Extension of the 1981 Arakawa and Lamb Scheme to Arbitrary Grids

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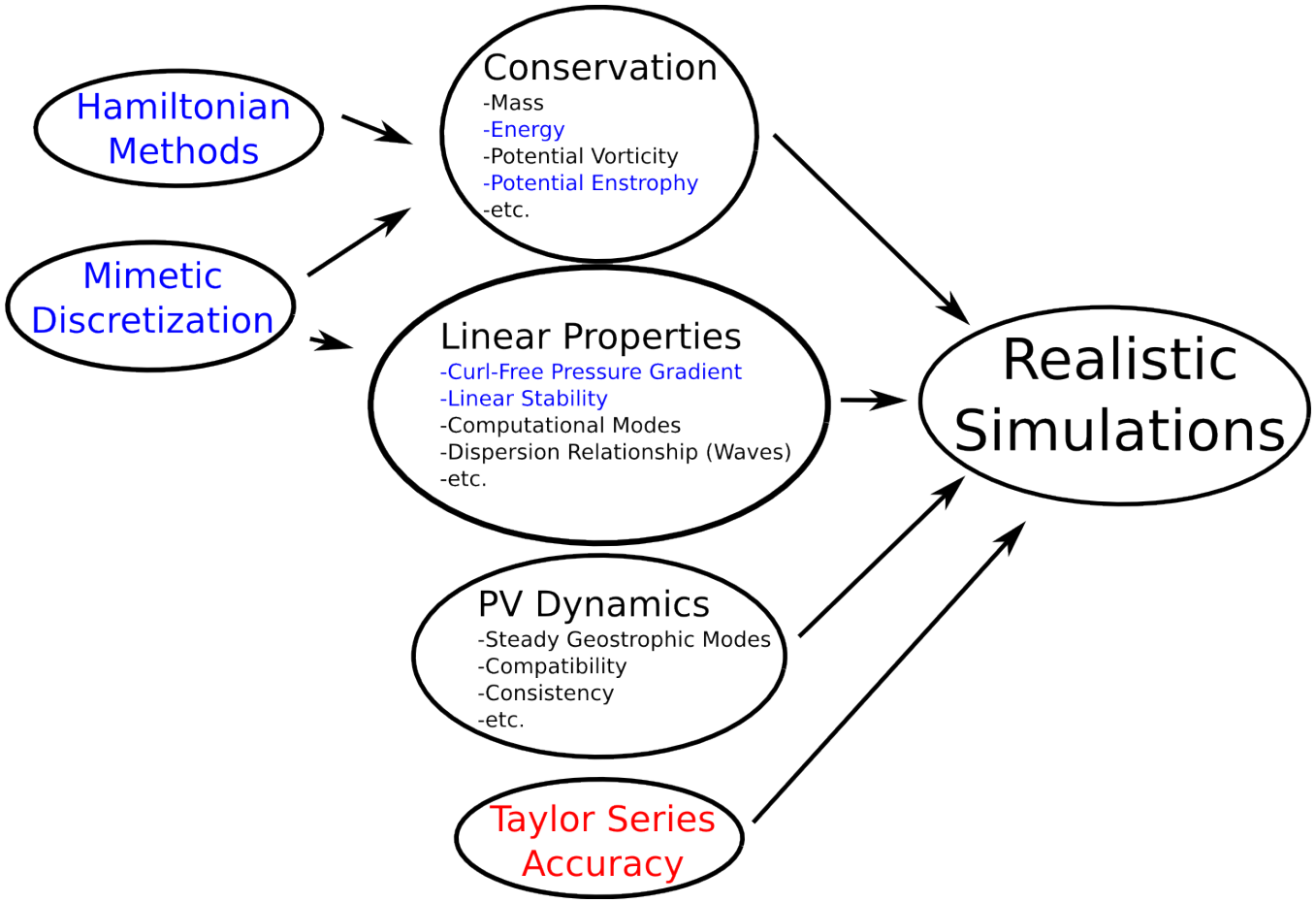
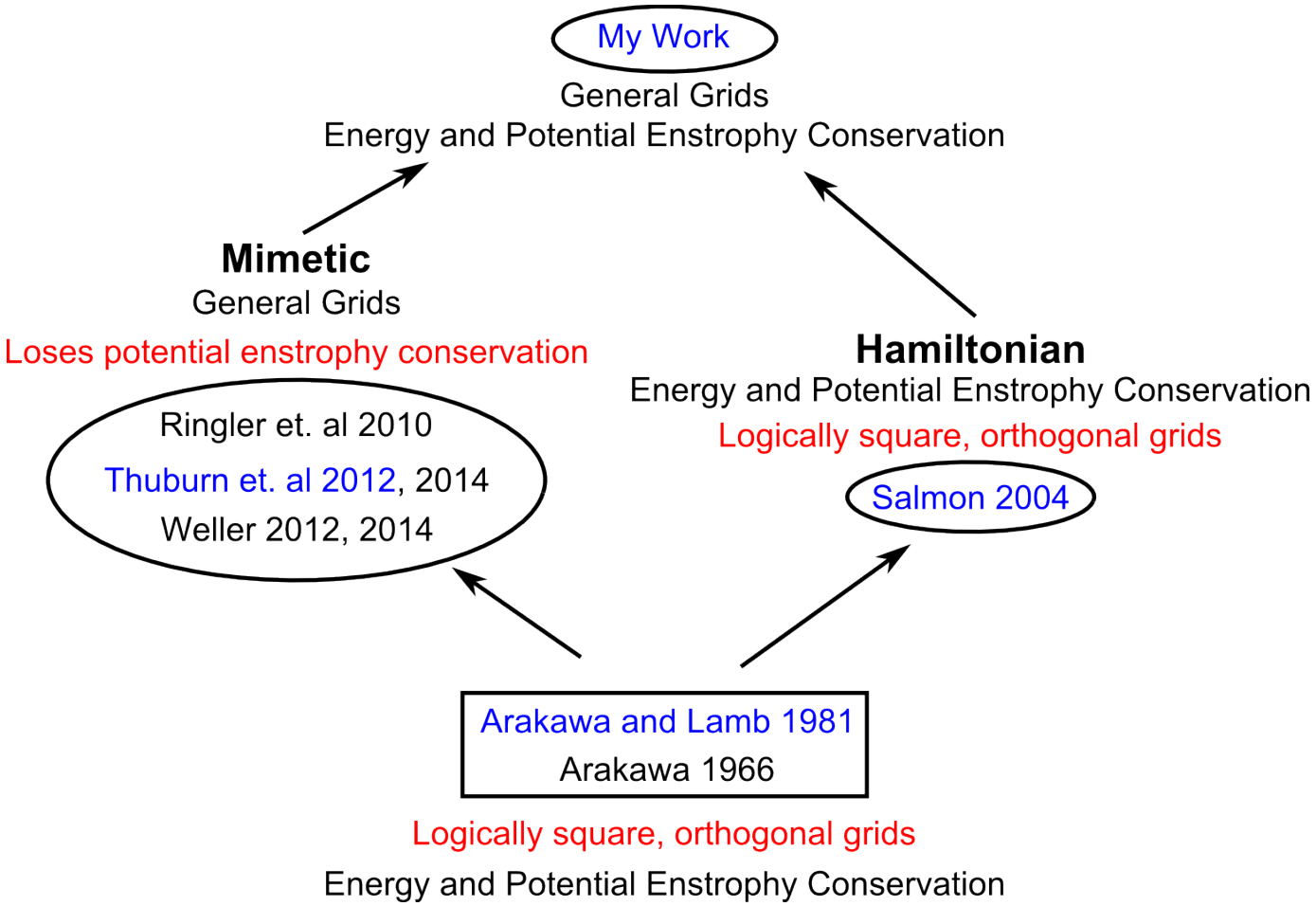
Intro

