A Hydrostatic Dynamical Core using Higher-Order Structure-Preserving Finite Elements

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(I) Desirable Model Properties

Realistic Simulations

Control Over Approximations
-Non-traditional/Deep
-Non-Spherical Geopotential
-Arbitrary Equations of State

Conservation
-Mass
-Energy
-Potential Vorticity
-Entropy

Mimetic Properties
-Discrete DeRham Cohmology
-Bounded Co-Chain Projections
$\vec{\nabla}\cdot\vec{\nabla}\times = 0 \quad \vec{\nabla}\times\vec{\nabla} = 0$
$(\vec{\nabla}\cdot)* = -\vec{\nabla}$

Linear Modes
-No Spurious Stationary Modes
-Good Dispersion Relationship
-No Spurious Branches of Waves
-No Inertial Modes

Stability
-No Hollingsworth Instability
-Nonlinear Stability

Efficiency
-Strong Scalability
-Weak Scalability
-Time to Solution

Accuracy
-Taylor Series Sense
-Convergence to Reference Solutions

PV Dynamics
-Steady Geostrophic Modes
-Compatibility
-Consistency

Get these properties through **Hamiltonian Formulation** and **Mimetic Discretization**

(II) Hamiltonian Formulation in a General Vertical Coordinate η

Prognose $(\mu, S, \vec{v})$ using

$$\frac{\partial \mu}{\partial t} + \vec{\nabla}\cdot\frac{\delta\mathcal{H}}{\delta\vec{v}} + \partial_\eta(\mu\dot{\eta}) = 0$$

$$\frac{\partial S}{\partial t} + \vec{\nabla}\cdot(s\frac{\delta\mathcal{H}}{\delta\vec{v}}) + \partial_\eta(s\mu\dot{\eta}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\vec{\nabla}\times\vec{v}}{\mu}\times\frac{\delta\mathcal{H}}{\delta\vec{v}} + \vec{\nabla}\frac{\delta\mathcal{H}}{\delta\mu} + s\vec{\nabla}\frac{\delta\mathcal{H}}{\delta S} + \frac{\mu\dot{\eta}}{\mu}\partial_\eta(\vec{v}) = 0$$

with Hamiltonian $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}[\mu,\vec{v},S,z] = \int \mu K(\vec{v},z) + \mu U(\frac{1}{\mu}\frac{\partial z}{\partial\eta},\frac{S}{\mu}) + \mu\Phi(z)$$

$$\frac{\delta\mathcal{H}}{\delta\vec{v}} = \mu\vec{u}$$

$$\frac{\delta\mathcal{H}}{\delta\mu} = K + \Phi + U + \mu\frac{\partial U}{\partial\mu}$$

$$\frac{\delta\mathcal{H}}{\delta S} = \mu\frac{\partial U}{\partial S}$$

and diagnose z using

$$\frac{\delta\mathcal{H}}{\delta z} = \mu\frac{\partial K}{\partial z} + \mu\frac{\partial\Phi}{\partial z} - \frac{\partial}{\partial\eta}(\mu\frac{\partial U}{\partial(\frac{\partial z}{\partial\eta})}) = 0 \text{ (Hydrostatic balance)}$$

where $\vec{v} = \vec{u} + \vec{R}(z)$ is the horizontal (covariant) absolute velocity, $\mu = \frac{1}{\alpha}\frac{\partial z}{\partial\eta}$ is the pseudo-density, z the height, $\mu\dot{\eta}$ is the vertical mass flux and $S = \mu s$ the mass-weighted entropy. These equations work for a deep, non-spherical atmosphere with an arbitrary equation of state $U(\alpha, s)$. Details are in [3].

Lagrangian Vertical Coordinate:
Defined by $\dot{\eta} = 0$.

