Stationary and Propagating Modes in Discrete Models of the Linear Shallow Water Equations

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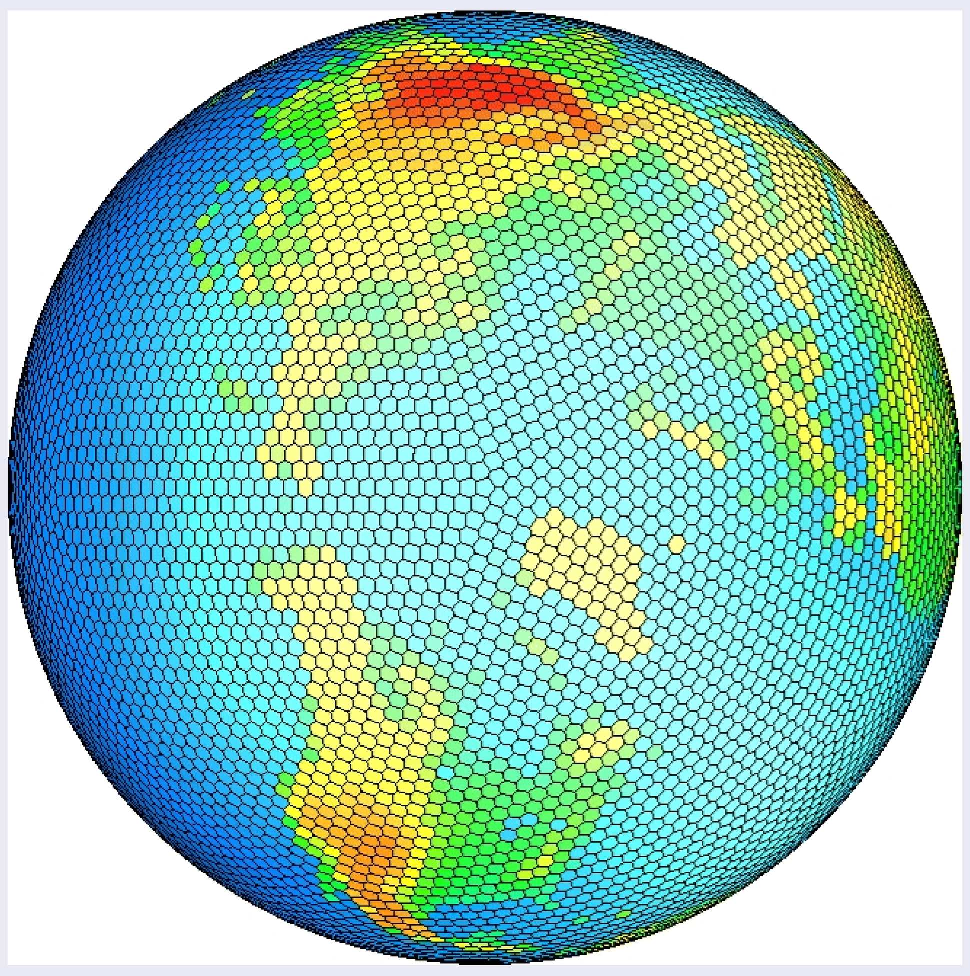
Abstract

- Shallow water equations are a useful analogue of the fully compressible Euler equations for atmospheric model development
- Linear properties (propagating and stationary modes) play an important physical role in the behaviour of the atmosphere
- Using the Atmospheric Dynamical Core Testbed (ADCoT, described below), the linear properties of two finite-difference schemes (TRiSK: Ringler, Thuburn, Klemp & Skamarock 2010 and HR95: Heikes & Randall 1995) on the **f-sphere** are compared to those of the continuous equations

ADCOT: Design & Implementation

- Horizontal meshes represented using MOAB mesh library
- Currently supported meshes: Perfect planar square and hexagonal; geodesic (tweaked) spherical meshes (working is ongoing to add cubed-sphere and additional types of geodesic meshes)

Figure: Sample geodesic grid



- Variables (scalar, vector, vector component) placed arbitrarily on mesh elements
- Operators defined as sparse matrices (linear) or algebraic combinations of vector operators and (sparse) matrix multiplication (non-linear)

$$\vec{\nabla} \cdot u \rightarrow \mathrm{D}\vec{u}$$
$$\vec{\nabla}(\frac{u^2}{2} + gh) \rightarrow \mathrm{G}(\mathrm{K}\vec{u}^2 + g\vec{h})$$

