Extension of the 1981 Arakawa and Lamb Scheme to Arbitrary Grids

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Intro

Introduction

Key Papers

Arakawa and Lamb 1981

A Potential Enstrophy and Energy Conserving Scheme for the Shallow Water Equations

AKIO ARAKAWA AND VIVIAN R. LAMB[1]

Salmon 2004

Poisson-Bracket Approach to the Construction of Energy- and Potential-Enstrophy-Conserving Algorithms for the Shallow-Water Equations

RICK SALMON

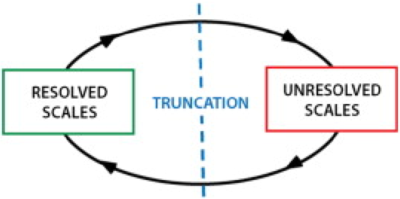
Thuburn, Cotter and Dubos 2012

A FRAMEWORK FOR MIMETIC DISCRETIZATION OF THE ROTATING SHALLOW-WATER EQUATIONS ON ARBITRARY POLYGONAL GRIDS*

J. THUBURN† AND C. J. COTTER‡

Broad Research Overview: Dynamical Cores

1. Develop discrete models of the atmosphere
2. Dynamical core: deals with "resolved processes"
3. Parameterizations: deal with "unresolved processes"
4. Model: dynamical core + parameterizations



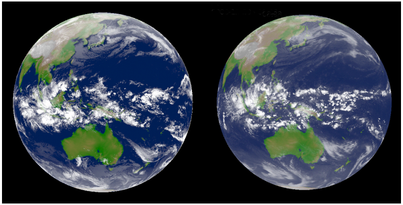
Key Principles of Numerical Modeling

1. Not solving arbitrary PDEs- physical system
2. No analytic solutions
3. Differential equations $\rightarrow$ algebraic equations
4. **Do algebraic solutions have the same properties as the differential solutions?**

Philosophy of Dynamical Core Design

1. Models should respect underlying physics
   1. Energetics
   2. PV dynamics
   3. Wave dynamics
   4. **Conserved quantities- energy, potential enstrophy**
   5. **Mimetic properties**

$$\vec{\nabla} \times \vec{\nabla}\phi = 0$$



Shallow Water Equations

Vector Invariant Shallow Water Equations

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{u}) = 0$$

$$\frac{\partial \vec{u}}{\partial t} + q\hat{k} \times (h\vec{u}) + \vec{\nabla}\Phi = 0$$

$$\vec{x} = (h, \vec{u})$$

$$\frac{\partial \vec{x}}{\partial t} = \mathbb{J}\frac{\delta \mathcal{H}}{\delta \vec{x}}$$

$$\mathbb{J} = \begin{pmatrix} 0 & -\vec{\nabla}\cdot \\ -\vec{\nabla} & -q\hat{k}\times \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2}g(h, h) + \frac{1}{2}(\vec{F}, \vec{u})$$

$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \Phi \\ h\vec{u} \end{pmatrix}$$

Shallow Water Equations: (Subset of) Properties

Mimetic

No Spurious Vorticity Production:

$$\vec{\nabla} \times \vec{\nabla} \phi = 0$$

Pressure Gradient is Energy Conserving:

$$(\vec{\nabla})^* = -\vec{\nabla} \cdot$$

Conserved Quantities

Total Energy $\int_\Omega \frac{gh^2}{2} + \frac{h|\vec{u}|^2}{2}$

Potential Enstrophy $\int_\Omega h \frac{q^2}{2}$

Arakawa and Lamb 1981 Scheme: Desirable Properties

(A Subset of) Desirable Properties

1. No spurious vorticity production (curl-free gradient)
2. Energy-conserving pressure gradient force (divergence and gradient are adjoints)
3. Total energy and potential enstrophy conservation

Mimetic

$$\bar{D}_2 \bar{D}_1 = 0$$

$$D_2^T = -\bar{D}_1$$

Conservation

$$\mathbb{J} = -\mathbb{J}^T$$

$$\mathbb{J} \frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

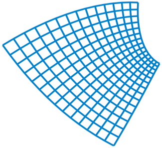
Arakawa and Lamb 1981 Scheme: Limitations