Mass-based Vertical Coordinate:
Defined by $M(\eta) = \int_0^1 \mu d\eta = A(\eta)M_s + A(\eta)M_0$ and prognoses M_s instead of $\mu \rightarrow \mu = -\frac{\partial A}{\partial\eta}M_s - \frac{\partial B}{\partial\eta}M_0$, this also gives the equation for $\mu\dot{\eta}$:

$$\partial_\eta(\mu\dot{\eta}) = -\frac{\partial A}{\partial\eta}\int_0^1(\vec{\nabla}\cdot\frac{\delta\mathcal{H}}{\delta\vec{v}})d\eta - \vec{\nabla}\cdot\frac{\delta\mathcal{H}}{\delta\vec{v}}$$

Also must redefine $\frac{\delta\mathcal{H}}{\delta\mu}$ to conserve energy (vertical relabelling invariance):

$$\mu\partial_\eta\frac{\delta\mathcal{H}}{\delta\mu} = \partial_\eta\vec{v}\frac{\delta\mathcal{H}}{\delta\vec{v}} - S\partial_\eta\frac{\delta\mathcal{H}}{\delta S}$$

(III) General Compatible Galerkin Discretizations

- Select any 1D Spaces $\mathcal{A} = H^1$ and $\mathcal{B} = L_2$ such that: $\mathcal{A} \xrightarrow{\frac{d}{dx}} \mathcal{B}$
- Use tensor products to extend to n-dimensions
- Our (novel) choices of $\mathcal{A}$ and $\mathcal{B}$ are guided by linear mode properties and coupling to physics/tracer transport

$$\mathbb{W}_0 \xrightarrow[\vec{\nabla}]{d} \mathbb{W}_1 \xrightarrow[\vec{\nabla}\times]{d} \mathbb{W}_2 \xrightarrow[\vec{\nabla}\cdot]{d} \mathbb{W}_3$$

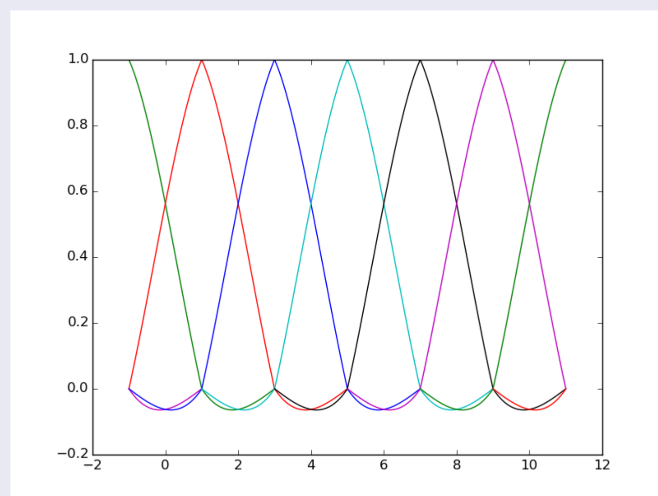
$$\mathbb{W}_0 \xleftarrow[\delta]{\vec{\nabla}\cdot} \mathbb{W}_1 \xleftarrow[\delta]{\vec{\nabla}\times} \mathbb{W}_2 \xleftarrow[\delta]{\vec{\nabla}} \mathbb{W}_3$$

$$(da^k, b^{k+1}) = (a^k, \delta b^{k+1})$$

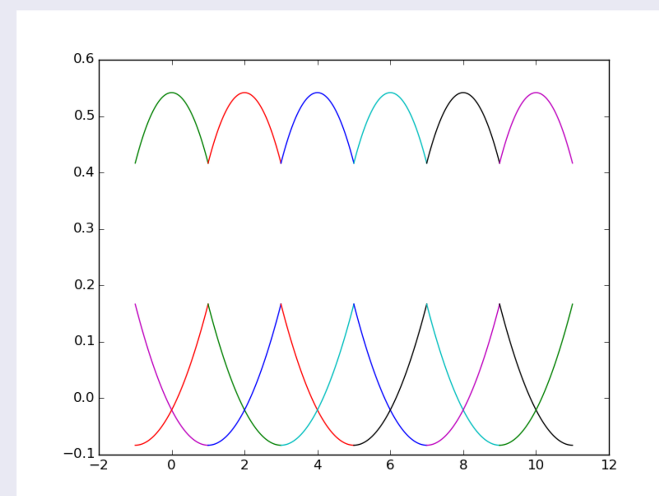
Primal deRham Complex
Exact sequence/bounded co-chain projection ($\vec{\nabla}\times\vec{\nabla} = 0$,etc)

(IV) Mimetic Galerkin Differences (MGD)

H^1 space defined following [1], with L_2 defined to be compatible following [4]. This is an arbitrary order extension of [2]. For 3rd order gives:



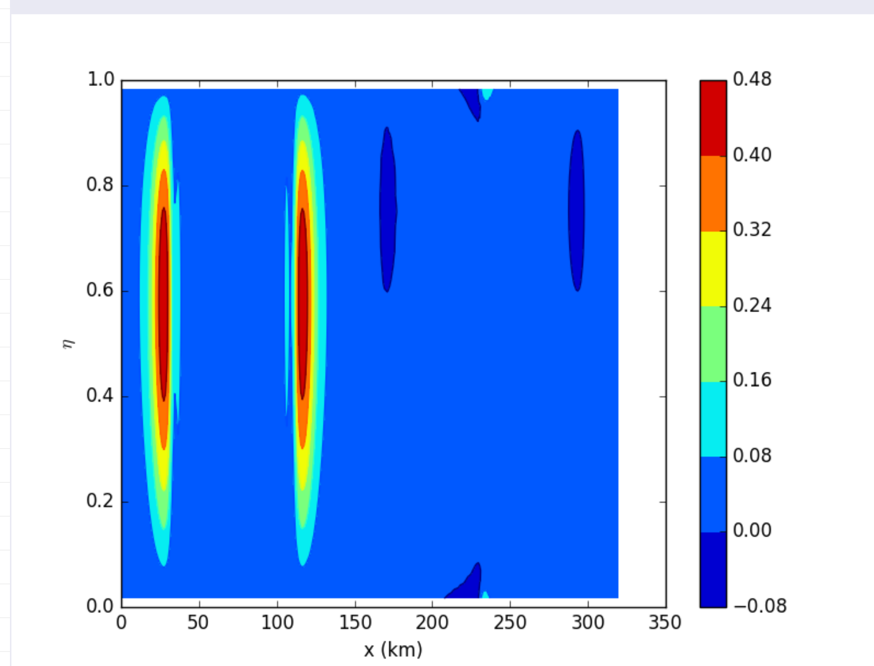
$\mathcal{A} = H_1$ Space (1D)



$\mathcal{B} = L_2$ Space (1D)

Single degree of freedom per geometric entity with higher order through larger stencils $\rightarrow$ no spectral gaps, easy coupling to physics/tracer transport, less local

(VII) Preliminary Results



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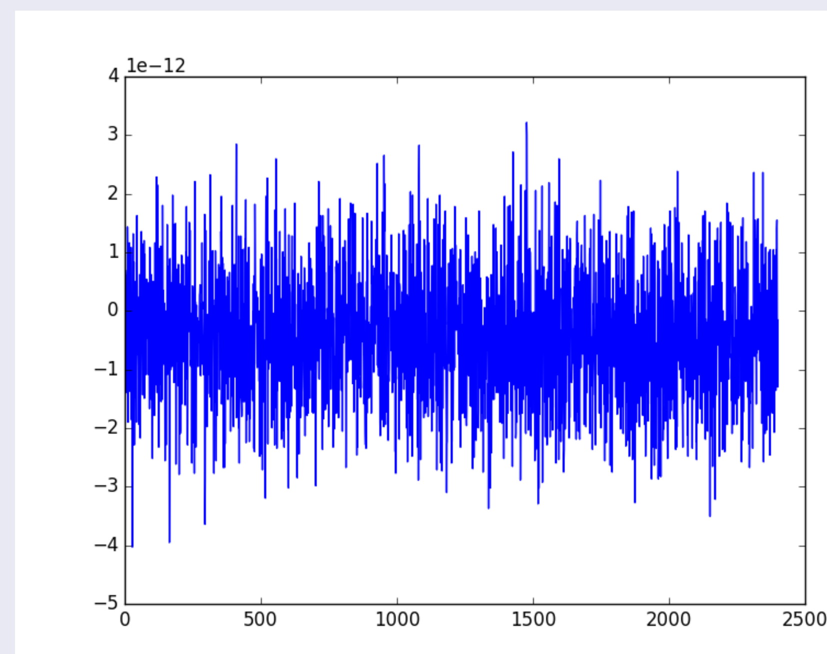
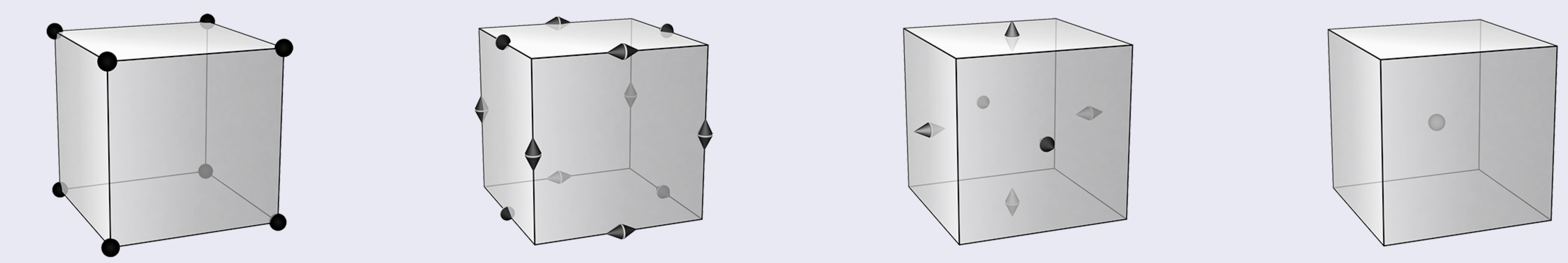


