Propagating Modes in Discrete Models of the Linear Shallow Water Equations

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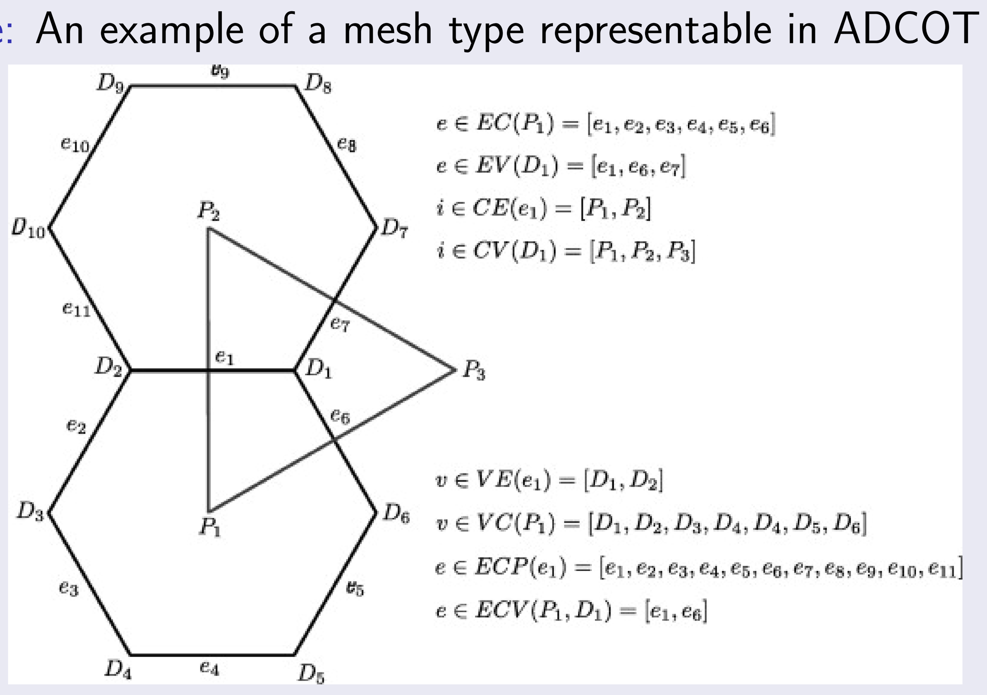
Abstract

- Shallow water equations are a useful analogue of the fully compressible Euler equations for atmospheric model development
- Linear properties (propagating and stationary modes) play an important physical role in the behaviour of the atmosphere
- Using the Atmospheric Dynamical COre Testbed (ADCOT) built on top of Morphe, the linear properties of two finite-difference schemes (TRiSK: Ringler, Thuburn, Klemp & Skamarock 2010 and HR95: Heikes & Randall 1995) are compared to those of the continuous equations

ADCOT: Design & Implementation

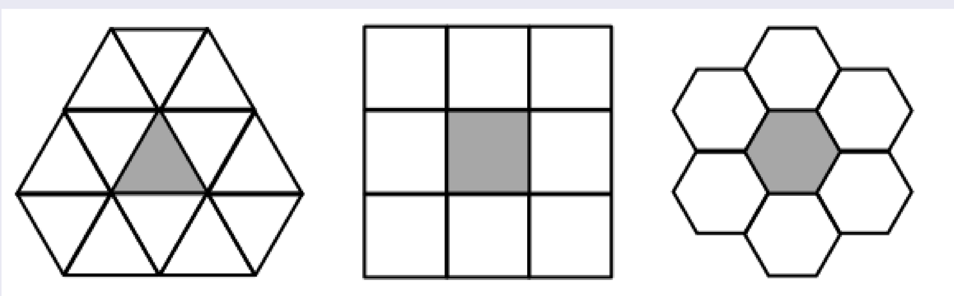
- Horizontal meshes represented as sets of relationships between elements

