + Z} \right) | X_r = 0 \right] - E^1 \left[\Phi(X) | X_r = 0 \right] \right) dM^r(Z),$$

Now, as a consequence of condition (P) in Definition 5.3.1, by dominated convergence, for all $Z \in C_+([0,1])$ we have

$$\lim_{d,n,k \to \infty} E^1 \left[\Phi^d_{n,k} \left(\sqrt{X^2 + Z} \right) | X_r = 0 \right] - E^1 \left[\Phi^d_{n,k}(X) | X_r = 0 \right] =$$
$$E^1 \left[\Phi \left(\sqrt{X^2 + Z} \right) | X_r = 0 \right] - E^1 \left[\Phi(X) | X_r = 0 \right],$$

and by condition (I_1), all three convergences happen with uniform domination by $||Z||_1$. Since, by Lemma 5.4.4, we have

$$\int_0^1 \mathrm{dr} \frac{|h(r)|}{(r(1-r))^{1/2}} \int ||Z||_1 M^r(\mathrm{dZ}) \leq \frac{1}{3} \int_0^1 \mathrm{dr} \frac{|\mathrm{h(r)}|}{(\mathrm{r(1-r)})^{1/2}} < \infty,$$

by dominated convergence, we deduce that

$$\lim_{d,n,k \to \infty} \int_0^1 dr \frac{h(r)}{(r(1-r))^{1/2}} \int \left(E^1 \left[\Phi^d_{n,k} \left(\sqrt{X^2 + Z} \right) | X_r = 0 \right] - E^1 \left[\Phi^d_{n,k}(X) | X_r = 0 \right] \right) \mathrm{d}M^r(Z)$$
$$= \int_0^1 dr \frac{h(r)}{(r(1-r))^{1/2}} \int \left(E^1 \left[\Phi \left(\sqrt{X^2 + Z} \right) | X_r = 0 \right] - E^1 \left[\Phi(X) | X_r = 0 \right] \right) \mathrm{d}M^r(Z),$$

i.e.

$$\lim_{d,n,k \to \infty} \int_0^1 dr \frac{h(r)}{(r(1-r))^{1/2}} \frac{d^2}{db^2} E^1[\Phi^d_{n,k}(X) | X_r = b]\bigg|_{b=0} =$$
$$\int_0^1 dr \frac{h(r)}{(r(1-r))^{1/2}} \frac{d^2}{db^2} E^1[\Phi(X) | X_r = b]\bigg|_{b=0}.$$

We thus obtain the claim. □

5.5.3 Extension of the IbPF for $\delta \in (0,1)$

Proof. Let $\Phi \in \mathcal{S}C_b^{1,1}(L^1(0,1))$, and consider $(\Phi^d_{n,k})_{d,n,k\geq 1}$ approximating Φ as in Proposition 5.3.6. Then, for all $d,n,k \geq 1$, since $\Phi^d_{n,k} \in \mathscr{S}$, by (5.4) and (5.36), we have

$$\begin{aligned} &E^\delta\left[\partial_h \Phi^d_{n,k}(X)\right] = -E^\delta\left[\langle h'', X\rangle \Phi^d_{n,k}(X)\right] \qquad (5.39)\\ &- \kappa(\delta)\int_0^1 dr\ h(r)\int_0^\infty \mathrm{d}b\ \frac{p^\delta_r(b)}{b^3}\mathcal{T}^2_{0,b}E^\delta[\Phi^d_{n,k}(X)|X_r=\cdot]\\ &- \frac{\kappa(\delta)\Gamma(\frac{\delta-3}{2})}{\Gamma(\frac{\delta}{2})}\int_0^1 \mathrm{d}r\frac{h(r)}{(2r(1-r))^{3/2}}E^\delta[\Phi^d_{n,k}(X)|X_r=0]\\ &- \frac{\kappa(\delta)\Gamma(\frac{\delta-1}{2})}{2\Gamma(\frac{\delta}{2})}\int_0^1 \mathrm{d}r\frac{h(r)}{(2r(1-r))^{1/2}}\frac{\mathrm{d}^2}{\mathrm{d}x^2}E^\delta[\Phi^d_{n,k}(X)|X_r=x]\Big|_{x=0} \end{aligned}$$

Reasoning exactly as in the proofs of Theorems 5.1.3 and 5.1.6, we obtain that

$$\lim_{d,n,k\to\infty} E^\delta(\partial_h \Phi^d_{n,k}(X)) = E^\delta(\partial_h\Phi(X)),$$

$$\lim_{d,n,k\to\infty} E^\delta(\langle h'',X\rangle \Phi^d_{n,k}(X)) = E^\delta(\langle h'',X\rangle\Phi(X)),$$

$$\lim_{d,n,k\to\infty}\int_0^1 dr\frac{h(r)}{(r(1-r))^{3/2}}E^\delta[\Phi^d_{n,k}(X)|X_r=0] = \int_0^1 dr\frac{h(r)}{(r(1-r))^{3/2}}E^\delta[\Phi(X)|X_r=0],$$

and

$$\begin{aligned}\lim_{d,n,k\to\infty}\int_0^1 dr\frac{h(r)}{(r(1-r))^{1/2}}\frac{d^2}{db^2}E^\delta[\Phi^d_{n,k}(X)|X_r=b]\Big|_{b=0} =\\ \int_0^1 dr\frac{h(r)}{(r(1-r))^{1/2}}\frac{d^2}{db^2}E^\delta[\Phi(X)|X_r=b]\Big|_{b=0}.\end{aligned}$$

Hence, there only remains to treat the second term in the right-hand side of (5.39).

Reasoning as before, we see that, for all $r \in (0,1)$, $b \geq 0$, we have

$$\lim_{d,n,k\to\infty}\mathcal{T}^2_{0,b}E^\delta[\Phi^d_{n,k}(X)|X_r=\cdot] = \mathcal{T}^2_{0,b}E^\delta[\Phi^d(X)|X_r=\cdot],$$

and, as a consequence of 5.3.6, by Proposition 5.4.8, all three limits happen with uniform domination by b^4. Since

$$\int_0^1 dr\ |h(r)|\int_0^\infty db\ \frac{p^\delta_r(b)}{b^3}b^4 < \infty,$$

by dominated convergence, we deduce that

$$\lim_{d,n,k\to\infty} \int_0^1 dr\ h(r) \int_0^\infty db\ \frac{p_r^\delta(b)}{b^3} \mathcal{T}_{0,b}^{2k} E^\delta[\Phi_{n,k}^d(X)|X_r = \cdot]$$
$$= \int_0^1 dr\ h(r) \int_0^\infty db\ \frac{p_r^\delta(b)}{b^3} \mathcal{T}_{0,b}^{2k} E^\delta[\Phi^d(X)|X_r = \cdot].$$

We thus obtain the claim. □

Remark 5.5.2 (An open question)**.** As mentioned in Remark 5.4.10, it is still unknown whether Theorems 5.1.6 and 5.1.8 apply for any $\Phi \in C_b^1(L^2(0,1))$. Answering this question would require to obtain either sharpness statements or refinements of the estimates obtained in Section 5.4.

5.6 Proofs of the approximation results

We now give a proof of the approximation results we used, which state the possibility of approximating regular enough functions on $\mathbb{R}_+^d$ by linear combinations of exponential functions. The main idea is simply to proceed to a change of variable using the exponential, so that we are led to the problem of approximating functions on $[0,1]^d$ by polynomials; this, in turn, is done using Bernstein polynomials. Note that while Lemma 5.3.4 is a consequence of Theorem 1.1.2 in [Lla86], we could not find in the litterature a version of the Weierstrass approximation Theorem yielding the particular type of convergence needed in Lemma 5.3.7. We therefore propose an elementary construction of the approximating sequences which works for both lemmas.

Proof of Lemma 5.3.4. Define

$$f : \begin{cases} [0,1]^d & \to \mathbb{R} \\ (y_1, \dots, y_d) & \mapsto h\left(-\ln(y_1), \dots, -\ln(y_d)\right) \end{cases}$$

For all $k \geq 0$, define the polynomial function $P_k f$ on $[0,1]^d$ by

$$P_k f(y) := \sum_{\substack{\ell=(\ell_1,\dots,\ell_d) \\ 0\leq \ell_1,\dots,\ell_d \leq k}} f\left(\frac{\ell}{k}\right) \prod_{i=1}^d B_{\ell_i}^k(y_i), \quad y \in [0,1]^d.$$

where we use the notation $\frac{\ell}{k} := (\frac{\ell_1}{k}, \dots, \frac{\ell_d}{k})$ and, for all $0 \leq m \leq k$, B_m^k is the Bernstein polynomial defined by

$$B_m^k(X) := \binom{k}{m} X^m (1-X)^{k-m}.$$

Note that these polynomials form a partition of unity

$$\forall k \geq 0, \quad \sum_{m=0}^{k} B_m^k(X) = 1. \tag{5.40}$$

We claim that the following holds:

- for all $y \in [0,1]^d$, $P_k f(y) \underset{k\to\infty}{\longrightarrow} f(y)$, and

$$\forall k \geq 0, \quad ||P_k f||_\infty \leq ||f||_\infty$$

- for all $y \in [0,1]^d$, $\nabla P_k f(y) \underset{k\to\infty}{\longrightarrow} \nabla f(y)$, and

$$\forall k \geq 0, \forall i = 1, \dots, d, \quad ||\partial_i P_k f||_\infty \leq ||\partial_i f||_\infty.$$

To prove the first point, note that, for all $y \in [0,1]^d$, we have

$$P_k f(y) = \mathbb{E}\left[f\left(\frac{S_k^1}{k}, \dots, \frac{S_k^d}{k}\right)\right] \tag{5.41}$$

where, for $1 \leq i \leq d$

$$S_k^i := \sum_{j=1}^{k} X_j^i,$$

the X_j^i being independent random variable, with X_j^i a Bernoulli variable of parameter y_i, for all $i = 1, \dots, d$ and $j = 1, \dots, k$. As a consequence of the weak law of large numbers, we have the convergence in probability

$$\frac{S_k^i}{k} \underset{k\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} y_i$$

for all $i = 1, \dots, d$. Hence, we have

$$\left(\frac{S_k^1}{k}, \dots, \frac{S_k^d}{k}\right) \underset{k\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} (y_1, \dots, y_d),$$

so that, since f is bounded and continuous on $[0,1]^d$, we deduce that $P_k f(y) \underset{k\to\infty}{\longrightarrow} f(y)$. Moreover, from the representation (5.41), we see that

$$||P_k f||_\infty \leq ||f||_\infty.$$

We now establish the second point. For all $i = 1, \dots, d$ and $y \in [0,1]^d$, we have

$$\partial_i P_k f(y) = \sum_{0 \leq \ell_1, \dots, \ell_d \leq k} f\left(\frac{\ell}{k}\right) B_{\ell_1}^k(y_1) \dots B_{\ell_i}^{k\,\prime}(y_i) \dots B_{\ell_d}^k(y_d).$$

But, for all $\ell = 1, \dots, k$, we have

$$B_\ell^{k\prime} = k\left(B_{\ell-1}^{k-1} - B_\ell^{k-1}\right)$$

(with the convention $B_m^n = 0$ if $n < 0$, $m < 0$ or $m > n$). Therefore, we have

$$\partial_i P_k f(y) = \sum_{0 \le \ell_1, \dots, \ell_d \le k} f\left(\frac{\ell}{k}\right) k\left(B_{\ell_i - 1}^{k-1}(y_i) - B_{\ell_i}^{k-1}(y_i)\right) \prod_{j \ne i} B_{\ell_j}^k(y_j),$$

which, after a discrete summation by parts, yields

$$\partial_i P_k f(y) = \sum_{0 \le \ell_1, \dots, \ell_d \le k} k\left(f\left(\frac{\ell}{k} + \frac{1}{k}e_i\right) - f\left(\frac{\ell}{k}\right)\right) B_{\ell_i}^{k-1}(y_i) \prod_{j \ne i} B_{\ell_j}^k(y_j),$$

where we have denoted by $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$. Now, for all $\ell \in \{0, \dots, k\}^d$, we have

$$\left| f\left(\frac{\ell}{k} + \frac{1}{k}e_i\right) - f\left(\frac{\ell}{k}\right) \right| \le \frac{1}{k} \|\partial_i f\|_\infty,$$

so that, recalling (5.40), we obtain

$$\begin{aligned} |\partial_i P_k f(y)| &\le \|\partial_i f\|_\infty \sum_{0 \le \ell_1, \dots, \ell_d \le k} B_{\ell_i}^{k-1}(y_i) \prod_{j \ne i} B_{\ell_j}^k(y_j) \\ &= \|\partial_i f\|_\infty. \end{aligned}$$

Moreover, we can write

$$\begin{aligned} \partial_i P_k f(y) = &\sum_{0 \le \ell_1, \dots, \ell_d \le k} \partial_i f\left(\frac{\ell}{k}\right) B_{\ell_i}^{k-1}(y_i) \prod_{j \ne i} B_{\ell_j}^k(y_j) \\ &+ \sum_{0 \le \ell_1, \dots, \ell_d \le k} R(k, \ell) B_{\ell_i}^{k-1}(y_i) \prod_{j \ne i} B_{\ell_j}^k(y_j), \end{aligned} \tag{5.42}$$

where, for all $\ell \in \{0, \dots, k\}^d$

$$R(k, \ell) := k\left(f\left(\frac{\ell}{k} + \frac{1}{k}e_i\right) - f\left(\frac{\ell}{k}\right)\right) - \partial_i f\left(\frac{\ell}{k}\right)$$

Since $\partial_i f$ is continuous on $[0,1]^d$, reasoning as for f, we obtain that the first term in the RHS of (5.42) converges, as $k \to \infty$, to $\partial_i f(y)$. Regarding the second term, note that

$$\begin{aligned}|R(k,\ell)| &= k\left|\int_{\frac{\ell_i}{k}}^{\frac{\ell_i+1}{k}} \left(\partial_i f\left(\frac{\ell_1}{k},\dots,t,\dots,\frac{\ell_d}{k}\right) - \partial_i f\left(\frac{\ell_1}{k},\dots,\frac{\ell_i}{k},\dots,\frac{\ell_d}{k}\right)\right) dt\right| \\ &\leq k\int_{\frac{\ell_i}{k}}^{\frac{\ell_i+1}{k}} \omega\left(\partial_i f, \frac{1}{k}\right) dt \\ &= \omega\left(\partial_i f, \frac{1}{k}\right),\end{aligned}$$

where $\omega(\partial_i f, \cdot)$ denotes the modulus of continuity of $\partial_i f$ on $[0,1]^d$. Therefore, the second term in the RHS of (5.42) is dominated by

$$\omega\left(\partial_i f, \frac{1}{k}\right) \sum_{0\leq \ell_1,\dots,\ell_d \leq k} B_{\ell_i}^{k-1}(y_i) \prod_{j\neq i} B_{\ell_j}^{k}(y_j) = \omega\left(\partial_i f, \frac{1}{k}\right),$$

which converges to 0 as $k \to \infty$. Therefore, sending $k \to \infty$ in (5.42), we deduce that

$$\partial_i P_k f(y) \underset{k\to\infty}{\longrightarrow} \partial_i f(y).$$

This proves the second point.
We can now conclude the proof of the lemma. Indeed, setting, for all $k \in \mathbb{N}$ and $x \in \mathbb{R}_+^d$,

$$h_k(x) := P_k f\left(e^{-x_1},\dots,e^{-x_d}\right),$$

it follows that h_k has the requested form. Moreover, by the first point above, we have

$$\forall x \in \mathbb{R}_+^d, \quad h_k(x) \underset{k\to\infty}{\longrightarrow} f\left(e^{-x_1},\dots,e^{-x_d}\right) = h(x_1,\dots,x_d),$$

together with the domination

$$\forall k \in \mathbb{N}, \quad \|h_k\|_\infty \leq \|P_k f\|_\infty \leq \|f\|_\infty.$$

On the other hand, by the second point above, for all $i = 1,\dots,d$ and $x \in \mathbb{R}_+^d$, we have

$$\begin{aligned}\partial_i h_k(x) =& - e^{-x_i}\partial_i P_k f\left(e^{-x_1},\dots,e^{-x_d}\right) \\ \underset{k\to\infty}{\longrightarrow}& - e^{-x_i}\partial_i f\left(e^{-x_1},\dots,e^{-x_d}\right) \\ =&\partial_i h(x).\end{aligned}$$

Moreover, we have

$$\forall k \in \mathbb{N}, \quad \|\partial_i h_k\|_\infty \leq \|\partial_i f\|_\infty$$

But, for all $y \in [0,1]^d$, we have

$$\partial_i f(y) = \frac{1}{y_i} \partial_i h \left(-\ln(y_1), \dots, -\ln(y_d)\right).$$

Now, since $\partial_i h$ is supported in $[0,M]^d$, $\partial_i f$ is supported in $[e^{-M},1]^d$. Therefore

$$\begin{aligned}\|\partial_i f\|_\infty &= \sup_{y\in[e^{-M},1]^d} \left| \frac{1}{y_i} \partial_i h \left(-\ln(y_1), \dots, -\ln(y_d)\right) \right| \\ &\le e^M \|\partial_i h\|_\infty,\end{aligned}$$

and, therefore, we have

$$\forall k \in \mathbb{N}, \quad \|\partial_i h_k\|_\infty \le e^M \|\partial_i h\|_\infty,$$

which gives the requested bound (with $C(M) := e^M$) . The lemma is proved. □

We now prove Lemma 5.3.7:

Proof of Lemma 5.3.7. To obtain the claim, it suffices to show that, as a consequence of the estimate (5.22), the sequence of functions $(h_k)_{k\ge 0}$ constructed in the proof of Lemma 5.3.4 satisfies, for all $k \ge 0$ and $i = 1, \dots, d$

$$\forall x, y \in \mathbb{R}_+^d, \quad |\partial_i h_k(x) - \partial_i h_k(y)| \le C'(M) \left(L' + \|\partial_i h\|_\infty\right) \sum_{j=1}^d |x_j - y_j|$$

for some constant $C'(M) > 0$.
From now on, let $k \ge 0$ and $i = 1, \dots, d$ be fixed. First note that, for all $u, v \in [e^{-M-1},1]^d$, we have

$$\begin{aligned}|\partial_i f(u) - \partial_i f(v)| &= \left| \frac{1}{u_i} \partial_i h \left(-\ln(u_1), \dots, -\ln(u_d)\right) - \frac{1}{v_i} \partial_i h \left(-\ln(v_1), \dots, -\ln(v_d)\right) \right| \\ &\le \frac{1}{u_i} \left| \partial_i h \left(-\ln(u_1), \dots, -\ln(u_d)\right) - \partial_i h \left(-\ln(v_1), \dots, -\ln(v_d)\right) \right| \\ &+ \left| \frac{1}{u_i} - \frac{1}{v_i} \right| \left| \partial_i h \left(-\ln(v_1), \dots, -\ln(v_d)\right) \right| \\ &\le e^{M+1} L' \sum_{j=1}^d |\ln(u_j) - \ln(v_j)| + \frac{|u_i - v_i|}{u_i v_i} \|\partial_i h\|_\infty \\ &\le e^{2(M+1)} L' \sum_{j=1}^d |u_j - v_j| + e^{2(M+1)} \|\partial_i h\|_\infty |u_i - v_i| \\ &\le e^{2(M+1)} \left(L' + \|\partial_i h\|_\infty\right) \sum_{j=1}^d |u_j - v_j|.\end{aligned}$$

Moreover, since f is supported in $[0,M]^d$, for all $u,v \notin [e^{-M},1]^d$, we have

$$|\partial_i f(u) - \partial_i f(v)| = 0$$

Finally, for all $u \in [e^{-n},1]^d$ and $v \notin [e^{-n-1},1]^d$, we have

$$\begin{aligned}|\partial_i f(u) - \partial_i f(v)| &= |\partial_i f(u)| \\ &\leq \|\partial_i f\|_\infty \\ &\leq e^M \|\partial_i h\|_\infty,\end{aligned}$$

and, since $\sum_{j=1}^d |u_j - v_j| \geq e^{-M}(1-e^{-1}) \geq e^{-M-2}$ by our assumption on u and v, we deduce that

$$|\partial_i f(u) - \partial_i f(u)| \leq e^{2M+2} \|\partial_i h\|_\infty \sum_{j=1}^d |u_j - v_j|.$$

Thus, we deduce that, for all $u,v \in [0,1]^d$, we have

$$|\partial_i f(u) - \partial_i f(u)| \leq e^{2(M+1)} \left(L' + \|\partial_i h\|_\infty\right) \sum_{j=1}^d |u_j - v_j|. \tag{5.43}$$

We will use this estimate to bound the second-order partial derivatives of $P_k f$.

Let first $j \in \{1,\dots,d\}$ such that $j \neq i$, and suppose, for example, that $j > i$. Then recall from the proof of Lemma 5.3.4 that, for all $u \in [0,1]^d$

$$\partial_i P_k f(u) = \sum_{0 \leq \ell_1,\dots,\ell_d \leq k} k \left(f\left(\frac{\ell}{k} + \frac{1}{k} e_i\right) - f\left(\frac{\ell}{k}\right)\right) B_{\ell_i}^{k-1}(u_i) \prod_{j \neq i} B_{\ell_j}^k(u_j).$$

By the same computations, we get

$$\partial_{i,j}^2 P_k f(y) = \sum_{0 \leq \ell_1,\dots,\ell_d \leq k} D(k,\ell) B_{\ell_i}^{k-1}(u_i) B_{\ell_j}^{k-1}(u_i) \prod_{m \neq i,j} B_{\ell_m}^k(u_m).$$

where, for all $\ell \in \{0,\dots,k\}^d$

$$\begin{aligned}D(k,\ell) := k^2 \Bigg[& f\left(\frac{\ell}{k} + \frac{1}{k} e_i + \frac{1}{k} e_j\right) - f\left(\frac{\ell}{k} + \frac{1}{k} e_i\right) \\ & - f\left(\frac{\ell}{k} + \frac{1}{k} e_j\right) + f\left(\frac{\ell}{k}\right)\Bigg].\end{aligned}$$

Hence, as a consequence of (5.43), we have

$$
\begin{aligned}
|D(k,\ell)| &= k^2 \left| \int_0^{\frac{1}{k}} \left(\partial_j f\left(\frac{\ell}{k} + \frac{1}{k}e_i + te_j\right) - \partial_j f\left(\frac{\ell}{k} + te_j\right)\right) dt \right| \\
&\leq k^2 e^{2(M+1)} \int_0^{\frac{1}{k}} \left(L' + \|\partial_j h\|_\infty\right) \frac{1}{k} dt \\
&= e^{2(M+1)} \left(L' + \|\partial_j h\|_\infty\right).
\end{aligned}
$$

Therefore, for all $u \in [0,1]^d$, recalling (5.40), we have

$$
\begin{aligned}
|\partial^2_{i,j} P_k f(u)| &\leq e^{2(M+1)} \left(L' + \|\partial_j h\|_\infty\right) \sum_{0\leq \ell_1,\ldots,\ell_d \leq k} B^{k-1}_{\ell_i}(u_i) B^{k-1}_{\ell_j}(u_i) \prod_{m\neq i,j} B^k_{\ell_m}(u_m) \\
&= e^{2(M+1)} \left(L' + \|\partial_i h\|_\infty\right).
\end{aligned}
$$

In a similar way we obtain that, for all $u \in [0,1]^d$, we have

$$
|\partial^2_{i,i} P_k f(u)| \leq e^{2(M+1)} \left(L' + \|\partial_i h\|_\infty\right).
$$

Recall now that h_k is defined, for all $x \in \mathbb{R}^d_+$, by

$$
h_k(x) = P_k f\left(e^{-x_1}, \ldots, e^{-x_d}\right).
$$

Hence, for all $j \neq i$ and $x \in \mathbb{R}^d_+$, we have

$$
\begin{aligned}
|\partial^2_{i,j} h_k(x)| &= |e^{-x_i} e^{-x_j} \partial^2_{i,j} P_k f\left(e^{-x_1}, \ldots, e^{-x_d}\right)| \\
&\leq e^{2(M+1)} \left(L' + \|\partial_i h\|_\infty\right).
\end{aligned}
$$

On the other hand, we have, for all $x \in \mathbb{R}^d_+$

$$
\partial^2_{i,i} h_k(x) = e^{-x_i} \partial_i P_k f\left(e^{-x_1}, \ldots, e^{-x_d}\right) + e^{-2x_i} \partial^2_{i,i} P_k f\left(e^{-x_1}, \ldots, e^{-x_d}\right),
$$

so that

$$
\begin{aligned}
|\partial^2_{i,i} h_k(x)| &\leq \|\partial_i P_k f\|_\infty + \|\partial^2_{i,i} P_k f\|_\infty \\
&\leq e^M \|\partial_i h\|_\infty + e^{2(M+1)} \left(L' + \|\partial_i h\|_\infty\right) \\
&\leq 2e^{2(M+1)} \left(L' + \|\partial_i h\|_\infty\right).
\end{aligned}
$$

We have thus proved that, for all $j = 1, \ldots, d$ and all $x \in \mathbb{R}^+_d$

$$
|\partial^2_{i,j} h_k(x)| \leq 2e^{2(M+1)} \left(L' + ||\partial_i h|_\infty\right).
$$

Therefore, for all $x, y \in \mathbb{R}^+_d$, we have

$$
|\partial_i h_k(x) - \partial_i h_k(y)| \leq 2e^{2(M+1)} \left(L' + \|\partial_i h\|_\infty\right) \sum_{j=1}^d |x_j - y_j|,
$$

which yields the desired bound. $\square$

Chapter 6

The case of integer dimensions

In this chapter we relate the integration by parts formulae (IbPF) for Bessel bridges obtained in Theorem 3.2.4 above with the formulae obtained in 2005 by Zambotti (see [Zam05]) for the law of a reflected Brownian motion, and later by Grothaus and Vosshall, (see [GV16] for the law of the modulus of a Brownian bridge. Note that those latter IbPF rely on the representation of a 1-Bessel process in terms of a Brownian motion. A similar representation holds for a δ-Bessel process for any integer δ, so that similar formulae can be obtained also in those cases: this is the content of Section 6.3 below. Finally, in Section 6.4, we propose a dynamical interpretation of these IbPF in terms of Itô-Tanaka formulae for SPDEs.

6.1 Link with an already known formula

In Theorem 3.2.5 above, we have proved an IbPF for P^1 for elements of $\mathcal{S}$. This formula contains, beside the usual term corresponding to the additive stochastic heat equation, additional terms of the form:

$$\frac{d^2}{dx^2}\Sigma_r^1(\,\mathrm{d}X|x)\bigg|_{x=0},$$

where $r \in (0,1)$, and the measures $\Sigma_r^1(\,\mathrm{d}X|x)$, $x \geq 0$, are given by (2.21) with $a = a' = 0$. Previous works have already explored the case $\delta = 1$. Thus, in [Zam05], the second author obtained an IbPF for the law of a reflected Brownian motion, and in [GV16], Grothaus and Vosshall treated the case of a reflected Brownian bridge. In these works, the additional term appearing in the IbPF is, at first sight, very different from what we obtained in the above section. For instance, in the case of a Brownian bridge on $[0,1]$, which we denote by β, setting $X = |\beta|$, the additional term can be written as some infinite-dimensional generalized functional in the sense of Schwartz defined in terms of a Hida's renormalization of the squared derivative of β and in terms of the local time of X.

We first introduce some notations. For all $\epsilon > 0$, let ρ_ϵ be a smooth mollifier as in (4.17). Then, for any function $w \in C([0,1])$, we define the functions w_ϵ and $\dot{w}_\epsilon$ by

$$w_{\epsilon,r} := \int_0^1 \rho_\epsilon(r-s)\, w_s \, \mathrm{d}s,$$

$$\dot{w}_{\epsilon,r} := w'_{\epsilon,r} = \int_0^1 \rho'_\epsilon(r-s)\, w_s \, \mathrm{d}s,$$

for all $r \in [0,1]$. Then, the last term in the IbPF derived in the previous works can be written as follows:

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\Phi(X) \int_0^1 h_r \left(: \dot{\beta}^2_{\epsilon,r} : -1\right) d\ell^0_r\right] \tag{6.1}$$

for all $h \in C^2_c(0,1)$ and all $\Phi \in \mathcal{FC}^\infty_b(L)$ (we refer to [GV16] for the precise statement and definitions). Here, $(\ell^0_r)_{r\in[0,1]}$ denotes the local time process at 0 of X. Moreover for all $\epsilon > 0$ and $r \in [0,1]$, $: \dot{\beta}^2_{\epsilon,r} :$ is defined by

$$: \dot{\beta}^2_{\epsilon,r} := \dot{\beta}^2_{\epsilon,r} - \mathbb{E}\left[\dot{\beta}^2_{\epsilon,r}\right], \tag{6.2}$$

In other words, to obtain the random variable $: \dot{\beta}^2_{\epsilon,r} :$, we regularize β, differentiate, take the square, and finally center the r.v. by substracting the mean. Note that, in [GV16], the authors provide a more direct description of (6.1), by constructing the associated generalized functional as an integral in the space of Hida distributions. That construction presents the advantage of not requiring a limit procedure, but it relies on highly sophisticated white noise analysis. We shall not present these details in this section, but only stress the fact that both descriptions coincide by Theorem 3.2 in [GV16] (see also Remark 1.3 in that article).