- Uses MOAB, PETSc and SLEPc to provide grid management, linear/eigenvalue solvers and I/O; main code written in Fortran 95
- Code generation using Cheetah enables fast prototyping and flexibility
- Analysis packages are written in Python/Fortran 95 using the NFFT, PyNGL, Numpy, Scipy and Matplotlib libraries
- Adams-Bashford and Runge-Kutta explicit time stepping (implicit and semi-implicit time stepping is planned as well)
- TRiSK and HR95 horizontal discretizations (more to come!)
- Intended primarily for single moment discretizations

Linear Shallow Water Equations on an f-sphere

- Momentum Form

$$\frac{\partial \vec{u}}{\partial t} = -f\hat{k} \times \vec{u} - g\vec{\nabla} h$$
$$\frac{\partial h}{\partial t} = -H(\vec{\nabla} \cdot u)$$

- Vorticity-Divergence Form

$$\frac{\partial \zeta}{\partial t} = -f\delta$$
$$\frac{\partial \delta}{\partial t} = f\zeta - g\vec{\nabla}^2 h$$
$$\frac{\partial h}{\partial t} = -H\delta$$

- Propagating Modes (Inertia-Gravity Waves)

$$\left(\frac{\sigma}{f}\right)^2 = 1 + \frac{\lambda^2}{a^2} n(n+1)$$

Results: Propagating Modes

- Dispersion relationship calculated as $\frac{d\vec{x}}{dt} = \mathbb{L}\vec{x} \rightarrow i\omega\vec{x} = \mathrm{A}\vec{x}$ (eigenvalue problem)
- Spherical harmonic transforms (NFFT package) are used to determine which spatial wavenumbers each eigenvector/eigenvalue pair is associated with
- Geodesic grid with n_e edges, n_i faces and n_v vertices; n_{ig} inertia-gravity modes and n_{stat} stationary modes (geostrophic + spurious)
- Vlevel=1 (42 faces) geodesic grid with $f = 0.0001 s^{-1}$, $g = 9.81 m s^{-1}$, $H = 56334m$ (A) or $H = 140m$ (B), $a = 6371220m$
- $\frac{\lambda}{d} \approx 2.0$ (A) or $\frac{\lambda}{d} \approx 0.1$ (B)
- Both TRiSK (C-grid) and HR95 (Z-Grid) investigated

Figure: HR95 Results: Case A, red circles are theoretical, black crosses are numerical

