Extension of the 1981 Arakawa and Lamb Scheme to Arbitrary Grids

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Intro

Introduction

Key Papers

Arakawa and Lamb 1981

A Potential Enstrophy and Energy Conserving Scheme for the Shallow Water Equations

AKIO ARAKAWA AND VIVIAN R. LAMB[1]

Salmon 2004

Poisson-Bracket Approach to the Construction of Energy- and Potential-Enstrophy-Conserving Algorithms for the Shallow-Water Equations

RICK SALMON

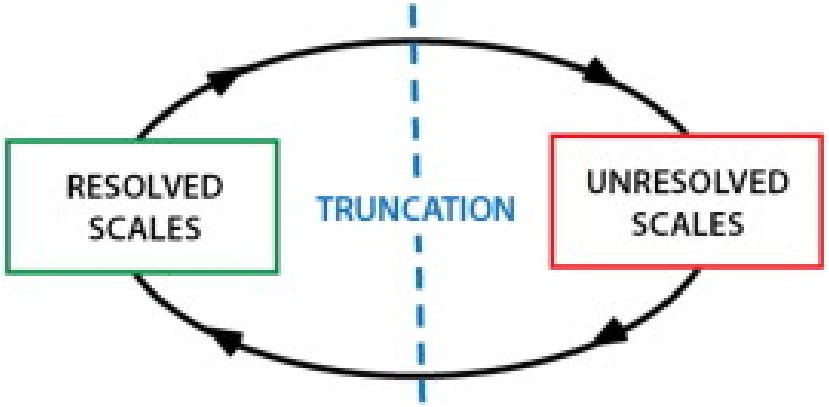
Thuburn, Cotter and Dubos 2012

A FRAMEWORK FOR MIMETIC DISCRETIZATION OF THE ROTATING SHALLOW-WATER EQUATIONS ON ARBITRARY POLYGONAL GRIDS*

J. THUBURN† AND C. J. COTTER‡

Broad Research Overview: Dynamical Cores

1. Develop discrete models of the atmosphere
2. Dynamical core: deals with "resolved processes"
3. Parameterizations: deal with "unresolved processes"
4. Model: dynamical core + parameterizations



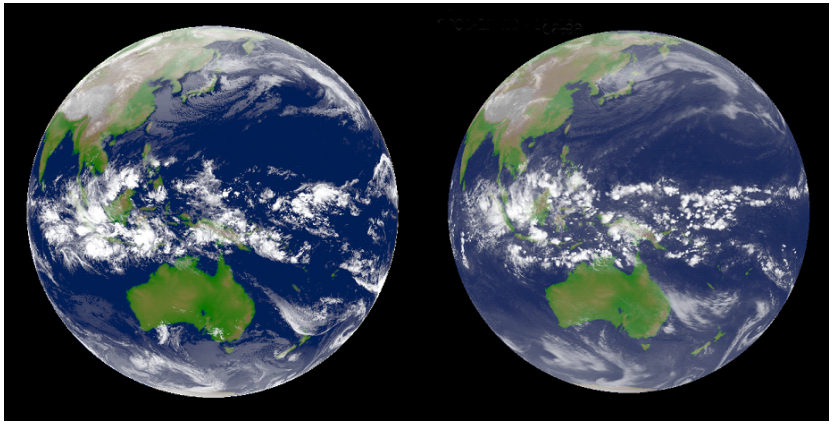
Key Principles of Numerical Modeling

1. Not solving arbitrary PDEs- physical system
2. No analytic solutions
3. Differential equations $\rightarrow$ algebraic equations
4. **Do algebraic solutions have the same properties as the differential solutions?**

Philosophy of Dynamical Core Design

1. Models should respect underlying physics
   1. Energetics
   2. PV dynamics
   3. Wave dynamics
   4. **Conserved quantities- energy, potential enstrophy**
   5. **Mimetic properties**

$$\vec{\nabla} \times \vec{\nabla}\phi = 0$$



Shallow Water Equations

Vector Invariant Shallow Water Equations

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + q\hat{k} \times (h\vec{u}) + \vec{\nabla}\Phi = 0$$

$$\mathbb{J} = \begin{pmatrix} 0 & -\vec{\nabla}\cdot \\ -\vec{\nabla} & -q\hat{k}\times \end{pmatrix}$$

$$\vec{x} = (h, \vec{u})$$

$$\frac{\partial \vec{x}}{\partial t} = \mathbb{J}\frac{\delta \mathcal{H}}{\delta \vec{x}}$$

$$\mathcal{H} = \frac{1}{2}g(h,h) + \frac{1}{2}(\vec{F}, \vec{u})$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ h\vec{u} \end{pmatrix}$$

Shallow Water Equations: (Subset of) Properties

Mimetic

No Spurious Vorticity Production:

$$\vec{\nabla} \times \vec{\nabla}\phi = 0$$

Pressure Gradient is Energy Conserving:

$$(\vec{\nabla})^* = -\vec{\nabla}\cdot$$

Conserved Quantities

Total Energy $\int_\Omega \frac{gh^2}{2} + \frac{h|\vec{u}|^2}{2}$

Potential Enstrophy $\int_\Omega h\frac{q^2}{2}$

Arakawa and Lamb 1981 Scheme: Desirable Properties

(A Subset of) Desirable Properties

1. No spurious vorticity production (curl-free gradient)
2. Energy-conserving pressure gradient force (divergence and gradient are adjoints)
3. Total energy and potential enstrophy conservation

Mimetic

$$\bar{D}_2 \bar{D}_1 = 0$$

$$D_2^T = -\bar{D}_1$$

Conservation

$$\mathbb{J} = -\mathbb{J}^T$$

$$\mathbb{J} \frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

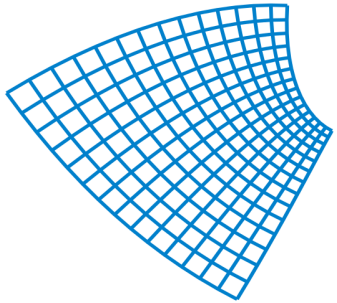
Arakawa and Lamb 1981 Scheme: Limitations