In view of these already existing results, it is natural to ask how they relate to our formula. We claim that the following holds:

Proposition 6.1.1. *Let β denote a Brownian bridge over $[0,1]$ and let $X = |\beta|$, so that $X \overset{(d)}{=} P^1_{0,0}$. Then, for all $\Phi \in \mathcal{S}$ and $h \in C^2_c(0,1)$, we have*

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\Phi(X) \int_0^1 h_r \left(: \dot{\beta}^2_{\epsilon,r} : -1\right) d\ell^0_r\right] = \frac{1}{4} \int_0^1 h_r \frac{\mathrm{d}^2}{\mathrm{d}b^2} \Sigma^{1,r}_a(\Phi(X) \,|\, b)\Big|_{b=0} \mathrm{d}r$$

Note that this result does not follow immediately from the computations done in [Zam05] and [GV16]. Indeed, there, the authors rely on the Cameron-Martin formula in order to compute (6.1) for all Φ of the form

$$\Phi(X) = \exp\left(\langle k, X\rangle\right), \quad X \in L^2(0,1),$$

where $k : [0,1] \to \mathbb{R}$ is some regular enough function. By contrast, elements of $\mathcal{S}$ are linear combinations of functionals of the form

$$\Phi(X) = \exp\left(-\langle m, X^2\rangle\right), \quad X \in L^2(0,1),$$

where m is a finite Borel measure on $[0,1]$. Thus, proving Proposition 6.1.1 is non-trivial and requires switching from the former class of functionals to the latter.

6.2 Proof of Proposition 6.1.1

Let $(\mathcal{F}_t)_{t\in[0,1]}$ denote the canonical filtration, and $\mathbb{W}$ denote the law of a standard Brownian motion on $C([0,1])$. The proof of Proposition 6.1.1 will rely on the following variation of Lemma 2.2.3 above.

Lemma 6.2.1. *Let m be a finite Borel measure on $[0,1]$, and let ϕ be the function thereto associated via the relation (2.13). Then, the measure R defined on $(C([0,1]), \mathcal{F}_1)$ by*

$$R := \phi_1^{-1/2} \exp\left(-\langle m, X^2\rangle\right) \mathbb{W}(\,\mathrm{d}X)$$

is a probability measure, and corresponds to the law of the process

$$(\phi_t \, W_{\varrho_t})_{t\in[0,1]},$$

where $W \overset{(d)}{=} \mathbb{W}$ and ϱ is the same time change as in Proposition 2.2.3.

Proof. Under $\mathbb{W}$, the coordinate process B is a standard Brownian motion started from 0. Therefore, by Itô's lemma, we have

$$B_t^2 = 2\int_0^t B_s dB_s + t.$$

In particular, $M_t := B_t^2 - t$ is a martingale, so the process

$$Y_t := \mathscr{E}\left(\frac{1}{2}\int_0^{\cdot} \frac{\phi_s'}{\phi_s}\,\mathrm{d}M_s\right)_t$$

is a local martingale. Reasoning as in the proof of Theorem (3.2) in Chapter XI of [RY13], we obtain the following expression for all $t \in [0,1]$:

$$Y_t = \exp\left[\frac{1}{2}\frac{\phi_t'}{\phi_t}B_t^2 - \int_0^t \theta_s B_s^2 ds - \frac{1}{2}\log\phi_t\right].$$

In particular, we deduce that $(Y_t)_{t\in[0,1]}$ is a martingale with respect to the canonical filtration $(\mathcal{F}_t)_{t\in[0,1]}$. Thus, the measure

$$R := \phi_1^{-1/2} \exp\left(-\langle m, X^2\rangle\right) \mathbb{W} = Y_1.\mathbb{W}$$

is a probability measure on $\mathcal{F}_1$. Moreover, by Girsanov's theorem, under R, the canonical process B satisfies the following SDE on $[0,1]$:

$$B_t = \tilde{B}_t + \int_0^t \frac{\phi_s'}{\phi_s} B_s ds,$$

where $\tilde{B}$ is a standard Brownian motion. Solving that equation yields the following expression for B:

$$B_t = \phi_t \int_0^t \phi_s^{-1}\, \mathrm{d}\tilde{B}_s, \quad t \in [0,1].$$

The claim follows upon invoking Lévy's theorem. □

We now turn to the proof of Proposition 6.1.1

Proof of Prop. 6.1.1. By linearity, we may assume that Φ is of the form (3.1). Let $\epsilon > 0$ be fixed. We start by rewriting

$$\mathbb{E}\left[\Phi(X) \int_0^1 h_r \left(: \dot{\beta}_{\epsilon,r}^2 : -1\right) \mathrm{d}\ell_r^0\right]$$

in terms of the laws of pinned Brownian bridges. Let $(L_t^a)_{a\in\mathbb{R},t\geq 0}$ denote a bicontinuous version of the local time process of β. Note that, for all $f : \mathbb{R} \to \mathbb{R}$ bounded and Borel, we have

$$\int_{-\infty}^{+\infty} \mathbb{E}\left[\Phi(X) \int_0^1 h_r \left(: \dot{\beta}_{\epsilon,r}^2 : -1\right) \mathrm{d}L_r^a\right] f(a) da =$$
$$\mathbb{E}\left[\Phi(X) \int_{-\infty}^{+\infty} f(a) \int_0^1 h_r \left(: \dot{\beta}_{\epsilon,r}^2 : -1\right) dL_r^a\; da\right] =$$
$$\mathbb{E}\left[\Phi(X) \int_0^1 h_r f(B_r) \left(: \dot{\beta}_{\epsilon,r}^2 : -1\right) dr\right] =$$
$$\int_{-\infty}^{+\infty} \int_0^1 h_r \frac{\exp\left(-\frac{a^2}{2r(1-r)}\right)}{\sqrt{2\pi r(1-r)}} \mathbb{E}\left[\Phi(X) \left(: \dot{\beta}_{\epsilon,r}^2 : -1\right) | \beta_r = a\right] dr\; f(a) da.$$

Here, we applied an extension of the occupation times formula (see exercise 1.15 in Chapter VI of [RY13]) to obtain the third line, and performed a conditioning to obtain the fourth one. Since the function f was arbitrary, we deduce that the equality

$$\mathbb{E}\left[\Phi(X)\int_0^1 h_r\left(:\dot{\beta}_{\epsilon,r}^2:-1\right)\,\mathrm{d}L_r^a\right]=$$
$$\int_0^1 h_r\frac{\exp\left(-\frac{a^2}{2r(1-r)}\right)}{\sqrt{2\pi r(1-r)}}\mathbb{E}\left[\Phi(X)\left(:\dot{\beta}_{\epsilon,r}^2:-1\right)|\beta_r=a\right]dr$$

holds for Lebesgue a.e. $a\in\mathbb{R}$, hence for every $a\in\mathbb{R}$ by continuity. As a consequence, recalling that $\ell_r^0=2L_r^0$, we deduce that

$$\mathbb{E}\left[\Phi(X)\int_0^1 h_r\left(:\dot{\beta}_{\epsilon,r}^2:-1\right)\,\mathrm{d}\ell_r^0\right]= \tag{6.3}$$
$$\int_0^1\frac{2\,h_r}{\sqrt{2\pi r(1-r)}}\mathbb{E}\left[\Phi(X)\left(:\dot{\beta}_{\epsilon,r}^2:-1\right)|\beta_r=0\right]dr.$$

Therefore, it suffices to compute, for all $r\in(0,1)$, the quantity

$$\mathbb{E}\left[\Phi(X)\left(:\dot{\beta}_{\epsilon,r}^2:-1\right)|\beta_r=0\right].$$

By our choice of Φ, and since $X=|\beta|$, this can be rewritten as a difference of two terms as follows:

$$\mathbb{E}\left[\exp\left(-\langle m,\beta^2\rangle\right)\dot{\beta}_{\epsilon,r}^2|\beta_r=0\right]-(c_{\epsilon,r}+1)\mathbb{E}\left[\exp\left(-\langle m,X^2\rangle\right)|X_r=0\right], \tag{6.4}$$

where $c_{\epsilon,r}:=\mathbb{E}\left[\dot{\beta}_{\epsilon,r}^2\right]$. Noting that

$$c_{\epsilon,r}+1=\frac{||\rho||_{L^2(\mathbb{R})}^2}{\epsilon},$$

(see [GV16]), and recalling Lemma 2.2.6, we see that the second term in (6.4) equals

$$\frac{||\rho||_{L^2(\mathbb{R})}^2}{\epsilon}\left[\frac{r(1-r)}{\psi_r\hat{\psi}_r}\right]^{\frac{1}{2}}.$$

So there remains to compute the first term, which we can rewrite as

$$\mathbb{E}\left[\exp\left(-\langle m,B^2\rangle\right)\dot{B}_{\epsilon,r}^2|B_r=B_1=0\right].$$

where B denotes a standard Brownian motion on $[0,1]$. By Fubini, this equals

$$\int_0^1\int_0^1\rho_\epsilon'(s-r)\rho_\epsilon'(t-r)\underbrace{\mathbb{E}\left[\exp\left(-\langle m,B^2\rangle\right)B_sB_t|B_r=B_1=0\right]}_{\xi(r,s,t)}ds\;dt. \tag{6.5}$$

Note that $\xi(r,s,t) = \xi(r,t,s)$ for all $s,t \in (0,1)$, so it suffices to compute $\xi(r,s,t)$ for $0 < s \le t < 1$. For such s,t, as a consequence of Lemma 6.2.1, we have

$$\begin{aligned}\xi(r,s,t) &= \sqrt{\phi_1}\frac{\phi_s\phi_t}{\phi_r\phi_1}\sqrt{\frac{r(1-r)}{\varrho_r(\varrho_1-\varrho_r)}}\mathbb{E}\left[B_{\varrho_s}B_{\varrho_t}|B_{\varrho_r} = B_{\varrho_1} = 0\right] \\ &= \phi_s\phi_t\sqrt{\frac{r(1-r)}{\psi_r\hat{\psi}_r}}\mathbb{E}\left[B_{\varrho_s}B_{\varrho_t}|B_{\varrho_r} = B_{\varrho_1} = 0\right].\end{aligned}$$

But, for all s,t as above, we have

$$\begin{aligned}&\mathbb{E}\left[B_{\varrho_s}B_{\varrho_t}|B_{\varrho_r} = B_{\varrho_1} = 0\right] = \\ &\mathbf{1}_{s\le t<r}\frac{\varrho_s(\varrho_r-\varrho_t)}{\varrho_r} + \mathbf{1}_{r<s\le t}\frac{(\varrho_s-\varrho_r)(\varrho_1-\varrho_t)}{\varrho_1-\varrho_r}.\end{aligned} \tag{6.6}$$

Hence

$$\xi(r,s,t) = \phi_s\phi_t\sqrt{\frac{r(1-r)}{\psi_r\hat{\psi}_r}}\left(\mathbf{1}_{s\le t<r}\frac{\varrho_s(\varrho_r-\varrho_t)}{\varrho_r} + \mathbf{1}_{r<s\le t}\frac{(\varrho_s-\varrho_r)(\varrho_1-\varrho_t)}{\varrho_1-\varrho_r}\right).$$

Note in particular that $\xi(r,s,t) = 0$ if $s < r < t$. We can simplify this expression by rewriting all the terms in ϱ using only the functions ϕ, ψ and $\hat{\psi}$. To do so, note that, for all $0 \le u \le v \le 1$

$$\phi_u\phi_v\left(\varrho_v - \varrho_u\right) = \frac{\psi_v\hat{\psi}_u - \psi_u\hat{\psi}_v}{\psi_1}.$$

Exploiting this identity, we finally obtain

$$\begin{aligned}\xi(r,s,t) = &\sqrt{\frac{r(1-r)}{\psi_r\hat{\psi}_r}}. \\ &\left(\mathbf{1}_{s\le t<r}\frac{\psi_s}{\psi_1}\left(\hat{\psi}_t - \frac{\psi_t\hat{\psi}_r}{\psi_r}\right) + \mathbf{1}_{r<s\le t}\frac{\hat{\psi}_t}{\psi_1}\left(\psi_s - \frac{\psi_r\hat{\psi}_s}{\hat{\psi}_r}\right)\right).\end{aligned} \tag{6.7}$$

We are now in position to compute the double integral (6.5). By the symmetry and vanishing properties of $\xi(r,\cdot,\cdot)$, this can be rewritten as $2(I_{<r} + I_{>r})$, where

$$I_{<r} = \int\int_{0<s\le t<r} \rho'_\epsilon(s-r)\rho'_\epsilon(t-r)\xi(r,s,t)\,\mathrm{d}s\,\mathrm{d}t,$$

and

$$I_{>r} = \int\int_{r<s\le t<1} \rho'_\epsilon(s-r)\rho'_\epsilon(t-r)\xi(r,s,t)\,\mathrm{d}s\,\mathrm{d}t.$$

Performing two successive integration by parts, we obtain

$$I_{<r} = \int_0^r \rho_\epsilon'(t-r)\rho_\epsilon(t-r)\xi(r,t,t)dt + \int_0^r \rho_\epsilon(s-r)^2 \frac{\partial \xi}{\partial s}(r,s,s+)ds$$
$$+ \int\int_{0<s\le t<r} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\partial^2 \xi}{\partial s \partial t}(r,s,t)ds\ dt,$$

where, for all $s \in (0,r)$

$$\frac{\partial \xi}{\partial s}(r,s,s+) := \lim_{t \downarrow s} \frac{\partial \xi}{\partial s}(r,s,t).$$

Note that we have exploited the cancellations

$$\xi(r,0,t) = 0, \quad \lim_{t \uparrow r} \frac{\partial \xi}{\partial s}(r,s,t) = 0$$

for all $0 < s \le t < r$. Remarking that

$$\rho_\epsilon'(t-r)\rho_\epsilon(t-r) = \frac{1}{2}\frac{d}{dt}\rho_\epsilon(t-r)^2, \quad t \in (0,r),$$

we thus obtain

$$I_{<r} = \frac{1}{2}\int_0^r \rho_\epsilon(s-r)^2 \left(\frac{\partial \xi}{\partial s}(r,s,s+) - \frac{\partial \xi}{\partial t}(r,s,s+) \right) ds$$
$$+ \int\int_{0<s\le t<r} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\partial^2 \xi}{\partial s \partial t}(r,s,t)ds\ dt,$$

where

$$\frac{\partial \xi}{\partial t}(r,s,s+) := \lim_{t \downarrow s} \frac{\partial \xi}{\partial t}(r,s,t).$$

Reasoning similarly, we obtain

$$I_{>r} = \frac{1}{2}\int_r^1 \rho_\epsilon(s-r)^2 \left(\frac{\partial \xi}{\partial s}(r,s,s+) - \frac{\partial \xi}{\partial t}(r,s,s+) \right) ds$$
$$+ \int\int_{r<s\le t<1} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\partial^2 \xi}{\partial s \partial t}(r,s,t)ds\ dt.$$

Hence, the double integral (6.5) equals

$$\int_0^1 \rho_\epsilon(s-r)^2 \left(\frac{\partial \xi}{\partial s}(r,s,s+) - \frac{\partial \xi}{\partial t}(r,s,s+) \right) ds \tag{6.8}$$
$$+ 2\int\int_{0<s\le t<r} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\partial^2 \xi}{\partial s \partial t}(r,s,t)ds\ dt \tag{6.9}$$
$$+ 2\int\int_{r<s\le t<1} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\partial^2 \xi}{\partial s \partial t}(r,s,t)ds\ dt. \tag{6.10}$$

Now, by the expression (6.7), we have, for all $s < r$

$$\frac{\partial \xi}{\partial s}(r,s,s+) - \frac{\partial \xi}{\partial t}(r,s,s+) = \sqrt{\frac{r(1-r)}{\psi_r \hat{\psi}_r}} \frac{\psi'_s \hat{\psi}_s - \psi_s \hat{\psi}'_s}{\psi_1},$$

which, in virtue of relation (3.10), yields

$$\frac{\partial \xi}{\partial s}(r,s,s+) - \frac{\partial \xi}{\partial t}(r,s,s+) = \sqrt{\frac{r(1-r)}{\psi_r \hat{\psi}_r}}.$$

Reasoning similarly, we see that this equality holds also for $s > r$. Therefore, the integral (6.8) above equals

$$\sqrt{\frac{r(1-r)}{\psi_r \hat{\psi}_r}} \int_0^1 \rho_\epsilon(s-r)^2 ds = \sqrt{\frac{r(1-r)}{\psi_r \hat{\psi}_r}} \frac{||\rho||^2_{L^2(\mathbb{R})}}{\epsilon},$$

so it exactly compensates with the diverging second term in (6.4). We now turn to the term (6.9) above. By (6.7), this equals

$$2\sqrt{\frac{r(1-r)}{\psi_r \hat{\psi}_r}} \int\int_{0<s\le t<r} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\psi'_s}{\psi_1}\left(\hat{\psi}'_t - \frac{\psi'_t \hat{\psi}_r}{\psi_r}\right) \mathrm{d}s \ \mathrm{d}t.$$

As for the term (6.10), we obtain the expression

$$2\sqrt{\frac{r(1-r)}{\psi_r \hat{\psi}_r}} \int\int_{r<s\le t<1} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\hat{\psi}'_t}{\psi_1}\left(\psi'_s - \frac{\psi_r \hat{\psi}'_s}{\hat{\psi}_r}\right) ds \ dt.$$

In conlusion, for all $\epsilon > 0$ and $r \in (0,1)$ we have

$$\begin{aligned}
&\mathbb{E}\left[\Phi(X)\left(:\dot{\beta}^2_{\epsilon,r}:-1\right)|\beta_r = 0\right] = \\
&2\sqrt{\frac{r(1-r)}{\psi_r \hat{\psi}_r}} \int\int_{0<s\le t<r} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\psi'_s}{\psi_1}\left(\hat{\psi}'_t - \frac{\psi'_t \hat{\psi}_r}{\psi_r}\right) \mathrm{d}s \ \mathrm{d}t + \\
&2\sqrt{\frac{r(1-r)}{\psi_r \hat{\psi}_r}} \int\int_{r<s\le t<1} \rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\hat{\psi}'_t}{\psi_1}\left(\psi'_s - \frac{\psi_r \hat{\psi}'_s}{\hat{\psi}_r}\right) \mathrm{d}s \ \mathrm{d}t.
\end{aligned}$$

Plugging this expression in (6.3), we obtain

$$\mathbb{E}\left[\Phi(X)\int_0^1 h_r\left(:\dot{\beta}^2_{\epsilon,r}:-1\right)d\ell^0_r\right]=$$

$$\int_0^1 \frac{4\ h_r}{\sqrt{2\pi\psi_r\hat{\psi}_r}}\int\int_{0<s\leq t<r}\rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\psi'_s}{\psi_1}\left(\hat{\psi}'_t-\frac{\psi'_t\hat{\psi}_r}{\psi_r}\right)ds\ dt+$$

$$\int_0^1 \frac{4\ h_r}{\sqrt{2\pi\psi_r\hat{\psi}_r}}\int\int_{r<s\leq t<1}\rho_\epsilon(s-r)\rho_\epsilon(t-r)\frac{\hat{\psi}'_t}{\psi_1}\left(\psi'_s-\frac{\psi_r\hat{\psi}'_s}{\hat{\psi}_r}\right)\mathrm{d}s\ \ \mathrm{d}t.$$

We now send ϵ to 0. Note that, since

$$\int\int_{u\leq v<0}\rho(u)\rho(v)\ du\ dv=\int\int_{0<u\leq v}\rho(u)\rho(v)\ \ \mathrm{d}u\ \ \mathrm{d}v=\frac{1}{8},$$

in both double integrals above, the kernel $\rho_\epsilon(s-r)\rho_\epsilon(t-r)$ acts as an approximation of $\frac{1}{8}\delta_{(r,r)}(s,t)$. Since, moreover, ψ and $\hat{\psi}$ are C^1, we deduce that

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\Phi(X)\int_0^1 h_r\left(:\dot{\beta}^2_{\epsilon,r}:-1\right)d\ell^0_r\right]=$$

$$\int_0^1\frac{h_r}{2\sqrt{2\pi\psi_r\hat{\psi}_r}}\left(\frac{\psi'_r}{\psi_1\psi_r}-\frac{\hat{\psi}'_r}{\psi_1\hat{\psi}_r}\right)\left(\psi_r\hat{\psi}'_r-\psi'_r\hat{\psi}_r\right)\mathrm{d}r=$$

$$-\frac{1}{2\sqrt{2\pi}}\int_0^1\psi_1\ \ \mathrm{d}r\ h_r\ \ \left(\psi_r\hat{\psi}_r\right)^{-\frac{3}{2}},\tag{6.11}$$

where we used relation (3.10) to obtain the third line. Recalling equality (3.16), the claim follows. □

6.3 IbPF for the laws of integer-dimensional Bessel bridges

The IbPF obtained in [GV16] concerns the law of a one-dimensional Bessel bridge. In this section we aim at generalizing that result to the laws of d-dimensional Bessel bridges, for all integer $d\geq 2$. To do so, we will follow the approach and techniques used in [Zam05]. The formulae we obtain here thus provide a natural extension of the results of [Zam05] and [GV16] to higher integer dimensions, and involve a similar type of renormalization.

In the sequel we fix an integer $d \geq 2$. As above, we consider a d- dimensional Brownian bridge $(\beta_r)_{0\leq t\leq 1}$ on $[0,1]$ and $X = ||\beta||$. We denote by C the set

$$C := \{k : [0,1] \to \mathbb{R}^d \text{ continuous}, k_0 = k_1 = 0\}.$$

Note that any $k \in C$ can be written as $k = (k^1, \ldots, k^d)$, where $k^1, \ldots, k^d$ are continuous functions on $[0,1]$. In this section, the space of test functionals we shall consider is given by

$$\text{Exp}(C) := \text{Span}\{\exp(\langle \cdot, k\rangle), k \in C\}.$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product on $L^2([0,1], \mathbb{R}^d)$:

$$\langle f, g\rangle := \sum_{i=1}^d \int_0^1 f_r^i g_r^i \, dr, \quad f, g \in L^2([0,1], \mathbb{R}^d).$$

For all $u \in \mathbb{R}^d \setminus \{0\}$, we denote by $\Pi_{u^\perp} : \mathbb{R}^d \to \mathbb{R}^d$ the orthogonal projection on $u^\perp$, i.e.

$$\Pi_{u^\perp}(v) = v - \left\langle v, \frac{u}{\|u\|} \right\rangle \frac{u}{\|u\|}, \qquad v \in \mathbb{R}^d.$$

Then, for all $\epsilon > 0$ and $r \in (0,1)$, the quantity

$$\Pi_{\beta_r^\perp}\left(\dot{\beta}_{\epsilon,r}\right)$$

is almost-surely well-defined, since $\mathbb{P}(\beta_r = 0) = 0$. On the negligible set where β_r vanishes, we can extend the above definition arbitrarily by setting

$$\Pi_{\beta_r^\perp}\left(\dot{\beta}_{\epsilon,r}\right) = 0.$$

Moreover, we set

$$: \|\Pi_{\beta_r^\perp}\left(\dot{\beta}_{\epsilon,r}\right)\|^2 : = \|\Pi_{\beta_r^\perp}\left(\dot{\beta}_{\epsilon,r}\right)\|^2 - \mathbb{E}\left[\|\Pi_{\beta_r^\perp}\left(\dot{\beta}_{\epsilon,r}\right)\|^2\right].$$

Furthermore, for any function $\varphi \in C_b^2(\mathbb{R}^d)$, and all $y \in \mathbb{R}^d$, we denote by $D^2\varphi(y) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the Hessian of φ at the point y, and we set

$$\begin{aligned} : D^2\varphi(y)\left(\dot{\beta}_{\epsilon,r}, \dot{\beta}_{\epsilon,r}\right) : &= D^2\varphi(y)\left(\dot{\beta}_{\epsilon,r}, \dot{\beta}_{\epsilon,r}\right) - \mathbb{E}\left[D^2\varphi(y)\left(\dot{\beta}_{\epsilon,r}, \dot{\beta}_{\epsilon,r}\right)\right] \\ &= D^2\varphi(y)\left(\dot{\beta}_{\epsilon,r}, \dot{\beta}_{\epsilon,r}\right) - \Delta\varphi(y)(c_{\epsilon,r} - 1), \end{aligned}$$

where $c_{\epsilon,r} = \frac{||\rho||_{L^2}^2}{\epsilon}$ is the same diverging constant as in [Zam05].

Finally, for all $F \in C_b^1(L^2(0,1))$, $\varphi \in C_b^2(\mathbb{R}^d)$, and $h \in C_c(0,1)$ we define

$$\partial_{h\nabla\varphi(\zeta)}F(\zeta) := \lim_{\epsilon\to 0}\frac{\Phi(\zeta+\epsilon h\nabla\varphi(\zeta))-\Phi(\zeta)}{\epsilon}, \qquad \zeta \in C,$$

thus adapting the notations of [Zam05] to the multi-dimensional setting.

