Discrete Potential Enstrophy Conservation in the Nonlinear Shallow Water Equations

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Continuous Hamiltonian Formulation

- Consider the nonlinear shallow water equations in vector invariant form with $\vec{x} = (h, \vec{u})$
- Their evolution is governed by $\frac{\partial \vec{x}}{\partial t} = \mathbb{J}\frac{\delta \mathcal{H}}{\delta \vec{x}}$ with

$$\mathbb{J} = \begin{pmatrix} 0 & \vec{\nabla}\cdot \\ \vec{\nabla} & q\hat{k}\times \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2}(h\vec{u}, \vec{u}) + \frac{1}{2}(gh, h)$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ h\vec{u} \end{pmatrix}$$

1. Energy conservation comes from $\mathbb{J}$ being anti-symmetric
2. Potential enstrophy conservation comes from $\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}}$ lying in the nullspace of $\mathbb{J}$

In general, having a discrete analogue of (1) is "easy", while a discrete analogue of (2) is "hard". Having both (1) and (2) on a C grid has not been done, outside of Arakawa and Lamb 1981.

Discrete Hamiltonian Formulation (Arbitrary C Grid)

- Based on work of Thuburn, Cotter, Ringler, Skamarok, Klemp, Dubos, Weller, Salmon and others (drawing heavily from Thuburn & Cotter 2013)
- Hybrid of Hamiltonian and Discrete Exterior Calculus approaches
- Discrete variables are $m_i = \int h dA$ (discrete primal 2-form) and $u_e = \int \vec{u} \cdot \vec{dl}$ (discrete dual 1-form), staggered in a C grid on arbitrary primal-dual polygonal grids (low-order finite difference/finite volume)

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & Q \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathrm{I}} + \frac{1}{2}(F_e, u_e)_{\mathrm{H}}$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi_i \\ F_e \end{pmatrix}$$

- D_1, D_2 are incidence matrices on primal grid; $\bar{D}_1, \bar{D}_2$ are incidence matrices on dual grid
- $D_2 D_1 = 0$, $\bar{D}_2 \bar{D}_1 = 0$ (discrete analogues of $\vec{\nabla} \cdot \vec{\nabla}^T = 0$ and $\vec{\nabla}^T \cdot \vec{\nabla} = 0$)
- $-\bar{D}_1{}^T = D_2, D_1^T = \bar{D}_2$ (ensures adjointness of discrete $\vec{\nabla}\cdot$ and $\vec{\nabla}$ operators)
- Q is the discrete transpose nonlinear PV flux operator
- Φ_i and F_e are the discrete Bernoulli function and mass flux
- $(,)_{\mathrm{I}}, (,)_{\mathrm{H}}$ are discrete inner products induced by Hodge stars (see Thuburn & Cotter 2013 for more details)
- There many possible choices for these operators
- Careful selection will give steady geostrophic modes; conservative, consistent and compatible PV advection; linear energy and potential enstrophy conservation; mass conservation, a discrete Hodge decomposition and other desirable mimetic properties, even on non-orthogonal grids
- **However, so far it has not been possible to obtain BOTH total energy and potential enstrophy conservation**

Potential Enstrophy Conservation (Continuous)

- Consider the potential enstrophy $\mathcal{Z}_{\mathcal{C}} = \frac{1}{2}(hq, q)$
- Note that

$$\mathbb{J}\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} 0 & \vec{\nabla}\cdot \\ \vec{\nabla} & q\hat{k}\times \end{pmatrix} \begin{pmatrix} -\frac{q^2}{2} \\ -\vec{\nabla}^T q \end{pmatrix} = 0$$

$$-\vec{\nabla} \cdot \vec{\nabla}^T q = 0$$

$$\vec{\nabla}\frac{q^2}{2} + q\hat{k} \times \vec{\nabla}^T q = 0$$

- Relies on $\vec{\nabla} \cdot \vec{\nabla}^T = 0$, product rule for $\vec{\nabla} q^2$ and "transformation" between $\hat{k} \times \vec{\nabla}^T$ and $\vec{\nabla}$

Potential Enstrophy Conservation (Discrete)