(A Subset of) AL81 Shortcomings

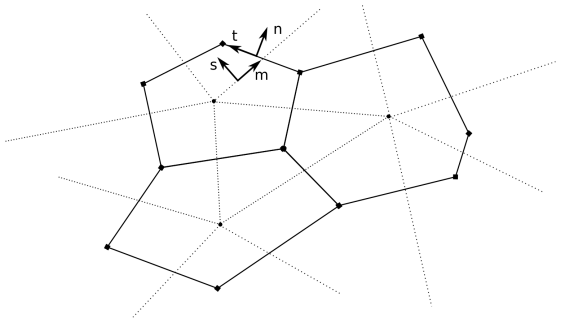
1. Restricted to logically square, orthogonal grids

TRiSK: Ringler, Skamarok, Klemp, Thuburn, Cotter, Dubos

1. General, non-orthogonal polygonal grids
2. **Choose between total energy and potential enstrophy conservation**



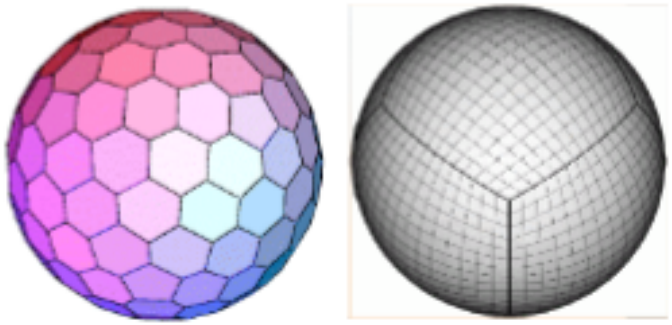
Logically square, orthogonal grid



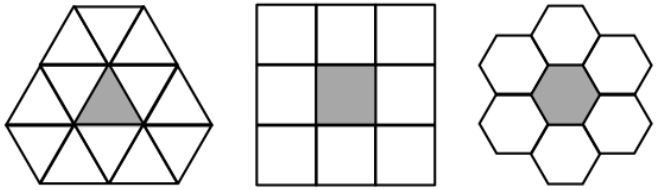
General, non-orthogonal grid

Recap: What am I trying to accomplish?

1. How can Arakawa and Lamb 1981 be extended to **arbitrary, non-orthogonal polygonal grids**?
2. In a way that preserves all of its desirable properties, and does not add new limitations?



Icosahedral-hexagons Cubed Sphere



AL81 on arbitrary grids

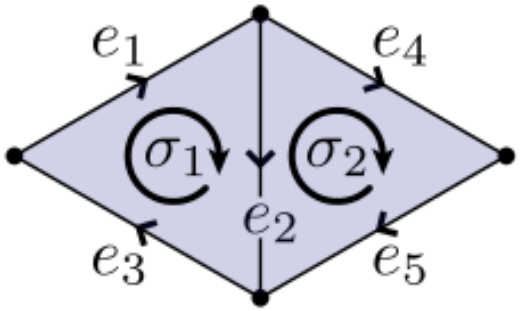
Extension of AL81 to Arbitrary Grids

Mimetic Methods (Discrete Exterior Calculus)

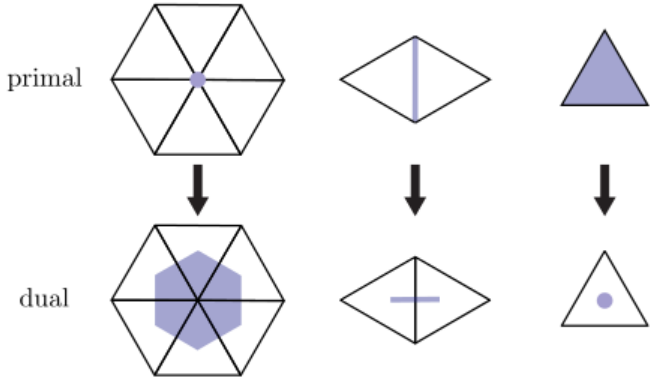
Mimetic Operators

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \iff \bar{D}_2 \bar{D}_1 = 0$$

$$(\vec{\nabla})^* = -\vec{\nabla}\cdot \iff D_2 = -\bar{D}_1{}^T$$



(Discrete) Exterior Derivative



Primal-Dual Grid

Conservation Laws (Hamiltonian Mechanics)

Hamiltonian (Energy)

$$\mathbb{J}^T = -\mathbb{J}$$

$\mathcal{H}$ is positive definite

Casimirs (Potential Enstrophy)

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

Discrete Conservation

Enforce these conditions in discrete case $\rightarrow$ conservation!

Recap: Conservative, Mimetic Methods

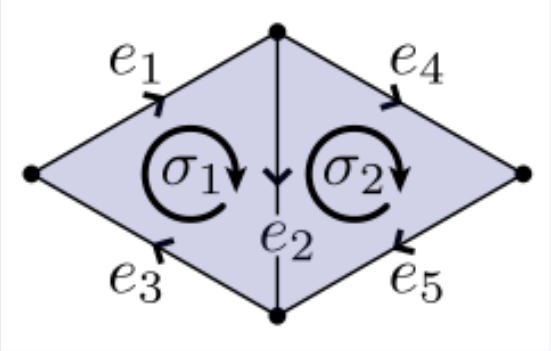
Conservation Laws

$\mathcal{H}$ is positive definite

$$\mathbb{J} = -\mathbb{J}^T$$

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

Mimetic Properties



(Discrete) Exterior Derivative

Conservative, Mimetic Methods

1. Use mimetic operators to build a discrete (quasi-)Hamiltonian system
2. Unifies two important lines of research

Generalized C Grid Discretization: Hamiltonian-DEC

- Discrete variables are $m_i = \int h dA$ and $u_e = \int \vec{u} \cdot \vec{dl}$
- C grid staggering (m_i at cell centers, u_e at edges)