With all these definitions at hand, we can now state the following:

Proposition 6.3.1. *For all $\Psi \in Exp(C)$, $\varphi \in C_b^2(\mathbb{R}^d)$ and $h \in C_c^2(0,1)$, we have*

$$\begin{aligned}&\mathbb{E}\left[\partial_{h\nabla\varphi(\beta)}\Psi(\beta)\right] = -\mathbb{E}\left[\langle h'', \varphi(\beta)\rangle \Psi(\beta)\right] \\ &+\lim_{\epsilon\to 0}\mathbb{E}\left[\Psi(\beta)\int_0^1 h_r\left(: D^2\varphi(\beta_r)\left(\dot{\beta}_{\epsilon,r},\dot{\beta}_{\epsilon,r}\right) : -\Delta\varphi(\beta_r)\right)\,\mathrm{d}r\right].\end{aligned}$$

Motivated by this Proposition, we formulate the following conjecture, which should follow by choosing φ to be the convex function given by $\varphi(x) := \|x\|$, $x \in \mathbb{R}^d$, and by using similar approximation techniques as for the proof of Theorem 2.3 in [Zam05]:

Conjecture 6.3.2. *Let $X := \|\beta\|$. Then, for all $\Phi \in C_b^1(L^2(0,1))$ and $h \in C_c^2(0,1)$, we have*

$$\begin{aligned}&\mathbb{E}\left[\partial_h\Phi(X)\right] = -\mathbb{E}\left[\langle h'', X\rangle \Phi(X)\right] \\ &+\lim_{\epsilon\to 0}\mathbb{E}\left[\Phi(X)\int_0^1 h_r\, X_r^{-1}\left(: \|\Pi_{\beta_r^\perp}\left(\dot{\beta}_{\epsilon,r}\right)\|^2 : -(d-1)\right)\,\mathrm{d}r\right].\end{aligned}$$

Proof of Prop 6.3.1. The arguments follow the same lines as the proof of Theorem 2.2 in [Zam05]. We first compute the limit

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\Psi(\beta)\int_0^1 h_r\left(: D^2\varphi(\beta_r)\left(\dot{\beta}_{\epsilon,r},\dot{\beta}_{\epsilon,r}\right) : -\Delta\varphi(\beta_r)\right)\,\mathrm{d}r\right].$$

By linearity, we may assume that $\Phi = \exp(\langle\cdot, k\rangle)$, with $k \in C$. Then, by the Cameron-Martin formula, for all $\epsilon > 0$ and $r \in (0,1)$ we have

$$\begin{aligned}&\mathbb{E}\left[\Psi(\beta)D^2\varphi(\beta_r)\left(\dot{\beta}_{\epsilon,r},\dot{\beta}_{\epsilon,r}\right)\right] \\ &= e^{1/2\langle Qk,k\rangle}\mathbb{E}\left[D^2\varphi(\beta_r+K_r)\left((\beta+K)'_{\epsilon,r},(\beta+K)'_{\epsilon,r}\right)\right].\end{aligned}$$

Now we fix $\epsilon > 0$ such that h is supported in $(\epsilon, 1-\epsilon)$, and we fix $r \in (\epsilon, 1-\epsilon)$. We can write

$$\beta_\sigma = \ell_\sigma\beta_r + \gamma_\sigma, \quad \sigma \in (0,1),$$

where $\ell_\sigma = \frac{\sigma}{r}\mathbf{1}_{\sigma\le r} + \frac{1-\sigma}{1-r}\mathbf{1}_{\sigma>r}$, $\sigma \in (0,1)$, and γ is a centered Gaussian process independent from β_r. Therefore, we have

$$\mathbb{E}\left[\Psi(\beta)D^2\varphi(\beta_r)\left(\dot\beta_{\epsilon,r},\dot\beta_{\epsilon,r}\right)\right]$$
$$= e^{1/2\langle Qk,k\rangle}\int_{\mathbb{R}^d}\mathcal{N}(0,r(1-r))(\,\mathrm{d}y)\sum_{i,j=1}^d \partial^2_{i,j}\varphi(y+K_r)\mathbb{E}\left[(\gamma_i+\ell y_i+K_i)'_{\epsilon,r}\,(\gamma_j+\ell y_j+K_j)'_{\epsilon,r}\right].$$

Now, for all $1\le i,j\le d$, we have

$$\mathbb{E}\left[(\gamma_i+\ell y_i+K_i)'_{\epsilon,r}\,(\gamma_j+\ell y_j+K_j)'_{\epsilon,r}\right] = \delta_{i,j}\mathbb{E}[(\dot\gamma_i)^2_{\epsilon,r}] + (\ell y_i+K_i)'_{\epsilon,r}(\ell y_j+K_j)'_{\epsilon,r},$$

where, for all $i=1,\dots,d$, we denoted by $(\dot\gamma_i)_{\epsilon,r}$ the derivative of $(\gamma_i)_{\epsilon,r}$. We easily check that

$$\mathbb{E}[(\dot\gamma_i)^2_{\epsilon,r}] = c_{\epsilon,r} - \frac{1}{4r(1-r)},$$

and that

$$\ell'_{\epsilon,r} = \frac{1-2r}{2r(1-r)}.$$

Hence, setting for all $1\le i,j\le d$ and $x,y\in\mathbb{R}^d$

$$\lambda_{i,j}(r,x,y) := \left(x_i + \frac{1-2r}{2r(1-r)}y_i\right)\left(x_j+\frac{1-2r}{2r(1-r)}y_j\right) - \delta_{i,j}\,\frac{1}{4r(1-r)},$$

we obtain

$$\mathbb{E}\left[\Psi(\beta)D^2\varphi(\beta_r)\left(\dot\beta_{\epsilon,r},\dot\beta_{\epsilon,r}\right)\right]$$
$$= e^{1/2\langle Qk,k\rangle}\int_{\mathbb{R}^d}\mathcal{N}(0,r(1-r))(\,\mathrm{d}y)\sum_{i,j=1}^d \partial^2_{i,j}\varphi(y+K_r)\left(\delta_{i,j}c_{\epsilon,r} + \lambda_{i,j}(r,K'_{\epsilon,r},y)\right)$$
$$= \mathbb{E}\left[\Psi(\beta)\left(c_{\epsilon,r}\Delta\varphi(\beta_r) + \sum_{i,j=1}^d \partial^2_{i,j}\varphi(\beta_r)\lambda_{i,j}(r,K'_{\epsilon,r},\beta_r-K_r)\right)\right].$$

Hence, we deduce that

$$\mathbb{E}\left[\Psi(\beta)\int_0^1 h_r\left(:D^2\varphi(\beta_r)\left(\dot\beta_{\epsilon,r},\dot\beta_{\epsilon,r}\right): -\Delta\varphi(\beta_r)\right)\mathrm{d}r\right]$$
$$= \mathbb{E}\left[\Psi(\beta)\int_0^1 h_r\left(\sum_{i,j=1}^d \partial^2_{i,j}\varphi(\beta_r)\,\lambda_{i,j}(r,K'_{\epsilon,r},\beta_r-K_r)\right)\mathrm{d}r\right],$$

which converges, as $\epsilon \to 0$, to

$$\mathbb{E}\left[\Psi(\beta)\int_0^1 h_r\left(\sum_{i,j=1}^d \partial^2_{i,j}\varphi(\beta_r)\,\lambda_{i,j}(r,K'_r,\beta_r-K_r)\right)\mathrm{d}r\right].$$

We now compute

$$\mathbb{E}\left[\partial_{h\nabla\varphi(\beta)}\Psi(\beta)\right]+\mathbb{E}\left[\langle h'',\varphi(\beta)\rangle\Psi(\beta)\right].$$

Using Itô's formula, and proceeding as in the proof of Lemma 4.1 of [GV16], we obtain, for all $r\in(0,1)$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\mathbb{E}(\varphi(\beta_r+K_r)) &= -\sum_{i=1}^d k^i_r\,\mathbb{E}[\partial_i\varphi(\beta_r+K_r)]\\ &\quad+\sum_{i,j=1}^d \mathbb{E}[\partial^2_{i,j}\varphi(\beta_r+K_r)\lambda_{i,j}(r,K'_r,\beta_r)]\end{aligned}$$

Hence, multiplying by h_r and integrating in r, we obtain

$$\begin{aligned}\int_0^1 h''_r\,\mathbb{E}(\varphi(\beta_r+K_r))\,\mathrm{d}r &= -\mathbb{E}\left[\sum_{i=1}^d\int_0^1 h_rk^i_r\,\partial_i\varphi(\beta_r+K_r)\,\mathrm{d}r\right]\\ &\quad+\sum_{i,j=1}^d\int_0^1 h_r\,\mathbb{E}[\partial^2_{i,j}\varphi(\beta_r+K_r)\lambda_{i,j}(r,K'_r,\beta_r)]\,\mathrm{d}r.\end{aligned}$$

By the Cameron-Martin formula, we deduce that

$$\begin{aligned}\int_0^1 h''_r\,\mathbb{E}[\Psi(\beta)\varphi(\beta_r)]\,\mathrm{d}r &= -\mathbb{E}\left[\Psi(\beta)\sum_{i=1}^d\int_0^1 h_rk^i_r\,\partial_i\varphi(\beta_r+K_r)\,\mathrm{d}r\right]\\ &\quad+\sum_{i,j=1}^d\int_0^1 h_r\,\mathbb{E}[\Psi(\beta)\partial^2_{i,j}\varphi(\beta_r)\lambda_{i,j}(r,K'_r,\beta_r-K_r)]\,\mathrm{d}r,\end{aligned}$$

that is

$$\begin{aligned}&\mathbb{E}\left[\partial_{h\nabla\varphi(\beta)}\Psi(\beta)\right]+\mathbb{E}\left[\langle h'',\varphi(\beta)\rangle\Psi(\beta)\right]\\ &=\mathbb{E}\left[\Psi(\beta)\int_0^1 h_r\left(\sum_{i,j=1}^d\partial^2_{i,j}\varphi(\beta_r)\lambda_{i,j}(r,K'_r,\beta_r-K_r)\right)\mathrm{d}r\right].\end{aligned}$$

The claim follows. □

6.4 Conjectures for the dynamics

What can we conjecture for integer dimensions ? Note that, if w is a solution to the additive stochastic heat equation with Dirichlet boundary conditions

$$\partial_t w = \frac{1}{2}\partial_x^2 w + \xi,$$

then, by the Itô-Tanaka formula obtained in [Zam06], the process $v := |w|$ solves the following equation:

$$\partial_t v = \frac{1}{2}\partial_x^2 v - \,:\partial_x w^2 : dL_t^0(x) + \text{sign}(w)\xi$$

where, for all $x \in (0,1)$, $(L_t^0(x))_{t\geq 0}$ denotes the local time process at 0 of the process $(w(t,x))_{t\geq 0}$. The latter equation appears as a dynamical version of the IbPF obtained in [Zam05] and [GV16]. A still open question is whether v is actually a Markov process. If it is case, then, by the distinction result 4.2.9 of Chapter 4 above, this process *does not* coincide with the process constructed in Section 4.2, and which corresponds to the SPDE (4.7). More generally, by the results of Section 6.1 above, for all $d \geq 1$, considering d independent space-time white noises $\xi_1, \ldots, \xi_d$, denoting by $w^1, \ldots, w^d$ the solutions to the corresponding heat equations, and setting

$$v := \sqrt{\sum_{i=1}^{d} (w^i)^2},$$

then we conjecture that v should satisfy an equation of the type

$$\partial_t v = \frac{1}{2}\partial_x^2 v - \frac{1}{2v} : \|\Pi_{w(t,x)^\perp} \partial_x w(t,x)\|^2 : + \frac{1}{u}\sum_{i=1}^{d} w^i \xi_i.$$

where $: \|\Pi_{w(t,x)^\perp} \partial_x w(t,x)\|^2 :$ is a singular term defined by the limit :

$$\lim_{\epsilon \to 0} \left[\|\Pi_{w(t,x)^\perp} \partial_x w_\epsilon(t,x)\|^2 - (d-1)C_\epsilon \right],$$

where $w_\epsilon(t,x) := \int_0^t \int_0^1 g_\epsilon(t-s, x-y) w(s,y) \, \mathrm{d}s \, \mathrm{d}y$, with g the fundamental solution of the heat equation, and where C_ϵ is an appropriate renormalization constant. Here also, it is unknown whether such a process is Markovian, and what is the relation with a solution u to the Bessel SPDE of parameter d. However, the formulae above suggest that there may be non-trivial relations between the quantity

$$: \|\Pi_{w(t,x)^\perp} \partial_x w(t,x)\|^2 :$$

and the quantity

$$\frac{c(d)}{u^3}.$$

Such relations are still to explore.

Chapter 7

Strong Feller property in the case of non-dissipative drifts: the example of Bessel processes

As mentioned in Section 4.5 above, one fundamental open question is whether the solutions of the Bessel SPDEs of parameter $\delta < 3$ discussed in Chapter 4 above have the strong Feller property. Recall that the strong Feller property does hold for the Bessel SPDEs of parameter $\delta \geq 3$, because of the dissipativity of their drift (see Section 5.4 in [Zam17]). On the other hand, when $\delta < 3$, the drift becomes highly non-dissipative as suggested by (4.6) , (4.7) and (4.8) above. This however does not rule out the possibility that the corresponding semigroup have the strong Feller property.

In this chapter, we tackle the easier case of Bessel processe, and show that their semigroup does indeed have the strong Feller property, regardless of the dimension. More precisely, for all $\delta \geq 0$ and $T > 0$, we compute the derivative of the function $x \mapsto \mathbf{P}^{\delta}_{T} F(x)$, where $(\mathbf{P}^{\delta}_{t})_{t \geq 0}$ is the transition semi-group associated with the δ - dimensional Bessel process, and F is any bounded Borel function on $\mathbb{R}_+$. The obtained expression shows a nice interplay between the transition semi-groups of the δ - and the $(\delta+2)$-dimensional Bessel processes. As a consequence, we deduce that the Bessel processes satisfy the strong Feller property, with a continuity modulus which is independent of the dimension, despite the lack of dissipativity in the case $\delta \leq 1$. Moreover, we provide a probabilistic interpretation of this expression as a Bismut-Elworthy-Li formula. The content of this chapter is based on the publication [EA18].

7.1 Classical Bismut-Elworthy-Li formula for one-dimensional diffusions

In this section we recall very briefly the Bismut-Elworthy-Li fomula in the case of one-dimensional diffusions, and the way this formula implies the strong Feller property.

Consider an SDE on $\mathbb{R}$ of the form

$$dX_t = b(X_t)dt + dB_t, \quad X_0 = x \tag{7.1}$$

where $b : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies

$$\begin{aligned} |b(x) - b(y)| &\leq C|x - y|, && x, y \in \mathbb{R} \\ b'(x) &\leq L, && x \in \mathbb{R} \end{aligned} \tag{7.2}$$

where $C > 0$, $L \in \mathbb{R}$ are some constants. By the classical theory of SDEs, for all $x \in \mathbb{R}$, there exists a unique continuous, square-integrable process $(X_t(x))_{t\geq 0}$ satisfying (7.1). Actually, by the Lipschitz assumption on b, there even exists a bi-continuous process $(X_t(x))_{t\geq 0, x\in\mathbb{R}}$ such that, for all $x \in \mathbb{R}$, $(X_t(x))_{t\geq 0}$ solves (7.1).

Let $x \in \mathbb{R}$. Consider the solution $(\eta_t(x))_{t\geq 0}$ to the variation equation obtained by formally differentiating (7.1) with respect to x

$$d\eta_t(x) = b'(X_t)\eta_t(x)dt, \quad \eta_0(x) = 1$$

Note that this is a (random) linear ODE with explicit solution given by

$$\eta_t(x) = \exp\left(\int_0^t b'(X_s)ds\right)$$

It is easy to prove that, for all $t \geq 0$ and $x \in \mathbb{R}$, the map $y \to X_t(y)$ is a.s. differentiable at x and

$$\frac{dX_t}{dx} \overset{a.s.}{=} \eta_t(x) \tag{7.3}$$

Remark 7.1.1. Note that $\eta_t(x) > 0$ for all $t \geq 0$ and $x \in \mathbb{R}$. This reflects the fact that, for all $x \leq y$, by a comparison theorem for SDEs (see Theorem 3.7 in Chapter IX in [RY13]), one has $X_t(x) \leq X_t(y)$.

Recall that a Markovian semi-group $(\mathbf{P}_t)_{t\geq 0}$ on a Polish space E is said to satisfy the strong Feller property if, for all $t > 0$ and $\varphi : E \to \mathbb{R}$ bounded and Borel, the function $\mathbf{P}_t\varphi : E \to \mathbb{R}$ defined by

$$\mathbf{P}_t\varphi(x) = \int \varphi(y)\mathbf{P}_t(x, dy), \quad x \in \mathbb{R}$$

is continuous.

The strong Feller property is very useful in the study of SDEs and SPDEs, namely for the proof of ergodicity (see, e.g., the monographs [Cer01], [DPZ96] and [Zam17], as well as the recent articles [HM18] and [TW18], for applications of the strong Feller property in the context of SPDEs).

Let $(\mathbf{P}_t)_{t\geq 0}$ be the Markovian semi-group associated to the SDE (7.6). We are interested in proving the strong Feller property for $(\mathbf{P}_t)_{t\geq 0}$. Note that, by assumption (7.2), $\eta_t(x) \leq e^{Lt}$ for all $t \geq 0$ and $x \in \mathbb{R}$. Therefore, by (7.3) and the dominated convergence theorem, for all $\varphi : \mathbb{R} \to \mathbb{R}$ differentiable with a bounded derivative, one has

$$\frac{d}{dx}\left(\mathbf{P}_t\varphi\right)(x) = \frac{d}{dx}\mathbb{E}\left[\varphi(X_t(x))\right] = \mathbb{E}\left[\varphi(X_t(x))\eta_t(x)\right]$$

As a consequence, for all $t \geq 0$, $\mathbf{P}_t$ preserves the space $C_b^1(\mathbb{R})$ of bounded, continuously differentiable functions on $\mathbb{R}$ with a bounded derivative. It turns out that, actually, for all $t > 0$, $\mathbf{P}_t$ maps the space $C_b(\mathbb{R})$ of bounded and continuous functions into $C_b^1(\mathbb{R})$. This is a consequence of the following, nowadays well-known, result:

Theorem 7.1.2 (Bismut-Elworthy-Li formula)**.** *For all $T > 0$ and $\varphi \in C_b(\mathbb{R})$, the function $\mathbf{P}_T\varphi$ is differentiable and we have*

$$\frac{d}{dx}\mathbf{P}_T\varphi(x) = \frac{1}{T}\mathbb{E}\left[\varphi(X_T(x))\int_0^T \eta_s(x)dB_s\right] \tag{7.4}$$

Proof. See [EL94], Theorem 2.1, or [Zam17], Lemma 5.17 for a proof. □

Corollary 7.1.3. *The semi-group $(\mathbf{P}_t)_{t\geq 0}$ satisfies the strong Feller property and, for all $T > 0$ and $\varphi : \mathbb{R} \to \mathbb{R}$ bounded and Borel, one has*

$$\forall x, y \in \mathbb{R}, \quad |\mathbf{P}_T\varphi(x) - \mathbf{P}_T\varphi(y)| \leq e^L \frac{||\varphi||_\infty}{\sqrt{T \wedge 1}}|x-y|, \tag{7.5}$$

where $||\cdot||_\infty$ denotes the supremum norm.

The following remark is crucial.

Remark 7.1.4. Inequality (7.5) involves only the dissipativity constant L, not the Lipschitz constant C. This makes the Bismut-Elworthy-Li formula very useful in the study of SPDEs with a dissipative drift.

Proof of Corollary 7.1.3. By approximation, it suffices to prove (7.5) for $\varphi \in C_b(\mathbb{R})$. For such a φ and for all $T > 0$, by the Bismut-Elworthy-Li formula, one has

$$\left|\frac{d}{dx}\mathbf{P}_T\varphi(x)\right| \leq \frac{||\varphi||_\infty}{T}\mathbb{E}\left[\left|\int_0^T \eta_s(x)dB_s\right|\right]$$

Remark that the process $(\eta_t(x))_{t\geq 0}$ is locally bounded since it is dominated by $(e^{Lt})_{t\geq 0}$, so that the stochastic integral $\left(\int_0^t \eta_s(x)dB_s\right)_{t\geq 0}$ is an L^2 martingale. Hence using Jensen's inequality as well as Itô's isometry formula, we obtain

$$\begin{aligned}\mathbb{E}\left[|\int_0^T \eta_s(x)dB_s|\right] &\leq \sqrt{\mathbb{E}\left[\int_0^T \eta_s(x)^2 ds|\right]} \\ &\leq \sqrt{\int_0^T e^{2Ls}\, ds}\end{aligned}$$

and the last quantity is bounded by $\sqrt{e^{2L}T} = e^L\sqrt{T}$ for all $T \in (0,1]$. Therefore, we deduce that

$$\forall x \in \mathbb{R}, \qquad \left|\frac{d}{dx}\mathbf{P}_T\varphi(x)\right| \leq e^L \frac{||\varphi||_\infty}{\sqrt{T}}$$

so that

$$\forall x, y \in \mathbb{R}, \quad |\mathbf{P}_T\varphi(x) - \mathbf{P}_T\varphi(y)| \leq e^L \frac{||\varphi||_\infty}{\sqrt{T}}|x-y|$$

for all $\varphi \in C_b(\mathbb{R})$ and $T \in (0,1]$. The case $T > 1$ follows at once by using the semi-group property of $(\mathbf{P}_t)_{t\geq 0}$:

$$\begin{aligned}|\mathbf{P}_T\varphi(x) - \mathbf{P}_T\varphi(y)| &= |\mathbf{P}_1\left(\mathbf{P}_{T-1}\varphi\right)(x) - \mathbf{P}_1\left(\mathbf{P}_{T-1}\varphi\right)(y)| \\ &\leq e^L \frac{||\mathbf{P}_{T-1}\varphi||_\infty}{\sqrt{1}}|x-y| \\ &\leq e^L ||\varphi||_\infty |x-y|\end{aligned}$$

The claim follows. □

Remark 7.1.5 (A brief history of the Bismut-Elworthy-Li formula)**.** A particular form of this formula had originally been derived by J.M. Bismut in [Bis84] using Malliavin calculus in the framework of the study of the logarithmic derivative of the fundamental solution of the heat equation on a compact manifold. In [EL94], K.D. Elworthy and X.-M. Li used a martingale approach, instead of a Malliavin calculus method, to generalize this formula to a large class of diffusion processes on noncompact manifolds with smooth coefficients, and gave also variants of this formula to higher-order derivatives. The key to their proof is to select a stochastic process, which in this case is the stochastic flow, to give a probabilistic representation for the derivative of the semigroup.

The key property allowing the analysis performed in this section is the dissipativity property (7.2). Without this property being true, one would not even expect the Bismut-Elworthy-Li formula to hold. However, in the sequel, we shall prove that results such as Theorem 7.1.2 and Corollary 7.1.3 above can also be obtained for a family of diffusions with a non-dissipative drift (informally $L = +\infty$) , namely for the Bessel processes of dimension smaller than 1.

7.2 Bessel processes: notations and basic facts

Recall that, for any subinterval I of $\mathbb{R}_+$, $C(I)$ denotes the set of continuous functions $I \to \mathbb{R}$. We shall consider this set endowed with the topology of uniform convergence on compact sets, and will denote by $\mathcal{B}(C(I))$ the corresponding Borel σ-algebra.

Consider the canonical measurable space $(C(\mathbb{R}_+), \mathcal{B}(C(I)))$ endowed with the canonical filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(B_t)_{t\geq 0}$ be a standard linear $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. For all $x \geq 0$ and $\delta \geq 0$, there exists a unique continuous, predictable, nonnegative process $(X_t^\delta(x))_{t\geq 0}$ satisfying

$$X_t = x^2 + 2\int_0^t \sqrt{X_s}dB_s + \delta t. \tag{7.6}$$

$(X_t^\delta(x))_{t\geq 0}$ is a squared Bessel process of dimension δ started at x^2, and the process $\rho_t^\delta(x) := \sqrt{X_t^\delta(x)}$ is a δ-dimensional Bessel process started at x. In the sequel, we will also write the latter process as $(\rho_t(x))_{t\geq 0}$, or ρ, when there is no risk of ambiguity.

We recall the following monotonicity property of the family of Bessel processes:

Lemma 7.2.1. *For all couples $(\delta, \delta'), (x, x') \in \mathbb{R}_+$ such that $\delta \leq \delta'$ and $x \leq x'$, we have, a.s.*

$$\forall t \geq 0, \qquad \rho_t^\delta(x) \leq \rho_t^{\delta'}(x').$$

Proof. By Theorem (3.7) in [RY13], Section IX, applied to the equation (7.6), the following property holds a.s.:

$$\forall t \geq 0, \qquad X_t^\delta(x) \leq X_t^{\delta'}(x').$$

Taking the square root on both sides above, we deduce the result. □

For all $a \geq 0$, let $T_a(x)$ denote the $(\mathcal{F}_t)_{t\geq 0}$ stopping time defined by

$$T_a(x) := \inf\{t > 0, \rho_t(x) \leq a\}$$

(we shall also write T_a). We recall the following fact, (see e.g. Proposition 3.6 of [Zam17]):

Proposition 7.2.2. *The following dichotomy holds:*

- $T_0(x) = +\infty$ *a.s., if* $\delta \geq 2$,
- $T_0(x) < +\infty$ *a.s., if* $0 \leq \delta < 2$.

Applying Itô's lemma to $\rho_t = \sqrt{X_t^\delta(x)}$, we see that ρ satisfies the following relation on the interval $[0, T_0)$:

$$\forall t \in [0, T_0), \qquad \rho_t = x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\rho_s} + B_t. \tag{7.7}$$

7.3 Derivative in space of the Bessel semi-group

Let $\delta \geq 0$. We denote by P_x^δ the law, on $(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+)))$, of the δ-dimensional Bessel process started at x, and we write E_x^δ for the corresponding expectation operator. We also denote by $(\mathbf{P}_t^\delta)_{t\geq 0}$ the family of transition kernels associated with the δ-dimensional Bessel process, defined by

$$\mathbf{P}_t^\delta F(x) := E_x^\delta(F(\rho_t))$$

for all $t \geq 0$ and all $F : \mathbb{R}_+ \to \mathbb{R}$ bounded and Borel. The aim of this section is to prove the following:

Theorem 7.3.1. *For all* $T > 0$ *and all* $F : \mathbb{R}_+ \to \mathbb{R}$ *bounded and Borel, the function* $x \to \mathbf{P}_t^\delta F(x)$ *is differentiable on* $\mathbb{R}_+$*, and for all* $x \geq 0$

$$\frac{d}{dx} \mathbf{P}_T^\delta F(x) = \frac{x}{T} \left(\mathbf{P}_T^{\delta+2} F(x) - \mathbf{P}_T^\delta F(x) \right). \tag{7.8}$$

In particular, the function $x \to \mathbf{P}_t^\delta F(x)$ *satisfies the Neumann boundary condition at* 0*:*

$$\frac{d}{dx} \mathbf{P}_T^\delta F(x) \Big|_{x=0} = 0.$$

Remark 7.3.2. By Theorem 7.3.1, the derivative of the function $x \mapsto \mathbf{P}_T^\delta F(x)$ is a smooth function of $\mathbf{P}_T^{\delta+2} F(x)$ and $\mathbf{P}_T^\delta F(x)$. Hence, reasoning by induction, we deduce that the function $x \mapsto \mathbf{P}_T^\delta F(x)$ is actually smooth on $\mathbb{R}_+$.

Proof. The proof we propose here relies on the explicit formula for the transition semi-group of the Bessel processes. We first treat the case $\delta > 0$.

Given $\delta > 0$, let $\nu := \frac{\delta}{2} - 1$, and denote by I_ν the modified Bessel function of index ν. We have (see, e.g., Chap. XI.1 in [RY13])

$$\mathbf{P}_t^\delta F(x) = \int_0^\infty p_T^\delta(x, y) F(y) dy$$

where, for all $y \geq 0$

$$p_t^\delta(x,y) = \frac{1}{T}\left(\frac{y}{x}\right)^\nu y \exp\left(-\frac{x^2+y^2}{2T}\right) I_\nu\left(\frac{xy}{T}\right), \quad \text{if} \quad x>0,$$
$$p_t^\delta(0,y) = \frac{2^{-\nu}T^{-(\nu+1)}}{\Gamma(\nu+1)} y^{2\nu+1}\exp\left(-\frac{y^2}{2T}\right)$$

where Γ denotes the gamma function. By the power series expansion of the function I_ν we have, for all $x, y \geq 0$

$$p_T^\delta(x,y) = \frac{1}{T}\exp\left(-\frac{x^2+y^2}{2T}\right)\tilde{p}_T^\delta(x,y) \tag{7.9}$$

with

$$\tilde{p}_T^\delta(x,y) := \sum_{k=0}^\infty \frac{y^{2k+2\nu+1}\ x^{2k}\ (1/2T)^{2k+\nu}}{k!\ \Gamma(k+\nu+1)}.$$

Note that $\tilde{p}_T^\delta(x,y)$ is the sum of a series with infinite radius of convergence in x, hence we can compute its derivative by differentiating under the sum. We have

$$\begin{aligned}\frac{\partial}{\partial x}\tilde{p}_T^\delta(x,y) &= \frac{\partial}{\partial x}\left(\sum_{k=0}^\infty \frac{x^{2k}\ y^{2k+2\nu+1}\ (1/2T)^{2k+\nu}}{k!\ \Gamma(k+\nu+1)}\right)\\ &= \sum_{k=0}^\infty \frac{2k\ x^{2k-1}\ y^{2k+2\nu+1}\ (1/2T)^{2k+\nu}}{k!\ \Gamma(k+\nu+1)}\\ &= \frac{x}{T}\ \sum_{k=1}^\infty \frac{x^{2k-2}\ y^{2k+2\nu+1}\ (1/2T)^{2k+\nu-1}}{(k-1)!\ \Gamma(k+\nu+1)}.\end{aligned}$$

Hence, performing the change of variable $j = k-1$, and remarking that $\nu+1 = \frac{\delta+2}{2}-1$, we obtain

$$\begin{aligned}\frac{\partial}{\partial x}\tilde{p}_T^\delta(x,y) &= \frac{x}{T}\ \sum_{j=0}^\infty \frac{x^{2(j+1)-2}\ y^{2(j+1)+2\nu+1}\ (1/2T)^{2(j+1)+\nu-1}}{j!\ \Gamma((j+1)+\nu+1)}\\ &= \frac{x}{T}\ \sum_{j=0}^\infty \frac{x^{2j}\ y^{2j+2(\nu+1)+1}\ (1/2T)^{2j+(\nu+1)}}{j!\ \Gamma(j+(\nu+1)+1)}\\ &= \frac{x}{T}\ \tilde{p}_T^{\delta+2}(x,y).\end{aligned}$$

As a consequence, differentiating equality (7.9) with respect to x, we obtain

$$\begin{aligned}\frac{\partial}{\partial x}p_T^\delta(x,y) &= \left(-\frac{x}{T}\ \tilde{p}_T^\delta(x,y) + \frac{\partial}{\partial x}\tilde{p}_T^\delta(x,y)\right)\frac{1}{T}\exp\left(-\frac{x^2+y^2}{2T}\right)\\ &= \frac{x}{T}\ \left(-p_T^\delta(x,y) + p_T^{\delta+2}(x,y)\right).\end{aligned}$$

Hence, we deduce that the function $x \mapsto \mathbf{P}^{\delta}_{T} F(x)$ is differentiable, with a derivative given by (7.8).