Figure: An example of a mesh type representable in ADCOT



- Perfect planar square, hexagonal and triangular meshes are currently implemented (more to come, especially geodesic and cubed-sphere)

Figure: Supported planar meshes in ADCOT



- Variables (scalar, vector, array, vector component) placed arbitrarily on mesh elements
- Operators defined as sparse matrices (linear) or algebraic combinations of vector and sparse matrix operations (non-linear)

$$\vec{\nabla} \cdot u \rightarrow \mathbf{D}\vec{u}$$

$$\vec{\nabla}(KE + gh) \rightarrow \mathbf{G}(\mathbf{K}u_e^2 + g\vec{h})$$

- Compile and run-time polymorphism provide flexibility
- Models are written in Fortran 90 using Cheetah for compile time polymorphism; code is heavily shared between linear and non-linear models
- Uses PETSc and SLEPc to provide grid management, linear/eigenvalue solvers and I/O
- Analysis packages are written in Python using the Numpy, Scipy and Matplotlib libraries
- Adams-Bashford and Runge-Kutta explicit time stepping
- TRiSK and HR95 horizontal discretizations (more to come!)

Linear Shallow Water Equations on an f-plane

- Momentum Form Linear Shallow Water Equations

$$\frac{\partial \vec{u}}{\partial t} = -f\hat{k} \times \vec{u} + g\vec{\nabla}h$$

$$\frac{\partial h}{\partial t} = -H(\vec{\nabla} \cdot u)$$

- Vorticity-Divergence Form Linear Shallow Water Equations

$$\frac{\partial \zeta}{\partial t} = -f\delta$$

$$\frac{\partial \delta}{\partial t} = f\zeta - g\vec{\nabla}^2 h$$

$$\frac{\partial h}{\partial t} = -H\delta$$

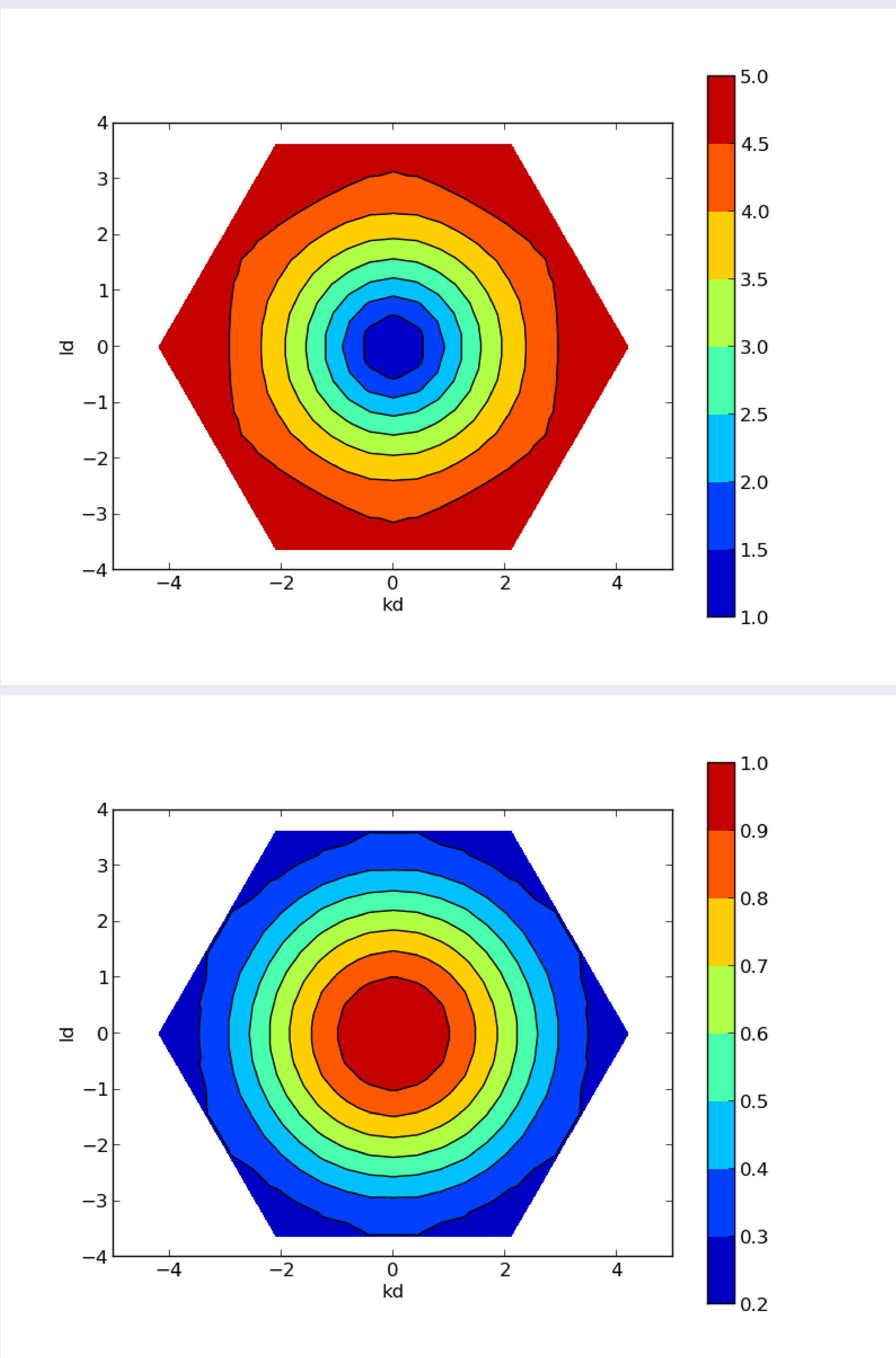
- Propagating Modes (Inertia-Gravity Waves)