Introduction

Elements of Dynamical Core Design



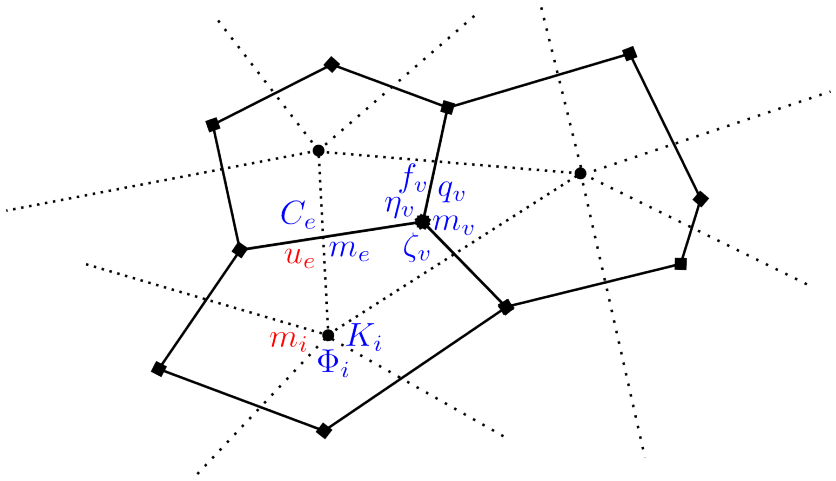
Standing on the Shoulders of Giants



AL81 on arbitrary grids

Extension of AL81 to Arbitrary Grids

Generalized C Grid Scheme



Grid Staggering

Discrete Generalized C Grid Formulation

$$\frac{\partial m_i}{\partial t} + D_2 F_e = 0$$

$$\frac{\partial u_e}{\partial t} - \mathbf{Q}(q_v, F_e) + \bar{D}_1 \Phi_i = 0$$

- Discrete variables are $m_i = \int h dA$ and $u_e = \int \vec{u} \cdot \vec{dl}$
- C grid staggering (m_i at cell centers, u_e at edges)
- Different choices for F_e, Φ_i and $\mathbf{Q}$ gives rise to a wide range of schemes (TRiSK, Thuburn et. al, Weller, AL81)

Q operator is the remaining hurdle

What is **Q**?

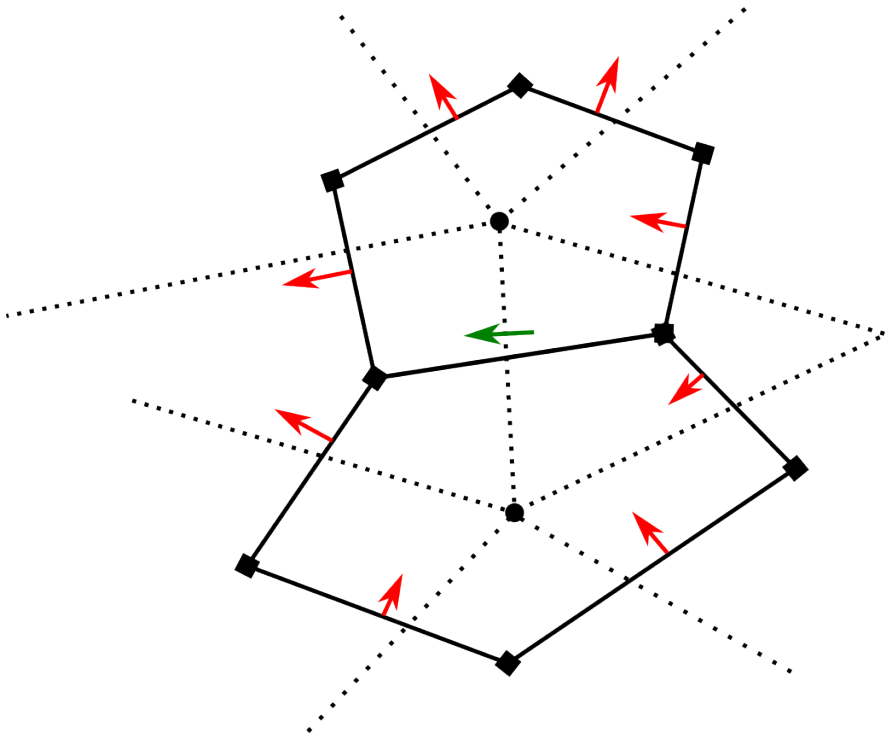


Diagram of Q operator action

1. $$q\hat{k}\times \rightarrow \mathbf{Q}$$
2. Given mass fluxes normal to primal edges and potential vorticities at vertices (black diamonds), produces PV fluxes normal to dual edges

General Form of **Q**

Following Salmon 2004, set

$$\mathbf{Q}F_e = \sum_{e' \in ECP(e)} \sum_{v \in VC(i)} q_v \alpha_{e,e',v} F_{e'}$$

What are $\alpha_{e,e',v}$'s?
Each $\alpha_{e,e',v}$ is associated with one green/red edge pair; and one blue vertex