Now suppose that $\delta = 0$. We have, for all $x \geq 0$

$$P^0_T F(x) = \exp\left(-\frac{x^2}{2T}\right) F(0) + \int_0^\infty p_T(x,y) F(y) dy \tag{7.10}$$

where, for all $y \geq 0$

$$p_T(x,y) = \frac{1}{T} \exp\left(-\frac{x^2+y^2}{2T}\right) \tilde{p}_T(x,y)$$

with

$$\tilde{p}_T(x,y) := x \ I_1\left(\frac{xy}{T}\right) = \sum_{k=0}^{\infty} \frac{x^{2k+2} \ (y/2T)^{2k+1}}{k!(k+1)!}.$$

Here again, we can differentiate the sum term by term, so that, for all $x, y \geq 0$

$$\begin{aligned}
\frac{\partial}{\partial x} \tilde{p}_T(x,y) &= \frac{x}{T} \sum_{k=0}^{\infty} \frac{x^{2k} y^{2k+1} (1/2T)^{2k}}{k!^2} \\
&= \frac{x}{T} \ \tilde{p}^2_T(x,y).
\end{aligned}$$

Therefore, for all $x, y \geq 0$, we have

$$\begin{aligned}
\frac{\partial}{\partial x} p_T(x,y) &= \left(-\frac{x}{T} \tilde{p}_T(x,y) + \frac{\partial}{\partial x} \tilde{p}_T(x,y)\right) \frac{1}{T} \exp\left(-\frac{x^2+y^2}{2T}\right) \\
&= \frac{x}{T} \left(-\tilde{p}_T(x,y) + \tilde{p}^2_T(x,y)\right) \frac{1}{T} \exp\left(-\frac{x^2+y^2}{2T}\right) \\
&= \frac{x}{T} \left(-p_T(x,y) + p^2_T(x,y)\right)
\end{aligned}$$

Hence, differentiating (7.10) with respect to x, and using the dominated convergence theorem to differentiate inside the integral, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x} P^0_T F(x) &= -\frac{x}{T} \exp(-\frac{x^2}{2T}) F(0) + \frac{x}{T} \int_0^\infty \left(-p_T(x,y) + p^2_T(x,y)\right) F(y) dy \\
&= \frac{x}{T} \ \left(-P^0_T F(x) + P^2_T F(x)\right),
\end{aligned}$$

which yields the claim. □

Remark 7.3.3. Formula (7.8) can also be derived using the Laplace transform of the one-dimensional marginals of the squared Bessel processes. Indeed, denote by $(\mathbf{Q}^{\delta}_t)_{t \geq 0}$ the family of transition kernels of the δ-dimensional squared Bessel process.

Then for all $\delta \geq 0$, $x \geq 0$, $T > 0$, and all function f of the form $f(x) = \exp(-\lambda x)$ with $\lambda \geq 0$, one has

$$\mathbf{Q}_T^\delta f(x) = \exp\left(-\frac{\lambda x}{1+2\lambda T}\right)(1+2\lambda T)^{-\delta/2}$$

(see [RY13], Chapter XI, Cor. (1.3)). For such test functions f, we check at once that the following equality holds:

$$\frac{d}{dx}\mathbf{Q}_T^\delta f(x) = \frac{1}{2T}\left(\mathbf{Q}_T^{\delta+2} f(x) - \mathbf{Q}_T^\delta f(x)\right).$$

By linearity and by the Stone-Weierstrass theorem, we deduce that this equality holds for all bounded, continuous functions f . Then an approximation argument enables to deduce the equality for all functions $f : \mathbb{R}_+ \to \mathbb{R}$ Borel and bounded. Finally, remarking that for all bounded Borel function F on $\mathbb{R}_+$ we have

$$\mathbf{P}_T^\delta F(x) = \mathbf{Q}_T^\delta f(x^2)$$

with $f(x) := F(\sqrt{x})$, we deduce that

$$\begin{aligned}\frac{d}{dx}\mathbf{P}_T^\delta F(x) &= 2x\,\frac{d}{dx}(\mathbf{Q}_T^\delta f)(x^2)\\ &= \frac{x}{T}\left(\mathbf{Q}_T^{\delta+2} f(x^2) - \mathbf{Q}_T^\delta f(x^2)\right)\\ &= \frac{x}{T}\left(\mathbf{P}_T^{\delta+2} F(x) - \mathbf{P}_T^\delta F(x)\right)\end{aligned}$$

which yields the equality (7.8).

Corollary 7.3.4. *The semi-group* $(\mathbf{P}_t^\delta)_{t\geq 0}$ *has the strong Feller property. More precisely, for all* $T > 0$*,* $R > 0$*,* $x, y \in [0, R]$ *and* $F : \mathbb{R}_+ \to \mathbb{R}$ *bounded and Borel, we have*

$$|\mathbf{P}_T^\delta F(x) - \mathbf{P}_T^\delta F(y)| \leq \frac{2R||F||_\infty}{T}|y-x|. \tag{7.11}$$

Proof. By Theorem 7.3.1, for all $x, y \in [0, R]$ such that $x \leq y$, we have

$$\begin{aligned}|\mathbf{P}_t^\delta F(x) - \mathbf{P}_t^\delta F(y)| &= \left|\int_x^y \frac{u}{T}\left(\mathbf{P}_T^{\delta+2} F(u) - \mathbf{P}_T^\delta F(u)\right) du\right|\\ &\leq \frac{2||F||_\infty}{T}\int_x^y u\, du\\ &\leq \frac{2R||F||_\infty}{T}|y-x|.\end{aligned}$$

□

Remark 7.3.5. The bound (7.11) is in $1/T$, which is not very satisfactory for T small. However, in the sequel, we will improve this bound by getting a better exponent on T, at least for $\delta \geq 2(\sqrt{2}-1)$ (see inequality (7.25) below).

7.4 Differentiability of the flow

In the following, we are interested in finding a probabilistic interpretation of Thm 7.3.1, in terms of the Bismut-Elworthy-Li formula. To do so we study, for all $\delta \geq 0$, and all couple $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+^*$, the differentiability at x of the function

$$\begin{aligned}\rho_t \colon \mathbb{R}_+ &\to \mathbb{R}_+ \\ y &\mapsto \rho_t^\delta(y).\end{aligned}$$

In this endeavour, we first need to choose an appropriate modification of the process $(\rho_t(x))_{t\geq 0, x>0}$. We have the following result:

Proposition 7.4.1. *Let $\delta \geq 0$ be fixed. There exists a modification $(\tilde{\rho}_t^\delta(x))_{x,t\geq 0}$ of the process $(\rho_t^\delta(x))_{x,t\geq 0}$ such that, a.s., for all $x, x' \in \mathbb{R}_+$ with $x \leq x'$, we have*

$$\forall t \geq 0, \qquad \tilde{\rho}_t^\delta(x) \leq \tilde{\rho}_t^\delta(x'). \tag{7.12}$$

Proof. For all $q, q' \in \mathbb{Q}_+$, such that $q \leq q'$, by Lemma 7.2.1, the following property holds a.s.:

$$\forall t \geq 0, \qquad \rho_t^\delta(q) \leq \rho_t^\delta(q').$$

For all $x \in \mathbb{R}_+$, we define the process $\tilde{\rho}^\delta(x)$ by

$$\forall t \geq 0, \qquad \tilde{\rho}_t^\delta(x) := \inf_{q\in\mathbb{Q}_+, q\geq x} \rho_t^\delta(q).$$

Then $(\tilde{\rho}_t^\delta(x))_{x,t\geq 0}$ yields a modification of the process $(\rho_t^\delta(x))_{x,t\geq 0}$ with the requested property. □

In the sequel, when $\delta \geq 0$ is fixed and there is no ambiguity, we shall write $\tilde{\rho}$ instead of $\tilde{\rho}^\delta$.

Remark 7.4.2. Given $\delta \geq 0$, we may not have, almost-surely, joint continuity of all the functions $t \mapsto \tilde{\rho}_t(x)$, $x \geq 0$. Note however that, by definition, for all $x \geq 0, x \in \mathbb{Q}$, we have a.s.

$$\forall t \geq 0, \qquad \tilde{\rho}_t(x) = \rho_t(x),$$

so that, a.s., $t \mapsto \tilde{\rho}_t(x)$ is continuous and satisfies

$$\forall t \in [0, T_0(x)), \qquad \tilde{\rho}_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\tilde{\rho}_s(x)} + B_t.$$

As a consequence, by countability of $\mathbb{Q}$, there exists an almost sure event $\mathcal{A} \in \mathcal{F}$ on which, for all $x \in \mathbb{Q}_+$, the function $t \mapsto \tilde{\rho}_t(x)$ is continuous and satisfies

$$\forall t \in [0, T_0(x)), \qquad \tilde{\rho}_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\tilde{\rho}_s(x)} + B_t.$$

Actually, in Corollary 7.7.2 of the Appendix, we will prove the stronger fact that, almost-surely, we have

$$\forall x \geq 0, \quad \forall t \in [0, \tilde{T}_0(x)), \qquad \tilde{\rho}_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{\tilde{\rho}_s(x)} + B_t,$$

where, for all $x \geq 0$

$$\tilde{T}_0(x) := \inf\{t > 0, \tilde{\rho}_t(x) = 0\}$$

In this section, as well as the Appendix, we always work with the modification $\tilde{\rho}$. Similarly, we work with $\tilde{T}_0(x)$ instead of $T_0(x)$, for all $\delta, x \geq 0$. We will write again ρ and T_0 instead of $\tilde{\rho}$ and $\tilde{T}_0$. Note that, a.s., the function $x \mapsto T_0(x)$ is non-decreasing on $\mathbb{R}_+$.

Proposition 7.4.3. *Let $\delta \geq 0$, $t > 0$ and $x > 0$. Then, a.s., the function ρ_t is differentiable at x, and its derivative there is given by*

$$\left.\frac{d\rho_t(y)}{dy}\right|_{y=x} \overset{a.s.}{=} \eta_t(x) := \mathbf{1}_{t<T_0(x)} \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2}\right). \tag{7.13}$$

The proof of this proposition is quite technical. Since, moreover, the result will not be necessary in the sequel, we prefer to postpone the proof to Section 7.7 below.

Remark 7.4.4. In particular, when $\delta = 1$, the above formula reduces to

$$\left.\frac{d\rho_t(y)}{dy}\right|_{y=x} \overset{a.s.}{=} \mathbf{1}_{t<T_0(x)} \tag{7.14}$$

a formula which was already well-known (see e.g. [AL17], Lemma A.1).

Remark 7.4.5. Note that the indicator function $\mathbf{1}_{t<T_0(x)}$ in the right-hand side of (7.13) is related to the behavior of the Bessel process at the boundary 0. It is reminiscent of Theorem 1 in [DZ05] , where a similar indicator function appears in the expression of the spatial derivative of the flow of vector-valued solutions to SDEs with reflection.

Remark 7.4.6. Proposition 7.4.3 shows that, for all $t, x > 0$, the function ρ_t is almost-surely differentiable at x. We may, however, ask if, a.s., the function ρ_t is differentiable on the whole of $\mathbb{R}_+^*$. The case where $\delta > 1$ was treated in detail in [Vos09], where it was shown that, a.s., for all $t \geq 0$ the function $x \mapsto \rho_t(x)$ is differentiable on $\mathbb{R}_+^*$, and that the derivative $\frac{d\rho_t(x)}{dx}$ is continuous in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^*$. However, as δ gets smaller than 1, the regularity of the process $(\rho_t(x))_{t \geq 0, x > 0}$ becomes much worse. Note that $\delta = 1$ corresponds to the case of the flow of reflected Brownian motion on the half-line; in that case the flow is no longer continuously differentiable as suggested by (7.14). Many works have been carried out on the study of the flow of reflected Brownian motion on domains in higher dimension (see e.g. [Bur09] and [VW85]) or on manifolds with boundary (see e.g. [AL17]). By contrast, the regularity of Bessel flows of dimension $\delta < 1$ seems to be a very open problem.

In the remainder of this chapter, however, we shall not need any regularity results on the Bessel flow. Instead, for all fixed $x > 0$, we shall study the process $(\eta_t(x))_{t\geq 0}$ defined above as a process in itself.

7.5 Properties of η

In the sequel, for all $x \geq 0$, we shall consider the process $(\eta_t(x))_{t\geq 0}$ defined as above:

$$\eta_t(x) := \mathbf{1}_{t<T_0(x)} \exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right). \tag{7.15}$$

When there is no ambiguity we shall drop the x from our notation and denote this process by η.

7.5.1 Regularity of the sample paths of η

We are interested in the continuity property of the process η. It turns out that, as δ decreases, η becomes more and more singular, as shown by the following result.

Proposition 7.5.1. *If $\delta > 1$, then a.s. η is bounded and continuous on $\mathbb{R}_+$.*
If $\delta = 1$, then a.s. η is constant on $[0, T_0)$ and $[T_0, +\infty)$, but has a discontinuity at T_0.
If $\delta \in [0, 1)$, then a.s. η is continuous away from T_0, but it diverges to $+\infty$ as $t \uparrow T_0$.

Proof. When $\delta \geq 2$, $T_0 = \infty$ almost-surely, so that, by (7.15), the following equality of processes holds:

$$\eta_t = \exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right).$$

Hence, a.s., η takes values in $[0, 1]$ and is continuous on $\mathbb{R}_+$. To treat the case $\delta < 2$ we need a lemma:

Lemma 7.5.2. *Let $\delta < 2$ and $x > 0$. Then the integral*

$$\int_0^{T_0} \frac{ds}{(\rho_s(x))^2}$$

is infinite a.s.

We admit this result for the moment. Then, when $\delta \in (1,2)$, η takes values in $[0,1]$, is continuous away from T_0 and, almost-surely, as $t \uparrow T_0$

$$\eta_t = \exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right) \longrightarrow 0.$$

Since, $\eta_t = 0$ for all $t \geq T_0$, η is continuous and the claim follows. When $\delta = 1$,

$$\eta_t(x) := \mathbf{1}_{t<T_0(x)}$$

so the claim follows at once. Finally, if $\delta \in [0,1)$, then η is continuous away from T_0, but by the above lemma, a.s., as $t \uparrow T_0$

$$\eta_t = \exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right) \longrightarrow +\infty$$

so the claim follows. □

We now prove Lemma 7.5.2

Proof of Lemma 7.5.2. The proof is in two steps. In a first step we prove the lemma when ρ is replaced with a Brownian motion started at some positive point, and in a second step we invoke a representation theorem of Bessel processes as time-changes of some power of the Brownian motion to conclude.

First step: Let (β_t) be a Brownian motion started from some $y > 0$, and let T_0 denote its hitting time of the origin. Then the integral

$$\int_0^{T_0} \frac{ds}{(\beta_s(y))^2}$$

is a.s. infinite. Indeed, denote by $h : [0,\infty) \to \mathbb{R}_+$ the function given by

$$h(t) := \begin{cases} \sqrt{t|\log(1/t)|}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

Let $A > 0$. By Levy's modulus of continuity (see Theorem (2.7), Chapter I, in [RY13]), there exists a $\kappa > 0$, such that the event

$$\mathcal{M} := \{\ \forall s,t \in [0,1], \quad |\beta_t - \beta_s| \leq \kappa\ h(|s-t|)\ \}$$

has probability one. Therefore, by scale invariance of Brownian motion, setting $\kappa_A := \sqrt{A}\kappa$, one deduces that the event

$$\mathcal{M}_A := \{\ \forall s,t \in [0,A], \quad |\beta_t - \beta_s| \leq \kappa_A\ h(|s-t|)\ \}$$

also has probability one. Moreover, under the event $\{T_0 < A\} \cap \mathcal{M}_A$, we have, for small $h > 0$.

$$\beta_{T_0-h}^2 = |\beta_{T_0-h} - \beta_{T_0}|^2 \leq {\kappa_A}^2 \; h \log(1/h).$$

Since $\frac{1}{h\log(1/h)}$ is not integrable as $h \to 0^+$, we deduce that, under the event $\{T_0 < A\} \cap \mathcal{M}_A$, we have $\int_0^{T_0} \frac{ds}{(\beta_s)^2} = +\infty$. Therefore

$$\mathbb{P}[T_0 < A] = \mathbb{P}[\{T_0 < A\} \cap \mathcal{M}_A] \leq \mathbb{P}\left(\int_0^{T_0} \frac{ds}{(\beta_s)^2} = +\infty\right).$$

Since $T_0 < +\infty$ a.s., we have $\lim_{A\to\infty} \mathbb{P}[T_0 < A] = 1$. Hence, letting $A \to \infty$ in the above, we deduce that

$$\mathbb{P}\left(\int_0^{T_0} \frac{ds}{(\beta_s)^2} = +\infty\right) = 1$$

as claimed.

Second step: Now consider the original Bessel process $(\rho_t(x))_{t\geq 0}$. Suppose that $\delta \in (0,2)$. Then, by Thm 3.5 in [Zam17], the process $(\rho_t(x))_{t\geq 0}$ is equal in law to $(|\beta_{\gamma(t)}|^{\frac{1}{2-\delta}})_{t\geq 0}$, where β is a Brownian motion started from $y := x^{2-\delta}$, and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse of the increasing function $A : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\forall u \geq 0, \qquad A(u) = \frac{1}{(2-\delta)^2} \int_0^u |\beta_s|^{\frac{2(\delta-1)}{2-\delta}} ds.$$

Therefore, denoting by T_0^β the hitting time of 0 by the Brownian motion β, we have

$$\begin{aligned}
\int_0^{T_0} \frac{ds}{(\rho_s(x))^2} &\overset{(d)}{=} \int_0^{A(T_0^\beta)} \frac{ds}{|\beta_{\gamma(s)}|^{\frac{2}{2-\delta}}} \\
&= \int_0^{T_0^\beta} \frac{1}{|\beta_u|^{\frac{2}{2-\delta}}} \frac{1}{(2-\delta)^2} |\beta_u|^{\frac{2(\delta-1)}{2-\delta}} du \\
&= \frac{1}{(2-\delta)^2} \int_0^{T_0^\beta} \frac{du}{{\beta_u}^2}
\end{aligned}$$

where we have used the change of variable $u = \gamma(s)$ to get from the first line to the second one. By the first step, the last integral is infinite a.s., so the claim follows.

There still remains to treat the case $\delta = 0$. By Thm 3.5 in [Zam17], in that case, the process $(\rho_t(x))_{t\geq 0}$ is equal in law to $\left(\left(\beta_{\gamma(t)\wedge T_0^\beta}\right)^{1/2}\right)_{t\geq 0}$, where β is a Brownian motion started from $y := x^2$, T_0^β is its hitting time of 0 and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is the inverse of the increasing function $A : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\forall u \geq 0, \qquad A(u) = \frac{1}{4} \int_0^{u \wedge T_0^\beta} \beta_s^{-1} ds.$$

Then, the same computations as above yield the equality in law

$$\int_0^{T_0} \frac{ds}{(\rho_s(x))^2} \overset{(d)}{=} \frac{1}{4} \int_0^{T_0^\beta} \frac{du}{{\beta_u}^2}$$

so the result follows as well. □

7.5.2 Study of a martingale related to η

Let $\delta \in [0, 2)$ and $x > 0$ be fixed. In the previous section, we have shown that, a.s.

$$\int_0^t \frac{ds}{\rho_s(x)^2} \underset{t \to T_0(x)}{\longrightarrow} +\infty$$

As a consequence, for $\delta \in [0, 1)$, a.s., the modification η_t of the derivative at x of the stochastic flow ρ_t diverges at $T_0(x)$:

$$\eta_t(x) = \mathbf{1}_{t < T_0(x)} \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2} \right) \underset{t \uparrow T_0(x)}{\longrightarrow} +\infty.$$

However, since $\rho_t(x) \longrightarrow 0$ as $t \to T_0(x)$, this does not exclude the possibility that the product $\rho_t(x)\eta_t(x)$ converges as $t \to T_0(x)$. This motivates to study the process

$$D_t := \rho_t(x)\eta_t(x) = \mathbf{1}_{t < T_0(x)} \rho_t(x) \exp\left(\frac{1-\delta}{2} \int_0^t \frac{ds}{\rho_s(x)^2} \right). \tag{7.16}$$

As a matter of fact, we will show that $(D_t)_{t \geq 0}$ is an L^p continuous martingale for some $p \geq 1$.

Remark 7.5.3. The process $(D_t)_{t \geq 0}$ appears as (one half times) the derivative of the stochastic flow associated with the squared Bessel process $X_t(x) = (\rho_t(x))^2$. Indeed, by applying formally the chain rule, we have, for all $t \geq 0$ and $x > 0$

$$\frac{dX_t(x)}{dx} = 2\rho_t(x)\eta_t(x).$$

7.5.3 Continuity of $(D_t)_{t\geq 0}$

In this subsection we show that the process $(D_t)_{t\geq 0}$ has a.s. continuous sample paths. By the expression (7.16), continuity holds as soon as $T_0(x) = \infty$ a.s., i.e. as soon as $\delta \geq 2$. On the other hand, if $\delta \in [0,2)$ it suffices to prove that, a.s., $D_t \to 0$ as $t \uparrow T_0(x)$. This is the content of the following proposition.

Proposition 7.5.4. *For all $\delta \in [0,2)$ and $x > 0$, with probability one:*

$$\rho_t(x) \exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right) \underset{t\to T_0(x)}{\longrightarrow} 0.$$

Proof. If $\delta \in [1,2)$, then $\exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right) \leq 1$ for all $t \geq 0$. Since $\rho_t \longrightarrow 0$ as $t \to T_0(x)$, the claim follows at once.

If $\delta \in [0,1)$, on the other hand, $\exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right) \underset{t\uparrow T_0(x)}{\longrightarrow} +\infty$ whereas $\rho_t \underset{t\to T_0(x)}{\longrightarrow} 0$ so a finer analysis is needed. We have

$$\log\left[\frac{\rho_t}{x}\exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right)\right] = \log\frac{\rho_t}{x} + \frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s^2}$$

Now, recall that a.s., for all $t < T_0$, we have

$$\rho_t = x + \frac{\delta-1}{2}\int_0^t \frac{ds}{\rho_s} + B_t$$

Hence, defining for all integer $n \geq 1$ the $(\mathcal{F}_t)_{t\geq 0}$-stopping time τ_n as

$$\tau_n := \inf\{t > 0,\ \rho_t \leq 1/n\} \wedge n,$$

we have

$$\rho_{t\wedge\tau_n} = x + \frac{\delta-1}{2}\int_0^{t\wedge\tau_n} \frac{ds}{\rho_s} + B_{t\wedge\tau_n}.$$

Hence, by Itô's lemma, we deduce that

$$\log\frac{\rho_{t\wedge\tau_n}}{x} = \frac{\delta-1}{2}\int_0^{t\wedge\tau_n}\frac{ds}{\rho_s^2} + \int_0^{t\wedge\tau_n}\frac{dB_s}{\rho_s} - \frac{1}{2}\int_0^{t\wedge\tau_n}\frac{ds}{\rho_s^2}$$

so that

$$\log\frac{\rho_{t\wedge\tau_n}}{x} + \frac{1-\delta}{2}\int_0^{t\wedge\tau_n}\frac{ds}{\rho_s^2} = \int_0^{t\wedge\tau_n}\frac{dB_s}{\rho_s} - \frac{1}{2}\int_0^{t\wedge\tau_n}\frac{ds}{\rho_s^2}. \tag{7.17}$$

Consider now the random time change

$$A\colon [0,T_0) \to \mathbb{R}_+$$
$$t \mapsto A_t := \int_0^t \frac{ds}{\rho_s^2}.$$

Note that A is differentiable with strictly positive derivative. Moreover, since $A_t \underset{t\to T_0}{\longrightarrow} +\infty$ a.s. by Lemma 7.5.2, we deduce that A is a.s. onto. Hence, a.s., A is a diffeomorphism $[0,T_0) \to \mathbb{R}_+$, the inverse of which we denote by

$$\begin{aligned} C\colon \mathbb{R}_+ &\to [0,T_0) \\ u &\mapsto C_u. \end{aligned}$$

Let $\beta_u := \int_0^{C_u} \frac{dB_r}{\rho_r}$, $u \geq 0$. Then β is a local martingale started at 0 with quadratic variation $\langle \beta, \beta\rangle_u = u$, so by Lévy's theorem it is a Brownian motion. The equality (7.17) can now be rewritten

$$\log \frac{\rho_{t\wedge\tau_n}}{x} + \frac{1-\delta}{2}\int_0^{t\wedge\tau_n} \frac{ds}{\rho_s^2} = \beta_{A_{t\wedge\tau_n}} - \frac{1}{2}A_{t\wedge\tau_n}.$$

Letting $n\to\infty$, we obtain, for all $t < T_0$

$$\log \frac{\rho_t}{x} + \frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s^2} = \beta_{A_t} - \frac{1}{2}A_t.$$

By the asymptotic properties of Brownian motion (see Corollary (1.12), Chapter II in [RY13]), we know that, a.s.

$$\limsup_{s\to+\infty} \frac{\beta_s}{h(s)} = 1$$

where $h(s) := \sqrt{2s\log\log s}$. In particular, a.s., there exists $T > 0$ such that, for all $t \geq T$, we have $\beta_t \leq 2h(t)$. Since, a.s., $A_t \underset{t\to T_0}{\longrightarrow} +\infty$, we deduce that

$$\begin{aligned} \limsup_{t\to+\infty} \left(\beta_{A_t} - \frac{1}{2}A_t\right) &\leq \limsup_{t\to+\infty} \left(2h(A_t) - \frac{1}{2}A_t\right) \\ &= -\infty. \end{aligned}$$

Hence, a.s.

$$\log\left[\frac{\rho_t}{x}\exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right)\right] \underset{t\uparrow T_0(x)}{\longrightarrow} -\infty$$

i.e.

$$\rho_t \exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right) \underset{t\uparrow T_0(x)}{\longrightarrow} 0$$

as claimed. □

7.5.4 Martingale property of $(D_t)_{t\geq 0}$

Let $\delta \geq 0$ and $x > 0$ be fixed. We show in this section that $(D_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$ martingale which, up to a positive constant, corresponds to a Girsanov-type change of probability measure.

Recall that, by definition

$$D_t = \mathbf{1}_{t<T_0(x)} \rho_t(x) \exp\left(-\frac{\delta-1}{2}\int_0^t \frac{ds}{\rho_s^2}\right). \tag{7.18}$$

Notation 7.5.5. For all $a \geq 0$ and $t \geq 0$, we denote by $P_x^a\big|_{\mathcal{F}_t}$ the image of the probability measure P_x^a under the restriction map

$$\begin{aligned}(C(\mathbb{R}_+), \mathcal{B}(C(\mathbb{R}_+))) &\to (C([0,t]), \mathcal{F}_t)\\ w &\mapsto w\big|_{[0,t]}\end{aligned}$$

The following proposition is a generalization of the absolute continuity results obtained in [PY81].