(A Subset of) AL81 Shortcomings

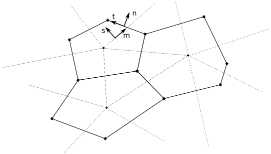
1. Restricted to logically square, orthogonal grids

TRiSK: Ringler, Skamarok, Klemp, Thuburn, Cotter, Dubos

1. General, non-orthogonal polygonal grids
2. **Choose between total energy and potential enstrophy conservation**



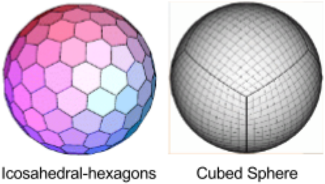
Logically square, orthogonal grid

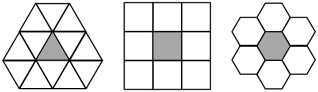


General, non-orthogonal grid

Recap: What am I trying to accomplish?

1. How can Arakawa and Lamb 1981 be extended to **arbitrary, non-orthogonal polygonal grids**?
2. In a way that preserves all of its desirable properties, and does not add new limitations?





AL81 on arbitrary grids

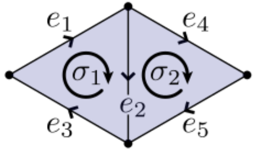
Extension of AL81 to Arbitrary Grids

Mimetic Methods (Discrete Exterior Calculus)

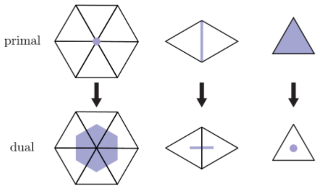
Mimetic Operators

$$\vec{\nabla} \times \vec{\nabla}\phi = 0 \iff \bar{D}_2 \bar{D}_1 = 0$$

$$(\vec{\nabla})^* = -\vec{\nabla}\cdot \iff D_2 = -\bar{D}_1{}^T$$



(Discrete) Exterior Derivative



Primal-Dual Grid

Conservation Laws (Hamiltonian Mechanics)

Hamiltonian (Energy)

$$\mathbb{J}^T = -\mathbb{J}$$

$\mathcal{H}$ is positive definite

Casimirs (Potential Enstrophy)

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

Discrete Conservation

Enforce these conditions in discrete case $\rightarrow$ conservation!

Recap: Conservative, Mimetic Methods

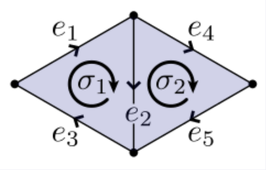
Conservation Laws

$\mathcal{H}$ is positive definite

$$\mathbb{J} = -\mathbb{J}^T$$

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0$$

Mimetic Properties



(Discrete) Exterior Derivative

Conservative, Mimetic Methods

1. Use mimetic operators to build a discrete (quasi-)Hamiltonian system
2. Unifies two important lines of research

Generalized C Grid Discretization: Hamiltonian-DEC

- Discrete variables are $m_i = \int h dA$ and $u_e = \int \vec{u} \cdot \vec{dl}$
- C grid staggering (m_i at cell centers, u_e at edges)

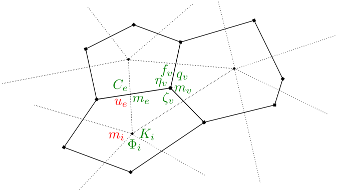
$$\vec{x} = (m_i, u_e)$$

$$\mathbb{J} = \begin{pmatrix} 0 & D_2 \\ \bar{D}_1 & \mathbf{Q} \end{pmatrix}$$

$$\mathcal{H} = \frac{1}{2} g (m_i, m_i)_{\mathbf{I}} + \frac{1}{2} (C_e, u_e)_{\mathbf{H}}$$

$$\mathcal{Z} = \frac{1}{2} (\zeta_v, \eta_v)_{\mathbf{J}}$$

$$q_v = \frac{\eta_v}{m_v} = \frac{\zeta_v + f_v}{m_v} = \frac{\bar{D}_2 u_e + f_v}{\mathbf{R} m_i}$$