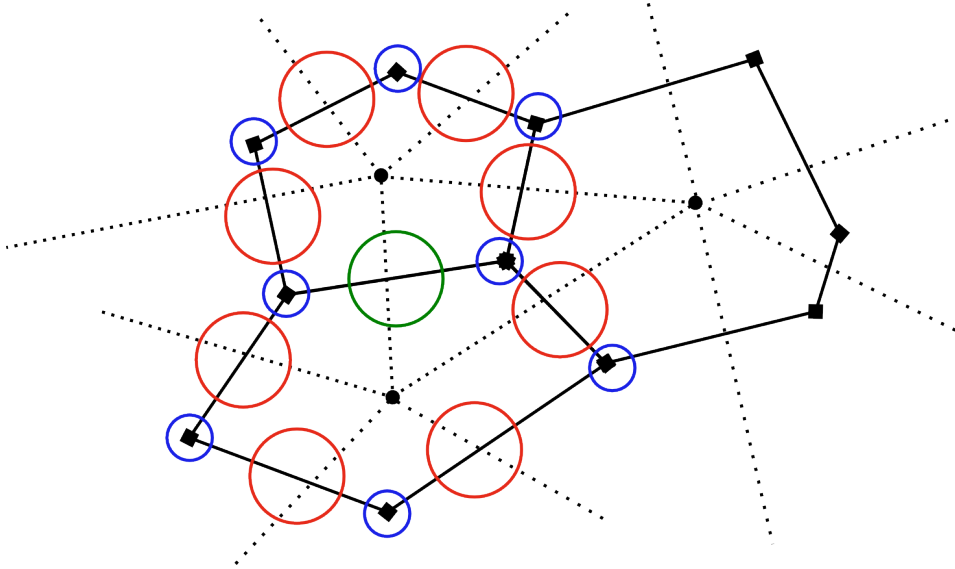


Diagram of Q operator stencil

Discrete Conservation

Can use Hamiltonian methods to derive:

Energy

$$\mathbf{Q} = -\mathbf{Q}^T \longrightarrow \alpha_{e,e',v} = -\alpha_{e',e,v}$$

Potential Enstrophy

$$\mathbf{Q}(q_v, D_1 q_v) - \bar{D}_1 \mathbf{R}^T \frac{q_v^2}{2} = 0 \qquad \forall q_v \longrightarrow$$

$$\text{linear system of equations} \longrightarrow \mathbf{A}\vec{\alpha} = \vec{b}$$

Also want $\mathbf{Q}$ to give steady geostrophic modes when q_v is constant (equivalent to PV compatibility)

Solving $\mathbf{A}\vec{\alpha} = \vec{b}$

Issue: System is too large

- Geodesic grid: 90 coefficients per cell, all coefficients are interdependent $\rightarrow$ not feasible for realistic grids
- Cubed sphere grid is similar (24 coefficients per cell)

Solution: Subsystem Splitting

$$\mathbf{A}\vec{\alpha} = \vec{b} \longrightarrow \sum_i \mathbf{A}_i \vec{\alpha}_i = \vec{b}_i$$

Split into independent subsystems for each cell!

System has been solved for various planar and spherical grids
Gives AL81 on a uniform square grid

Recap: What have I accomplished?

What has been done?

- Arakawa and Lamb 1981 extended to arbitrary grids via new **Q**
- Coefficients for new **Q** can be precomputed (efficiently)
- Unfortunately, **Q** inherits the (in)accuracy of **W** (TRiSK reconstruction operator)

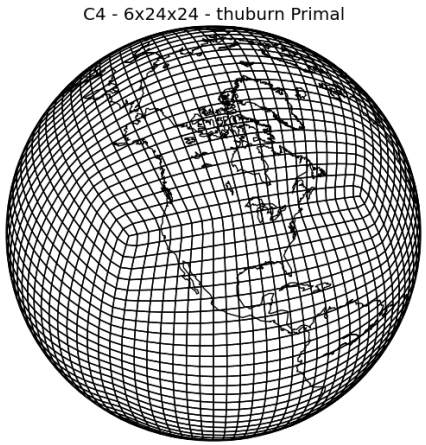
Test Results

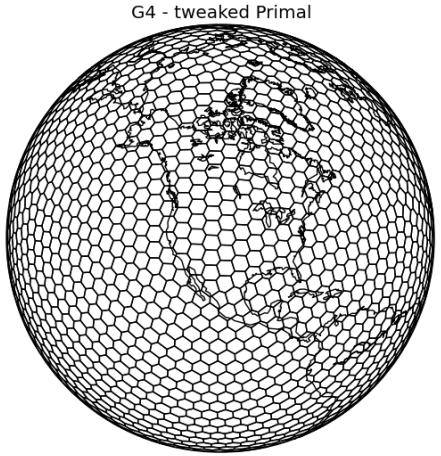
Test Case Results

Model Configuration

Settings

- Cubed Sphere: 6x384x384, 880K cells (25km resolution)
- Geodesic: G8, 640K cells (30km resolution), HR95 opt.
- 3rd Order Adams Bashford (15s CS, 22.5s Geodesic)
- 3 Variants of $\mathbf{Q}$: Energy conserving, Enstrophy conserving, Doubly conservative