Proposition 7.5.6. *Let $\delta \geq 0$ and $x > 0$. Then, for all $t \geq 0$, the law $P_x^{\delta+2}{}_{|\mathcal{F}_t}$ is absolutely continuous w.r.t. the law $P^\delta_{x\,|\mathcal{F}_t}$, and the corresponding Radon-Nikodym derivative is given by*

$$\frac{dP_x^{\delta+2}}{dP_x^\delta}\bigg|_{\mathcal{F}_t}(\rho) \overset{a.s.}{=} \mathbf{1}_{t<T_0(x)}\frac{\rho_t(x)}{x}\exp\left(-\frac{\delta-1}{2}\int_0^t\frac{ds}{\rho_s^2}\right).$$

Proof. Fix $\epsilon > 0$. Under $P^\delta_{x\,|\mathcal{F}_t}$, the canonical process ρ stopped at T_ϵ satisfies the following SDE on $[0,t]$:

$$\rho_{s\wedge T_\epsilon} = x + \frac{\delta-1}{2}\int_0^{s\wedge T_\epsilon}\frac{ds}{\rho_s} + B_{s\wedge T_\epsilon}.$$

Consider the process M^ϵ defined on $[0,t]$ by

$$M_s^\epsilon := \int_0^{s\wedge T_\epsilon}\frac{dB_u}{\rho_u}$$

M^ϵ is an L^2 martingale on $[0,t]$. The exponential local martingale thereto associated is

$$\mathcal{E}(M^\epsilon)_s = \exp\left(\int_0^{s\wedge T_\epsilon}\frac{dB_u}{\rho_u} - \frac{1}{2}\int_0^{s\wedge T_\epsilon}\frac{du}{\rho_u^2}\right).$$

Since, by Itô's lemma

$$\log\left(\frac{\rho_{s\wedge T_\epsilon}}{x}\right) = \int_0^{s\wedge T_\epsilon}\frac{dB_u}{\rho_u} + \left(\frac{\delta}{2}-1\right)\int_0^{s\wedge T_\epsilon}\frac{du}{\rho_u^2},$$

we have

$$\mathcal{E}(M^\epsilon)_s = \exp\left[\log\left(\frac{\rho_{s\wedge T_\epsilon}}{x}\right) - \frac{\delta-1}{2}\int_0^{s\wedge T_\epsilon}\frac{du}{\rho_u^2}\right]$$
$$= \frac{\rho_{s\wedge T_\epsilon}}{x}\exp\left[-\frac{\delta-1}{2}\int_0^{s\wedge T_\epsilon}\frac{du}{\rho_u^2}\right].$$

Note that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle M^\epsilon, M^\epsilon\rangle_t\right)\right] \leq \exp\left(\frac{t}{2\epsilon}\right)$$
$$< \infty$$

so that, by Novikov's criterion, $\mathcal{E}(M^\epsilon)$ is a uniformly integrable martingale on $[0,t]$. So we may consider the probability measure $\mathcal{E}(M^\epsilon)_t \; P_x^\delta\big|_{\mathcal{F}_t}$.
Note also that

$$\langle M^\epsilon, B\rangle_t = \int_0^{s\wedge T_\epsilon}\frac{du}{\rho_u}.$$

Hence, by Girsanov's theorem, under the probability measure $\mathcal{E}(M^\epsilon)_t \; P_x^\delta\big|_{\mathcal{F}_t}$, the process

$$\rho_{s\wedge T_\epsilon} - x - \frac{\delta+1}{2}\int_0^{s\wedge T_\epsilon}\frac{du}{\rho_u}$$

is a local martingale, with quadratic variation given by $s\wedge T_\epsilon$. Therefore, by Theorem (1.7) in Chapter V of [RY13], there exists, on some enlarged probability space, a Brownian motion β such that, a.s.

$$\forall s\in[0,t], \;\; \rho_{s\wedge T_\epsilon} = x + \frac{\delta+1}{2}\int_0^{s\wedge T_\epsilon}\frac{du}{\rho_u} + \beta_{s\wedge T_\epsilon}.$$

Denote by $\bar{\rho}$ the unique strong solution on $[0,t]$ of the SDE

$$\bar{\rho}_s = x + \frac{\delta+1}{2}\int_0^s\frac{du}{\bar{\rho}_u} + \beta_s.$$

Then, by strong uniqueness of the solution to this SDE, we deduce that, under $\mathcal{E}(M^\epsilon)_t \; P_x^\delta\big|_{\mathcal{F}_t}$, a.s.

$$\forall s\in[0,t], \;\; s < T_\epsilon \implies \rho_s = \bar{\rho}_s.$$

Since $\bar{\rho}$ has the law of a $\delta+2$-dimensional Bessel process started at x, we deduce that, for all $F : C([0,T],\mathbb{R}_+)\to\mathbb{R}_+$ Borel, we have

$$E_x^\delta\left[\mathcal{E}(M^\epsilon)_t F(\rho)\mathbf{1}_{t<T_\epsilon}\right] = E_x^{\delta+2}[F(\rho)\mathbf{1}_{t<T_\epsilon}]$$

i.e.

$$E_x^\delta\left[\frac{\rho_t}{x}\exp\left(-\frac{\delta-1}{2}\int_0^t\frac{ds}{\rho_s^2}\right)F(\rho)\mathbf{1}_{t<T_\epsilon}\right]=E_x^{\delta+2}[F(\rho)\mathbf{1}_{t<T_\epsilon}].$$

Letting $\epsilon\to 0$, by the monotone convergence theorem, we obtain

$$E_x^\delta\left[\frac{\rho_t}{x}\exp\left(-\frac{\delta-1}{2}\int_0^t\frac{ds}{\rho_s^2}\right)F(\rho)\mathbf{1}_{t<T_0}\right]=E_x^{\delta+2}[F(\rho)\mathbf{1}_{t<T_0}].$$

But, since $P_x^{\delta+2}[T_0<+\infty]=0$, this yields

$$E_x^\delta\left[\frac{\rho_t}{x}\exp\left(-\frac{\delta-1}{2}\int_0^t\frac{ds}{\rho_s^2}\right)F(\rho)\mathbf{1}_{t<T_0}\right]=E_x^{\delta+2}[F(\rho)]$$

as stated. □

Remark 7.5.7. Proposition 7.5.6 is actually a particular case of a more general result. Indeed, for all $x>0$, $t\geq 0$, and $\delta'\geq\delta\geq 0$, such that $\delta'\geq 2$, $P_x^{\delta'}{}_{|\mathcal{F}_t}$ is absolutely continuous w.r.t. the law $P^\delta_{x\,|\mathcal{F}_t}$, and the corresponding Radon-Nikodym derivative is given by

$$\left.\frac{dP_x^{\delta'}}{dP_x^\delta}\right|_{\mathcal{F}_t}(\rho)\overset{a.s.}{=}\mathbf{1}_{t<T_0(x)}\left(\frac{\rho_t(x)}{x}\right)^{\frac{\delta'-\delta}{2}}\exp\left[-\frac{\delta'-\delta}{2}\left(\frac{\delta'+\delta}{4}-1\right)\int_0^t\frac{ds}{\rho_s^2}\right].\quad(7.19)$$

The proof of this fact is in all respect similar to that of Proposition 7.5.6 above.

Corollary 7.5.8. $(D_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t)_{t\geq 0}$ *continuous martingale*

Proof. The process $(D_t)_{t\geq 0}$ is continuous. Moreover, for all $t\geq 0$, $\frac{1}{x}D_t$ is the Radon-Nikodym derivative of $P_x^{\delta+2}{}_{|\mathcal{F}_t}$ w.r.t. $P^\delta_{x\,|\mathcal{F}_t}$. Therefore $(\frac{1}{x}D_t)_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$ martingale, so $(D_t)_{t\geq 0}$ is a martingale as well, and the claim follows. □

7.5.5 Moment estimates for the martingale $(D_t)_{t\geq 0}$

In this section, we prove that the martingale $(D_t)_{t\geq 0}$ is actually in L^p for some $p\geq 1$. We first recall the following fact:

Lemma 7.5.9. *For all* $a\geq 0$, $t\geq 0$, *and* $m\geq 0$, *we have*

$$E_x^a(\rho_t^m)<\infty.$$

Proof. Denote by d any integer such that $d\geq a$. By Lemma 7.2.1, we have

$$E_x^a(\rho_t^m)\leq E_x^d(\rho_t^m)$$

Since P_x^d is the law of $(||B_s||)_{s\geq 0}$, where $(B_s)_{s\geq 0}$ is a d-dimensional Brownian motion and $||\cdot||$ is the Euclidean norm in $\mathbb{R}^d$ (see [RY13], Chapter 11), this inequality can be rewritten as

$$E_x^a(\rho_t^m) \leq \mathbb{E}\left(||B_t||^m\right)$$

Since B_t is a Gaussian random variable, $\mathbb{E}\left(||B_t||^m\right)$ is finite, and the result follows. □

Proposition 7.5.10. *$(D_t)_{t\geq 0}$ is an L^p martingale for all finite positive number p such that $p \leq p(\delta)$, where $p(\delta) \in [1, +\infty]$ is given by*

$$p(\delta) := \begin{cases} \frac{(2-\delta)^2}{4(1-\delta)} & \text{if } \delta < 1, \\ +\infty & \text{if } \delta \geq 1. \end{cases} \tag{7.20}$$

Moreover the above statement is sharp: for $\delta < 1$ and $t > 0$, the random variable D_t is not in L^p for $p > p(\delta)$.

Remark 7.5.11. We emphasize that p is *finite* in the above result. Indeed D_t is never in L^∞ even if $\delta \geq 1$; for example, when $\delta = 1$, $D_t = \rho_t \mathbf{1}_{t<T_0(x)}$ which is clearly not bounded a.s. .

Proof of Prop 7.5.10. If $\delta \geq 1$, then, for all $t \geq 0$, $D_t \leq \rho_t$. Hence, for all $p \in (0, +\infty)$

$$\mathbb{E}\left(D_t^p\right) \leq E_x^\delta({\rho_t}^p)$$

which is finite by Lemma 7.5.9.

On the other hand, if $\delta \in [0, 1)$, then, for all $t > 0$ and $p > 0$, we have

$$\mathbb{E}\left(D_t^p\right) = E_x^\delta\left[\mathbf{1}_{t<T_0}\ {\rho_t}^p\ \exp\left(-p\frac{\delta-1}{2}\int_0^t \frac{ds}{\rho_s^2}\right)\right].$$

By the absolute continuity relation (7.19) applied with $\delta' := 2$, the latter equals

$$E_x^2\left[x^{\frac{2-\delta}{2}}\ {\rho_t}^{p+\frac{\delta-2}{2}}\ \exp\left(\underbrace{\left(-p\frac{\delta-1}{2} - \frac{(\delta-2)^2}{8}\right)}_{:=A(p)}\int_0^t \frac{ds}{\rho_s^2}\right)\right].$$

For $p = p(\delta)$, $A(p) = 0$, so that

$$\begin{aligned}\mathbb{E}\left[D_t^{p(\delta)}\right] &= E_x^2\left[x^{\frac{2-\delta}{2}}\,{\rho_t}^{p(\delta)+\frac{\delta-2}{2}}\right] \\ &= x^{1-\frac{\delta}{2}}\ E_x^2\left[{\rho_t}^{p(\delta)+\frac{\delta}{2}-1}\right].\end{aligned}$$

Since $\frac{\delta}{2} + p(\delta) - 1 \geq 0$, by Lemma 7.5.9, the last quantity is finite. Hence D_t is indeed in $L^{p(\delta)}$.

Suppose now that $p = p(\delta) + r$ for some $r > 0$. We show that $D_t \notin L^p$. We have

$$\begin{aligned} \mathbb{E}\left[D_t^p\right] &= E_x^2\left[x^{\frac{2-\delta}{2}} \ {\rho_t}^{p+\frac{\delta-2}{2}} \ \exp\left(\left(-p\frac{\delta-1}{2} - \frac{(\delta-2)^2}{8}\right)\int_0^t \frac{ds}{{\rho_s}^2}\right)\right] \\ &= x^{1-\frac{\delta}{2}} E_x^2\left[{\rho_t}^{p+\frac{\delta}{2}-1} \ \exp\left(\frac{1-\delta}{2} r \int_0^t \frac{ds}{{\rho_s}^2}\right)\right] \end{aligned}$$

We claim that the last quantity is infinite. Indeed, first note that by Jensen's inequality and Fubini, for any $C > 0$ we have

$$E_x^2\left[\exp\left(C\int_0^t \frac{ds}{{\rho_s}^2}\right)\right] \geq \exp\left(C\int_0^t E_x^2\left(\rho_s^{-2}\right) ds\right)$$

and the right-hand side is infinite since, for all $s > 0$, $E_x^2\left(\rho_s^{-2}\right) = +\infty$ (indeed, by formula (7.9), the transition density $p_s^2(x,y)$ does not integrate y^{-2} as $y \to 0$). Therefore

$$E_x^2\left[\exp\left(C\int_0^t \frac{ds}{{\rho_s}^2}\right)\right] = +\infty \tag{7.21}$$

Consider now any $c > 0$ and $a, b > 0$ such that $\frac{1}{a} + \frac{1}{b} = 1$. By (7.21) and Hölder's inequality, we have

$$\begin{aligned} +\infty &= E_x^2\left[\exp\left(\frac{1-\delta}{2a} r \int_0^t \frac{ds}{{\rho_s}^2}\right)\right] \\ &\leq E_x^2\left[{\rho_t}^{ac}\exp\left(\frac{1-\delta}{2} r \int_0^t \frac{ds}{{\rho_s}^2}\right)\right]^{1/a} E_x^2\left[\rho_t^{-bc}\right]^{1/b} \end{aligned}$$

Set $c = \frac{\frac{\delta}{2}+p-1}{\frac{\delta}{2}+p}$, $a = \frac{\delta}{2} + p$, and $b = \frac{\frac{\delta}{2}+p}{\frac{\delta}{2}+p-1}$. Remark that $\frac{\delta}{2} + p - 1 > 0$ since $p > p(\delta) \geq 1$, so that this choice for c, a, and b makes sense. We obtain

$$E_x^2\left[{\rho_t}^{\frac{\delta}{2}+p-1} \ \exp\left(\frac{1-\delta}{2} r \int_0^t \frac{ds}{{\rho_s}^2}\right)\right]^{\frac{1}{\frac{\delta}{2}+p}} E_x^2\left[\rho_t^{-1}\right]^{\frac{\frac{\delta}{2}+p-1}{\frac{\delta}{2}+p}} = +\infty$$

By the comparison lemma 7.2.1 and the expression (7.9) for the transition density of the Bessel process, we have

$$E_x^2\left[\rho_t^{-1}\right] \leq E_0^2\left[\rho_t^{-1}\right] = \int_0^\infty \frac{1}{t}\exp\left(-\frac{y^2}{2t}\right) dy$$

so that $E_x^2\left[\rho_t^{-1}\right] < +\infty$. Therefore, we deduce that

$$E_x^2\left[{\rho_t}^{\frac{\delta}{2}+p-1} \ \exp\left(\frac{1-\delta}{2} r \int_0^t \frac{ds}{{\rho_s}^2}\right)\right] = +\infty$$

as claimed. Hence $D_t \notin L^p$ for $p > p(\delta)$. □

7.6 A Bismut-Elworthy-Li formula for the Bessel processes

We are now in position to provide a probabilistic interpretation of the right-hand-side of equation (7.8) in Theorem 7.3.1.

Let $\delta > 0$, and $x > 0$. As we saw in the previous section, the process $(\eta_t(x))_{t\geq 0}$ may blow up at time T_0, so that the stochastic integral $\int_0^t \eta_s(x) dB_s$ is a priori ill-defined, at least for $\delta \in (0,1)$. However, it turns out that we can define the latter process rigorously as a local martingale.

Proposition 7.6.1. *Suppose that $\delta > 0$. Then the stochastic integral process $\int_0^t \eta_s dB_s$ is well-defined as a local martingale and is indistinguishable from the continuous martingale $D_t - x$.*

Proof. We first treat the case $\delta \geq 2$, which is much easier to handle. In that case, $\eta_t \in [0,1]$ for all $t \geq 0$, so that the stochastic integral $\int_0^t \eta_s dB_s$ is clearly well-defined as an L^2 martingale. Moreover, since $T_0 = +\infty$ a.s., by Itô's lemma we have

$$\begin{aligned} D_t = \rho_t \eta_t &= x + \int_0^t \eta_s \, d\rho_s + \int_0^t \rho_s \, d\eta_s \\ &= x + \int_0^t \eta_s \left(\frac{\delta - 1}{2} \frac{ds}{\rho_s} + dB_s \right) - \int_0^t \rho_s \frac{\delta - 1}{2} \frac{\eta_s}{{\rho_s}^2} ds \\ &= x + \int_0^t \eta_s dB_s \end{aligned}$$

so the claim follows.

Now suppose that $\delta \in (0,2)$ and fix an $\epsilon > 0$. Recall that $T_\epsilon(x) := \inf\{t \geq 0, \rho_t(x) \leq \epsilon\}$ and note that, since $T_\epsilon < T_0$, the stopped process η^{T_ϵ} is continuous on $\mathbb{R}_+$, so that the stochastic integral $\int_0^{t \wedge T_\epsilon(x)} \eta_s(x) dB_s$ is well-defined as a local martingale. Using as above Itô's lemma, but this time with the stopped processes ρ^{T_ϵ} and η^{T_ϵ}, we have

$$\int_0^{t \wedge T_\epsilon} \eta_s dB_s = D_{t \wedge T_\epsilon} - x. \tag{7.22}$$

Our aim would be to pass to the limit $\epsilon \to 0$ in this equality. By continuity of D, as $\epsilon \to 0$, $D_{t \wedge T_\epsilon}$ converges to $D_{t \wedge T_0} = D_t$ almost-surely. So the right-hand side of (7.22) converges to $D_t - x$ almost-surely.

The convergence of the left-hand side to a stochastic integral is more involved, since we first have to prove that the stochastic integral $\int_0^t \eta_s dB_s$ is indeed well-defined as a local martingale. For this, it suffices to prove that, almost-surely

$$\forall t \geq 0, \qquad \int_0^t \eta_s^2 \, ds < \infty.$$

We actually prove the following stronger fact. For all $t \geq 0$

$$\mathbb{E}\left[\left(\int_0^t \eta_s^2 ds\right)^{p/2}\right] < \infty \tag{7.23}$$

for all finite positive number p such that $p \in (1, p(\delta)]$. Indeed, applying successively the Burkholder-Davis-Gundy (BDG) inequality and Doob's inequality to the martingale $\int_0^{T_\epsilon \wedge \cdot} \eta_s dB_s$, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^{t\wedge T_\epsilon} \eta_s^2 ds\right)^{p/2}\right] &\leq C_p \ \mathbb{E}\left[\sup_{s\leq t\wedge T_\epsilon}\left|\int_0^s \eta_u dB_u\right|^p\right] \\
&= C_p \ \mathbb{E}\left[\sup_{s\leq t\wedge T_\epsilon} |D_s - x|^p\right] \\
&\leq C_p \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|D_{t\wedge T_\epsilon} - x|^p\right]
\end{aligned}$$

where C_p is a constant depending only on p. Now, since $(D_t - x)_{t\geq 0}$ is a continuous martingale, by the optional stopping theorem and Jensen's inequality, we have

$$\mathbb{E}\left[|D_{t\wedge T_\epsilon} - x|^p\right] \leq \mathbb{E}(|D_t - x|^p)$$

and the right-hand side is finite because D_t is in L^p . Hence, letting $\epsilon \to 0$ in the above, by the monotone convergence theorem we deduce that

$$\mathbb{E}\left[\left(\int_0^{t\wedge T_0} \eta_s^2 ds\right)^{p/2}\right] < \infty$$

But since $\eta_t = 0$ for all $t \geq T_0$, this implies the bound (7.23), and hence the stochastic integral $\int_0^t \eta_s dB_s$ is well-defined as a local martingale. Moreover, for all $t \geq 0$, by the BDG inequality, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^t \eta_s dB_s - \int_0^{t\wedge T_\epsilon} \eta_s dB_s\right)^p\right] &= \mathbb{E}\left[\left(\int_{t\wedge T_\epsilon}^{t\wedge T_0} \eta_s dB_s\right)^p\right] \\
&\leq c_p \ \mathbb{E}\left[\left(\int_{t\wedge T_\epsilon}^{t\wedge T_0} \eta_s^2 ds\right)^{p/2}\right]
\end{aligned}$$

where c_p is some constant depending only on p. Now, by the dominated convergence theorem, the last quantity above goes to 0 as $\epsilon \to 0$, and hence

$$\int_0^{t\wedge T_\epsilon} \eta_s dB_s \underset{\epsilon\to 0}{\longrightarrow} \int_0^t \eta_s dB_s$$

in L^p. Hence, the left-hand side of equality (7.22) converges in L^p to the stochastic integral $\int_0^t \eta_s dB_s$. Letting $\epsilon \to 0$ in that equality, we thus obtain

$$\int_0^t \eta_s dB_s = D_t - x$$

as claimed. □

Using the above proposition, Theorem 7.3.1 can now be interpreted probabilistically as a Bismut-Elworthy-Li formula.

Theorem 7.6.2 (Bismut-Elworthy-Li formula)**.** *Let $\delta > 0$. Then, for all $T > 0$, and all $F : \mathbb{R}_+ \to \mathbb{R}$ bounded and Borel, the function $x \to \mathbf{P}_t^\delta F(x)$ is differentiable on $\mathbb{R}_+$, and for all $x > 0$*

$$\frac{d}{dx}\mathbf{P}_T^\delta F(x) = \frac{1}{T}\,\mathbb{E}\left[F(\rho_t(x))\left(\int_0^T \eta_s(x)dB_s\right)\right]. \tag{7.24}$$

Proof. By Theorem 7.3.1, the differentiablity property holds, and we have

$$\frac{d}{dx}\mathbf{P}_T^\delta F(x) = \frac{x}{T}\left[\mathbf{P}_T^{\delta+2}F(x) - \mathbf{P}_T^\delta F(x)\right].$$

Moreover, by Proposition 7.5.6, for all $x > 0$

$$\mathbf{P}_T^{\delta+2}F(x) - \mathbf{P}_T^\delta F(x) = E_x^\delta\left[F(\rho_T)\left(\frac{D_T}{x} - 1\right)\right]$$

and, by Proposition 7.6.1, we have

$$E_x^\delta\left[F(\rho_T)\left(\frac{D_T}{x} - 1\right)\right] = \frac{1}{x}\mathbb{E}\left[F(\rho_T(x))\left(\int_0^T \eta_s(x)dB_s\right)\right]$$

so equality (7.24) follows. □

Using the Bismut-Elworthy-Li formula, we are now able to sharpen the Strong Feller estimate obtained in equation (7.11) above.

Corollary 7.6.3. *Let $T > 0$ and $\delta \geq 2(\sqrt{2}-1)$. Then, for all $R > 0$, there exists a constant $C > 0$ such that, for all $x, y \in [0, R]$ and $F : \mathbb{R}_+ \to \mathbb{R}$ bounded and Borel, we have*

$$|\mathbf{P}_T^\delta F(x) - \mathbf{P}_T^\delta F(y)| \leq \frac{C||F||_\infty}{T^{\alpha(\delta)}}\,|y - x| \tag{7.25}$$

where the exponent $\alpha(\delta) \in [\frac{1}{2}, 1)$ is given by

$$\alpha(\delta) := \begin{cases} \frac{1}{2} + \frac{1-\delta}{2-\delta} & \text{if } \delta \in [2(\sqrt{2}-1), 1], \\ 1/2 & \text{if } \delta \geq 1. \end{cases}$$

Proof. Let $x > 0$. By Theorem 7.6.2, we have

$$\frac{d}{dx}\mathbf{P}_T^\delta F(x) = \frac{1}{T}\,\mathbb{E}\left[F(\rho_t(x))\left(\int_0^T \eta_s(x)dB_s\right)\right].$$

so that

$$\left|\frac{d}{dx}\mathbf{P}_T^\delta F(x)\right| \leq \frac{||F||_\infty}{T}\,\mathbb{E}\left[\left|\int_0^T \eta_s(x)dB_s\right|\right].$$

We now bound the quantity $\mathbb{E}\left[\left|\int_0^T \eta_s(x)dB_s\right|\right]$. If $\delta \geq 1$, then the process $(\eta_s(x))_{s\geq 0}$ takes values in $[0,1]$, so that, using the Cauchy-Schwarz inequality and Itô's isometry formula, we have

$$\mathbb{E}\left[\left|\int_0^T \eta_s(x)dB_s\right|\right] \leq \sqrt{\mathbb{E}\left(\int_0^T \eta_s(x)^2 ds\right)} \leq \sqrt{T}.$$

Therefore

$$\left|\frac{d}{dx}\mathbf{P}_T^\delta F(x)\right| \leq \frac{||F||_\infty}{\sqrt{T}}$$

and the claim follows with $C = 1$.

Suppose now that $\delta \in [2(\sqrt{2}-1), 1)$. Letting $p := p(\delta)$ as in (7.20), we have, by Jensen's inequality

$$\mathbb{E}\left[\left|\int_0^T \eta_s(x)dB_s\right|\right] \leq \left(\mathbb{E}\left|\int_0^T \eta_s(x)dB_s\right|^p\right)^{1/p}$$

Now, applying successively the BDG inequality, Jensen's inequality and the absolute continuity relation (7.19) between P_x^2 and P_x^δ, we have, for some constant c_p depending only on p

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^T \eta_s(x)dB_s\right|^p\right] &\leq c_p\,\mathbb{E}\left[\left(\int_0^T \eta_s(x)^2 ds\right)^{p/2}\right] \\
&\leq c_p\,T^{p/2-1}\,\mathbb{E}\left(\int_0^T \eta_s(x)^p\,ds\right) \\
&\leq c_p\,T^{p/2-1}\int_0^T E_x^\delta(\eta_s^p)\,ds \\
&= c_p\,T^{p/2-1}\int_0^T E_x^2\left[\left(\frac{\rho_s}{x}\right)^{\frac{\delta-2}{2}}\exp\left(\left(\frac{1-\delta}{2}p - \frac{(2-\delta)^2}{8}\right)\int_0^s \frac{du}{\rho_u^2}\right)\right] ds \\
&= c_p\,T^{p/2-1}\int_0^T E_x^2\left[\left(\frac{\rho_s}{x}\right)^{\frac{\delta-2}{2}}\right] ds
\end{aligned}$$

where the last equality follows from the fact that $\frac{1-\delta}{2}p - \frac{(2-\delta)^2}{8} = 0$ for $p = p(\delta)$. Now, since $\frac{\delta-2}{2} \le 0$, by the comparison lemma 7.2.1, as well as the scaling property of the Bessel processes (see, e.g., Remark 3.7 in [Zam17]), for all $s \in [0,T]$, we have

$$E_x^2\left[\rho_s^{\frac{\delta-2}{2}}\right] \le E_0^2\left[\rho_s^{\frac{\delta-2}{2}}\right] = s^{\frac{\delta-2}{4}} E_0^2\left[\rho_1^{\frac{\delta-2}{2}}\right].$$

Let $c := E_0^2\left[\rho_1^{\frac{\delta-2}{2}}\right]$. Using formula (7.9), we have

$$c = \int_0^\infty y^{\delta/2} \exp\left(-\frac{y^2}{2}\right)\, dy < \infty.$$

Hence

$$\begin{aligned}\int_0^T E_x^2\left[\left(\frac{\rho_s}{x}\right)^{\frac{\delta}{2}-1}\right] ds &\le c\, x^{1-\frac{\delta}{2}} \int_0^T s^{\frac{\delta-2}{4}} ds \\ &\le \frac{4\, c}{\delta+2}\, x^{1-\frac{\delta}{2}}\, T^{\frac{\delta+2}{4}}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^T \eta_s(x) dB_s\right|^p\right] &\le K\, x^{1-\frac{\delta}{2}}\, T^{\frac{p}{2}-1}\, T^{\frac{\delta+2}{4}} \\ &\le K\, x^{1-\frac{\delta}{2}}\, T^{\frac{p}{2}+\frac{\delta-2}{4}}\end{aligned}$$

where K is a constant depending only on δ. Hence

$$\mathbb{E}\left[\left|\int_0^T \eta_s(x) dB_s\right|\right] \le K^{1/p}\, x^{\frac{1}{p}(1-\frac{\delta}{2})}\, T^{\frac{1}{2}+\frac{\delta-2}{4p}}.$$

Note that, since $p = p(\delta)$, we have $\frac{1}{p}(1-\frac{\delta}{2}) = \frac{2(1-\delta)}{2-\delta}$, and $\frac{\delta-2}{4p} = -\frac{1-\delta}{2-\delta}$. Therefore, we obtain

$$\left|\frac{d}{dx}\mathbf{P}_T^\delta F(x)\right| \le K^{1/p}\, x^{2\frac{1-\delta}{2-\delta}}\, ||F||_\infty\, T^{-\frac{1}{2}-\frac{1-\delta}{2-\delta}}.$$

Therefore, given $R > 0$, one has for all $x \in [0,R]$

$$\left|\frac{d}{dx}\mathbf{P}_T^\delta F(x)\right| \le C\frac{||F||_\infty}{T^{\alpha(\delta)}}$$

with $C := K^{1/p}\, R^{2\frac{1-\delta}{2-\delta}}$. This yields the claim.

□

Remark 7.6.4. In the above proposition, the value $2(\sqrt{2}-1)$ that appears is the smallest value of δ for which η is in L^2. For $\delta < 2(\sqrt{2}-1)$, η is no longer in L^2 but only in L^p for $p = p(\delta) < 2$, so that we cannot apply Jensen's inequality to bound the quantity $\mathbb{E}\left(\int_0^T \eta_s(x)^2 ds\right)^{p/2}$ anymore. It seems reasonable to expect that the bound (7.25) holds also for $\delta < 2(\sqrt{2}-1)$, although we do not have a proof of this fact.

7.7 Proof of a technical result

In this section, we prove Proposition 7.4.3. Recall that we still denote by $(\rho_t(x))_{t,x\geq 0}$ the process $(\tilde{\rho}_t^\delta(x))_{t,x\geq 0}$ constructed in Proposition 7.4.1.

Lemma 7.7.1. *For all rational numbers $\epsilon, \gamma > 0$, let*

$$\mathcal{U}_\gamma^\epsilon := [0, T_\epsilon(\gamma)) \times (\gamma, +\infty)$$

and set

$$\mathcal{U} := \bigcup_{\epsilon,\gamma\in\mathbb{Q}_+^*} \mathcal{U}_\gamma^\epsilon.$$

Then, a.s., the function $(t,x) \mapsto \rho_t(x)$ is continuous on the open set $\mathcal{U}$.

Proof. By patching, it suffices to prove that, a.s., the function $(t,x) \mapsto \rho_t(x)$ is continuous on each $\mathcal{U}_\gamma^\epsilon$, where $\epsilon, \gamma \in \mathbb{Q}_+^*$.

