Extension of the 1981 Arakawa and Lamb Scheme to Arbitrary Grids

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Key Papers

Arakawa and Lamb 1981

A Potential Enstrophy and Energy Conserving Scheme for the Shallow Water Equations

AKIO ARAKAWA AND VIVIAN R. LAMB[1]

Salmon 2004

Poisson-Bracket Approach to the Construction of Energy- and Potential-Enstrophy-Conserving Algorithms for the Shallow-Water Equations

RICK SALMON

Thuburn and Cotter 2012

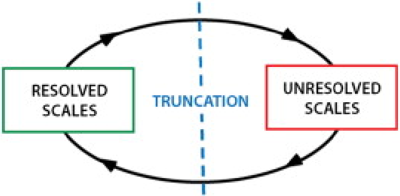
A FRAMEWORK FOR MIMETIC DISCRETIZATION OF THE ROTATING SHALLOW-WATER EQUATIONS ON ARBITRARY POLYGONAL GRIDS*

J. THUBURN† AND C. J. COTTER‡

1. AL81 is original paper of scheme
2. S04 provided a very useful framework for the development of schemes
3. TRISK papers extended scheme in different direction
4. This work is largely about a merger of these two approaches

Broad Research Overview: Dynamical Cores

1. Develop discrete models of the atmosphere
2. Dynamical core: deals with "resolved processes"
3. Parameterizations: deal with "unresolved processes"
4. Model: dynamical core + parameterizations



Key Principles of Numerical Modeling

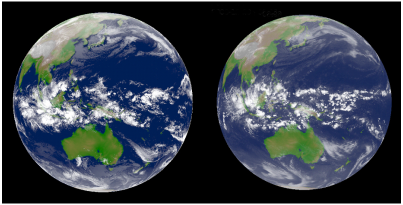
1. Not solving arbitrary PDEs- physical system
2. No analytic solutions
3. Differential equations $\rightarrow$ algebraic equations
4. **Do algebraic solutions have the same properties as the differential solutions?**

1. Convergence: Algebraic equations -¿ differential equations when dx,dt-¿0
2. Consistent: Solutions of algebraic equations approach solutions of differential equations as dx,dt-¿0
3. Ex. forward in time, centered in space for advection: convergent but not consistent
4. But convergence and consistently can still be wrong: ex. mean state of BVE equations using different discretizations

Philosophy of Dynamical Core Design

1. Models should respect underlying physics
    1. Energetics
    2. PV dynamics
    3. Wave dynamics
    4. **Conserved quantities- energy, potential enstrophy**
    5. **Mimetic properties**

$$\vec{\nabla} \times \vec{\nabla}\phi = 0$$



1. Energetics: conservation, conversion
2. PV dynamics: balanced flow, invertibility, etc.
3. Wave dynamics: IG, Rossby, Kelvin, etc.
4. Mass, momentum, PV, tracer conservation, etc.
5. Mimetic properties have a both physical interpretation and a mathematical one

Shallow Water Equations

Vector Invariant Shallow Water Equations

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + q\hat{k} \times (h\vec{u}) + \vec{\nabla}\Phi = 0$$

$$\vec{x} = (h, \vec{u})$$

$$\frac{\partial \vec{x}}{\partial t} = \mathbb{J}\frac{\delta \mathcal{H}}{\delta \vec{x}}$$

$$\mathbb{J} = \begin{pmatrix} 0 & -\vec{\nabla}\cdot \\ -\vec{\nabla} & -q\hat{k}\times \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2}g(h, h) + \frac{1}{2}(\vec{F}, \vec{u})$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ h\vec{u} \end{pmatrix}$$

1. Discretize J and H- independent!

Shallow Water Equations: (Subset of) Properties

Mimetic

No Spurious Vorticity Production:

$$\vec{\nabla} \times \vec{\nabla}\phi = 0$$

Pressure Gradient is Energy Conserving:

$$(\vec{\nabla})^* = -\vec{\nabla}\cdot$$

Conserved Quantities

Total Energy $\int_\Omega \frac{gh^2}{2} + \frac{h|\vec{u}|^2}{2}$

Potential Enstrophy $\int_\Omega h\frac{q^2}{2}$

1. Both physical and mathematical interpretation of mimetic properties

Arakawa and Lamb 1981 Scheme: Desirable Properties

(A Subset of) Desirable Properties

1. No spurious vorticity production (curl-free gradient)
2. Energy-conserving pressure gradient force (divergence and gradient are adjoints)
3. Total energy and potential enstrophy conservation

1. First two are mimetic properties
2. Last one can be done using Hamiltonian methods

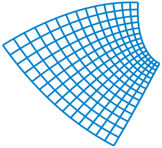
Arakawa and Lamb 1981 Scheme: Limitations