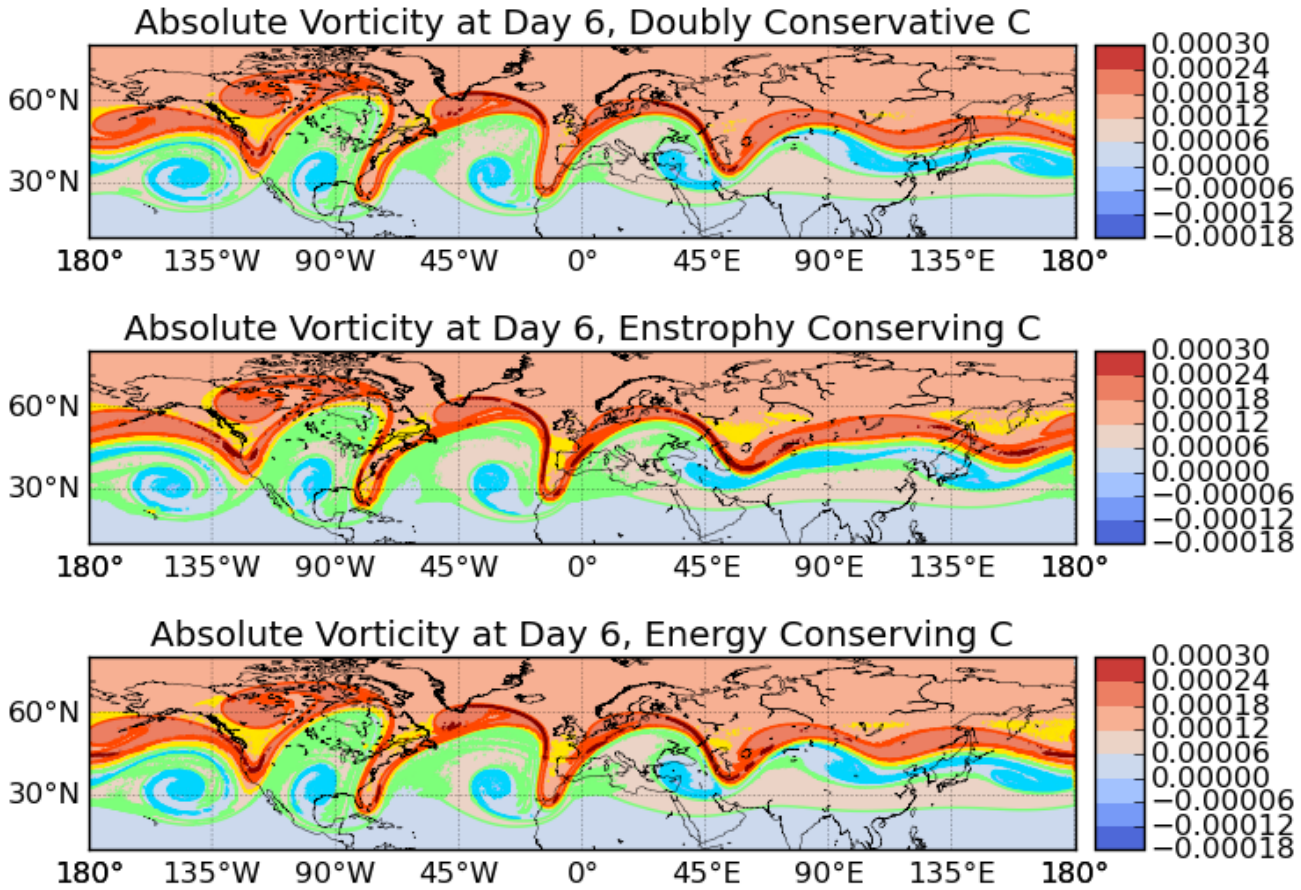




Galewsky et. al

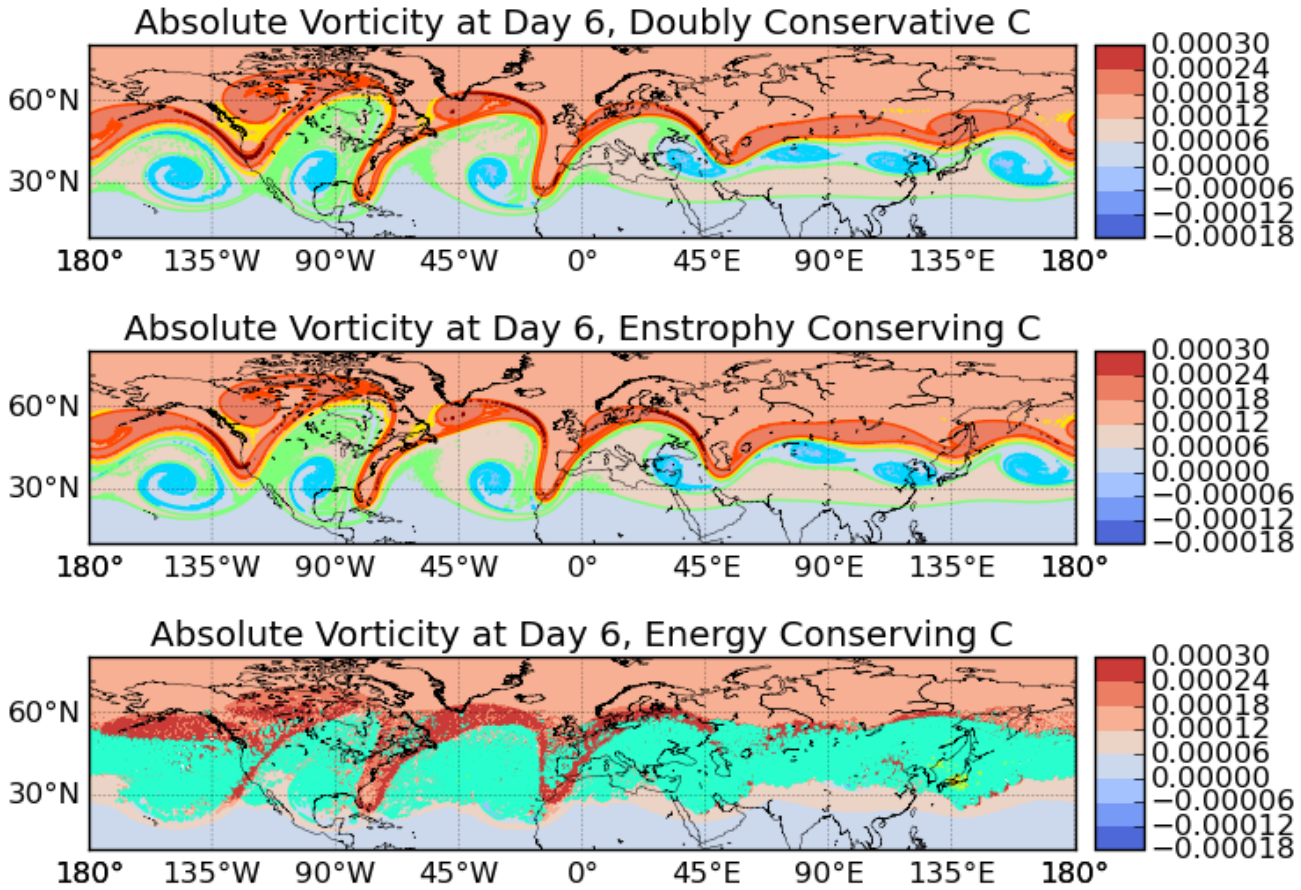
Galewsky (Unstable Jet)