Fix $\epsilon, \gamma \in \mathbb{Q}_+^*$, and let $x, y \in (\gamma, +\infty) \cap \mathbb{Q}$. We proceed to show that, a.s., for all $t \leq s < T_\epsilon(\gamma)$ the following inequality holds

$$|\rho_t(x) - \rho_s(y)| \leq |x-y| \exp\left(\frac{|\delta-1|}{2\epsilon^2} t\right) + \frac{|\delta-1|}{2\epsilon}|s-t| + |B_s - B_t|. \qquad (7.26)$$

Since $T_\epsilon(\gamma) < T_0(\gamma)$, a.s., for all $t \leq s \leq T_\epsilon(\gamma)$, we have

$$\forall \tau \in [0,t], \qquad \rho_\tau(x) = x + \frac{\delta-1}{2}\int_0^\tau \frac{du}{\rho_u(x)} + B_\tau$$

as well as

$$\forall \tau \in [0,s], \qquad \rho_\tau(y) = y + \frac{\delta-1}{2}\int_0^\tau \frac{du}{\rho_u(y)} + B_\tau$$

and hence

$$\forall \tau \in [0,t], \qquad |\rho_\tau(x) - \rho_\tau(y)| \leq |x-y| + \frac{|\delta-1|}{2}\int_0^\tau \frac{|\rho_u(x)-\rho_u(y)|}{\rho_u(x)\rho_u(y)} du.$$

By the monotonicity property of ρ, we have, a.s., for all t, s as above and $u \in [0, s]$

$$\rho_u(x) \wedge \rho_u(y) \geq \rho_u(\gamma) \geq \epsilon \tag{7.27}$$

so that

$$\forall \tau \in [0,t], \quad |\rho_\tau(x) - \rho_\tau(y)| \leq |x-y| + \frac{|\delta - 1|}{2} \int_0^\tau \frac{|\rho_u(x) - \rho_u(y)|}{\epsilon^2} du,$$

which, by Grönwall's inequality, implies that

$$|\rho_t(x) - \rho_t(y)| \leq |x-y| \exp\left(\frac{|\delta-1|}{2\epsilon^2} t\right). \tag{7.28}$$

Moreover, we have

$$\rho_s(y) - \rho_t(y) = \frac{\delta - 1}{2} \int_t^s \frac{du}{\rho_u(y)} + B_s - B_t$$

which, by (7.27), entails the inequality

$$|\rho_s(y) - \rho_t(y)| \leq \frac{|\delta - 1|}{2\epsilon} |s-t| + |B_s - B_t|. \tag{7.29}$$

Putting inequalities (7.28) and (7.29) together yields the claimed inequality (7.26). Hence, we have, a.s., for all rationals $x, y > \gamma$ and all $t \leq s < T_\epsilon(\gamma)$

$$|\rho_t(x) - \rho_s(y)| \leq |x-y| \exp\left(\frac{|\delta-1|}{2\epsilon^2} t\right) + \frac{\delta-1}{2} |s-t| + |B_s - B_t|$$

and, by density of $\mathbb{Q} \cap (\gamma, +\infty)$ in $(\gamma, +\infty)$, this inequality remains true for all $x, y > \gamma$. Since, a.s., $t \mapsto B_t$ is continuous on $\mathbb{R}_+$, the continuity of ρ on $\mathcal{U}_\gamma^\epsilon$ is proved. □

Corollary 7.7.2. *Almost-surely, we have*

$$\forall x \geq 0, \quad \forall t \in [0, T_0(x)), \qquad \rho_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{du}{\rho_u(x)} + B_t. \tag{7.30}$$

Remark 7.7.3. We have already remarked in Section 7.2 that, for all fixed $x \geq 0$, the process $(\rho_t(x))_{t \geq 0}$ satisfies the SDE (7.7). By contrast, the above Corollary shows the stronger fact that, considering the modification $\tilde{\rho}$ of the Bessel flow constructed in Proposition 7.4.1 above, a.s., for *each* $x \geq 0$, the path $(\tilde{\rho}_t(x))_{t \geq 0}$ still satisfies relation (7.7).

Proof. Consider an almost-sure event $\mathcal{A} \in \mathcal{F}$ as in Remark 7.4.2. On the event $\mathcal{A}$, for all $r \in \mathbb{Q}_+$, we have

$$\forall t \in [0, T_0(r)), \qquad \rho_t(r) = r + \frac{\delta - 1}{2} \int_0^t \frac{du}{\rho_u(r)} + B_t.$$

Denote by $\mathcal{B} \in \mathcal{F}$ any almost-sure event on which ρ satisfies the monotonicity property (7.12). We show that, on the event $\mathcal{A} \cap \mathcal{B}$, the property (7.30) is satisfied.

Suppose $\mathcal{A} \cap \mathcal{B}$ is fulfilled, and let $x \geq 0$. Then for all $r \in \mathbb{Q}$ such that $r \geq x$, we have

$$\forall t \geq 0, \qquad \rho_t(x) \leq \rho_t(r)$$

so that $T_0(r) \geq T_0(x)$. Hence, for all $t \in [0, T_0(x))$, we have in particular $t \in [0, T_0(r))$, so that

$$\rho_t(r) = r + \frac{\delta - 1}{2} \int_0^t \frac{du}{\rho_u(r)} + B_t.$$

Since, for all $u \in [0, t]$, $\rho_u(r) \downarrow \rho_u(x)$ as $r \downarrow x$ with $r \in \mathbb{Q}$, by the monotone convergence theorem, we deduce that

$$\int_0^t \frac{du}{\rho_u(r)} \longrightarrow \int_0^t \frac{du}{\rho_u(x)}$$

as $r \downarrow x$ with $r \in \mathbb{Q}$. Hence, letting $r \downarrow x$ with $r \in \mathbb{Q}$ in the above equation, we obtain

$$\rho_t(x) = x + \frac{\delta - 1}{2} \int_0^t \frac{du}{\rho_u(x)} + B_t.$$

This yields the claim. □

One of the main difficulties for proving Proposition 7.4.3 arises from the behavior of $\rho_t(x)$ at $t = T_0(x)$. However we will circumvent this problem by working away from the event $t = T_0(x)$. To do so, we will make use of the following property.

Lemma 7.7.4. *Let $\delta < 2$ and $x \geq 0$. Then the function $y \mapsto T_0(y)$ is a.s. continuous at x.*

Proof. The function $y \mapsto T_0(y)$ is nondecreasing over $\mathbb{R}_+$. Hence, if $x > 0$, it has left- and right-sided limits at x, $T_0(x^-)$ and $T_0(x^+)$, satisfying

$$T_0(x^-) \leq T_0(x) \leq T_0(x^+). \tag{7.31}$$

Similarly, if $x = 0$, there exists a right-sided limit $T_0(0^+)$ satisfying $T_0(0) \leq T_0(0^+)$. Suppose, e.g., that $x > 0$. Then we have

$$\mathbb{E}\left(e^{-T_0(x^+)}\right) \leq \mathbb{E}\left(e^{-T_0(x)}\right) \leq \mathbb{E}\left(e^{-T_0(x^-)}\right). \tag{7.32}$$

Now, by the scaling property of the Bessel processes (see, e.g., Remark 3.7 in [Zam17]), for all $y \geq 0$, the following holds:

$$(y\rho_t(1))_{t\geq 0} \overset{(d)}{=} (\rho_{y^2 t}(y))_{t\geq 0},$$

so that $T_0(y) \overset{(d)}{=} y^2 T_0(1)$. Therefore, using the dominated convergence theorem, we have

$$\begin{aligned}\mathbb{E}\left(e^{-T_0(x^+)}\right) &= \lim_{y\downarrow x} \mathbb{E}\left(e^{-T_0(y)}\right) \\ &= \lim_{y\downarrow x} \mathbb{E}\left(e^{-y^2 T_0(1)}\right) \\ &= \mathbb{E}\left(e^{-x^2 T_0(1)}\right) \\ &= \mathbb{E}\left(e^{-T_0(x)}\right).\end{aligned}$$

Similarly, we have $\mathbb{E}\left(e^{-T_0(x^-)}\right) = \mathbb{E}\left(e^{-T_0(x)}\right)$. Hence the inequalities (7.32) are actually equalities; recalling the original inequality (7.31), we deduce that $T_0(x^-) = T_0(x) = T_0(x^+)$ a.s.. Similarly, if $x = 0$, we have $T_0(0) = T_0(0^+)$ a.s. □

Before proving Proposition 7.4.3, we need a coalescence lemma, which will help us prove that the derivative of ρ_t at x is 0 if $t > T_0(x)$:

Lemma 7.7.5. *Let $x, y \geq 0$, and let τ be a nonnegative $(\mathcal{F}_t)_{t\geq 0}$-stopping time. Then, almost-surely*

$$\rho_\tau(x) = \rho_\tau(y) \quad \Rightarrow \quad \forall s \geq \tau,\ \rho_s(x) = \rho_s(y).$$

Proof. On the event $\{\rho_\tau(x) = \rho_\tau(y)\}$, the processes $(X_t^\delta(x))_{t\geq 0} := (\rho_t(x)^2)_{t\geq 0}$ and $(X_t^\delta(y))_{t\geq 0} := (\rho_t(y)^2)_{t\geq 0}$ both satisfy, on $[\tau, +\infty)$, the SDE

$$X_t = \rho_\tau(x)^2 + 2\int_\tau^t \sqrt{X_s} dB_s + \delta(t - \tau).$$

By pathwise uniqueness of this SDE (see [RY13], Theorem (3.5), Chapter IX), we deduce that, a.s. on the event $\{\rho_\tau(x) = \rho_\tau(y)\}$, $X_t(x) = X_t(y)$, hence $\rho_t(x) = \rho_t(y)$ for all $t \geq \tau$. □

Now we are able to prove Prop. 7.4.3.

Proof of Proposition 7.4.3. Let $t > 0$ and $x > 0$ be fixed. First remark that

$$\mathbb{P}(T_0(x) = t) = 0.$$

Indeed, if $\delta > 0$, then

$$\mathbb{P}(T_0(x) = t) \leq \mathbb{P}(\rho_t(x) = 0)$$

and the RHS is zero since the law of $\rho_t(x)$ has no atom on $\mathbb{R}_+$ (it has density $p_t^\delta(x, \cdot)$ w.r.t. Lebesgue measure on $\mathbb{R}_+$, where p_t^δ was defined in equation (7.9) above). On the other hand, if $\delta = 0$, then 0 is an absorbing state for the process ρ, so that, for all $s \geq 0$

$$\mathbb{P}(T_0(x) \leq s) = \mathbb{P}(\rho_s(x) = 0)$$

and the RHS is continuous in s on $\mathbb{R}_+$, since it is given by $\exp(-\frac{x^2}{2s})$ (see [RY13], Chapter XI, Corrolary 1.4). Hence, also in the case $\delta = 0$ the law of $T_0(x)$ has no atom on $\mathbb{R}_+$. Hence, a.s., either $t < T_0(x)$ or $t > T_0(x)$.

First suppose that $t < T_0(x)$. A.s., the function $y \mapsto T_0(y)$ is continuous at x, so there exists a rational number $y \in [0, x)$ such that $t < T_0(y)$; since, by Remark (7.4.2), $t \mapsto \rho_t(y)$ is continuous, there exists $\epsilon \in \mathbb{Q}_+^*$ such that $t < T_\epsilon(y)$. By monotonicity of $z \mapsto \rho(z)$, for all $s \in [0, t]$ and $z \geq y$, we have

$$\rho_s(z) \geq \rho_s(y) \geq \epsilon.$$

Hence, recalling Corollary 7.7.2, for all $s \in [0, t]$ and $h \in \mathbb{R}$ such that $|h| < |x - y|$

$$\rho_s(x + h) = x + h + \int_0^s \frac{\delta - 1}{2} \frac{du}{\rho_u(x + h)} + B_s.$$

Hence, setting $\eta_s^h(x) := \frac{\rho_s(x+h) - \rho_s(x)}{h}$, we have

$$\forall s \in [0, t], \qquad \eta_s^h(x) = 1 - \frac{\delta - 1}{2} \int_0^t \frac{\eta_u^h(x)}{\rho_u(x)\rho_u(x + h)} du$$

so that

$$\eta_t^h(x) = \exp\left(\frac{1 - \delta}{2} \int_0^t \frac{ds}{\rho_s(x)\rho_s(x + h)} \right).$$

Note that, for all $s \in [0, t]$ and $h \in \mathbb{R}$ such that $|h| < |x - y|$, we have $(s, x + h) \in [0, T_\epsilon(y)) \times (y, +\infty) \subset \mathcal{U}$. Hence, by Lemma 7.7.1, we have, for all $s \in [0, t]$

$$\rho_s(x + h) \underset{h \to 0}{\longrightarrow} \rho_s(x)$$

with the domination property

$$\frac{1}{\rho_s(x)\rho_s(x + h)} \leq \epsilon^{-2}$$

valid for all $|h| < |x-y|$. Hence, by the dominated convergence theorem, we deduce that

$$\eta_t^h(x) \underset{h\to 0}{\longrightarrow} \exp\left(\frac{1-\delta}{2}\int_0^t \frac{ds}{\rho_s(x)^2}\right)$$

which yields the claimed differentiability of ρ_t at x.

We now suppose that $t > T_0(x)$. Since the function $y \mapsto T_0(y)$ is a.s. continuous at x, a.s. there exists $y > x$, $y \in \mathbb{Q}$, such that $t > T_0(y)$. By Remark (7.4.2), the function $t \mapsto \rho_t(y)$ is continuous, so that $\rho_{T_0(y)}(y) = 0$. By monotonicity of $z \mapsto \rho(z)$, we deduce that, for all $z \in [0, y]$, we have

$$\rho_{T_0(y)}(z) = 0.$$

By Lemma 7.7.5, we deduce that, leaving aside some event of proability zero, all the trajectories $(\rho_t(z))_{t\geq 0}$ for $z \in [0, y] \cap \mathbb{Q}$ coincide from time $T_0(y)$ onwards. In particular, we have

$$\forall z \in [0, y] \cap \mathbb{Q}, \qquad \rho_t(z) = \rho_t(x).$$

Since, moreover, the function $z \mapsto \rho_t(z)$ is nondecreasing, we deduce that it is constant on the whole interval $[0, y]$:

$$\forall z \in [0, y], \qquad \rho_t(z) = \rho_t(x).$$

In particular, the function $z \mapsto \rho_t(z)$ has derivative 0 at x. This concludes the proof. □

Chapter 8

Towards local times for solutions to SPDEs

In this section, we aim at proposing methods to obtain the existence of local times associated with solutions to SPDEs driven by space-time white noise. To do so, we shall exploit the criteria provided in [GH80].

8.1 Local times of the solution to the stochastic heat equation

We first consider the simplest case, given by the additive stochastic heat equation on $\mathbb{R}_+ \times [0,1]$, with homogeneous Dirichlet boundary conditions:

$$\begin{cases} \partial_t u(t,x) = \dfrac{1}{2}\partial_x^2 u + \xi \\[2ex] u(t,0) = u(t,1) = 0 \end{cases}$$

Consider u a solution to this equation. Given $x \in (0,1)$ fixed, we are interested in the existence and regularity of a family of occupation times $(\ell^a_{t,x})_{t\geq 0, a\in\mathbb{R}}$ associated with the process $(u(t,x))_{t\geq 0}$. Note that, for all $s,t \geq 0$, we have

$$\Delta(s,t) := \mathbb{E}[|u(s,x) - u(t,x)|^2] \geq c\,|t-s|^{1/2}$$

for some constant $c > 0$, see e.g. the proof of Lemma 3.1 in [Zam06]. Therefore, for all $p < 3/2$, we have

$$\int_0^1 \int_0^1 \frac{ds\ dt}{\Delta(s,t)^{p+1/2}} < \infty$$

By Theorem (28.1) in [GH80], we deduce that, for all $t \geq 0$ fixed, there exists a process $(\ell^a_{t,x})_{a\in\mathbb{R},t\geq 0}$ such that, for all Borel function $f : \mathbb{R} \to \mathbb{R}_+$

$$\int_0^t f(u(s,x))\,\mathrm{d}s = \int_{\mathbb{R}} f(a)\,\ell^a_{t,x}\,\mathrm{d}a.$$

Moreover, by Theorem (2.8.5) in [GH80], there exists a jointly continuous modification of the process $(\ell^a_{t,x})_{a\in\mathbb{R},t\geq 0}$ such that, a.s., the function $a \mapsto \ell^a_{t,x}$ is absolutely continuous in a for all $t \geq 0$, with derivative in $L^2(\mathbb{R}, \mathrm{d}a)$. Actually, arguing as in the proof of Theorem (28.1), we can check that a.s., for all $T > 0$ the function $(t, a) \mapsto \partial_a \ell^a_{t,x}$ is in $L^2([0, T] \times \mathbb{R}, \mathrm{d}t\ \mathrm{d}a)$.

8.2 A perturbation method

We now consider an SPDE of the form

$$\begin{cases}\partial_t u(t,x) = \frac{1}{2}\partial_x^2 u(t,x) + f(u)(t,x) + W(t,x)\\ u(t,0) = u(t,1) = 0, u(0,x) = u_0(x)\end{cases} \tag{8.1}$$

where $f : \mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, and where we denote by $f(u)(t,x)$ the quantity $f(t,x,u(t,x))$. Fix $x \in (0,1)$. Our aim would be to obtain a family of local times for the process $(u(t,x))_{t\geq 0}$. This problem is non-trivial since the latter process is *not* Gaussian in general. We propose a pathwise approach based on a perturbation result.

Note that the solution to (8.1) satisfies the relation

$$u(t,x) = v(t,x) + \int_0^t \int g_{t-s}(x,y) f\left(s,y,u(s,y)\right) dy\ ds \tag{8.2}$$

where v is the solution of the additive stochastic heat equation on $[0,1]$ with Dirichlet boundary condition started at u_0, and where $(g_t(x,y)_{t\geq 0,x,y\in[0,1]}$ is the fundamental solution of the stochastic heat equation with Dirichlet boundary conditions, which is defined by (4.11). By Section 8.1, for all $x \in (0,1)$ fixed, the process $(v(t,x))_{t\geq 0}$ admits a bi-continuous family $\ell^a_{t,x}$ of local times such that, for all $t > 0$, $a \to \ell^a_t(x)$ is absolutely continuous on $\mathbb{R}$, and the function $(t,a) \to \partial_a \ell^a_{t,x}$ is a.s. in $L^2\left([0,T] \times \mathbb{R}\right)$ for all $T > 0$.

Thus, heuristically, if the function f in the drift term of (8.1) above is sufficiently regular, so that the second term in the right-hand side of (8.2) is sufficiently smooth, the process $u(\cdot, x)$ should inherit the existence of local times from v. The following perturbation lemma, which is a slight variation of Theorem (12.1) in [GH80], makes this idea rigorous.

Lemma 8.2.1. *Let $(X_t)_{t\geq 0}$ be a real-valued stochastic process admitting a bi-continuous family $(\ell_t^a)_{t\geq 0, a\in\mathbb{R}}$ of local times such that, a.s., for all $t > 0$, $a \to \ell_t^a$ is absolutely continuous on $\mathbb{R}$, and the function $(t,a) \to \partial_a \ell_t^a$ is in $L^2([0,T]\times\mathbb{R})$ for all $T > 0$. Then, for all $h \in W^{1,1}(\mathbb{R})$, $T > 0$ and $\varphi \in C_b(\mathbb{R})$, we have*

$$\int_0^T \varphi(X_s + h(s))ds = \int_{\mathbb{R}} \varphi(b) \left(\ell_T^{b-h(T)} + \int_0^T h'(s)\partial_b \ell_s^{b-h(s)} ds \right) db$$

In particular, the process $(X_t + h(t))_t$ admits $(\tilde{\ell}_t^b)_{t\geq 0, b\in\mathbb{R}}$ as a family of local times, where

$$\tilde{\ell}_t^b := \ell_t^{b-h(t)} + \int_0^t h'(s)\partial_b \ell_s^{b-h(s)} \,\mathrm{d}s.$$

Proof. By density it suffices to prove the above formula for $\varphi \in C_b^1(\mathbb{R})$. For such a φ, we have

$$\begin{aligned}\int_0^T \varphi(X_s + h(s))ds &= \int_{\mathbb{R}} \int_0^T \varphi(a + h(s))\,\mathrm{d}\ell_s^a\,\mathrm{d}a \\ &= \int_{\mathbb{R}} \left(\varphi(a + h(T))\ell_T^a - \int_0^T \ell_s^a\, \varphi'(a + h(s))\, h'(s)\,\mathrm{d}s \right) \mathrm{d}a\end{aligned}$$

where we performed an integration by parts w.r.t. s to obtain the second line. The right-hand side can be split into the difference of two integrals, the first of which we rewrite as

$$\int_{\mathbb{R}} \varphi(b)\, \ell_T^{b-h(T)}\, \mathrm{d}b$$

by doing the change of variable $b := a + h(T)$. We now treat the second integral. By the occupation times formula and the bi-continuity of ℓ, there exists $M(T) > 0$ such that, for all $t \in [0,T]$, $a \mapsto \ell_t^a$ has support in $[-M(T), M(T)]$, so that, in particular, ℓ_t^a is bounded uniformly on $(t,a) \in [0,T]\times\mathbb{R}$ by $C(T) := \sup\{\ell_t^a,\ 0 \leq t \leq T\,,\ |a| \leq M(T)\}$. Therefore

$$\int_{\mathbb{R}}\int_0^T \ell_s^a |\varphi'(a + h(s))||h'(s)|ds\ da \leq 2M(T)C(T)\|\varphi'\|_\infty \|h'\|_{L^1} < \infty$$

Hence we can apply Fubini's theorem to obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_0^T \ell_s^a \varphi'(a + h(s))h'(s)\,\mathrm{d}s\ \ \mathrm{d}a &= \int_0^T h'(s) \left(\int_{\mathbb{R}} \ell_s^a\, \varphi'(a + h(s))\,\mathrm{d}a \right) \mathrm{d}s \\ &= \int_0^T h'(s) \left(\int_{\mathbb{R}} \ell_s^{b-h(s)} \varphi'(b)\,\mathrm{d}b \right) \mathrm{d}s\end{aligned}$$

Integrating by parts in b, we see that the latter equals

$$-\int_0^T h'(s)\left(\int_{\mathbb{R}} \partial_b \ell_s^{b-h(s)} \varphi(b)\,\mathrm{d}b\right)\,\mathrm{d}s. \tag{8.3}$$

Now, by our assumption on $\partial_a \ell$ and the above remark on the support of the process ℓ, we have

$$\int_0^T \int_{\mathbb{R}} |h'(s)|\,|\partial_b \ell_s^{b-h(s)}|\,|\varphi(b)|\,\mathrm{d}b\ \mathrm{d}s = \int_0^T \int_{-M(T)}^{M(t)} |h'(s)|\,|\partial_a \ell_s^a|\,|\varphi(a+h(s))|\,\mathrm{d}a\ \mathrm{d}s$$
$$\leq \sqrt{\int_0^T \int_{-M(T)}^{M(t)} \left(h'(s)\,\varphi(a+h(s))\right)^2\,\mathrm{d}a\ \mathrm{d}s}\ \sqrt{\int_0^T \int_{\mathbb{R}} |\partial_a \ell_s^a|^2\,\mathrm{d}a\ \mathrm{d}s} < \infty$$

Therefore, we can apply Fubini again to deduce that (8.3) equals

$$-\int_{\mathbb{R}} \varphi(b) \int_0^T h'(s)\,\partial_b \ell_s^{b-h(s)}\ \mathrm{d}b\ \mathrm{d}s.$$

Finally, we thus obtain

$$\int_0^T \varphi(X_s + h(s))\,\mathrm{d}s = \int_{\mathbb{R}} \varphi(b) \ell_T^{b-h(T)}\ \mathrm{d}b + \int_{\mathbb{R}} \varphi(b) \left(\int_0^T h'(s)\,\partial_b \ell_s^{b-h(s)}\,\mathrm{d}s\right)\,\mathrm{d}b,$$

which yields the claim.

□

The perturbation result above would thus not only allow to prove the existence of local times for SPDEs such as (8.1), but also to express them in terms of local times of the (SHE). In order to complete this program, there would remain to find appropriate conditions on f ensuring that Lemma (8.2.1) does indeed apply. This is a future possible research direction.

8.3 A more probabilistic approach

In the above section, we considered the very specific case of an SPDE with additive noise. Here, we consider more general SPDEs, with multiplicative noise:

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + f(u)(t,x) + g(u)(t,x) W(t,x) \tag{8.4}$$

for $(t,x) \in \mathbb{R}_+ \times [0,1]$, where $f : (t,x,r) \mapsto f(t,x,r)$ and $g : (t,x,r) \mapsto g(t,x,r)$ are Borel functions $\mathbb{R}_+ \times [0,1] \times \mathbb{R} \to \mathbb{R}$. We assume that:

1. for all $(t,x) \in \mathbb{R}_+ \times [0,1]$, the functions $f(t,x,\cdot)$ and $g(t,x,\cdot)$ are in $W^{2,\infty}(\mathbb{R})$,

2. the functions $f, \frac{\partial f}{\partial r}, \frac{\partial^2 f}{\partial^2 r}, |g|^{-1}, g, \frac{\partial g}{\partial r}, \frac{\partial^2 g}{\partial^2 r}$ are bounded on $\mathbb{R}_+ \times [0,1] \times \mathbb{R}$.

Well-posedness results for SPDEs of the type (8.4) were proved in [GP92].

Let us consider a solution u. By p. 499 in [BGP94], for all $(t,x) \in \mathbb{R}_+ \times (0,1)$, the random variable $u(t,x)$ admits a density $p_{t,x}(y)$ w.r.t. Lebesgue measure $\mathrm{d}y$ on $\mathbb{R}$, and the following bound holds

$$\forall \alpha \in (0,1), \forall \epsilon > 0, \qquad p_{t,x}(y) \leq K(f,g)\left(1 + t^{-\left(\epsilon + \frac{1+\alpha}{4}\right)}\right)(x \wedge (1-x))^{-\left(\epsilon + \frac{2-\alpha}{2}\right)},$$

where $K(f,g) > 0$ is a constant independent of t, x and y. Note that, in the estimate above, we can choose the exponent $\beta := \epsilon + \frac{1+\alpha}{4}$ so that $\beta \in (1/4, 1/4+\delta)$, for any $\delta > 0$ arbitrarily small ($\delta = 3/4$ will suffice for our purpose). In particular, setting $A(x) := (x \wedge (1-x))^{-\left(\epsilon + \frac{2-\alpha}{2}\right)}$ for all $x \in (0,1)$, we deduce that

$$\mathbb{P}\left(|a - u(r,x)| \leq \epsilon\right) \leq 2\epsilon\ K(f,g) A(x)\left(1 + r^{-\beta}\right)$$

for all $\epsilon > 0$, $a \in \mathbb{R}$ and $r > 0$. Now, let us denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration given by

$$\mathcal{F}_t := \sigma\left(\{u_s,\ s \leq t\}\right), \qquad t \geq 0.$$

Moreover, for all $u_0 \in C([0,1])$, we denote by $\mathbb{P}_{u_0}$ the law, on $C(\mathbb{R}_+, C([0,1]))$, of the solution to (8.4) started from u_0. Then, for all $0 \leq s < t$ and $\epsilon > 0$, making use of the Markov property for the process u, we have

$$\begin{aligned}
\mathbb{P}\left(|u(s,x) - u(t,x)| \leq \epsilon\right) &= \mathbb{E}\left(\mathbb{P}\left(|u(s,x) - u(t,x)| \leq \epsilon | \mathcal{F}_s\right)\right) \\
&= \mathbb{E}\left(\mathbb{P}_{u(s,\cdot)}\left(|u(0,x) - u(t-s,x)| \leq \epsilon\right)\right) \\
&\leq 2\epsilon\ K(f,g) A(x)\left(1 + |t-s|^{-\beta}\right)
\end{aligned}$$

Therefore, we deduce that, for all $T > 0$

$$\liminf_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^T \int_0^T \mathbb{P}\left(|u(s,x) - u(t,x)| \leq \epsilon\right) ds\ dt \lesssim \int_0^T \int_0^T \left(1 + |t-s|^{-\beta}\right) ds\ dt$$

and the right-hand side is finite for $\beta \in (1/4, 1)$. Therefore, by Theorem (21.15) in [GH80], there exists a local time process $\left(\ell^a_{t,x}\right)_{t \geq 0, a \in \mathbb{R}}$ such that $\left(\ell^a_{t,x}\right)_{a \in \mathbb{R}} \in L^2\left(\mathrm{d}a \times \mathbb{P}\right)$ for all $t > 0$ (where $\mathrm{d}a$ stands for the Lebesgue measure on $\mathbb{R}$).

8.4 Comparison of the two methods

Note that the method proposed in Section 8.3 enables to treat the case of a multiplicative noise, so a fortiori the case of an additive noise. In a sense, it is thus stronger than the perturbation technique used in Section 8.2, but it is less explicit and pathwise in spirit. On the other hand, the former method, which seems more restrictive, is closer in spirit to pathwise theories such as rough paths or regularity structures. We finally stress the existence of a new technique proposed in [Lê18] for the construction of local times of Markov processes, and which mixes probabilistic estimates with rough path techniques. It may be interesting to try to apply such techniques in the context of SPDEs as above.

