# Atmospheric Dynamical Core Development

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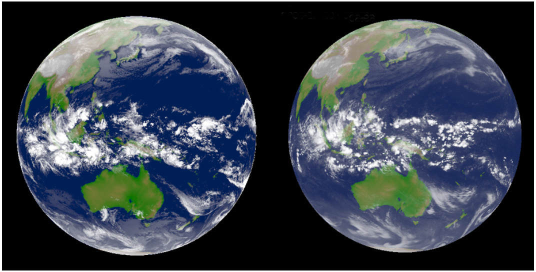
# Who am I?

1. PhD Student in Atmospheric Science at Colorado State University
2. Work for Dave Randall at Center for Multiscale Modelling of Atmospheric Processes (CMMAP)
3. Did undergraduate work at Carnegie Mellon University

# Broad Research Overview: What are dynamical cores?

1. Develop discrete models of the atmosphere (dynamical cores)
2. Dynamical core: deals with "resolved processes"
3. Parameterizations: deals with "unresolved processes"
4. Model: dynamical core + parameterizations



Which is the model, and which is reality? (from Miura et. al 2007)

# Philosophy of Dynamical Core Development

1. Model development is a series of choices: equation sets, predicted variables, numerical methods, many others
2. Numerical model: Continuous equations $\rightarrow$ algebraic equations
3. Are solutions between these two the same?
4. $\rightarrow$ "Mimetic" methods, conservation properties
5. Need to use "expertise" to develop models

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} = 0 \Rightarrow \frac{A_i^{n+1} - A_i^n}{\Delta t} + \frac{A_{i+1}^n - A_{i-1}^n}{2\Delta x} = 0$$

# Conservation Laws

1. Equations of fluid dynamics come from conservation laws: mass, momentum, energy, etc.
2. Conserved quantities provide a powerful constraint on the dynamics
3. Different types of conserved quantities: integral quantities (total energy,...), parcel quantities (mass, potential vorticity,...)
4. **Conservation laws are FUNDAMENTAL to atmospheric dynamics $\rightarrow$ Discretization should preserve (an analogue) of them**

# A Simple Example, part 1

## 1-D Advection

Consider 1-D advection of $A$:

$$\frac{\partial A}{\partial t} + \frac{\partial A}{\partial x} = 0$$

This system conserves (with period B.C.'s)

$$\int_{\Omega} A dx$$

and

$$\int_{\Omega} A^2 dx$$

(among other things)

# A Simple Example, part 2

## 1-D Advection, Discretized

Now consider 2 different spatial discretizations of this system:

$$\frac{\partial A_i}{\partial t} + \frac{A_i - A_{i-1}}{\Delta x} = 0$$

and

$$\frac{\partial A_i}{\partial t} + \frac{A_{i+1} - A_{i-1}}{2\Delta x} = 0$$

Both will conserve $\sum_i A_i$ (discrete analogue of $\int_\Omega A dx$)
Both are consistent
What about conservation of $\sum_i A_i^2$ (discrete analogue of $\int_\Omega A^2 dx$)? (Let's assume $\Delta x = 1$ for convenience)

# A Simple Example, part 3

## First Discretization

$$\frac{\partial}{\partial t}\sum_i A_i^2 = \sum_i \frac{\partial}{\partial t}A_i^2 = \sum_i A_i \frac{\partial}{\partial t}A_i = \sum_i A_i(A_i - A_{i-1}) \neq 0$$

Doesn't conserve $\sum_i A_i^2$

## Second Discretization

$$\frac{\partial}{\partial t}\sum_i A_i^2 = \sum_i \frac{\partial}{\partial t}A_i^2 = \sum_i A_i \frac{\partial}{\partial t}A_i = \sum_i A_i(A_{i+1} - A_{i-1})$$

$$= \sum_i A_i A_{i+1} - \sum_i A_i A_{i-1} = 0$$

Conserves $\sum_i A_i^2$

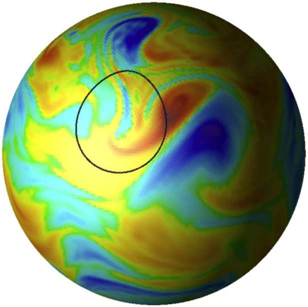
# A Simple Example, part 4

## Not so fast...

1. Second discretization can be unstable with certain time stepping schemes
2. Only first discretization is monotonic and sign-preserving
3. Tradeoffs...
4. Real models must make choices like this all the time