(A Subset of) AL81 Shortcomings

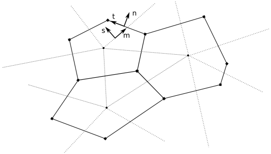
1. Restricted to logically square, orthogonal grids

TRiSK: Ringler, Skamarok, Klemp, Thuburn, Cotter, Dubos

1. General, non-orthogonal polygonal grids
2. **Choose between total energy and potential enstrophy conservation**



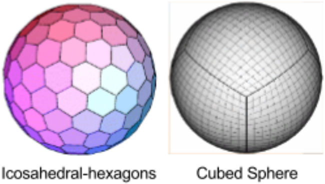
Logically square, orthogonal grid

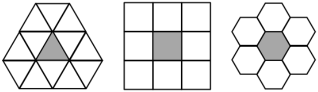


General, non-orthogonal grid

Recap: What am I trying to accomplish?

1. How can Arakawa and Lamb 1981 be extended to on **arbitrary, non-orthogonal polygonal grids**?



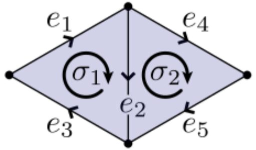


Mimetic Methods (Discrete Exterior Calculus)

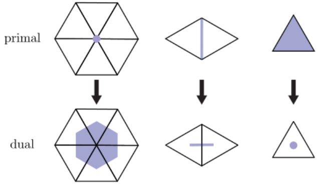
Mimetic Operators

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \iff \bar{D}_2 \bar{D}_1 = 0$$

$$(\vec{\nabla})^* = -\vec{\nabla} \cdot \iff D_2 = -\bar{D}_1^{\,T}$$



(Discrete) Exterior Derivative



Primal-Dual Grid

1. De-Rham cohomology

Conservation Laws (Hamiltonian Mechanics)

Hamiltonian (Energy)

$$\mathbb{J}^T = -\mathbb{J}$$

$\mathcal{H}$ is positive definite

Casimirs (Potential Enstrophy)

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

Discrete Conservation

Enforce these conditions in discrete case $\rightarrow$ conservation!

1. J and H are indep
2. Tricky to do both at the same time- this is my contribution

Recap: Conservative, Mimetic Methods

Conservation Laws

1. Hamiltonian Mechanics
2. Salmon, Dubos, Gassmann, Sommer, Nevir, others

Mimetic Properties

1. Discrete Exterior Calculus
2. Thuburn, Cotter, others

Conservative, Mimetic Methods

1. Use mimetic operators to build a discrete (quasi-)Hamiltonian system
2. Unifies two important lines of research

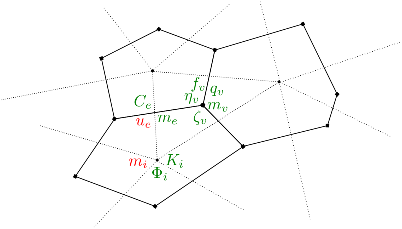
Generalized C Grid Discretization: Hamiltonian-DEC

- Discrete variables are $m_i = \int hdA$ and $u_e = \int \vec{u} \cdot \vec{dl}$
- C grid staggering (m_i at cell centers, u_e at edges)
- General formulation is:

$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & \mathbf{Q} \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathbf{I}} + \frac{1}{2} (C_e, u_e)_{\mathbf{H}}$$



Q operator is the remaining hurdle

General Form of Q

Following Salmon 2004, set

$$\mathbf{Q} F_e = \sum_{e' \in ECP(e)} \sum_{v \in VC(i)} q_v \alpha_{e,e',v} F_{e'}$$

What are $\alpha_{e,e',v}$'s?

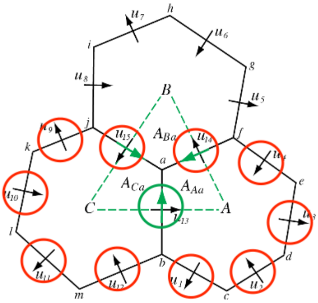


Diagram of Q operator (from Bill Skamarock)

1. This form is identical to AL81 and S04 (for given stencil)

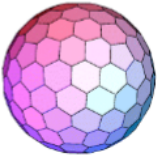
Discrete Conservation

Energy

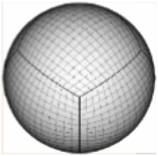
$$\mathbb{J}^T = -\mathbb{J} \longrightarrow \mathbf{Q} = -\mathbf{Q}^T \longrightarrow \alpha_{e,e',v} = -\alpha_{e',e,v}$$

Potential Enstrophy

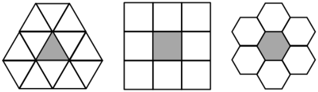
$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0 \longrightarrow \text{ linear system of equations } \longrightarrow \mathbf{A}\vec{\alpha} = \vec{b}$$