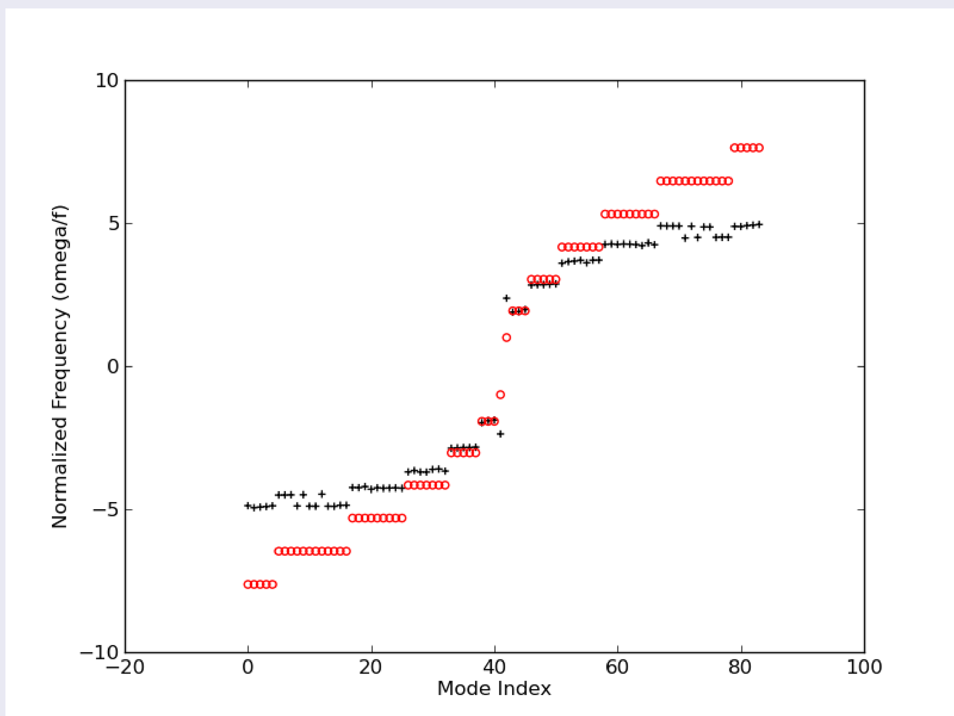
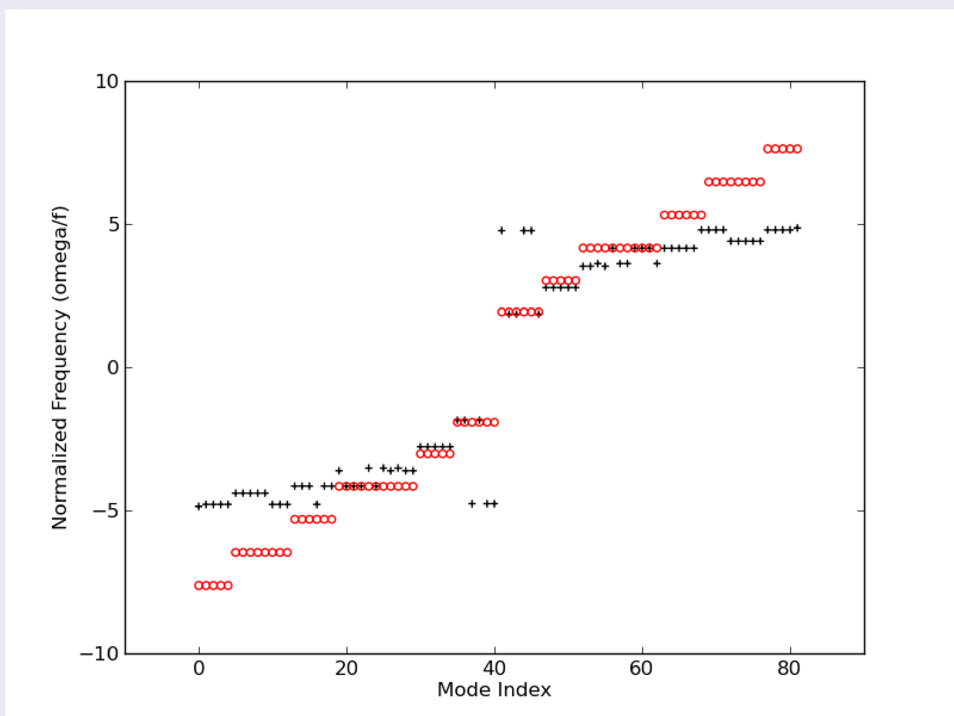


Figure: TRiSK Results: Case A, red circles are theoretical, black crosses are numerical