Galewsky et. al (Unstable Jet)- C Grid Geodesic



1. Inactive portion of jet differs
2. Minor differences in active portion of jet

Galewsky et. al (Unstable Jet)- C Grid Cubed Sphere



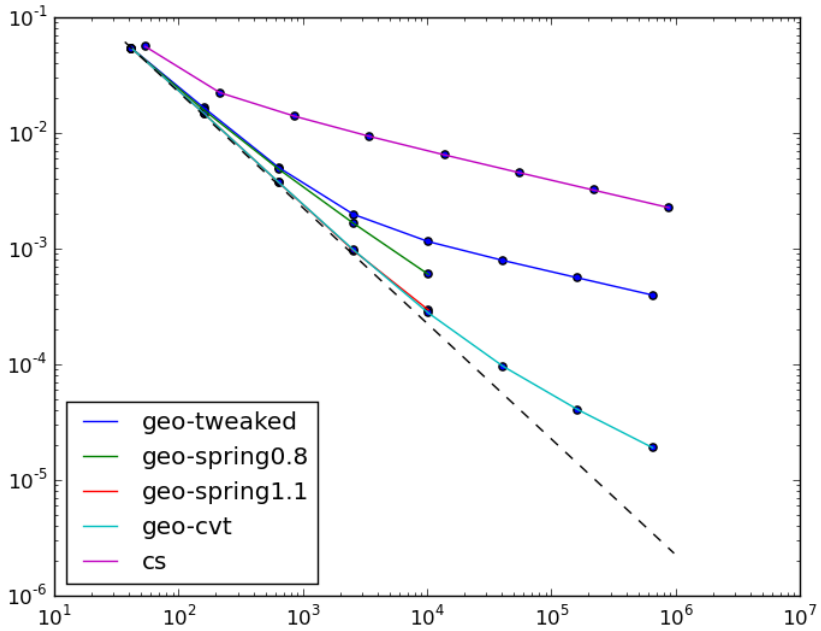
1. Inactive portion of jet differs
2. Total energy variant is unstable

Order of Accuracy

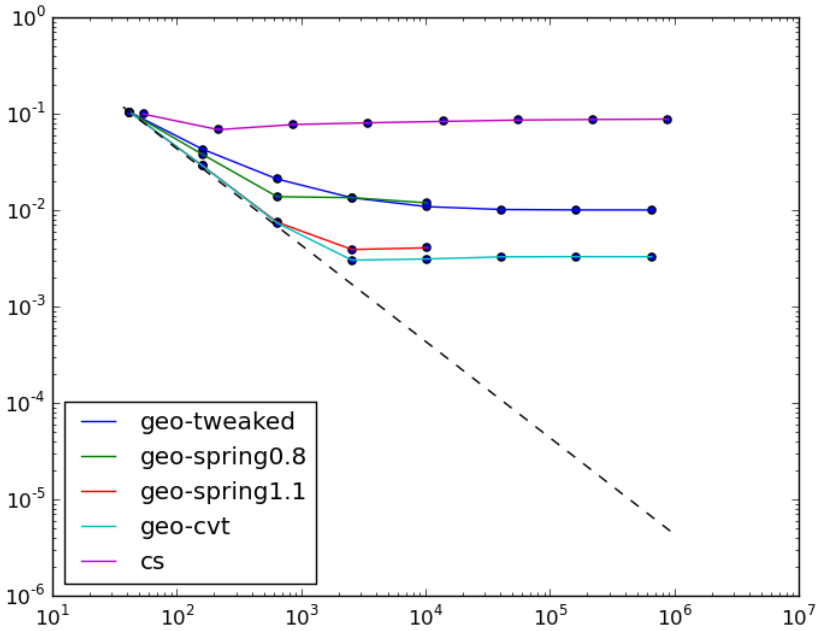
Order of Accuracy (Taylor Series Sense)

Order of Accuracy- **Q**

RMS Error



Maximum Error



Computed for $\psi = q = \cos(\theta)\sin(\lambda)$

error $= q_e \bar{D}_1 \psi_e - \mathbf{Q}(q_v, D_1 \psi_v)$

error = exact PV flux from streamfunction vs computed PV flux

All grids are inconsistent

Thuburn Test

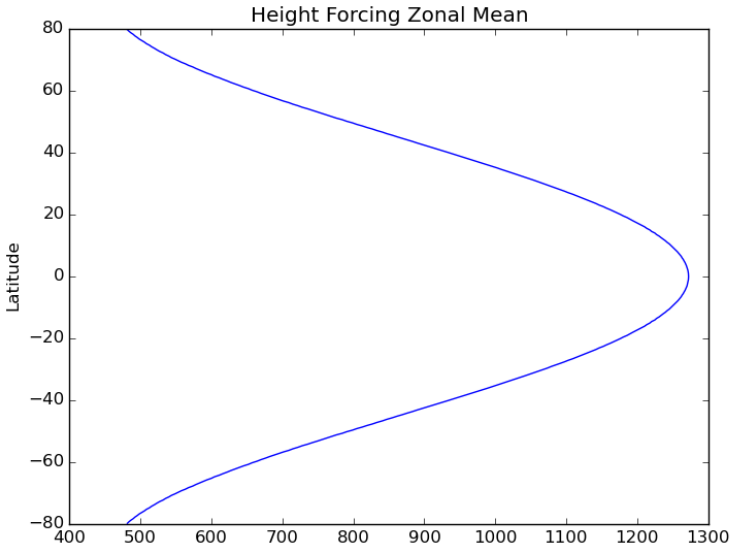
Thuburn (Forced Turbulence)

Thuburn (Forced Turbulence)- Description

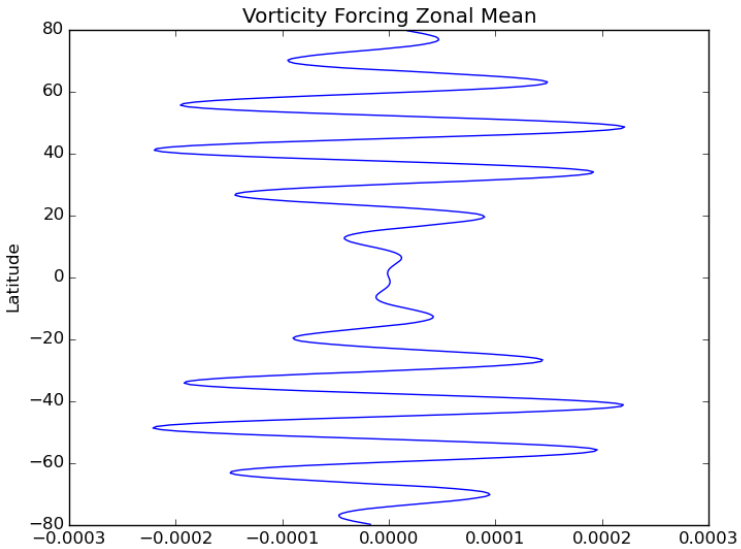
$$\frac{\partial h}{\partial t} = \cdots + \frac{h - h_{eqm}}{\tau_h}$$

$$\frac{\partial \vec{u}}{\partial t} = \cdots + \frac{\vec{u} - \vec{u}_{eqm}}{\tau_u}$$