$$\frac{\delta \mathcal{H}}{\delta \vec{x}} = \begin{pmatrix} \mathbf{I} \Phi_i \\ F_e \end{pmatrix} = \begin{pmatrix} \mathbf{I} K_i + g \mathbf{I} m_i \\ \mathbf{H} C_e \end{pmatrix}$$

$$\frac{\delta \mathcal{Z}}{\delta \vec{x}} = \begin{pmatrix} -\mathbf{R}^T \frac{q_v^2}{2} \\ D_1 q_v \end{pmatrix}$$

Q operator is the remaining hurdle

General Form of **Q**

Following Salmon 2004, set

$$\mathbf{Q} F_e = \sum_{e' \in ECP(e)} \sum_{v \in VC(i)} q_v \alpha_{e,e',v} F_{e'}$$

What are $\alpha_{e,e',v}$'s?

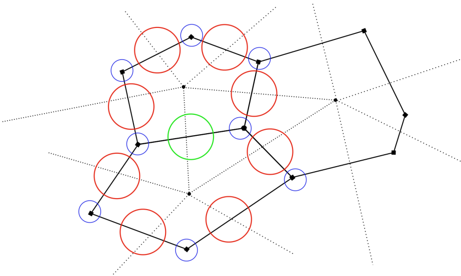


Diagram of Q operator stencil

Discrete Conservation

Energy

$$\mathbb{J}^T = -\mathbb{J} \longrightarrow \mathbf{Q} = -\mathbf{Q}^T \longrightarrow \alpha_{e,e',v} = -\alpha_{e',e,v}$$

Potential Enstrophy

$$\mathbb{J}\frac{\delta \mathcal{Z}}{\delta \vec{x}} = 0 \longrightarrow \text{ linear system of equations } \longrightarrow \mathbf{A}\vec{\alpha} = \vec{b}$$

Also want $\mathbf{Q} \rightarrow \mathbf{W}$ when $q_v = \text{const}$ (steady geostrophic modes)

Solving $\mathbf{A}\vec{\alpha} = \vec{b}$

Issue: System is too large

- Geodesic grid: 90 coefficients per cell, all coefficients are interdependent $\rightarrow$ not feasible for realistic grids
- Cubed sphere grid is similar (24 coefficients per cell)

Solution: Subystem Splitting

$$\mathbf{A}\vec{\alpha} = \vec{b} \longrightarrow \sum_i \mathbf{A}_i \vec{\alpha}_i = \vec{b}_i$$

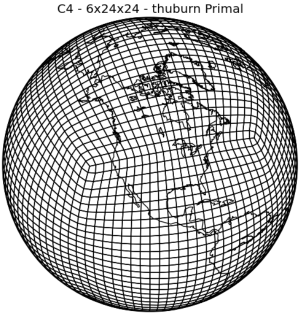
Split into independent subsystems for each cell!

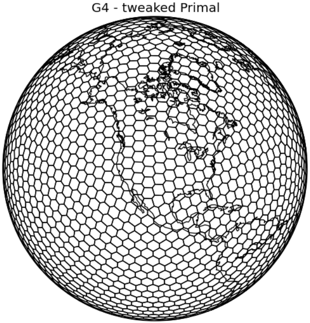
System has been solved for various planar and spherical grids

Recap: What have I accomplished?

What has been done?

- Arakawa and Lamb 1981 extended to arbitrary grids via new **Q**
- Coefficients can be precomputed (efficiently)





Test Results

Test Case Results

Variants of **Q** Operator

Total Energy

$$\mathbf{Q}F_e = \frac{1}{2}q_e\mathbf{W}F_e + \frac{1}{2}\mathbf{W}q_eF_e$$

Works for ANY choice of q_e (APVM, CLUST, etc.)