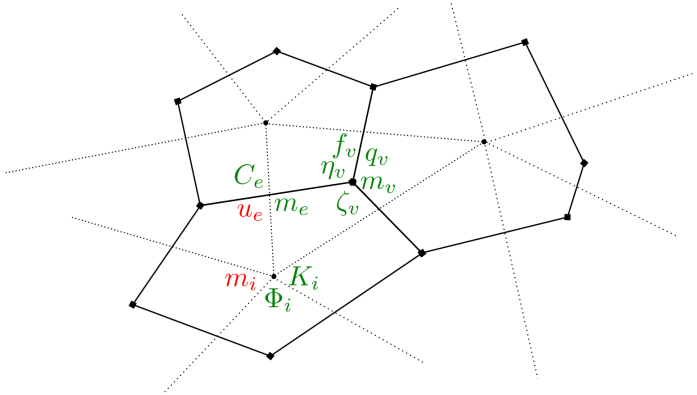
$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & \mathbf{Q} \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g (m_i, m_i)_{\mathbf{I}} + \frac{1}{2} (C_e, u_e)_{\mathbf{H}}$$

$$\mathcal{Z} = \frac{1}{2} (\zeta_v, \eta_v)_{\mathbf{J}}$$

$$q_v = \frac{\eta_v}{m_v} = \frac{\zeta_v + f_v}{m_v} = \frac{\bar{D}_2 u_e + f_v}{\mathbf{R} m_i}$$



$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I} \Phi_i \\ F_e \end{pmatrix} = \begin{pmatrix} \mathbf{I} K_i + g \mathbf{I} m_i \\ \mathbf{H} C_e \end{pmatrix}$$

$$\frac{\delta \mathcal{Z}}{\delta \vec{x}} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$

Q operator is the remaining hurdle

General Form of **Q**

Following Salmon 2004, set

$$\mathbf{Q}F_e = \sum_{e' \in ECP(e)} \sum_{v \in VC(i)} q_v \alpha_{e,e',v} F_{e'}$$

What are $\alpha_{e,e',v}$'s?

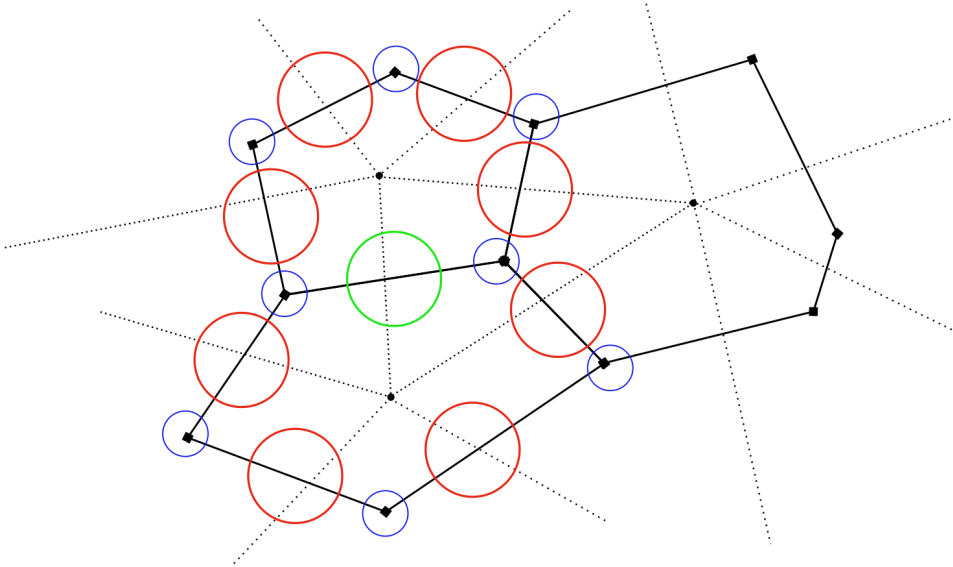


Diagram of Q operator stencil

Discrete Conservation

Energy

$$\mathbb{J}^T = -\mathbb{J} \longrightarrow \mathbf{Q} = -\mathbf{Q}^T \longrightarrow \alpha_{e,e',v} = -\alpha_{e',e,v}$$

Potential Enstrophy

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0 \longrightarrow \text{ linear system of equations} \longrightarrow \mathbf{A}\vec{\alpha} = \vec{b}$$

Also want $\mathbf{Q} \rightarrow \mathbf{W}$ when $q_v = \text{const}$ (steady geostrophic modes)

Solving $\mathbf{A}\vec{\alpha} = \vec{b}$

Issue: System is too large

- Geodesic grid: 90 coefficients per cell, all coefficients are interdependent $\rightarrow$ not feasible for realistic grids
- Cubed sphere grid is similar (24 coefficients per cell)

Solution: Subystem Splitting

$$\mathbf{A}\vec{\alpha} = \vec{b} \longrightarrow \sum_i \mathbf{A}_i \vec{\alpha}_i = \vec{b}_i$$

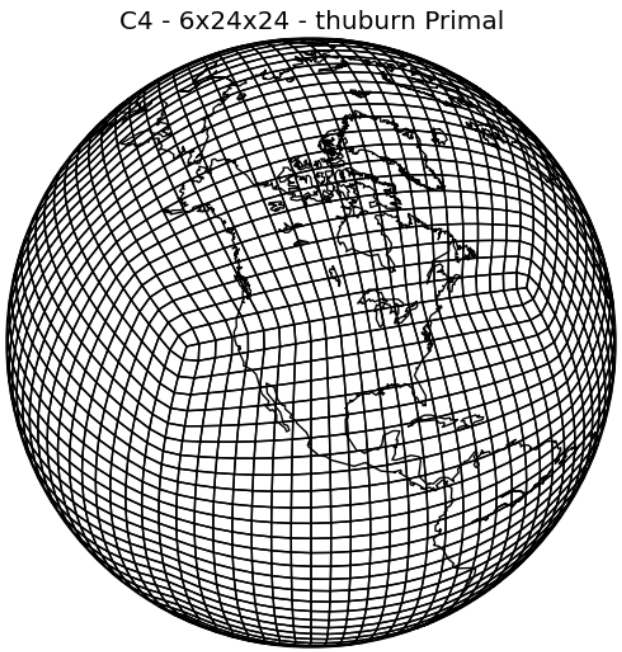
Split into independent subsystems for each cell!

