Dynamico-FE: A Hydrostatic Dynamical Core using Higher-Order Structure-Preserving Finite Elements

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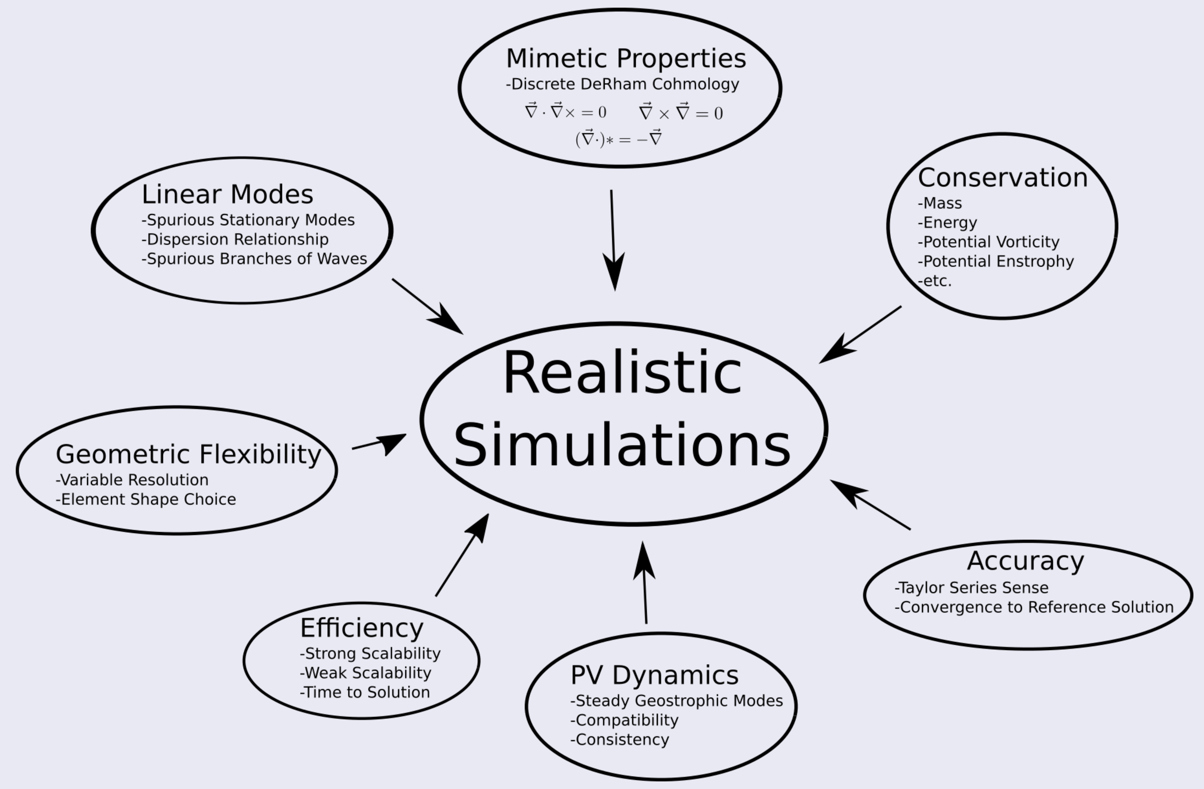
(I) Desirable Model Properties



How do we get these properties?

(II) Hamiltonian Formulation in a General Vertical Coordinate

Prognose $(\mu, S, \vec{v})$ using

$$\frac{\partial \mu}{\partial t} + \vec{\nabla}\cdot\frac{\delta \mathcal{H}}{\delta \vec{v}} + \partial_\eta(W) = 0$$

$$\frac{\partial S}{\partial t} + \vec{\nabla}\cdot\left(s\frac{\delta \mathcal{H}}{\delta \vec{v}}\right) + \partial_\eta(sW) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\vec{\nabla}\times\vec{v}}{\mu}\times\frac{\delta \mathcal{H}}{\delta \vec{v}} + \vec{\nabla}\frac{\delta \mathcal{H}}{\delta \mu} + s\vec{\nabla}\frac{\delta \mathcal{H}}{\delta S} + \frac{W}{\mu}\partial_\eta(\vec{v}) = 0$$