Chapter 9

Application to the scaling limit of dynamical critical pinning models

In this chapter, we consider several critical wetting models, in the discrete as well as the continuum. These probability laws are known to converge (after an appropriate rescaling in the discrete case) to the law of a reflecting Brownian motion (or of the modulus of a Brownian bridge, according to the boundary conditions). On the other hand, to these laws, one can associate reversible Markov processes, the dynamics of which are encoded by integration by parts formulae. One shows, in the discrete case, that the associated reversible dynamics are tight. One provides a conjecture on the limiting process, which we believe to satisfy an SPDE of the form (1.25), or at least its weaker version (1.28). This chapter is based on joint work with Jean-Dominique Deuschel and Tal Orenshtein.

9.1 The δ-pinning case

We first recall the model considered in [DGZ05]. For all $N \geq 1$ and $\beta \in \mathbb{R}$, we consider the measure $\mathbb{P}^f_{\beta,N}$ on $\mathbb{R}^N_+$ defined by

$$\mathbb{P}^f_{\beta,N}(\mathrm{d}\phi) = \frac{1}{Z^f_{\beta,N}} \rho\left(\phi\right) \prod_{i=1}^{N} \left(\mathrm{d}\phi_i \mathbf{1}_{[0,\infty)} + e^{\beta} \delta_0(\mathrm{d}\phi_i) \right),$$

where

$$\rho(\phi) = \exp\left(- \sum_{i=0}^{N-1} V(\phi_{i+1} - \phi_i) \right).$$

with $\phi(0) := 0$, and where $V : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is such that $\exp(-V)$ is continuous, $V(0) < \infty$, and

$$\int_{\mathbb{R}} e^{-V} < \infty.$$

Here, for simplicity, we shall consider $V(x) = \frac{x^2}{2}$ for all $x \in \mathbb{R}$. Above, the superscript f stands for "free", meaning that we do not constrain the value of x_N

We also recall the main theorem in [DGZ05]. Let $H := L^2([0,1])$. For all $N \geq 1$, we let $\Phi_N : \mathbb{R}^N \to H$ denote the rescaling and interpolation map defined by

$$\Phi_N(\phi)(y) = \frac{1}{\sqrt{N}} \phi_{\lfloor Ny \rfloor} + \frac{1}{\sqrt{N}} (Ny - \lfloor Ny \rfloor) \left(\phi_{\lfloor Ny \rfloor + 1} - \phi_{\lfloor Ny \rfloor} \right), \quad y \in [0,1], \quad (9.1)$$

where we use the convention that $\phi_0 := 0$ in the right-hand side above. We will denote by H_N the vector space $\Phi_N(\mathbb{R}^N) \subset H$, which coincides with the space of continuous piecewise affine functions adapted to the partition $[\frac{i-1}{N}, \frac{i}{N}), 1 \leq i \leq N$. We finally denote by $\mathbf{P}^f_{\beta,N}$ the image of the measure $\mathbb{P}^f_{\beta,N}$ under Φ_N. We then have the following:

Theorem 9.1.1. *There exists $\beta_c \in (0, \infty)$ such that:*

- *if $\beta < \beta_c$ (subcritical case), then $\mathbf{P}^f_{\beta,N} \underset{N\to\infty}{\longrightarrow} m$ in law, where m is the law of a Brownian meander on $[0,1]$*
- *if $\beta = \beta_c$ (critical case), then $\mathbf{P}^f_{\beta,N} \underset{N\to\infty}{\longrightarrow} P^1_0$ in law, where we recall that P^1_0 is the law of a reflecting Brownian motion started from 0 on $[0,1]$*
- *if $\beta > \beta_c$ (supercritical case), then $\mathbf{P}^f_{\beta,N}$ converges in law, as $N \to \infty$, to the measure concentrated on the function identically equal to 0 on $[0,1]$.*

One can ask whether it is possible to build a Markov process on $\mathbb{R}^N_+$ admitting the measure $\mathbb{P}^f_{\beta,N}$ as a reversible measure. Due to the presence of Dirac masses in the definition of $\mathbb{P}^f_{\beta,N}$, this problem is highly non-trivial. For instance, in the particular case $N = 1$, a natural candidate is given by a sticky reflecting Brownian motion, which is a solution to

$$\begin{cases} dX_t = \frac{1}{2} \, \mathrm{d}\ell^0_t + \mathbf{1}_{\{X_s > 0\}} \, \mathrm{d}B_s \\ \\ \mathbf{1}_{\{X_t = 0\}} \, \mathrm{d}t = \frac{e^{-\beta}}{2} \, \mathrm{d}\ell^0_t. \end{cases}$$

This stochastic equation is known to possess a weak solution, but no strong solutions (see e.g. [EP14]). In [FGV16] and [GV18] some diffusions having $\mathbb{P}^f_{\beta,N}$ as a reversible measure were constructed and studied using sophisticated Dirichlet form methods.

9.2 Case of the wetting model with a shrinking strip

[DO18] introduced a variant of the above wetting model. Namely, for all $N \geq 1$, and $a > 0$, one considers the measure $\mathbb{P}^f_{\varphi_a,N}$ on $\mathbb{R}^N_+$ defined by

$$\mathbb{P}^f_{\varphi_a,N}(\mathrm{d}\phi) = \frac{1}{Z^f_{\varphi_a,N}} \rho(\phi) \prod_{i=1}^N e^{\varphi_a(\phi_i)} \, \mathrm{d}\phi_i \,,$$

where ρ is as before, and where, for all $a > 0$, $\varphi_a : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function supported in $[0, a]$, satisfying the conditions of Def. 1.1 in [DO18]. Thus, the measure $\mathbb{P}^f_{\beta,N}$ above, which had some atoms, has been replaced by a measure which is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^N_+$. With this new version with a "strip", one can however recover a scaling limit result as in the critical regime above.

Let us denote by $\mathbf{P}^f_{\varphi_a,N}$ the image of $\mathbb{P}^f_{\varphi_a,N}$ under the map Φ_N. Then, [DO18] showed - see Theorem 1.5 therein - that if we choose $(a_N)_{N\geq 1}$ such that $a_N = o(N^{-1/2})$, then $\mathbf{P}^f_{\varphi_{a_N},N}$ converges in law, as $N \to \infty$, to the law of a reflecting Brownian motion on $[0, 1]$. Here and in the sequel, we fix such a sequence $(a_N)_{N\geq 1}$, and we write for concision $\mathbb{P}^f_N$ instead of $\mathbb{P}^f_{\varphi_a,N}$, and $\mathbf{P}^f_N$ instead of $\mathbf{P}^f_{\varphi_{a_N},N}$, for all $N \geq 1$. Our aim is to show a tightness result for the dynamics associated with $\mathbf{P}^f_N$, $N \geq 1$.

The advantage of this new model with respect to the δ-pinning measure is the fact that $\mathbb{P}^f_N$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^N_+$, so it is traightforward to construct an associated reversible Markov process. It suffices indeed to consider the corresponding gradient SDE

$$\begin{cases} X^i_t = -\displaystyle\int_0^t \partial_i H_N(X(s)) + \ell^i_t + \sqrt{2} W^i_t, \quad i = 1, \ldots, N \\ X^i_t \geq 0, \ \mathrm{d}\ell^i_t \geq 0, \ \int_0^\infty X^i_t \, \mathrm{d}\ell^i_t = 0, \end{cases} \tag{9.2}$$

with a random initial condition X_0 distributed as $\mathbb{P}^f_N$. Above, we have denoted by $H_N : \mathbb{R}^N_+ \to \mathbb{R}$ the potential defined by:

$$e^{-H_N(\phi)} = \rho(\phi) \prod_{i=1}^N e^{\varphi_a(\phi_i)}, \quad \phi \in \mathbb{R}^N_+.$$

It was conjectured in [DO18] that the family of processes X^N properly rescaled is tight. Here we prove this claim.

9.2.1 A tightness result

In Section 1.5 of [DO18], the authors considered the processes $(X_t^N)_{t\geq 0}, N \geq 1$, where $X_t^N = \Phi_N(X_t)$, with $(X_t)_{t\geq 0}$ the reversible evolution in $\mathbb{R}_+^N$ for the pinning measure $\mathbb{P}^f_{\varphi_{a_N},N}$ as given by (9.2), and Φ_N as in (9.1) above. For all $N \geq 1$ and $t \geq 0$, let $Y_t^N := X^N_{N^2 t}$. For all $\gamma > 0$, we introduce the space $H^{-\gamma}(0,1)$, completion of $H = L^2(0,1)$ w.r.t. the norm $\|\cdot\|_{-\gamma}$ defined by

$$\|f\|^2_{-\gamma} := \sum_{n=1}^{\infty} n^{-2\gamma} |\langle f, e_n\rangle|^2, \qquad f \in H,$$

where $e_n(\theta) := \sqrt{2}\sin(n\pi\theta), \qquad \theta \in [0,1]$.

Theorem 9.2.1. *For all $T > 0$, the family of processes $(Y_t^N)_{t\in[0,T]}$, $N \geq 1$, is tight in $C([0,T], H^{-1}(0,1))$.*

Proof. Let us denote by $(x_N^i)_{1\leq i\leq N}$ the image of the canonical basis of $\mathbb{R}^N$ under Φ_N. It then follows that, for all $N \geq 1$, the process $\left(Y_t^N\right)_{t\geq 0}$ coincides in law with the reversible Markov process associated with the Dirichlet form $(\mathrm{Dom}(\mathcal{E}_N^f), \mathcal{E}_N^f)$ which is the closure of the form

$$\mathcal{E}_N^f(u,v) = N^2 \int_{K_N} \sum_{i=1}^{N} \langle \nabla u(x), x_N^i\rangle \langle \nabla v(x), x_N^i\rangle \, \mathrm{d}\mathbf{P}_N^f(x), \quad u, v \in C_b^1(H_N),$$

where $K_N := \{u \in H_N, u \geq 0\}$, and where we recall that $\mathbf{P}_N^f$ is the image of the measure $\mathbb{P}_N^f$ under Φ_N. Then, for all $T > 0$ and $h \in H$, by the Lyons-Zheng decomposition, see e.g. Thm 5.7.1 in [FOT10], we have

$$\langle Y_t^N, h\rangle - \langle Y_0^N, h\rangle = \frac{1}{2} M_t^1 - \frac{1}{2}\left(M_T^2 - M_{T-t}^2\right),$$

where, for $i = 1, 2$, M^i is an H_N-valued $(\mathcal{F}_t^i)_{t\geq 0}$ martingale, with $\mathcal{F}_t^1 = \sigma(Y_s^N, s \leq t)$ and $\mathcal{F}_t^2 = \sigma(Y_{T-s}^N, s \leq t)$. More precisely, defining $\psi \in C_b^1(H_N)$ by $\psi := \langle h, \cdot\rangle$, by Theorem 5.7.1 in [FOT10], we have the above decomposition with

$$M_t^1 := M_t^{[\psi]}, \quad M_t^2(\omega) := M_t^{[\psi]}(r_T\omega),$$

where r_T is the time-reversing operator on the canonical space $\Omega := C([0,T], K_N)$:

$$(r_T\omega)_t = \omega_{T-t}, \qquad \omega \in \Omega,\ t \in [0,T].$$

Moreover, $M^{[\psi]}$ denotes the martingale additive functional appearing in the Fukushima decomposition of the continuous additive functional (CAF) given by

$$A_t^{[\psi]} := \psi(Y_t^N) - \psi(Y_0^N) = \langle Y_t^N, h\rangle - \langle Y_0^N, h\rangle, \quad t \geq 0,$$

see Theorem 5.2.2 in [FOT10]. Hence, the quadratic variation of the martingales M^1 and M^2 is given by the sharp bracket $\langle M^{[\psi]}\rangle_t$ of the martingale additive functional $M^{[\psi]}$. The latter is a positive continuous additive functional with Revuz measure μ satisfying

$$\int_{K_N} g(x)\,\mathrm{d}\mu(x) = 2\mathcal{E}_N^f(\psi g,\psi) - \mathcal{E}_N^f(\psi^2, g),$$

for all $g \in \mathrm{Dom}(\mathcal{E}_N^f)$, see Thm 5.2.3 in [FOT10]. But, for all $g \in C_b^1(H_N)$, by the Leibniz rule, and recalling that $\nabla\varphi = h$, we have

$$2\,\mathcal{E}_N^f(\psi g,\psi) - \mathcal{E}_N^f(\psi^2, g) = 2N^2 \int_K g(x) \sum_{k=1}^N \langle h, x_i^N\rangle^2\,\mathbf{P}_N^f(\mathrm{d}x).$$

Therefore, for all $g \in C_b^1(H_N)$, it holds

$$\int_K g(x)\mu(\mathrm{d}x) = \int_K g(x) \left(2N^2 \sum_{k=1}^N \langle h, x_i^N\rangle^2\right) \mathbf{P}^f_{\varphi_a,N}(\mathrm{d}x).$$

Therefore, we deduce that

$$\mu(\mathrm{d}x) = \left(2N^2 \sum_{k=1}^N \langle h, x_i^N\rangle^2\right) \mathbf{P}_N^f(\mathrm{d}x).$$

Hence, by the Revuz correspondence, we deduce the equality

$$\langle M^{[\psi]}\rangle_t = \left(2N^2 \sum_{k=1}^N \langle h, x_i^N\rangle^2\right) t, \quad t \geq 0$$

in the sense of additive functionals, which implies that, for all $i = 1, 2$

$$\langle M^i\rangle_t = 2N^2 \sum_{k=1}^N \langle h, x_i^N\rangle^2\, t, \quad t \geq 0. \tag{9.3}$$

But recall that, for all $k = 1,\ldots,N$, $x_k^N = \Phi_N(e_k)$, where $(e_1,\ldots,e_N)$ is the canonical basis of $\mathbb{R}^N$. In words, x_k^N is the function in H_N which takes the value $\frac{1}{\sqrt{N}}$ at the point k/N, and the value 0 at the points j/N, $j \neq k$. Note in particular that

$$0 \leq x_k^N \leq \frac{1}{\sqrt{N}} \mathbf{1}_{[\frac{k-1}{N},\frac{k+1}{N}]},$$

so that we have the bound $\|x_k^N\|^2 \leq 2N^{-2}$. Using the Cauchy-Schwarz inequality followed by the latter bound in (9.3), we obtain

$$\langle M^i \rangle_t \leq 4\|h\|^2\, t.$$

Hence, by the BDG inequality, for all $p \geq 1$, there exists a constant $C_p > 0$ (depending only on p) such that, for all $t \geq s \geq 0$

$$\left(\mathbb{E}\left[\langle Y_t^N - Y_s^N, h\rangle^p\right]\right)^{1/p} \leq C_p (t-s)^{1/2}\,\|h\|$$

The above being true for any $h \in H$, we deduce that, for all $p \geq 2$

$$\begin{aligned}
\left(\mathbb{E}\left[\|Y_t^N - Y_s^N\|^p_{H^{-1}(0,1)}\right]\right)^{1/p} &= \left(\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\langle Y_t^N - Y_s^N, e_k\rangle^2 k^{-2}\right)^{p/2}\right]\right)^{1/p} \\
&\leq \zeta(2)^{\frac{1}{2}-\frac{1}{p}}\left(\mathbb{E}\left[\sum_{k=1}^{\infty}\langle Y_t^N - Y_s^N, e_k\rangle^p k^{-2}\right]\right)^{1/p} \\
&\leq \zeta(2)^{1/2}\, C_p\,(t-s)^{1/2} \\
&\leq C_p'(t-s)^{1/2},
\end{aligned}$$

where $C_p' > 0$ depends only on $p > 0$, and where we applied Jensen's inequality to obtain the second line. We have thus obtained, for all $p \geq 2$, the following bound holding uniformly in $t, s \in [0, T]$:

$$\left(\mathbb{E}\left[\|Y_t^N - Y_s^N\|^p_{H^{-1}(0,1)}\right]\right)^{1/p} \leq C_p'\,|t-s|^{1/2}.$$

Moroever, for all $t \geq 0$, $Y_t^N \overset{(d)}{=} \mathbf{P}_N^f$ and, by Theorem 1.5 in [DO18], $\mathbf{P}_N^f \underset{N\to\infty}{\longrightarrow} P_0^1$ in law, where P_0^1 is the law of a reflecting Brownian bridge started from 0 on $[0,1]$. Hence the sequence of processes $(Y_t^N)_{t\in[0,T]}, N \geq 1$, is tight in $C([0,T], H^{-1}(0,1))$. □

9.2.2 An integration by parts formula

For any differentiable function $g : \mathbb{R}_+^N \to \mathbb{R}$ and any $h \in \mathbb{R}^N$, we denote by $\partial_h g$ the derivative of g in the direction h:

$$\partial_h g(\phi) := \lim_{\epsilon\to 0}\frac{g(\phi+\epsilon h) - g(\phi)}{\epsilon}, \quad \phi \in \mathbb{R}_+^N.$$

We then have the following IbPF for the measure $\mathbb{P}^f_{\varphi_a,N}$ on $\mathbb{R}_+^N$:

Proposition 9.2.2. *For all $g \in C_b^1(\mathbb{R}_+^N)$ and $h \in \mathbb{R}^N$, we have:*

$$\int_{\mathbb{R}_+^N} \partial_h g(\phi)\, \mathbb{P}^f_{\varphi_a,N}(\mathrm{d}x) = \sum_{i=1}^N h_i \left(\int_0^a e^{\varphi_a(b)} \frac{\mathrm{d}}{\mathrm{d}b} \sigma_i^N(g|b)\, \mathrm{d}b - \sigma_i^N(g|a) \right),$$
$$- \int_{\mathbb{R}_+^N} g(\phi) \sum_{i=1}^N \phi_i\ (h_{i+1} + h_{i-1} - 2h_i)\ \mathbb{P}^f_{\varphi_a,N}(\mathrm{d}\phi).$$

where, for all $b \geq 0$ and $i = 1, \dots, N$

$$\sigma_i^N(g|b) := e^{-\varphi_a(b)}\, \mathbb{P}^f_{\varphi_a,N}(x_i \in \mathrm{d}b) \int_{\mathbb{R}_+^N} g(x)\, \mathbb{P}^f_{\varphi_a,N}(\,\mathrm{d}\phi|\phi_i = b)$$
$$= \frac{1}{Z^f_{\varphi_a,N}} \int_{\mathbb{R}_+^N} g(\phi)\rho(\phi)\delta_b(\mathrm{d}\phi_i) \prod_{n \neq i} e^{\varphi_a(\phi_n)}\, \mathrm{d}\phi_n.$$

Heuristically, the measures $\sigma_i^N(\,\mathrm{d}\phi|b)$ are meant to be a discrete analog of the measures $\Sigma_0^{1,r}(\mathrm{d}X|b)$, $r \in (0,1), b \geq 0$ defined by (3.17) above.

Proof. Recalling the definition of $\mathbb{P}^f_{\varphi_a,N}$, and integrating by parts with respect to the Lebesgue measure on $\mathbb{R}_+^N$, we have

$$\int_{\mathbb{R}_+^N} \partial_h g(\phi)\, \mathbb{P}^f_{\varphi_a,N}(\mathrm{d}\phi) = -\frac{1}{Z^f_{\varphi_a,N}} \int_{\mathbb{R}_+^N} g(\phi)\, \partial_h \rho(\phi) \prod_{n=1}^N e^{\varphi_a(\phi_n)}\, \mathrm{d}\phi_n$$
$$- \sum_{i=1}^N h_i \frac{1}{Z^f_{\varphi_a,N}} \int_{\mathbb{R}_+^N} g(\phi)\rho(\phi) e^{\varphi_a(0)} \delta_0(\mathrm{d}\phi_i) \prod_{\substack{n=1 \\ n\neq i}}^N e^{\varphi_a(\phi_n)}\, \mathrm{d}\phi_n$$
$$- \sum_{i=1}^N h_i \frac{1}{Z^f_{\varphi_a,N}} \int_{\mathbb{R}_+^N} g(\phi)\rho(\phi)\varphi_a'(\phi_i) \prod_{n=1}^N e^{\varphi_a(\phi_n)}\, \mathrm{d}x_n.$$

We recognize in the first term of the right-hand side above the quantity

$$- \int_{\mathbb{R}_+^N} g(\phi) \sum_{i=1}^N (h_{i+1} + h_{i-1} - 2h_i)\, \phi_i\, \mathbb{P}^f_{\varphi_a,N}(\mathrm{d}\phi)$$

On the other hand, the second term can be rewritten

$$- \sum_{i=1}^N h_i\, e^{\varphi_a(0)}\, \sigma_i^N(g|0).$$

Finally, the third term can be rewritten

$$-\sum_{i=1}^{N} h_i \int_0^a \frac{d}{db}\left(e^{\varphi_a(b)}\right) \sigma_i^N(g|b)\, \mathrm{d}b,$$

or, after an integration by parts:

$$-\sum_{i=1}^{N} h_i \left\{ \sigma_i^N(g|a) - e^{\varphi_a(0)}\, \sigma_i^N(g|0) - \int_0^a e^{\varphi_a(b)} \frac{d}{db}\left(\sigma_i^N(g|b)\right)\, \mathrm{d}b \right\}.$$

Adding up all three quantities, and noting the cancellation of

$$\sum_{i=1}^{N} h_i\, e^{\varphi_a(0)}\, \sigma_i^N(g|0),$$

we obtain the claim. □

9.2.3 Conjecture for the scaling limit

Above we have shown the tightness of the family of processes $(Y_t^N)_{t\in[0,T]}$, $N \geq 1$. We make the following conjecture for the corresponding limit in law. Let us denote by $(\tilde{u}_t)_{t\geq 0}$ the reversible Markov process associated with the Dirichlet form $\tilde{\mathcal{E}}$ generated by the bilinear form

$$\tilde{\mathcal{E}}(f,g) := \frac{1}{2}\int \langle \nabla f, \nabla g\rangle\, \mathrm{d}\tilde{\nu}, \qquad f, g \in \mathcal{FC}_b^\infty(K),$$

where $\tilde{\nu}$ denotes the law, on K, of a reflecting Brownian motion on $[0,1]$. Such a process can be constructed using exactly the same techniques as in Section 4.2. We consider the process $(\tilde{u}_t)_{t\geq 0}$ started from equilibrium, i.e. $\tilde{u}_0 \overset{(d)}{=} \tilde{\nu}$. Note that, arguing as in Theorem 4.2.8 of Section 4.2, one can show that $(\tilde{u}_t)_{t\geq 0}$ satisfies an equation of the form (1.28).

Conjecture 9.2.3. *For all $T > 0$, as $N \to \infty$, $(Y_t^N)_{t\in[0,T]}$ converges in law in $C([0,T], H^{-1}(0,1))$ to the process $(\tilde{u}_t)_{t\in[0,T]}$.*

A natural route to prove the above conjecture would be to show that, after rescaling, the IbPF for $\mathbb{P}_N^f$ obtained in Prop. 9.2.2 above converges to the IbPF (3.21) for the law of the reflecting Brownian motion, and use the same techniques as in [Zam04a] to deduce therefrom the convergence of the associated evolutions. Unfortunately, in spite of some similarities between the two IbPF, it is not clear at all that the former converges to the latter. Another problem that arises is related, again, with the distributional nature of the last term appearing in these IbPF. Finally, an important feature exploited in [Zam04a] is the uniform continuity of the Markov semi-groups. In our case, this feature is highly non-trivial, and would in particular imply the strong Feller property for the Markov semigroup associated with $(\tilde{u}_t)_{t\geq 0}$: as argued in Chapter 7, this is a very open problem.

9.3 A wetting model in the continuum

In this section we introduce an analog of the wetting model in the continuum, which corresponds to the law of a Brownian meander tilted by a functional of its local times.

9.3.1 Motivation: wetting model and local times

Recall that, in the discrete setting described above, for the case of a wetting model with a strip, we have

$$\mathbb{P}^f_{\varphi_a,N}(\mathrm{d}\phi) = \frac{1}{Z^+_{\varphi_a,N}} \exp\left(\sum_{i=1}^N \varphi_a(\phi_i)\right) \mathbb{P}^+_N(\mathrm{d}\phi),$$

where $\mathbb{P}^+_N$ is the law of a standard Gaussian walk on $\mathbb{R}^N$ conditioned to remain nonnegative, and $Z^+_{\varphi_a,N}$ is a normalisation constant. For all $a > 0$, φ_a is a smooth function. For the sake of simplicity, let us however assume here the simple expression

$$\varphi_a = \beta_a \mathbf{1}_{[0,a]}, \tag{9.4}$$

where the sequence $(\beta_a)_{a>0}$ is such that

$$a e^{\beta_a} \underset{a\to 0}{\longrightarrow} e^{\beta_c}.$$

Note that this condition ensures that the following convergence of measures holds in the weak sense on $\mathbb{R}_+$:

$$e^{\varphi_a(x)}\,\mathrm{d}x \underset{a\to 0}{\longrightarrow} e^{\beta_c}\delta_0(\mathrm{d}x) + \mathrm{d}x.$$

With the ansatz (9.4), we can rewrite the wetting measure with strip as

$$\mathbb{P}^f_{\varphi_a,N}(\mathrm{d}\phi) = \frac{1}{Z^+_{\varphi_a,N}} \exp\left(\beta_{a_N} \sum_{i=1}^N \mathbf{1}_{[0,a]}(\phi_i)\right) \mathbb{P}^+_N(\mathrm{d}\phi).$$

Thus, the wetting measure corresponds to the measure $\mathbb{P}^+_N$ tilted by the local time in the strip $[0, a]$ of the random walk. We could hope that such a description be stable under taking the scaling limit : in the continuum, the law P_0^1 of a reflecting Brownian motion started from 0 would correspond to the law m of a Brownian meander tilted by some appropriate functional of its continuous local time process. As such, this claim is false, since the probability measures P_0^1 is not absolutely continuous with respect to m. However, me may ask whether

$$P_0^1(\mathrm{d}X) = \lim_{\eta \to 0} \exp(\Phi_\eta(L(X)))\, m(\mathrm{d}X), \tag{9.5}$$

where $L(X) = \left(L_t^b(X)\right)_{b \geq 0, t \geq 0}$ is the local time process associated with $(X_t)_{0 \leq t \leq 1}$, when $X \overset{(d)}{=} m$, and $\Phi_\eta, \eta > 0$, are appropriate functionals on $C([0,1] \times \mathbb{R}_+, \mathbb{R})$. Note that we could not hope to choose Φ_η to depend solely on $L^0(X)$, the local time at 0 of X, since this identically vanishes when $X \overset{(d)}{=} m$: we need instead to make it depend on the process $L^\eta(X)$ for η close to 0.

9.3.2 A continuous wetting model

To obtain a representation of the form (9.5), we shall proceed as follows. Recall that, for all $a \geq 0$, P_a^3 denotes the law of a 3-dimensional Bessel process started from a. Recall that P_0^3 and m are mutually absolutely continuous on $C([0,1], \mathbb{R})$. Moreover, under P_a^3, the canonical process $(X_t)_{0 \leq t \leq 1}$ satisfies the SDE

$$X_t = a + \int_0^t \frac{\mathrm{d}s}{X_s} + B_t.$$

On the other hand, under P_a^1, the canonical process satisfies the equation

$$X_t = a + \frac{1}{2} L^0(X)_t + B_t.$$

Hence, the idea is to approximate the latter equation by an SDE of the form

$$X_t^\eta = a + \int_0^t f_\eta(X_s)\, \mathrm{d}s + B_t,$$

where f_η is a smooth function such that $\int_0^t f_\eta(X_s)\, \mathrm{d}s$ approximates $\frac{1}{2} L^0(X)_t$ as $\eta \to 0$. Then, for any fixed $\eta > 0$, one could apply Girsanov's theorem to obtain

$$P_a^{1,\eta}(\mathrm{d}X) = \exp(\Phi_\eta(L(X)))\, P_a^3(\mathrm{d}X),$$

where $P_a^{1,\eta}$ is the law of X^η, and Φ_η is some appropriate functional. This strategy is implemented in the next theorem.

Theorem 9.3.1. *The convergence* $P_a^{1,\eta} \underset{\eta\to 0}{\longrightarrow} P_a^1$ *holds in the sense of weak topology for probability measures on* $C([0,1])$*, where, for all* $\eta > 0$

$$P_a^{1,\eta}(\mathrm{d}X) = \frac{X_1 \wedge \eta}{X_1} \frac{a}{a \wedge \eta} \exp\left(\frac{1}{2\eta} L_1^\eta\right) P_a^3(\mathrm{d}X).$$

Here, for a 3-Bessel process X*,* $(L_t^b)_{b\geq 0, t\geq 0}$ *denotes the semimartingale local time process associated with* X*. In particular* $P_0^{1,\eta} \underset{\eta\to 0}{\longrightarrow} P_0^1$*, where*

$$P_0^{1,\eta}(\mathrm{d}X) = \sqrt{\frac{2}{\pi}}(X_1 \wedge \eta) \exp\left(\frac{1}{2\eta} L_1^\eta\right) m(\mathrm{d}X),$$

and where m *is the law of a Brownian meander on* $[0,1]$.