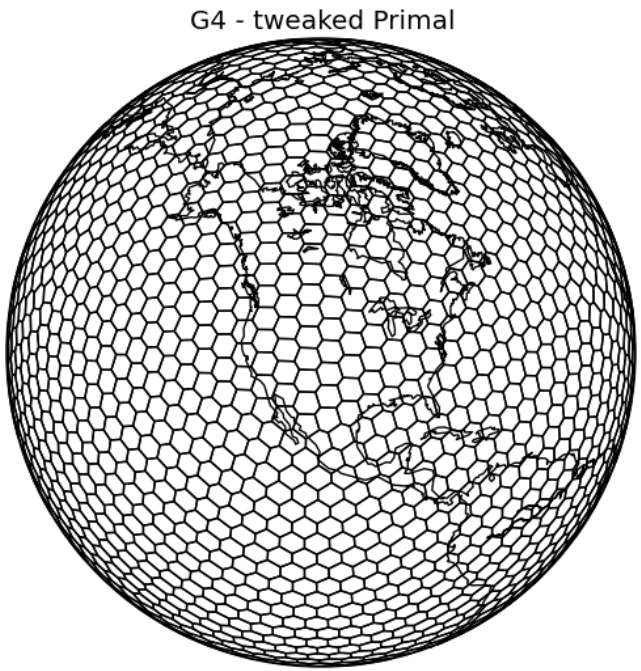
System has been solved for various planar and spherical grids

Recap: What have I accomplished?

What has been done?

- Arakawa and Lamb 1981 extended to arbitrary grids via new **Q**
- Coefficients can be precomputed (efficiently)





Test Results

Test Case Results

Variants of **Q** Operator

Total Energy

$$\mathbf{Q}F_e = \frac{1}{2}q_e\mathbf{W}F_e + \frac{1}{2}\mathbf{W}q_eF_e$$

Works for ANY choice of q_e (APVM, CLUST, etc.)

Potential Enstrophy

$$\mathbf{Q}F_e = q_e\mathbf{W}F_e$$

Requires that q_e is arithmetic mean

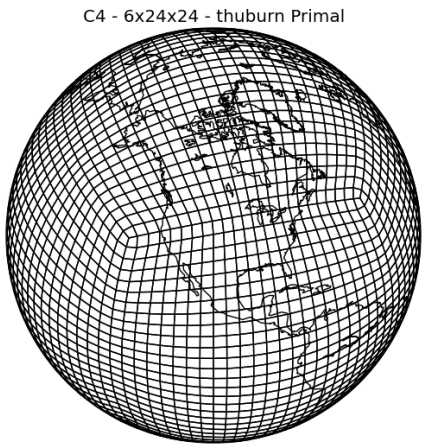
Total Energy and Potential Enstrophy

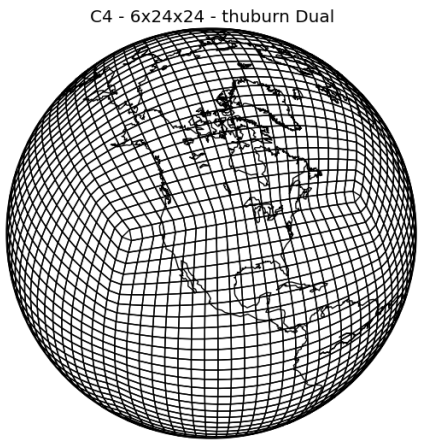
$$\mathbf{Q}F_e = Q(q_v, F_e)$$

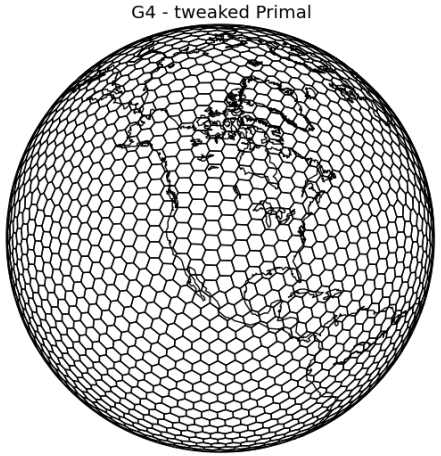
Grids

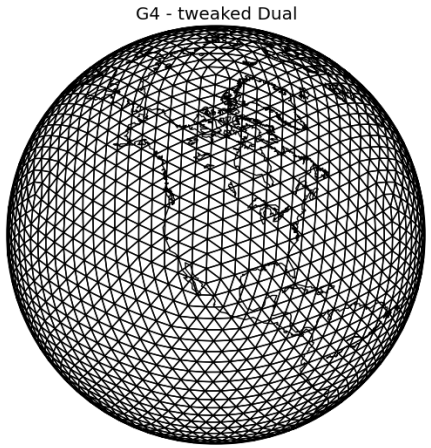
Grids

- Cubed Sphere: 6x384x384, 880K cells (25km resolution)
- Geodesic: G8, 640K cells (30km resolution)