$$\left(\frac{\sigma}{f}\right)^2 = 1 + \lambda^2(k^2 + l^2)$$

Results: Numerical Dispersion Relationships (Propagating Modes)

- Dispersion relationship calculated as $\frac{d\vec{x}}{dt} = \mathbb{L}\vec{x} \rightarrow i\omega\vec{x} = \mathbf{A}\vec{x}$ (eigenvalue problem)
- Fourier transforms are used to determine which spatial wavenumbers each eigenvector/eigenvalue pair is associated with
- Well resolved Rossby radii ($\frac{\lambda}{d} = 2.0$) and poorly resolved Rossby radii ($\frac{\lambda}{d} = 0.1$) tested
- Both TRiSK (C-grid) and HR95 (Z-Grid) perfect planar square and hexagonal meshes investigated (only hexagonal grid results shown; for a 15x15 mesh)
- Dispersion relations are mapped to the Brillouin zone associated with a perfect planar hexagonal mesh
- Results are identical to theoretical dispersion relations (within numerical bounds)

Figure: C-Grid dispersion relations for $\frac{\lambda}{d} = 2.0$ (top) and $\frac{\lambda}{d} = 0.1$ (bottom) where $\omega = \frac{\sigma}{f}$



Results: Propagating Modes

Figure: Z-Grid dispersion relations for $\frac{\lambda}{d} = 2.0$ (top) and $\frac{\lambda}{d} = 0.1$ (bottom) where $\omega = \frac{\sigma}{f}$