Height Forcing

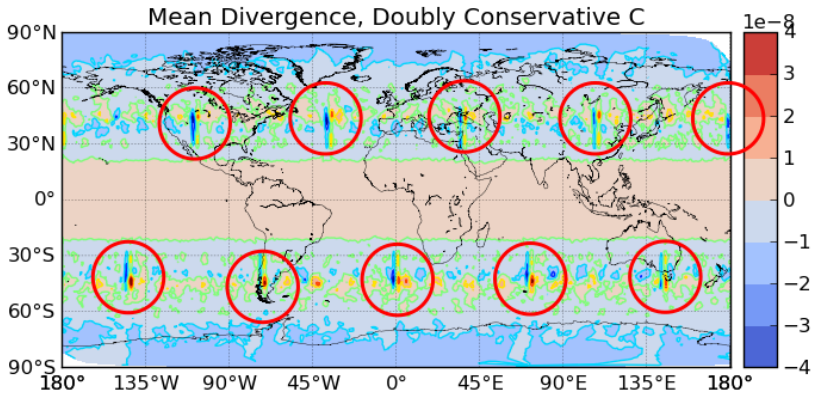


Vorticity Forcing

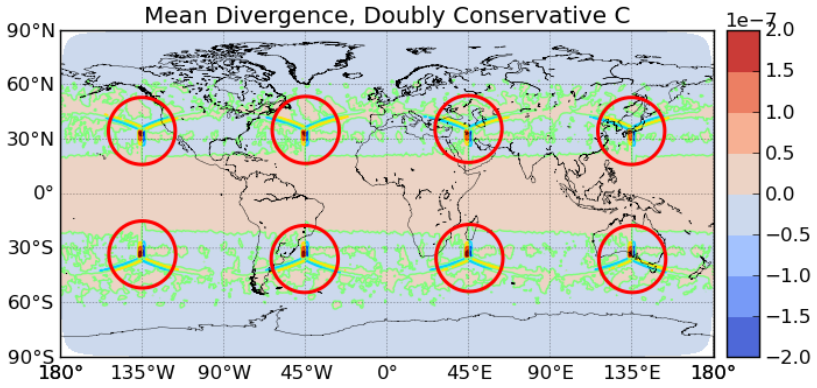


2400 days total run time: 400 days spin-up, 2000 days simulation
Run at C6/G6 resolution ($\approx$ 120km resolution)

Thuburn (Forced Turbulence)- Divergence (C Grid)



C Grid Geodesic



C Grid Cubed Sphere

-Strong grid imprinting for both geodesic and cubed-sphere grids

-Same issues seen in energy conserving and enstrophy conserving variants on both geodesic and cubed-sphere grids $\rightarrow$ strongly suggests this is due to accuracy in **W**

-Issue not seen in a Z grid model with same mimetic and conservation properties

Summary and Conclusions

Conclusions

Summary

Summary

1. Galewsky results indicate all geodesic variants, and all but total energy cubed-sphere variants are producing good results
2. **Q** is inconsistent for all grids
3. Long time tests (Thuburn forced tubrulence) show strong grid imprinting NOT present in shorter (Galewsky) tests

Future Work

1. Apply these techniques to primal-dual FE (Thuburn et. al 2015) to develop doubly conservative version of this
2. Cubed sphere accuracy- Purser optimized grid

Appendix

Fixing the C Grid Scheme

Primal-Dual Finite Elements

1. Recent work by John Thuburn and Colin Cotter
2. Low order, compound polygonal finite elements (same degrees of freedom) $\rightarrow$ all operators are consistent
3. Current version does not conserve energy or potential enstrophy
4. Can fix this with Hamiltonian approach

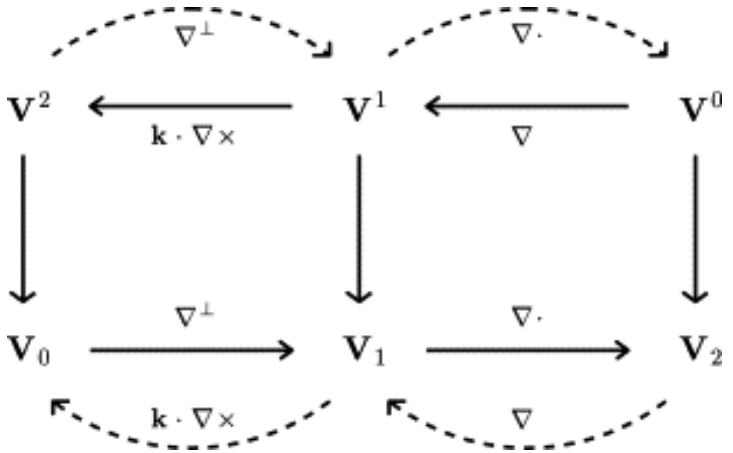


Figure from Thuburn and Cotter 2015. Bottom row is primal function spaces, top row is dual function spaces.

Discrete Hamiltonian Framework

$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & -D_2 \\ -\bar{D}_1 & \mathbf{Q} \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g(m_i, m_i)_{\mathbf{I}} + \frac{1}{2}(C_e, u_e)_{\mathbf{H}}$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I}\Phi_i \\ F_e \end{pmatrix} = \begin{pmatrix} \mathbf{I}K_i + g\mathbf{I}m_i \\ \mathbf{H}C_e \end{pmatrix}$$

$$m_v = \mathbf{R} m_i$$

$$m_v q_v = \eta_v = \zeta_v + f_v = \bar{D}_2 u_e + f_v$$

$$C_e = m_e u_e$$

$$K_i = \phi^T \frac{u_e^T \mathbf{H} u_e}{2}$$

$$\mathcal{Z} = \frac{1}{2}(q_v, \zeta_v)_{\mathbf{J}}$$

$$\frac{\delta \mathcal{Z}}{\delta \vec{x}} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$