Operators

Operators

$$\mathbf{R} = \sum_{v \in VC(i)} \frac{A_{iv}}{A_i}$$

$$\mathbf{\Phi} = \frac{A_{ie}}{A_e} + \frac{A_{je}}{A_e}$$

$$\mathbf{I} = \frac{1}{A_i}$$

$$\mathbf{J} = \frac{1}{A_v}$$

$$\mathbf{H} = \frac{le}{de}$$

H different for cubed-sphere

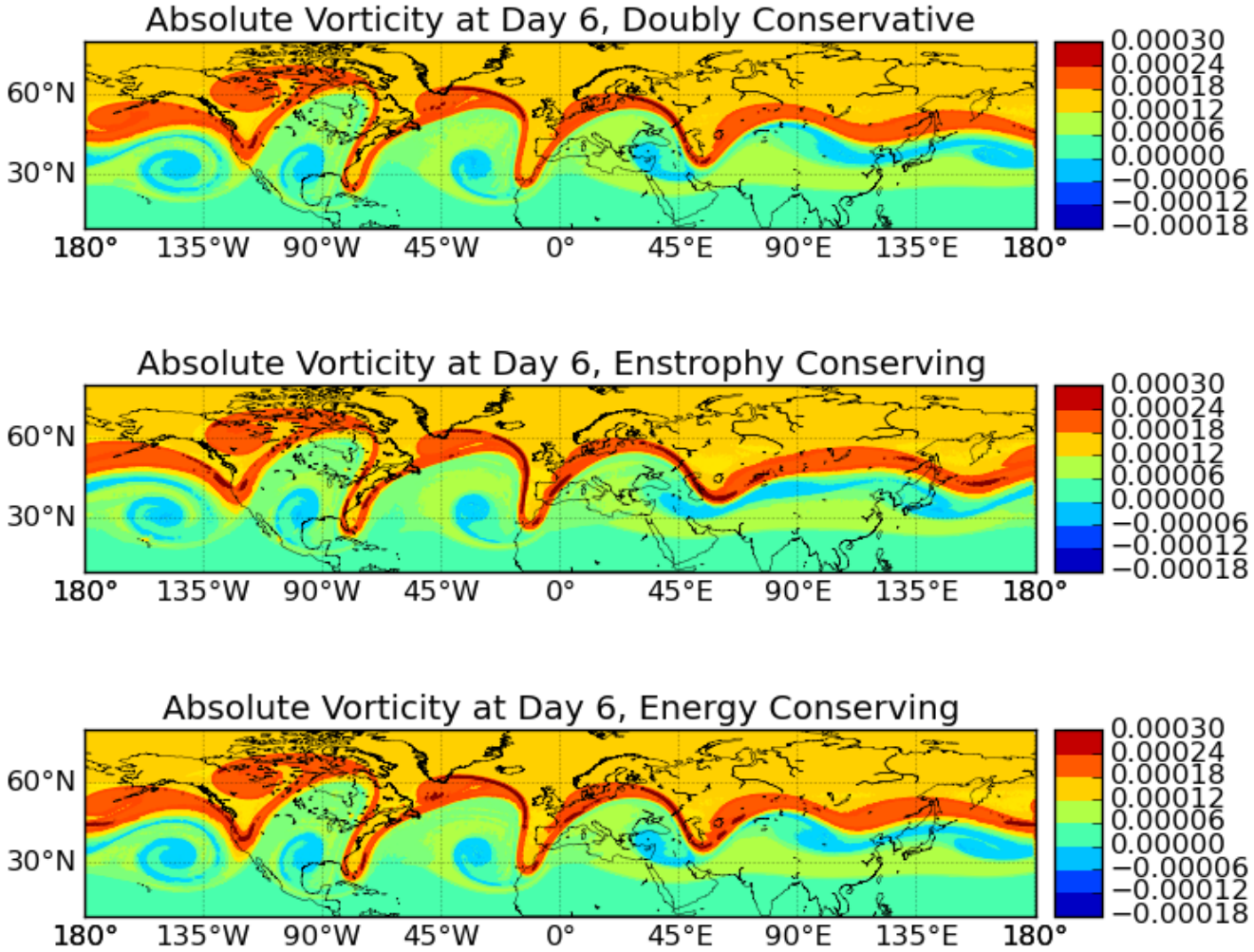
Time Stepping

- Adams-Bashford 3rd Order (15s cubed-sphere, 22.5s geodesic)
- 10 simulated days, output every 6 hours