Potential Enstrophy

$$\mathbf{Q}F_e = q_e\mathbf{W}F_e$$

Requires that q_e is arithmetic mean

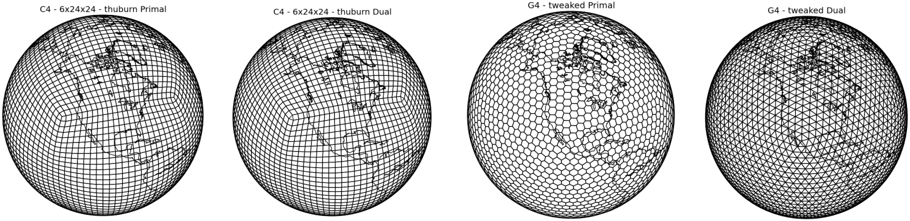
Total Energy and Potential Enstrophy

$$\mathbf{Q}F_e = Q(q_v, F_e)$$

Grids

Grids

- Cubed Sphere: 6x192x192, 220K cells (49km resolution)
- Geodesic: G7, 160K cells (60km resolution)



Operators

Operators

$$\mathbf{R} = \sum_{v \in VC(i)} \frac{A_{iv}}{A_i}$$

$$\mathbf{\Phi} = \frac{A_{ie}}{A_e} + \frac{A_{je}}{A_e}$$

$$\mathbf{I} = \frac{1}{A_i}$$

$$\mathbf{J} = \frac{1}{A_v}$$

$$\mathbf{H} = \frac{le}{de}$$

H different for cubed-sphere

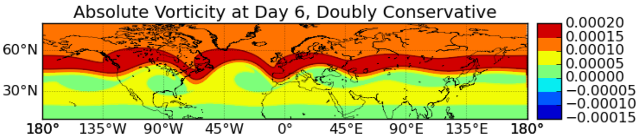
Time Stepping

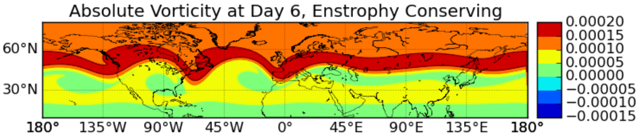
- Adams-Bashford 3rd Order (30s cubed-sphere, 45s geodesic)
- 10 simulated days, output every 6 hours

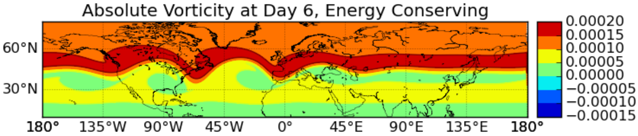
Dissipation

- $\vec{\nabla}^2 \vec{u}$ specified in Galewsky et. al added to that test case
- Schemes stable without it

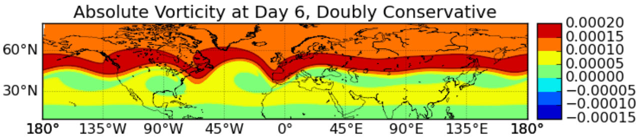
Galewsky et. al (Unstable Jet)- Geodesic (Tweaked)

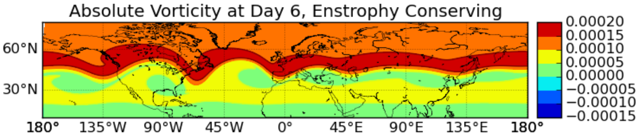


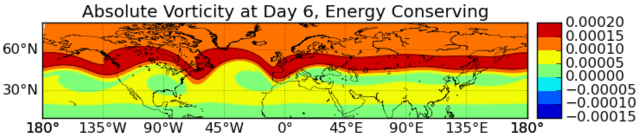




Galewsky et. al (Unstable Jet)- Cubed Sphere (Thuburn)