$$\frac{\partial m_v q_v}{\partial t} - \bar{D}_2 \mathbf{Q}(q_v, F_e) = 0$$

$$\frac{\partial m_v}{\partial t} + \mathbf{R} D_2 F_e = 0$$

$$\mathbf{Q} = \frac{1}{2}\mathbf{Q_e}\mathbf{W} + \frac{1}{2}\mathbf{W}\mathbf{Q_e}$$

$$\mathbf{Q} = \mathbf{Q_e}\mathbf{W}$$

Continuous Hamiltonian Formulation

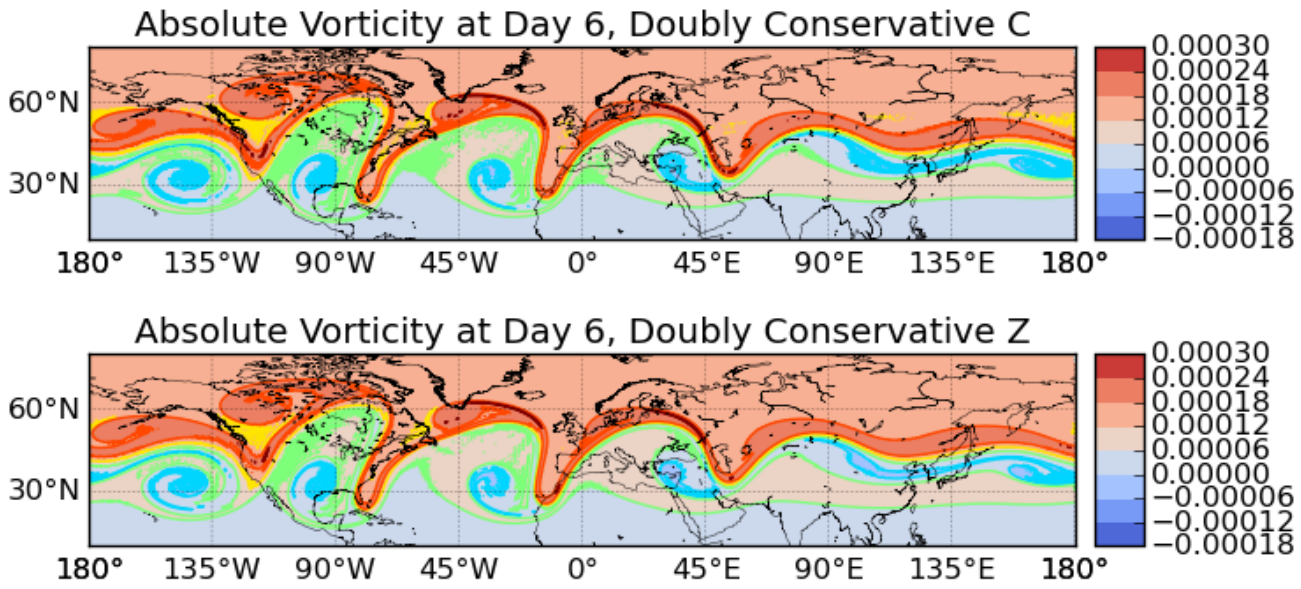
$$\vec{x} = (h, \vec{u})$$

$$\mathbb{J} = \begin{pmatrix} 0 & \vec{\nabla}\cdot \\ \vec{\nabla} & q\hat{k}\times \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2}g(h,h) + \frac{1}{2}(h\vec{u}, \vec{u})$$

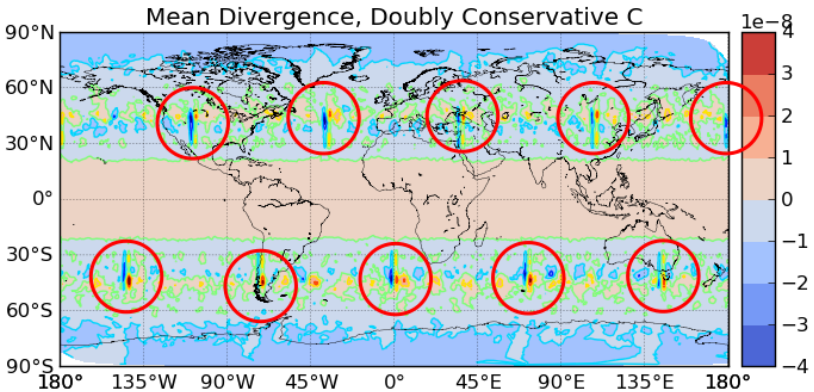
$$\frac{\delta\mathcal{H}}{\delta\vec{x}} = \begin{pmatrix} \Phi \\ \vec{F} \end{pmatrix} = \begin{pmatrix} gh + gh_s + K \\ h\vec{u} \end{pmatrix} = \begin{pmatrix} gh + gh_s + \frac{\vec{u}\cdot\vec{u}}{2} \\ h\vec{u} \end{pmatrix}$$

Galewsky et. al (Unstable Jet)- Z Grid Geodesic

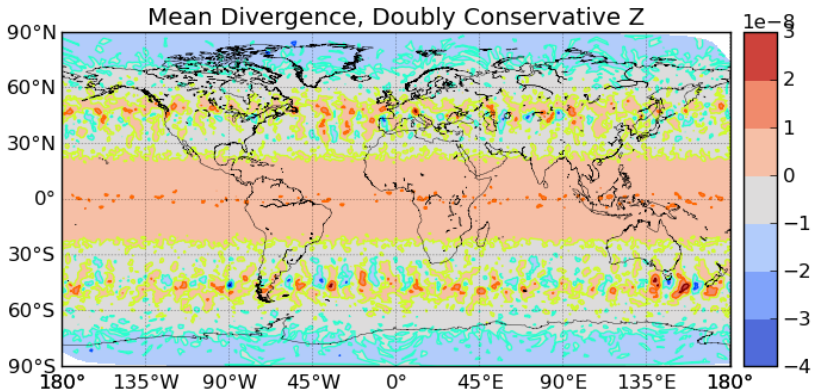


1. Almost identical results

Thuburn (Forced Turbulence)- Divergence (Z Grid)



C Grid Geodesic



Z Grid Geodesic

Z grid does not show the same grid imprinting

Possibly due to better behavior of inconsistent Z grid operator (Jacobian)?