Results: Propagating Modes

Figure: HR95 Results: Case B

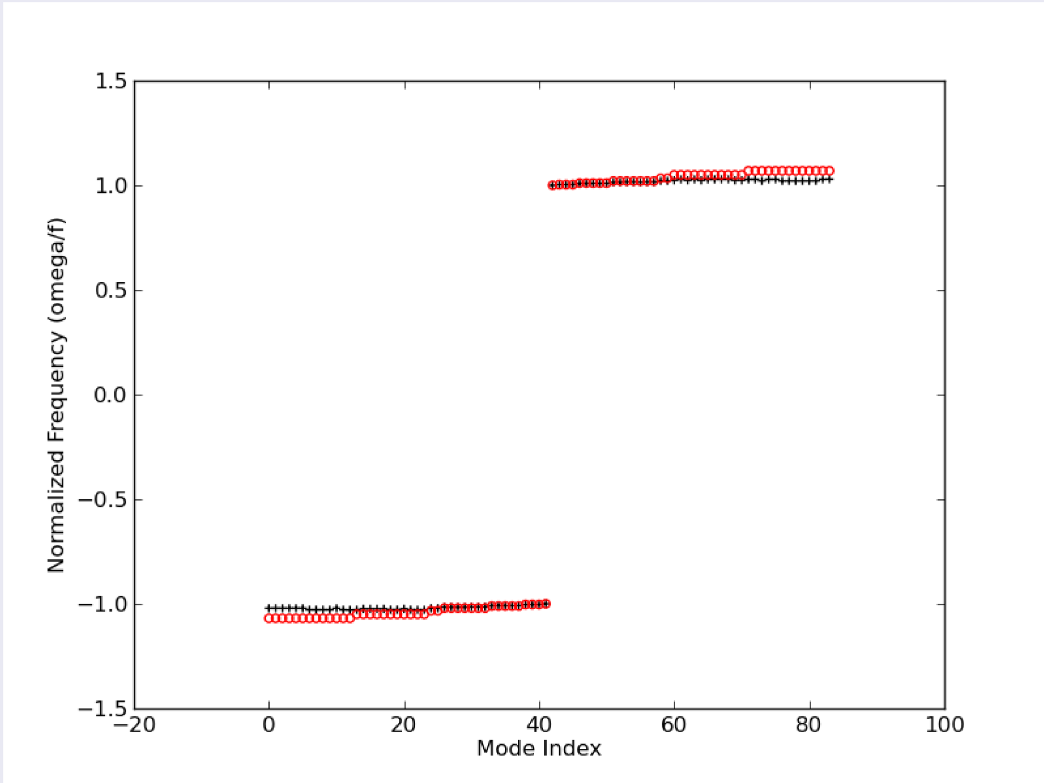
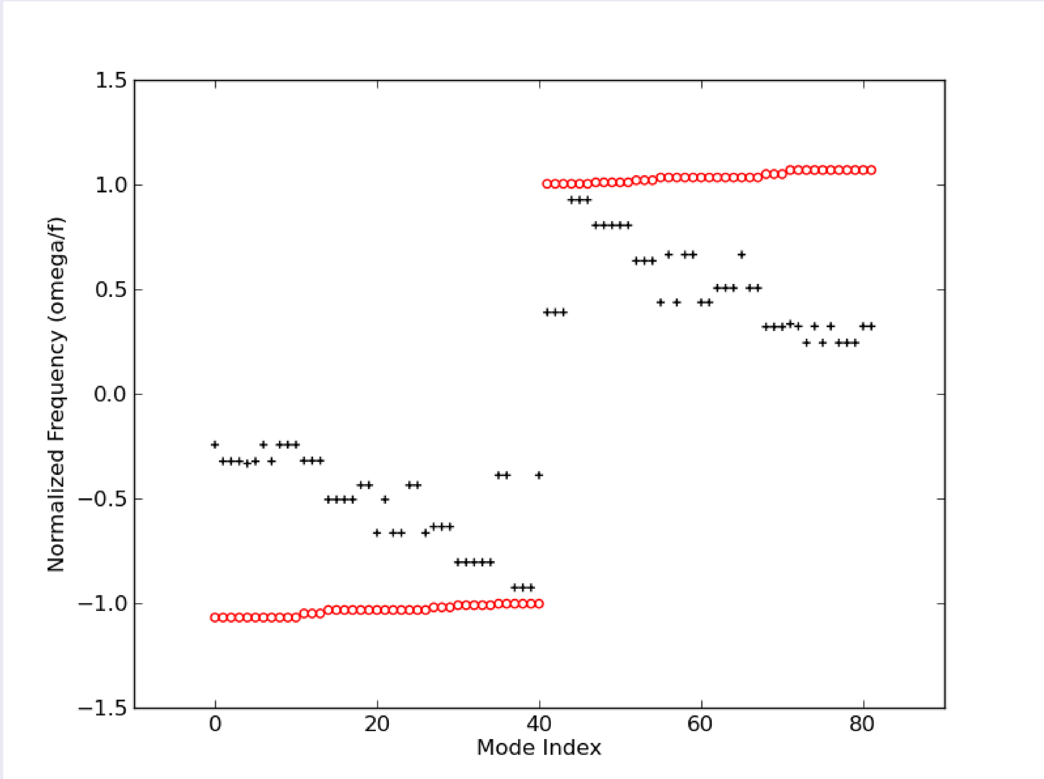


Figure: TRiSK Results: Case B



- Both grids show good agreement for lower-frequency modes
- HR95 does an excellent job when Rossby radius is poorly resolved; TRiSK has issues
- Geodesic grid means that high frequency eigenvectors do not resemble higher-frequency spherical harmonics
- Mode numbers are as expected:

Stationary (TRiSK)	n_v
Stationary (HR95)	n_i
Inertia-Gravity (TRiSK)	$2n_i - 2$
Inertia-Gravity (HR95)	$2n_i$

Results: Discrete Operator Null Space Dimension

- Operator null space calculated as $0 = \mathbb{L}\vec{x} \rightarrow 0 = \mathrm{A}\vec{x}$ (SVD problem)
- Null spaces are connected to stationary modes

Divergence (TRiSK)	1
Gradient (TRiSK)	1
Vector Reconstruction (TRiSK)	n_e - n_{ig}
Laplacian (HR95)	1

Conclusions

- ADCOT provides a useful framework for inter-comparison of various numerical schemes for the nonlinear shallow water equations
- Two very different schemes (TRiSK and HR95) can be analysed under the same code framework
- Initial results indicate the f-sphere dispersion relationships on geodesic grids for TRiSK and HR95 are similar in structure and character to analytic dispersion relationships for the f-plane perfect hexagonal grid