Icosahedral-hexagons



Cubed Sphere



Planar Grids

Solving $\mathbf{A}\vec{\alpha} = \vec{b}$

Issue: System is too large

Geodesic grid: O(100) coefficients per cell, all coefficients are interdependent $\rightarrow$ not feasible for realistic grids

Solution: Subsystem Splitting

$$\mathbf{A}\vec{\alpha} = \vec{b} \longrightarrow \sum_i \mathbf{A}_i\vec{\alpha}_i = \vec{b}_i$$

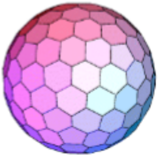
Split into independent subsystems for each cell!

System has been solved for various planar and geodesic grids

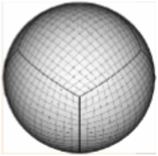
1. **A** is a rectangular matrix of integers (0,1,-1)
2. Underdetermined system- free parameters
3. $\vec{\alpha}$ are the coefficients
4. $\vec{b}$ is known (function of grid geometry)

Recap: What have I accomplished?

- Arakawa and Lamb 1981 extended to arbitrary grids
- Coefficients can be precomputed (efficiently)



Icosahedral-hexagons



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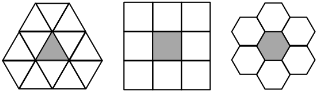
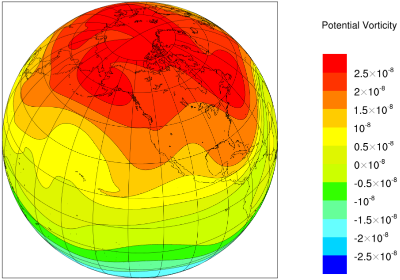


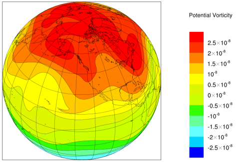
Planar Grids

Flow over an Isolated Mountain (Williamson Test Case 5)

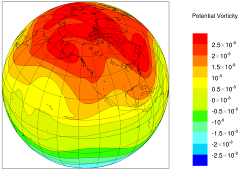
Potential Vorticity at Day 50, Doubly Conservative



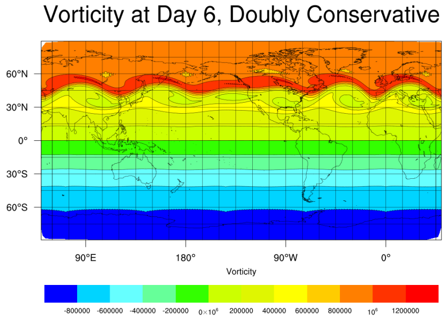
Potential Vorticity at Day 50, Enstrophy Conserving

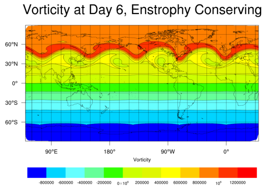


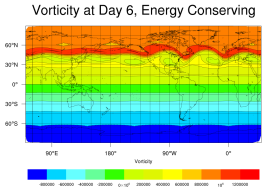
Potential Vorticity at Day 50, Energy Conserving



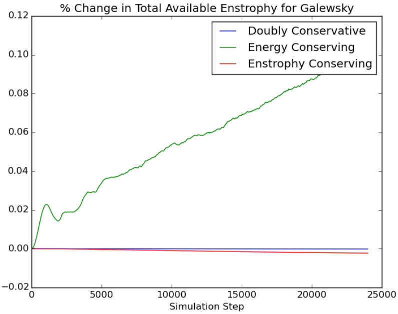
Galewsky et. al (Unstable Jet)

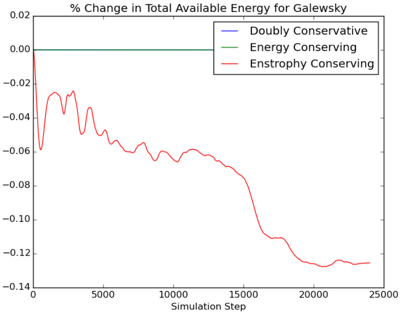






Conservation Properties (Galewsky Test Case)





These results are WITHOUT any added dissipation

Summary and Conclusions

Conclusions

1. Combined Salmon 2004 (Hamiltonian) and Thuburn, Cotter and Dubos 2012 (Mimetic/Discrete Exterior Calculus) approaches to develop an extension of Arakawa and Lamb 1981 to arbitrary polygonal grids
2. Preserves all desirable properties of AL81

Future Work

1. Detailed evaluation of scheme
2. Comparison to Z-grid scheme (Salmon 2007)
3. Test approach on other grids (cubed sphere)