with Hamiltonian $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}[\mu,\vec{v},S,z] = \int \mu K(\vec{v},z) + \mu U\left(\frac{1}{\mu}\frac{\partial z}{\partial \eta},\frac{S}{\mu}\right) + \mu\Phi(z)$$

$$\frac{\delta \mathcal{H}}{\delta \vec{v}} = \mu\vec{u}$$

$$\frac{\delta \mathcal{H}}{\delta \mu} = K + \Phi + U + \mu\frac{\partial U}{\partial \mu}$$

$$\frac{\delta \mathcal{H}}{\delta S} = \mu\frac{\partial U}{\partial S}$$

and diagnose z using

$$\frac{\delta \mathcal{H}}{\delta z} = \mu\frac{\partial K}{\partial z} + \mu\frac{\partial \Phi}{\partial z} - \frac{\partial}{\partial \eta}\left(\mu\frac{\partial U}{\partial(\frac{\partial z}{\partial \eta})}\right) = 0 \text{ (Hydrostatic balance)}$$

where $\vec{v} = \vec{u} + \vec{R}(z)$ is the horizontal (covariant) absolute velocity, $\mu = \frac{1}{\alpha}\frac{\partial z}{\partial \eta}$ is the pseudo-density, z the height and $S = \mu s$ the mass-weighted entropy. These equations work for a deep, non-spherical atmosphere with an arbitrary equation of state $U(\alpha, s)$. Lagrangian vertical coordinate defined by $W = 0$. Mass-based vertical coordinate prognoses M_s and specifies $M(\eta) = \int \mu = a(\eta)M_s + b(\eta)M_0 \rightarrow \mu$ is no longer prognostic (it gives the equation for W) $\rightarrow$ redefine $\frac{\delta \mathcal{H}}{\delta \mu}$ to conserve energy (vertical remapping invariance). Details are in [3].

(III) General Mimetic Discretizations: Primal deRham Complex

- Select any 1D Spaces $\mathcal{A}$ and $\mathcal{B}$ such that: $\mathcal{A} \xrightarrow{\frac{d}{dx}} \mathcal{B}$
- Use tensor products to extend to n-dimensions
- Our (novel) choices of $\mathcal{A}$ and $\mathcal{B}$ are guided by linear mode properties and coupling to physics/tracer transport

$$\begin{tikzcd}
\mathbb{W}_0 \arrow[r, "d"', "\vec{\nabla}"] & \mathbb{W}_1 \arrow[l, bend left, "\vec{\nabla}\cdot"] \arrow[r, "d"', "\vec{\nabla}\times"] & \mathbb{W}_2 \arrow[l, bend left, "\vec{\nabla}\times"] \arrow[r, "d"', "\vec{\nabla}\cdot"] & \mathbb{W}_3 \arrow[l, bend left, "\vec{\nabla}"]
\end{tikzcd}$$

$\delta \qquad \delta \qquad \delta$

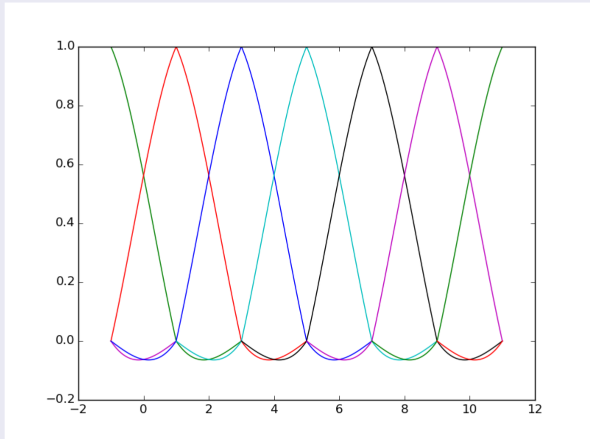
$$(da^k, b^{k+1}) = (a^k, \delta b^{k+1})$$