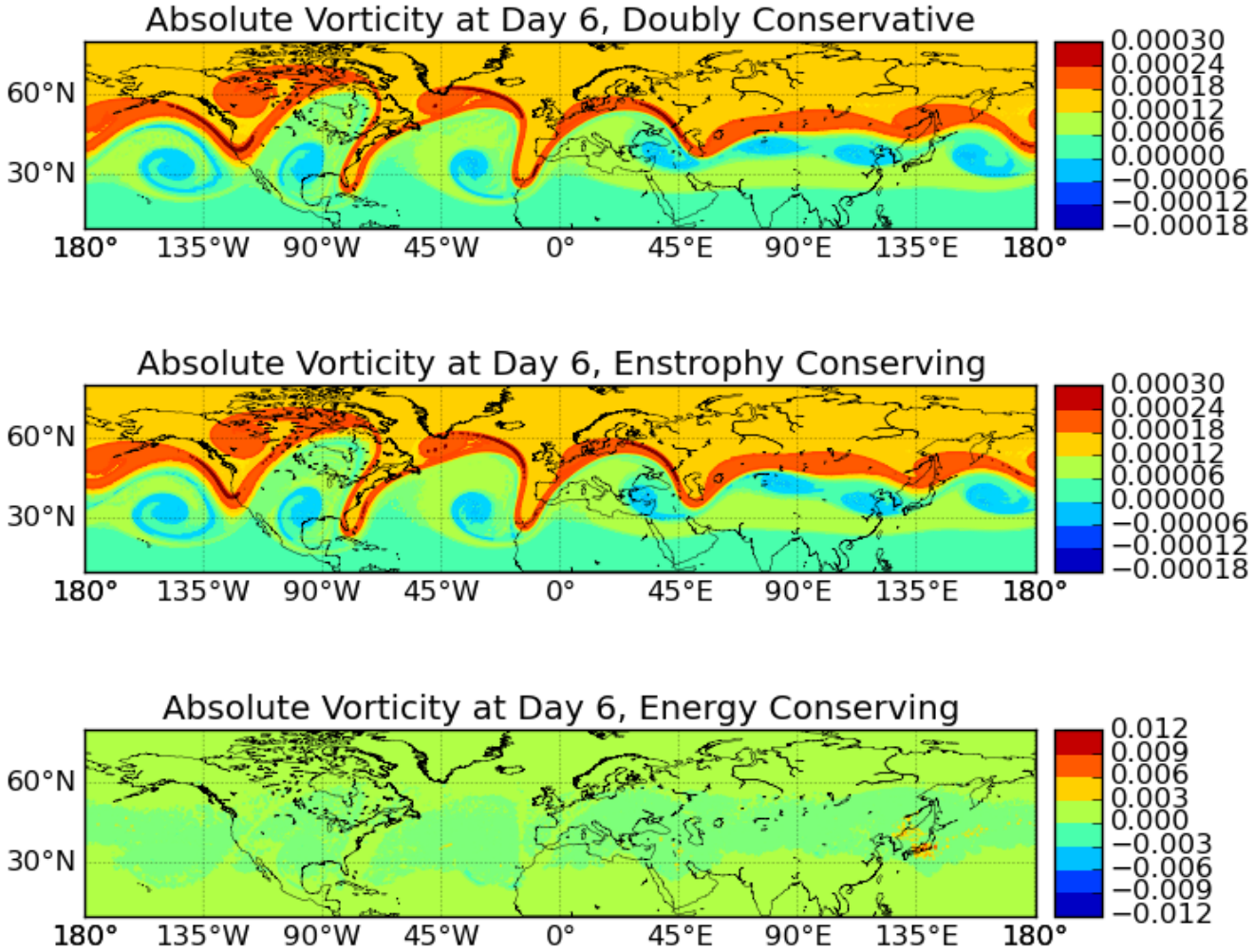
Dissipation

- $\vec{\nabla}^2 \vec{u}$ specified in Galewsky et. al added to that test case
- Schemes stable without it

Galewsky et. al (Unstable Jet)- Geodesic (Tweaked)

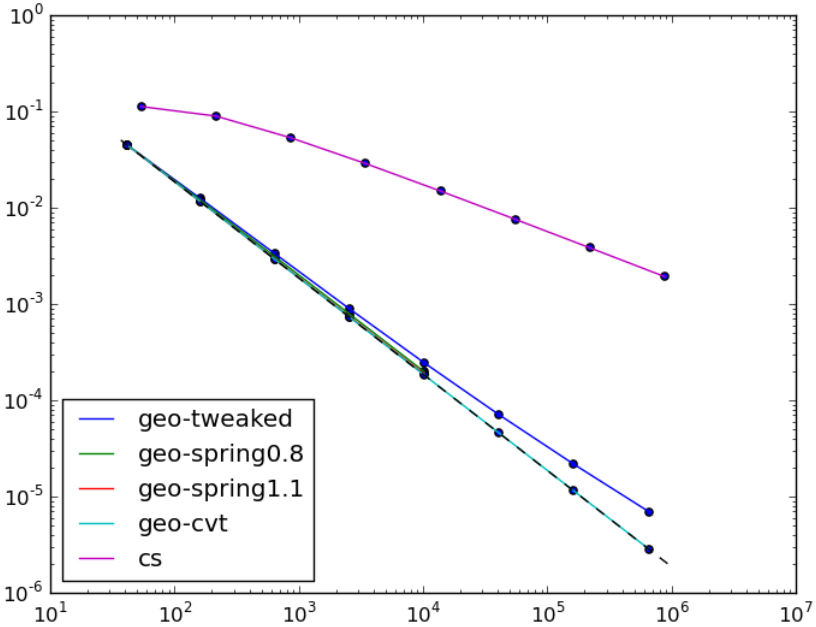


Galewsky et. al (Unstable Jet)- Cubed Sphere (Thuburn)