# What do I do?

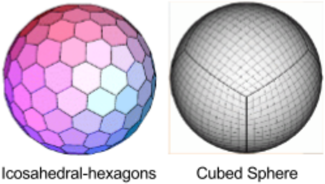
1. How can we develop numerical methods that conserve discrete analogues of important physical quantities: **mass, potential vorticity, total energy, potential enstrophy**?
2. What other **mimetic properties** should our models have?

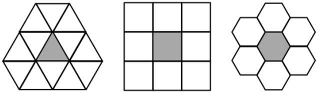


Shallow water simulation from Ringler et. al 2010

# Specifically?

1. How can we do these things (conservation, other mimetic properties) on **arbitrary, non-orthogonal polygonal grids**?



Planar Grids

# Conservation Laws: Non Canonical Hamiltonian Structure

1. Equations of atmospheric dynamics have a special structure
2. **Numerical models should reflect this structure**
3. Non-canonical infinite dimensional Hamiltonian mechanics:

$$\frac{\partial \vec{x}}{\partial t} = \mathbb{J}\frac{\delta \mathcal{H}}{\delta \vec{x}}$$

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

$$\frac{d\mathcal{F}}{dt} = -(\frac{\delta \mathcal{H}}{\delta \vec{x}}, \mathbb{J}\frac{\delta \mathcal{F}}{\delta \vec{x}})$$

$\mathbb{J}$ - symplectic operator
$\mathcal{H}$ - Hamiltonian functional
$\mathcal{Z}$ - Casimir functional
$\mathcal{F}$ - arbitrary functional
$(,)$ - inner product
$\vec{x}$ - set of variables

# Conservation Properties

## Hamiltonian (Energy)

Requires only the anti-symmetry of $\mathbb{J}$ and the positive definiteness of $(,)$ (and therefore $\mathcal{H}$). Fluid dynamical systems: $\mathcal{H}$ is **total energy**.

## Casimirs (Mass, Potential Vorticity, Potential Enstrophy)

Functionals $\mathcal{Z}$ such that

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

will be conserved. Some important examples are the **mass**, **potential vorticity** and **potential enstrophy**.

# Shallow Water Equations

## Vector Invariant Shallow Water Equations

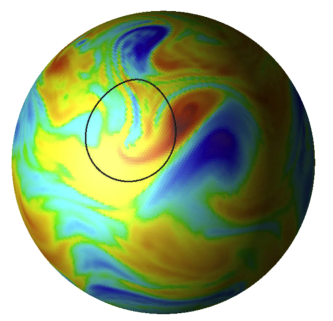
$$\vec{x} = (h, \vec{u})$$

$$\mathbb{J} = \begin{pmatrix} 0 & \vec{\nabla}\cdot \\ \vec{\nabla} & q\hat{k}\times \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(h, h) + \frac{1}{2}(\vec{F}, \vec{u})$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ \vec{F} \end{pmatrix} = \begin{pmatrix} gh + K \\ h\vec{u} \end{pmatrix}$$

$$\mathcal{Z} = \int_{\Omega} d\Omega h C(q)$$



Shallow water simulation from Ringler et. al 2010

# Generalized C Grid Discretization

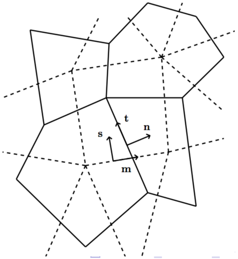
- Discrete variables are $m_i = \int hdA$ (discrete primal 2-form) and $u_e = \int \vec{u} \cdot \vec{dl}$ (discrete dual 1-form)
- C grid staggering ($m_i$ at cell centers, $u_e$ at edges)
- General formulation is:

$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & Q \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathbf{I}} + \frac{1}{2}(F_e, u_e)_{\mathbf{H}}$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I}\Phi_i \\ \mathbf{H}F_e \end{pmatrix}$$



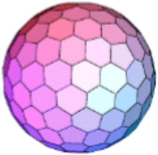

Non-orthogonal primal-dual C grid (from Dubos 2012)

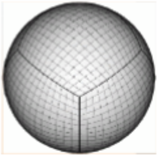
# Discrete Energy Conservation

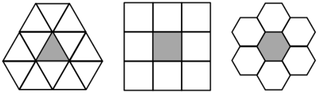
Requires two things:

1. $\mathbb{J}$ is anti-symmetric: $D_2^T = -\bar{D}_1$, and $\mathbf{Q} = \mathbf{Q}^T$
2. $\mathcal{H}$ is positive definite: $\mathbf{I}$ and $\mathbf{H}$ are symmetric positive definite

This gives semi-discrete energy conservation (fully discrete is a different matter).


Icosahedral-hexagons


Cubed Sphere


Planar Grids

## Summary

1. General discrete framework can conserve **mass, potential vorticity, total energy and potential enstrophy** on **general, non-orthogonal polygonal meshes**
2. Framework cleanly splits **topological** ($\bar{D}_1$,$D_2$, etc.) and **metrical** (**I**,**H**,etc.) aspects; can change one component without changing others
3. Getting total energy and potential enstrophy conservation together is tricky (having only one is "easy")
4. Hamiltonian/DEC framework also has useful mimetic properties (linear stability, no spurious vorticity production, etc.)

# Mass and Potential Vorticity Conservation

## Mass Conservation

$$\frac{\partial m_i}{\partial t} + D_2 \mathbf{H} F_e = 0$$

Local and global conservation by form alone ("flux form", independent of how $F_e$ is formulated)

## Potential Vorticity Conservation

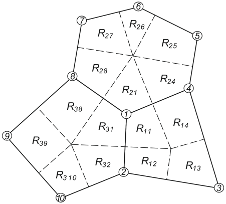
$$\frac{\partial m_v q_v}{\partial t} + \bar{D}_2 \mathbf{Q} \mathbf{H} F_e = 0$$

Local and global conservation by form alone ("flux form", independent of how $\mathbf{Q}$ is formulated)

# Discrete Potential Enstrophy

$$\mathcal{Z}_{\mathcal{C}} = \frac{1}{2}(\eta_v, \mathbf{J}^{-1} q_v)_{\mathbf{J}}$$

$$\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ \bar{D}_2{}^T q_v \end{pmatrix} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$




Construction of **R** from Thuburn et. al 2009

- Discrete vorticity is $\zeta_v = \bar{D}_2 u_e$
- Mass-weighted potential vorticity $m_v q_v = \zeta_v + f = \eta_v$
- $m_v = \mathbf{R} m_i$; **R** maps primal 2-forms to dual 2-forms

# Discrete Potential Enstrophy Conservation

## Conservation

Casimir conservation requires

$$\mathbb{J}\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = 0$$

which gives

$$D_2 D_1 q_v = 0 \qquad (1)$$

$$-\bar{D}_1 \mathbf{R}^T \frac{q_v^2}{2} + \mathbf{Q} D_1 q_v = 0 \qquad (2)$$

## Operator Requirements

(1) is automatic ($D_2 D_1 = 0$)
(2) is tricky- depends on $R^T$

- TRiSK and Arakawa and Lamb 1981 schemes both construct $\mathbf{Q}$ such that (2) is satisfied
- Only Arakawa and Lamb 1981 also has $\mathbf{Q} = \mathbf{Q}^T$ (energy conservation)

- Form of (2) gives hope that Arakawa and Lamb 1981 scheme can be extended to non-orthogonal, arbitrary polygonal meshes