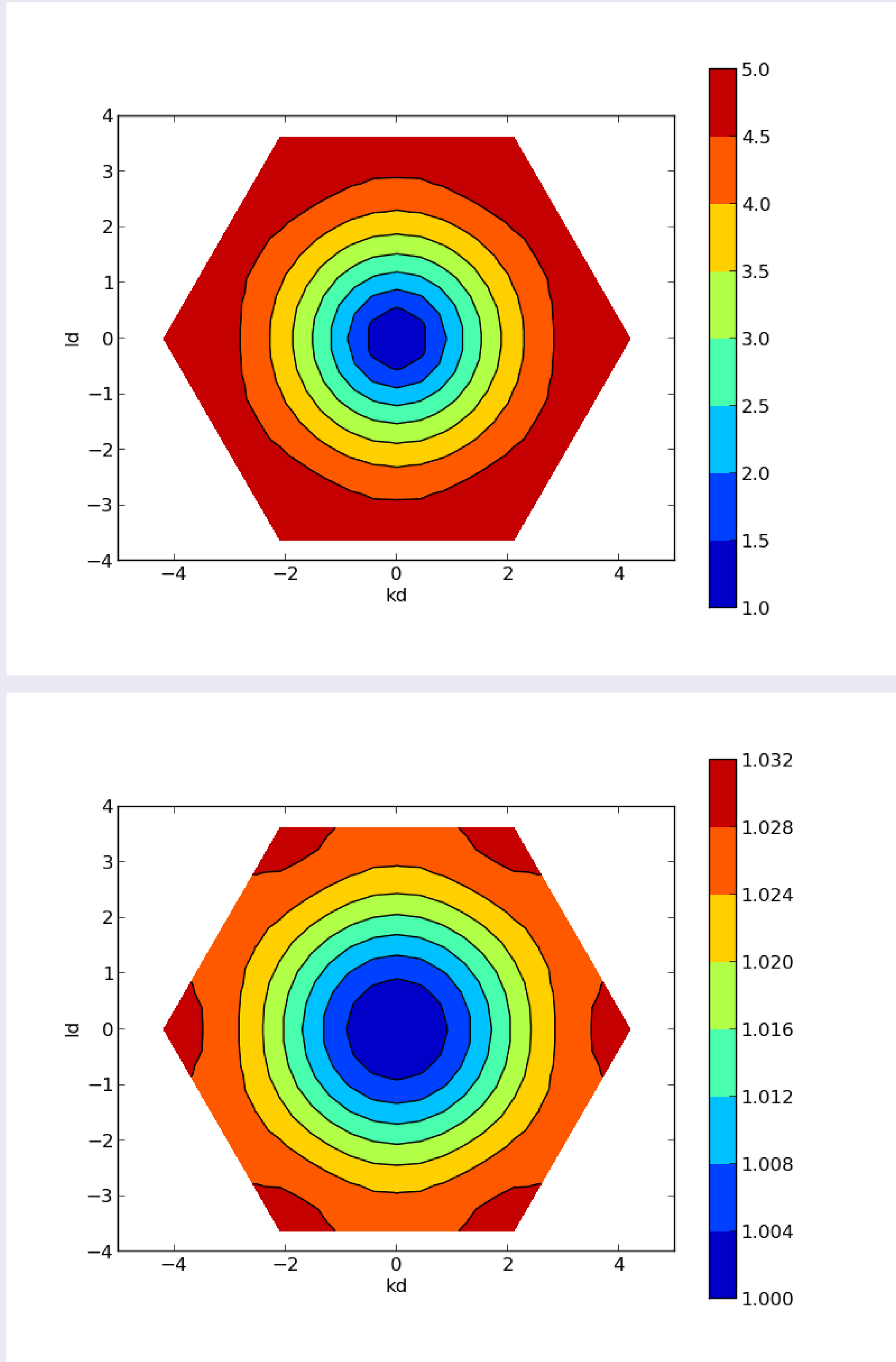
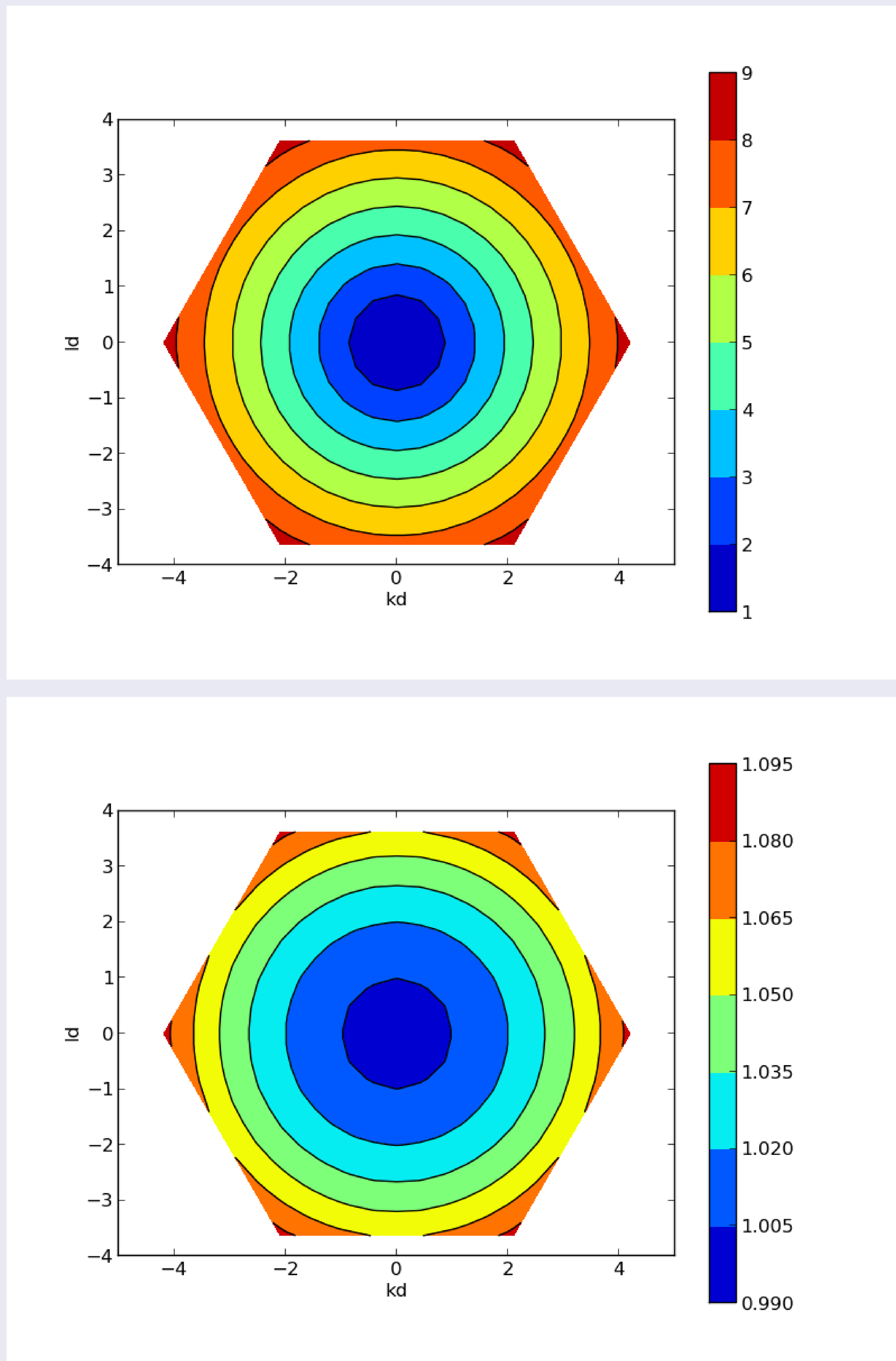


Figure: Continuous dispersion relations for $\frac{\lambda}{d} = 2.0$ (top) and $\frac{\lambda}{d} = 0.1$ (bottom) where $\omega = \frac{\sigma}{f}$



Conclusions

- ADCOT provides a useful framework for inter-comparison of various numerical schemes for the nonlinear shallow water equations
- Two very different schemes (TRiSK and HR95) can be implemented under the same code framework; previous results for these scheme are reproduced