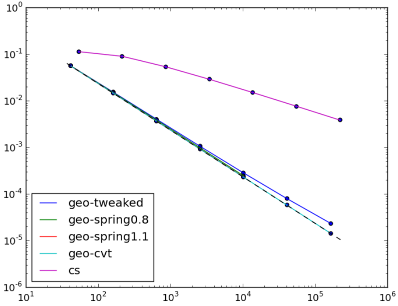




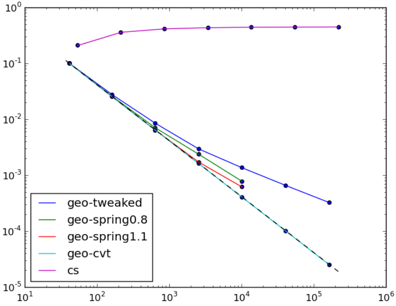


Order of Accuracy- Laplacian on Primal

L_2 Error



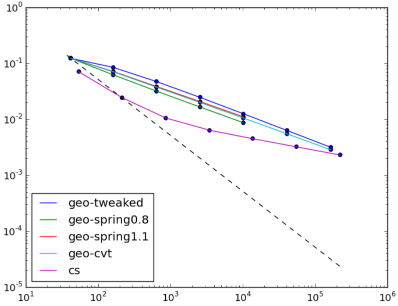
L_∞ Error



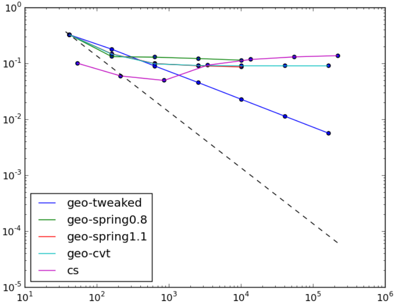
Computed for $\psi = \cos(\theta)\sin(\lambda)$

Order of Accuracy- Laplacian on Dual

L_2 Error



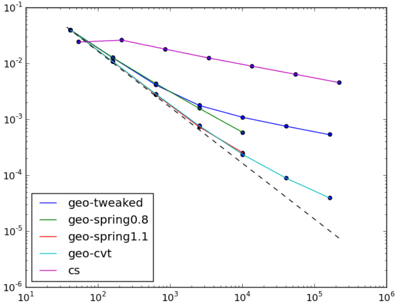
L_∞ Error



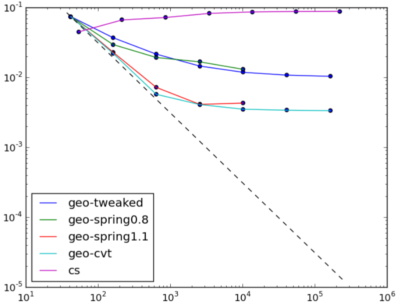
Computed for $\psi = \cos(\theta)\sin(\lambda)$

Order of Accuracy- **Q** (also **W**)

L_2 Error



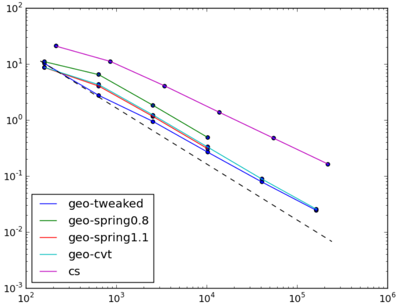
L_∞ Error



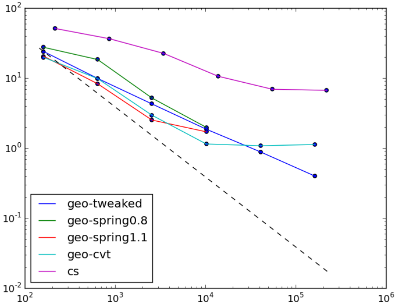
Computed for $\psi = q = \cos(\theta)\sin(\lambda)$
error $= q_e \bar{D}_1 \psi_e - \mathbf{Q}(q_v, D_1 \psi_v)$
W errors are very similar

Order of Accuracy- TC2

L_2 Error



L_∞ Error



Height errors

Conclusions

Conclusions

Summary and Conclusions

Conclusions

1. Hamiltonian + Discrete Exterior Calculus → AL81 on arbitrary polygonal grids
2. Preserves all desirable properties of AL81

Future Work

1. Consistent **W** (and **Q**)
2. Anticipated Potential Vorticity + other dissipation options
3. Optimization of grid properties (esp. cubed sphere)
4. Effects of extra Rossby modes on geodesic grid?
5. Comparison to Z-grid scheme (Salmon 2007)