Structure Preserving Discretization of the Rotating Shallow Water Equations

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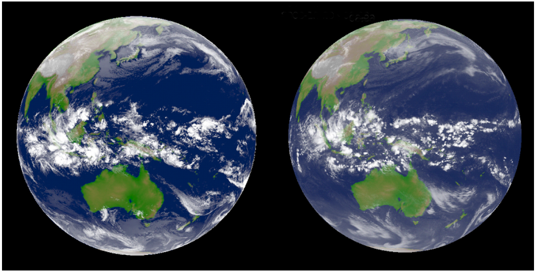
Outline

Who am I?

1. PhD Student in Atmospheric Science at Colorado State University
2. Work for Dave Randall at Center for Multiscale Modelling of Atmospheric Processes (CMMAP)
3. Did undergraduate work at Carnegie Mellon University

Broad Research Overview

1. Develop discrete models of the atmosphere (dynamical cores)
2. Interested in numerical methods that preserve Hamiltonian structure of the equations
3. Also interested in mimetic methods (discrete exterior calculus)



Which is the model, and which is reality? (from Miura et. al 2007)

Non Canonical Hamiltonian Structure

1. Equation of atmospheric dynamics have a special structure
2. Structure underlies most of the important theory from atmospheric dynamics (conservation laws, balanced models, disturbance invariants, stability theorems, etc.)
3. **Numerical models should reflect this structure**
4. Non-canonical infinite dimensional Hamiltonian mechanics:

$$\frac{\partial \vec{x}}{\partial t} = \mathbb{J}\frac{\delta \mathcal{H}}{\delta \vec{x}}$$

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

$$\frac{d\mathcal{F}}{dt} = -(\frac{\delta \mathcal{H}}{\delta \vec{x}}, \mathbb{J}\frac{\delta \mathcal{F}}{\delta \vec{x}})$$

$\mathbb{J}$ - symplectic operator
$\mathcal{H}$ - Hamiltonian functional
$\mathcal{Z}$ - Casimir functional
$\mathcal{F}$ - arbitrary functional
$(,)$ - inner product
$\vec{x}$ - set of variables

Shallow Water Equations

Vector Invariant Shallow Water Equations

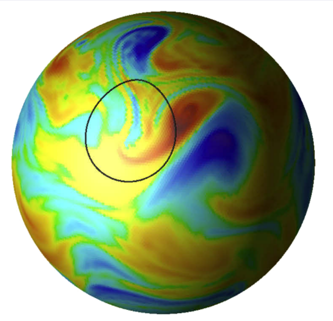
$$\vec{x} = (h, \vec{u})$$

$$\mathbb{J} = \begin{pmatrix} 0 & \vec{\nabla}\cdot \\ \vec{\nabla} & q\hat{k}\times \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2}g(h,h) + \frac{1}{2}(\vec{F},\vec{u})$$

$$\frac{\delta\mathcal{H}}{\delta\vec{x}} = \begin{pmatrix} \Phi \\ \vec{F} \end{pmatrix} = \begin{pmatrix} gh + K \\ h\vec{u} \end{pmatrix}$$

$$\mathcal{Z} = \int_\Omega d\Omega h C(q)$$



Shallow water simulation from Ringler et. al 2010

Conservation Properties

Hamiltonian (Energy)

Requires only the anti-symmetry of $\mathbb{J}$ and the positive definiteness of $(,)$ (and therefore $\mathcal{H}$). Fluid dynamical systems: $\mathcal{H}$ is **total energy**.

Casimirs (Mass, Potential Vorticity, Potential Enstrophy)

Functionals $\mathcal{Z}$ such that

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

will be conserved. Important examples are the **mass**, **potential vorticity** ($q = \frac{\eta}{h} = \frac{\zeta + f}{h}$) and **potential enstrophy** ($hq^2 = \frac{\eta^2}{h}$).