Primal deRham Complex

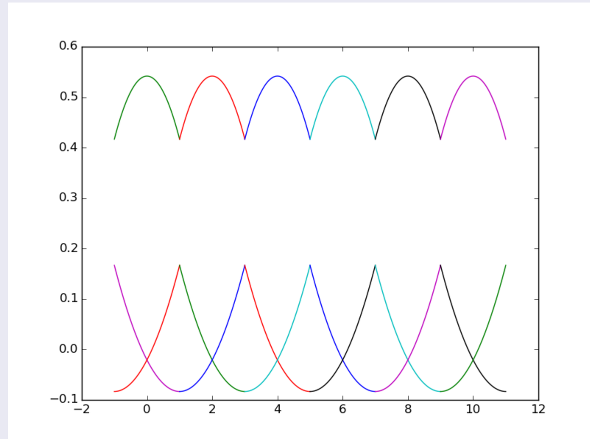
Integration by parts plus $\delta = *d*$ implicitly defines $*$

(IV) Mimetic Galerkin Differences

H^1 space defined following [1], with L_2 defined to be compatible following [4]. This is an arbitrary order extension of [2]. For 3rd order gives:



$\mathcal{A} = H_1$ Space (1D)

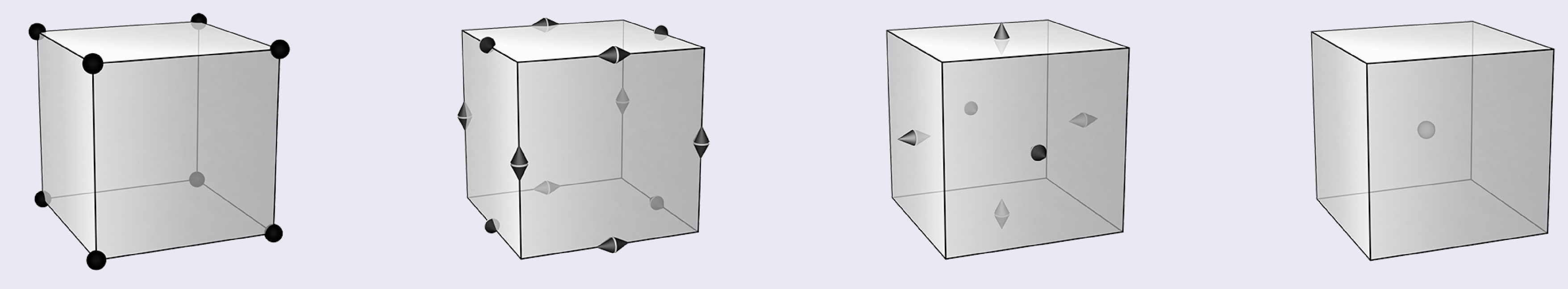


$\mathcal{B} = L_2$ Space (1D)

Single degree of freedom per geometric entity with higher order through larger stencils $\rightarrow$ no spectral gaps, easy coupling to physics/tracer transport, less local

(V) Compatible Spaces and Discretization

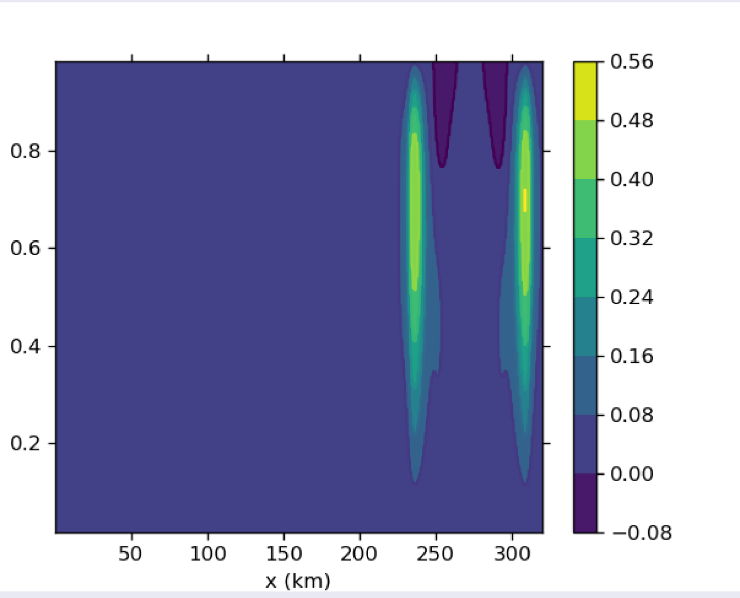
Restrict variables to finite dimensional subspaces, and multiply by test functions to obtain to obtain discrete weak form equations. Choose spaces and staggering as:



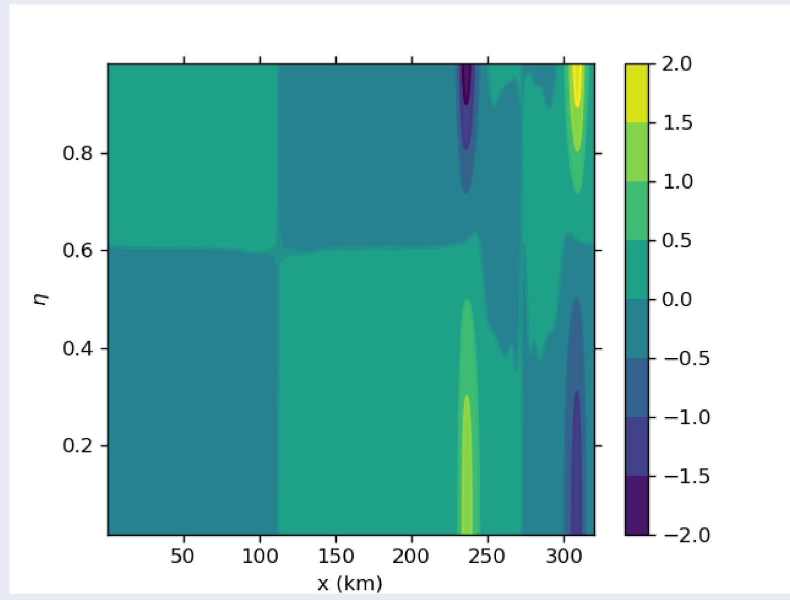
$\psi \in H^1 \qquad \vec{\zeta} \in H(curl) \qquad \vec{u},\vec{v} \in H(div) \qquad \mu,\delta,\chi,S \in L_2$

This approach gives a (quasi-)Hamiltonian semi-discretization that conserves mass, entropy and total energy, independent of the choice of specific spaces!

(VI) Preliminary Results



θ'



u'

DCMIP 3.1 (Gravity Wave) adapted to slice. $\Delta x = 1km$, 30 vertical levels with a Lagrangian vertical coordinate, $u_0 = 100ms^{-1}$. 4th-order Runge Kutta time stepping. Shown at $t = 1125s$. Prognoses Θ.

(VII) Conclusions

1. Hydrostatic dynamical core can run with mass-based or Lagrangian vertical coordinates, prognosing Θ or S, on slice or planar grid.
2. Possible to obtain most of the desirable properties by combining **mimetic discretization** with a **Hamiltonian formulation**
3. Hamiltonian formulation enables treatment of many approximations (deep/shallow, traditional/non-traditional, spherical/non-spherical) with a **unified framework**

(VIII) Future and Ongoing Work

1. Extension to the cubed sphere grid
2. Corrected $\frac{\delta \mathcal{H}}{\delta \mu}$ for mass-based vertical coordinate
3. Extending conservation to time integration (Poisson integrators)
4. Introduction of moisture in a consistent way (Multicomponent formulation)
5. Incorporation of dissipation and parameterization in a consistent way (Metriplectic formulation, stochastic pdes, higher order moments)
6. Computational efficiency
7. Extension to non-hydrostatic or semi-hydrostatic (sound-proof) equations

(IX) Themis: Accelerated Computational Science

Themis is a PETSc-based software framework for parallel, high-performance discretization of variational forms through mimetic, tensor-product Galerkin methods.

Available online at `https://bitbucket.org/chris_eldred/themis`

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