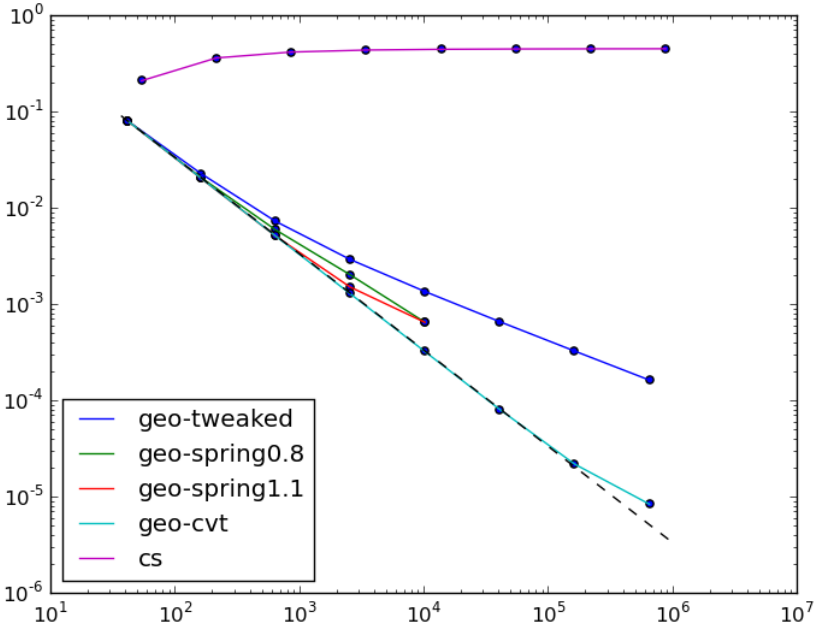


Order of Accuracy- Laplacian on Primal

L_2 Error



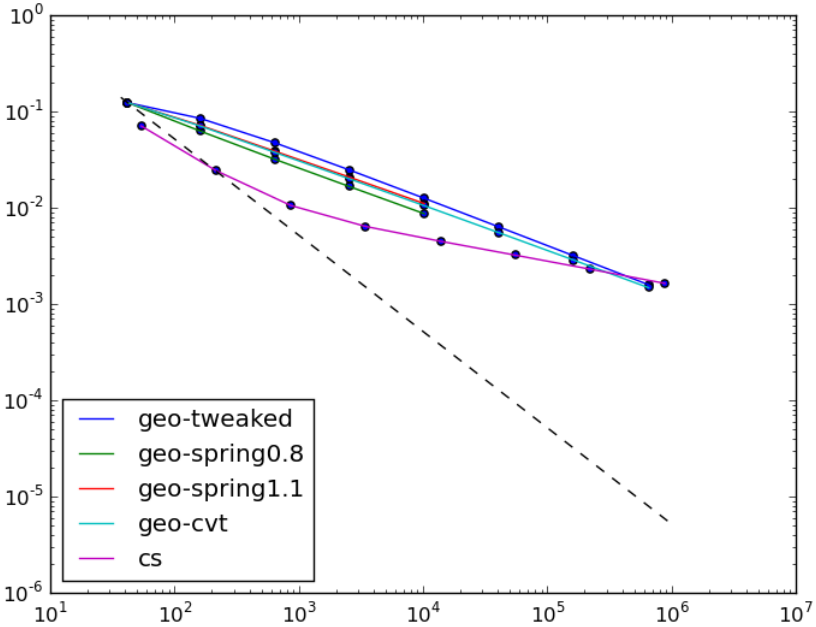
L_∞ Error



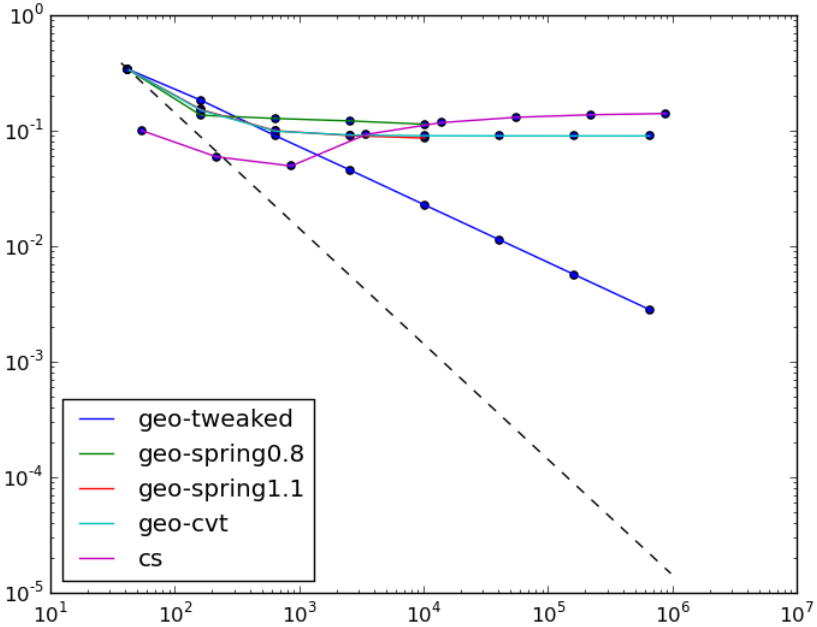
Computed for $\psi = \cos(\theta)\sin(\lambda)$