Generalized C Grid Discretization

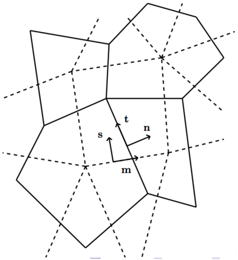
- Discrete variables are $m_i = \int hdA$ (discrete primal 2-form) and $u_e = \int \vec{u} \cdot \vec{dl}$ (discrete dual 1-form)
- C grid staggering (m_i at cell centers, u_e at edges)
- General formulation is:

$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & Q \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathbf{I}} + \frac{1}{2}(F_e, u_e)_{\mathbf{H}}$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I}\Phi_i \\ \mathbf{H}F_e \end{pmatrix}$$



Non-orthogonal primal-dual C grid (from Dubos 2012)

Mass and Potential Vorticity Conservation

Mass Conservation

$$\frac{\partial m_i}{\partial t} + D_2 \mathbf{H} F_e = 0$$

Local and global conservation by form alone ("flux form", independent of how F_e is formulated)

Potential Vorticity Conservation

$$\frac{\partial m_v q_v}{\partial t} + \bar{D}_2 \mathbf{Q} \mathbf{H} F_e = 0$$

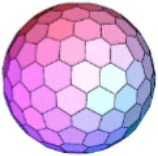
Local and global conservation by form alone ("flux form", independent of how $\mathbf{Q}$ is formulated)

Discrete Energy Conservation

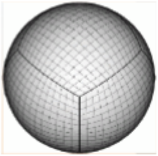
Requires two things:

1. $\mathbb{J}$ is anti-symmetric: $D_2^T = -\bar{D}_1$, and $\mathbf{Q} = \mathbf{Q}^T$
2. $\mathcal{H}$ is positive definite: $\mathbf{I}$ and $\mathbf{H}$ are symmetric positive definite

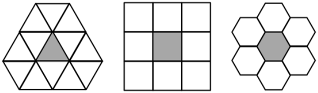
This gives semi-discrete energy conservation (fully discrete is a different matter).



Icosahedral-hexagons



Cubed Sphere

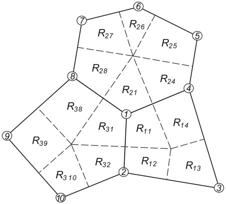


Planar Grids

Discrete Potential Enstrophy

$$\mathcal{Z}_{\mathcal{C}} = \frac{1}{2}(\eta_v, \mathbf{J}^{-1} q_v)_{\mathbf{J}}$$

$$\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ \bar{D}_2{}^T q_v \end{pmatrix} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$



Construction of **R** from Thuburn et. al 2009

- Discrete vorticity is $\zeta_v = \bar{D}_2 u_e$
- Mass-weighted potential vorticity $m_v q_v = \zeta_v + f = \eta_v$
- $m_v = \mathbf{R} m_i$; **R** maps primal 2-forms to dual 2-forms

Discrete Potential Enstrophy Conservation

Conservation

Casimir conservation requires

$$\mathbb{J}\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = 0$$

which gives

$$D_2 D_1 q_v = 0 \qquad (1)$$

$$-\bar{D}_1 \mathbf{R}^T \frac{q_v^2}{2} + \mathbf{Q} D_1 q_v = 0 \qquad (2)$$

Operator Requirements

(1) is automatic ($D_2 D_1 = 0$)
(2) is tricky- depends on R^T

- TRiSK and Arakawa and Lamb 1981 schemes both construct $\mathbf{Q}$ such that (2) is satisfied
- Only Arakawa and Lamb 1981 also has $\mathbf{Q} = \mathbf{Q}^T$ (energy conservation)

- Form of (2) gives hope that Arakawa and Lamb 1981 scheme can be extended to non-orthogonal, arbitrary polygonal meshes

Summary

1. General discrete framework can conserve **mass, potential vorticity, total energy and potential enstrophy** on **general, non-orthogonal polygonal meshes**
2. Framework cleanly splits **topological** ($\bar{D}_1$,D_2, etc.) and **metrical** ($\mathbf{I}$,$\mathbf{H}$,etc.) aspects; can change one component without changing others
3. Getting total energy and potential enstrophy conservation together is tricky (having only one is "easy")
4. Hamiltonian/DEC framework also has useful mimetic properties (linear stability, no spurious vorticity production, etc.)

Future Work

1. **Extend Arakawa and Lamb 1981 to non-square and non-orthogonal grids (ongoing)**
2. **Develop an analogous framework for the vorticity-divergence formulation of the equations (ongoing)**
3. Fully discrete conservation (ie including time discretization)

Thanks

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Any Questions?