(E[t] - E[0])/E[0]*100.

DCMIP3.1 (adapted to plane), 320x30 mesh (320km x 10km domain, $\Delta x = 1$km), $\Delta t = 3$s, Lagrangian vertical coordinate, MGD-1, shown at $T = 3600$s, xz slice, 2nd order Poisson integrator

(V) "Grid Staggering" and Discretization

Choose variables in appropriate finite dimensional subspaces, and let $F = \int \hat{\mu}\mu$ (and other variants) in Poisson brackets (not shown) to obtain to obtain discrete weak form equations. Equivalent to multiplying by test functions and integrating over domain. Choose spaces and staggering as:



$s \in H^1$ | $\vec{\zeta} \in H(curl)$ | $\vec{u}, \vec{v}, \mu\dot{\eta} \in H(div)$ | $\mu, S, M_s \in L_2$

Galkerin analogue of a C/Lorenz grid
This approach gives a discretization that conserves mass, entropy and total energy to machine precision (when combined with time integrator below)

(VI) Energy-Conserving Time Integrator

Given system of ODEs in the form:

$$\frac{\partial\vec{x}}{\partial t} = \mathcal{J}(\vec{x})\frac{\delta\mathcal{H}}{\delta\vec{x}}(\vec{x})$$

where $\mathcal{J} = -\mathcal{J}^T$ (all energy-conserving spatial semi-discretizations can be written in this form), use 2nd order energy-conserving Poisson integrator from [5] (higher-order variants also available):

$$\frac{\vec{x}^{n+1} - \vec{x}^n}{\Delta t} = \mathcal{J}(\frac{x^{n+1} + x^n}{2})\int_0^1 \frac{\delta\mathcal{H}}{\delta\vec{x}}(x^n + \tau(x^{n+1} - x^n))d\tau$$

Evaluate integral on rhs via quadrature rule $\rightarrow$ exact for polynomial Hamiltonians, can be made practically exact for non-polynomial by increasing order of quadrature
Energy conservation holds even if Jacobian is simplified to yield semi-implicit method $\rightarrow$ final scheme is very similar to Crank-Nicholson semi-implicit

(VIII) Conclusions

1. Developed a hydrostatic model in Lagrangian and mass-based vertical coordinates, for slice and planar domains
2. Obtain (most of) desirable properties by combining **mimetic spatial discretization** with a **Hamiltonian formulation** and an **energy-conserving time integrator**
3. Hamiltonian formulation enables treatment of many approximations (deep/shallow, traditional/non-traditional, spherical/non-spherical) within a **unified framework**

(IX) Ongoing and Future Work

1. Hollingsworth Instability
2. Extension to block-structure grids (cubed sphere)
3. Computational efficiency: simplified Jacobian, preconditioners
4. Look at replacing S by s (Lorenz $\rightarrow$ Charney-Phillips)
5. Nonhydrostatic equations in Lagrangian and mass-based vertical coordinates
6. Vertical dispersion analysis for compatible Galerkin methods
7. Incorporation of moisture, chemistry/tracers, dissipation and associated parameterizations in a consistent way (Multicomponent, metriplectic formulation)

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