- Using the DEC approach, the discrete vorticity dual 2-form is given by $\zeta_v = \bar{d}_2 u_e$
- A mass-weighted potential vorticity can then be defined as

$$m_v q_v = \zeta_v + f = \eta_v$$

where $m_v = \mathrm{R} m_i$ and R is a mapping from primal 2-forms to dual 2-forms
- This motivates the definition of discrete potential enstrophy as

$$\mathcal{Z}_{\mathcal{C}} = \frac{1}{2}(m_v q_v, q_v)_{\mathrm{J}}$$

- Functional derivatives are then given as

$$\frac{\delta \mathcal{Z}_{\mathcal{C}}}{\delta \vec{x}} = \begin{pmatrix} -\mathrm{R}^T \frac{q_v^2}{2} \\ \bar{D}_2{}^T q_v \end{pmatrix} = \begin{pmatrix} -\mathrm{R}^T \frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$

- Combine with discrete $\mathbb{J}$ to get

$$D_2 D_1 q_v = 0$$

$$-\bar{D}_1 \mathrm{R}^T \frac{q_v^2}{2} + \mathrm{Q} D_1 q_v = 0$$

- The first equation is satisfied INDEPENDENT of choice of R (due to properties of DEC primal-dual formulation)
- The second equation is more complicated, and requires careful construction of Q in combination with R. Note that is must hold for ARBITRARY values of q_v.
- The first term of the second equation can be rewritten as

$$-\bar{D}_1 \mathrm{R}^T \frac{q_v^2}{2} = -\bar{D}_1 \mathrm{R}^T q_v q_e$$

where $q_e = \sum_{VE} \frac{q_v}{2}$,
- Thus we see that the TRiSK choice of $\mathrm{Q} = q_e \mathrm{W}$ does in fact, preserve potential enstrophy
- For the AL1981 scheme, the second equation can also be written out as

$$-\sum_{CE} n_{ei} \sum_{VC} R_{iv} \frac{q_v^2}{2} + \sum_{ECP} Q_{e,e'} \sum_{VE} t_{ev} q_v = 0$$

- Careful choice of the $Q_{e,e'}$ coefficients as weighted sums of q_v gives cancellation, and also that $\mathrm{Q} = \mathrm{Q}^T$ (for total energy conservation)
- This has been done ONLY for a square grid
- However, the form suggests it might be possible on a general polygonal grid
- Important Question: **Are there other forms for Q that might be more amenable to a potential enstrophy conserving discretization?**

TRiSK (2010) Scheme

- Developed for orthogonal, Voronoi grids (Ringler et. al 2010 is the primary reference)
- Both total energy and potential enstrophy conserving variants
- R is defined as $R_{iv} = \frac{A_{iv}}{A_i}$, with $-\bar{D}_2 \mathrm{W} = \mathrm{R} D_2$ where $\mathrm{W} = \mathrm{W}^T$ is the anti-symmetric discrete vector reconstruction operator- it maps from primal 1-forms to dual 1-forms
- $\mathrm{Q} F_e = \frac{1}{2}\mathrm{W}(q_e F_e) + \frac{q_e}{2}\mathrm{W} F_e$ for the energy conserving variant, where q_e is some arbitrary potential vorticity value at the edge
- $\mathrm{Q} F_e = q_e \mathrm{W} F_e$ for the potential enstrophy conserving variant, where $q_e = \sum_{VE} \frac{q_v}{2}$
- There are other variants of TRiSK (Weller 2013, Thuburn et. al 2013) that change H, ϕ, Q or other operators in order to allow for better PV advection, non-Voronoi and/or non-orthogonal grids or other desirable properties

Arakawa and Lamb 1981 Scheme

- Potential enstrophy and total energy conserving scheme for logically square, orthogonal grids
- Uses SAME formulation as TRiSK (2010) when the latter is applied to a logically square, orthogonal grid; EXCEPT for the discretization of the Q term
- The stencil of Q is the same as TRiSK (the ECP stencil)
- Q is constructed such that $\mathrm{Q} = \mathrm{Q}^T$ (this ensures energy conservation)

Remaining Challenges

- Constructive method for potential enstrophy conserving Q on arbitrary grids, such that $\mathrm{Q} = \mathrm{Q}^T$ and the other desirable linear mimetic properties are retained
- Extension of this generalized framework to the vorticity-divergence form of the equations (will probably require the continuous $\zeta - \delta$ equations in exterior calculus and Hamiltonian form, work underway)
- Important question: **Are there alternative definitions of discrete potential enstrophy that are more amenable to constructive methods for Q?**
- Important question: **Can the generalized framework be extended to collocated (A grid) methods?**
- Related: **Can an A grid method that preserves total energy and potential enstrophy; and has all of the other desirable mimetic properties; be developed for arbitrary polygonal grids?**