Energy Conservation

Hamiltonian mechanics says that energy conservation requires ONLY

$$\mathbb{J}^T = -\mathbb{J}$$

which implies

$$\mathbf{Q}^T = -\mathbf{Q}$$

$$D_2^T = -\bar{D}_1{}^T$$

Only the first needs to be enforced, the second is built into DEC operators by construction. Also need $\mathcal{H}$ to be positive definite $\rightarrow \mathbf{I}, \mathbf{J}, \mathbf{H}$ symmetric positive-definite

Potential Enstrophy Conservation

Hamiltonian mechanics says that Casimir $\mathcal{Z}$ is conserved when

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

which implies that

$$D_2 D_1 q_v = 0$$

$$\mathbf{Q}(q_v, D_1 qv) - \bar{D}_1 \mathbf{R}^T \frac{q_v^2}{2} = 0$$

The first is an automatic feature of DEC operators, while the 2nd must be enforced.

DeRham Cohomology

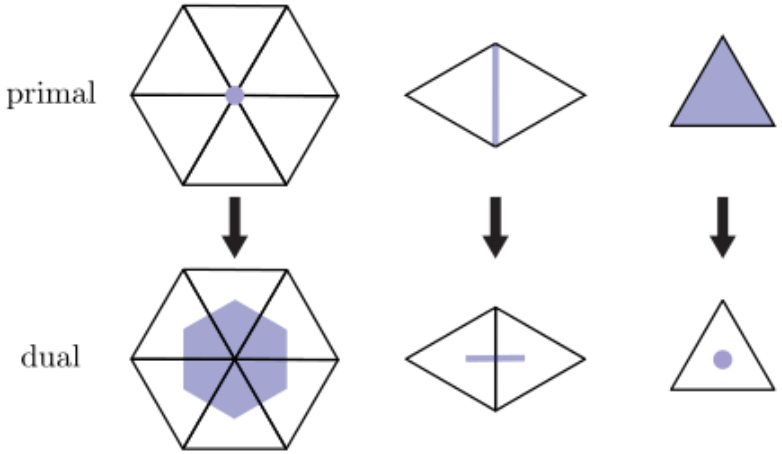
$$0 \xrightarrow{\mathbf{R}^T} 0$$

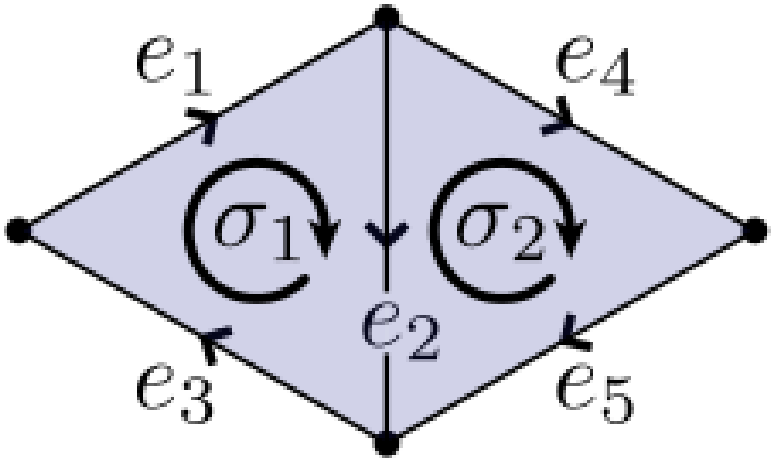
$$1 \underset{\mathbf{Q}}{\overset{\mathbf{W}}{\rightleftarrows}} 1$$

$$\underset{\text{Primal}}{2} \xrightarrow{\mathbf{R}} \underset{\text{Dual}}{2}$$

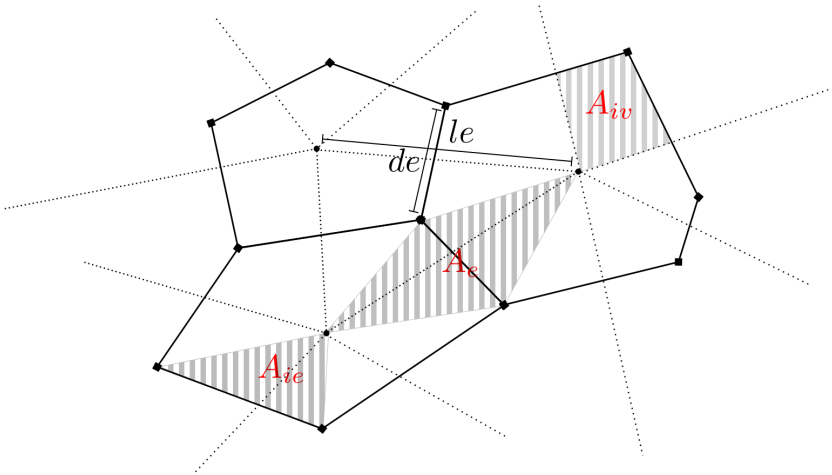
$$\begin{tikzcd}
0 \arrow[r, shift left, "\mathbf{J}^{-1}"] \arrow[d, "D_1"'] & 2 \arrow[l, shift left, "\mathbf{J}"] \\
1 \arrow[r, shift left, "\mathbf{H}^{-1}"] \arrow[d, "D_2"'] & 1 \arrow[l, shift left, "\mathbf{H}"] \arrow[u, "\bar{D}_2"'] \\
\underset{\text{Primal}}{2} \arrow[r, shift left, "\mathbf{I}"] & \underset{\text{Dual}}{0} \arrow[l, shift left, "\mathbf{I}^{-1}"] \arrow[u, "\bar{D}_1"']
\end{tikzcd}$$

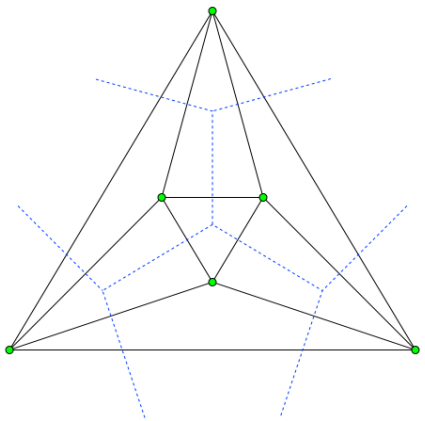
DEC Operators

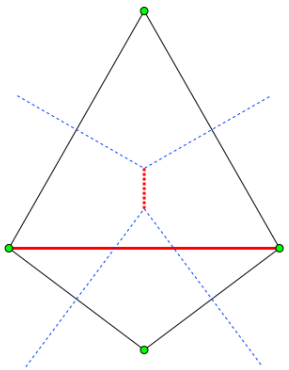




Grid Geometry







Linear Modes