Order of Accuracy- Laplacian on Dual

L_2 Error



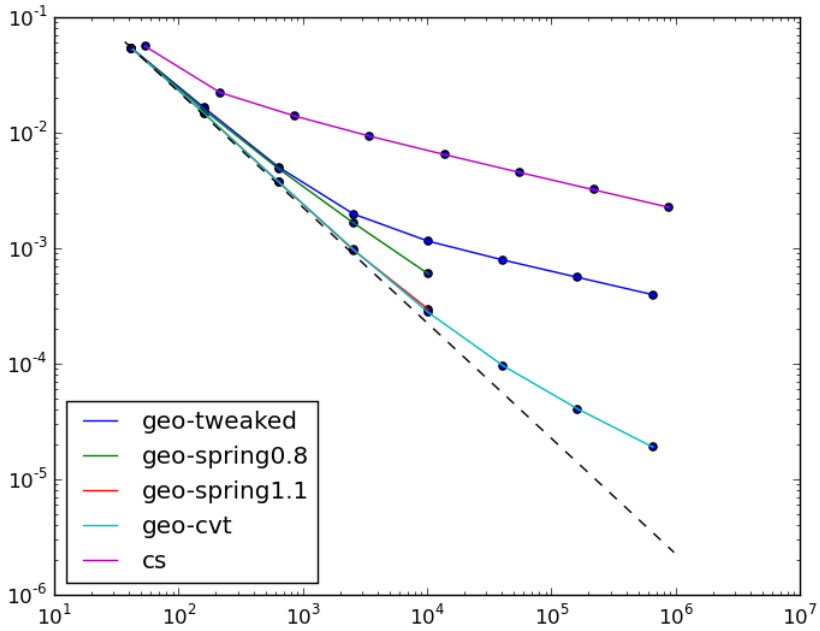
L_∞ Error



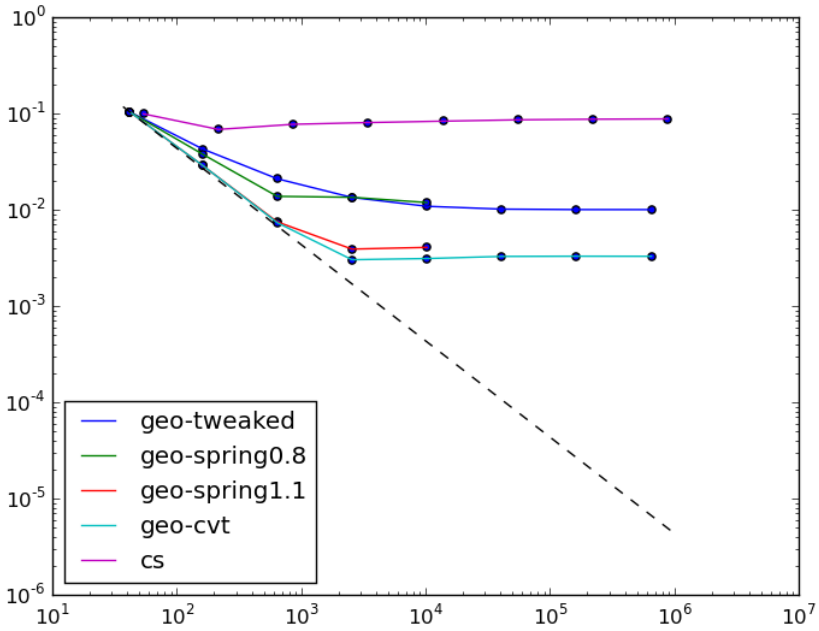
Computed for $\psi = \cos(\theta)\sin(\lambda)$

Order of Accuracy- **Q** (also **W**)

L_2 Error



L_∞ Error



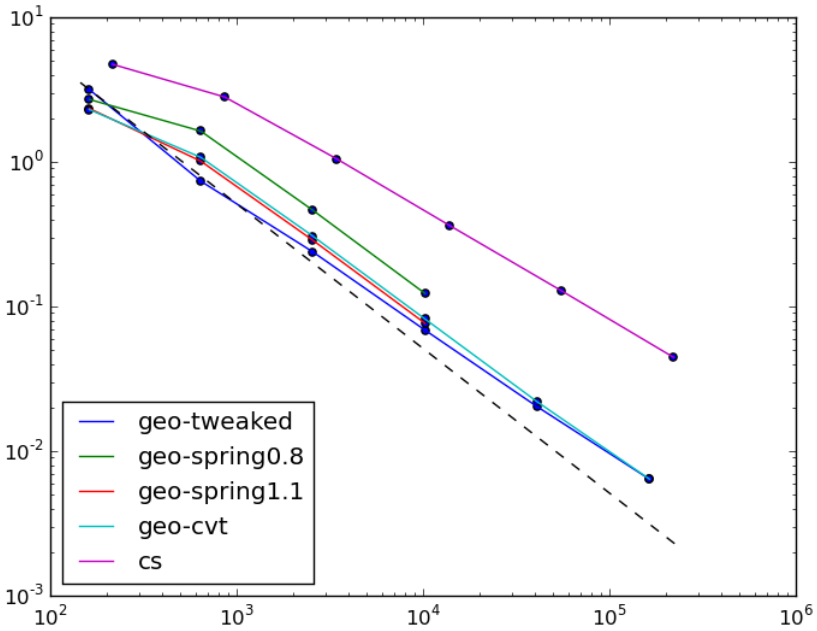
Computed for $\psi = q = \cos(\theta)\sin(\lambda)$

error $= q_e \bar{D}_1 \psi_e - \mathbf{Q}(q_v, D_1 \psi_v)$

W errors are very similar

Order of Accuracy- Zonal Jet (Williamson Test Case 2)

L_2 Error



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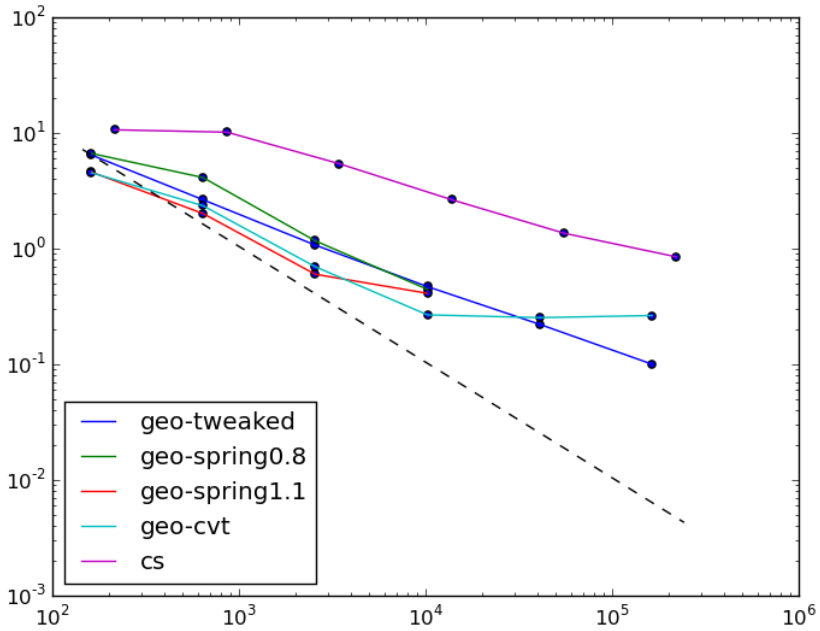

Height errors

Conclusions

Conclusions

Summary and Conclusions

Conclusions

1. Hamiltonian + Discrete Exterior Calculus $\rightarrow$ AL81 on arbitrary polygonal grids
2. Preserves all desirable properties of AL81

Future Work

1. Consistent $\mathbf{W}$ (and $\mathbf{Q}$)
2. Anticipated Potential Vorticity + other dissipation options
3. Cubed sphere accuracy- Hodge stars, grid optimization
4. Effects of extra Rossby modes on geodesic grid?
5. Comparison to Z-grid scheme (Salmon 2007)