Proof. The second claim follows from the first one, since, by Imhof's relation (see Exercise 4.18 in [RY13, ChapterXII]), we have

$$P_0^3(\mathrm{d}X) = \sqrt{\frac{2}{\pi}} X_1 \, m(\mathrm{d}X).$$

So it suffices to prove the first claim. Under P_a^3, the canonical process on $C([0,1])$ satisfies the SDE

$$X_t = a + \int_0^t \frac{\mathrm{d}s}{X_s} + B_t.$$

For all $\eta > 0$, denote by $P_a^{1,\eta}$ the law, on $C([0,1])$, of the unique strong solution to the SDE

$$X_t = a + \int_0^t \frac{\mathbf{1}_{X_s \leq \eta}}{X_s} \, \mathrm{d}s + B_t. \tag{9.6}$$

Note that the latter SDE admits a unique strong solution since the function $x \mapsto \frac{\mathbf{1}_{x\leq\eta}}{x}$ is non-increasing on $\mathbb{R}_+$. We will first prove that

$$P_a^{1,\eta} = \frac{X_1 \wedge \eta}{X_1} \frac{a}{a \wedge \eta} \exp\left(\frac{1}{2\eta} L_1^\eta\right) P_a^3(\mathrm{d}X), \tag{9.7}$$

and then we will show that

$$P_a^1(\mathrm{d}X) = \lim_{\eta\to 0} P_a^{1,\eta}. \tag{9.8}$$

This will yield the claim.

Equality (9.7) is proven using Girsanov's Theorem. Indeed, consider the local martingale

$$M_t = -\int_0^t \frac{\mathbf{1}_{X_s > \eta}}{X_s} dB_s, \quad t \geq 0.$$

The corresponding exponential local martingale is given by

$$\mathscr{E}(M)_t = \exp\left(M_t - \frac{1}{2}\langle M, M\rangle_t\right), \quad t \geq 0.$$

Now, we have

$$M_t - \frac{1}{2}\langle M, M\rangle_t = -\int_0^t \frac{\mathbf{1}_{X_s>\eta}}{X_s}\,\mathrm{d}B_s - \frac{1}{2}\int_0^t \frac{\mathbf{1}_{X_s>\eta}}{X_s^2}\,\mathrm{d}s.$$

We intend to re-express this quantity without stochastic integral. To do so, consider the function $F : \mathbb{R}_+^* \to \mathbb{R}_+$ defined by

$$F(x) := \log\left(\frac{x \wedge \eta}{x}\right), \quad x > 0.$$

F is the difference of two convex functions on $\mathbb{R}_+^*$. Therefore, by Itô-Tanaka's formula, we have

$$F(X_t) = F(a) + \int_0^t F'(X_s)\,\mathrm{d}X_s + \frac{1}{2}\int_{\mathbb{R}} F''(\mathrm{d}x) L_t^x.$$

Since

$$F'(x) = -\frac{\mathbf{1}_{x>\eta}}{x}, \quad x > 0,$$

and

$$F''(\mathrm{d}x) = \frac{\mathbf{1}_{x>\eta}}{x^2}\,\mathrm{d}x - \frac{1}{\eta}\delta_\eta(\mathrm{d}x),$$

under the law P_a^3, the canonical process thus satisfies

$$\log\left(\frac{X_t \wedge \eta}{X_t}\right) = \log\left(\frac{a \wedge \eta}{a}\right) - \int_0^t \frac{\mathbf{1}_{X_s>\eta}}{X_s}\left(\frac{1}{X_s}\,\mathrm{d}s + \mathrm{d}B_s\right) + \frac{1}{2}\int_0^t \frac{\mathbf{1}_{X_s>\eta}}{X_s^2}\,\mathrm{d}s - \frac{1}{2\eta}L_t^\eta,$$

whence we obtain

$$-\int_0^t \frac{\mathbf{1}_{X_s>\eta}}{X_s}\,\mathrm{d}B_s - \frac{1}{2}\int_0^t \frac{\mathbf{1}_{X_s>\eta}}{X_s^2}\,\mathrm{d}s = \log\left(\frac{X_t \wedge \eta}{X_t}\frac{a}{a \wedge \eta}\right) + \frac{1}{2\eta}L_t^\eta.$$

Therefore

$$\mathscr{E}(M)_t = \frac{X_t \wedge \eta}{X_t}\frac{a}{a \wedge \eta}\exp\left(\frac{1}{2\eta}L_t^\eta\right).$$

In particular, for all $t \in [0, 1]$, we have the bound

$$\mathscr{E}(M)_t \leq \frac{a}{a \wedge \eta}\exp\left(\frac{1}{2\eta}L_1^\eta\right),$$

which, since L_1^η has finite exponential moments, shows that $(\mathscr{E}(M)_t)_{0\le t\le 1}$ is a bona fide martingale. Therefore, by Girsanov's theorem, under the probability law $\mathscr{E}(M)_1\, P_a^3$ on $C([0,1])$, the canonical process satisfies (9.6). By weak uniqueness of this SDE, we deduce that equality (9.7) holds.

There remains to establish the convergence (9.8). To do so, for all $\eta > 0$, denote by $(X_t^\eta)_{t\ge 0}$ the unique strong solution of the SDE (9.6). By comparison, a.s., for all $\eta, \bar\eta > 0$, we have $X^\eta \le X^{\bar\eta}$. Since moreover $X^\eta \ge 0$, we deduce the existence of a limiting process $X_t = \lim\limits_{\eta\to 0} \downarrow X_t^\eta$, $t \ge 0$. There remains to identify X. To do so, we set

$$Z_t := X_t^2, \quad t \ge 0$$

and, for all $\eta > 0$

$$Z_t^\eta := (X_t^\eta)^2, \quad t \ge 0.$$

Then, by Itô's lemma

$$Z_t^\eta = a^2 + 2\int_0^t \sqrt{Z_s^\eta}\,\mathrm{d}B_s + 2\int_0^t \mathbf{1}_{Z_s^\eta \le \eta^2} + t. \tag{9.9}$$

From this equation, we deduce that the sequence of probability measures $(P_a^{1,\eta})_{\eta>0}$ is tight on $C([0,1])$. Indeed, by (9.9), for all $0 \le s < t \le 1$

$$\mathbb{E}\left[|Z_t^\eta - Z_s^\eta|^4\right] \le C\left(\left(\int_s^t \mathbb{E}(Z_u^\eta)\,\mathrm{d}u\right)^2 + (t-s)^2\right),$$

where $C > 0$ is some universal constant. Since, by comparison, $Z^\eta \le Z^\infty$, where Z^∞ is a 3-dimensional squared Bessel process, we deduce that

$$\mathbb{E}\left[|Z_t^\eta - Z_s^\eta|^4\right] \le C'(t-s)^2,$$

for some (other) universal constant $C' > 0$, whence

$$\mathbb{E}\left[|X_t^\eta - X_s^\eta|^8\right] \le C'(t-s)^2,$$

and the claimed tightness follows. On the other hand, by the comparison theorem (3.7) in [RY13, Chapter IX], we deduce from (9.9) that, a.s., for all $\eta > 0$, $Z^\eta \ge Z^0$, where Z^0 is the unique strong solution of

$$Z_t^0 = a^2 + 2\int_0^t \sqrt{Z_s^0}\,\mathrm{d}B_s + t.$$

In particular, sending $\eta \to 0$, we deduce that, a.s., $Z \ge Z^0$. Note that Z^0 is a one-dimensional squared Bessel process. Therefore, almost-surely, $Z_t \ge Z_t^0 > 0$ for a.e. $t \ge 0$. Hence, a.s., for a.e. $t \ge 0$

$$\mathbf{1}_{Z_t^\eta \le \eta^2} \underset{\eta\to 0}{\longrightarrow} 0.$$

Hence, by dominated convergence, letting $\eta \to 0$ in (9.9), we deduce that Z satisfies the SDE

$$Z_t = a^2 + 2\int_0^t \sqrt{Z_s}\,\mathrm{d}B_s + t$$

By strong uniqueness of this SDE, we deduce that $Z = Z^0$. Hence, in particular, $X \overset{(d)}{=} P_a^1$. This uniquely determines the limit, as $\eta \to 0$ of the sequence $(P_a^{1,\eta})_{\eta>0}$. The convergence (9.8) follows. □

We now aim at obtaining an integration by parts formula for the probability measures $P_a^{1,\eta}$, for all $\eta > 0$. Let us denote by $E_a^{1,\eta}$ the expectation operator associated with $P_a^{1,\eta}$. We conjecture the following result:

Conjecture 9.3.2. *For all functional $\Phi \in C_b^1(L^2(0,1))$, and all $h \in C_c^2(0,1)$, we have*

$$\begin{aligned} E_a^{1,\eta}(\partial_h \Phi(X)) &= -E_a^{1,\eta}(\langle h'', X\rangle \Phi(X)) \\ &\quad - \frac{1}{2}\int_0^1 \mathrm{d}r h_r \left(\mathbf{\Sigma}_a^{1,\eta,r}(\Phi|0) - \frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}b}\Sigma_a^{1,\eta,r}(\Phi|b)\Big|_{b=\eta} \right), \end{aligned}$$

where, for all $r \in (0,1)$ and $b \geq 0$, the measure $\Sigma_a^{1,\eta,r}(\mathrm{d}X|b)$ on $C([0,1])$ is defined by

$$\Sigma_a^{1,\eta,r}(\mathrm{d}X|b) = p_r^{1,\eta}(a,b) P_a^{1,\eta}(\mathrm{d}X|X_r = b),$$

with $(p_r^{1,\eta}(x,y))_{r\geq 0, x,y\geq 0}$ denoting the family of transition densities of the Markov process associated with $P_a^{1,\eta}$, and where

$$\mathbf{\Sigma}_a^{1,\eta,r}(\Phi|0) := \lim_{b\to 0} \frac{1}{b^2}\Sigma_a^{1,\eta,r}(\Phi|b).$$

Note the similarity of this IbPF with the formula of Proposition 9.2.2. Note also that, for all $b > 0$ $\Sigma_a^{1,\eta,r}(\mathrm{d}X|b)$ should be the Revuz measure associated with the local time process, at level b, of the process $(u^\eta(t,r))_{t\geq 0}$, where u^η is a hypothetical Markov process associated with $P_a^{1,\eta}$, see Conjecture 9.3.3 below. We may thus think of $\Sigma_a^{1,\eta,r}(\mathrm{d}X|b)$ as representing these local times in the above IbPF.

A partial proof. We take an approximation parameter $\epsilon > 0$, and construct $\rho_\epsilon : \mathbb{R} \to \mathbb{R}$, an approximation of δ_0 as in (4.17) above. By the Leibniz formula, we have

$$\begin{aligned} &E_a^3\left[\partial_h\Phi(X)\left(\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}\int_0^1 \rho_\epsilon(X_s-\eta)\,\mathrm{d}s\right)\right)\right] = \\ &E_a^3\left[\partial_h\Psi_\epsilon(X)\right] - E_a^3\left[\Phi(X)\,\partial_h\left(\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}\int_0^1\rho_\epsilon(X_s-\eta)\,\mathrm{d}s\right)\right)\right]. \end{aligned} \tag{9.10}$$

where Ψ_ϵ denotes the functional

$$\Psi_\epsilon(X) = \frac{X_1 \wedge \eta}{X_1} \frac{a}{a \wedge \eta} \exp\left(\frac{1}{2\eta}\int_0^1 \rho_\epsilon(X_s - \eta)\,\mathrm{d}s\right)\Phi(X).$$

We will obtain the requested IbPF by sending $\epsilon \to 0$ in the equality (9.10). We first consider the left-hand side. As we send $\epsilon \to 0$, the integral $\int_0^1 \rho_\epsilon(X_s - \eta)\,\mathrm{d}s$ converges to L_1^η a.s. Morover, by the occupation times formula, we have

$$\int_0^1 \rho_\epsilon(X_s - \eta)\,\mathrm{d}s = \int_{-\epsilon+\eta}^{\epsilon+\eta} \rho_\epsilon(b) L_1^b\,\mathrm{d}b \le \sup_{b\in[-1+\eta,1+\eta]} L_1^b,$$

for all $\epsilon \in (0,1)$. Since the last r.v. has exponential moments for the law P_a^3, by the dominated convergence theorem, we deduce that

$$E_a^3\left[\partial_h\Phi(X)\left(\frac{X_1 \wedge \eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}\int_0^1 \rho_\epsilon(X_s-\eta)\,\mathrm{d}s\right)\right)\right] \underset{\epsilon\to 0}{\longrightarrow} E_a^{1,\eta}(\partial_h\Phi(X)).$$

We now consider the RHS of (9.10). For the first term, the IbPF (3.20) for P_a^3 yields

$$E_a^3(\partial_h\Psi_\epsilon(X)) = -E_a^3(\langle h'', X\rangle \Psi_\epsilon(X)) - \frac{1}{2}\int_0^1 \mathrm{d}r\, h_r\, \Sigma_a^{3,r}\,(\Psi_\epsilon|0) \qquad (9.11)$$

where we recall that, for all $b \ge 0$

$$\Sigma_a^{3,r}\,(\,\mathrm{d}X|b) := \frac{p_r^3(a,b)}{b^2} P_a^3(\mathrm{d}X|X_r = b),$$

with $(p_r^3(x,y))_{r\ge 0, x,y\ge 0}$ denoting the family of transition densities of a 3-Bessel process. Now, we have

$$E_a^3(\langle h'', X\rangle\Psi_\epsilon(X)) = E_a^3\left[\langle h'', X\rangle\left(\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}\int_0^1 \rho_\epsilon(X_s-\eta)\,\mathrm{d}s\right)\right)\Phi(X)\right],$$

which, when $\epsilon \to 0$, converges to

$$E_a^{1,\eta}(\langle h'', X\rangle\Phi(X)).$$

Similarly, the term

$$-\frac{1}{2}\int_0^1 \mathrm{d}r\, h_r\, \Sigma_{3,r}^a\,(\Psi_\epsilon(X)|0)$$

converges to

$$-\frac{1}{2}\int_0^1 \mathrm{d}r\, h_r\, \Sigma_a^{3,r}\left(\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}L_1^\eta\right)\Phi(X)\middle|0\right).$$

Now, by a conditioning argument, for all $r \in (0,1)$, we have the relation

$$\Sigma_a^{3,r}\left(\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}L_1^\eta\right)\Phi(X)\Big|b\right) = b^{-2}\,\Sigma_a^{1,\eta,r}(\Phi(X)|b), \tag{9.12}$$

for all $b>0$, as well as

$$\Sigma_a^{3,r}\left(\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}L_1^\eta\right)\Phi(X)\Big|0\right) = \lim_{b\downarrow 0} b^{-2}\,\Sigma_a^{1,\eta,r}(\Phi(X)|b) =: \boldsymbol{\Sigma}_a^{1,\eta,r}(\Phi(X)|0).$$

Hence, the first term in the right-hand side of (9.10) is thus shown to converge to

$$E_a^{1,\eta}(\langle h'',X\rangle\Phi(X)) - \frac{1}{2}\int_0^1 \mathrm{d}r\ h_r\,\boldsymbol{\Sigma}_a^{1,\eta,r}(\Phi(X)|0).$$

Finally, for the last term in (9.10), we have

$$\begin{aligned}&E_a^3\left[\Phi(X)\,\partial_h\left(\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}\int_0^1\rho_\epsilon(X_s-\eta)\,\mathrm{d}s\right)\right)\right] = \\ &E_a^3\left[\Phi(X)\,\frac{1}{2\eta}\langle h,\rho_\epsilon'(X-\eta)\rangle\left(\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}\int_0^1\rho_\epsilon(X_s-\eta)\,\mathrm{d}s\right)\right)\right],\end{aligned}$$

which by Fubini and conditioning, we can rewrite as

$$\begin{aligned}&\frac{1}{2\eta}\int_0^1\mathrm{d}r\,h_r\int_0^\infty\mathrm{d}b\,\rho_\epsilon'(b-\eta)\\ &\left(p_r^3(a,b)E_a^3\left[\Phi(X)\,\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}\int_0^1\rho_\epsilon(X_s-\eta)\,\mathrm{d}s\right)\Big|X_r=b\right]\right) = \\ =&-\frac{1}{2\eta}\int_0^1\mathrm{d}r\,h_r\int_0^\infty\mathrm{d}b\,\rho_\epsilon(b-\eta)\\ &\frac{\mathrm{d}}{\mathrm{d}b}\left(p_r^3(a,b)E_a^3\left[\Phi(X)\,\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}\int_0^1\rho_\epsilon(X_s-\eta)\,\mathrm{d}s\right)\Big|X_r=b\right]\right),\end{aligned}$$

where we performed an integration by parts to obtain the second line; note that the bracket term vanishes due to the fact that $p_r^3(a,0)=0$. Note also that here we claimed that the derivative in b in the right-hand side is well-defined, a claim which would require a justification. As we send $\epsilon\to 0$, the above expression should converge to

$$-\frac{1}{2\eta}\int_0^1\mathrm{d}r\,h_r\,\frac{\mathrm{d}}{\mathrm{d}b}\left(p_r^3(a,b)E_a^3\left[\Phi(X)\,\frac{X_1\wedge\eta}{X_1}\frac{a}{a\wedge\eta}\exp\left(\frac{1}{2\eta}L_1^\eta\right)\Big|X_r=b\right]\right)\Big|_{b=\eta},$$

Note that this step would require further justification: it does not just follow from the weak convergence of the kernel ρ_ϵ to δ_0, since the approximation parameter

ϵ appears also in the test function against which we integrate it. Now, for all $r \in (0,1)$ and $b > 0$, by (9.12), we have

$$p_r^3(a,b)\, E_a^3 \left[\Phi(X)\, \frac{X_1 \wedge \eta}{X_1} \frac{a}{a \wedge \eta} \exp\left(\frac{1}{2\eta} L_1^\eta \right) \Big| X_r = b \right] = \Sigma_a^{1,\eta,r}(\Phi(X)|b).$$

We therefore deduce that the last term in (9.10) converges to

$$-\frac{1}{2\eta} \int_0^1 \mathrm{d}r\, h_r\, \frac{\mathrm{d}}{\mathrm{d}b} \Sigma_a^{1,\eta,r}(\Phi(X)|b) \Big|_{b=\eta},$$

as requested. Hence the claim. □

9.3.3 The corresponding dynamics

Let $\eta > 0$ be fixed. We assume for simplicity that $a = 0$. We recall the definitions $H := L^2([0,1])$, and $K := \{f \in L^2([0,1]),\, f \geq 0\}$. Then, with $\mathcal{FC}_b^\infty(K)$ denoting the space of functionals introduced in Section 4.2.2 above, we consider the form $\mathcal{E}^{1,\eta}$ defined on $\mathcal{FC}_b^\infty(K)$ as follows:

$$\mathcal{E}^{1,\eta}(u,v) = \frac{1}{2} \int_K \nabla u(x).\nabla v(x)\, \mathrm{d}P_0^{1,\eta}(x), \quad u, v \in \mathcal{FC}_b^\infty(K).$$

We conjecture the following:

Conjecture 9.3.3. *The form $(\mathcal{FC}_b^\infty(K), \mathcal{E}^{1,\eta})$ is closable. Its closure $(Dom(\mathcal{E}^{1,\eta}), \mathcal{E}^{1,\eta})$ is a quasi-regular Dirichlet form.*

By Conjecture 9.3.2, the K-valued stationary Markov process $(u_t^\eta)_{t\geq 0}$ associated with $\mathcal{E}^{1,\eta}$ should formally satisfy the following Nualart-Pardoux type equation:

$$\begin{cases} \dfrac{\partial u^\eta}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u^\eta}{\partial x^2} + \xi + \zeta - \dfrac{1}{4\eta} \dfrac{\partial}{\partial b} \ell_{t,x}^{\eta,b} \Big|_{b=\eta}, \\ \\ u^\eta \geq 0, \quad \mathrm{d}\zeta \geq 0, \quad \int_{\mathbb{R}_+ \times [0,1]} u^\eta \, \mathrm{d}\zeta = 0, \end{cases} \tag{9.13}$$

where ξ is space-time white noise, and $\left(\ell_{t,x}^{\eta,b} \right)_{b,\geq 0}$ is a family of local times for $(u^\eta(t,x))_{t\geq 0}$. More concretely, noting that $P_0^{1,\eta}$ is the limit, as $\epsilon \to 0$, of the approximating measures

$$P_0^{1,\eta,\epsilon} = \frac{X_1 \wedge \eta}{X_1} \frac{a}{a \wedge \eta} \exp\left(\frac{1}{2\eta} \int_0^1 \rho_\epsilon(X_s - \eta) \right) P_0^3(\mathrm{d}X),$$

where $\rho_\epsilon = \frac{1}{\epsilon}\rho(\frac{x}{\epsilon})$, with ρ a smooth, even function supported in $[-1,1]$ such that

$$\rho \geq 0, \qquad \int_{\mathbb{R}} \rho = 1, \qquad \rho' \leq 0 \quad \text{on } \mathbb{R}_+,$$

then (9.13) should be the limit as $\epsilon \to 0$ of the following, well-posed, SPDEs of Nualart-Pardoux type:

$$\begin{cases} \dfrac{\partial u^{\eta,\epsilon}}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^{\eta,\epsilon}}{\partial x^2} + \xi + \zeta + \dfrac{1}{4\eta}\rho'_\epsilon(u^{\eta,\epsilon} - \eta), \\ \\ u^{\eta,\epsilon} \geq 0, \quad \mathrm{d}\zeta \geq 0, \quad \int_{\mathbb{R}_+\times[0,1]} u^{\eta,\epsilon}\, \mathrm{d}\zeta = 0. \end{cases} \tag{9.14}$$

Note that, for all $\epsilon \in (0,\eta)$, the term $\rho'_\epsilon(u^{\eta,\epsilon} - \eta)$ is zero, except when $|u^{\eta,\epsilon} - \eta| \leq \epsilon$. Moreover, it is positive when $\eta - \epsilon \leq u^{\eta,\epsilon} \leq \eta$, and negative when $\eta \leq u^{\eta,\epsilon} \leq \eta + \epsilon$. Thus, equation (9.13) could be interpreted as an SPDE with reflection at 0 and stickiness at η.

9.3.4 Convergence of the whole dynamics

For all $\eta > 0$, we consider the K-valued stationary Markov process $(u^\eta_t)_{t\geq 0}$ associated with the Dirichlet form $\mathcal{E}^{1,\eta}$, and started from equilibrium : $u^\eta_0 \overset{(d)}{=} P^{1,\eta}_0$. Recall that $H^{-1}(0,1)$ denotes the completion of $H = L^2(0,1)$ w.r.t. the norm

$$\|f\|^2_{-1} := \sum_{n=1}^{\infty} n^{-2}\, |\langle f, e_n\rangle|^2,$$

where $e_n(\theta) := \sqrt{2}\sin(n\pi\theta)$, $\theta \in [0,1]$, $n \geq 1$.

Conjecture 9.3.4. *For all $T > 0$, $(u^\eta_t)_{t\in[0,T]}$ weakly converges to $(\tilde{u}_t)_{t\in[0,T]}$ in $C([0,T], H^{-1}(0,1))$ as $\eta \to 0$, where $\tilde{u}$ is the Markov process considered in Section 9.2.3 above.*

Sketch of proof for the tightness. Let $T > 0$. We show that the sequence of processes $(u^\eta_t)_{t\in[0,T]}$, $\eta > 0$, is tight in $C([0,T], H^{-1}(0,1))$ as $\eta \to 0$. To do so we proceed as in the proof of Theorem 9.2.1 above by invoking the Lyons-Zheng decomposition, which enables to write, for all $h \in H$

$$\langle u^\eta_t, h\rangle - \langle u^\eta_0, h\rangle = \frac{1}{2}M^1_t - \frac{1}{2}\left(M^2_T - M^2_{T-t}\right) \quad \text{a.s.},$$

where M^i, $i = 1,2$, are martingales. More precisely, setting $\mathcal{F}^1_t = \sigma(u^\eta_s, s \leq t)$ and $\mathcal{F}^2_t = \sigma(u^\eta_{T-s}, s \leq t)$, then for $i = 1,2$, M^i is an $(\mathcal{F}^i_t)_{t\geq 0}$ martingale with quadratic variation $\langle M^i \rangle_t = t\|h\|^2$. Indeed, denoting by φ the element of $\mathrm{Dom}(\mathcal{E}^{1,\eta})$ given by $\varphi(x) = \langle h, x\rangle$, $x \in K$, by Theorem 5.7.1 in [FOT10], we have the above decomposition with

$$M_t^1 := M_t^{[\varphi]}, \quad M_t^2(\omega) := M_t^{[\varphi]}(r_T\omega),$$

where r_T is the time-reversing operator on the canonical space $\Omega := C([0,T], K)$:

$$(r_T\omega)_t = \omega_t, \quad \omega \in \Omega,\, t \in [0,T].$$

Moreover, $M^{[\varphi]}$ denotes the martingale additive functional appearing in the Fukushima decomposition of the continuous additive functional given by

$$A_t^{[\varphi]} := \varphi(u_t^\eta) - \varphi(u_0^\eta) = \langle u_t^\eta, h\rangle - \langle u_0^\eta, h\rangle, \quad t \geq 0.$$

Hence, the quadratic variations of the martingales $(M_t^i)_{t\geq 0}$ $(i = 1,2)$ are given by the sharp bracket $\langle M^{[\varphi]}\rangle_t$. The latter is a positive CAF with Revuz measure μ satisfying

$$\int_K f(x)\mu(\mathrm{d}x) = 2\mathcal{E}^{1,\eta}(\varphi f, \varphi) - \mathcal{E}^{1,\eta}(\varphi^2, f),$$

for all $f \in \mathrm{Dom}(\mathcal{E}^{1,\eta})$ bounded, see Thm 5.2.3 in [FOT10]. But, for all $f \in \mathcal{FC}_b^\infty(K)$, we have, by the Leibniz rule

$$2\mathcal{E}^{1,\eta}(\varphi f, \varphi) - \mathcal{E}^{1,\eta}(\varphi^2, f) = \int_K f(x)\|\nabla\varphi(x)\|^2 P_0^{1,\eta}(\mathrm{d}x).$$

Therefore, for all $f \in \mathcal{FC}_b^\infty(K)$, it holds

$$\int_K f(x)\mu(\mathrm{d}x) = \int_K f(x)\|\nabla\varphi(x)\|^2 P_0^{1,\eta}(\mathrm{d}x) = \|h\|^2 \int_K f(x) P_0^{1,\eta}(\mathrm{d}x).$$

Since $\mathcal{FC}_b^\infty(K)$ is dense in $C_b(K)$, we deduce that $\mu(\mathrm{d}x) = \|h\|^2 P_0^{1,\eta}(\mathrm{d}x)$. Since the latter measure is the Revuz measure of the additive functional $\|h\|^2\, t$, by the Revuz correspondence, we deduce the equality

$$\langle M^{[\varphi]}\rangle_t = \|h\|^2\, t,$$

in the sense of additive functionals, so that

$$\langle M^1\rangle_t = \langle M^2\rangle_t = \|h\|^2\, t,$$

as claimed.

Hence, by the BDG inequality, for all $p \geq 1$, there exists a constant $C_p > 0$ (depending only on p) such that, for all $t \geq s \geq 0$

$$\left(\mathbb{E}\left[\langle u_t^\eta - u_s^\eta, h\rangle^p\right]\right)^{1/p} \leq C_p(t-s)^{1/2}||h||$$

Hence, we obtain, for all $p \geq 2$

$$\begin{aligned}\left(\mathbb{E}\left[\|u_t^\eta - u_s^\eta\|^p_{H^{-1}(0,1)}\right]\right)^{1/p} &= \left(\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\langle u_t^\eta - u_s^\eta, e_k\rangle^2 k^{-2}\right)^{p/2}\right]\right)^{1/p} \\ &\leq \zeta(2)^{\frac{1}{2}-\frac{1}{p}}\left(\mathbb{E}\left[\sum_{k=1}^{\infty}\langle u_t^\eta - u_s^\eta, e_k\rangle^p k^{-2}\right]\right)^{1/p} \\ &\leq \zeta(2)^{1/2}\, C_p\, (t-s)^{1/2} \\ &\leq C_p'\,(t-s)^{1/2},\end{aligned}$$

where the second line follows by Jensen's inequality, and where $C_p' = \zeta(2)^{1/2} C_p$. Since, moreover, for all $t \geq 0$, law $P_0^{1,\eta}$ of u_t^η converges to P_0^1 as $\eta \to 0$, the claimed tightness follows. □

To obtain the conjecture above, there would remain to identify the subsequential limits in probability of u^η as $\eta \to 0$. For the same reasons as mentioned in Section 9.2.3 above this question is, still, very open.

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